Delay Gains from Network Coding in Wireless Networks

by

Ebad Ahmed

S.B. Electrical Engineering and Computer Science, MIT (2006)

Submitted to the Department of Electrical Engineering and Computer Science

in partial fulfillment of the requirements for the degree of

Master of Engineering in Electrical Engineering and Computer Science

at the

MASSACHUSETTS INSTITUTE OF TECHNOLOGY

June 2007

Author ....

Department of Electrical Engineering and Computer Science

May 25, 2007

Certified by ....

Muriel Médard

Professor of Electrical Engineering

Associate Director, Laboratory for Information and Decision Systems

Thesis Supervisor

Certified by ....

Asuman Ozdaglar

Assistant Professor

Laboratory for Information and Decision Systems

Thesis Supervisor

Certified by ....

Atilla Eryilmaz

Post-doctoral Associate

Laboratory for Information and Decision Systems

Thesis Co-Supervisor

Accepted by ....

Arthur C. Smith

Chairman, Department Committee on Graduate Students

Delay Gains from Network Coding in Wireless Networks

by

Ebad Ahmed

Submitted to the Department of Electrical Engineering and Computer Science
on May 25, 2007, in partial fulfillment of the
requirements for the degree of
Master of Engineering in Electrical Engineering and Computer Science

Abstract

We consider a single-hop cellular wireless system with a single source (base station) broadcasting a stream of incoming files to multiple receivers over stochastic time-varying channels with non-zero erasure probabilities. The base station charges a price per receiver per file with the aim of maximizing its profit. Customers who wish to transmit files to the receivers decide to enter the system based on the price, the queuing delay, and the utility derived from the transaction. We look at network coding and scheduling as possible strategies for file transmission, and obtain approximate characterizations of the optimal customer admission rate, optimal price and the optimal base-station profit as functions of the first and second moments of the service time processes under mild assumptions. We show that network coding leads to significant gains in the base station profits as compared to scheduling, and also demonstrate that the optimal network coding window size is highly insensitive to the number of receivers, which suggests that pricing and coding decisions can be decoupled. We also investigate the behavior of network coding in the case where the number of receivers is sufficiently large, and derive scaling laws for the asymptotic gains from network coding. We subsequently propose a way to extend our analysis of single-source, multiple-receiver systems to multiple-source, multiple-receiver systems in general network topologies and obtain explicit characterizations of the file download completion time under network coding and scheduling, also taking into account the effects of collisions and interference among concurrent packet transmissions by two or more sources. Our formulation allows us to model multi-hop networks as a series of single-hop multiple-source, multiple-receiver systems, which provides a great deal of insight into the workings of larger and denser multi-hop networks such as overlay networks and peer-to-peer systems, and appears to be a promising application of network coding in such networks in the future.

Thesis Supervisor: Muriel Médard
Title: Professor of Electrical Engineering
Associate Director, Laboratory for Information and Decision Systems

Thesis Supervisor: Asuman Ozdaglar
Title: Assistant Professor
Laboratory for Information and Decision Systems

Thesis Co-Supervisor: Atilla Eryilmaz
Title: Post-doctoral Associate
Laboratory for Information and Decision Systems

Acknowledgments

I owe my deepest gratitude to my advisers Professor Muriel Médard and Professor Asuman Ozdaglar, who have been a wonderful source of support and guidance throughout my graduate education. Their contribution to my professional and personal growth has been immense, and their invaluable insights have been indispensable in my development as a researcher. They have not only taught me a great deal about communication networks but have also given me invaluable advice on my career beyond MIT, and I consider myself extremely lucky to have been groomed under their supervision. I have been a member of Professor Médard's group since my sophomore year, and have been greatly inspired by the breadth of her knowledge and her dedication to her students.

I owe a profound debt to Atilla Eryilmaz, without whose help this thesis would not be been possible. I have not only learned vastly from his technical insights and advice on my research work and my thesis but also from our general discussions on various aspects of graduate life. I am also very grateful to Atilla for introducing me to LATEX, which eventually proved vital during the writing of this thesis.

My thanks go to my former group members Desmond Lun and Fang Zhao, who were instrumental in my progress when I first joined our research group as a UROP student. As graduate student supervisors, Desmond and Fang saw to it that I kept moving forward with my research, and made working on my UROP project a pleasure. I am also indebted to my colleagues MinJi Kim, Xiaomeng Shi, Vishal Doshi, Daniel Lucani and Ali Parandehgheibi. It has been a delight to work with them, and our experiences as members of the organizing committee of the LIDS Student Conference are among my most cherished.

I also owe much gratitude to my 6.041 instructor and former housemaster Jinane Abounadi who introduced me to this area of research and the ongoing work in LIDS.

There are many others to thank, and I wish to acknowledge a few individuals for their help and support. A very special word of thanks to Zeeshan Syed and Murtaza Zafer. They were always available for advice when I needed it, and even went out of their way to help me at times. As a LIDS alumnus, Murtaza also offered me insightful ideas on the workings of wireless networks, from which I have benefited greatly. I have known these amazing individuals since my very first year at MIT, and have come to truly value their friendship and support.

I thank Faisal Kashif, Rehan Tahir, Adnan Dosani, Zaid Samar and Omair Malik for all their help and support. I would like to thank Omair in particular, since we have come to know each other very well during our undergraduate and graduate education at MIT and I have thoroughly enjoyed working with him on the numerous undergraduate classes we have taken together.

I thank Daanish Maqbool and Saad Zaheer for being truly wonderful friends. Daanish also happens to be one of the few people who surpass me in their passion for ice-cream, and our late-night ice-cream sessions during the writing of this thesis went a long way into making the writing process stress-free and enjoyable.

I thank Saif Khan for his remarkable friendship and unwavering support throughout my undergraduate years at MIT. I also owe Saif for redounding to my linguistic skills by teaching me Hindi and listening patiently and avidly to my incessant ramblings about Sanskrit and Latin. I thank Siddhartha Jain for all his help, support and advice. The fervor with which he is willing to help others never ceases to astound me. I thank Hasan Ansari for his support and good wishes. I have known him since high school, and his presence and cheerful disposition made MIT a truly memorable experience. I am also grateful to Wajahat Khan, Yaser Khan and Asif Khan for their encouragement.

Finally, I would like to express my deepest gratitude to my family and my parents, without whom this would not have been possible. They have made many sacrifices and have invested a great deal to let me accomplish all that I have been able to. I am eternally indebted to them for all that they have done for me.

This work was supported in part through the Control-Based Mobile Ad-Hoc Networking Program (CBMANET) sponsored by the BAE Systems National Security Solution, Inc. under Subcontract 060786

Contents

List of Figures

Chapter 1

Introduction

The advent of wireless networks has revolutionized the whole domain of packet networks. The arrival of third-generation cellular systems has resulted in an increased demand for wireless networks for the purpose of data transmission, and has also expedited the growth in the capacity of wireless networks. Wireless networks are now being employed for a vast variety of purposes including data transmission in multicast settings such as video-conferencing and file transfer.

Traditional approaches such as routing from hop to hop, which work reasonably well in wireline networks, fail to exploit the full capacity of wireless networks: first, because wireless links are considerably more unreliable than wireline links, which results in a higher likelihood of packet erasure; and second, because traditional routing is incapable of utilizing the wireless broadcast phenomenon, i.e., in a wireless network, transmission to all nodes within a given radius can be achieved without incurring additional cost. Moreover, in tree-based multicast settings, hop-by-hop routing translates into solving the directed Steiner tree problem, which is known to be NP-complete [5, 41].

In light of the inefficacy of the hop-by-hop routing approach, it has been recognized that broadcasting to multiple destinations may be accomplished much more efficiently if *network coding* is used ([2, 27, 29]). Network coding is a recent idea in the domain of communication networks; however, ever since network coding was first proposed by Ahlswede et al. [2], the growth in the range of its applications to networking has been phenomenal. In the next section, we describe the fundamental idea of network coding and discuss the research work that has subsequently been done on network coding. We outline the body of the thesis and state our main contribution in Section 1.2.

1.1 Network Coding: Background

In traditional routing, each node is constrained to only relaying and replicating incoming packets. The idea behind network coding is to relax the constraints on nodes and allow them to perform a much broader range of functions including algebraic operations on incoming packets. This notion of performing coding on incoming packets so that outgoing packets are arbitrary, causal functions of incoming packets was proposed by Ahlswede et al. [2]. Ahlswede et al. were one of the first to consider the problem of multicast. Previous work had predominantly focused on unicast settings in which packets were transmitted from multiple sources to a single node [30]. The work of Ahlswede et al. was met with considerable interest in the networking community, and other work quickly followed. In particular, Li et al. [29] and Koetter and Médard [27] showed that capacity could be achieved in the multicast setting using simple linear codes. Further capacity-achieving codes were provided by Jaggi et al. [22], Ho et al. [20], and Fragouli and Soljanin [12]. As the benefits of network coding were begun to be realized, research on network coding grew at a rapid pace. The utility of network coding was investigated in a wide variety of applications: network management [19], overlay networks [14, 23, 44], wireless networks (e.g. [15, 39, 24, 25, 42, 43]), capacity of random networks [36], network security against Byzantine adversaries [6, 7, 18, 21, 11] and multiple multicast [24, 34, 28, 37, 38, 9].

