Multiresolution Laser Radar Range Profiling of Real Imagery

by

Donald Reed Greer

Submitted to the Department of Electrical Engineering and Computer Science in partial fulfillment of the requirements for the degrees of Master of Engineering in Electrical Engineering and Computer Science and Bachelor of Science in Electrical Engineering at the MASSACHUSETTS INSTITUTE OF TECHNOLOGY January 23, 1996

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Abstract

Coherent laser radars are capable of collecting range images by raster-scanning a field of view in pulsed-imager operation. In previous research work, the expectation-maximization (EM) algorithm has been used as a procedure for maximum-likelihood (ML) fitting of the multiresolution Haar-wavelet basis to simulated range imagery. This thesis continues the development by extending range profiling to real 3-D laser radar range imagery. In particular, a more powerful EM algorithm is designed using the special structure of the Haar-wavelet basis to reduce the computational complexity of the ML equation from quadratic to linear growth as a function of resolution and image size. Moreover, this new design results in an algorithm which is numerically robust. Given the dramatic increase in computational speed, real 3-D range imagery is profiled at various resolutions. Error analysis shows that range profiles at high resolutions are approximately unbiased and have error variances that approach the complete data bound.

Further improvements include developing modified weighting schemes which increase the computational speed of the algorithm while achieving similar performance in terms of the profile results and likelihood measurements. The robustness of the algorithm is analyzed by studying the effects of varying laser radar and image parameters. Finally, the multiresolution Haar-wavelet basis is examined in terms of its effectiveness for ML fitting of range imagery. For all work, the performance and analysis of this estimation scheme is evaluated via run-time, weighting, and likelihood measurements.

Thesis Supervisor: Jeffrey H. Shapiro
Title: Professor and Associate Head of Electrical Engineering
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Chapter 1

Introduction

Coherent laser radars are capable of collecting highly resolved range, intensity, and Doppler images by raster scanning a field of view [1], [2]. In this thesis, we focus on the image processing of range data. In pulsed-imager mode, the range image is produced by measuring for each pixel the time-of-flight between the transmitted laser pulse and the peak intensity of the video-detected return waveform. Range imagery is subject to fluctuations arising from the combined effects of laser speckle and local-oscillator shot noise. The former is due to the rough-surfaced nature of most reflecting surfaces measured on the scale of a laser wavelength which causes constructive and destructive interference in the laser pulses resulting in measurement anomalies [3]; the latter is the fundamental noise encountered in optical heterodyne detection [4]. The nature of these degradation processes has suggested a statistical approach to laser radar image processing.

In previous work, statistical modeling of peak-detecting coherent laser radars [5] has led to the development of techniques for performing target detection for 2-D imagers [6]-[8] and 3-D imagers [9]-[11]. In a recently completed Master's thesis by Irene Fung, maximum-likelihood (ML) estimation was used to fit the multiresolution Haar-wavelet basis—at a sequence of increasingly fine resolutions—to simulated 2-D laser radar range data [12]. In particular, the expectation-maximization (EM) algorithm was employed to find the ML range profile and the weights associated with the EM iterations were shown to provide a reliable indicator for terminating the
coarse-to-fine resolution progression. Simulation results confirmed that this algorithm showed good noise suppression. However, the application of this multiresolution ML imaging scheme to 3-D laser radar range data was significantly hindered by two major issues: computational load and numerical robustness. Consequently, range profiling was limited to relatively small images at low resolutions.

This thesis extends the multiresolution range profiling work to 3-D real laser radar range imagery. In particular, a more powerful EM algorithm is designed using the special structure of the Haar-wavelet basis to reduce the computational complexity of the ML equation from quadratic to linear growth as a function of resolution and image size. Moreover, this new design results in an algorithm which is numerically robust. The general theory for the fast ML/EM algorithm is presented. Given the dramatic increase in computational speed, real 3-D range imagery is profiled at various resolutions. The performance of the estimation scheme is evaluated via the weights associated with EM iterations, likelihood measurements, and error analysis.

Further work includes developing modified weighting schemes to increase the computational speed of the algorithm. The parameter robustness of the EM algorithm is also examined. For instance, the local range accuracy in real laser radar range imagery depends on the type of scenery and may thus be non-uniform across an image. This thesis looks at how variations in image parameters affect the ML range estimate. Finally, the multiresolution Haar-wavelet basis is examined in terms of its effectiveness for ML fitting of range imagery.

The remainder of this thesis is organized as follows. Chap. 2 describes the single-pixel statistical model and scene geometry assumed for the derivation of the multiresolution range profiler. Chap. 3 discusses maximum-likelihood (ML) range profile estimation via the expectation-maximization (EM) algorithm. In Chap. 4, the fast ML/EM algorithm is derived and its run-time measurements compared to the conventional EM algorithm. In Chap. 5, multiresolution range profiling is performed on real laser radar range imagery and the results are analyzed. Chap. 6 presents two modified weighting schemes and compares their performance in terms of profile results and likelihood measurements. Chap. 7 analyzes the parameter robustness
of the ML/EM algorithm in terms of how variations in local range accuracy affect the range estimate. Chap. 8 examines the effectiveness of the Haar-wavelet basis in representing or fitting laser radar range imagery. In Chap. 9, the major conclusions of the work are summarized.
Chapter 2

Measurement Models

In this chapter, we present the measurement models upon which the range profiling algorithm is based. In particular, we describe a single-pixel statistical model for laser radar range measurements and a model for the assumed geometry of the scene.

2.1 Single-Pixel Statistics

In pulsed imager mode, the laser radar produces range imagery by the following process. A coherent laser radar transmits a series of pulses – one for each pixel in a raster scan. Each laser pulse travels through the atmosphere until it reaches an object and is reflected back. The reflected light that is collected undergoes optical heterodyne detection, intermediate-frequency filtering, video detection, and peak detection as shown in Fig. 2-1. For each pixel, the delay between the peak of the transmitted pulse and the peak of the return waveform is recorded as the range measurement for that particular laser pulse. The complete set of measurements for the raster scan constitutes the range image of the scene.

Range measurements are subject to fluctuations arising from the combined effects of laser speckle and local-oscillator shot noise. Laser speckle is due to the rough-surfaced nature of most reflecting surfaces on the scale of a laser wavelength [3] and degrades imagery by creating range anomalies. This occurs when a deep target-return fade and a strong noise peak conspire to produce a range measurement far from the
true range [5] as shown in Fig. 2-2. Local-oscillator shot noise is due to fundamental noise encountered in optical heterodyne detection [4] and results in Gaussian noise in the local accuracy of range measurements.

![Block diagram of a monostatic, shared-optics coherent laser radar.](image)

Figure 2-1: Block diagram of a monostatic, shared-optics coherent laser radar.

![Peak detection range measurement examples showing nonanomalous and anomalous behavior.](image)

Figure 2-2: Peak detection range measurement examples showing nonanomalous and anomalous behavior. Herein, \( R^* \) is the true range value, \( R \) is the measured range, \( R_{\text{res}} \) is the range resolution, and \( \Delta R \equiv R_{\text{max}} - R_{\text{min}} \) is the range uncertainty interval.

For the single-pixel statistical model, we seek to describe the conditional probability density function (PDF) that a measured range value, \( r = R \), occurs, given a true range value, \( r^* = R^* \). This is obtained in the following derivation. Consider a radar system with range uncertainty interval, \( \Delta R \equiv R_{\text{max}} - R_{\text{min}} \), and nominal range resolution, \( R_{\text{res}} \approx cT/2 \), where \( c \) is the speed of light and \( T \) is the laser pulse duration. The range uncertainty interval contains \( N \equiv \Delta R/R_{\text{res}} \) non-overlapping
range resolution cells (see Fig. 2-2), \( N - 1 \) containing only noise and one, denoted \( N* \), containing noise and the reflector return. An anomaly (event \( A \)) is defined as a peak-detected range measurement located in one of the \( N - 1 \) noise cells while a non-anomaly (event \( \bar{A} \)) is defined as a peak-detected range measurement located in \( N* \). Since \( A \) and \( \bar{A} \) form a mutually exclusive, collectively exhaustive set of outcomes, Bayes’ rule can be applied to construct \( p_r(R) \), the PDF of range value, \( r \), for sample value \( R \), given by

\[
p_r(R) = p_{r|\bar{A}}(R|\bar{A})\Pr(\bar{A}) + p_{r|A}(R|A)\Pr(A)
\]

(2.1)

where \( \Pr(\bar{A}) = 1 - \Pr(A) \). For event \( A \), range values are assumed to be uniformly distributed over the uncertainty interval, \( \Delta R \) [5], given by

\[
p_{r|A}(R|A) = \frac{1}{\Delta R} \text{ for } R_{\text{min}} \leq R \leq R_{\text{max}}.
\]

(2.2)

For event \( \bar{A} \), range values are assumed to be Gaussianly distributed about the true range value, \( R* \), given by

\[
p_{r|\bar{A}}(R|\bar{A}) = \frac{\exp\left(-\frac{(R - R*)^2}{2\delta R^2}\right)}{\sqrt{2\pi R^2}} \text{ for } R_{\text{min}} \leq R, R* \leq R_{\text{max}}
\]

(2.3)

where \( \delta R \) is the local range accuracy and \( \Delta R \gg \delta R \). Combining Eqs. 2.1 - 2.3 results in the desired equation describing the conditional probability density function (PDF) that \( r = R \) given \( r* = R* \), given by

\[
p_{r|r*}(R|R*) = [1 - \Pr(A)]\frac{\exp\left(-\frac{(R - R*)^2}{2\delta R^2}\right)}{\sqrt{2\pi R^2}} + \Pr(A)\frac{1}{\Delta R} \text{ for } R_{\text{min}} \leq R, R* \leq R_{\text{max}}.
\]

(2.4)

The first term on the right in Eq. 2.4 is the contribution from nonanomalous pixels and represents the local range behavior. It is equal to the probability that the pixel is not anomalous times a Gaussian with mean equal to the true range, \( R* \), and standard deviation equal to the local range accuracy, \( \delta R \). The second term on the right is the
contribution from anomalous pixels and represents the global range behavior. It is equal to the probability the pixel is anomalous times a uniform probability density over the range uncertainty interval, \( \Delta R \).

In previous target detection studies, the local accuracy and range anomaly behavior incorporated in Eq. 2.4 have been demonstrated through theory, simulations, and experiment [5]. In terms of the range resolution \( R_{\text{res}} \), number of range bins \( N \), and carrier-to-noise ratio,

\[
\text{CNR} \equiv \frac{\text{average radar return power}}{\text{average local-oscillator shot noise power}},
\]

we can estimate the local range accuracy and probability of anomaly via

\[
\delta R \approx \frac{R_{\text{res}}}{\sqrt{\text{CNR}}}
\]

and

\[
\Pr(A) = \frac{1}{\text{CNR}} \left( \ln(N) - \frac{1}{N} + 0.577 \right), \text{ for CNR} \gg 1 \text{ and } N \gg 1.
\]

### 2.2 Scene Geometry

In previous work, the processors used in background range-plane estimation and target detection [10] and range profiling [12] assumed a downlooking geometry as shown in Fig. 2-3. This scene geometry is again assumed in multiresolution range profiling. For this work, the objective is to range profile an arbitrary scene so there is no restriction on the presence of any particular target or type of scenery.

The measured data is a 3-D range image, \( r = \{ r_{jk} : 1 \leq j \leq J, 1 \leq k \leq K \} \), where the value of \( r_{jk} \) represents the depth of the pixel. For notational convenience,
the range image is assembled into a single column vector of length $JK$:

$$
\mathbf{r} \equiv \begin{bmatrix}
\mathbf{r}_1 \\
\mathbf{r}_2 \\
\vdots \\
\mathbf{r}_j \\
\vdots \\
\mathbf{r}_J 
\end{bmatrix} \quad \text{where} \quad \mathbf{r}_j \equiv \begin{bmatrix}
\mathbf{r}_{j1} \\
\mathbf{r}_{j2} \\
\vdots \\
\mathbf{r}_{jK}
\end{bmatrix} \quad \text{for } 1 \leq j \leq J. \quad (2.8)
$$

The angular spacing between pixels is assumed to be sufficiently large to ensure statistical independence. Hence, the joint PDF of the complete range image, $\mathbf{r}$, can be represented as the product of the single-pixel PDF’s as described in section 3.1. In the next chapter, we develop our multiresolution range profiler based upon this statistical model of the range data.
Chapter 3

Range Profile Estimation

In this chapter, we present a framework for maximum-likelihood (ML) range profile estimation via the expectation-maximization (EM) algorithm. This estimation procedure was previously developed for multiresolution range profiling of low-dimensional, simulated imagery [12], and is described here for clarity and as a basis for further work. In Sec. 4.4, the EM algorithm is modified and optimized so that it can meet the computational demands of range profiling much larger range imagery.

3.1 Range Profile Estimation Problem

The objective of laser radar range profiling is to find the optimal estimate of the true range image, $r^*$, given a range data image, $r$, and resolution constraint. This resolution constraint serves the purpose of suppressing anomalies in the range data while at the same time giving the desired resolution of image features. This will be described in more detail in Sec. 3.2.

Suppose we collect a $Q$-pixel range image of some field of view, $\{r_q : 1 \leq q \leq Q\}$, having the respective true range values, $\{r^*_q : 1 \leq q \leq Q\}$. As discussed in Sec. 2.2, we assume that the pixel spacing is large enough to ensure uncorrelated speckle on each radar return [13], so that the range measurements are statistically independent. Thus the joint probability density that $r = R$, given $r^* = R^*$, is equal to the product
of the single-pixel PDF’s (see Eq. 2.4) for each pixel, given by

\[
p_{Pr|R^*}(R|R^*) = \prod_{q=1}^{Q} \left[ 1 - Pr(A) \right] \frac{\exp \left( \frac{-(R_q - R_q^*)^2}{2\delta R^2} \right)}{\sqrt{2\pi\delta R^2}} + \frac{Pr(A)}{\Delta R}. \]  

(3.1)

The laser radar range profiling problem is to find the optimal estimate of the true range image, \( r^* \), given the range image, \( r \). To do this, we employ maximum-likelihood (ML) estimation. Given a particular data vector, \( R \), the maximum-likelihood estimate, \( \hat{R}_{ML} \), of the range data is, by definition, the \( R^* \) that maximizes \( p_{Pr|R^*}(R|R^*) \). However, this also implies that the best ML estimate that can be obtained is the range data itself. Mathematically,

\[
\hat{r}_{ML}(R) = \arg \max_{R^*} \left( \frac{p_{Pr|R^*}(R|R^*)}{Pr(A)} = R. \right) \]  

(3.2)

Thus ML estimation without any resolution constraint does not suppress anomalies at all. This would be a serious problem since our objective is to find the ML image while suppressing anomalies which may include 10% or more of the range pixels. In the next section, we show how the EM algorithm solves this anomaly problem.

