Finite Element Output Bounds for a Stabilized Discretization of Incompressible Stokes Flow

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Abstract— **We introduce a new method for computing *a posteriori* bounds on engineering outputs from finite element discretizations of the incompressible Stokes equations. The method results from recasting the output problem as a minimization statement without resorting to an error formulation. The minimization statement engenders a duality relationship which we solve approximately by Lagrangian relaxation. We demonstrate the method for a stabilized equal-order approximation of Stokes flow, a problem to which previous output bounding methods do not apply. The conceptual framework for the method is quite general and shows promise for application to stabilized nonlinear problems, such as Burger's equation and the incompressible Navier-Stokes equations, as well as potential for compressible flow problems.**

Keywords— **Finite Element, Stabilization, Output Bounds, Error Estimation, Stokes Equations**

I. Introduction

A PRIOR error estimates inform us of the asymptotic rates of convergence, but cannot answer the ever present engineering question, "can I trust the current approximation?" Such questions often revolve around concerns of mesh fidelity and feature resolution – issues of numerical uncertainty which erode confidence in the simulation. As confidence erodes, so does the utility of the simulation in the engineering design process: either the simulation is not trusted, or it is more costly than necessary. We propose an implicit *a posteriori* method for computing rigorous constant-free upper and lower bounds for outputs from finite element discretizations of incompressible fluid flows. The error bounds have the potential to significantly reduce numerical uncertainty by providing confirmation of accuracy as well as allowing for the effective trade-off between accuracy and computational cost.

Error estimates, even *a posteriori* error estimates, are not new to finite element approximations (see [1] for a review). Work on explicit methods was begun by Babuska and Rheinboldt in the late 1970's. In the late 1980's Zienkiewicz and Zhu proposed a recovery based method which has experienced some popularity due to its simplicity. These methods have been difficult to extend to more complex problems and fail to provide confirmation of accuracy as they contain undetermined constants or lack rigor in their construction (for example, assuming a smoother solution to be a better solution). Such limitations relegate the error estimator to merely serving as an oracle for balancing error contributions (of unknown magnitude) in mesh adaptivity, and undermine their effectiveness as methods for confirmation and building adaptive meshes with guaranteed error tolerances.

II. Framework

We begin with a brief and abstract overview of the underlying structure of our new output bounding framework, which differs from our previous general frameworks[2] in working with the complete solution, instead of the solution error, and in maintaining the Lagrangian formulation throughout[3], where previously we had resorted to an algebraic formulation. Consider the following continuous variational-weak problem

$$\text{find } u \in U : \mathcal{A}(v, u) = \ell(v), \quad \forall v \in V, \tag{1}$$

for U an essential condition satisfying subset of the appropriate Hilbert space, usually $H^1(\Omega)$, with an associated homogeneous space V. For our conceptual overview, we assume that $\mathcal{A}(v, u)$ is linear in $v \in V$, the test slot, and nonlinear in $u \in U$, the solution slot.

Furthermore, we assume that we can define an energy like quantity, $\mathcal{E}(w)$ for $w \in U$, by choosing some $w_0 \in U$ so that $w - w_0 \in V$ and defining

$$\mathcal{E}(w) \equiv \mathcal{A}(w - w_0, w) - \ell(w - w_0).$$

Actually, we have more freedom to exercise in our choice for $\mathcal{E}$ (and work on nonlinear problems suggests that we may need to exercise such freedom), but an essential property of this energy form is that $\mathcal{E}(u) = 0$.

We are not interested directly in u, but in outputs of engineering interest such as mass flow rate or drag, which are functionals of u. We represent the output abstractly as $s(u)$.

We begin constructing the method by recasting the above problem as a minimization statement. Actually, we recast the problem as a pair of seemingly meaningless minimization statements

$$\begin{aligned} \mp s(u) = \inf \quad & \kappa\mathcal{E}(w) \mp s(w) \\ \text{s.t.} \quad & \mathcal{A}(v, w) = \ell(v), \quad \forall v \in V, \\ & w \in U, \end{aligned}$$

for an arbitrary nonnegative scalar κ, which we can later use to minimize the resulting finite element bound gap, if desired. The feasible set of this minimization consists trivially of a single function u, the assumed to be unique solution of Equation (1), for which the objective function obtains the value of $\mp s(u)$.

Our goal being to relax the minimization in a manner which allows us to compute inexpensive bounds on s, we form the Lagrangian of the above trivial minimization problem for $(w, \phi) \in U \times V$

$$\mathcal{L}^{\pm}(w, \phi) \equiv \kappa\mathcal{E}(w) \mp s(w) + \ell(\phi) - \mathcal{A}(\phi, w). \tag{2}$$

The dual function of this Lagrangian can be solved by inspection

$$\mathcal{L}^{*,\pm}(w) \equiv \sup_{\phi\in V} \mathcal{L}^{\pm}(w,\phi) = \begin{cases} \mp s(u) & \text{if } w = u, \\ +\infty & \text{otherwise,} \end{cases} \quad (3)$$

from which it is clear that

$$\mp s = \inf_{w\in U} \sup_{\phi\in V} \mathcal{L}^{\pm}(w,\phi) \quad (4)$$

For an arbitrary candidate Lagrange multiplier $\psi^* \in V$, it is always true that $\mathcal{L}^{\pm}(w,\psi^*) \le \sup_{\phi\in V} \mathcal{L}^{\pm}(w,\phi)$ for $w \in U$, so that with (4) we have the lower bound

$$s^- = \inf_{w\in U} \mathcal{L}^-(w,\psi^*) \le \inf_{w\in U} \sup_{\phi\in V} \mathcal{L}^-(w,\phi) = s. \quad (5)$$