More recently, it was shown by Lun et al. [31] that in addition to providing an immediate increase in the capacity of the network, network coding allows efficient computation of a minimum-cost subgraph for a single multicast session given a fixed (i.e., inelastic) rate demand. Lun et al. proposed decentralized algorithms for minimum-cost subgraph construction in coded packet networks. Previously, finding a minimum-cost multicast tree without network coding was done using approximation algorithms for solving the directed Steiner tree problem, which were suboptimal and required full knowledge of the network topology. The coding-based decentralized approach proposed in [31] is not only optimal but is also fully decentralized and does not assume full network knowledge. This approach has also been extended to solve the problem of minimum-energy multicast in wireless networks [32]. More recent work by Sundarajan et al. [40] has characterized throughput gains from network coding in crossbar switches serving multicast traffic flow.

Most of the work described above has focused on capacity, throughput, and energy gains from network coding, and has demonstrated that network coding does indeed provide significant advantages in these areas. However, one aspect of network coding that is still not well understood is its delay performance. The single-most important reason for this apparent difficulty in analyzing the delay performance of network coding is the way network coding operates: packets that are encoded have to be decoded, and this must be carried out in 'bulk', i.e., a certain number of packets must be successfully transferred before decoding can be done. In such a scenario, the concept of a 'rate of packet flow' (i.e., the number of bits transferred per unit time) is rendered meaningless. This phenomenon proves to be the inherent difficulty in any analysis that attempts to compare the delay performance of network coding with traditional rate-based schemes such as routing and scheduling.

One of the first steps taken in this direction - to study the delay performance of network coding - was by Eryilmaz et al. [10]. They proposed a model to analyze and compare expected delays in file downloads using various transmission schemes including network coding and scheduling. In order to overcome the analytical difficulty posed by the bulk encoding and decoding of packets in network coding, they used a *rateless* transmission scenario to assess the delay performance of network coding. In other words, the delay performance of network coding is analyzed after all the data has been successfully transmitted. The logical metric to quantify delay in this case is the completion time of the file download. The work of Eryilmaz et al. is distinct in that it is the first to explicitly model and quantify delay gains from network coding.

They consider the cellular downlink scenario with time-varying stochastic channels. This is a standard model for wireless networks, and often serves as the underlying topology of wireless systems used for diverse applications such as video-conferencing and satellite communication. Eryilmaz et al. investigate the delay performance of various file transmission schemes under a variety of conditions, including the presence/lack of channel side information (CSI), and broadcast and multiple unicast. Among the various coding and scheduling strategies, they use random linear coding, which is the optimal coding strategy for network coding (as shown by Li et al. [29]), and round robin, which is the optimal scheduling strategy in this case.

The work of Eryilmaz et al. demonstrates that there are significant delay gains to be had from network coding in wireless systems as compared to traditional scheduling methods. The two scenarios which are studied are single file broadcast (i.e., one file was to be broadcast to all the end nodes), and multiple unicast (i.e., different files were to be sent to different end nodes). In the absence of CSI, network coding performs significantly better than scheduling in both scenarios. In the presence of CSI, however, network coding yields gains in the broadcast case, but performs worse than scheduling in the case of multiple unicast. In practice, nevertheless, the assumption that channel side information is available is unrealistic at best, and therefore, the results of this work illustrate that network coding does indeed seem to be a promising approach in real-world delay-constrained systems with no CSI.

1.2 Thesis Goals and Outline

Our goal in this thesis is to further our understanding of the delay performance of network coding in wireless systems as compared to traditional transmission schemes such as scheduling. We start with a simple extension of the cellular downlink scenario to a dynamic setting - a single base station broadcasting a continuous stream of incoming files (users) to a number of receivers - and rigorously analyze the various aspects of the dynamic model in order to gain a better insight into the delay characteristics of network coding. The main contribution of this thesis is to show that

- delay gains from network coding can be translated into economic gains in a dynamic setting,
- network coding can provide significant delay gains in the asymptotic case where a sufficiently large number of receivers exist in the cellular downlink,
- the single-hop cellular downlink model can be readily extended to a layered multi-hop topology which can be used to model general multicast settings, and
- significant delay gains from network can be realized in the multi-hop topology.

In Chapter 2, we introduce economic motives (such as profit maximization on the base station's part and delay minimization on the users' part) into the model and quantify the economic gains from network coding in the resulting dynamic setting. The economic aspects of queueing systems have been studied in other works [1, 33, 35]. None of these works, however, considers the broadcast scenario with the possibility of network coding. Moreover, we are interested in the effect of practical parameters such as the coding window and the number of receivers will have on the performance of the system.

In Chapter 3, we consider the case in which the number of receivers is sufficiently large, and analyze the delay performance of network coding in this asymptotic case. The assumption of a large number of receivers is realistic in that many real-world networks spanning a large area often involve packet transmission to a large number of nodes. We first derive scaling laws for a large number of receiver, a fixed coding window, and elastic traffic. Subsequently, we look at the case in which the user traffic is inelastic, and compare the number of users that can be supported by the system under network coding and round robin scheduling.

In Chapter 4, we delineate a way to extend the single base station cellular downlink case to one in which there are multiple base stations. Furthermore, we consider a scheme whereby a general multi-hop network topology can be arranged as a chain of single-hop cellular downlink networks, which can then by analyzed using the results for the single-hop cellular downlink scenario.

In Chapter 5, we provide a summary of our work and discuss future research directions.

Chapter 2

Economic Gains from Network Coding

2.1 The Cellular Downlink Scenario

We consider the downlink of a base station broadcasting a sequence of incoming files (users) to N receivers over time varying channels. This model has been presented in [10]. We extend the model in the following way: upon arrival to the system, each user decides whether it will enter the queue for service based on its valuation of the service and the price charged by the service provider. The files that have entered the queue are served in a First-In-First-Out (FIFO) fashion. Thus, the transmission of the next file starts after the current file has been received by every receiver.

Each file is assumed to be composed of K *packets*, where Packet-k of a given file is referred to as $\mathbf{P}_k$, which is a vector of length m over a finite field $\mathbb{F}_q$, for some $q \in \mathcal{Z}_+$. Transmissions take place in regularly arranged time slots with each slot long enough to accommodate a single packet transmission. The channel between the base station and each receiver has a time varying nature to capture the influence of changing channel conditions, possible interference effects and the mobility of the receivers. Specifically, we assume that the channel condition in slot t between the base station and the n^{th} receiver is captured by a Bernoulli distributed random variable $C_n[t]$ with mean c_n that is independent across users and time slots. When $C_n[t] = 1$, the channel is assumed to be ON and the transmission of the base station is successfully received by the n^{th} receiver. If, on the other hand, $C_n[t] = 0$, the transmitted packet does not reach receiver n. We will refer to c_n as the mean channel rate for channel n.

2.1.1 The Scheduling Mode

Let $\mathbf{P}[t]$ denote the packet chosen for transmission in slot t. If the base station is not allowed to code, then at any given slot it must transmit a single packet from one of the files. Thus, we have $\mathbf{P}[t] \in \{\mathbf{P}_k\}_{\{k=1,\cdots,K\}}$. This is the typical mode of transmission considered in literature. We will refer to this mode as the *Scheduling Mode* (or simply *Scheduling*).

2.1.2 The Coding Mode

If coding is allowed, then in one time- slot, say t, any linear combination of the packets can be transmitted. Specifically, we have

$$\mathbf{P}[t] = \sum_{k=1}^{K} a_k[t]\mathbf{P}_k,$$

where $a_k[t] \in \mathbb{F}_q$ for each $k \in \{1, \cdots, K\}$. The transmitter chooses the coefficients $\{a_k[t]\}$ at every time slot t. This mode of transmission will henceforth be referred to as the *Coding Mode* (or simply *Coding*).

2.1.3 Channel Side Information

The strategy employed by the base station to broadcast the head-of-the-line file to the receivers has a critical effect on the service time distribution of the base station. In [10], an extensive analysis of the delay performance of such a file download is provided under Network Coding/Scheduling, and the presence/lack of Channel-Side-Information (CSI). The authors propose a randomized coding strategy, where at every time slot $a_k[t]$ is chosen uniformly at random from $\mathbb{F}_q \backslash \{0\}$. It is shown that such a policy is delay optimal both in the presence and lack of CSI. Unless N is very small, the assumption of the availability of CSI is impractical because of the requirements of frequent feedback and training signals. Therefore we focus on the realistic scenario where no CSI is available at the base station, and feedback is sent only when a receiver gets the whole file. Such a system is not only simpler to implement, but also dissipates less energy and bandwidth resources.

2.2 System Dynamics

Throughout, we will use the terms *file* and *user* interchangeably. We assume that users arrive according to a Poisson process of rate $\gamma > 0$ to be broadcast to all the receivers. The base station charges each user a price p per receiver for the file transfer. Thus, broadcasting to N receivers costs a total amount Np to the user. Each user has the option of either accepting or refusing the services provided by the base station. This model is depicted in Figure 2-1. The decision is based on the utility derived by the user on accepting the service, the delay it will experience before the completion of the download, and the price it will pay to the base station. In particular, we assume that each user will derive a utility $U^{(N,K)}$ from transferring a single file of size K to the N receivers, where $U^{(N,K)}$ is a random variable with distribution function $F_{U^{(N,K)}}(\cdot)$.

