Lecture 13: Extreme Events, Lévy Stability and Fractional Calculus

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1 Extreme Events

Consider \( N \) i.i.d. random variables with PDF \( p(x) \), i.e., an \( N \)-step random walk. The question we ask now is: How is the largest step up to time \( N \) distributed? Here we focus on fat-tail random variables with PDF

\[
p(x) \sim \frac{A}{x^{1+\alpha}} \quad \text{as } x \to \infty.
\]

For example, we may take \( p(x) = l_\alpha(x) \).

The outcomes can now be ordered

\[
\Delta x_{(1)} \leq \Delta x_{(2)} \leq \ldots \leq \Delta x_{(N)},
\]

so that the largest value cumulative distribution function is defined as

\[
F_N(x) \equiv \text{Prob}\left(\Delta x_{(N)} \leq x\right).
\]

If the number of steps \( N \) is large enough, the largest outcome is always sampled in the tail, where the CDF is

\[
P(x) \simeq 1 - \frac{A}{\alpha x}\alpha.
\]

Then, for the largest step we obtain

\[
F_N(x) = [P(x)]^N.
\]

This equation merely states the simple fact that the largest step is smaller than \( x \) if and only if all steps are smaller than \( x \). For \( N \to \infty \), and \( \Delta x_{(N)} \to \infty \),

\[
F_N(x) \simeq \left[1 - \frac{A}{\alpha x}\alpha\right]^N \simeq \exp\left[-\frac{NA}{\alpha x}\alpha\right].
\]

The expression in the exponent suggests the following rescaling:

\[
z_N = \frac{\Delta x_{(N)}}{(AN/\alpha)^{1/\alpha}}. \tag{7}
\]
Figure 1: Fréchet distribution is characterized by an essential singularity at the origin and a power-law tail.

Then we rewrite (6) as

$$F_N(z) \sim \exp \left[-z^{-\alpha}\right].$$

(8)

This is called Fréchet distribution\(^1\) (introduced in 1926).

Recall now that in a Lévy flight with \(p(x) = l_\alpha(x, a) \propto ax^{-(1+\alpha)}\), the position after \(N\) steps is distributed like

$$P_N(x) = \frac{1}{N^{1/\alpha}} l_\alpha \left( \frac{x}{N^{1/\alpha}}, a \right),$$

(9)

i.e., it has a characteristic width of \(N^{-1/\alpha}\). Behold that the extremal value \(z_N\) also scales like \(N^{-1/\alpha}\). This interesting phenomenon is a direct consequence of the power-law distribution which has no scale and therefore in a typical Lévy flight, all sizes of steps are present. The total displacement is therefore of the same order of magnitude as the largest step in the walk (see Fig. 2).

2 Lévy Stability

What are the possible limiting distributions for a sum of i.i.d. random variables? By now, we know the answer to this question for a quite broad class of distributions, with \(\text{var}(\Delta x) < \infty\), that converge to a Gaussian in accordance with the Central Limit Theorem. The analog of the CLT for fat-tail distributions are the so-called Lévy Stability Laws, that are the subject of the present section.

\(^1\)A nice summary of extreme values distributions, also called Fisher-Tippett distributions can be found online at http://mathworld.wolfram.com/Fisher-TippettDistribution.html
Figure 2: A typical Lévy flight: the maximal step and the entire walk extent are of the same order of magnitude.

For i.i.d. variables,

$$P_N(k) = [P(k)]^N.$$

As we have seen in several case studies before, short-time correlations do not influence the long-time behavior of $P_N(x)$; their main effect is the rescaling of the time unit in terms of “correlation time” $n_c$, so that the sum of $N$ r. v. acts like a sum of $N/n_c$ independent r. v. More generally,

$$P_{n \times m}(k) \sim [P_n(k)]^m, \quad n \gg n_c. \tag{11}$$

For i.i.d. variables, this equation is exact.

In what follows, we assume zero mean for $\Delta x_n$. Then, rescaling

$$z_N = \frac{X_N}{a_N}, \tag{12}$$

we find that $F_N(z)$, which is the PDF for $z_N$, satisfies

$$\hat{F}_{n \times m}(a_{mn}k) = [\hat{F}_n(a_n k)]^m, \tag{13}$$

or

$$\hat{F}_{n \times m}(\frac{a_{mn}}{a_n} k) = [\hat{F}_n(k)]^m. \tag{14}$$

Let $\hat{F}_N(k)$ converge to some limit $\hat{F}(k)$ as $N \to \infty$, where $\hat{F}(k)$ is a nontrivial characteristic function. Then

$$\lim_{n \to \infty} \frac{a_{mn}}{a_n} = c_m \quad (m \text{ is fixed}). \tag{15}$$

Thus, (14) reduces to the following functional equation

$$\hat{F}_m(c_m k) = [\hat{F}(k)]^m. \tag{16}$$
What are the possible solutions of this equation? To answer this, we use a scaling argument. The scaling constants $a_N$ can be expressed as

$$a_N = N^{1/\alpha} L_N,$$

where $\alpha$ is some constant and $L_N$ is a slowly varying function of $N$, e. g. $L_N = (\log N)^\mu$.