Similarly, we also have an upper bound

$$s^+ = -\inf_{w\in U} \mathcal{L}^+(w,\psi^*) \ge -\inf_{w\in U} \sup_{\phi\in V} \mathcal{L}^+(w,\phi) = s. \quad (6)$$

The basic strategy for constructing output bounding methods will be to compute approximations of the dual variables, usually solved for on a coarse "working" mesh, which are then used as data in solving the minimizations of (5) and (6).

Our new framework requires the solution of nonlinear bounding subproblems, a departure from previous efforts based on Taylor expansions of any nonlinearities (either in the governing equations or output).

In the face of indefinite terms in the Lagrangian relaxation, we rely on the the energy form $\mathcal{E}(w)$ to ensure the existence and finiteness of the above minimizations, a property which is at least partially obtained by the very nature of stabilization schemes.

III. Problem Statement

The solution to the Stokes flow equations consists of a velocity vector field, $u \in U$, and a scalar pressure field, $p \in Q$. We will work with the "skew-symmetric" form of the Stokes equations, which can be written as a "Stokes tableau"

$$\begin{aligned} \text{find } u\in U:&\quad a(v,u) - d(v,p) = \ell(v), && \forall v\in V, \\ p\in Q:&\quad d(u,q) = 0, && \forall q\in Q, \end{aligned}$$

where, for the symmetric strain tensor

$$\varepsilon_{ij}(w) = \frac{1}{2}\left(\frac{\partial w_i}{\partial x_j} + \frac{\partial w_j}{\partial x_i}\right)$$

we have the following definitions

$$a(v,w) = \nu \int_\Omega \varepsilon(v) : \varepsilon(w)\,\mathrm{d}\Omega, \qquad \ell(v) = \ell_f(v) + \ell_N(v),$$

$$\ell_f(v) = \int_\Omega v\cdot f\,\mathrm{d}\Omega, \qquad \ell_N(v) = \int_{\Gamma_N} v\cdot t\,\mathrm{d}s.$$

$$d(q,w) = \int_\Omega q(\operatorname{div} w)\,\mathrm{d}\Omega,$$

for the set of essential condition satisfying functions

$$U = \left\{ v \in \left(H^1(\Omega)\right)^d \,\middle|\, u|_{\Gamma_D} = u_D \right\},$$

and the spaces

$$V = \left\{ v \in \left(H^1(\Omega)\right)^d \,\middle|\, v|_{\Gamma_D} = 0 \right\}$$
$$Q = L^2(\Omega),$$

in d spatial dimensions. For all-Dirichlet problems (for which $\Gamma_D = \Gamma$), we substitute the pressure space $Q = L^2(\Omega)\backslash\mathbb{R}$.

A. Stabilized Finite Element Formulation

Traditionally, the Stokes equations are approximated with stable mixed finite element formulations, but more recently, stabilized methods have evolved to solve them with equal-order finite element interpolations. In this section, we will consider the first-order Algebraic Subgrid Scale (ASGS) stabilization [4], written for the Stokes equations on a given triangulation $\mathcal{T}_h(\Omega)$ as

$$\begin{aligned} \text{find } u_h\in U_h:&\quad \tilde{a}_h(v,u_h) - d(v,p_h) = \ell(v), && \forall v\in V_h, \\ p_h\in Q_h:&\quad d(u_h,q) + \Theta_h(q,p_h) = \tilde{\ell}_h(q), && \forall q_h\in Q_h, \end{aligned}$$

where

$$\tilde{a}_h(v,w) = a(v,w) + \tau_2 \int_{\Omega'} \frac{\partial v_i}{\partial x_i}\frac{\partial w_j}{\partial x_j}\,\mathrm{d}\Omega,$$

for, $\tau_2 = c_2\nu$, and

$$\Theta_h(q,r) = \tau_1 \int_{\Omega'} \frac{\partial q}{\partial x_k}\frac{\partial p}{\partial x_k}\,\mathrm{d}\Omega, \quad \tilde{\ell}_h(q) = \tau_1 \int_{\Omega'} \frac{\partial q}{\partial x_k} f_k\,\mathrm{d}\Omega,$$

for $\tau_1 = \dfrac{h^2}{c_1\nu}$.