### 3.2 Parametric Range Profiling

In previous work, range profiling was developed to estimate a planar background in an image [10]. This was then extended to the more general parametric profiling developed to estimate the true range image for any type of scenery [12]. Here we present the theory behind parametric range profiling.

Given a true range vector, \( r^* \), of length \( Q \),

\[
r^* = \begin{bmatrix} r^*_1 \\ \vdots \\ r^*_Q \end{bmatrix}, \]  

(3.3)
which we wish to represent by the parameter vector, \( \mathbf{x} \), also of length \( Q \),

\[
\mathbf{x} \equiv \begin{bmatrix} x_1 \\ \vdots \\ x_Q \end{bmatrix}.
\] (3.4)

We define \( \{ \Phi_q : 1 \leq q \leq Q \} \) to be an arbitrary orthonormal column-vector basis for the \( Q \)-length vector space. These vectors are used to construct a \( Q \times Q \) transformation matrix, \( \mathbf{H} \),

\[
\mathbf{H} = [\Phi_1 \ \Phi_2 \ \cdots \ \Phi_Q].
\] (3.5)

By multiplying \( \mathbf{H}^T \) by \( \mathbf{R}^* \), the true range vector can be transformed into the parameter vector, \( \mathbf{x} \),

\[
\mathbf{x} = \mathbf{H}^T \mathbf{r}^*.
\] (3.6)

Since \( \{ \Phi_q \} \) forms an orthonormal basis, \( \mathbf{H} \) is orthonormal and therefore \( \mathbf{H}^{-1} = \mathbf{H}^T \). Thus, by multiplying \( \mathbf{H} \) to each side of Eq. 3.6 and simplifying, we can write

\[
\mathbf{r}^* = \mathbf{H} \mathbf{x}.
\] (3.7)

Suppose we know that the true range profile, \( \mathbf{r}^* \), can be characterized by a parameter vector, \( \mathbf{x} \), of length \( P \) where \( P < Q \) (i.e., only the first \( P \) rows of \( \mathbf{x} \) are non-zero.) Thus we can write

\[
\mathbf{x}_P = \mathbf{H}_P^T \mathbf{r}^*,
\] (3.8)

where

\[
\mathbf{x}_P = \begin{bmatrix} x_1 \\ \vdots \\ x_P \end{bmatrix}
\] (3.9)

and

\[
\mathbf{H}_P = [\Phi_1 \ \Phi_2 \ \cdots \ \Phi_P].
\] (3.10)
By letting $\mathbf{H}_P^c$ equal the remaining components of $\mathbf{H}$,

$$
\mathbf{H}_P^c = [\Phi_{P+1} \; \Phi_{P+2} \; \cdots \; \Phi_Q],
$$

(3.11)

we have

$$
\mathbf{H} = [\mathbf{H}_P \; \mathbf{H}_P^c].
$$

(3.12)

Thus for the parameter vector $\mathbf{x}$ with $P$ non-zero components, we can write

$$
\mathbf{x} = \begin{bmatrix} \mathbf{x}_P \\ 0 \end{bmatrix} = \mathbf{H}^T \mathbf{r}^* = \begin{bmatrix} \mathbf{H}_P^T \mathbf{r}^* \\ \mathbf{H}_P^c \mathbf{r}^* \end{bmatrix}.
$$

(3.13)

The last $Q - P$ columns in $\mathbf{H}$ are not used in characterizing the range truth and hence are not used in finding the range estimate. Furthermore, this shows that the resolution, $P$, of the range estimate can be selected by constructing a $Q \times P$ transformation matrix, $\mathbf{H}_P$, from $P$ orthonormal column vectors.

The process of representing a range estimate at a lower resolution ($P < Q$) than the original data involves suppressing certain data. In our problem, this is quite useful since our the objective is to suppress the anomalous data. Thus, given an estimate for the number of anomalous pixels in the range data, we seek to find a resolution that suppresses approximately that number of data measurements. This procedure is known as the stopping rule [12] and will be applied to real imagery in Ch. 5 to find the optimal range estimate.

Given this design, we have derived a representation for the range truth, $\mathbf{r}^* = \mathbf{H}_P \mathbf{X}_P$, and now substitute this into Eq. 3.1. The conditional probability that $\mathbf{r} = \mathbf{R}$, given $\mathbf{x}_P = \mathbf{X}_P$, is given by

$$
p_{\mathbf{r}|\mathbf{x}_P}(\mathbf{R}|\mathbf{X}_P) = \prod_{q=1}^{Q} \left[ 1 - \Pr(A) \right] \frac{\exp \left( -\frac{(R_q - (\mathbf{H}_P \mathbf{X}_P)_q)^2}{2\delta R^2} \right)}{\sqrt{2\pi\delta R^2}} + \frac{\Pr(A)}{\Delta R}
$$

(3.14)

where $R^*_q = (\mathbf{H}_P \mathbf{X}_P)_q$ is the $q$th component of $\mathbf{R}^*$. In the next section, we will solve
for the maximum likelihood estimate, $\hat{x}_{P_{ML}}$.

### 3.3 Maximum-Likelihood Estimation

The maximum-likelihood estimate of the parameter vector, $\hat{x}_{P_{ML}}$, is by definition, the $X_P$ that maximizes $p_{R|X_P}(R|X_P)$, given a particular range data vector, $R$. Thus, in Eq. 3.14, we seek to maximize the product of the $Q$ single-pixel PDF’s, given a particular $R$. Since probability densities are always non-negative, it is equivalent to maximize the logarithm of the likelihood function and is analytically much easier to do so. Thus the ML estimate satisfies

$$\frac{\partial}{\partial X_P} \ln[p_{R|X_P}(R|X_P)]_{X_P=\hat{x}_{P_{ML}}} = 0, \quad (3.15)$$

the necessary condition for an extremum at $X_P = \hat{x}_{P_{ML}}$. Substituting Eq. 3.14 into Eq. 3.15 and doing the indicated differentiation leads to the following nonlinear vector equation in $\hat{x}_P$,

$$\frac{\partial}{\partial X_P} \ln[p_{R|X_P}(R|X_P)]_{X_P=\hat{x}_{P_{ML}}} = \frac{1}{\delta R^2} H_P^T W(X_P)(R - H_P X_P)_{X_P=\hat{x}_{P_{ML}}} = 0, \quad (3.16)$$

where $W(X_P)$ is a $Q \times Q$ diagonal matrix where the $qq$th element is the $q$th-pixel weight, $w_q$, defined as

$$w_q(x_P) = \frac{\exp\left(-\frac{(R_q - (H_P X_P)_q)^2}{2\delta R^2}\right)}{[1 - Pr(A)]^{\frac{\sqrt{2\pi}\delta R^2}{2\delta R^2}} + \frac{Pr(A)}{\Delta R}}. \quad (3.17)$$

Note that the weights are proper fractions, $0 \leq w_q \leq 1$, representing the conditional probability that the associated pixel is not anomalous. Furthermore, if the anomaly probability is very small such that $w_q \approx 1$ for all $q$, then $W(X_P)$ is approximately...
the identity matrix, \( \mathbf{I} \). Then Eq. 3.16 becomes linear and its solution is easily shown to be [14]
\[
\hat{x}_{PM} = (H_P^T H_P)^{-1} H_P^T R, \tag{3.18}
\]
which, since \( \mathbf{H} \) is orthogonal, can be further reduced to become
\[
\hat{x}_{PM} = H_P^T R. \tag{3.19}
\]

Unfortunately, laser radar operation is often in the regime in which the anomaly probability, \( \Pr(A) \), is substantial. This prevents us from taking advantage of the above simplification and requires that we solve the nonlinear estimation problem of Eq. 3.16. In the next section, we present the expectation-maximization (EM) algorithm as our route to solving this nonlinear problem.

### 3.4 Expectation-Maximization Algorithm

The EM algorithm is well suited to ML estimation problems wherein the observed vector constitutes incomplete data [16]. In these situations, only part of the complete data vector is observed and there is a degree of modeling freedom available in establishing the complete data vector. In our ML range-profiling context, the complete data vector, \( \mathbf{y} \), is given by
\[
\mathbf{y} \equiv \begin{bmatrix} \mathbf{r} \\ \mathbf{a} \end{bmatrix} \tag{3.20}
\]
where \( \mathbf{r} \) is the range-data vector—our observations—and
\[
\mathbf{a} \equiv \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_Q \end{bmatrix} \tag{3.21}
\]
is the anomaly data—the missing part of the complete data. Here

$$a_q = \begin{cases} 
0, & \text{if } r_q \text{ is anomalous}, \\
1, & \text{if } r_q \text{ is not anomalous}.
\end{cases} \quad (3.22)$$

Were the complete data, $y$, available, we could easily identify and suppress the anomalous pixels thus reducing the ML estimation task to a linear problem involving only the nonanomalous pixels. However, since $a$ is not directly observed, we deal with the possibility of anomalies in a statistical fashion by applying a pixel-weighting scheme. As a result, the estimation problem is inevitably nonlinear, but the linearity of the complete data estimation task makes the EM algorithm computationally simple.

### 3.4.1 EM Algorithm

Starting from any initial estimate of the parameter vector, $\hat{x}_p(0)$, the expectation-maximization algorithm produces a sequence of estimates, $\{\hat{x}_p(n) : n = 1, 2, 3, \ldots\}$, through an iterative sequence of expectation and maximization steps. Moreover, the corresponding likelihood sequence, $\{p_{R|X_p}(R|\hat{x}_p(n)) : 1, 2, 3, \ldots\}$, is monotonically nondecreasing. Hence, the EM algorithm will climb a hill on the surface, $\{p_{R|X_p}(R|X_p) : X_p \in \chi\}$, where $\chi$ is the set of possible parameter vectors.

If the initial estimate is good enough to place the EM algorithm on the highest hill, the global maximum will be achieved. For imagery with very low anomaly probabilities, a linear least-squares estimate may be sufficient for initializing the EM algorithm. Unfortunately, for most imagery, such initialization will not reliably locate the global maximum. The recursive expectation-maximization (REM) algorithm is suggested as an alternative initialization process [10] and is discussed in section 3.4.2.

**Initialization**

For presentation purposes, let us use the linear least-squares estimate to initialize the EM algorithm. In other words, we initialize by assuming that all of the range data is nonanomalous. This corresponds to solving Eq. 3.16 with $W \equiv I$ which is
precisely what we solved for in Eq. 3.19. Thus as an initial estimate, \( \hat{x}_p(0) \), we use

\[
\hat{x}_p(0) = H_P^T R. \tag{3.23}
\]

**Update Procedure**

In the EM algorithm, after the \( n \)th estimate, for \( n = 0, 1, 2, \ldots \), we have calculated the current estimate, \( \hat{x}_p(n) \), and its corresponding weighting matrix, \( W(n) \). The EM algorithm updates the current estimate to \( \hat{x}_p(n+1) \) by the following two-step procedure:

1. The *expectation* step updates the weights via

\[
w_q(n + 1) = \frac{\exp \left( -\frac{[R_q - (H_P \hat{x}_p(n) )_q]^2}{2\delta R^2} \right)}{[1 - \Pr(A)] \sqrt{2\pi \delta R^2}} + \frac{\Pr(A)}{\Delta R}
\]

for \( n = 0, 1, 2, \ldots \) and \( 1 \leq q \leq Q \).

2. The *maximization* step then updates the estimate via

\[
\hat{x}_p(n + 1) \equiv (H_P^T W(n + 1) H_P)^{-1} H_P^T W(n + 1) R, \text{ for } n = 0, 1, 2, \ldots \tag{3.25}
\]

Basically, we use the latest estimate to update the weights and then solve the linear estimation problem using the new weights. As discussed earlier, the EM algorithm produces a series of estimates which are monotonically nondecreasing in likelihood. Thus, it is natural to terminate the iterative procedure when the difference between successive likelihoods is within some acceptable tolerance. The final estimate, \( \hat{x}_p(n + 1) \), is the ML estimate.

In theory, the inverse of the matrix \( H_P^T W(n + 1) H_P \) in Eq. 3.25 should always exist [12]. For \( \Pr(A) < 1 \), \( w_q > 0 \) for \( 1 \leq q \leq Q \) so the matrix \( W \) is positive definite. This theoretically guarantees that the matrix \( H_P^T W(n + 1) H_P \) has rank = \( P \) and is therefore invertible. In practice, however, computational underflow occurs when
\[ w_q(n) \text{ is many } \delta R \text{ from the range estimate so } w_q(n) \approx 0. \] This may result in the matrix \( H_p^T W(n + 1) H_p \) having rank \( < P \) making the matrix non-invertible and causing the algorithm to fail. The fast EM algorithm described in Sec. 4.4 overcomes this difficulty in solving for the ML range estimate.

### 3.4.2 Recursive EM Algorithm

The REM algorithm applies a recursive approach to produce successively better initial estimates for the EM algorithm. It has been demonstrated in planar [10] and parametric [12] range profiling work that the REM algorithm obtains a more reliable estimate than the least-squares (LS) initiated EM algorithm.

The REM algorithm begins by setting the local range accuracy, \( \delta R \), in Eq. 3.14, equal to the range uncertainty interval, \( \Delta R \), so \( \delta R_0 \equiv \Delta R \). The resulting density is employed in an LS-initialized EM algorithm to obtain a zeroth-order estimate, \( \hat{x}_{REM}(0) \). The REM algorithm then resets the local range accuracy to \( \delta R_1 \equiv \delta R_0 / 2 = \Delta R / 2 \). The resulting density is employed in an \( \hat{x}_{REM}(0) \)-initialized EM algorithm to obtain a first-order estimate, \( \hat{x}_{REM}(1) \). This recursive procedure is continued until, in some final stage, \( m \), the local accuracy is set to \( \delta R_m \equiv \delta R \). The output of this stage is the final REM estimate, \( \hat{x}_{REM} = \hat{x}_{REM}(m) \).

The REM algorithm described above has significant improvements over the LS-initialized EM algorithm. In the REM algorithm, the local accuracy is gradually decreased from a very large value, \( \delta R_0 = \Delta R \), to its true value, \( \delta R_m = \delta R \). As a result, only a few pixels which appear most likely be anomalous are discarded in each stage. This reduces the chance that a lot of nonanomalous pixels are suppressed from the estimate because of the quality of the LS initial estimate. Thus, it is more likely that the REM estimate will coincide with the ML estimate.

### 3.5 Error Performance

Our objective in multiresolution range profiling is to find the ML range estimate. Error performance plays an important role in determining the quality of the estimate.
In this section, we describe various measures of error and then describe the Cramér-Rao inequality and find related bounds on error performance.

