Auction Theory

Yossi Sheffi
Mass Inst of Tech
Cambridge, MA

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Outline

- Introduction to auctions
- Private value auctions
  - 1st price auctions
  - 2nd price auctions
  - Revenue equivalence
  - Other auctions
  - Reservation price
- Interdependent values and the winner's curse
- Extensions

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Auctions - Examples

- As old as the hills…
  - Fixed price is only 100+ years old

John Glenn – Rocket and module built by the lowest bidder

Babilonian times (500 BC) sale of women

In 193 AD the Pretorian Guard auctioned off the entire Roman Empire. Winning bid: 25,000 sesterces/man. The winner: Didius Julianus was declared emperor but as broke and was beheaded after two month (winner’s curse)

Auctions – What and Why?

- An auction is an allocation pricing mechanism
- An auction determines:
  -
  -
- Auctions elicit information about how much buyers are willing to pay.
  - Universality
  - Anonymity
- The framework:
  - Each bidder has a value for the item
  - If he wins his surplus is the price paid minus the value.
- Auctions
  - Avoid dishonest “smoke-filled-room” dealings
  - Determine the value
  - Give it to the buyer who wants it most (efficiency)

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Simple Auctions (Single Item)

- Open bids:
  - English auction – bidder calls increasing price until one bidder left. Bidder pays the price at that point (Japanese auction).
  - Dutch auction – bidder starts high and lower price. First bidder to call gets the item
- Sealed bids:
  - First price – highest bid wins
  - Second price – highest bid wins but pays the second-highest bid

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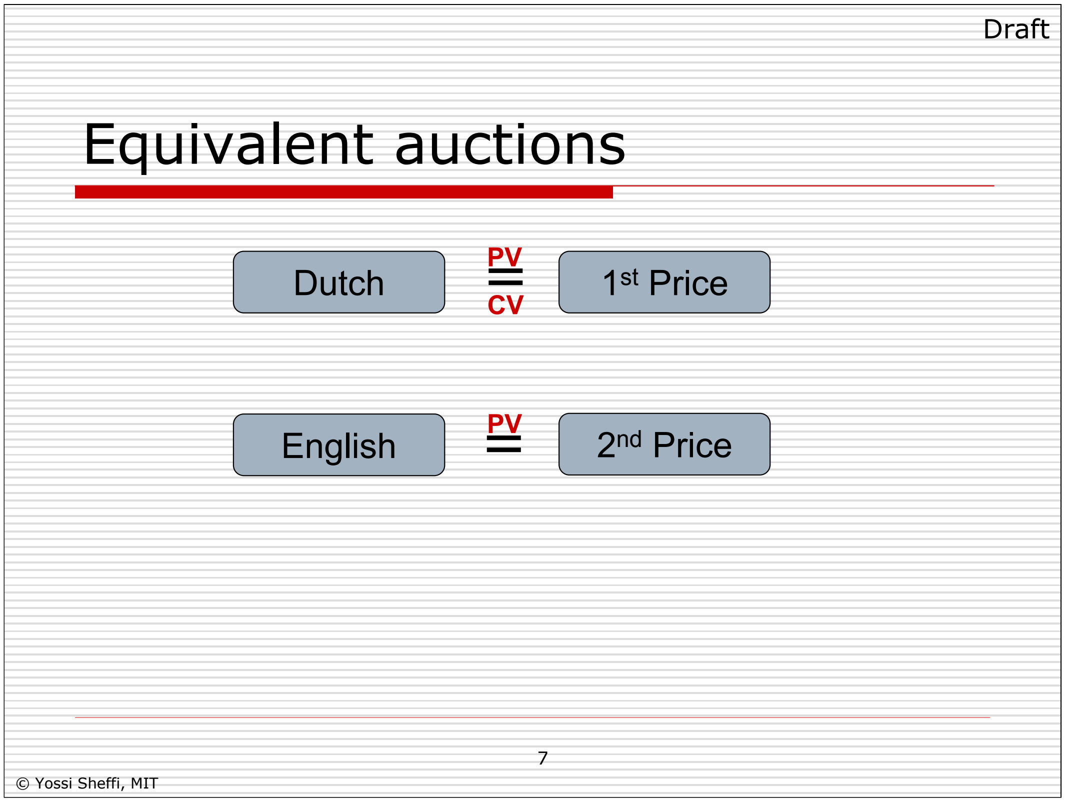
Information distribution