We have introduced the usual linear Lagrange finite element discretizations

$$U_h = \left\{ u\in U \mid u|_{T_h} \in \mathbb{P}^1(T_h),\ \forall T_h \in \mathcal{T}_h \right\},$$
$$V_h = \left\{ v\in V \mid v|_{T_h} \in \mathbb{P}^1(T_h),\ \forall T_h \in \mathcal{T}_h \right\}$$
$$Q_h = \left\{ q\in Q \mid q|_{T_h} \in \mathbb{P}^1(T_h),\ \forall T_h \in \mathcal{T}_h \right\}.$$

B. Force Output Reformulation

The output of particular interest for fluid problems is the force output. We consider the force acting on a portion of the boundary, Γ_F, which we reformulate from the obvious boundary integral for the boundary force exerted by the stress tensor field $\sigma_{ij} = \nu\varepsilon_{ij}(u) - p\delta_{ij}$ engendered by the solution pair (u,p):

$$F_k = \int_{\Gamma_F} \sigma_{kj} n_j\,\mathrm{d}\Gamma,$$

which, for any $\chi_i|_{\Gamma_F} = \delta_{ik}$,

$$= \int_{\Gamma_F} \chi_i \sigma_{ij} n_j\,\mathrm{d}\Gamma,$$

and by the divergence theorem is equivalent to

$$= - \int_\Omega D_j \left(\chi_i \sigma_{ij}\right) \, \mathrm{d}\Omega.$$

This simplifies to

$$\begin{aligned} &= - \int_\Omega (D_j \chi_i)\sigma_{ij} + \chi_i (D_j \sigma_{ij}) \, \mathrm{d}\Omega, \\ &= \ell(\chi) - a(\chi, u) + d(p, \chi). \end{aligned}$$

We then write the force output associated with a particular χ as

$$s^\chi(u, p) = \ell(\chi) - a(\chi, u) + d(p, \chi).$$

The purpose of the reformulation is to obtain a bounded functional, a property of our output functional required to maintain optimal theoretical and computed convergence rates.

IV. Bounds Formulation

We will now develop the details of a force output bounding algorithm for the ASGS stabilized linear finite-element discretization of incompressible Stokes flow. The method shares several features of earlier methods, such as [5], but results from a new conceptual framework and applies to stabilized equal-order interpolations.

A. Elemental Decomposition

A practical bounds algorithm must decompose the subproblems into local subproblems in order to be less expensive than computing the fine mesh solution which we are trying to avoid. We partition the domain with a coarse triangulation, $\mathcal{T}_H$, on which we define the elementally broken analogue of a Hilbert space Y as

$$\hat{Y} = \prod_{T_H \in \mathcal{T}_H} Y(T_H).$$

We define a continuity bilinear form $b : \hat{Y} \times B^Y$ on the edges of $\mathcal{T}_H$ such that

$$Y = \left\{ \hat{v} \in \hat{Y}(\Omega) \mid b(\hat{v}, \beta) = 0, \quad \forall \beta \in B^Y \right\},$$

where B spans the traces of $\hat{v} \in \hat{Y}(\Omega)$ on the edges of the coarse triangulation $\mathcal{T}_H$. For our purposes, the space Y is either the velocity space V or the pressure space Q and the bilinear form $b(v, \beta)$ is understood to be overloaded for the appropriate space, a specialization which can be committed without ambiguity. The continuity multipliers act as equilibrating fluxes across elementally decomposed edge boundaries, in the spirit of which, are referred to as *hybrid fluxes*.

B. Minimization Statement and Lagrangian

We recast the stabilized Stokes problem as a minimization statement posed on the fine "truth" mesh $\mathcal{T}_h(\Omega)$, a proper refinement of a prototype coarse working mesh $\mathcal{T}_H(\Omega)$, with continuity across coarse mesh edges enforced through continuity constraints (hybrid fluxes)

$$\begin{aligned} \inf \quad & \kappa\{\tilde{a}_h(\hat{w}_h^\pm - w_0, \hat{w}_h^\pm) + d(w_0, \hat{r}_h^\pm) + \\ & \quad \Theta_h(\hat{r}_h^\pm, \hat{r}_h^\pm) - \ell(\hat{w}_h^\pm - w_0) - \tilde{\ell}_h(\hat{r}_h^\pm)\} \\ & \quad \mp \{\ell(\chi) - a(\chi, \hat{w}_h^\pm) + d(\chi, \hat{r}_h^\pm)\} \\ \text{s.t.} \quad & \tilde{a}_h(v, \hat{w}_h^\pm) - d(v, \hat{r}_h^\pm) = \ell(v), \quad \forall v \in V_h, \\ & d(\hat{w}_h^\pm, q) + \Theta_h(q, \hat{r}_h^\pm) = \tilde{\ell}_h(q), \quad \forall q \in Q_h, \\ & b(\hat{w}_h^\pm, \beta^u) = 0, \quad \forall \beta^u \in B_h^V, \\ & b(\hat{r}_h^\pm, \beta^p) = 0, \quad \forall \beta^p \in B_h^Q, \\ & \hat{w}_h^\pm \in \hat{V}_h, \\ & \hat{r}_h^\pm \in \hat{Q}_h, \end{aligned}$$

for which the optimal objective function is our output of interest (modulo a sign), $\mp s^\chi(u_h, p_h)$, obtained at the singleton solution pair (u_h, p_h).

