

Chapter 8

Present Value Mathematics for Real Estate

Real estate deals almost
always involve ***cash amounts***
at ***different points in time.***

Examples:

- Buy a property now, sell it later.
- Sign a lease now, pay rents monthly over time.
- Take out a mortgage now, pay it back over time.
- Buy land now for development, pay for construction and sell the building later.

Present Value Mathematics

- In order to be successful in a real estate career (not to mention to succeed in subsequent real estate courses), you need to know how to *relate cash amounts across time*.
- In other words, you need to know how to do “Present Value Mathematics”,
 - On a business *calculator*, and
 - On a *computer* (e.g., using a spreadsheet like “Excel”®).

*Why is \$1 today not equivalent to \$1
a year from now? ...*

- Dollars at different points in time are related by the “*opportunity cost of capital*” (OCC), expressed as a rate of return.
- We will typically label this rate, “*r*”.

Example

If $r = 10\%$, then \$1 a year from now is worth:

$$\frac{\$1}{1+0.10} = \frac{\$1}{1.1} = \$0.909$$

today.

Two major types of PV math problems:

- Single-sum problems
- Multi-period cash flow problems
- Correspond to two equations hardwired into your business calculator . . .

$$0 = -PV + \left(\frac{PMT}{(1+r)} + \frac{PMT}{(1+r)^2} + \dots + \frac{PMT}{(1+r)^N} \right) + \frac{FV}{(1+r)^N}$$

$$NPV = CF_0 + \frac{CF_1}{(1+IRR)} + \frac{CF_2}{(1+IRR)^2} + \dots + \frac{CF_T}{(1+IRR)^T}$$

$$(r = i / m)$$

8.1 Single-sum formulas...

Your tenant owes \$10,000 in rent. He wants to postpone payment for a year. You are willing, but only for a 15% return. How much will your tenant have to pay?

$$FV = (1.15) \$10,000 = \$11,500$$

This is the basic building block.

8.1 Single-sum formulas...

8.1.1 Single-period discounting & growing

$$FV = (1 + r) PV$$

$$PV = \frac{FV}{1 + r}$$

$$0 = -PV + \left(\frac{PMT}{(1+r)} + \frac{PMT}{(1+r)^2} + \dots + \frac{PMT}{(1+r)^N} \right) + \frac{FV}{(1+r)^N}$$

With $PMT = 0$, $N = 1$.

8.1.2 Single-sums over multiple periods.

$$PV = \frac{FV}{(1+r)^N}$$

$$FV = (1+r)^N PV$$

$$0 = -PV + \left(\frac{PMT}{(1+r)} + \frac{PMT}{(1+r)^2} + \dots + \frac{PMT}{(1+r)^N} \right) + \frac{FV}{(1+r)^N}$$

With $PMT = 0$

Single-sums over multiple periods

You're interested in a property you think is worth \$1,000,000. You think you can sell this property in five years for that same amount. Suppose you're wrong, and you can only really expect to sell it for \$900,000 at that time. How much would this reduce what you're willing to pay for the property today, if 15% is the required return?

Single-sums over multiple periods

Answer:

$$\frac{\$1,000,000}{(1.15)^5} = \$497,177, \quad \frac{\$900,000}{(1.15)^5} = \$447,459$$

$\$497,177 - \$447,459 = \$49,718$, down to \$950,282.

On calculator:

N	I/YR	PV	PMT	FV
5	15	CPT	0	100000 Or 900000

Or equivalently:

5	15	CPT	0	100000
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Solving for the return or the time it takes
to grow a value...

You can buy a piece of land for \$1,000,000.

You think you will be able to sell it to a
developer in about 5 years for twice that
amount. You think an investment with this
much risk requires an expected return of
20% per year. Should you buy the land?...

Solving for **the return** or the time it takes to grow a value...

$$\text{\$1,000,000}(1+r)^5 = \text{\$2,000,000} \quad \textit{What's r?}$$

$$r = \left(\frac{FV}{PV} \right)^{1/N} - 1 = \left(\frac{\text{\$2,000,000}}{\text{\$1,000,000}} \right)^{1/5} - 1 = 14.87\%$$

Answer: No.

N	I/YR	PV	PMT	FV
5	CPT	-1000000	0	2000000

Solving for the return or **the time** it takes to grow a value...

Your investment policy is to try to buy vacant land when you think its ultimate value to a developer will be about twice what you have to pay for the land. You want to get a 20% return on average for this type of investment. You need to wait to purchase such land parcels until you are within **how many years** of the time the land will be “*ripe*” for development?...

Solving for the return or **the time** it takes to grow a value...

$$N = \frac{LN(FV) - LN(PV)}{LN(1 + r)} = \frac{LN(2) - LN(1)}{LN(1.20)} = 3.8$$

3.8 years.