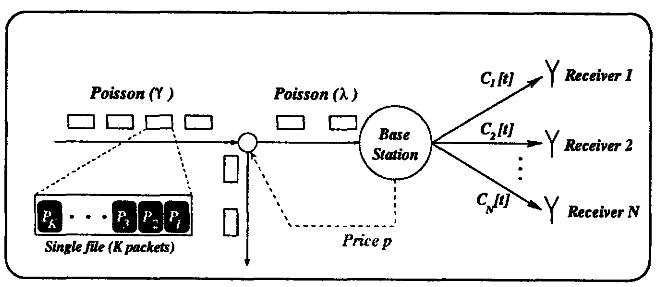

Figure 2-1: System model.

2.2.1 Expected Queueing Delay

The system can be effectively modeled as an $M/G/1$ queue, and each user will experience a delay $D(\gamma, p, N)$ depending on the transmission strategy used by the base-station (network coding or scheduling) and the number of users waiting in the queue (dictated by the arrival rate γ). The expression for the expected delay is given by the celebrated *Pollaczek-Khinchin* formula:

$$\mathbb{E}[Delay] = \frac{\lambda \mathbb{E}[Z^2]}{2(1 - \lambda \mathbb{E}[Z])}, \tag{2.1}$$

where Z is the service time of a single file broadcast. The distribution of Z will depend on the transmission strategy employed at the base station.

2.2.2 Utility Theory

The utility theory from microeconomics dictates that a user will decide to enter if and only if its net utility from the file transfer is non-negative. More specifically, a user will enter the system if and only if

$$U^{(N,K)} - Np - q\mathbb{E}[D(\gamma, p, N)] \geq 0, \tag{2.2}$$

where $\mathbb{E}[D(\gamma, p, N)]$ is the expected delay experienced by the user, and $q > 0$ is a constant, which we introduce to change the units of delay from time units to monetary units.

This implies that the *effective input rate* λ is given by

$$\begin{aligned} \lambda &= \gamma \mathbb{P}\left(U^{(N,K)} - Np - q\mathbb{E}[D(\gamma, p, N)] \geq 0\right) && (2.3) \\ &= \gamma \int_{U^{(N,K)} \geq Np + q\mathbb{E}[D]} f_{U^{(N,K)}}(u)\, du, && (2.4) \end{aligned}$$

where $f_{U^{(N,K)}}(\cdot)$ gives the probability density function of $U^{(N,K)}$. We drop the dependence of D on γ, p and N for ease of exposition. Throughout this paper, we adopt the following assumption on the utility $U^{(N,K)}$.

Assumption 1. *$U^{(N,K)}$ is uniformly distributed over the interval $[0, Nb(K)]$, where $b(K)$ is a non-decreasing concave function of the file size K.*

The concave dependence of the upper support of the utility value on the file size K suggests that the utility derived from file transfer has diminishing returns. This is a standard assumption in the literature and leads to a tractable analysis.

Under Assumption 1, the relation in (2.4) simplifies to

$$\frac{1}{Nb(K)}\left(Nb(K) - Np - q\mathbb{E}[D]\right) = \frac{\lambda}{\gamma}. \tag{2.5}$$

The revenue π generated by the base station per unit time is the amount each user pays, times the rate at which users enter the system, i.e., $\pi = Np\lambda$. Therefore, the base station's profit maximization problem can be written as

$$\begin{aligned} \max_{\lambda\geq 0, p\geq 0} \quad & Np\lambda, && (2.6)\\ \text{subject to} \quad & \frac{1}{Nb(K)}(Nb(K) - Np - q\mathbb{E}[D]) = \frac{\lambda}{\gamma}, && (2.7)\\ & \lambda \leq \frac{1}{X_1}. && (2.8) \end{aligned}$$

The constraint $\lambda \leq 1/X_1$ is necessary in order for the expected delay to be nonnegative [cf. (3.10)]. The model we have outlined corresponds to a dynamic game with the following timing of events:

- The base-station sets an entry price p.

- Incoming users decide whether or not to accept the services of the base-station given p.

Characterizing the optimal price p_{opt} and the optimal file size K_{opt} from the perspective of the base-station corresponds to finding the subgame perfect equilibrium of this dynamic two-stage game. Here, every p defines a different subgame. The subgame perfect equilibrium of this game is given by the optimal solution of problem (2.6) and the corresponding input effective rate λ [cf. (2.4)]. The above game can also be viewed as a Stackelberg game [4], with the base-station as the leader and potential users as the followers.

2.3 Revenue Maximization

We next characterize the optimal solution to problem (2.6). It can be seen that the objective function is continuous and the constraint set is compact, and therefore there exists an optimal solution to (2.6) denoted by λ_{opt}. Also note that in order to have a finite $\mathbb{E}[D]$ in (2.7), λ should satisfy $\lambda < 1/X_1$ and thus $\lambda_{opt} < 1/X_1$. This implies that the optimal Lagrange multiplier associated with (2.8) must be zero due to the slackness constraint. Therefore, it is omitted from the subsequent discussion. In order to find λ_{opt}, we first construct the Lagrangian function $\mathcal{L}(\lambda, p, \mu)$ for problem (2.6), which is given by

$$\begin{aligned}\mathcal{L}(\lambda, p, \mu) &= Np\lambda \\ &\quad +\mu\left[1 - \frac{q\lambda X_2}{2Nb(K)(1-\lambda X_1)} - \frac{p}{nb(K)} - \frac{\lambda}{\gamma}\right],\end{aligned}$$

where $b(K)$ defines the utility of incoming users (cf. Section 2), μ is the Lagrange multiplier for constraint (2.7), and X_1 and X_2 are the first and second moments of the service time distribution at the base-station, respectively. Then, the first order optimality conditions for problem (2.6) yield the following relation between p and λ:

$$p = \lambda\left(\frac{qX_2}{2N(1-\lambda X_1)^2} + \frac{b(K)}{\gamma}\right). \tag{2.9}$$

Together with the feasibility constraint [cf. (2.5)], we obtain the following cubic equation in λ:

$$a_1\lambda^3 + a_2\lambda^2 + a_3\lambda + a_4 = 0, \tag{2.10}$$

where

$$\begin{aligned}a_1 &= 4Nb(K)X_1^2, \\ a_2 &= -(8Nb(K)X_1 + 2Nb\gamma X_1^2 + \gamma qX_1X_2), \\ a_3 &= (4Nb(K) + 2\gamma qX_2 + 4Nb(K)\gamma X_1), \\ a_4 &= -2Nb(K)\gamma.\end{aligned}$$

Since there exists an optimal solution to problem (2.6), the optimal admission rate λ_{opt} is a solution to the above equation. Our goal is to understand the dependence of λ_{opt} on the number of receivers N and the file size K. In the next section, we show that, when network coding is used, X_1 and X_2 can be expressed as functions of N and K through the use of extreme value theory.

2.3.1 Extreme Value Theory

In order to better understand the behavior of λ_{opt} that is described by (2.10), we need to characterize X_1 and X_2 as functions of N and K. To that end, we use results from Extreme Value Theory, which is stated next.

Theorem 1 ([8]). *Let $h_1, \ldots, h_N$ be i.i.d. real random variables with a common distribution function $F(h)$ and density $f(h)$ satisfying the following conditions:*

(a) $F(h)$ is twice differentiable for all h.

(b) $f(h)$ is such that

$$\lim_{h \to \infty} \frac{d}{dh} \left[\frac{1 - F(h)}{f(h)} \right] = 0. \tag{2.11}$$

Let l_N be such that $F(l_N) = 1 - \frac{1}{N}$. Then, the random variable given by

$$\max_{1 \leq i \leq N} K f(l_N)(h_i - l_N), \tag{2.12}$$

converges in distribution to a random variable as $N \to \infty$ with cumulative distribution function $exp(-e^{-x})$, and mean 0.5772, which is the Euler-Mascheroni *constant.*

Let Z denote the completion time of a single file broadcast. It was shown in [10] that when network coding is used, Z is the maximum of N Pascal variables. It is difficult to find exact, closed-form expressions for the first and second moments of Z. The Pascal distribution is a discrete-valued distribution and does not have a continuous, invertible density function. Our current formulation, therefore, does not readily lend itself to extreme value theory. However, a Pascal distribution of order K describing the number of experiments until K successes are achieved can be approximated by an Erlang distribution of order K if the probability of success c in every experiment is sufficiently small [?]. In the following, we adopt this approximation and make the following assumption:

Assumption 2. *The mean channel rate $c_n = \frac{\mu_n}{h(N)}$, where $h(N)$ is some monotonically increasing function of N with $\lim_{N\to\infty} h(N) = \infty$, and $\mu_n > 0$ is a constant.*

Assumption 2 implies that as the number of receivers N increases, channel conditions between the base-station and the receiver deteriorate, which in turn implies that the probability of a successful packet transmission, i.e. c_n for channel n, becomes smaller. This scenario is particularly relevant in the case where multiple transmitters are situated in the vicinity of the base station, and it is not possible to disregard the possibility of packet erasure due to interference with another transmission. The transmission probability c_n is proportional to the constant μ_n. Therefore, μ_n can be considered to be a measure of the reliability of the channel. In general, the larger μ_n is, the better the chances of a successful transmission are.