We form the Lagrangians $\mathcal{L}_h^\pm : \hat{U}_h \times \hat{Q}_h \times V_h \times Q_h \times B_h^V \times B_h^Q$

$$\begin{aligned} \mathcal{L}_h^\pm(\hat{u}_h^\pm, \hat{p}_h^\pm; \psi^\pm, \pi^\pm, \gamma^{u,\pm}, \gamma^{p,\pm}) = \\ & \kappa\{\tilde{a}_h(\hat{u}_h^\pm - w_0, \hat{u}_h^\pm) + d(w_0, \hat{p}_h^\pm) \\ & \quad + \Theta_h(\hat{p}_h^\pm, \hat{p}_h^\pm) - \ell(\hat{u}_h^\pm - w_0) - \tilde{\ell}_h(\hat{p}_h^\pm)\} \\ & \mp \{\ell(\chi) - a(\chi, \hat{u}_h^\pm) + d(\chi, \hat{p}_h^\pm)\} \\ & + \ell(\psi^\pm) - \tilde{a}_h(\psi^\pm, \hat{u}_h^\pm) + d(\psi^\pm, \hat{p}_h^\pm) \\ & + \tilde{\ell}_h(\pi^\pm) - d(\hat{u}_h^\pm, \pi^\pm) - \Theta_h(\pi^\pm, \hat{p}_h^\pm) \\ & - b(\hat{u}_h^\pm, \gamma^{u,\pm}) - b(\hat{p}_h^\pm, \gamma^{p,\pm}), \end{aligned}$$

whose velocity gradient condition is

$$\begin{aligned} & \kappa\{\tilde{a}_h(\hat{v}, \hat{u}_h^\pm) + \tilde{a}_h(\hat{u}_h^\pm - w_0, \hat{v}) - \ell(\hat{v})\} \pm \\ & \quad a(\chi, \hat{v}) - \tilde{a}_h(\psi^\pm, \hat{v}) - d(\hat{v}, \pi^\pm) - b(\hat{v}, \gamma^{u,\pm}) = 0, \\ & \qquad \forall \hat{v} \in \hat{V}_h, \end{aligned} \tag{7}$$

and pressure gradient condition is

$$\begin{aligned} & \kappa\{2\Theta_h(\hat{q}, \hat{p}_h^\pm) + d(w_0, \hat{q}) - \tilde{\ell}_h(\hat{q})\} \mp d(\chi, \hat{q}) \\ & \quad + d(\psi^\pm, \hat{q}) - \Theta_h(\pi^\pm, \hat{q}) - b(\hat{q}, \gamma^{p,\pm}) = 0, \\ & \qquad \forall \hat{q} \in \hat{Q}_h. \end{aligned} \tag{8}$$

Now that we have established the gradient (stationarity) conditions of our Lagrangian reformulation of the original Stoke's flow problem, we can relax the Lagrangian by constructing candidate (but sub-optimal) dual variables $psi^\pm$, $\pi^\pm$, $\gamma^{u,\pm}$, and $\gamma^{p,\pm}$.

C. Coarse Mesh Adjoint

Our candidate dual multipliers will be approximations computed from the gradient conditions on a coarse working mesh $\mathcal{T}_H$– the prototype mesh of which $\mathcal{T}_h$ is a proper refinement.

Utilizing the definitions $\psi_H^\pm = \pm\psi_H + \kappa(u_H - w_0)$ and $\pi_H^\pm = \pm\pi_H + \kappa p_H$, we can write the coarse mesh adjoint equation from (7) and (8), substituting $\hat{u}_H^\pm = u_H$, as

$$\begin{aligned}
&\text{find } (\psi_H, \pi_H) \in V_H \times Q_H : \\
&\qquad \tilde{a}_H(\psi_H, v) + d(v, \pi_H) = a(\chi, v), \\
&\qquad -d(\psi_H, q) + \Theta_H(q, \pi_H) = -d(\chi, q), \\
&\qquad\qquad \forall (v, q) \in V_H \times Q_H, \quad (9)
\end{aligned}$$

where we have evoked primal feasibility on the coarse mesh.

D. Coarse Mesh Hybrid Fluxes

In addition to the adjoint (which is the Lagrange multiplier for the equilibrium constraint), we must produce a candidate Lagrange multiplier for the continuity constraint (hybrid fluxes), which we again compute from the coarse mesh instantiation of the gradient conditions with data $\hat{u}_H^\pm = u_H$ and $\psi = \psi_H^\pm$.

By choosing $\gamma_H^{u,\pm} = -\kappa\gamma_H^u \pm \gamma_H^\psi$ and $\gamma_H^{p,\pm} = -\kappa\gamma_H^p \pm \gamma_H^\pi$, the equilibration problem can be decomposed into two κ independent problems

$$\begin{aligned}
&\text{find } (\gamma_H^u, \gamma_H^p) \in B_H^V \times B_H^Q : \\
&\qquad b(\hat{v}, \gamma_H^u) = \ell(\hat{v}) - \tilde{a}_H(\hat{v}, u_H) + d(\hat{v}, p_H), \\
&\qquad b(\hat{q}, \gamma_H^p) = \tilde{\ell}_H(\hat{q}) - d(u_H, \hat{q}) - \Theta_H(\hat{q}, p_H), \\
&\qquad\qquad \forall (\hat{v}, Y) \in \hat{V}_H \times \hat{Q}_H, \quad (10)
\end{aligned}$$

and

$$\begin{aligned}
&\text{find } (\gamma_H^\psi, \gamma_H^\pi) \in B_H^V \times B_H^Q : \\
&\qquad b(\hat{v}, \gamma_H^\psi) = a(\chi, \hat{v}) - \tilde{a}_H(\psi_H, \hat{v}) - d(\hat{v}, \pi_H), \\
&\qquad b(\hat{q}, \gamma_H^\pi) = -d(\chi, \hat{q}) + d(\psi_H, \hat{q}) - \Theta_H(\hat{q}, \pi_H), \\
&\qquad\qquad \forall (\hat{v}, \hat{q}) \in \hat{V}_H \times \hat{Q}_H, \quad (11)
\end{aligned}$$

which we can solve efficiently with the Ladevèze method[6].