N	I/YR	PV	PMT	FV
CPT	20	-1	0	2

Simple & Compound Interest...

15% interest for two years, compounded:

For every \$1 you start with, you end up with:

$$(1.15)(1.15) = (1.15)^2 = \$1.3225.$$

The “15%” is called “**compound interest**”.

(In this case, the compounding interval is *annual*).

Simple & Compound Interest...

$$(1.15)(1.15) = (1.15)^2 = \$1.3225$$

Note: you ended up with 32.25% more than you started with.

$$32.25\% / 2 \text{ yrs} = 16.125\%.$$

So the same result could be expressed as:

- 16.125% “**simple annual interest for two years**” (no compounding).
- or 32.25% “**simple biennial interest**”.

Simple & Compound Interest...

- Suppose you get 1% simple interest each month. This is referred to as a “12% **nominal annual rate**”, or “equivalent nominal annual rate” (**ENAR**). We will use the label “*i*” (or “**NOM**”) to refer to the ENAR:

- **Nominal Annual Rate =**

$$\text{(Simple Rate Per Period)} * \text{(Periods/Yr)}$$

$$i = (r)(m) = \text{“NOM\%”}$$

$$12\% = (1\%)(12 \text{ mo/yr})$$

Simple & Compound Interest...

Suppose the 1% simple monthly interest is compounded at the end of every month. Then in 1 year (12 months) this 12% nominal annual rate gives you:

$$- (1.01)^{12} = \$1.126825$$

For every \$1 you started out with at the beginning of the year.

12.00% nominal rate = **12.6825% effective annual rate**

“**Effective Annual Rate**” (EAR) is aka EAY (“effective ann. yield”).

The relationship between **ENAR** and **EAR**:

$$EAR = (1 + i/m)^m - 1 = \text{"EFF\%"}$$

$$i = m [(1 + EAR)^{1/m} - 1] = ENAR = \text{"NOM\%"}$$

Rates are usually quoted in *ENAR* (nominal) terms.

Example:

- What is the EAR of a 12% mortgage?

$$EAR = (1 + 0.12/12)^{12} - 1 = (1.01)^{12} - 1 = 12.6825\%$$

- You don't have to memorize the formulas if you know how to use a business calculator:

HP-10B	TI-BAII PLUS
CLEAR ALL	I Conv
12 P/YR	NOM = 12 ENTER ↓ ↓
12 I/YR	C/Y = 12 ENTER ↑
EFF% gives 12.68	CPT EFF = 12.68

- Effective Rate = “EFF” = “EAR”
- Nominal Rate = “NOM” = ENAR

“Bond-Equivalent” & “Mortgage-Equivalent” Rates...

- Traditionally, bonds pay interest *semi-annually* (twice per year).
- Bond interest rates (and yields) are quoted in nominal annual terms (ENAR) assuming semi-annual compounding ($m = 2$).
- This is often called “bond-equivalent yield” (BEY), or “coupon-equivalent yield” (CEY). Thus:

$$EAR = (1 + BEY/2)^2 - 1$$

What is the EAR of an 8% bond?

Fast!

“Bond-Equivalent” & “Mortgage-Equivalent” Rates

- Traditionally, mortgages pay interest *monthly*.
- Mortgage interest rates (and yields) are quoted in nominal annual terms (ENAR) assuming monthly compounding ($m = 12$).
- This is often called “mortgage-equivalent yield” (MEY) Thus:

$$EAR = (1 + MEY/12)^{12} - 1$$

What is the EAR of an 8% mortgage?

Fast!

Example:

Yields in the bond market are currently 8% (CEY). What interest rate must you charge on a mortgage (MEY) if you want to sell it at par value in the bond market?

Answer:

7.8698%.

$$EAR = (1 + BEY/2)^2 - 1 = (1.04)^2 - 1 = 0.0816$$

$$MEY = 12 [(1 + EAR)^{1/12} - 1] = 12 [(1.0816)^{1/12} - 1] = 0.078698$$

HP-10B	TI-BAII PLUS
CLEAR ALL	I Conv
2 P/YR	NOM = 8 ENTER ↓ ↓
8 I/YR	C/Y = 2 ENTER ↑
EFF% gives 8.16	CPT EFF = 8.16 ↓
12 P/YR	C/Y = 12 ENTER ↑ ↑
NOM% gives 7.8698	CPT NOM = 7.8698

Example:

You have just issued a mortgage with a 10% contract interest rate (MEY). How high can yields be in the bond market (BEY) such that you can still sell this mortgage at par value in the bond market?