### 3.5.1 Measures of Error

For any estimator, $\hat{\mathbf{x}}_P$, of the unknown parameter vector, $\mathbf{x}_P$, the estimation error is defined, $\mathbf{e} \equiv \mathbf{x}_P - \hat{\mathbf{x}}_P$. The error performance of the estimator can be characterized by the bias vector,

$$ b(\mathbf{X}_P) \equiv E(\mathbf{e}|\mathbf{x}_P = \mathbf{X}_P), \quad (3.26) $$

and error covariance matrix,

$$ \Lambda_e(\mathbf{X}_P) \equiv E\{[\mathbf{e} - b(\mathbf{X}_P)][\mathbf{e} - b(\mathbf{X}_P)]^T|\mathbf{x}_P = \mathbf{X}_P\}. \quad (3.27) $$

Ideally, we would like to have an unbiased estimator,

$$ b(\mathbf{X}_P) = 0, \text{ for all } \mathbf{X}_P, \quad (3.28) $$

with minimum-variance estimation error, thus minimizing

$$ \text{var}(\mathbf{x}_p - \hat{\mathbf{x}}_p) = [\Lambda_e(\mathbf{X}_P)]_{pp}, \text{ for } p = 1, 2, ..., P \text{ and all } \mathbf{X}_P. \quad (3.29) $$

If the estimate vector is unbiased, the minimum-variance estimation error is equivalent to the minimum mean-squared error, $\xi^2_p(\mathbf{X}) \equiv E[(\mathbf{x}_p - \hat{\mathbf{x}}_p)^2]$ for $p = 1, 2, ..., P$.

### 3.5.2 The Cramér-Rao Inequality

The Cramér-Rao inequality is a theorem that places a lower bound on the error covariance of an unbiased estimator, thus serving as a standard for the performance of our estimate. In particular, the Cramér-Rao inequality states that the error covariance matrix of any unbiased estimator satisfies

$$ \Lambda_e(\mathbf{X}_P) \geq \mathbf{I}_r(\mathbf{X}_P)^{-1} \quad (3.30) $$
where

\[ I_r(X_p) = E \left\{ \left[ \frac{\partial}{\partial X_p} \ln[p_r|X_p(R|X_p)] \right] \left[ \frac{\partial}{\partial X_p} \ln[p_r|X_p(R|X_p)] \right]^T \right\} x_p = X_p \right\}, \quad (3.31) \]

is the Fisher information matrix for estimating \( x_p \) from \( r \). Several important observations can be made about this inequality [14]. It shows that any unbiased estimator must have some minimum variance. Any unbiased estimate whose covariance matrix equals the inverse Fisher information matrix is called an \textit{efficient} estimate. In general, efficient estimators do not exist, but if one does exist, it is the maximum-likelihood estimate, \( \hat{x}_{p_{ML}} \). An estimator is unbiased and efficient if and only if [15]

\[
\hat{x}_p(R) = X_p + I_r^{-1}(X_p) \frac{\partial}{\partial X_p} \ln[p_r|X_p(R|X_p)] \quad (3.32)
\]

where the right-hand side of this equation must be independent of \( X_p \). Thus we can determine if an efficient, unbiased estimator exists by evaluating Eq. 3.32 and checking whether or not it is independent of \( X_p \).

### 3.5.3 Bounds on Error Performance

For our problem, we obtain the information matrix by substituting Eq. 3.16 into Eq. 3.31:

\[ I_r(X_p) = \frac{1}{(\delta R^2)^2} H_P^T E \left\{ W(X_p)(R - H_pX_p)(R - H_pX_p)^T W(X_p) \right\} x_p = X_p \right\} H_P. \quad (3.33) \]

For this general case, it is difficult to evaluate the Fisher information matrix for estimating \( x_p \) from \( r \), due to the log-likelihood’s nonlinear dependence on \( R \). However, for some special cases, it can be evaluated quite readily.

When \( Pr(A) = 0 \), all range measurements are nonanomalous and hence, \( W = I \). Then the estimation problem reduces to Gaussian estimation with the maximum-likelihood estimate given in Eq. 3.19. For \( W = I \), the information matrix is given
by
\[
I_r(X_P) = \frac{1}{(\delta R^2)^2} H_P^T E \left\{ (R - H_P X_P)(R - H_P X_P)^T \Big| X_P = X_P \right\} H_P.
\] (3.34)

In the above equation, since \( R^* = H_P X_P \), the expectation is equal to the covariance matrix, \( \Lambda_r = \delta R^2 I_P \). Substituting this into the above equation and substituting the result into Eq. 3.30, we find the Cramér-Rao bound,

\[
\Lambda_e(X_P) \geq \delta R^2 I_P
\] (3.35)

In fact, this estimator, \( \hat{x}_{PML} = H_P R \), is unbiased and efficient and has an error covariance that equals the lower bound, \( \Lambda_e(X) = \delta R^2 I_P \) [14].

When \( \text{Pr}(A) > 0 \), the Fisher information matrix can be readily calculated for estimating \( x_P \) from the complete data vector, \( y \), because we are back to a linear problem. In actuality, we do not have the complete data vector, which entails knowing precisely which pixels of the range data are anomalous, but presuming the availability of this extra information only weakens the bound. Thus the resulting complete-data bound for \( \hat{x}_P(r) \) is [10],

\[
\Lambda_e(X_P) \geq I_r(X_P)^{-1} \geq I_y(X_P)^{-1} = \frac{\delta R^2}{1 - \text{Pr}(A)} I_P.
\] (3.36)

The range estimate, \( \hat{r} \), is a linear transformation of the parameter-vector estimate, \( \hat{x}_P \), viz., \( \hat{r} = H_P \hat{x}_P \). Thus the information matrix, \( I_y(R) \), can be readily calculated from \( I_y(X_P) \),

\[
I_y(R) = H_P I_y(X_P) H_P^T.
\] (3.37)

Then the complete data bound in terms of the range estimate \( R \) is given by

\[
\Lambda_e(R) \geq I_y^{-1}(R) = \frac{\delta R^2}{1 - \text{Pr}(A)} H_P H_P^T.
\] (3.38)

Note that the complete data bound becomes the Cramér-Rao bound when \( \text{Pr}(A) = 0 \).

It has been shown for planar range profiling that ML estimation performance can
approach the complete data bound for $\Pr(A) \geq 0$ [10], [11]. In our work, we will look at how closely multiresolution range profiling approaches this bound.
Chapter 4

Range Profiling Using the Haar-Wavelet Basis

In this chapter, the EM algorithm is applied to range imagery using the Haar-wavelet basis as the orthonormal transformation matrix, \( H \), described in Sec. 3.2. Results are shown for a sample range image profiled at various resolutions.

The Haar-wavelet ML/EM algorithm has been shown to be an effective method for profiling simulated range imagery [12]. However, it also has some significant limitations. First, due to the computational complexity of the algorithm, its application is limited to profiling relatively small imagery at low resolutions. Second, the algorithm fails if the matrix \( H^T WH_P \) in Eq. 3.25 becomes numerically non-invertible.

In the latter part of the chapter, the computational complexity of the EM algorithm is analyzed by looking at the run-times to profile various images of different sizes, \( Q \), at different resolutions, \( P \). A more efficient and powerful version of the EM algorithm is then presented. This algorithm takes advantage of the structure of the Haar-wavelet basis to dramatically improve computational speed and to eliminate the problem of non-invertibility of the matrix \( H^T WH_P \). Run-time comparisons show the speed up to be several orders of magnitude. This fast EM algorithm makes it possible to extend multiresolution range profiling to much larger real imagery at much higher resolutions.
4.1 Haar-Wavelet Basis

The Haar-wavelet basis [17] is used to construct the orthonormal transformation matrix, \( H \), described in Sec. 3.2. This basis is composed of the set of orthonormal Haar-wavelets, \( \{ \Phi_p \in \mathbb{R}^Q \} \), placed in order of increasingly high resolution. Thus, the multiresolution nature of the Haar-wavelet basis makes it possible to profile data at any particular resolution, \( P \leq Q \), by constructing \( H \) from \( \{ \Phi_p : 1 \leq p \leq P \} \). For this work, both \( P \) and \( Q \) are defined to be powers of 2 such that \( P \leq Q \).

In this section, first we describe the one-dimensional Haar-wavelet basis and then use this to construct the two-dimensional Haar-wavelet basis. This basis is used to profile the 3-D laser radar range imagery.

4.1.1 One-dimensional Haar-Wavelets

The set of 1-D Haar wavelets, \( \{ \Phi_p \} \), are defined:

\[
\Phi_p = \begin{bmatrix}
\phi_{p1} \\
\vdots \\
\phi_{pQ}
\end{bmatrix}, \quad \text{for} \quad 1 \leq p \leq P, \tag{4.1}
\]

where

\[
\phi_{1q} = \frac{1}{\sqrt{Q}}, \quad \text{for} \quad 1 \leq q \leq Q, \tag{4.2}
\]

and

\[
\phi_{pq} = \frac{\psi_n[qQ[(p-1)2^{-(n-1)} - 1]]}{\sqrt{Q}}, \quad \text{for} \quad 1 \leq q \leq Q \quad \text{and} \quad 1 \leq 2^{n-1} < p \leq 2^n \leq Q. \tag{4.3}
\]

In the above equation,

\[
\psi_n[q] \equiv 2^{(n-1)/2} \psi[2^{n-1}q], \quad \text{for} \quad 1 \leq q \leq Q, \tag{4.4}
\]
where

\[
\psi[q] = \begin{cases} 
0 : & q \leq 0, \\
1 : & 1 \leq q \leq Q/2, \\
-1 : & Q/2 < q \leq Q, \\
0 : & q > Q.
\end{cases}
\] (4.5)

To illustrate the nature of the Haar-wavelet basis, the set of \(\{\Phi_p : 1 \leq p \leq P\}\) are shown below for \(P = 8\) and \(Q = 8\):

\[
\begin{align*}
\Phi_1^T &= \sqrt{\frac{1}{8}} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}, \\
\Phi_2^T &= \sqrt{\frac{1}{8}} \begin{bmatrix} 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \end{bmatrix}, \\
\Phi_3^T &= \sqrt{\frac{2}{8}} \begin{bmatrix} 1 & 1 & -1 & -1 & 0 & 0 & 0 & 0 \end{bmatrix}, \\
\Phi_4^T &= \sqrt{\frac{2}{8}} \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 1 & -1 & -1 \end{bmatrix}, \\
\Phi_5^T &= \sqrt{\frac{4}{8}} \begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \\
\Phi_6^T &= \sqrt{\frac{4}{8}} \begin{bmatrix} 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \end{bmatrix}, \\
\Phi_7^T &= \sqrt{\frac{4}{8}} \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \end{bmatrix}, \\
\Phi_8^T &= \sqrt{\frac{4}{8}} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix}.
\end{align*}
\] (4.6)

The structure of these wavelets shows a progression in fine scale behavior. Since the range estimate is composed of some linear combination of the wavelets \(\{\Phi_p\}\), the multiresolution nature of the Haar-wavelet basis allows for increasingly fine piecewise approximations of the range data. At the lowest resolution \(P = 1\), \(H_1 = [\Phi_1]\) and the range estimate is a single constant-valued \(Q\)-length interval. At \(P = 2\), \(H_2 = [\Phi_1 \Phi_2]\) and the range estimate is composed of two piecewise constant \(Q/2\)-length intervals. In general, \(H_P = [\Phi_1 \Phi_2 \cdots \Phi_P]\) and the range estimate is composed of \(P\) piecewise constant \(Q/P\)-length intervals.

### 4.1.2 Two-dimensional Haar-Wavelets

The extension from a 1-D to 2-D Haar-wavelet basis is readily achieved by generating two 1-D bases and letting their product equal the 2-D Haar-wavelet basis. Thus,
given two 1-D bases, \( H_{P_j} \) which is \( Q_j \times P_j \) and \( H_{P_K} \) which is \( Q_K \times P_K \),

\[
H_{P_j} \equiv [\Phi_1 \cdots \Phi_j \cdots \Phi_{P_j}] \text{ where } \Phi_j = \begin{bmatrix} \phi_{j1} \\ \vdots \\ \phi_{jQ_j} \end{bmatrix}, \text{ for } 1 \leq j \leq P_j, \tag{4.7}
\]

and

\[
H_{P_K} \equiv [\Phi_1 \cdots \Phi_k \cdots \Phi_{P_K}] \text{ where } \Phi_k = \begin{bmatrix} \phi_{k1} \\ \vdots \\ \phi_{kQ_K} \end{bmatrix}, \text{ for } 1 \leq k \leq P_K, \tag{4.8}
\]

the 2-D Haar-wavelet basis, \( H_{P_jP_K} \), is given by

\[
H_{P_jP_K} \equiv [\Psi_{11} \Psi_{12} \cdots \Psi_{jk} \cdots \Psi_{P_jP_K}], \text{ for } 1 \leq j \leq P_j \text{ and } 1 \leq k \leq P_K, \tag{4.9}
\]

where \( \{\Psi_{jk}\} \) is the separable column-vector basis for the \( Q = Q_jQ_K \)-length vector space, given by

\[
\Psi_{jk} = \begin{bmatrix} \Phi_j \phi_{k1} \\ \Phi_j \phi_{k2} \\ \vdots \\ \Phi_j \phi_{kQ_K} \end{bmatrix}. \tag{4.10}
\]

Note that different values can be associated with the resolutions of the two 1-D bases. Thus it is possible to estimate the range data using different resolutions in elevation and azimuth directions. Both \( P_j \) and \( P_K \) are defined to be powers of 2, \( P_j = 2^j \) and \( P_K = 2^K \), and represent the number of non-zero parameters in the elevation and azimuth respectively. For notational clarity, we define \( P \equiv P_jP_K \) and \( \Phi_p \equiv \Psi_{jk} \) where \( 1 \leq p \leq P, 1 \leq j \leq J, \) and \( 1 \leq k \leq K \). Thus, the \( Q \times P \) 2-D Haar-wavelet basis, \( H_P \), in Eq. 4.9 can be written as

\[
H_P \equiv H_{P_jP_K} \equiv [\Psi_{11} \Psi_{12} \cdots \Psi_{P_jP_K}]
\equiv [\Phi_1 \Phi_2 \cdots \Phi_P]. \tag{4.11}
\]
4.2 Range Profiling Results

The EM algorithm is now applied to the range image of a truck, provided by MIT Lincoln Laboratory, shown in Fig. 4-1 to illustrate range profiling using the 2-D Haar-wavelet transformation matrix, $H_P$. In Sec. 5.1, we will discuss the laser radar range imagery in detail, but in this section, it suffices to describe some general features of the tank image. The shade of each pixel represents the distance in range bins (see Fig. 2-2) where one range bin equals 1.1 meters. The image is contained in the region from 760 to 850 range bins as shown in the calibration on the right. Since there is an additional range offset of 430 meters, the actual distance is approximately 1260 to 1360 meters. The range image is a 32 x 32-pixel ($Q = 1024$) section of real imagery which is taken to be the range truth, $R^*$, for this demonstration of ML range profiling.