E. Fine Mesh Elemental Subproblems

The subproblems corresponding to the minimization of (5) and (6) result from the substitution of $\psi_H^\pm = \pm\psi_H + \kappa(u_H - w_0)$, $\pi_H^\pm = \pm\pi_H + \kappa p_H$, $\gamma_H^{u,\pm} = -\kappa\gamma_H^u \pm \gamma_H^\psi$ and $\gamma_H^{p,\pm} = -\kappa\gamma_H^p \pm \gamma_H^\pi$ into the stationarity conditions:

$$\begin{aligned}
&\text{find } (\hat{u}_h^\pm, \hat{p}_h^\pm) \in \hat{U}_h \times \hat{Q}_h : \\
&\qquad 2\kappa\tilde{a}_h(\hat{v}, \hat{u}_h^\pm) = \kappa\{\ell(\hat{v}) + \tilde{a}_h(u_H, \hat{v}) \\
&\qquad\qquad + d(\hat{v}, p_H) - b(\hat{v}, \gamma_H^u)\} \\
&\qquad\qquad \mp \{a(\chi, \hat{v}) - \tilde{a}_h(\psi_H, \hat{v}) \\
&\qquad\qquad - d(\hat{v}, \pi_H) - b(\hat{v}, \gamma_H^\psi)\}, \\
&\qquad 2\kappa\Theta_h(\hat{q}, \hat{p}_h^\pm) = \kappa\{\tilde{\ell}_h(\hat{q}) - d(u_H, \hat{q}) \\
&\qquad\qquad + \Theta_h(p_H, \hat{q}) - b(\hat{q}, \gamma_H^p)\} \\
&\qquad\qquad \mp \{-d(\chi, \hat{q}) + d(\psi_H, \hat{q}) \\
&\qquad\qquad - \Theta_h(\pi_H, \hat{q}) - b(\hat{q}, \gamma_H^\pi)\}, \\
&\qquad\qquad \forall (\hat{v}, \hat{q}) \in \hat{V}_h \times \hat{Q}_h. \quad (12)
\end{aligned}$$

We can solve these problems as two κ independent problems if we choose $\hat{u}_h^\pm = \hat{z}_h^u \mp \frac{1}{\kappa}\hat{z}_h^\psi$ and $\hat{p}_h^\pm = \hat{z}_h^p \mp \frac{1}{\kappa}\hat{z}_h^\pi$ to obtain

$$\begin{aligned}
&\text{find } (\hat{z}_h^u, \hat{z}_h^p) \in \hat{U}_h \times \hat{Q}_h : \\
&\qquad 2\tilde{a}_h(\hat{v}, \hat{z}_h^u) = \ell(\hat{v}) + \tilde{a}_h(u_H, \hat{v}) \\
&\qquad\qquad + d(\hat{v}, p_H) - b(\hat{v}, \gamma_H^u), \\
&\qquad 2\Theta_h(\hat{q}, \hat{z}_h^p) = \tilde{\ell}_h(\hat{q}) - d(u_H, \hat{q}) \\
&\qquad\qquad + \Theta_h(p_H, \hat{q}) - b(\hat{q}, \gamma_H^p), \\
&\qquad\qquad \forall (\hat{v}, \hat{q}) \in \hat{V}_h \times \hat{Q}_h. \quad (13)
\end{aligned}$$

and

$$\begin{aligned}
&\text{find } (\hat{z}_h^\psi, \hat{z}_h^\pi) \in \hat{V}_h \times \hat{Q}_h : \\
&\qquad 2\tilde{a}_h(\hat{v}, \hat{z}_h^\psi) = a(\chi, \hat{v}) - \tilde{a}_h(\psi_H, \hat{v}) \\
&\qquad\qquad - d(\hat{v}, \pi_H) - b(\hat{v}, \gamma_H^\psi), \\
&\qquad 2\Theta_h(\hat{q}, \hat{z}_h^\pi) = -d(\chi, \hat{q}) + d(\psi_H, \hat{q}) \\
&\qquad\qquad - \Theta_h(\pi_H, \hat{q}) - b(\hat{q}, \gamma_H^\pi), \\
&\qquad\qquad \forall (\hat{v}, \hat{q}) \in \hat{V}_h \times \hat{Q}_h. \quad (14)
\end{aligned}$$