Answer:

10.21%.

$$EAR = (1 + MEY/12)^{12} - 1 = (1.00833)^{12} - 1 = 0.1047$$

$$BEY = 2 [(1 + EAR)^{1/2} - 1] = 2 [(1.1047)^{1/2} - 1] = 0.1021$$

HP-10B	TI-BAII PLUS
CLEAR ALL	I Conv
12 P/YR	NOM = 10 ENTER ↓↓
10 I/YR	C/Y = 12 ENTER ↑
EFF% gives 10.47	CPT EFF = 10.47 ↓
2 P/YR	C/Y = 2 ENTER ↑↑
NOM% gives 10.21	CPT NOM = 10.21

8.2 Multi-period problems...

- Real estate typically lasts many years.
 - And it pays cash each year.
- Mortgages last many years and have monthly payments.
- Leases can last many years, with monthly payments.
- We need to know how to do PV math with multi-period cash flows.
- In general,
The *multi-period* problem is just the *sum* of a bunch of individual *single-sum* problems:

Example:

PV of \$10,000 in 2 years @ 10%, and
\$12,000 in 3 years at 11% is:

$$PV = \frac{\$10,000}{(1+0.10)^2} + \frac{\$12,000}{(1+0.11)^3} = \$8,264.46 + \$8,774.30 = \$17,038.76$$

“Special Cases” of the Multi-Period PV Problem...

- Level Annuity
- Growth Annuity
- Level Perpetuity
- Constant-Growth Perpetuity

“Special Cases” of the Multi-Period PV Problem...

- These “*special cases*” of the multi-period PV problem are particularly interesting to consider.
- They equate to (or approximate to) many practical situations:
 - Leases
 - Mortgages
 - Apartment properties
- And they have simple multi-period PV formulas.
- This enables general relationships to be observed in the PV formulas.

Example:

The relationship between the total return, the current cash yield, and the cash flow growth rate:

→ “Cap Rate” \approx Return Rate - Growth Rate.

8.2.2 The Level Annuity in Arrears

The PV of a regular series of cash flows, all of the same amount, occurring at the end of each period, the first cash flow to occur at the end of the first period (one period from the present).

$$PV = \frac{CF_1}{(1+r)} + \frac{CF_2}{(1+r)^2} + \dots + \frac{CF_N}{(1+r)^N}$$

where $CF_1 = CF_2 = \dots = CF_N$.

Label the constant periodic cash flow amount:
“***PMT***”:

$$PV = \frac{PMT}{(1+r)} + \frac{PMT}{(1+r)^2} + \dots + \frac{PMT}{(1+r)^N}$$

On your calculator . . .

$$0 = -PV + \left(\frac{PMT}{(1+r)} + \frac{PMT}{(1+r)^2} + \dots + \frac{PMT}{(1+r)^N} \right) + \frac{FV}{(1+r)^N}$$

With $FV = 0$.

Example:

What is the present value of a 20-year mortgage that has monthly payments of \$1000 each, and an interest rate of 12%/year?

$$PV = \frac{\$1000}{1.01} + \frac{\$1000}{1.01^2} + \frac{\$1000}{1.01^3} + \Lambda + \frac{\$1000}{1.01^{240}} = \$90,819$$

(Note: 12% nominal rate is 1% per month simple interest.)

This can be found either by applying the “*Annuity Formula*”

$$PV = \frac{PMT}{(1+r)} + \frac{PMT}{(1+r)^2} + \dots + \frac{PMT}{(1+r)^N}$$

$$= PMT \frac{1 - 1/(1+r)^N}{r}$$

$$= \$1000 \frac{1 - 1/(1.01)^{240}}{0.01} = \$90,819$$

Or by using your calculator . . .

N	I/YR	PV	PMT	FV
240	12	CPT	1000	0

Answer (cont'd):

In a computer spreadsheet like *Excel*® , there are functions you can use to solve for these variables. The *Excel*® functions equivalent to the HP-10B calculator registers are:

N	I/YR	PV	PMT	FV
=NPER()	=RATE()	=PV()	=PMT()	=FV()

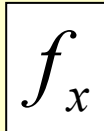
However, the computer does not convert automatically from nominal annual terms to simple periodic terms. For example, a 30-year, 12% interest rate mortgage with monthly payments in *Excel*® is a 360-month, 1% interest mortgage.

For example:

The *Excel* formula below:

=PMT(.01,240,-90819)

will return the value \$1000.00 in the cell in which the formula is entered.

You can use the *Excel*  key on the toolbar to get help with financial functions.

8.2.3 The Level Annuity in *Advance*

The PV of a regular series of cash flows, all of the same amount, occurring at the *beginning* of each period, the first cash flow to occur *at the present* time.