![Figure 4-1: Range truth, $R^*$, of a sample image of a truck.](image)

The range data, $R$, is generated from the range truth based on the single-pixel statistical model of Sec. 2.1. In Sec. 5.1, we will discuss why the range data, $R$, is simulated from the range image, but in this section, it suffices to describe some general properties of the simulated range image. In particular, zero-mean Gaussian noise with a standard deviation, $\delta R = 2$, is added to each pixel of the range truth. Then anomalies are simulated with a 5% probability rate, and if an anomaly does occur, the resulting measurement has a uniform probability density across the range gate, $\Delta R = 1524$. The resulting range image, $R$, is shown in Fig. 4-2.

The range data is profiled at various resolutions, $P = P_j \times P_k$, using the fast EM
Figure 4-2: Range data, $R$, of a sample image of a truck where local range accuracy, $\delta R = 2$, $Pr(A) = 0.05$, and range gate, $\Delta R = 1524$.

algorithm (see Sec. 4.4.) Fig. 4-3 is profiled at $P = 64$ where $P_j = 8$ and $P_k = 8$; Fig. 4-4 is profiled at $P = 128$ where $P_j = 8$ and $P_k = 16$; and Fig. 4-5 is profiled at $P = 256$ where $P_j = 16$ and $P_k = 16$.

Figure 4-3: Haar-fitted range profile at $P = 64$ where $P_j = 8$ and $P_k = 8$.

In the resulting profiles, there appears to be an almost complete absence of anomalies, which are glaringly obvious as black and white speckles in the range data, $R$, in Fig. 4-2. Thus, the EM algorithm is clearly successful in suppressing anomalies. As the resolution increases from $P = 64$ to $P = 256$ in Figs. 4-3 to 4-5, the details of the truck gradually appear more clearly defined. In this section, some general features of multiresolution range profiling have been presented, but in Ch. 5 a more extensive analysis will be done.
Figure 4-4: Haar-fitted range profile at $P = 128$ where $P_j = 8$ and $P_k = 16$.

Figure 4-5: Haar-fitted range profile at $P = 256$ where $P_j = 16$ and $P_k = 16$. 
4.3 Computational Complexity of EM Algorithm

The computational complexity of the EM algorithm can be analyzed as a function of image size, \( Q \), and profile resolution, \( P \). The calculational load of the EM algorithm is dominated by the maximization step given by Eq. 3.25,

\[
\hat{x}_P(n) \equiv (H_P^T W(n) H_P)^{-1} H_P^T W(n) R. \tag{4.12}
\]

First, computational complexity is estimated as a function of image size, \( Q \). In Eq. 4.12, the number of rows in the data vector \( R \), the weighting matrix \( W \), and the transformation matrix \( H \) increase proportionally with \( Q \) while the inversion matrix remains \( P \times P \). This results in approximately linear growth in run-time as a function of \( Q \) for constant \( P \).

Next, computational complexity is estimated as a function of profile resolution, \( P \). In Eq. 4.12, as the \( P \times P \) inversion matrix increases in size, the inversion calculation quickly dominates the computational run-time (i.e., for \( P > 16 \)). The computational effort to invert a \( P \times P \) matrix is proportional to the square of \( P \). This results in approximately quadratic growth in run-time as a function of \( P \) for constant \( Q \) when the inversion calculation dominates the run-time.

Given these results, it is informative to compare how much longer it would take to range profile a 16 \( \times \) 16-pixel image (\( Q_1 = 256 \)) than a 8 \( \times \) 8-pixel image (\( Q_2 = 64 \)) at a resolution so the block-sizes, \( Q/P \), are the same. Note that an image is made up of \( Q \) pixels and \( P \) blocks where each block has \( Q/P \) pixels. Since \( Q_1 \) is four times larger than \( Q_2 \), \( P_1 \) must also be four times larger than \( P_2 \) so the block sizes are the same, \( Q_1/P_1 = Q_2/P_2 \). Run-time is proportional to \( QP^2 \) as determined above. Thus, it takes approximately 64 times longer to profile a 16 \( \times \) 16-pixel image than a 8 \( \times \) 8-pixel image. Given this rapid growth, the computational run-time quickly becomes very large, as image size is increased, if constant block size is desired.

In Figs. 4-6 to 4-9, actual run-time measurements versus resolution are presented. All computations were performed on a Sun Sparc Station 10 computer. Fig. 4-6 contains the run-times to profile a 32 \( \times \) 32-pixel image. Fig. 4-7 is the same
plot on a log-log scale to give a better spread of the run-time measurements. At lower resolutions ($P < 16$), the run-time is quite small and is mostly the result of overhead setting up the algorithm. Thus, there is no clear order of growth in run-time as a function of $P$. At higher resolutions when the inversion of the $H_P^T W H_P$ matrix dominates the computational load, the run-time increases with approximately quadratic growth. For this relatively small image, at $P = 512$ the run-time is already nearly 40,000 seconds (or over 10 hours.)

Fig. 4-8 contains the run-times to profile a 128 × 128-pixel image. Fig. 4-9 is the same plot on a log-log scale. Similar to the previous case, for lower resolutions ($P < 16$), the run-time is quite small with no clear order of growth, but at higher resolutions, the run-time increases with approximately quadratic growth. At the relatively low resolution, $P = 128$, the measured run-time is already nearly 40,000 seconds (or over 10 hours.) By calculating the predicted run-times using approximately quadratic growth in resolution ($O(P) \approx 2.2$), at $P = 4096$ the run-time is found to be about 76,000,000 seconds (or 2.4 years!)

4.4 Fast Haar-Wavelet ML/EM Algorithm

The conventional EM algorithm of Sec. 3.4 can range profile small imagery at low resolutions in reasonable run-times. However, the computational load to profile large imagery at high resolutions becomes too great due to calculational complexity of the algorithm. For instance, as shown in the previous section it would require a predicted run-time of about 2.4 years on a Sun Sparc Station 10 computer to profile 128 × 128-pixel imagery ($Q = 16384$) at resolution, $P = 4096$. This is clearly well beyond what could be profiled in a reasonable time period. This situation has necessitated the development of a more efficient range profiling algorithm.

In the EM algorithm, the maximization step, Eq. 3.25, dominates the computational load as mentioned earlier:

$$\hat{x}_P(n) \equiv (H_P^T W(n) H_P)^{-1} H_P^T W(n) R.$$ (4.13)
Figure 4-6: Run-time versus resolution to range profile a 32 × 32-pixel image ($Q = 1024$).

Figure 4-7: Run-time versus resolution on a log-log scale to range profile a 32 × 32-pixel image ($Q = 1024$).
Figure 4-8: Run-time versus resolution to range profile a 128 × 128-pixel image (Q = 16384).

Figure 4-9: Run-time versus resolution on log-log scale to range profile a 128 × 128-pixel image (Q = 16384).
The range estimate can be directly calculated from the estimate vector,

\[ \hat{r}(n) = H_P \hat{x}_P(n) = H_P (H_P^T W(n) H_P)^{-1} H_P^T W(n) R \]  

(4.14)

The calculation of the range estimate can be very computationally demanding since it involves the inversion of the \( P \times P \) matrix \( H_P^T W(n) H_P \) where \( P \) can be very large (i.e., \( P = 4096 \)). However, by taking advantage of the structure of the Haar-wavelet transformation matrix, Eq. 4.14 can be written in a much simpler and more efficient form. This is called the fast EM algorithm and its derivation is presented here.

Consider a \( Q \)-pixel range image and let \( H \) be the separable Haar-wavelet basis for the associated \( Q \)-length vector space. For convenience, let \( Q \) be a power of two, and let us consider range profiling the image at resolution \( P \) which is also a power of two. Furthermore, we shall assume that the Haar-wavelet basis \( \{ \Phi_q \} \) comprising \( H \) is arranged in order of increasingly fine resolution. It then follows that the range space of \( H_P \) has a basis composed of \( P \) orthonormal \( Q \)-length vectors, \( \{ \Phi'_p : 1 \leq p \leq P \} \), with non-overlapping support, viz., \( \phi'_{pq} \phi'_{rq} = 0 \) for \( p \neq r \) and \( 1 \leq q \leq Q \). The \( \{ \Phi'_p \} \) are vectors where all components are zero except a set of \( Q/P \) components in positions \((p-1)[Q/P]+1 \text{ to } p[Q/P] \) which have value \( \sqrt{P/Q} \). For future reference, let \( Q_s(p) \equiv \{ q : \phi_{pq} \neq 0 \} \) denote the support set for \( \Phi'_p \). The non-overlapping support of the \( \{ \Phi'_p \} \) imbues the singular value decomposition of \( H_P \) with a property that obviates the numerical difficulties and greatly relieves the computational burden of doing Haar-wavelet EM/ML range profiling, as will be shown now.

For the Haar-wavelet matrix, \( H_P \),

\[ H_P = [\Phi_1 \Phi_2 \cdots \Phi_P], \]  

(4.15)

the range space of \( H_P \), \( \{ \Phi'_p \} \), can be written as the matrix, \( H'_P \),

\[ H'_P = [\Phi'_1 \Phi'_2 \cdots \Phi'_P]. \]  

(4.16)
Here, \( H_P = H'_P U \) where \( U \) is the unitary matrix which satisfies this transformation equation. Substituting \( H_P = H'_P U \) into the right-hand side of Eq. 4.14, we have

\[
\hat{f}(n) = H_P (H'_P W(n) H_P)^{-1} H'_P W(n) R
\]

\[
= H'_P U (U^T H'_P W(n) H'_P U)^{-1} U^T H'_P W(n) R
\]

\[
= H'_P U U^T (H'_P W(n) H'_P)^{-1} U U^T H'_P W(n) R
\]

\[
= H'_P (H'_P W(n) H'_P)^{-1} H'_P W(n) R
\]

(4.17)

Since the vectors \( \{\Phi'_p\} \) are orthonormal with non-overlapping support and \( W(n) \) is diagonal, \( H'_P W(n) H'_P \) is a diagonal matrix (containing the eigenvalues of \( H'_P W(n) H'_P \)) and hence can easily be inverted. Thus, we have

\[
\Lambda = H'_P W(n) H'_P
\]

\[
= \sum_{p=1}^{P} \Phi'_P W(n) \Phi'_p
\]

(4.18)

\[
= \text{diag}[\lambda_1 \lambda_2 \cdots \lambda_P]
\]

where

\[
\lambda_p = \frac{P}{Q} \sum_{q' \in Q_s(p)} w_{q'}. \tag{4.19}
\]

Substituting this result into Eq. 4.17, we have

\[
\hat{f}(n) = H'_P (H'_P W(n) H'_P)^{-1} H'_P W(n) R
\]

\[
= H'_P \Lambda^{-1} H'_P W(n) R. \tag{4.20}
\]

This is the key to the fast ML/EM algorithm. Eq. 4.20 is much easier to calculate than Eq. 4.14 because it is simply a matrix multiplication instead of a more complicated matrix inversion. Finally, Eq. 4.20 can be further simplified by multiplying out the matrices. The \( q \)th component of the ML range estimate equals

\[
\hat{r}_q(n) = \left[ H'_P \Lambda^{-1} H'_P W(n) R \right]_q
\]

\[
= \frac{\sum_{q' \in Q_s(p)} w_{q'}(n) R_{q'}}{\sum_{q' \in Q_s(p)} w_{q'}(n)}, \text{ for } p \text{ such that } q \in Q_s(p). \tag{4.21}
\]
Eq. 4.21 represents an extraordinary simplification of the EM/ML range profiling equation, Eq. 4.14. First, there is no longer any matrix inversion to be performed. Moreover, by declaring \( \hat{r}_{q'}(n) \) to be anomalous for all \( q' \in Q_s(p) \) whenever numerical underflow is encountered on all elements of \( \{w_{q'}(n) : q' \in Q_s(p)\} \), we guarantee that our maximization step is numerically robust. Finally, the algorithm is fully parallelizable; the range estimate for each of the \( P \) blocks can be computed separately.

As an illustration, the fast EM algorithm described in Eq. 4.21 is demonstrated explicitly. Suppose we are given a range data vector, \( R \), of length \( Q = 8 \),

\[
R^T = [400 \ 400 \ 300 \ 600 \ 400 \ 400 \ 300 \ 300],
\]

(4.22)

and its corresponding weighting matrix, \( W \),

\[
W(n) = \text{diag}[1.0 \ 1.0 \ 0.5 \ 1.0 \ 1.0 \ 1.0 \ 1.0 \ 1.0].
\]

(4.23)

As described in Sec. 3.3, the weight, \( w_q \), represents the probability that its corresponding range pixel, \( R_q \), is considered non-anomalous.

Suppose we wish to resolve \( R \) by an estimate vector, \( \hat{r} \), of length \( P = 4 \). The Haar-wavelet transformation matrix, \( H_4 \), is composed from the first four column vectors of the \( 8 \times 8 \) transformation matrix, \( H \), of Eq. 4.6,

\[
H_4 = [\Phi_1 \ \Phi_2 \ \Phi_3 \ \Phi_4].
\]

(4.24)

Thus, \( H_4^T \) equals

\[
H_4^T = \begin{bmatrix}
0.35 & 0.35 & 0.35 & 0.35 & 0.35 & 0.35 & 0.35 & 0.35 \\
0.35 & 0.35 & 0.35 & -0.35 & -0.35 & -0.35 & -0.35 & -0.35 \\
0.50 & 0.50 & -0.50 & -0.50 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0.50 & 0.50 & -0.50 & -0.50
\end{bmatrix}.
\]

(4.25)
First, we find the range space of $H_4$ and construct the matrix $H'_{4}$,

$$H'_{4} = \begin{bmatrix}
0.71 & 0.71 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0.71 & 0.71 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0.71 & 0.71 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0.71 & 0.71
\end{bmatrix}. \quad (4.26)$$

The transformation matrix, $U$, which satisfies $H_4 = H'_{4}U$ is calculated and confirmed to be a unitary matrix (and hence, $U^T = U^{-1}$ to satisfy Eq. 4.17),

$$U = \begin{bmatrix}
0.50 & 0.50 & 0.71 & 0 \\
0.50 & 0.50 & -0.71 & 0 \\
0.50 & -0.50 & 0 & 0.71 \\
0.50 & -0.50 & 0 & -0.71
\end{bmatrix}. \quad (4.27)$$

The diagonal matrix, $\Lambda = H'^T_{4}W(n)H'_{4}$, is calculated using Eqs. 4.18 and 4.19 and confirmed to be the eigenvalues of $H'^T_{4}W(n)H_{4}$,

$$\Lambda = H'^T_{4}W(n)H'_{4} = \text{diag}[\text{eigs}(H'^T_{4}W(n)H_{4})] = \text{diag}[1.0 \quad 0.75 \quad 1.0 \quad 1.0]. \quad (4.28)$$

Finally, we solve for the range estimate, $\hat{r}(n)$, using Eq. 4.21 and verify that it gives the same solution as Eq. 4.14,

$$\hat{r}(n) = [400 \quad 400 \quad 500 \quad 500 \quad 400 \quad 400 \quad 300 \quad 300]. \quad (4.29)$$

Note that in components three and four of $\hat{r}(n)$, the range estimate is somewhat closer to $R_4$ than $R_3$ as expected since $w_4 = 1.0$ and $w_3 = 0.5$ placing greater weight on $R_4$.