- Both buyers and seller are uncertain what the value of the item sold is.
- Private values – each bidder knows the value to himself (no bidder knows the valuation of other bidders; in any case it will not affect the self valuation)
- Common values – the value is the same for all bidders (example: mineral rights – the real value becomes known later)
- Interdependent values – bidders modify their estimate during the bidding process. Both common and private elements



eBay is second price. The automatic bidders can be set to bid in set increments until a certain point. This means that one bids the value but pays the bid (value) of the next highest bidder (within the accuracy of the increment)

Auction Metrics

- Revenue (expected selling price) – the auctioneer wants the highest
- Efficiency – make sure that the winner is the bidder who values the item the most *ex post*.
  - In most procurement auctions there is no secondary markets
  - Secondary markets involve extra transaction costs
- Simplicity
- Time and effort

Assumptions

- Private values
- n bidders
- i.i.d. values from $F(V)$ with $f(V)$
  (symmetric, independent bidders)
- Risk neutrality
- No collusion or predatory behavior

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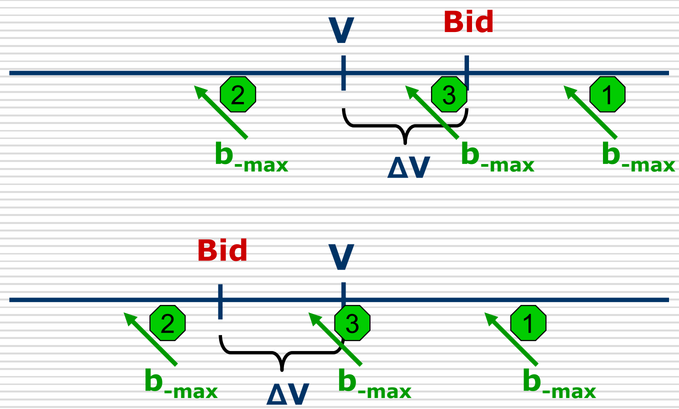
2nd Price – Bidding Strategies



Dominant strategy in 2nd price (and English) auctions:
Bid your value

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In English auctions a bidder should raise his bid "by a little" as long as the current price is lower than his valuation. The outcome is that the person with the highest valuation wins but pays the valuation of the second highest bidder (plus "a little")

If bidding higher than value – may win and pay a price higher than the value. Or may lose at a price still lower than the value (should have gone higher)

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2nd Price –
How Much will the Winner Pay?

- n bidders, iid $F(v)$ with density $f(v)$, PV:
  - Bidders' values: $\{V_1, V_2, \ldots, V_n,\}$
  - Order statistics: $\{V_{(1)}, V_{(2)}, \ldots, V_{(n)},\}$.
- Density of k^{th} lowest: $f(v_{(k)}) = \frac{n!}{(k-1)! \bullet (n-k)!} f(v) \bullet [F(v)]^k \bullet [1-F(v)]^{n-k}$
- Density of U(0,1): $f(v_{(k)}) = \frac{n!}{(k-1)! \bullet (n-k)!} v^{k-1} \bullet [1-v]^{n-k}$
- Mean value of k^{th} order statistic: $E[v_{(k)}] = \frac{k}{n+1}$
- Mean value of 2^{nd} order statistic: $E[v_{(n-1)}] = \frac{n-1}{n+1}$
  (expected revenue for the auctioneer)