The Level Annuity in **Advance**

PV just equals $(1+r)$ times the PV of the annuity in *arrears*:

$$PV = PMT + \frac{PMT}{(1+r)} + \dots + \frac{PMT}{(1+r)^{N-1}}$$

$$= (1+r) \left(\frac{PMT}{(1+r)} + \frac{PMT}{(1+r)^2} + \dots + \frac{PMT}{(1+r)^N} \right)$$

$$= (1+r)PMT \frac{1 - 1/(1+r)^N}{r}$$

$$= PMT(1+r) \frac{1 - 1/(1+r)^N}{r}$$

PMT just equals
 $1/(1+r)$ times PMT in
arrears, for same PV

Example:

What is the present value of a 20-year *lease* that has monthly rental payments of \$1000 each, due at the *beginning* of each month, where 12%/year is the appropriate cost of capital (OCC) for discounting the rental payments back to present value?

Answer:

$$PV = \$1000 + \frac{\$1000}{1.01} + \frac{\$1000}{1.01^2} + \Lambda + \frac{\$1000}{1.01^{239}}$$

$$= \$1000 \left((1.01) \frac{1 - 1/(1.01)^{240}}{0.01} \right)$$

$$= \$91,728 = 1.01 * \$90,819$$

(Note: This is slightly more than the \$90,819 PV of the annuity in arrears.)

(Remember: $r = i/m$, e.g.: $1\% = 12\%/yr. \div 12mo/yr.$)

Answer (cont'd):

In your calculator, set:

HP-10B: Set **BEG/END** key to **BEGIN**

TI-BA II+: Set **2nd BGN, 2nd SET, ENTER**

(Note: This setting will remain until you change it back.)

Then, as with the level annuity in advance:

N	I/YR	PV	PMT	FV
240	12	CPT	1000	0

8.2.9 Solving the Annuity for *Future Value*.

Suppose we want to know how much a level stream of cash flows (in arrears) will *grow* to, at a constant interest rate, over a specified number of periods.

The general formula is:

$$FV = (1 + r)^N PV = (1 + r)^N \left[\frac{PMT}{(1 + r)} + \frac{PMT}{(1 + r)^2} + \dots + \frac{PMT}{(1 + r)^N} \right]$$

$$= (1 + r)^N \left[PMT \frac{1 - 1/(1 + r)^N}{r} \right]$$

Example:

What is the future value after 20 years of \$1000 deposited at the end of every month, at 12% annual interest compounded monthly?

$$(1.01)^{239} \$1000 + (1.01)^{238} \$1000 + K + \$1000 = \$989,255$$

N	I/YR	PV	PMT	FV
240	12	0	1000	CPT

8.2.10 Solving for the Annuity Periodic *Payment* Amount.

What is the regular periodic payment amount (in arrears) that will provide a given present value, discounted at a stated interest rate, if the payment is received for a stated number of periods?

Just *invert* the annuity formula to solve for the **PMT** amount:

$$PMT = PV \frac{r}{1 - 1/(1+r)^N}$$

Example:

A borrower wants a 30-year, monthly-payment, fixed-interest mortgage for \$80,000. As a lender, you want to earn a return of 1 percent per month, compounded monthly. Then you would agree to provide the \$80,000 up front (i.e., at the present time), in return for the commitment by the borrower to make 360 equal monthly payments in the amount of?...

Answer:

$$PMT = \$80,000 \frac{0.01}{1 - 1/1.01^{360}} = \$822.89$$

\$822.89

Or, on the calculator (with P/Yr = 12):

N	I/YR	PV	PMT	FV
360	12	80000	CPT	0

8.2.11 Solving for the *Number* of Periodic Payments

How many regular periodic payments (in arrears) will it take to provide a given return to a given present value amount?

The general formula is:

$$N = -\frac{\text{LN}\left(1 - r\frac{PV}{PMT}\right)}{\text{LN}(1+r)}$$

where LN is the natural logarithm. *(But who cares!)*

Example:

How long will it take you to pay off a \$50,000 loan at 10% annual interest (compounded monthly), if you can only afford to pay \$500 per month?

Answer:

$$216 = - \frac{LN \left(1 - (0.10/12) \frac{50000}{500} \right)}{LN (1 + (0.10/12))}$$

Or solve it on the calculator:

N	I/YR	PV	PMT	FV
CPT	10	50000	-500	0

Another example:

Solving for the *number of periods* required to obtain a *future value*.

The general formula is:

$$N = \frac{\text{LN} \left(1 + r \frac{FV}{PMT} \right)}{\text{LN} (1 + r)}$$

Example:

You expect to have to make a capital improvement expenditure of \$100,000 on a property in five years. How many months before that time must you begin setting aside cash to have available for that expenditure, if you can only set aside \$2000 per month, at an interest rate of 6%, if your deposits are at the *ends* of each month?

Aka: “Sinking Fund”

Answer:

$$45 = \frac{LN \left(1 + (0.005) \frac{100000}{2000} \right)}{LN (1.005)}$$

45 months

Or solve it on the calculator:

N	I/YR	PV	PMT	FV
CPT	6	0	-2000	100000

8.2.4 The Growth Annuity in Arrears

The PV of a finite series of cash flows, each one a constant multiple of the preceding cash flow, all occurring at the *ends* of the periods.