F. Bounds Expression

The linearity of the Stokes equations allows the Lagrangian to be greatly simplified by substituting the gradient conditions into the Lagrangian to obtain

$$\begin{aligned}
&\mathcal{L}_h^\pm(\hat{u}_h^\pm, \hat{p}_h^\pm; \psi_H^\pm, \pi_H^\pm, \gamma_H^{u,\pm}, \gamma_H^{p,\pm}) = \\
&\qquad \kappa\{\ell(u_H) + \tilde{\ell}_h(p_H) - \tilde{a}_h(\hat{u}_h^\pm, \hat{u}_h^\pm) - \Theta_h(\hat{p}_h^\pm, \hat{p}_h^\pm)\} \\
&\qquad \mp \{\ell(\chi) - \ell(\psi_H) - \tilde{\ell}_h(\pi_H)\}.
\end{aligned}$$

Substituting $\hat{u}_h^\pm = \hat{z}_h^u \mp \frac{1}{\kappa}\hat{z}_h^\psi$ and $\hat{p}_h^\pm = \hat{z}_h^p \mp \frac{1}{\kappa}\hat{z}_h^\pi$

$$\begin{aligned}
\mathcal{L}_h^\pm(\cdot) &= \kappa\{\ell(u_H) + \tilde{\ell}_h(p_H) - \tilde{a}_h(\hat{z}_h^u, \hat{z}_h^u) - \Theta_h(\hat{z}_h^p, \hat{z}_h^p)\} \\
&\quad - \tfrac{1}{\kappa}\{\tilde{a}_h(\hat{z}_h^\psi, \hat{z}_h^\psi) - \Theta_h(\hat{z}_h^\pi, \hat{z}_h^\pi)\} \\
&\quad \mp \{\ell(\chi) - \ell(\psi_H) - \tilde{\ell}_h(\pi_H) \\
&\qquad + 2\tilde{a}_h(\hat{z}_h^\psi, \hat{z}_h^u) + 2\Theta_h(\hat{z}_h^\pi, \hat{z}_h^p)\}.
\end{aligned}$$

Defining the bound average as $\bar{s}_h^\pm = \frac{1}{2}\{s_h^+ + s_h^-\}$

$$\bar{s}_h^\pm = \ell(\chi) - \ell(\psi_H) - \tilde{\ell}_h(\pi_H) + 2\tilde{a}_h(\hat{z}_h^\psi, \hat{z}_h^u) + 2\Theta_h(\hat{z}_h^\pi, \hat{z}_h^p).$$

The bound gap is defined as $\Delta s_h^\pm = \frac{1}{2}\{s_h^+ - s_h^-\}$

$$\begin{aligned}
\Delta s_h^\pm &= \kappa\{\tilde{a}_h(\hat{z}_h^u, \hat{z}_h^u) + \Theta_h(\hat{z}_h^p, \hat{z}_h^p) - \ell(u_H) - \tilde{\ell}_h(p_H)\} \\
&\quad - \tfrac{1}{\kappa}\{\tilde{a}_h(\hat{z}_h^\psi, \hat{z}_h^\psi) - \Theta_h(\hat{z}_h^\pi, \hat{z}_h^\pi)\}.
\end{aligned}$$

The bound gap can be minimized over κ

$$\begin{aligned}
\Delta s_h^\pm &= 2\sqrt{PD} \\
P &= \tilde{a}_h(\hat{z}_h^u, \hat{z}_h^u) + \Theta_h(\hat{z}_h^p, \hat{z}_h^p) - \ell(u_H) - \tilde{\ell}_h(p_H) \\
D &= \tilde{a}_h(\hat{z}_h^\psi, \hat{z}_h^\psi) - \Theta_h(\hat{z}_h^\pi, \hat{z}_h^\pi).
\end{aligned}$$

Figure 1 summarizes the complete procedure.

1. Coarse Solution

$$\begin{aligned} \text{find}\ \ u_H \in V_H: &\ \ \tilde{a}_H(u_H, v_H) - d(v_H, p_H) &= \ell(v_H), &\quad \forall v_H \in V_H, \\ p_H \in Q_H: &\ \ d(u_H, q_H) + \Theta_H(q_H, p_H) &= \tilde{\ell}_H(q_H), &\quad \forall q_H \in Q_H, \end{aligned}$$

2. Coarse Adjoint

$$\begin{aligned} \text{find}\ \ \psi_H \in V_H: &\ \ \tilde{a}_H(\psi_H, v_H) + d(v_H, \pi_H) &= a(\chi, v_H), &\quad \forall v_H \in V_H, \\ \pi_H \in Q_H: &\ \ -d(\psi, q_H) + \Theta_H(q_H, \pi_H) &= -d(\chi, q_H), &\quad \forall q_H \in Q_H, \end{aligned}$$

3. Equilibration

$$\begin{aligned} \text{find}\ \ \gamma_H^u \in B_H^V: &\ \ b(\hat{v}_H, \gamma_H^u) = \ell(\hat{v}_H) - \tilde{a}_H(\hat{v}_H, u_H) + d(\hat{v}_H, p_H), &\quad \forall \hat{v}_H \in \hat{V}_H, \\ \gamma_H^p \in B_H^Q: &\ \ b(\hat{q}_H, \gamma_H^p) = \tilde{\ell}_H(\hat{q}_H) - d(u_H, \hat{q}_H) - \Theta_H(\hat{q}_H, p_H), &\quad \forall \hat{q}_H \in \hat{Q}_H, \end{aligned}$$

and

$$\begin{aligned} \text{find}\ \ \gamma_H^\psi \in B_H^V: &\ \ b(\hat{v}_H, \gamma_H^\psi) = a(\chi, \hat{v}_H) - \tilde{a}_H(\psi_H, \hat{v}_H) + d(\hat{v}_H, \pi_H), &\quad \forall \hat{v}_H \in \hat{V}_H, \\ \gamma_H^\pi \in B_H^Q: &\ \ b(\hat{q}_H, \gamma_H^\pi) = -d(\chi, \hat{q}_H) + d(\psi_H, \hat{q}_H) - \Theta_H(\psi_H, \hat{q}_H), &\quad \forall \hat{q}_H \in \hat{Q}_H, \end{aligned}$$