We will concentrate on symmetric channel conditions in this paper in order to avoid technical complications, i.e., we will set $\mu_n = \mu > 0$ for all $n \in \{1, \ldots, N\}$. Under Assumption 2, $c_n \to 0$ as $N \to \infty$, and the service time at the base station converges in distribution to a random variable T, which is the maximum of N Erlang variables. The analysis for the general case follows the same line of argument.

Lemma 1. *The Erlang distribution satisfies the conditions of Theorem 1.*

Proof. The Erlang distribution of order K and rate μ has probability density $f(x)$ and cumulative distribution function $F(x)$ given by

$$\begin{aligned} f(x) &= \frac{\mu^K x^{K-1} e^{-\mu x}}{(K-1)!}, \\ F(x) &= \frac{\gamma(K, \mu x)}{(K-1)!} = 1 - e^{-\mu x} \sum_{i=0}^{K-1} \frac{(\mu x)^i}{i!}, \end{aligned}$$

where $\gamma(\cdot)$ is the *incomplete gamma function.*

It follows that $F(x)$ is twice differentiable for all x. We next show that $f(x)$ and $F(x)$ satisfy (2.11). We have

$$\begin{aligned} \frac{1-F(x)}{f(x)} &= \frac{e^{-\mu x}\sum_{i=0}^{K-1}\frac{(\mu x)^i}{i!}}{\frac{\mu^K x^{K-1}e^{-\mu x}}{(K-1)!}}, \\ &= \frac{(K-1)!}{\mu^K}\sum_{i=0}^{K-1}\frac{\mu^i}{i!}x^{i-K+1}, \end{aligned}$$

which implies that

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{1-F(x)}{f(x)}\right] = \frac{(K-1)!}{\mu^K}\sum_{i=0}^{K-1}\frac{\mu^i}{i!}(i-K+1)x^{i-K}.$$

Since $i < K$ for all i in the summation above, each term in the summation goes to zero as $x \to \infty$. Since there are a finite number of terms in the summation, the whole expression goes to zero as $x \to \infty$. Thus

$$\lim_{x\to\infty}\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{1-F(x)}{f(x)}\right] = 0,$$

showing that (2.11) is satisfied, and completing the proof. □

We use Theorem 1 to characterize the distribution of the completion time of a single file Z. By definition, $Z = \max_{1\le i\le N} Y_i$, where Y_i (the completion time for receiver i) is an Erlang distributed random variable of order K and rate μ, representing the completion time of the file for Receiver-i. If we now use (2.12) and Theorem 1, a simple linear transformation of variables shows that as $x \to \infty$, Z converges in distribution to a limiting random variable with cumulative distribution function

$$\exp(-e^{-(z-l_N)Nf(l_N)}),$$

with first and second moments given by

$$
\begin{aligned}
X_1 \equiv \mathbb{E}[Z] &= l_N + \frac{0.5772}{Nf(l_N)}, \\
X_2 \equiv \mathbb{E}[Z^2] &= (\mathbb{E}[Z])^2 + \frac{1}{6}\frac{\pi^2}{(Nf(l_N))^2}.
\end{aligned}
$$

2.3.2 First and Second Moments of the Service Time

In order to completely characterize the first and second moments of Z, we need an explicit expression for l_N. Theorem 1 defines l_N implicitly as $F(l_N) = 1 - \frac{1}{N}$. Replacing $F(.)$ by the expression for the cumulative distribution function of the order K-Erlang, we obtain

$$
\begin{aligned}
F(l_N) &= 1 - \frac{1}{N}, \\
1 - e^{-\mu l_N} \sum_{i=0}^{K-1} \frac{(\mu l_N)^i}{i!} &= 1 - \frac{1}{N}, \\
e^{-\mu l_N} \sum_{i=0}^{K-1} \frac{(\mu l_N)^i}{i!} &= \frac{1}{N}.
\end{aligned}
$$

We assume that $\mu l_N \gg K$, which is a reasonable assumption since l_N diverges to infinity as N increases. Thus, for a sufficiently large number of receivers, this assumption holds. Then, the last term in the summation would dominate, and we can omit all the other terms and retain the last one. Therefore, the above summation simplifies to:

$$\frac{\mu^K l_N^{K-1} e^{-\mu l_N}}{(K-1)!} = \frac{1}{N}.$$

Even though this expression is still intractable, we can use it to obtain upper and lower bounds on μl_N. If we substitute $\mu l_N = \log N$ in the expression, we obtain

$$\frac{\mu}{(K-1)!}\frac{(\log N)^{(K-1)}}{N},$$

which is $\Omega(1/N)$. Therefore, μl_N can be lower-bounded by $\log N$. Furthermore, if we substitute $\mu l_N = c \log N$, where c is a positive constant greater than 1, we obtain

$$\frac{\mu c^{(K-1)}}{(K-1)!} \frac{(\log N)^{(K-1)}}{N^c},$$

which, for a sufficiently large c, is $O(1/N)$. Therefore, μl_N can be upper-bounded by $c \log N$. In other words, there exists an N_0 such that for all $N > N_0$, $\log N \leq \mu l_N \leq c \log N$. More precisely, l_N behaves as $\Theta(logN)$ as $N \to \infty$. Given the asymptotic behavior of l_N, upper and lower bounds can be obtained on the first and second moments of Z.

If we plot X_1 and X_2 as a function of the file size K, then, for N sufficiently large, we observe that X_1 is almost linear in K for a large range of values of K. In fact, X_1 and X_2 can be approximated as:

$$X_1 = \frac{1.15(K-1) + 2.42log(N)}{\mu}, \qquad X_2 = X_1^2.$$

These approximations are plotted for $N = 50$ and $\mu = 1$ in Figures 2-2 and 2-3 along with the actual X_1 and X_2.

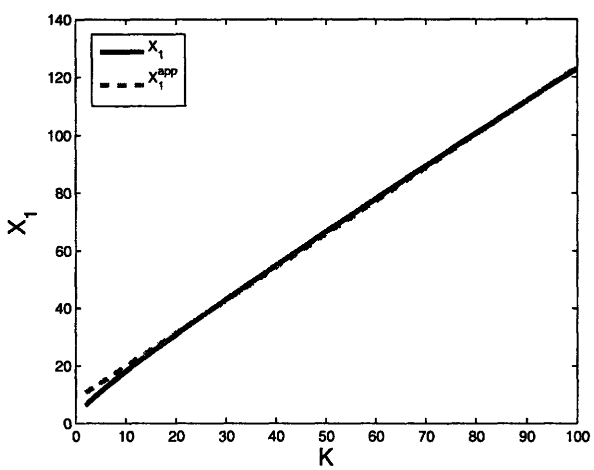

Figure 2-2: Approximation of X_1 as a function of file size K, $N = 50$, $\mu = 1$

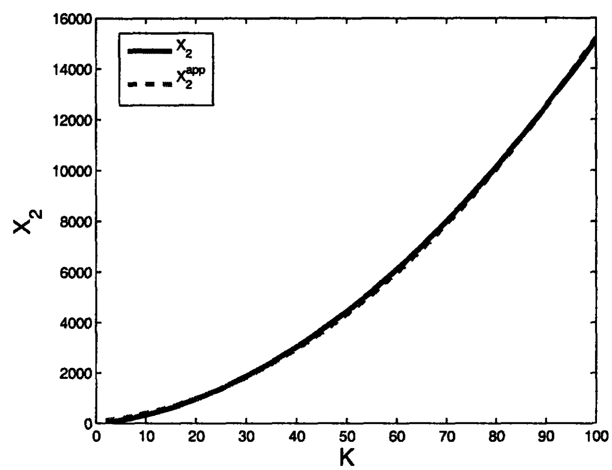

Figure 2-3: Approximation of X_2 as a function of file size K, $N = 50$, $\mu = 1$

2.3.3 Optimal Admission Rate in Low Traffic Regimes

Among the solutions of (2.10), we pick the one that yields the maximum revenue and is a feasible solution of (2.6).

We adopt the following assumption in our analysis:

Assumption 3. *The number of receivers N is sufficiently large, and the arrival rate γ is sufficiently small so that $\gamma q X_2 \ll 2b(K)N$.*

With this assumption, we focus our attention on the low-traffic regime with relatively dense network model. Note that this does not imply low throughput since the number of receivers is high and hence the aggregate throughput will typically be large. Also note that it is worthwhile to investigate the behavior of the optimal admission rate in the low-traffic mode, since in reality downlink systems are designed and deployed to avoid an inordinately large number of users demanding service, and the assumption that the base-station is operating in the low-traffic regime is a realistic one.