The simplification in Eq. 4.21 is referred to as the fast EM algorithm. Compared to the conventional EM algorithm in Eq. 4.14, the key advantage of the fast EM algorithm is the conversion of the matrix inversion problem to a matrix multiplication problem. This changes the order of growth in run-time to approximately linear growth as a function of resolution, $P$. Thus, the run-time grows both linearly for image size,
$Q$, and profile resolution, $P$. This allows profiling of much larger imagery at much higher resolutions and increases the calculation speed by several orders of magnitude.

In Figs. 4-10 to 4-13, actual run-time measurements versus resolution are presented for both the conventional and fast EM algorithms. Fig. 4-10 contains the run-times to profile a $32 \times 32$-pixel image ($Q = 1024$). Fig. 4-11 is the same plot on a log-log scale to give a better spread of the run-time measurements. At higher resolutions, the run-time increases with approximately quadratic growth for the conventional EM algorithm and approximately linear growth for the fast EM algorithm. For this relatively small image at $P = 512$, the run-time is more than 1,000 times faster using the fast EM algorithm, 40,000 seconds versus only 30 seconds.

Fig. 4-12 contains the run-times to profile a $128 \times 128$-pixel image ($Q = 16384$). Fig. 4-13 is the same plot on a log-log scale. At higher resolutions, the run-time increases with approximately quadratic growth for the conventional EM algorithm and at approximately linear growth for the fast EM algorithm ($O(P) \approx 1.3$). At $P = 4096$, the run-time is about 50,000 times faster using the the fast EM algorithm, 76,000,000 seconds versus only 15,000. Note that the run-time can and will be further speeded up by using different weighting schemes as discussed in Ch. 6. Thus, real range imagery can be profiled in manageable run-times which makes possible the range profiling results of the succeeding chapters.
Figure 4-10: Run-time versus resolution to range profile a $32 \times 32$-pixel image ($Q = 1024$).

Figure 4-11: Run-time versus resolution on a log-log scale to range profile a $32 \times 32$-pixel image ($Q = 1024$).
Figure 4-12: Run-time versus resolution to range profile a 128 x 128-pixel image ($Q = 16384$).

Figure 4-13: Run-time versus resolution on log-log scale to range profile a 128 x 128-pixel image ($Q = 16384$).
Chapter 5

Multiresolution Range Profiling of Real Imagery

In this chapter, we present results for multiresolution range profiling of real imagery using the fast EM algorithm of Sec. 4.4. First, we describe and find the laser radar and image parameters for a tank example. The range image of the tank is then profiled at various resolutions. Finally, the profiled results are analyzed by looking at the run-time, final weights, log-likelihood measurements, and error performance.

5.1 Laser Radar Range Imagery

The laser radar imagery used in this thesis has been supplied by MIT Lincoln Laboratory. The imagery was produced by airborne radar equipped with laser intensity, 1-m range sampling, Doppler sampling, and video recording [18]. As shown in Fig. 2-3, an airplane flies directly towards a target and takes numerous sets of laser radar measurements in framing mode. Each set of measurements comprises a video, range, intensity, and passive IR image. In this thesis, the main focus is on analyzing and processing the range data.

In this chapter, the sample image to be profiled contains an army tank as the target. The video image is shown in Fig. 5-1 and the corresponding range image is shown in Fig. 5-2. Note that the field of view of the laser radar range sampler is
much more focused than the field of view of the video recorder.

Figure 5-1: Video image of a tank in a field.

In Fig. 5-2, the shade of each pixel represents the distance in range bins (see Fig. 2-2) measured by the laser radar. According to the calibration bar on the right, most of the range image is between 400 and 500 range bins away. These readings can be converted to physical distances by knowing the various settings used by the laser radar to produce the imagery. Specifically, the size of a range bin is 1.1 meters, the range gate offset is 1400 feet = 427 meters, and the range gate width, $\Delta R$, is 5500 feet = 1676 meters. The distance, $D$, is calculated:

$$D = (\text{No. of Range Bins}) \times (1.1 \ m) + \text{Range Gate Offset (in m)}.$$  \hspace{1cm} (5.1)

Thus the range image is between approximately 870 and 980 meters away.
Figure 5-2: Range image of a tank.

Note that the top and left edges are shown as solid black lines. In the original imagery, the edges were splotched with many pixels where the laser radar had recorded 'no reading' which was assigned a zero in the data image. Thus to remove these edge effects and refine the image, the edge pixels were pre-set to zero producing the solid black lined edges shown in Fig. 5-2.

According to the statistical model of Sec. 2.1, the range image has a certain amount of Gaussian noise and some fraction of anomalous pixels. However, for this particular image, the fraction of anomalies seems to be very small. This is concluded from the fact that if certain pixels are anomalous, they take on values that are uniformly distributed across the range gate, $\Delta R = 5500$ ft = 1524 range bins, but this does not appear to happen. Since there are almost no anomalies, Fig. 5-2 is taken to be the range truth, $R^*$, for the purpose of testing our algorithm. In reality, this is not completely true since Gaussian noise is also present due to local-oscillator shot noise in the range measurements, but this approximation will be useful in testing the performance of the EM algorithm.

The range data, $R$, is generated from the range truth based on the single-pixel statistical model. In particular, local Gaussian noise with a standard deviation, $\delta R = 2$, is added to each pixel of the range truth. Then anomalies are simulated with a 5% probability rate, $\Pr(A) = 0.05$. If an anomaly does occur, the resulting measurement is produced from a uniform probability density with length, $\Delta R = 1524$. The resulting range image, $R$, is shown in Fig. 5-3.
5.2 Range Profile Results

Given the range data and the necessary image parameters, the EM algorithm is now used to find the maximum-likelihood image. As discussed in Sec. 3.4, the EM algorithm will converge to a local likelihood maximum. With appropriate initialization, this local maximum will be the global likelihood maximum. We used the recursive expectation-maximization (REM) to sequentially improve the initial estimate, $x_P$.

Since the algorithm has been designed to profile square images, the range data is made into a 128 $\times$ 128 pixel image ($Q = 16384$) by appending zeros to the end of the range data vector. After profiling, the latter part of the range estimate vector (corresponding to the appended zeros in the data vector) is discarded. This results in a range estimate which has the same dimensions as the original range image. The range data is profiled at several resolutions, $P = \{256, 512, 1024, 2048, 4096\}$, as shown in Figs. 5-4 to 5-8 respectively. In each figure, the block size is $\sqrt{Q}/P_j \times \sqrt{Q}/P_k$ pixels or a total of $Q/P$ pixels.

As noted earlier in Sec. 4.2, the resulting profiles in Figs. 5-4 to 5-8 suppress the anomalies, which appear as black and white speckles in the range data in Fig. 5-3. At low resolutions, the general shape of the tank can be observed against the planar background sloping towards the top-left part of the image. At higher resolutions, the block size decreases showing finer details in the range estimate and only at the
Haar-fitted Range Profiles

Figure 5-4: Haar-fitted range profile at $P = 256$ where $P_j = 16$ and $P_k = 16$.

Figure 5-5: Haar-fitted range profile at $P = 512$ where $P_j = 32$ and $P_k = 16$. 
Figure 5-6: Haar-fitted range profile at $P = 1024$ where $P_j = 32$ and $P_k = 32$.

Figure 5-7: Haar-fitted range profile at $P = 2048$ where $P_j = 64$ and $P_k = 32$.

Figure 5-8: Haar-fitted range profile at $P = 4096$ where $P_j = 64$ and $P_k = 64$. 
two highest resolutions can some parts of the tank's turret be seen. Note that at the highest resolution, \( P = 4096 \), a few of the 2 x 2-pixel blocks appear as black and white speckles. This occurs because there are so many anomalies in the block's pixels that the algorithm mistakenly calculates the maximum-likelihood block estimate to be in the range of the anomalous pixels.

### 5.3 Analysis of Range Profile Results

In this section, the range profiles of Figs. 5-4 to 5-8 are analyzed in terms of the weights, log-likelihood, and run-time measurements.

#### 5.3.1 Weighting Measurements

The EM algorithm produces a sequence of estimates, \( \{x_P(n) : n = 1, 2, 3, \ldots\} \), through an iterative process of expectation and maximization steps as described in Sec. 3.4. The expectation step involves updating the weights, \( w_q(n) : 1 \leq q \leq Q \), where \( w_q(n) \) represents the probability that the qth pixel of the range data is not anomalous in stage n. Thus the final weights for the parameter vector, \( \hat{x}_{P_{ML}} \), represent the probabilities that each pixel in the range data are not anomalous. Figs. 5-9 to 5-13 contain the final weighting matrix, \( W(n) \), arranged into a 45 x 128 matrix image. Note that the weights, \( \{0 \leq w_q \leq 1\} \), range from zero meaning completely anomalous to one meaning completely nonanomalous.

In Figs. 5-9 to 5-13, the weights on the top and left edges are all zeros. These edge pixels were preset to zero as described earlier because many dropouts appeared on the edges of the imagery. The pixel weights were also preset to zero indicating that the corresponding range values are not good readings.

Insight into how the Haar-wavelet EM algorithm works is provided by examining the final weights. Essentially, the algorithm divides the Q-pixel range image into a set of P blocks, each with \( Q/P \) pixels. The range estimate for each block is equal to a weighted average of its pixel values. Pixel values near the block's range estimate are weighted close to one, while those far away are weighted to zero as defined in Eq.
Final Pixel Weights, $W$

Figure 5-9: Final pixel weights, $W$, at $P = 256$.

Figure 5-10: Final pixel weights, $W$, at $P = 512$. 
Figure 5-11: Final pixel weights, $W$, at $P = 1024$.

Figure 5-12: Final pixel weights, $W$, at $P = 2048$.

Figure 5-13: Final pixel weights, $W$, at $P = 4096$. 
3.25. Maximum-likelihood is used to iteratively find the optimal weights and thus the optimal range estimate for the block.

In the weighting figures, there is a clumping of black pixels along the edges of the tank’s body and the background, especially at lower resolutions. This occurs when a large number of pixels in some block are far from the block’s range estimate. For instance, the pixels along the boundary between the top of the tank and the background have very different range values. The EM algorithm calculates the optimal range estimate for each block, but cannot possibly fit the wide variation in range values for the top of the tank and the nearby background pixels. Consequently, the block’s range estimate will either be close to range of the tank or the background, depending on the range of the majority of its pixels. Then pixel values near the block’s range estimate are considered nonanomalous and appear white, whereas distant pixel values are considered anomalous and appear black. Pixel values that are an intermediate distance from the block’s range estimate appear gray.

In Figs. 5-9 to 5-13, as the resolution increases, the clumping of black pixels along the edge of the tank decreases. This occurs because the range estimates for the blocks can better fit the range data as the blocks become smaller in size and larger in number. Thus at lower resolutions, there are many poorly fitted pixels which appear black and gray while at higher resolutions, there are fewer poorly fitted pixels and nearly all pixels appear black or white. This indicates that the algorithm becomes better at locating and deciding which pixels are anomalous and nonanomalous as resolution increases. In fact, an analysis of the range data suggests the algorithm is successfully locating the simulated anomalies.

5.3.2 Fraction of Low Weighted Pixels

The fraction of low weighted pixels is a useful measure for determining how successful the EM algorithm is at locating anomalies. In the original range data image, the relative frequency of anomalies was approximately equal to the anomaly probability, $\Pr(A) = 0.05$. Ideally, the algorithm would find this fraction of anomalies. However, there are a number of factors which affect or degrade this measurement as mentioned
in the previous section. In particular, blocks may contain pixels in different range intervals such as the tank and background. In addition, the blocks may simply be too large to appropriately fit a sloping range interval.

The low weighted pixels here are defined as those pixels which have weights less than 0.5, \( \{w_q < 0.5\} \). This is a reasonable approximation of the number of anomalous pixels because the vast majority of pixels are either nearly zero or one, especially at higher resolutions. Fig. 5-14 shows a plot of the fraction of low weight pixels as a function of the profile resolution.

![Fraction of low weight pixels versus resolution](image)

Figure 5-14: Fraction of low weight pixels versus resolution, \( P \).

At the lowest resolution, \( P = 256 \), the fraction of low weight pixels is about 0.15. This high fraction is largely a result of poorly fit pixels at the boundary between the tank and background. As the resolution increases, the fraction of low weighted pixels decreases significantly. At the highest resolution tested, \( P = 4096 \), the fraction of anomalies is just under 0.07 so this is closely approaching the actual anomaly probability, \( \Pr(A) = 0.05 \).

The stopping rule is a method used to determine the optimal resolution of a range estimate based on the statistics of the low weighted pixels [12]. Essentially, the objective is to find the maximum resolution which has sufficient anomaly suppression. The design of the stopping rule can be described as follows. Consider each pixel of the range data to have an associated independent, identically-distributed Bernoulli random variable with probability of success (success meaning non-anomalous) equal
to $1 - \Pr(A)$ [19]. Let $N_a$ equal the number of low weighted pixels or anomalies that occur in the $S$-pixel range data. Then the mean, $E(N_a) = S\Pr(A)$, and the standard deviation, $\sigma_{N_a} = \sqrt{S\Pr(A)(1 - \Pr(A))}$, can easily be computed. These statistics can also be represented in terms of the fraction of low weight pixels, $F_a$. This results in the mean, $E(F_a) = \Pr(A)$, and the standard deviation, $\sigma_{F_a} = \sqrt{\Pr(A)(1 - \Pr(A))/S}$. These statistics can be calculated for each resolution by profiling a large number of simulated range data with the same $\Pr(A)$. The optimal resolution would be that which produces the statistics closest to expected mean, $E(F_a)$, and standard deviation, $\sigma_{F_a}$.

For the range image used here, the anomaly probability is equal to $\Pr(A) = 0.05$ and the statistics are obtained from a rectangular block of $S = 3584$-pixels chosen from the range estimate. Thus, $E(F_a) = 0.05$ and $\sigma_{F_a} = 0.0036$ are the theoretical mean and standard deviation for the fraction of low weight pixels or anomalies. The measured mean and standard deviation are found by calculating the statistics for many profiles. To do this, several simulated range images were produced from the true-range image as described in Sec. 5.1 and then profiled at the various resolutions. The resulting experimental values of $E(F_a)$ and $\sigma_{F_a}$ are shown in Table 5.1.