4. Subproblems

$$\begin{aligned} \text{find}\ \ \hat{z}_h^u \in \hat{U}_h: &\ \ 2\tilde{a}_h(\hat{v}, \hat{z}_h^u) = \ell(\hat{v}) + \tilde{a}_h(u_H, \hat{v}) + d(\hat{v}, p_H) - b(\hat{v}, \gamma_H^u), &\quad \forall \hat{v} \in \hat{V}_h, \\ \hat{z}_h^p \in \hat{Q}_h: &\ \ 2\Theta_h(\hat{q}, \hat{z}_h^p) = \tilde{\ell}_h(\hat{q}) - d(u_H, \hat{q}) + \Theta_h(p_H, \hat{q}) - b(\hat{q}, \gamma_H^p), &\quad \forall \hat{q} \in \hat{Q}_h. \end{aligned}$$

and

$$\begin{aligned} \text{find}\ \ \hat{z}_h^\psi \in \hat{V}_h: &\ \ 2\tilde{a}_h(\hat{v}, \hat{z}_h^\psi) = a(\chi, \hat{v}) - \tilde{a}_h(\psi_H, \hat{v}) - d(\hat{v}, \pi_H) - b(\hat{v}, \gamma_H^\psi), &\quad \forall \hat{v} \in \hat{V}_h, \\ \hat{z}_h^\pi \in \hat{Q}_h: &\ \ 2\Theta_h(\hat{q}, \hat{z}_h^\pi) = -d(\chi, \hat{q}) + d(\psi_H, \hat{q}) - \Theta_h(\pi_H, \hat{q}) - b(\hat{q}, \gamma_H^\pi), &\quad \forall \hat{q} \in \hat{Q}_h. \end{aligned}$$

5. Bounds

$$s_h^\pm = \bar{s}_h^\pm \pm \Delta s_h^\pm,$$

$$\bar{s}_h^\pm = \ell(\chi) - \ell(\psi_H) - \tilde{\ell}_h(\pi_H) + 2\tilde{a}_h(\hat{z}_h^\psi, \hat{z}_h^u) + 2\Theta_h(\hat{z}_h^\pi, \hat{z}_h^p),$$

$$\Delta s_h^\pm = 2\left\{ \left[\tilde{a}_h(\hat{z}_h^u, \hat{z}_h^u) + \Theta_h(\hat{z}_h^p, \hat{z}_h^p) - \ell(u_H) - \tilde{\ell}_h(p_H)\right]\left[\tilde{a}_h(\hat{z}_h^\psi, \hat{z}_h^\psi) - \Theta_h(\hat{z}_h^\pi, \hat{z}_h^\pi)\right]\right\}^{\frac{1}{2}}.$$

Fig. 1. Summary of Bounds Procedure For Force Ouputs

V. Numerical Example

As a demonstration of the effectiveness of this procedure, we present numerical results for two outputs from a single example problem. The example is the symmetric flow through a channel containing a square obstruction. Figure 2 details the geometry and boundary conditions as well as shows the coarsest mesh, containing 88 elements. The two outputs considered are the drag on the obstruction, detailed above, and the mass flow rate through the channel, defined as

$$s^{\text{mfr}}(u) = \int_\Omega u \, \mathrm{d}\Omega.$$

VI. Conclusions

We have demonstrated the application of a new finite element output bound framework for a previously unsolved problem, namely a stabilized equal-order finite element discretization of the incompressible Stokes flow equations. The method relies on the existence of an appropriate energy form to ensure the well-posedness of the bounding subproblems, a requirement which is at least partially obtained through the very nature of stabilization schemes. We are currently exploring the applicability of the method to the nonlinear Burger's equation, with the real target being the incompressible Navier-Stokes equations.

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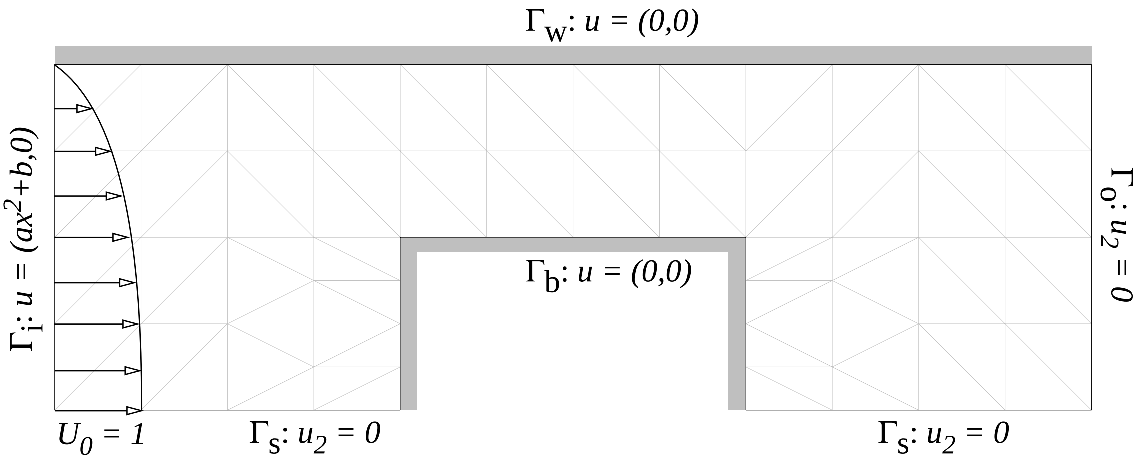