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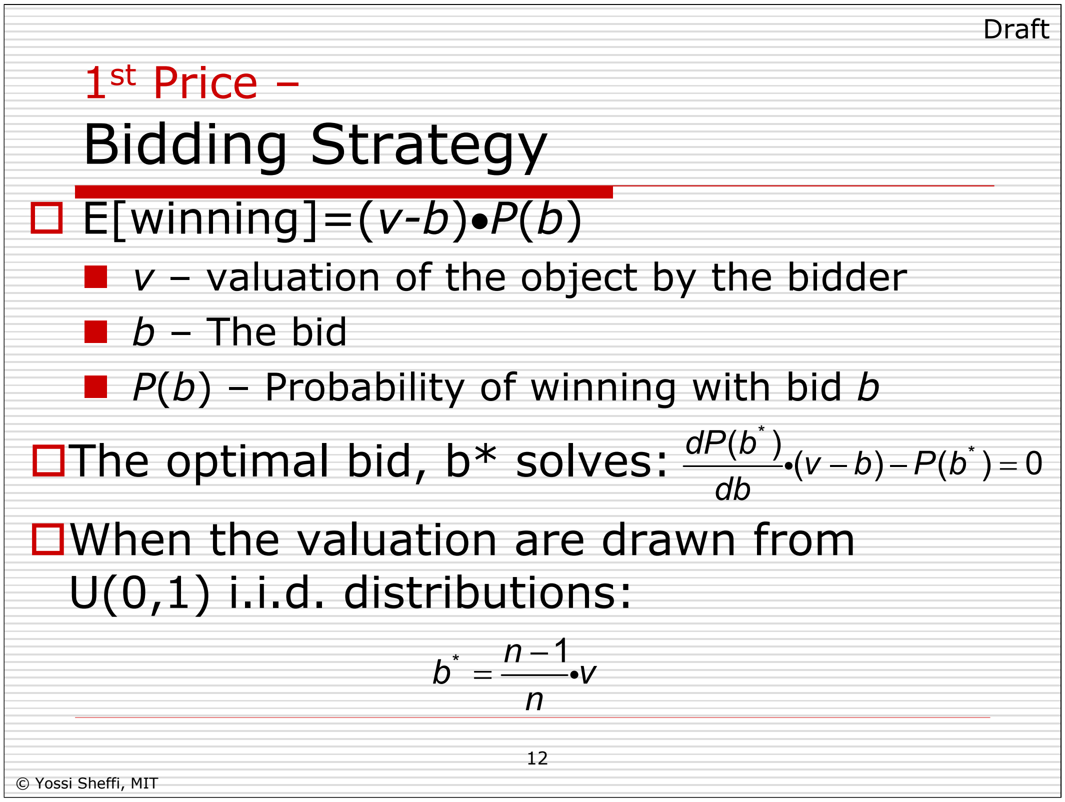
To see this result:



"shading" by 1/n to account for the need to get surplus

To develop the strategy, assume that each bidder wants to maximize the expected value of winning, which is the surplus associated with winning times the probability of winning:

E[winning]=(*v*-*b*)•*P*(*b*)

Where:

V = the bidder's valuation of the object auctioned off

b = the bid

P(*b*) = the probability that a bid *b* will win the auction.

The optimal bid is the one that maximizes the expectation in Eq. . Thus the optimal bid is the solution to:

Consider the case with *n* bidders whose valuations are drawn from independently from identical distribution. The probability that *b* is the winning bid is the probability that (*n*-1) valuations will be lower than *b*[1] or: *p(b) = [F(b)]n-1*. In the special case in which the valuations are drawn from a U(0,1) density function, *P(b) = bn-1* the optimal bid of a participant whose value is v would be:

Thus each bidder has to "shave" his bid by (1/*n*)·*v*.

[1] No bidders would bid higher than his value in a first price auction since this will cause him to have negative surplus in case he wins.

1st Price – The expected Payment

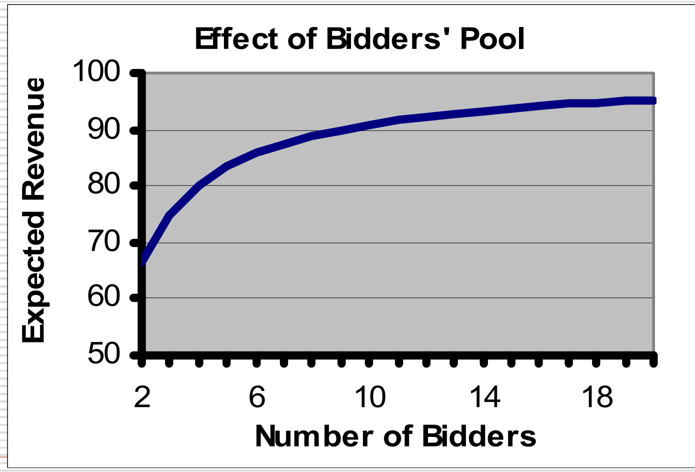
- The winning (highest) bid is the bid of the person with the highest order statistic: $V_{(n)}$.
- For U(0,1), this person bids: $\frac{n-1}{n} \cdot E[V_{(n)}]$
- In this case: $E[V_{(n)}] = \frac{n}{n+1}$
- So the payment is: $\frac{n-1}{n+1}$
- Same result as before (!)