<table>
<thead>
<tr>
<th>Resolution, $P$</th>
<th>No. of Trials</th>
<th>$E(F_a)$</th>
<th>$\sigma_{F_a}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>256</td>
<td>80</td>
<td>0.1475</td>
<td>0.0046</td>
</tr>
<tr>
<td>512</td>
<td>40</td>
<td>0.0868</td>
<td>0.0039</td>
</tr>
<tr>
<td>1024</td>
<td>20</td>
<td>0.0729</td>
<td>0.0029</td>
</tr>
<tr>
<td>2048</td>
<td>10</td>
<td>0.0641</td>
<td>0.0032</td>
</tr>
<tr>
<td>4096</td>
<td>5</td>
<td>0.0589</td>
<td>0.0012</td>
</tr>
</tbody>
</table>

Table 5.1: Resolution, $P$, versus no. of trials, $E(F_a)$, and $\sigma_{F_a}$

A smaller number of trials was used at higher resolutions due to the greater run-time requirement per trial. As a result, the statistics are not quite as accurate at higher resolutions. The measured and theoretical standard deviations are quite similar in value (except for $P = 4096$ which is probably due to the small number of trials used.) The measured mean for the number of low weight pixels approaches the
theoretical mean as the resolution is increased, but does not quite reach it as shown in Table 5.1. As mentioned earlier, the slightly high fraction of anomalies is largely due to edge effects at the boundary between the tank and background. Thus the stopping rule is partially limited in its effectiveness for imagery with large range variations.

5.3.3 Log-Likelihood Measurements

The likelihood equation, Eq. 3.1, is defined as the likelihood of obtaining the range-data vector, \( \mathbf{r} \), given the true-range vector, \( \mathbf{r}^* \). The log-likelihood, \( L(\mathbf{R}^*) \), is defined as:

\[
L(\mathbf{R}^*) = \sum_{q=1}^{Q} \ln \left[ \left( 1 - \Pr(A) \right) \frac{\exp \left( -\frac{(R_q - R_q^*)^2}{2\delta R^2} \right)}{\sqrt{2\pi\delta R^2}} + \frac{\Pr(A)}{\Delta R} \right].
\]  

(5.2)

The EM algorithm maximizes the log-likelihood, \( L(\mathbf{R}^*) \), and returns the maximum-likelihood estimate as described in Sec. 3.4. The log-likelihood values for Figs. 5-4 to 5-8 are shown in Fig. 5-15. The log-likelihood is basically a measure of how closely the range estimate matches the range data vector. Thus at higher resolutions, it is expected that the likelihood is greater since there is greater dimensionality (i.e., more blocks) in the range estimate. This is shown in Fig. 5-15 where \( L \) increases as a function of resolution.

![Figure 5-15: Log-likelihood, \( L(\mathbf{R}^*) \), versus resolution, \( P \).](image)
The objective of the range profiling problem is to obtain the maximum likelihood estimate that has sufficient anomaly suppression. Thus the weighting statistics also play an important role in determining the optimal resolution. If our sole objective was to obtain the maximum likelihood range estimate, this would be the range data itself as described in Eq. 3.2 of Sec. 3.1.

5.3.4 Run-Time Measurements

The run-time measurements for the range profiles from Figs. 5-4 to 5-8 are shown in Fig. 5-16. The run-time is equal to the amount of computational time required to profile the range image on a Sun Sparc Station 10 computer. Here analog weights are used as the weighting scheme, \( 0 \leq w_q \leq 1 \). (In the next chapter, different weighting schemes will be presented which significantly increase the computational speed of the algorithm.) Note that the run-times closely mirror the results for the fast EM algorithm shown in Sec. 4.4, but these run-times are slightly different because the raw range image was different.

![Figure 5-16: Run-time measurements versus resolution, \( P \).](image-url)
5.4 Error Performance

The error performance for range profiling is measured in terms of the bias and error variance of the range estimate. As discussed in Sec. 3.5.1, the estimation error for any estimator, \( \hat{x}_P \), of the unknown parameter vector, \( x_P \), is defined, \( e \equiv x_P - \hat{x}_P \). The error performance of the estimator can be characterized by the bias vector,

\[
b(X_P) \equiv E(e|x_P = X_P),
\]

and error covariance matrix,

\[
\Lambda_e(X_P) \equiv E\{[e - b(X_P)][e - b(X_P)]^T|x_P = X_P\}.
\]

The estimation error for the range estimate is defined, \( e \equiv r^* - \hat{r} \). Note that the range estimate, \( \hat{r} \), is a linear transformation of the parameter vector, \( x_P \), defined \( \hat{r} = H_P X_P \). Thus the error performance of the range estimate can be characterized by the bias vector,

\[
b(R) \equiv E(e|r = R),
\]

and error covariance matrix,

\[
\Lambda_e(R) = H_P \Lambda_e(X_P) H_P^T.
\]

5.4.1 Bias of Range Estimate

The bias and error variance are calculated from a set of estimates, the bias equaling the average estimation error and the error variance equaling the mean-squared difference between the estimation error and the bias. Since these calculations require a set of estimates, range profiling was performed numerous times on simulated data for each resolution. However, the number of trials decreased as resolution was increased because the range profiling time is much larger at higher resolutions. Table 5.2 shows the resolution and number of trials performed at each resolution.
<table>
<thead>
<tr>
<th>Resolution, $P$</th>
<th>No. of Trials</th>
</tr>
</thead>
<tbody>
<tr>
<td>256</td>
<td>80</td>
</tr>
<tr>
<td>512</td>
<td>40</td>
</tr>
<tr>
<td>1024</td>
<td>20</td>
</tr>
<tr>
<td>2048</td>
<td>10</td>
</tr>
<tr>
<td>4096</td>
<td>5</td>
</tr>
</tbody>
</table>

Table 5.2: Resolution, $P$, versus no. of trials

The magnitude of the bias is calculated for each resolution and is shown in Figs. 5-17 to 5-21. White represents high bias while black represents low bias. In each of the figures, there is a large amount of bias along the upper edge of the tank and the adjacent background. As described earlier, this happens because the Haar-wavelet EM algorithm cannot possibly find a range estimate that accommodates the wide variation in range measurements here. The amount of bias along the edges of the tank decreases at higher resolutions as the block-size decreases.

In general, the EM algorithm can better fit the range estimate blocks at higher resolution because there are a larger number of smaller-sized blocks. Thus at higher resolutions shown in Figs. 5-20 and 5-21, the range estimate becomes approximately unbiased, except along the edges of the tank and background. Fig. 5-21 has some more biased regions than Fig. 5-20, which may happen for two possible reasons. First, the resolution may have become so high (i.e. the block-size so small) that the EM algorithm has incorporated some anomalies into the range estimate. Secondly, there may have been too few trials performed at the highest resolution resulting in a noisier estimate of the bias.

### 5.4.2 Error Variance of Range Estimate

The error variance is calculated for each resolution and is shown in Figs. 5-22 to 5-26. White represents high error variance and black represents low error variance.

As mentioned earlier, few trials were run for higher resolutions due to large runtime required to profile the range imagery. If a large numbers of trials had been
Absolute Bias of Range Estimate

Figure 5-17: Absolute bias of range estimate at $P = 256$.

Figure 5-18: Absolute bias of range estimate at $P = 512$. 
Figure 5-19: Absolute bias of range estimate at $P = 1024$.

Figure 5-20: Absolute bias of range estimate at $P = 2048$.

Figure 5-21: Absolute bias of range estimate at $P = 4096$.
performed, the error variance would be expected to be approximately uniform—the error variance would be approximately the same value for each pixel of the range image. However at high resolutions such as Fig. 5-26, there is quite a bit of variation due to the relatively low number of trials performed (i.e. only five trials were performed at \( P = 4096 \) as shown in Table 5.2.)

At the lower resolutions, the error variance is very large in some blocks which are along the top edge of the tank and the nearby background. As the resolution increases, the average error variance increases as shown in Figs. 5-22 to 5-26. It turns out that the error variance approaches the complete data bound, which is slightly weaker or less restrictive than the Cramér-Rao bound as derived in Sec. 3.5.3. The complete data bound in terms of the range estimate \( \mathbf{R} \) is given by

\[
\Lambda_e(\mathbf{R}) \geq \mathbf{I}_y^{-1}(\mathbf{R}) = \frac{\delta R^2}{1 - \Pr(A)} \mathbf{H}_p \mathbf{H}_p^T.
\] (5.7)

The diagonal elements of the error covariance matrix are the error variances of the range-estimate pixels. For the Haar-wavelet basis, these error variances are all the same,

\[
[\Lambda_e(\mathbf{R})]_{qq} \geq \frac{\delta R^2}{1 - \Pr(A)} [\mathbf{H}_p \mathbf{H}_p^T]_{qq} = \frac{\delta R^2}{1 - \Pr(A)} \frac{P}{Q}
\] for \( 1 \leq q \leq Q \), (5.8)

where \( P \) is the resolution and \( Q = 16384 \) is the number of pixels in the square 128 \( \times \) 128-pixel image. Thus the complete data bound for the error variance is proportional to the resolution. For the simulated data used here, the anomaly probability was equal to five percent, \( \Pr(A) = 0.05 \). Table 5.3 shows the resolution, the complete data bound on the error variance, and the average error variance for the image. The latter was calculated by excluding the pixels values which were extremely far away from their averages (less than 1% of the pixels).

Table 5.3 shows that at lower resolutions, where the estimate is more biased, the error variance is significantly greater than the complete data bound. At higher resolutions, however, the error variance approaches the complete data bound. Thus at increasingly higher resolutions, the Haar-wavelet ML/EM algorithm calculates range
Table 5.3: Resolution, $P$, versus complete data bound and average error variance profiles that are nearly unbiased with error variances that approach the complete data bound. In other words, they approach the ultimate performance limit of unknown parameter estimation.
Error Variance of Range Estimate

Figure 5-22: Error variance of range estimate at $P = 256$.

Figure 5-23: Error variance of range estimate at $P = 512$. 


Figure 5-24: Error variance of range estimate at $P = 1024$.

Figure 5-25: Error variance of range estimate at $P = 2048$.

Figure 5-26: Error variance of range estimate at $P = 4096$. 
Chapter 6

Modified Weighting Schemes

The EM algorithm defined in Sec. 3.4 uses an iterative procedure to calculate the optimal analog weights, \( \{0 \leq w_q \leq 1\} \). In this chapter, two new weighting schemes are presented: discrete \((1/0)\) weights and hybrid discrete-and-analog weights. A comparison of the three weighting schemes shows that run-time measurements are significantly smaller for the new weighting schemes, yet the profiling results, fraction of low weight pixels, and log likelihood measurements are quite similar.

6.1 Discrete \((1/0)\) Weighting

In the expectation step of the EM algorithm, the weighting matrix, \( W(X_p) \equiv \text{diag}[w_q(X_p)] \), is calculated using Eq. 3.24. Each of the weights, \( w_q \), represents the conditional probability that the corresponding range data pixel, \( R_q \), is not anomalous given the parameter vector, \( \hat{x}_{PML} \). In general, these weights are either close to zero or close to one, especially at higher resolutions. This behavior is illustrated in Figs. 6-1 to 6-3 which show the weight distributions for the tank range profiles at \( P = \{1024, 2048, 4096\} \) from Sec. 4.2.

The EM algorithm calculates the maximum-likelihood estimate by an iterative procedure of expectation and maximization steps until the difference between successive likelihoods is within an acceptable tolerance. Finding the optimal weights requires numerous iterations because each small improvement to the analog weights
Figure 6-1: Weight distribution, $w_q$, versus $q$ at $P = 1024$.

Figure 6-2: Weight distribution, $w_q$, versus $q$ at $P = 2048$.

Figure 6-3: Weight distribution, $w_q$, versus $q$ at $P = 4096$. 
requires another iteration. However by using a discrete weighting scheme, the weights quickly reach their final values in relatively few iterations. This seems to be a reasonable approximation of the actual weights, because most weights are nearly zero or one, as shown in Figs. 6-1 to 6-3. Thus for the discrete weighting scheme, the weights are defined:

\[
  w_q^d = \begin{cases} 
    0 & \text{if } |R_q - (H_P\hat{x}_P)_q| > \gamma, \\
    1 & \text{if } |R_q - (H_P\hat{x}_P)_q| \leq \gamma, 
  \end{cases} \tag{6.1}
\]

where \( \gamma \) is the threshold value such that \( w_q(X_P) = 0.5 \). In Eq. 3.17, by setting \( w_q(X_P) = 0.5 \) and doing the necessary algebra to solve for \( \gamma = |R_q - (H_P\hat{x}_P)_q| \), we find

\[
  \gamma = |R_q - (H_P\hat{x}_P)_q| = \sqrt{-2\delta R^2 \ln \left[ \frac{\sqrt{2\pi\delta R^2}}{\Delta R} \frac{Pr(A)}{1 - Pr(A)} \right]} \tag{6.2}
\]

This simple threshold test decides whether each pixel of the range data is either anomalous or nonanomalous.

The range profiles at \( P = \{1024, 2048, 4096\} \) obtained using the REM algorithm with discrete weights are shown in Figs. 6-4 to 6-6. The discrete-weight range profiles are quite similar to the corresponding analog-weight range profiles in Figs. 5-6 to 5-8. However in Fig. 6-6 at \( P = 4096 \), there are several black blocks or holes appearing across the image. These holes result when all pixel weights in a block are less than 0.5 and are subsequently set to zero for the discrete weighting scheme. This will generally occur if there are both many anomalies clustered together and few pixels-per-block (i.e., here there are four pixels-per-block.) When all pixel weights of a block are zero, all range data for the block is considered anomalous, and the range estimate is set to zero. This problem has motivated the development of a hybrid weighting scheme, described in the next section. (For the conventional Haar-wavelet EM algorithm of Sec. 3.4, if all pixel weights in a block underflow, then the matrix \( H_P^T WH_P \) is non-invertible and the entire computation fails. However for the fast Haar-wavelet EM algorithm of Sec. 4.4, if all pixel weights in a block underflow, only the block's range estimate is set to be anomalous and the remainder of the range estimate is not affected.)
Discrete Weight Range Profiles

Figure 6-4: Haar-fitted range profile at $P = 1024$.

Figure 6-5: Haar-fitted range profile at $P = 2048$.

Figure 6-6: Haar-fitted range profile at $P = 4096$.
6.1.1 Run-Time Measurements

The run-time measurements for the discrete-weight case are shown in Fig. 6-7 as a function of resolution for $P = \{512, 1024, 2048, 4096\}$. These run-times are from two to five times faster than the analog-weight run-times shown in Fig. 5-16. At $P = 4096$, the discrete-weight computation time is less than 4,000 seconds, whereas the analog-weight computation time is greater than 19,000 seconds. The computational speed of the Haar-wavelet ML/EM algorithm could be increased further by calculating the maximum-likelihood block estimates in parallel. This can be achieved because the range estimate for each block is based only on the range values of its corresponding pixels. Hence, each block's range estimate is completely independent of the other $P - 1$ blocks so the ML range estimate for each of the $P$ blocks could be calculated in parallel. Ideally range profiling could be done in real time.