Each cash flow is the same multiple of the previous cash flow.

Example:

\$100.00, \$105.00, \$110.25, \$115.76, . . .

$\$105 = (1.05)\100 , $\$110.25 = (1.05)\105 ,
etc...

Continuing for a finite number of periods.

Example:

A 10-year lease with annual rental payments to be made at the end of each year, with the rent increasing by 2% each year. If the first year rent is \$20/SF and the OCC is 10%, what is the PV of the lease?

Answer:

$$PV = \frac{\$20}{1.10} + \frac{(1.02)\$20}{1.10^2} + \frac{(1.02)^2\$20}{1.10^3} + \Lambda + \frac{(1.02)^9\$20}{1.10^{10}}$$

\$132.51/SF.

How do we know...

The general formula is:

$$PV = \frac{CF_1}{(1+r)} + \frac{(1+g)CF_1}{(1+r)^2} + \frac{(1+g)^2 CF_1}{(1+r)^3} + \Lambda + \frac{(1+g)^{N-1} CF_1}{(1+r)^N}$$
$$= CF_1 \left(\frac{1 - ((1+g)/(1+r))^N}{r-g} \right)$$

So in the specific case of our example:

$$PV = \$20 \left(\frac{1 - (1.02/1.10)^{10}}{0.10 - 0.02} \right) = \$20(6.6253) = \$132.51.$$

Answer (cont'd):

Unfortunately, the typical business calculator is not set up to do this type of problem “automatically”. You either have to memorize the formula (or steps on your calculator), as below:

$$1.02 / 1.1 = 0.9273$$

$$y^x 10 = 0.47$$

$$- 1 = -0.53 +/-$$

$$.08 = 6.6253 \times 20 = 132.51$$

Answer (cont'd):

Or you can “*trick*” the calculator into solving this problem using its mortgage math keys, by transforming the level annuity problem:

1. Redefine the interest rate to be: $(1+r)/(1+g)-1$ (e.g., $1.1/1.02 - 1 = 0.078431 = 7.8431\%$).
2. Solve the level annuity *in advance* problem with this “fake” interest rate (e.g., BEG/END, N=10, I/YR=7.8431, PMT=20, FV=0, CPT PV=145.7568).
3. Then divide the answer by $1+r$ (e.g., $145.7568 / 1.1 = 132.51$).

(Note: you set the calculator for CFs in advance (BEG or BGN), but that’s just a “trick”. The actual problem you are solving is for CFs in arrears.)

A more complicated example (like Study Qu.#49):

A landlord has offered a tenant a 10-year lease with annual net rental payments of \$30/SF in arrears. The appropriate discount rate is 8%. The tenant has asked the landlord to come back with another proposal, with a lower initial rent in return for annual step-ups of 3% per year. What should the landlord's proposed starting rent be?

Answer

First solve the level-annuity PV problem:

$$PV = \$30 \left(\frac{1 - (1/1.08)^{10}}{0.08} \right) = \$201.30$$

→M	
STO	1

Then plug this answer into the growth-annuity formula and invert to solve for CF_1 :

$$CF_1 = \$201.30 \left(\frac{0.08 - 0.03}{1 - (1.03/1.08)^{10}} \right) = \$26.66$$

$1.03/1.08 = .9537 \quad y^x 10 = .6225 - 1 = -.3775 \quad +/- \div (.08 - .03) = 7.5501 \quad 1/x \rightarrow .13245 \quad X \text{ RM (or RCL 1)} = 26.66$
--

Or, using the “calculator trick” method, the steps are:

HP-10B	TI-BA II+
CLEAR ALL, BEG/END=END	BGN SET (=END) ENTER QUIT
1 P/YR	P/Y = 1 ENTER QUIT
10 N	10 N
8 I/Y	8 I/Y
30 PMT	30 PMT
PV gives 201.30	CPT PV = 201.30
$1.08/1.03-1=.0485 \times 100=$	$1.08/1.03-1=.0485 \times 100=$
4.85437 I/Y	4.85437 I/Y
BEG/END = BEG	BGN SET (=BGN) ENTER QUIT
PMT gives 24.68713	CPT PMT = 24.68713
$\times 1.08 = 26.66$	$\times 1.08 = 26.66$

Answer (cont'd):

Tenant gets an initial rent of \$26.66/SF instead of \$30.00/SF, but landlord gets same PV from lease (because rents grow).

8.2.5 The Constant-Growth Perpetuity (in arrears).

The PV of an *infinite* series of cash flows, each one a constant multiple of the preceding cash flow, all occurring at the *ends* of the periods.

Like the Growth annuity only it's a *perpetuity* instead of an annuity: an *infinite* stream of cash flows.