Revenue Equivalence Theorem

- In 2nd price participants bid their value and pay the highest losing bid
- In 1st price they shade their bid and pay what they bid
- In any particular case any given auction can give results that are better (worse) then any other auction
- **Revenue Equivalence**: All auction that allocate the item to the highest bidder and lead to the same bidder participation yield the same expected payoff.
  - Private values
  - Risk neutrality
  - iid valuation
  - No collusion

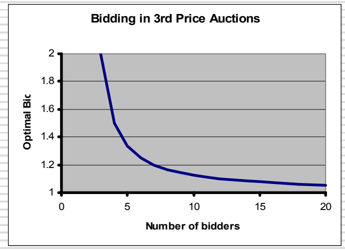
More Bidders=higher Expected Payoff

For n bidders with PV and V~U(50,100):



3rd Price Auctions

- For $V_i \sim U(0,1)$, iid with PV: $b^* = \frac{n-1}{n-2} \cdot v$
- Note:
  -
  -



- But the payment is still: $\frac{n-1}{n+1}$

Explain the bid: to really lose (pay too much), two other bids have to fall in the "delta" which is unlikely

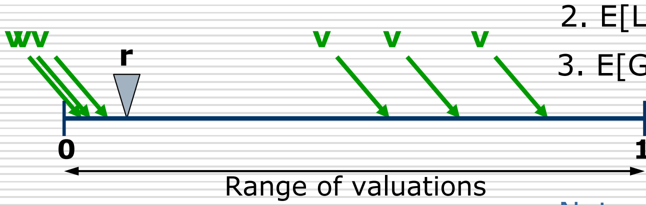
Reserve Prices

- A minimum price, *r*, below which the seller keeps the item
- "Excludes" some bidders with $v < r$
- Expected revenues in all auctions (iid, PV…) is the same
- A proper reserve price increases revenue

Reservation price

- Why set a reservation price?
- Consider two bidders (2nd price auction):
  (auctioneer's value = 0)

1. E[Gain]=No change
2. E[Loss] ≤ $r \cdot [F(r)]^2$
3. E[Gain]=$2(\frac{1}{2} \cdot r) \cdot F(r) \cdot [1-F(r)]$



Note: for small r, F(r) <<1

- So: in 2nd price auction, the benefit is from having the reserve price replace the 2nd and "bump" the price paid
- In 1st price, the benefit is from bidders tempering their shading not to bid just below the reserve price.

Optimal Reservation Price

- Given r^*, assume the seller raises it to $r^*+\delta$. (Assume value to seller is 0)
- Good move if there is exactly a single seller bidding above $(r^*+\delta)$.
  - $\Pr = n \cdot F(r^*)^{n-1} \cdot [1-F(r^*+\delta)]$. Gain = δ
- Bad move if the highest bid is between r^* and $(r^*+\delta)$.
  - $\Pr = n \cdot F(r^*)^{n-1} \cdot [F(r^*+\delta)-F(r^*)]$. Loss = r^*

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Optimal Reservation Price

- Net expected gain per increment in *r**:

$$\Delta(\delta) := n \bullet F(r^*)^{n-1} \bullet [1 - F(r^* + \delta)] - n \bullet F(r^*)^{n-1} \bullet \frac{F(r^* + \delta) - F(r^*)}{\delta} \bullet r^*$$

- Taking the limit:

$$\lim_{\delta \to 0} \Delta(\delta) = n \bullet F(r^*)^{n-1} \bullet \{[1 - F(r^*)] - f(r^*) \bullet r^*\}$$

- Setting Δ=0: $r^* = \frac{1 - F(r^*)}{f(r^*)}$
- Note: the optimal r does not depend on the # of bidders

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In the first part, there is del (the gain) divided by del (the increment)

Inverse hazard rate

Reservation Price

- Should be included in most auctions to avoid "nasty surprises."
- In procurement auctions
  - The auctioneer's value is the "next best" alternative:
    - "make" not "buy"
    - Stay with last year's contracts
  - In many cases not contracting is not an option (consequences too severe)

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Risk Aversion (PV)

- What happens is bidders are risk-averse?