![Figure 6-7: Discrete-weight run-time versus resolution, $P$.](image)

6.1.2 Fraction of Low Weight Pixels

The fraction of low discrete-weight pixels are shown in Fig. 6-8 as a function of resolution for $P = \{512, 1024, 2048, 4096\}$. The values shown in this figure are just slightly less than the analog-weight fraction of low weight pixels shown in Fig. 5-13. For the most part, the final weights are also very similar, with the discrete weights approximately equal to the analog weights processed by Eq. 6.1.
6.1.3 Log-Likelihood Measurements

The log-likelihood measurements for the discrete-weight case are shown in Fig. 6-9 as a function of resolution for \( P = \{512, 1024, 2048, 4096\} \). The discrete-weight log-likelihood measurements are just slightly less than but nearly the same as the analog-weight log-likelihood measurements shown in Fig. 5-15. This is indicative of how similar the range profiles are using either weighting scheme.
6.2 Hybrid Discrete-and-Analog Weighting

The motivation for the hybrid discrete-and-analog weighting scheme is to eliminate the *holes* that appear in the discrete-weight range profiles at high resolutions. As mentioned in the previous section, the *holes* appear as black blocks in the range profiles, such as in Fig. 6-6, indicating that all range data in these blocks were considered anomalous. The algorithm sets the range estimates for such blocks to zero. The analog weighting scheme does not face such a problem, because even when the pixel weights in a block are considered anomalous, they are still given some non-zero weight. Thus there is at least some information on which to base a range estimate for the block. In principle, it is possible that underflow could occur on all pixels in a block if the wrong initialization were used, but this is much less likely to happen using analog rather than discrete weighting.

The idea behind the hybrid discrete-and-analog weighting scheme is to use discrete weights for all non-*hole* blocks to take advantage of the speed of discrete weighting, but also to use analog weights for all *holes* to provide at least some information to produce a range estimate for the block. This is easily implemented by using discrete weights for all blocks and then by using analog weights for those blocks whose pixels are all zero-weighted (i.e. the eigenvalue for the block is zero.)

Hybrid-weight range profiles at $P = \{1024, 2048, 4096\}$ are shown in Figs. 6-10 to 6-12. These range profiles eliminate most of the *holes* in the discrete-weight range profiles. In particular, compare the discrete-weight Fig. 6-6 to the hybrid-weight Fig. 6-12 which shows the elimination of several bad blocks or *holes* throughout the image. The hybrid weight-range profiles appear almost identical to the analog-weight range profiles in Figs. 5-6 to 5-8. Thus the hybrid weighting scheme produces nearly as accurate maximum-likelihood range profiling as the analog weighting scheme, but is computationally much more efficient.
Hybrid Weight Range Profiles

Figure 6-10: Haar-fitted range profile at $P = 1024$.

Figure 6-11: Haar-fitted range profile at $P = 2048$.

Figure 6-12: Haar-fitted range profile at $P = 4096$. 
6.2.1 Run-Time Measurements

Hybrid-weight run-time measurements are shown in Fig. 6-13 as a function of resolution for \( P = \{512, 1024, 2048, 4096\} \). These run-times are just slightly longer than discrete-weight run-times. This is to be expected because the hybrid weighting scheme generally uses discrete weights, but in the case of a hole uses analog weights.

![Figure 6-13: Hybrid-weight run-time versus resolution, \( P \).](image)

6.2.2 Fraction of Low Weight Pixels

The fraction of low hybrid-weight pixels are shown in Fig. 6-14 as a function of resolution for \( P = \{512, 1024, 2048, 4096\} \). The fraction of low weight pixels shown here is nearly the same as the analog-weight fraction of low weight pixels. The only difference occurs at \( P = 4096 \) where the fraction of low hybrid-weight pixels is just slightly less.

6.2.3 Log-Likelihood Measurements

The log-likelihood measurements for the hybrid-weight profiles are shown in Fig. 6-15 as a function of resolution for \( P = \{512, 1024, 2048, 4096\} \). These log-likelihood measurements are nearly the same as the discrete-weight log-likelihood measurements. The only difference occurs at \( P = 4096 \), where the log-likelihood value using hybrid-weight is just slightly greater. This indicates, as expected, that the range estimate for
6.3 Comparison of Weighting Schemes

Both the discrete (1/0) weighting scheme and the hybrid weighting scheme have been shown to be effective in producing nearly the same ML range profiles as the more general analog weighting scheme. The principle advantage of the discrete and hybrid weighting schemes is that they are computationally much more efficient than the
analog weighting scheme. The hybrid discrete-and-analog weighting scheme largely eliminates the holes, or no information blocks, resulting in range profiles closer to the optimal ML range estimate. The run-times measurements, fraction of low weight pixels, and log-likelihood measurements are compared below for the three weighting schemes.

6.3.1 Run-Time Measurements

The run-time measurements are shown in Fig. 6-16 as a function of resolution for \( P = \{512, 1024, 2048, 4096\} \). This measure shows key the difference in the three weighting schemes. Discrete and hybrid weighting are from two to five times faster than analog weighting.

![Run-time versus resolution, P.](image)

6.3.2 Fraction of Low Weight Pixels

The fraction of low weight pixels are shown in Fig. 6-17 as a function of resolution for \( P = \{512, 1024, 2048, 4096\} \). The fraction of low weight pixels is highest for analog weighting, with the discrete and hybrid weighting behaving nearly identically. The fractional difference between analog weighting and discrete and hybrid weighting is equal to only about six additional low weight pixels out of 3,584 pixels for the image.
Figure 6-17: Fraction of low weight pixels versus resolution, \( P \).

6.3.3 Log-Likelihood Measurements

The log-likelihood measurements are shown in Fig. 6-18 as a function of resolution for \( P = \{512, 1024, 2048, 4096\} \). The log-likelihood measurements are nearly the same for the three weighting schemes. Only at the highest resolution do the discrete and hybrid weighting schemes have slightly lower likelihoods than the analog weighting scheme. This suggests that the hybrid discrete-and-analog weighting scheme is a very good approximation for the analog weighting scheme, but is computationally much faster and more efficient.

Figure 6-18: Log-likelihood, \( L(R^*) \), versus resolution, \( P \).
Chapter 7

Robustness of the ML/EM Algorithm

The objective of laser radar range profiling is to find the optimal estimate of the true range image, $r^*$, given the range image, $r$. This is achieved using maximum-likelihood estimation. The ML estimate, $\hat{R}_{ML}$, is by definition the $R^*$ that maximizes $p_{r|r^*}(R|R^*)$. As described in Sec. 3.1, the conditional probability density is given by

$$p_{r|r^*}(R|R^*) = \prod_{q=1}^{Q} \left[ \frac{\exp\left(-\frac{\left(R_q - R_q^*\right)^2}{2\delta R^2}\right)}{\sqrt{2\pi\delta R^2}} + \frac{\Pr(A)}{\Delta R} \right]. \quad (7.1)$$

It is important to have a good measure of the local range accuracy, $\delta R$, to accurately calculate the maximum log-likelihood, $L(R^*)$, defined in Eq. 5.2. This can present certain challenges because the local accuracy may not be uniform across the image. For instance, the radar’s carrier-to-noise (CNR) ratio can vary significantly for different types of scenery, which results in different values of $\delta R$ across the image. Since the EM algorithm uses a single value for the local range accuracy, an approximate $\delta R$ must be estimated for the image.

In this chapter, the robustness of the ML/EM algorithm is examined in terms of how the range estimate is affected by variations in the local range accuracy. First,
the range image of a transporter vehicle is profiled at various resolutions with an estimated value of $\delta R$. Then this image is profiled using various values of $\delta R$. The results are analyzed in terms of the weights and log-likelihood measurements.

### 7.1 Range Profiling Using Estimated $\delta R$

In this section, range profiling is performed at various resolutions using an estimated value for the local range accuracy, $\delta R$, of an image.

#### 7.1.1 Laser Radar Range Imagery

The range image to be profiled here contains a transporter vehicle in front of a dense growth of trees and underbrush. The video image of the transporter vehicle is shown in Fig. 7-1 and the corresponding range image is shown in Fig. 7-2. Note that the field of view of the range image is much more focused than the field of view of the video image.

As shown in Fig. 7-2, the foreground in the lower right of the image gradually slopes back towards the top left. The transporter vehicle was about 920 range bins or 1440 meters away from the laser radar when the image was scanned (See Sec. 5.1 which describes the distance conversion from range bins to meters.) The trees and brush, which appear as white and light gray regions just above the vehicle, are the most distant part of the image.

For this range image, the range gate width is given: $\Delta R = 5500$ ft. $= 1524$ bins. However, the local range accuracy, $\delta R$, must be estimated. As described in Sec. 2.1, nonanomalous laser radar range measurements have Gaussian noise with standard deviation, $\delta R$. Thus the local range accuracy can be estimated by calculating the average noise of several different regions in the image. In Fig. 7-2, ten $5 \times 10$-pixel regions were chosen which appeared to be approximately flat and uniform (i.e. made up of purely ground, target, or trees and brush.) A linear least-squares planar fit was performed for each region and the variance was calculated for the difference between the region and its planar fit. The average variance for the ten regions was found
Figure 7-1: Video image of a transporter vehicle alongside dense growth of trees and underbrush.

Figure 7-2: Range image of a transporter vehicle.
to equal 1.249 range bins. Thus the local range accuracy for the image is equal: 
\[ \delta R = \sqrt{1.249} = 1.118 \text{ range bins.} \]

Similar to the tank image in Sec. 5.1, the transporter vehicle image has almost no anomalies. However, it is assumed to have Gaussian shot noise. Thus Fig. 7-2 cannot be taken as the true range image, \( \mathbf{R}^* \), but is actually equal to \( \mathbf{R}^* + \mathbf{v} \) where \( \mathbf{v} \) is a Q-D column vector of independent, identically distributed, Gaussian random variables each with zero mean and variance \( \delta R^2 \). Anomalies are simulated with a 5\% probability rate, \( \Pr(A) = 0.05 \). For each anomaly, the resulting measurement is produced from a uniform probability density with length, \( \Delta R = 1524 \). The resulting range image, \( \mathbf{R} \), is shown in Fig. 7-3.

![Figure 7-3: Range data, \( \mathbf{R} \), of a transporter vehicle where local range accuracy, \( \delta R = 1.118 \), anomaly probability \( \Pr(A) = 0.05 \), and range gate, \( \Delta R = 1524 \).](image)

7.1.2 Range Profile Results

Given the range data and necessary image parameters, the fast EM algorithm is now used to find the maximum-likelihood range estimate. The range data is profiled at several resolutions, \( P = \{1024, 2048, 4096\} \), as shown in Figs. 7-4 to 7-6.

The resulting profiles in Figs. 7-4 to 7-6 show the outline of the transporter vehicle quite well for the estimated local accuracy, \( \delta R = 1.118 \). At each higher resolution, the range profile shows finer and more detailed features of the image.
Figure 7-4: Haar-fitted range profile at $P = 1024$.

Figure 7-5: Haar-fitted range profile at $P = 2048$.

Figure 7-6: Haar-fitted range profile at $P = 4096$. 
7.2 Effects of Varying $\delta R$

In this section, the robustness of the ML/EM algorithm is examined in terms of how strongly the range estimate depends on obtaining a good estimate of the local range accuracy, $\delta R$. As mentioned earlier, the laser's carrier-to-noise (CNR) ratio can vary significantly for different types of scenery resulting in different local range accuracies in different regions of the image. The EM algorithm is designed to use a single estimate for $\delta R$. In this section, the range image of the transporter vehicle is profiled using various values of $\delta R$ at the same resolution, $P = 4096$. The results are analyzed by looking at the weights and log-likelihood measurements.

7.2.1 Range Profile Results and Weights

The transporter vehicle image, Fig. 7-3, is profiled using local accuracies $\delta R = \{0.25, 0.5, 1.0, 2.0, 4.0\}$ at $P = 4096$. The range profile results and corresponding final weights are shown in Figs. 7-7 to 7-16.

The profile results are quite similar for each value of $\delta R$. In fact, there are few noticeable differences in each of the five images. In contrast, the final weight images change significantly for different values of $\delta R$. For small $\delta R$, the fraction of low weighted pixels is quite large but for large $\delta R$, the fraction of low weighted pixels is relatively small. This can be understood by looking at the equation for the pixel weights, $w_q$, described in Sec. 3.3,

$$w_q(X_P) = \frac{\exp\left(-\frac{(R_q - (H_P X_P)_q)^2}{2\delta R^2}\right)}{[1 - \Pr(A)]\sqrt{2\pi\delta R^2}} + \frac{\Pr(A)}{\Delta R}.$$

When $\delta R$ is small, the Gaussian component of Eq. 7.2 is small compared to $\Pr(A)/\Delta R$ so $w_q$ is near zero, and conversely when $\delta R$ is large, the Gaussian component is large compared to $\Pr(A)/\Delta R$ so $w_q$ is near one. Note that a weight of zero indicates an
Local Range Accuracy, $\delta R = 0.25$

Figure 7-7: Haar-fitted range profile at $P = 4096$ where $\delta R = 0.25$.

Figure 7-8: Final pixel weights, $\mathbf{W}$, at $P = 4096$ where $\delta R = 0.25$.

anomalous range measurement, whereas a weight of one indicates a nonanomalous range measurement.

In Eq. 7.2, the Gaussian density has mean, $R_q - R^*$, and standard deviation, $\delta R$, so the value of $\delta R$ effectively changes the width of the Gaussian density function. For small $\delta R$, the range measurement, $R_q$, must be fairly close to the true range, $R^*$, to be a high weighted pixel. For large $\delta R$, the range measurement can be relatively further away from the true range value and still be a high weighted pixel. Consequently, as $\delta R$ increases in the final weights of Figs. 7-7 to 7-16, there are many fewer low
Local Range Accuracy, $\delta R = 0.5$

Figure 7-9: Haar-fitted range profile at $P = 4096$ where $\delta R = 0.5$.

Figure 7-10: Final pixel weights, $W$, at $P = 4096$ where $\delta R = 0.5$.

weighted pixels.

As described in Sec. 5.3.1, low weight pixels result when the EM algorithm cannot find a good fit for the range measurements in the block’s range estimate. In the above figures, this tends to occur both along the boundary between the transporter vehicle and background and throughout the region which contains the trees and brush. In the latter region, there is greater variation in the range measurements which results in many low weight pixels as seen particularly well in Fig. 7-12. At high $\delta R$, the algorithm comes quite closely to locating only the truly anomalous range measure-
Local Range Accuracy, $\delta R = 1.0$

Figure 7-11: Haar-fitted range profile at $P = 4096$ where $\delta R = 1.0$.