The general formula is:

$$PV = \frac{CF_1}{(1+r)} + \frac{(1+g)CF_1}{(1+r)^2} + \frac{(1+g)^2 CF_1}{(1+r)^3} + \Lambda$$
$$= \frac{CF_1}{r-g}$$

The entire infinite collapses to this simple ratio!

This is a very useful formula, because:

- 1) It's simple.
- 2) It approximates a commercial building.
- 3) Especially if the building doesn't have long-term leases, and operates in a rental market that is not too *cyclical* (e.g., like most apartment buildings).

The *perpetuity formula* & the *cap rate*...

You've already seen the perpetuity formula.

It's the cap rate formula:

$$BldgVal = PV = \frac{CF_1}{r-g} \approx \frac{NOI}{caprate}$$

\Rightarrow

$$caprate = \frac{NOI}{PV} \approx r-g$$

$$= (r_f + RP) - g$$

$$= Tbill\ Rate + Risk\ Premium - GrowthRate$$

(Remember the three determinants of the cap rate...)

Example:

If the the required expected total return on the investment in the property (the OCC) is **10%**,

And the average annual growth rate in the rents and property value is **2%**,

Then the *cap rate* is approximately **8%** ($= 10\% - 2\%$).

(Or vice versa: If the cap rate is 8% and the average growth rate in rents and property values is 2%, then the expected total return on the investment in the property is 10%.)

Suppose Tbill yield is 6%, risk premium is 4%, and cap rate is 10%, then what is realistic LR growth rate expectation for property net cash flows? . . .

$$\text{Cap} = r - g \rightarrow g = r - \text{cap} \rightarrow g = (6\% + 4\%) - 10\% = 0\%.$$

Example:

An apartment building has 100 identical units that rent at \$1000/month with building operating expenses paid by the landlord equal to \$500/mo. On average, there is 5% vacancy. You expect both rents and operating expenses to grow at a rate of 3% per year (actually: 0.25% per month). The opportunity cost of capital is 12% per year (actually: 1% per month). How much is the property worth?

1) Calculate initial monthly cash flow:

$$\text{Potential Rent (PGI)} = \$1000 * 100 = \$100,000$$

$$\text{Less Operating Expenses} = -500 * 100 = -50,000$$

$$\begin{array}{r} \text{-----} \\ \$500 * 100 = \$50,000 \end{array}$$

$$\text{Less 5\% Vacancy} \rightarrow 0.95 * \$50,000 = \$47,500 \text{ NOI}$$

2) Calculate PV of Constant-Growth Perpetuity:

$$PV = \frac{\$47500}{1.01} + \frac{(1.0025)\$47500}{1.01^2} + \frac{(1.0025)^2 \$47500}{1.01^3} + \Lambda$$
$$= \frac{CF_1}{r - g} = \frac{\$47500}{0.01 - 0.0025} = \frac{\$47500}{0.0075} = \$6,333,333$$

$$\text{Cap Rate} = .01 - .0025 = .0075 / \text{mo.} \times 12 = 9\%$$

8.2.12 Combining the Single Lump Sum and the Level Annuity Stream: Classical Mortgage Mathematics

The typical calculations associated with mortgage mathematics can be solved as a *combination* of the *single-sum* and the *level-annuity (in arrears)* problems we have previously considered.

The classical mortgage is a level annuity
in arrears with:

Loan Principal = “PV” amount

Regular Loan Payments = “PMT” amount

Outstanding Loan Balance (OLB) can be found in either of two (mathematically equivalent) ways:

- 1) **OLB = PV** with “N” as the number of *remaining* (unpaid) regular payments.
- 2) **OLB = FV** with “N” as the number of *already paid* regular payments.

OLB

In other words, if Tm is the original number of payments in a fully-amortizing loan, and “ q ” is the number of payments that have already been made on the loan, then the OLB is given by the following general formula:

$$OLB = PMT \frac{1 - \left(\frac{1}{1 + \frac{i}{m}} \right)^{(Tm-q)}}{\frac{i}{m}}$$

Example:

Recall the \$80,000, 30-year, 12% mortgage with monthly payments that we previously examined (recall that $PMT = \$822.89$ per month). What is its OLB after 10 years of payments?

Answer:

$$74734 = 822.89 \frac{1 - 1/(1.01)^{240}}{.01}$$

\$74,734.

This can be computed in either of two ways on your calculator:

- 1) Solve for PV with N =number of *remaining* payments:

N	I/YR	PV	PMT	FV
240	12	CPT	822.89	0

- 2) Solve for FV with N =number of payments *already made*:

N	I/YR	PV	PMT	FV
120	12	80000	822.89	CPT

Another example: “*Balloon Mortgages*”...

Some mortgages are not “fully amortizing”, that is, they are required to be paid back before their OLB is fully paid down by the regular monthly payments.

The “*balloon payment*” (due at the maturity of the loan, when the borrower is required to pay off the loan), simply equals the ***OLB*** on the loan at that point in time.