An inspection of the coefficients in (2.10) shows that all the terms are multiples of N, except for the terms $\gamma q X_1 X_2$ in a_2 and $2\gamma q X_2$ in a_3. Interestingly, these two terms are also multiples of q. No other term contains q. Consequently, if q is small, then by using Assumption 3, we can modify the terms $\gamma q X_1 X_2$ in a_2 and $2\gamma q X_2$ in a_3 without significantly affecting the values of the coefficients a_2 and a_3. Our goal will be to use the roots of the modified cubic equation to obtain simpler expressions for the roots of the original cubic equation. If we change the term $2\gamma q X_2$ in a_3 to $\gamma q X_2$, without altering any other term, the resulting cubic equation admits the following three roots, which under Assumption 3 are close approximations to the roots of the original cubic equation:

$$
\begin{aligned}
\lambda_1 &= \frac{1}{X_1}, \\
\lambda_2 &= \frac{1}{2X_1} + \frac{\gamma}{4} + \frac{\gamma q X_2}{8b(K)NX_1} + \psi \\
\lambda_3 &= \frac{1}{2X_1} + \frac{\gamma}{4} + \frac{\gamma q X_2}{8b(K)NX_1} - \psi,
\end{aligned}
$$

where

$$\psi \triangleq \frac{\sqrt{(2b(K)N\gamma X_1 - 4b(K)N + \gamma q X_2)^2 + 16b(K)N\gamma q X_2}}{8b(K)NX_1}.$$

Out of these three roots, only $\lambda_3 < 1/X_1$ and is therefore the optimal solution of (2.6). In order to gain more insight into the expression for the optimal admission rate, we rewrite λ_3 as follows:

$$\lambda_3 = \frac{y - \sqrt{y^2 - z}}{8b(K)NX_1},$$

where $y \triangleq 4b(K)N + 2b(K)\gamma NX_1 + \gamma q X_2$, and $z \triangleq 32b(K)^2\gamma N^2 X_1$. The term within the square root can be expressed as

$\sqrt{y^2 - z}$

$$= y\sqrt{1 + \frac{(2+\gamma X_1)}{(2-\gamma X_1)^2}\frac{\gamma q X_2}{b(K)N} + (\frac{\gamma q X_2}{4b(K)N - 2b(K)\gamma NX_1})^2}$$

By Assumption 3, we can neglect the term in $(\frac{\gamma q X_2}{b(K)N})^2$. To simplify the expression for the optimal effective input rate, we further adopt the following assumption:

Assumption 4. *If Assumption 3 holds, then* $\frac{(2+\gamma X_1)}{(2-\gamma X_1)^2}\frac{\gamma q X_2}{b(K)N} \ll 1$.

Proposition 1. *Let Assumptions 3 and 4 hold. Then the optimal effective input rate is given by*

$$\lambda_{opt} = \lambda_3 = \begin{cases} \frac{\gamma}{2} - \frac{(\gamma)^2 q X_2}{4b(K)N(2-\gamma X_1)} & \gamma X_1 < 2 \\ \frac{1}{X_1} - \frac{\gamma q X_2}{2b(K)NX_1(\gamma X_1 - 2)} & \gamma X_1 > 2 \end{cases}$$

Proof. Use a first order Taylor approximation to write

$$\sqrt{1 + \frac{(2+\gamma X_1)}{(2-\gamma X_1)^2}\frac{\gamma q X_2}{b(K)N}} \approx 1 + \frac{1}{2}\frac{2+\gamma X_1}{(2-\gamma X_1)^2}\frac{\gamma q X_2}{b(K)N},$$

Under Assumptions 3 and 4, λ_3 can be written as

$$\frac{y - \sqrt{(4b(K)N - 2b(K)\gamma NX_1)^2}\left(1 + \frac{1}{2}\frac{2+\gamma X_1}{(2-\gamma X_1)^2}\frac{\gamma q X_2}{b(K)N}\right)}{8b(K)NX_1}.$$

We note that the term $(4b(K)N - 2b(K)\gamma N X_1)$ is positive for $\gamma X_1 < 2$, and is negative for $\gamma X_1 > 2$. Therefore, we consider the following two cases, and derive a piecewise expression for λ_3.

CASE 1. $\gamma X_1 < 2$.

In this situation, $(4b(K)N - 2b(K)\gamma N X_1)$ *is positive, and therefore*

$$\begin{aligned}\lambda_3 &= \frac{4b(K)N\gamma X_1 + \gamma q X_2 - \gamma q X_2\left(\frac{2+\gamma X_1}{2-\gamma X_1}\right)}{8b(K)NX_1} \\ &= \frac{\gamma}{2} - \frac{(\gamma)^2 q X_2}{4b(K)N(2-\gamma X_1)}.\end{aligned}$$

CASE 2. $\gamma X_1 > 2$.

In this case, $(4b(K)N - 2b(K)\gamma N X_1)$ *is negative, and we must negate it in order to take the positive square root. Therefore*

$$\begin{aligned}\lambda_3 &= \frac{8b(K)N + \gamma q X_2 - \gamma q X_2\left(\frac{2+\gamma X_1}{\gamma X_1 - 2}\right)}{8b(K)NX_1} \\ &= \frac{1}{X_1} - \frac{\gamma q X_2}{2b(K)NX_1(\gamma X_1 - 2)}.\end{aligned}$$

This completes the proof. □

Note that, λ_{opt} is constant at $\gamma/2$ for small values of K, i.e. for $\gamma X_1 < 2$, and decreases approximately as $1/X_1$ for larger values of K, i.e. $\gamma X_1 > 2$. Since the constraint $\gamma X_1 > 2$ is the same as $\gamma/2 > 1/X_1$, we can write

$$\lambda_{opt} = \begin{cases} \frac{\gamma}{2} - \frac{(\gamma)^2 q X_2}{4b(K)N(2-\gamma X_1)} & \gamma/2 < 1/X_1 \\ \frac{1}{X_1} - \frac{\gamma q X_2}{2b(K)NX_1(\gamma X_1 - 2)} & \gamma/2 > 1/X_1 \end{cases}$$

which can essentially be re-written as

$$\lambda_{opt} = \min\{\gamma/2, 1/X_1\} - f(\gamma, N, X_2),$$

for an appropriately defined $f(\cdot)$.

Figure 2-4 shows the optimal admission rate as a function of the file size K for $N = 50$ and $\mu = 2$.

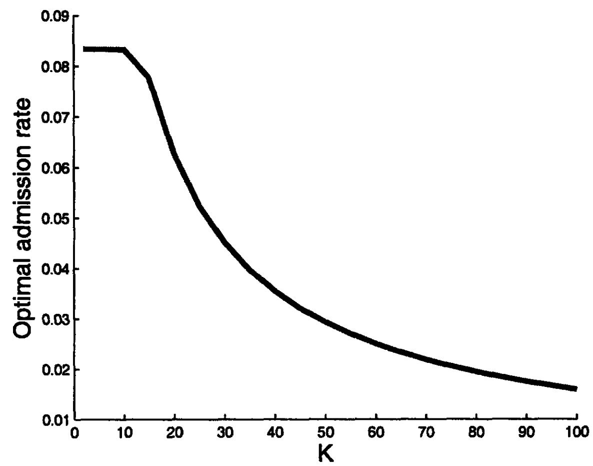

Figure 2-4: Optimal admission rate as a function of file size K, $N = 50$, $\mu = 2$

2.4 Optimal Price and Revenue

In this section, we study the optimal price and revenue behavior of our system with changing system parameters. Figure 2-5 plots the optimal revenue of the base station as a function of the corresponding file size K for varying values of N. We observe two features:

- The optimal revenue is a unimodal function of K, with a single stationary point
- The file size K_{opt} which maximizes the revenue does not change as the number of receivers N is varied.

We next characterize the optimal price and revenue under Assumptions 3 and 4.

Proposition 2. *Let Assumptions 3 and 4 hold. Let p_{opt} be the price that maximizes* *problem (2.6) and π_{opt} be the corresponding optimal profit. Then,*

$$p_{opt} = \begin{cases} \frac{b(K)}{2} + \frac{\gamma q X_2}{4N(1-\frac{\gamma X_1}{2})^2}, & \gamma X_1 < 2 \\ \frac{1}{X_1}\left[\frac{2Nb(K)^2(\gamma X_1 - 2)^2}{(\gamma)^2 q X_2} + \frac{b(K)}{\gamma}\right. & \\ \left. -\frac{b(K)(\gamma X_1 - 2)}{\gamma} - qX_2 2N(\gamma X_1 - 2)\right], & \gamma X_1 > 2 \end{cases}$$

$$\pi_{opt} = \begin{cases} \frac{(\gamma)^2 q X_2}{2(2-\gamma X_1)^2} + \frac{Nb(K)\gamma}{4}, & \gamma X_1 < 2 \\ \frac{1}{X_1^2}\left[\frac{2N^2 b(K)^2(\gamma X_1 - 2)^2}{(\gamma)^2 q X_2} + \frac{Nb(K)}{\gamma}\right], & \gamma X_1 > 2 \end{cases}$$

Proof.

CASE 1. *If $\gamma X_1 < 2$, then under Assumption 3, $\frac{(\gamma)^2 q X_2}{4b(K)N(2-\gamma X_1)}$ is negligible compared to $\gamma/2$ and $\lambda_{opt} \approx \gamma/2$. Therefore*

$$\begin{aligned} p_{opt} &= \lambda_{opt}\left[\frac{qX_2}{2N(1-\frac{\gamma X_1}{2})^2} + \frac{b(K)}{\gamma}\right] \\ &= \frac{b(K)}{2} + \frac{\gamma q X_2}{4N(1-\frac{\gamma X_1}{2})^2}. \\ \pi_{opt} &= N p_{opt}\lambda_{opt} \\ &= (\lambda_{opt})^2\left[\frac{qX_2}{2(1-\frac{\gamma X_1}{2})^2} + N\frac{b(K)}{\gamma}\right] \\ &= \frac{b(K)}{2} + \frac{\gamma q X_2}{4N(1-\frac{\gamma X_1}{2})^2} \\ &= \frac{(\gamma)^2 q X_2}{2(2-\gamma X_1)^2} + \frac{Nb(K)\gamma}{4}. \end{aligned}$$

We notice that when the file size K is small enough so that $\gamma X_1 < 2$, then under Assumption 3, $p_{opt} \approx b(K)/2$, and $\pi_{opt} \approx Nb(K)\gamma/4$.