Proof:

$$\frac{a_{mn}}{a_n} = \frac{(m \times n)^{1/\alpha}}{n^{1/\alpha}} \frac{L_{m \times n}}{L_n} \to c_m = m^{1/\alpha}.$$

Hence, (16) becomes

$$\hat{F}_m(m^{1/\alpha} k) = \left[ \hat{F}(k) \right]^{m}.$$

The only possible solutions of this equation satisfying $\hat{F}(0) = 1$ and $\hat{F}(\infty) = 0$ have the form

$$\hat{F}(k) = \exp \left[ -\nu \ \text{sign}(k) |k|^\alpha \right].$$

If the distribution is symmetric, then

$$\hat{F}(-k) = \hat{F}(k).$$

Since the characteristic function is in this case sign-independent, we obtain

$$\hat{F}(k) = \exp \left[ -a |k|^\alpha \right] = \hat{\lambda}_\alpha(a, k),$$

i. e. the Lévy distribution characteristic function. More generally, we have

$$\hat{F}(-k) = \hat{F}^*(k),$$

then rewriting $\nu \ \text{sign}(k) = c_1 + i c_2 \ \text{sign}(k)$ we find that the general form of the limiting distribution reads

$$\hat{F}(k) = \exp \left[ -a |k|^\alpha \left( 1 - i \beta \tan \left( \frac{\alpha \pi}{2} \right) \ \text{sign}(k) \right) \right] \equiv \hat{\lambda}_{\alpha, \beta}(a, k),$$

This is a three-parameter distribution; the parameters $0 < \alpha \leq 2$ and $-1 \leq \beta \leq 1$ determine the shape of the distribution and $a$ determines the width.

The function $\hat{\lambda}_{\alpha, \beta}(a, k)$ thus produces the limiting distribution for a much broader class of random walks.

2.1 Basins of Attraction

In the above context, several results have been obtained regarding the basins of attraction of $l_{\alpha, \beta}(a, x)$ that are worth mentioning here. We state these theorems without a proof.

**Theorem 1 (Gnedenko-Doeblin)** The distribution $p(x)$ is the basin of attraction of $l_{\alpha, \beta}(a, x)$, i. e.

$$\frac{1}{a_N} P_N \left( \frac{x}{a_N} \right) \to l_{\alpha, \beta}(a, x)$$

if and only if

the CDF $P(x) = \int_{-\infty}^{x} p(x') dx'$ satisfies
1. \[
\lim_{x \to \infty} \frac{p(-x)}{1 - p(x)} = \frac{1 - \beta}{1 + \beta}
\] (26)

2. \(\forall r > 0,\)
\[
\lim_{x \to \infty} \frac{1 - p(x) + p(-x)}{1 - p(rx) + p(-rx)} = r^\alpha
\] (27)

**Theorem 2 (Gnedenko)** The distribution \(p(x)\) is the basin of attraction of \(l_{\alpha,\beta}(a, x)\) with \(a_N = N^{1/\alpha}\) if
\[
p(x) \sim \frac{A_+}{|x|^{1+\alpha}} \quad x \to \pm \infty,
\] (28)

where
\[
\beta = \frac{A_+ - A_-}{A_+ + A_-}.
\] (29)

### 3 The Continuum Limit and Fractional Calculus

For a Lévy flight, we recall that
\[
\hat{P}_N(k) \sim \exp(-Na|k|^\alpha) = \exp\left[-\frac{a}{\tau}|k|^\alpha t\right],
\] (30)

where, as usual, we define the continuum limit by \(t = N\tau\). Then, \(\rho(x, t) = P_N(x)\) and the “diffusion equation” in the Fourier space reads
\[
\frac{\partial \hat{\rho}}{\partial t} = -\frac{a}{\tau}|k|^\alpha \hat{\rho}.
\] (31)

The RHS of this equation helps to define the Riesz fractional derivative of \(\rho\) as the inverse Fourier transform:
\[
\frac{\partial^\alpha \rho(x)}{\partial |x|^\alpha} \equiv \int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{-ikx} (-|k|^\alpha) \hat{\rho}(k).
\] (32)

This is a formal construction that is more or less frequently used in similar problems.