Example:

What is the balloon payment owed at the end of an \$80,000, 12%, monthly payment mortgage if the maturity of the loan is 10 years and the amortization rate on the loan is 30 years?

Answer:

$$74734 = 822.89 \frac{1 - 1/(1.01)^{240}}{.01}$$

\$74,734 (sounds familiar?)...

Found either as:

N	I/YR	PV	PMT	FV
240	12	CPT	822.89	0

Or:

N	I/YR	PV	PMT	FV
120	12	80000	822.89	CPT

Interest-only mortgages:

Some mortgages don't amortize at all. That is, they don't pay down their principal balance at all during the regular monthly payments. The OLB remains equal to the initial principal amount borrowed, throughout the life of the loan. This is called an "interest-only" loan.

You can represent interest-only loans in your calculator by setting the *FV* amount equal to the *PV* amount (only with an opposite sign), representing the contractual principal on the loan.

On your calculator . . .

$$0 = -PV + \left(\frac{PMT}{(1+r)} + \frac{PMT}{(1+r)^2} + \dots + \frac{PMT}{(1+r)^N} \right) + \frac{FV}{(1+r)^N}$$

With $FV = -PV$.

Example:

What is the monthly payment on an \$80,000, 12%, 10-year interest-only mortgage?

Answer:

$$\$800 = \$80000 (0.12/12) = \$80000 (0.01) = \$800.00.$$

Or compute it on your calculator as:

N	I/YR	PV	PMT	FV
120	12	80000	CPT	-80000

8.2.12 How the Present Value Keys on a Business Calculator Work

There are two types of PV Math keys on a typical business calculator:

1) The “*Mortgage Math*” keys.

They are useful for problems that involve a level annuity and/or at most one additional single-sum amount at the end of the annuity (e.g., the typical mortgage, even if it is not fully-amortizing). These keys solve the following equation:

$$PV = \left(\frac{PMT}{(1+r)} + \frac{PMT}{(1+r)^2} + \dots + \frac{PMT}{(1+r)^N} \right) + \frac{FV}{(1+r)^N}$$

Normally:

PV = Loan initial principal or OLB.

FV = Loan OLB or balloon.

PMT = Regular monthly payment.

$r = i/m$ = Periodic simple interest (per month).

N = Number of payment periods (remaining, or already paid, depending on how you're using the keys).

Note that this equation is equivalent to:

$$0 = -PV + \left(\frac{PMT}{(1+r)} + \frac{PMT}{(1+r)^2} + \dots + \frac{PMT}{(1+r)^N} \right) + \frac{FV}{(1+r)^N}$$

This makes it clearer why the “*PV*” amount is usually opposite sign to the “*PMT*” and “*FV*” amounts. (If you are solving for *FV* then *FV* may be of opposite sign to *PMT*, and same sign as *PV*.)

The second type of PV Math keys on your calculator are...

2) The “***DCF keys***”. They are useful for problems that are not a simple or single level annuity. The DCF keys can take an ***arbitrary stream*** of future cash flows, and convert them to **present value** or compute the **Internal Rate of Return (“IRR”)** of the cash flow stream (including an up-front negative cash flow representing the initial investment). The DCF keys solve the following equation:

$$NPV = CF_0 + \frac{CF_1}{(1 + IRR)} + \frac{CF_2}{(1 + IRR)^2} + \dots + \frac{CF_T}{(1 + IRR)^T}$$

The DCF Keys

$$NPV = CF_0 + \frac{CF_1}{(1+IRR)} + \frac{CF_2}{(1+IRR)^2} + \dots + \frac{CF_T}{(1+IRR)^T}$$

where the **IRR** is the discount rate that implies **NPV=0**. Typically, CF_0 is the up-front investment and therefore is of opposite sign to most of the subsequent cash flows (CF_1, CF_2, \dots). Each cash flow can be a different amount.

The DCF Keys

$$NPV = CF_0 + \frac{CF_1}{(1+IRR)} + \frac{CF_2}{(1+IRR)^2} + \dots + \frac{CF_T}{(1+IRR)^T}$$

In general, the IRR is the single rate of return (per period) that causes all the cash flows (including the up-front investment) to discount such that $NPV=0$.

The DCF Keys

$$NPV = CF_0 + \frac{CF_1}{(1+IRR)} + \frac{CF_2}{(1+IRR)^2} + \dots + \frac{CF_T}{(1+IRR)^T}$$

By setting $CF_0=0$, and entering the opportunity cost of capital (the discount rate, or required “going-in *IRR*”) as the interest rate (“I/YR” in the HP-10B, or “I” in the “NPV” program of the TI-BA2+), the DCF keys will solve the above equation (with *IRR* equal to the input interest rate) for the present value of the future cash flow stream.

Note: Make sure that the last cash flow amount, CF_T , includes the “reversion” (or asset terminal value) amount.