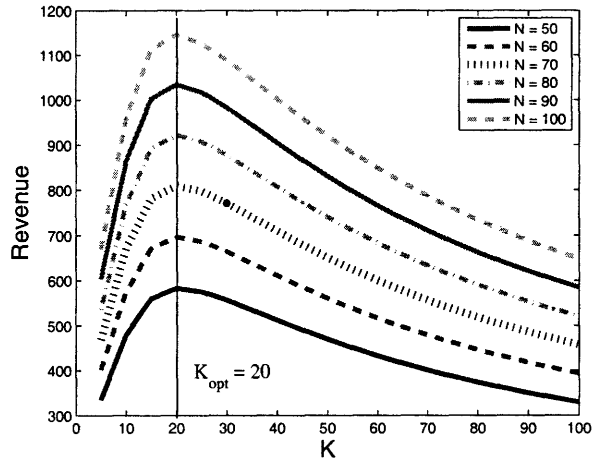

Figure 2-5: Revenue as a function of file size K for various values of N, $\mu = 2$.

CASE 2. *If* $\gamma X_1 > 2$, *then* $\lambda_{opt} = \frac{1}{X_1} - \frac{\gamma q X_2}{2b(K)NX_1(\gamma X_1 - 2)}$. *Therefore,*

$$
\begin{aligned}
p_{opt} &= \lambda_{opt}\left[\frac{qX_2}{2N(1-\frac{\gamma X_1}{2})^2} + \frac{b(K)}{\gamma}\right] \\
&= \left(\frac{1}{X_1} - \frac{\gamma q X_2}{2b(K)NX_1(\gamma X_1 - 2)}\right)\left[\frac{qX_2}{2N\left(\frac{\gamma q X_2}{2b(K)N(\gamma X_1 - 2)}\right)^2} + \frac{b(K)}{\gamma}\right] \\
&= \frac{1}{X_1}\left[\frac{2Nb(K)^2(\gamma X_1 - 2)^2}{(\gamma)^2 q X_2} + \frac{b(K)}{\gamma}\right. \\
&\qquad \left. - \frac{b(K)(\gamma X_1 - 2)}{\gamma} - qX_2 2N(\gamma X_1 - 2)\right]. \\
\pi_{opt} &= (\lambda_{opt})^2\left[\frac{qX_2}{2(1-\frac{\gamma X_1}{2})^2} + N\frac{b(K)}{\gamma}\right] \\
&= \left(\frac{1}{X_1} - \frac{\gamma q X_2}{2b(K)NX_1(\gamma X_1 - 2)}\right)\left[\frac{qX_2}{2\left(\frac{\gamma q X_2}{2b(K)N(\gamma X_1 - 2)}\right)^2} + \frac{Nb(K)}{\gamma}\right] \\
&= \frac{1}{X_1^2}\left(2b(K)N(\gamma X_1 - 2)^2 - \gamma q X_2\right)\left[\frac{1}{2(\gamma)^2 q X_2} + \frac{1}{4\gamma b(K)N(\gamma X_1 - 2)^2}\right] \\
&\approx \frac{1}{X_1^2}\left[\frac{2N^2 b(K)^2(\gamma X_1 - 2)^2}{(\gamma)^2 q X_2} + \frac{Nb(K)}{\gamma}\right].
\end{aligned}
$$

This completes the proof. □

These expressions are only approximate expressions because the revenue and price functions are extremely sensitive to λ. This is due to the presence of the $(1-\lambda X_1)$ term in the denominator and the fact that λ_{opt} is very close to $1/X_1$.

1. *The optimal revenue is a unimodal function of the file size:* We note that for $\gamma X_1 < 2$, the dependence of optimal revenue on K is given by $Nb(K)\gamma/4$. For $\gamma X_1 > 2$, the revenue function contains terms in $b(K)/X_1^2$ and $b(K)^2/X_1^4$. Since $b(K)$ is a concave function in K, and X_1 is linear in K (cf. Section 2.3.1), the revenue function is monotonically nonincreasing. We note further that since the revenue is monotonically increasing in the region $\gamma X_1 < 2$, the optimum file size K_{opt} which maximizes the revenue occurs in the range $\gamma X_1 > 2$.

2. *The file size that maximizes the revenue is insensitive to changes in N:* We next find the file size that maximizes the revenue, denoted by K_{opt} (cf. Figure 2-5). We know that K_{opt} occurs in the range $\gamma X_1 > 2$, so we need only look at the revenue function in that range. We assume $b(K) = c\log(K)$ where c is a constant. Using the approximations for X_1 and X_2 from Section 2.3.1 and taking first derivatives with respect to K, we obtain the following implicit expression in K and N, which characterizes K_{opt} in terms of N.

$$2\gamma^2\left(2X_1^3\frac{log(K)}{k} - 2.3X_1^2 log(K)^2\right) - 4\gamma\left(2X_1^2\frac{log(K)}{K} - 3.45X_1 log(K)^2\right) \\ +8\left(2X_1\frac{log(K)}{K} - 4.6 log(K)^2\right) - \frac{\gamma q}{Nc}\left(\frac{X_1^3}{K} - 2.3X_1^2 log(K)\right) = 0$$

The above equation is a transcendental equation, and does not admit a tractable solution. However, our primary objective is not to solve for K_{opt}, but to understand its relative stability as the number of receivers N changes. Towards this end, we differentiate the above equation implicitly with respect to N in order to obtain an expression for $\partial K_{opt}/\partial N$.

From Figure 2-5, we observe that $K_{opt} \approx 20$. We can calculate the values of $\partial K_{opt}/\partial N$ for $K_{opt} \approx 20$ and different values of N. If we let $N = 50$, the value of $\partial K_{opt}/\partial N$ turns out to be approximately 0.003. In other words, if N changes by 100, K_{opt} would change by only 0.3. In fact, for the range of N that is of interest to us, K_{opt} is constant as shown in Figure 2-5.

In the above analysis, we assumed that $b(K) = a \log K$. However, it can be shown that the same implications hold for other concave functions such as $\sqrt{K}$ and functions of the form $K^{1/p}$ where $p > 1$.

2.5 Sensitivity of System Variables

We next study through simulations the dependence of the optimal admission rate, the optimal price and the expected delay as functions of the file size K. Figures 2-4, 2-5 2-6 and 2-7 show plots of the optimal admission rate, the optimal revenue, the optimal price, and the expected delay respectively as functions of K.

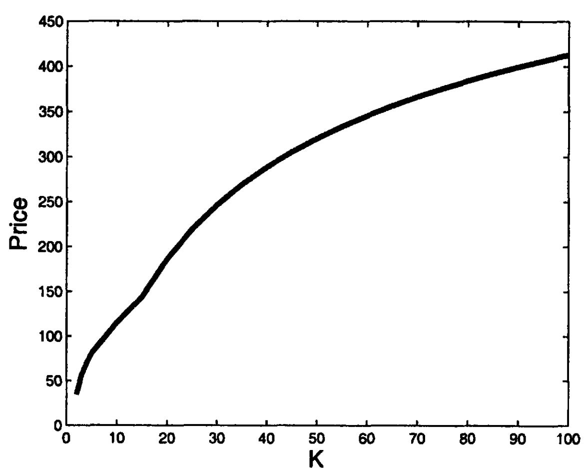

Figure 2-6: Optimal Price as a function of file size K, $N = 50$, $\mu = 2$

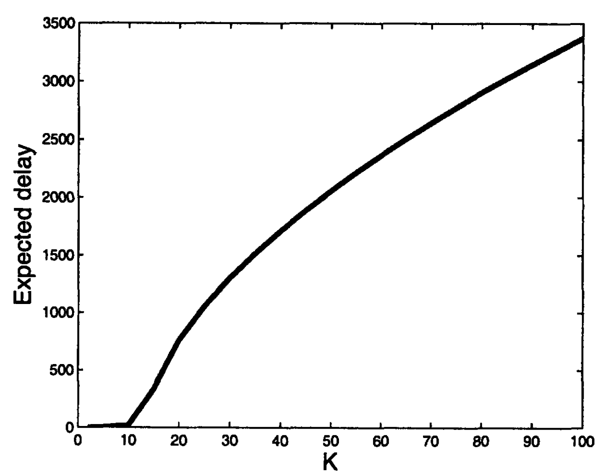

Figure 2-7: Expected queueing delay as a function of file size K, $N = 50$, $\mu = 2$

2.5.1 Optimal Admission Rate

The optimal admission rate offers a great deal of insight into the dynamics of the system. As noted in Section 2.3.3, the optimal admission rate is approximately $\gamma/2$ for small values of K, and approximately $1/X_1$ for larger values of K. Therefore, if the system is operating in the low-traffic regime, and the file size is small, the optimal admission rate is $\gamma/2$. Intuitively, this makes sense because the queueing delay is negligible (Figure 2-7), and the major cost experienced by the users is price. As noted earlier (cf. Section 2.4), price per receiver increases as $b(K)/2$ for small values of K, and given that the utility function is uniform between $[0, Nb(K)]$ (cf. Assumption 1), we would expect half of the arrivals to accept the service, and half to reject it.