Fig. 2. The flow geometry and coarsest mesh, containing 88 elements, for the example of symmetric flow through a channel containing a square obstruction.

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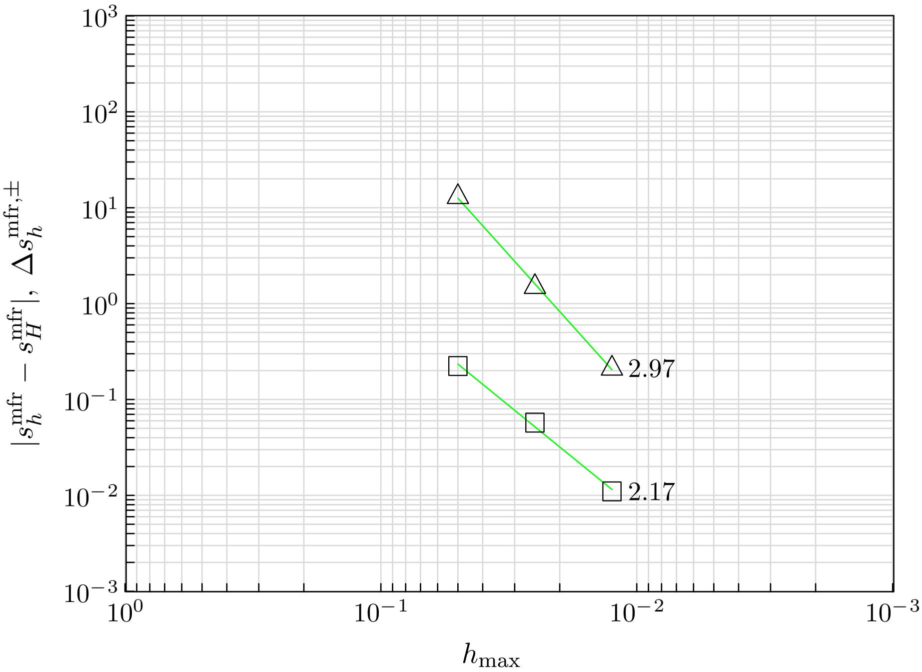

Fig. 3. The mass flow rate output error (□) and bound gap (△) convergence for the example of symmetric flow through a channel containing a square obstruction.

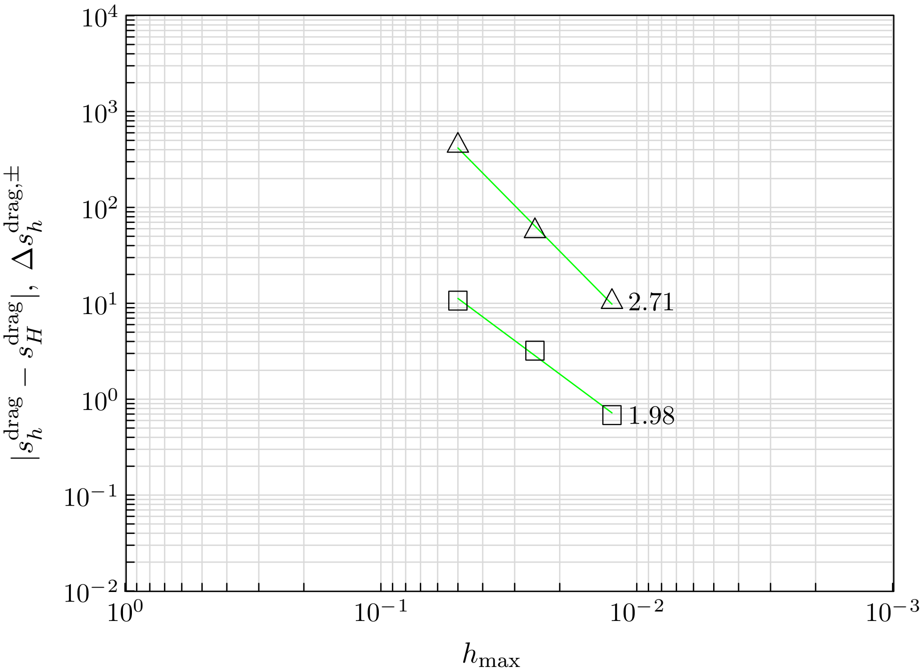

Fig. 4. The drag output error (□) and bound gap (△) convergence for the example of symmetric flow through a channel containing a square obstruction.