Interdependencies

- Interdependent values - a bidder's valuation is affected by knowing the valuation of other bidders
  - $V_i = v_i(S_1, S_2,\dots S_n)$; $v_i = E[V_i \mid s_1, s_2,\dots s_n]$;
- Pure common value – item has the same value for all bidders. Each bidder has only an (unbiased) estimate/signal of the value prior to the auction
  - $V_i = v(S_1, S_2,\dots S_n)$
  - Used to model oil drilling and mineral rights auctions

Winner’s Curse (CV Auctions)

- The winner is the bidder with highest signal.
- Winning means that everybody else had a lower estimate (“adverse selection bias”)
- So winning is “bad news” (cold feet make sense…)
- If bidders do not correct for this, the winner will overpay – bidders have to “shave” their bids further (1^{st} price “shave” + WC “shave”)

A Case Study

- Carolina Freight 1995 bid for K-Mart freight
- Overbid (*lowest* bidder in this case) and went bankrupt
- Bought by ABF, who probably overbid to acquire it

A Game
(or why most mergers fail)

- Corporate B wants to acquire A
- A knows its own true value
- B knows only that A's value is U(0,$100)
- B can make A worth 50% more than A's value after the acquisition
- How much should B offer?

A Game
(or why most mergers fail)

- Distribution of bids:
- Analysis:

Winner's Curse - Getting the Correct Expected value

- Common value U(0,1)
- Private signal: s_i drawn from: $U(V-\varepsilon, V+\varepsilon)$
  - $E[V \mid S_i = s_i] = s_i$
  - $E[V \mid S_i = s_{max}] = s_i - \varepsilon \cdot (n-1)/(n+1)$
- Essentially, a bidder should realize *a-piori* that if he wins, it is likely that his signal was unusually high.
- Thus, WC results strictly from judgment failure
- Note: the shading is higher (lower bids) with more bidders. This is the opposite of the 1st price shading which is lower (higher bids) with more bidders.
- Note: the existence of WC in practice is hotly debated among economists since it implies irrationality

The last point is a secondary effect. The argument: if values are affiliated and there are so many smart people bidding – "if the item is so good why doesn't somebody else bid higher?" This is the same reason that in large classroom people do not ask questions. "If my question is important, somebody else would have asked it…" Only after a few questions are asked, students learn that other students are not that smart and start asking questions…

Interdependent & Affiliated Auctions

- With interdependent values (signals): English Auction ≠ 2nd Price Auction
  - Bidders get information from those who dropped about the true value
- Affiliation: strong positive correlation between the valuations
- Ranking of expected revenue (with affiliation): {English} ≥ {2nd Price} ≥ {1st Price}
  - Openness of English auction may make participants more comfortable with their own estimates and thus bid higher
  - In a 1st price auction, auctioneers should release as much information as they have to get bidders to bid aggressively.

Revenue equivalence holds also with non-private value auctions interdependent as long as the signals are INDEPENDENT. Thus affiliation means that

Note that 2nd price sealed bid auctions are not usually conducted. Thus affiliation only mean that English auction will yield higher results.

Practical considerations

Asymmetric Valuations

- Asymmetric valuations – "strong" and "weak" bidders (valuations drawn from different distributions)
- Strong bidders prefer English – always win in an open format
- Weak bidders have a chance in sealed bids (1st price) which give them some chance of winning
- Since strong bidders will win in English, auctioneers may prefer it (possibly higher bids and higher auction efficiency)
- But:
  - weaker players may bid more aggressively (closer to their valuation)
  - More bidders, even weaker may mean more competition and keep the strong bidders "honest"

Practical considerations

Number of Bidders

- Auctioneers should make sure that there are enough bidders.
- English auction guarantees that that strong bidders will win, so it may deter weaker bidders and cause the strong bidders to win at a low price.
- **But**: a sealed bid auction allows weak bidders to win, thereby causing stronger bidders to bid more aggressively

Practical considerations

Predatory Behavior and Collusion

- English auctions are more susceptible to predatory behavior since buyers can bid aggressively in early rounds causing others to drop too early and win with a price that is too low.
- English auctions are more susceptible to collusion. In particular with multiple items bidders may signal each other in the early rounds, dividing the pie without driving the price too high. Also bidders can “punish” aggressive behavior by bidding high on something small that the other bidder really wants

Any Questions?

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