As the file size progressively increases, the expected queueing delay also increases. The threshold at which the effects of delay can no longer be ignored is $\frac{\gamma}{2} = \frac{1}{X_1}$. As this threshold is crossed, the effects of the delay become sufficiently appreciable, and the expected service time X_1 increases rapidly. The optimal admission rate is then constrained by the reciprocal of the mean service time. The second moment X_2 plays an important role in this case. It perturbs the optimal admission rate so that it is slightly below $1/X_1$. If X_2 were zero (physically impossible since this would mean that the service time has a negative variance), then λ_{opt} would be exactly $1/X_1$, which would lead to an infinite revenue, infinite price, and an infinite delay (all of which are physically impossible). The effect of X_2 implies that the users' decision is also affected by the variance of the service time. A higher variance will lead to a lower admission rate.

2.5.2 Optimal Revenue

The optimal revenue is proportional to the product of the optimal price and the optimal admission rate. For smaller values of K, the revenue of the base-station increases because the queuing delay is not very significant and the admission rate is constant. The base-station can therefore increase its price with the guarantee that the admission rate will not decrease as long as the file size is small. For larger file sizes, the delay becomes significant and the base-station is no longer able to increase its price. The optimal admission rate begins to decrease as a result of large queuing delays. Consequently, the revenue of the base-station reaches a maximum, and then begins to decrease as K increases. Qualitatively, this describes the unimodal shape of the optimal revenue function depicted in Figure 2-5.

2.5.3 Expected Delay

The expected delay increases rapidly (Figure 2-7), in fact almost linearly, as K increases. This increase in the expected delay is due to the fact that as the file size increases, it takes the base-station progressively longer to transmit the file to the receivers, which in turn increases the waiting time for other users in the queue.

2.5.4 Optimal Price

The optimal price (Figure 2-6) initially increases as K increases, but eventually tapers off in a sub-linear fashion. This observation can be understood if we think of the price and the expected delay as two different costs that users will experience upon entering the system. As the expected delay increases and the optimal admission rate begins to drop, the base-station cannot afford to keep increasing its price, since that would exacerbate the drop in the optimal admission rate. In order to mitigate the effect of the increased delay on the admission rate, the base-station must check its price in order to encourage more users to join the system.

2.5.5 Effect of Channel Conditions

The dynamics of the system are also affected by the channel conditions. The quality of the channel is captured by μ (cf. Assumption 1). From the analysis of Section 2.3.1, we know that X_1 is inversely proportional to μ. Therefore, if μ is larger (i.e. the channel conditions are better), the mean service time will be smaller, and files will be transferred more quickly. Since X_1 will be smaller, the threshold $\gamma/2 = 1/X_1$ will be crossed at a much larger value of K. In other words, the optimal admission rate will remain constant at $\gamma/2$ for larger file sizes as well. The base-station will be able to increase its revenue over a larger range of values of K. The optimum file size K_{opt} will also be larger. This result is shown in Figure 2-8. The optimum file size increases almost linearly with μ. The implication, therefore, is that given better channel conditions, the base station will choose to transfer larger files, and will attain higher profits at the same time.

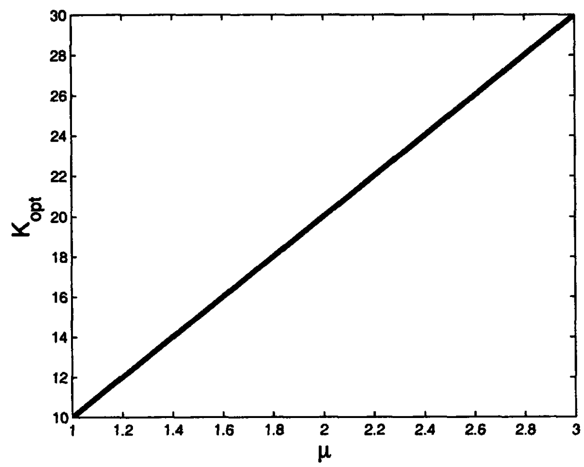

Figure 2-8: K_{opt} as a function of μ, $N = 50$

2.6 Coding vs. Scheduling

In order to quantify the economic gains from network coding, we must compare it with the traditional scheduling approach, which is currently the common mode of packet transmission. A comprehensive analysis of various scheduling policies with and without CSI is given in [10]. The authors find that the optimal scheduling policy for a system with no CSI is the *round robin* approach, in which the base station sends a single packet to every receiver in turn. We denote the first and second moments of the service time distribution in the round robin case by X_1^{RR} and X_2^{RR}, and the first and second moments of the service time in the network coding case by X_1^{NC} and X_2^{NC}.

It is shown in [10] that X_1^{RR} is lower-bounded as

$$X_1^{RR} \geq \frac{K}{2} + K\mathbb{E}[\max_{1 \leq k \leq K, 1 \leq n \leq N} U_n^k],$$

where U_n^k is a geometric random variable with parameter c, representing the number of slots until a given channel is ON. The geometric distribution converges to exponential distribution with rate μ as N gets large. Again, we use extreme value theory to compute the upper bound, and then use the fact that $X_2^{RR} \geq (X_1^{RR})^2$ in order to get a lower bound on X_2^{RR}. Since a lower bound ion both X_1^{RR} and X_2^{RR} improves performance, we use the above bounds to compute a lower bound on the expected delay for round robin. In the following analysis, we use this lower bound to compare the two transmission strategies.

Since the mean service time of round robin scheduler is larger [10], we would expect the threshold $\gamma/2 = 1/X_1^{RR}$ to be crossed at a much smaller file size than that for network coding, and the revenue of the base station would start to decrease for small K. In other words, K_{opt} for round robin will be very small, and thus the revenue earned by the base station will be much lower compared to the revenue from network coding. Figures 2-9 and 2-10 present a comparison of the optimal admission rate and revenue from network coding and round robin, respectively. As expected, the revenue in the round robin case begins to decrease at a much smaller value of K, and the difference between the revenues from network coding and round robin is also very significant. The optimal admission rate for round robin is also much lower than that for network coding, while the expected delay in the case of round robin is considerably larger.

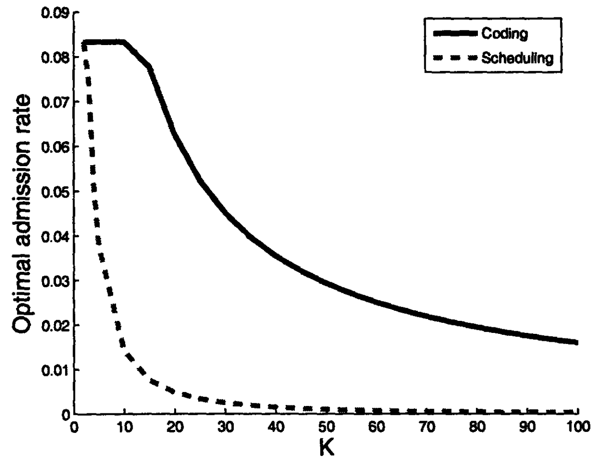

Figure 2-9: Comparison of optimal admission rate with network coding and round robin as a function of file size K, $N = 50$, $\mu = 2$

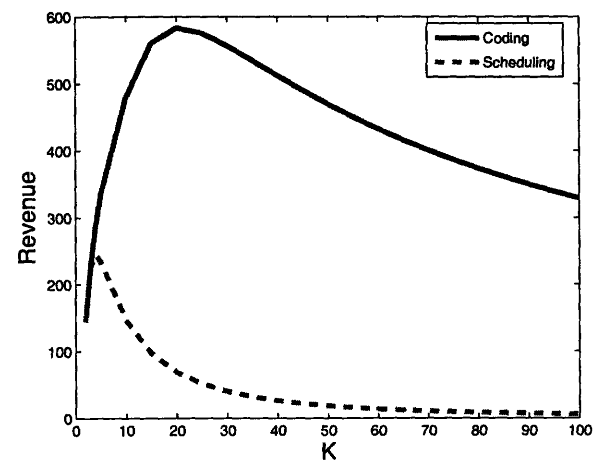

Figure 2-10: Comparison of revenue with network coding and round robin against file size K, $N = 50$, $\mu = 2$

Chapter 3

Asymptotic Delay Gains from Network Coding: Scaling Laws

Our goal in this chapter is to investigate the asymptotic mean delay performance of network coding versus scheduling as transmission strategies in cellular downlink wireless networks. Such an analysis is first provided in [10] in the case of single hop wireless networks with erasure channels. In the previous chapter, we had extended this model and had studied the economic gains from network coding in a dynamic setting. For the purpose of the asymptotic scenario which will be investigated in this chapter, we revert to the original cellular downlink model in [10]. The utility of this model is that it has the potential to serve as the fundamental building block of larger networks. For instance, its extension to tree-structured networks has recently been considered in [13].