Figure 7-12: Final pixel weights, $W$, at $P = 4096$ where $\delta R = 1.0$.

ments. Compare the low weight pixels of Fig. 7-16 to the truly anomalous range measurements which stand out as black and white pixels in Fig. 7-3.

The main result here is that the range profiles look quite similar for each value of $\delta R$ despite the wide variation in the final weights. This shows that range profiling using the ML/EM algorithm is not too strongly dependent on the estimate for the local range accuracy and shows that the algorithm is quite robust in terms of $\delta R$. 
Local Range Accuracy, $\delta R = 2.0$

![Figure 7-13: Haar-fitted range profile at $P = 4096$ where $\delta R = 2.0$.](image)

Figure 7-13: Haar-fitted range profile at $P = 4096$ where $\delta R = 2.0$.

![Figure 7-14: Final pixel weights, $W$, at $P = 4096$ where $\delta R = 2.0$.](image)

Figure 7-14: Final pixel weights, $W$, at $P = 4096$ where $\delta R = 2.0$.

### 7.2.2 Fraction of Low Weight Pixels

The fraction of low weight pixels are shown in Fig. 7-17 as a function of the local range accuracy for $\delta R = \{0.25, 0.5, 1.0, 2.0, 4.0\}$. The fraction of low weight pixels decreases from over 30% for $\delta R = 0.25$ to about 5% for $\delta R = 4.0$. The trend seems to be approaching the anomaly probability, $\Pr(A) = 0.05$, but will end up approaching 0. As $\delta R$ increases, the Gaussian density will gradually become so wide that the EM algorithm will erroneously conclude that true anomalies are low weight, nonanomalous measurements. This is already occurring to a small degree at $\delta R = 4.0$ in Fig. 7-16.
In relation to the weight-based stopping rule, this would suggest that the appropriate value of $\delta R$ should be chosen such that the fraction of low weight pixels is not less than $\text{Pr}(A)$.

### 7.2.3 Log-Likelihood Measurements

The log-likelihood measurements, $L(R^*)$, are shown in Fig. 7-18 as a function of the local range accuracy for $\delta R = \{0.25, 0.5, 1.0, 2.0, 4.0\}$. This figure is particularly illustrative in showing how varying $\delta R$ affects the quality of the range estimate.
Although the range profiles in Figs. 7-7 to 7-16 look quite similar, their log-likelihood measurements are quite different. In fact, $L(R^*)$ at $\delta R = 1.0$ is more than 40% higher than $L(R^*)$ at $\delta R = 0.25$. Thus this shows the importance of having a good estimate for the local range accuracy to find the highest likelihood range estimate.

In Fig. 7-18, the maximum log-likelihood versus $\delta R$ occurs somewhere between one and two range bins. In Sec. 7.1.1, the local range accuracy, $\delta R$, was directly determined to be 1.118 range bins by calculating the standard deviation for the difference between the range data and its least-squares planar fit in various regions throughout the image. Thus both methods for measuring the local range accuracy seem to agree and confirm the approximate value for $\delta R$. 

Figure 7-17: Fraction of low weight pixels versus resolution, $P$. 
Figure 7-18: Log-likelihood, $L(R^*)$, versus resolution, $P$. 
Chapter 8

Effectiveness of the Haar-Wavelet Basis

For parametric range profiling (described in Sec. 3.2), we assume that the $Q$-D true range vector, $\mathbf{r}^*$, can be characterized by a $P$-D parameter vector, $\mathbf{x}_P$, where $P < Q$. Once the parameter vector is found using the EM algorithm, the ML range estimate is calculated by multiplying $\mathbf{x}_{P_{ML}}$ by a $P \times Q$ transformation matrix, $\mathbf{H}_P$,

$$\hat{\mathbf{r}}_{ML} = \mathbf{H}_P \hat{\mathbf{x}}_{P_{ML}}$$  \hspace{1cm} (8.1)

For this work, the transformation matrix, $\mathbf{H}_P$, is constructed from the multiresolution Haar-wavelet basis. This results in range estimates that are composed of $P$ rectangular blocks arranged in a grid as shown in previous Haar-fitted range profiles.

In this chapter, Haar-wavelet range profiling is examined in terms of its effectiveness for different types of imagery. In particular, the Haar-wavelet basis is shown to be most effective at fitting horizontal and vertical image features and least effective at fitting diagonal features due to the block structure of the range estimate. Thus range profiling might be improved by initially rotating imagery so that main features, such as the terrain or target, better fit with the Haar-wavelet block structure. This is tested by profiling the range image of a truck.
8.1 Haar-fitted Range Profiling

Haar-fitted range profiling results in range estimates that are composed of rectangular blocks arranged in a grid. Thus the Haar-wavelet basis is most effective at fitting horizontal and vertical image features and least effective at fitting diagonal features due to the block structure of the range estimate. This is demonstrated in Figs. 8-1 to 8-3.

Fig. 8-1 is a 32 x 32-pixel simulated range image composed of two uniform regions separated along the diagonal. (There are no anomalies or Gaussian noise.) Fig. 8-2 shows the profile results at $P = 16$. For each block, the range estimate is equal to the range value of the region which has the majority of the pixels. The EM algorithm assigns these pixels a weight of one while the remainder are considered anomalous and are assigned a weight of zero.

![Figure 8-1: Simulated range image composed of two uniform regions separated along the diagonal.](image)

The range data in Fig. 8-1 is significantly different from the range estimate in Fig. 8-2 along the diagonal. This difference is not due to the presence of anomalies because $\Pr(A) = 0$, but is a result of the diagonal orientation of the two regions which the Haar-wavelet block structure cannot fit well. The range profile could be improved by increasing the resolution, but another solution might be to rotate the image to better align features with the Haar-wavelet block structure.

Fig. 8-3 is a 32 x 32-pixel simulated range image much like Fig. 8-1, but composed
of two uniform regions separated horizontally along the center. The range profile at $P = 16$ is equal to the range data. (In fact, the range profile at all but the lowest resolutions is equal to the range data.) This occurs because the Haar-wavelet block structure perfectly aligns with the range image. This suggests that had the range data in Fig. 8-1 been rotated to look something like Fig. 8-3, the range profile could have been much improved.

Figure 8-3: Simulated range image composed of two uniform regions separated horizontally along the center. Also the Haar-fitted range profile at $P = 16$. 

Figure 8-2: Haar-fitted range profile at $P = 16$. 
8.2 Range Profiling Rotated Imagery

As shown in the previous section on simulated data, Haar-wavelet range profiling can be improved by rotating imagery. This is now tested on real laser radar range imagery.

8.2.1 Laser Radar Range Imagery

The range image to be profiled here contains a truck in a large field. The video image of the truck is shown in Fig. 8-4 and the corresponding range image is shown in Fig. 8-5.

![Figure 8-4: Video image of a truck in a field.](image)

As shown in Fig. 8-5, the truck was about 300 bins or 760 meters away from the laser radar when the image was scanned (See Sec. 5.1 which describes the distance...
conversion from bins to meters.) For the range image, the range gate is given as \( \Delta R = 1676 \ m = 1524 \) bins. The local range accuracy is estimated to equal \( \delta R = 1.5909 \) bins, calculated as described in Sec. 7.1.1.

The range image of the truck has almost no anomalies so anomalies are simulated with a 5% probability rate, \( \text{Pr}(A) = 0.05 \). For each anomaly, the resulting measurement is produced from a uniform probability density with length, \( \Delta R = 1524 \) bins. The resulting range image, \( R \), is shown in Fig. 8-6.

As a test of whether range profiling is improved by aligning imagery with the Haar-wavelet block structure, the range image, \( R \), is rotated to align the roof of the truck with the horizontal pixel grid. This is achieved by rotating the image about 11 degrees. The rotated range image, \( R_r \), is shown in Fig. 8-7. The region above the
image is shaded black and the region below is shaded white so the actual range image shows up more pronounced.

Figure 8-7: Range data, $R_r$, of a truck in a field where local range accuracy, $\delta R = 1.5909$ bins, anomaly probability $Pr(A) = 0.05$, and range gate, $\Delta R = 1524$ bins.

### 8.2.2 Range Profile Results

Given the original (non-rotated) range image, $R$, and rotated range image, $R_r$, and necessary image parameters, the EM algorithm is now used to find the maximum-likelihood range estimate for each image. The range data are profiled at several resolutions, $P = \{1024, 2048, 4096\}$. The resulting range profiles for the original image are shown in Figs. 8-8 to 8-10 and the range profiles for the rotated image are shown in Fig. 8-11 to 8-13.

Comparing the figures for the two images, the range profiles appear quite similar in many respects. The main difference is that there appears to be a slightly better alignment of the features of the rotated image with the Haar-wavelet block structure. This is particularly noticeable at $P = 1024$—compare Figs. 8-8 and 8-14—but is also seen at higher resolutions.
Figure 8-8: Haar-fitted range profile of $R$ at $P = 1024$.

Figure 8-9: Haar-fitted range profile of $R$ at $P = 2048$.

Figure 8-10: Haar-fitted range profile of $R$ at $P = 4096$. 
Figure 8-11: Haar-fitted range profile of $R_r$ at $P = 1024$.

Figure 8-12: Haar-fitted range profile of $R_r$ at $P = 2048$. 
8.2.3 Log-Likelihood Measurements

Perhaps the most useful method for comparing the range profiles of the original (non-rotated) image and the rotated image is to look at their log-likelihood measurements. This calculation indicates how close the range profile is to the original data and thus is an approximate measure of the quality of the range profile. The log-likelihood measurements, $L(R^*)$, are shown in Fig. 8-14 as a function of resolution for $P = \{512, 1024, 2048, 4096\}$. This shows that the likelihoods for the rotated image, $R_r$, are slightly higher than for the original image, $R$. Since the likelihood measurements are only slightly higher, this suggests the quality of the range profiles are approximately the same, but just slightly better for the rotated image. In this case, the truck only needed to be rotated eleven degrees to align with the Haar-wavelet block structure. However, if the rotation angle needed to be greater, such as forty-five degrees, then presumably the rotated range profile would be much more accurate than the non-rotated range profile and the log-likelihood measurements would also be much higher for the rotated range profile. Thus rotating range imagery would produce improved range profiles due to better alignment of image features with the Haar-wavelet block structure.

Figure 8-13: Haar-fitted range profile of $R_r$ at $P = 4096$. 
Figure 8-14: Log-likelihood, $L(R^*)$, versus resolution, $P$. 
Chapter 9

Conclusion

Coherent laser radars are capable of collecting range images by raster scanning a field of view. Range imagery is subject to fluctuations arising from the combined effects of laser speckle and local-oscillator shot noise. The former causes range anomalies while the latter results in Gaussian noise spread throughout the image. The nature of these degradation processes has suggested a statistical approach to laser radar image processing. In previous work, the expectation-maximization (EM) algorithm was used to develop an explicit procedure for maximum-likelihood (ML) range profiling of simulated 2-D range data [12]. Our objective in this thesis was to extend ML/EM multiresolution range profiling to 3-D real laser radar range imagery.

Application of the conventional EM algorithm to 3-D range imagery was significantly hindered by two major issues: computational load and numerical robustness. Both of these issues arose from the maximization step in the EM iterations, namely, Eq. 3.25. This effectively limited range profiling to small (few-pixelled) imagery at relatively low resolutions. We developed a more powerful fast ML/EM algorithm using the special structure of the Haar-wavelet basis. This reduced the computational complexity of the ML equation from quadratic to linear growth as a function of resolution and increased the speed of the EM calculation tremendously. Thus, whereas it took about $3 \times 10^4$ seconds to range profile a $Q = 128 \times 128$-pixel image at $P = 128$ using the conventional EM algorithm, it took only about 90 seconds using the fast ML/EM algorithm. Given the dramatic increase in computational speed, this
made it possible to range profile laser radar range imagery of typical size (Q-value) at much high resolutions, \( P \). Furthermore, the problem of non-invertibility in the \( H^T_Q W_Q(n) H_P \) matrix in Eq. 3.25 was circumvented in the fast ML/EM algorithm using the special structure of the Haar-wavelet basis. This resulted in an algorithm which was numerically robust.

Multiresolution range profiling of real laser radar imagery using the fast Haar-wavelet EM/ML algorithm presented many important results. In general, the anomalies were almost completely suppressed at all resolutions. As the resolution increased, the fraction of low weight pixels in the range estimate steadily decreased toward its theoretical ensemble average, \( Pr(A) \). The gray-scale weighting images showed clumping of supposedly anomalous pixels along the edge between the target and background marking the location of the target. Although range profiling showed the general target shape, such as the tank's body, at low resolutions, smaller features, such as the gun barrel, were suppressed. However, the barrel appeared strongly silhouetted in the corresponding weight image. This suggests the notion that range-edge detection can be accomplished from the low-resolution weight images using a hybrid approach to Haar-wavelet EM/ML range profiling: use modest resolution to discern the body of the target, and recover fine features from the associated weight image. Next, it was shown that at high resolutions, range profiling is approximately unbiased and has error variances that approach the ultimate complete-data bound.

Further improvements to the fast ML/EM algorithm included developing modified weighting schemes. First, discrete weights were used in place of analog weights and shown to increase the profiling computational speed while producing nearly the same range estimate. However, certain range blocks were composed of purely zero-weighted anomalous pixels, resulting in holes in the range estimate. A hybrid discrete-and-analog weighting scheme was shown to eliminate the holes while taking advantage of the increased computational speed of discrete weighting.

The parameter robustness of the ML/EM algorithm was analyzed by studying the effects of varying the local range accuracy, \( \delta R \). It was shown that the range estimate calculated by the EM algorithm was not significantly affected by the value of the
local range accuracy, however, the maximum likelihood is achieved using an accurate estimate of $\delta R$.

Finally, the multiresolution Haar-wavelet basis was examined in terms of its effectiveness for ML fitting of range imagery. It was shown that the Haar-wavelet basis was most effective at fitting horizontal and vertical image features and least effective at fitting diagonal features due to the block structure of the range estimate. Thus range profiling was shown to have slightly greater likelihoods by initially rotating the sample range image of a truck to pre-align main features with the Haar-wavelet block structure.

There are many directions in which future work can proceed. The ML/EM algorithm could serve as a pre-processor from which the resulting range profiles would be used in object detection and recognition. The algorithm could be further optimized by parallel processing the blocks’ ML range estimates. Ideally, this could be done in real time. Finally, this method of multiresolution image processing could be applied to other types of sensor imagery, such as peak-detecting Doppler imagers, FLIR models, and SAR and ISAR models [12].
Bibliography