Example (like Ch.8 Study Qu.#53):

Suppose a certain property is expected to produce net operating cash flows annually as follows, at the end of each of the next five years: \$15,000, \$16,000, \$20,000, \$22,000, and \$17,000. In addition, at the end of the fifth year the property we will assume the property will be (or could be) sold for \$200,000. What is the net present value (NPV) of a deal in which you would pay \$180,000 for the property today assuming the required expected return or discount rate is 11% per year?

Answer:

+\$4,394 (the property is worth \$184,394.)

$$\text{NPV} = \text{PV}(\text{Benefit}) - \text{PV}(\text{Cost}) = \$184,394 - \$180,000 = +\$4,394.$$

$$4394 = -180000 + \left(\frac{15000}{1+0.11} \right) + \left(\frac{16000}{(1+0.11)^2} \right) + \left(\frac{20000}{(1+0.11)^3} \right) + \left(\frac{22000}{(1+0.11)^4} \right) + \left(\frac{(17000+200000)}{(1+0.11)^5} \right)$$

On the calculator this is solved using the DCF keys as:

HP-10B	TI-BAII+
CLEAR ALL	2 nd P/Y = 1 ENTER 2 nd QUIT
1 P/YR	CF
11 I/YR	2 nd CLR_Work
180000 +/- CFj	CF0 = 180000 +/- ENTER ↓
15000 CFj	C01 = 15000 ENTER ↓
16000 CFj	F01 = 1 ↓
20000 CFj	C02 = 16000 ENTER ↓
22000 CFj	F02 = 1 ↓
17000+200000=217000 CFj	C03 = 20000 ENTER ↓
NPV gives 4394	F03 = 1 ↓
	C04 = 22000 ENTER ↓
	F04 = 1 ↓
	C05 = 17000+200000=217000 ENTER
	F05 = 1
	NPV I= 11 ENTER ↓
	NPV CPT = 4394

Example (cont.):

If you could get the previous property for only \$170,000, what would be the expected IRR of your investment?

Answer:

$$0 = -170000 + \left(\frac{15000}{1+0.1315} \right) + \left(\frac{16000}{(1+0.1315)^2} \right) + \left(\frac{20000}{(1+0.1315)^3} \right) + \left(\frac{22000}{(1+0.1315)^4} \right) + \left(\frac{(17000+200000)}{(1+0.1315)^5} \right)$$

13.15%:

HP-10B	TI-BAI+
CLEAR ALL	2 nd P/Y = 1 ENTER 2 nd QUIT
1 P/YR	CF
170000 +/-	2 nd CLR_Work (if you want)
15000 CFj	CF0 = 170000 +/- ENTER ↓
16000 CFj	C01 = 15000 ENTER ↓
20000 CFj	F01 = 1 ↓
22000 CFj	C02 = 16000 ENTER ↓
17000+200000=217000 CFj	F02 = 1 ↓
IRR gives 13.15	C03 = 20000 ENTER ↓
	F03 = 1 ↓
	C04 = 22000 ENTER ↓
	F04 = 1 ↓
	C05 = 17000+200000=217000
	ENTER F05 = 1
	IRR CPT = 13.15

Example of Geometric Series long-term leased space PV problem (Ch.8 Study Qu.#55):

A simple model of an arch-typical commercial building space is as a perpetual series of long-term fixed-rent leases. Once signed, the lease CFs are relatively low risk, hence a low “intra-lease” discount rate is appropriate. But prior to lease signings, the future rent is more risky, hence, a higher “inter-lease” discount rate is appropriate. Rent may be expected to increase between leases, but not within...

Example (cont.):

Suppose expected first lease rent is \$20/SF/yr net, 100,000 SF building, first lease will be signed in one year with rent paid in advance annually. Leases will be 5 years, fixed rent, with expected rent growth between leases 2%/yr, no vacancy in between leases. Suppose intra-lease (low risk) discount rate is 8%/yr, inter-lease (high risk) discount rate is 12%. What is the PV of this space?

Answer:

The rent in the initial lease will be $20 \times 100000 = \$2,000,000$ per year. The first lease when it is signed will have a PV of \$8,624,254. This is a level annuity in advance:

$$2(1.08/0.08)(1 - (1/1.08)^5) = 8.624254.$$

The PV today of that first lease is $8624254 / 1.12 = \$7,700,227$.

The rent on each subsequent lease will be 1.02^5 higher than the rent on the previous lease, but it's PV will be discounted by 5 more years at 12%. This is a constant-growth perpetuity with a common ratio of: $(1.02/1.12)^5$.

Thus, apply the perpetuity formula of the geometric series (GM p.162):

$$PV = a/(1-d) = 7700227 / (1 - (1.02/1.12)^5) = 20615582.$$

The space is worth \$20,615,582 today.