In this model, the strategy employed by the base station to broadcast the head-of-the-line file to the receivers has a critical effect on the service time process at the base station. In [10], an extensive analysis of the delay performance of such a file download is provided under Network Coding/Scheduling, and the presence/lack of Channel-Side-Information (CSI). The authors propose a randomized coding strategy, where at every time slot $a_k[t]$ is chosen uniformly at random from a finite field $\mathbb{F}_q \backslash \{0\}$. It has been shown that such a policy is delay optimal both in the presence and lack of CSI. Unless N is very small, the assumption of the availability of CSI is impractical because of the requirements of frequent feedback and training signals. Therefore we focus on the realistic scenario where no CSI is available to the base station, and feedback is sent only when a receiver gets the whole file. Such a system is not only simpler to implement but also dissipates less energy and bandwidth resources. This downlink scenario can also be used to represent a multi-hop network where base station represents the source node, receivers represent end users, and each link represents a path to the corresponding end user. In this case, there appears to be an interesting situation where paths to different users may share links, and therefore the failure events of different links may be correlated.

The economic benefits of network coding versus scheduling in this specific scenario, as detailed in Chapter 2, have previously been studied in detail in [3]. Both [3] and [10] use the file download completion time as a measure of delay, and demonstrate that network coding provides significant delay and economic gains as compared to scheduling in such rateless transmission schemes.

We build upon the analysis in [3] and [10] and evaluate the asymptotic gains of network coding over scheduling as the number of receivers N increases without bound. In a subsequent work, Ghaderi et al. [13] have considered a network with a tree topology. They evaluate the performance of end-to-end and link-to-link error control techniques based on ARQ (Automatic Repeat Request) and FEC (Forward Error Coding) in tree-based multicast systems. While their model involves multi-hop trees, the building block in their analysis is the single-hop scenario that we focus on. Consequently, our arguments extend naturally to tree networks. In Chapter 4, we will describe a way to make this extension in the case of general network topologies.

3.1 File Download Completion Times for Network Coding and Scheduling

Let T_{NC} and T_{RR} denote the file download completion times for the case of network coding and scheduling (round-robin) respectively. It has been shown in [10] that

$$T_{NC} = \max_{1 \leq i \leq N} Y_i, \tag{3.1}$$

where Y_i follows a Pascal distribution of order K and parameter c, K is the file size, and

$$T_{RR} = \max_{1 \leq i \leq N} \max_{1 \leq k \leq K} KW_i^k + k, \tag{3.2}$$

where W_i^k is a geometric random variable with parameter c.

Following the analysis in [10], we obtain

$$\mathbb{E}[T_{NC}] = K + \sum_{t=K}^{\infty} \left[1 - \prod_{i=1}^{N} \left(\sum_{\tau=K}^{t} \begin{pmatrix} \tau - 1 \\ K - 1 \end{pmatrix} (1-c)^{(\tau-K)} c^K \right) \right], \tag{3.3}$$

where $\begin{pmatrix} n \\ m \end{pmatrix}$ gives the number of combinations of size m of n elements, and

$$\frac{\mathbb{E}[T_{RR}]}{K} = \gamma + \sum_{t=1}^{\infty} \left[1 - (1 - (1-c)^t)^{KN} \right], \tag{3.4}$$

for some $\gamma \in (1/2, 1)$.

3.2 Maximum Order Statistics for the Geometric and Pascal Distributions

Although the mean file completion times as given by equations (3.3) and (3.4) are exact, they are not explicit, closed-form functions of N and K, which makes it impossible to acquire a qualitative understanding of the dependence of $\mathbb{E}[T_{NC}]$ and $\mathbb{E}[T_{RR}]$ on N and K from these equations. In this section, we approximate (3.3) and (3.4) by more tractable expressions in order to gain a better understanding of how the mean completion times behave with N and K.

Using the analysis given in [16] for the maximum statistics of Pascal random variables of order K , we obtain

$$\mathbb{E}[T_{NC}] = \log_{\frac{1}{1-c}} N + (K-1)\log\log_{\frac{1}{1-c}} N + (K-1) - \log_{\frac{1}{1-c}} (K-1)! + o(\log N). \quad (3.5)$$

Furthermore, by observing that a geometric random variable is a Pascal random variable of order 1 and using equation (3.5), we obtain the maximum statistics of geometric random variables. Hence, from (3.4), we have

$$\frac{K}{2} + K\log_{\frac{1}{1-c}} KN \leq \mathbb{E}[T_{RR}] \leq K + K\log_{\frac{1}{1-c}} KN. \quad (3.6)$$

3.3 Asymptotic Performance Analysis of Network Coding and Scheduling

In this section, we provide an analysis of the asymptotic performance of network coding and scheduling. There are two cases to consider: elastic traffic and inelastic traffic. Elastic traffic corresponds to the case in which each user derives a fixed amount of utility from the service regardless of the time it takes for the service to be completed. In other words, all users are willing to join the system regardless of how large a queuing delay they will encounter upon accepting service. Inelastic traffic, on the other hand, refers to the case in which users have stringent delay constraints and enter the system only if their delay constraints are guaranteed to be met. The performance of network coding and scheduling for these two cases is analyzed in Sections 3.3.1 and 3.3.2.

3.3.1 Elastic Traffic

Let $G = \frac{\mathbb{E}[T_{NC}]}{\mathbb{E}[T_{RR}]}$. Hence G denotes the ratio of the file completion time of network coding to that of scheduling, and is a measure of the delay gains provided by network coding as compared to scheduling. Since there are no delay constraints, the behavior of G in the limit $N \rightarrow \infty$ will provide us with the asymptotic delay gains from network coding as compared to scheduling.

$$G = \frac{\mathbb{E}[T_{NC}]}{\mathbb{E}[T_{RR}]}.$$

In order to get an upper bound on G (i.e., delay gains from network coding in the worst-case scenario), we use the lower bound on $\mathbb{E}[T_{RR}]$ from (3.6). We therefore have

$$G \leq \frac{\log_{\frac{1}{1-c}} N + (K-1)\log\log_{\frac{1}{1-c}} N + (K-1) - \log_{\frac{1}{1-c}}(K-1)! + o(\log N)}{\frac{K}{2} + K\log_{\frac{1}{1-c}} KN}.$$

If we fix K and consider the limit $N \rightarrow \infty$, the dominant term in the numerator will be $\log N$, and the dominant term in the denominator will be $K \log N$. Therefore, in the limit $N \rightarrow \infty$,

$$G = \frac{\log N}{K \log N} \tag{3.7}$$

$$= \frac{1}{K}. \tag{3.8}$$

Figure 3-1 shows the behavior of G versus $\frac{1}{K}$. As N increases, the curves approach the line $G = 1/K$.

The asymptotic ratio of the the file completion time of network coding to that of scheduling is the reciprocal of K, which signifies that as the number of receivers N increases to a sufficiently large value, file downloads take K times longer to complete if scheduling instead of network coding is used. The value of K, i.e., the file size, can be chosen arbitrarily, and therefore larger asymptotic gains from network coding can

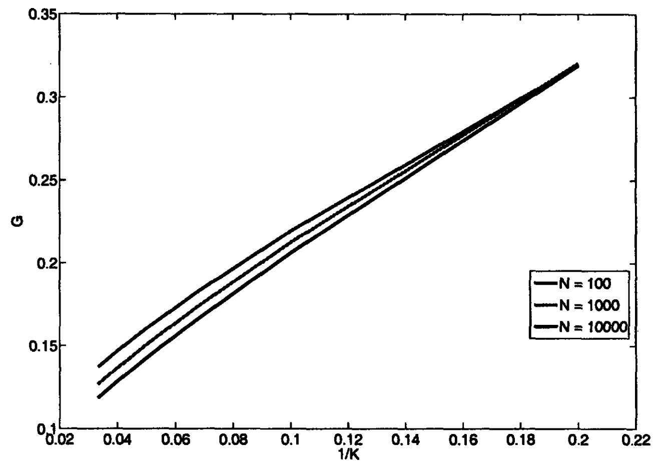

Figure 3-1: Behavior of G versus $\frac{1}{K}$

be realized for larger file sizes. Thus, although the gain is fixed for fixed values of K, it is essentially unbounded since the value of K can be chosen arbitrarily.

3.3.2 Inelastic Traffic

The gain expression in the previous section was obtained under the rateless transmission scenario, i.e., the expected delays for both network coding and scheduling were compared for the case when file transfer to all the receivers had been completed and the number of receivers was fixed. Furthermore, the demand for file downloads was assumed to be elastic, i.e., users were willing to join the system regardless of how large a queueing delay they had to encounter. This is often not the case in practice, and therefore, we must approach scaling laws such as the one above with caution. In the more realistic scenario where the demand for file download is inelastic[1], the above scaling law, $G = \frac{1}{K}$ ceases to hold. In such a scenario, the more relevant questions to consider are:

- Given a fixed queueing delay constraint of d time slots, a fixed file size K, and a fixed user admission rate λ, how many more receivers (i.e., N) can the system support with network coding than with scheduling?

[1]that is, users are willing to join the system as long as the queueing delay is below a certain threshold; if the queueing delay exceeds the threshold value, no user is willing to join the system

- How does N scale with d under the previous setting, i.e., how does relaxing the delay constraint affect the number of users the system can support?

- Given a fixed N and a fixed K, how does λ scale with d, i.e., how does the relaxing the delay constraint affect the user admission rate or the throughput of the system?

- Given a fixed N and a fixed d, how does λ scale with K, i.e., how does changing the file size affect the throughput of the system?

We expect that the system will be able to support a larger number of receivers with network coding than with scheduling. In order to illustrate this point better, let us consider the following scenario. Assume that users [2] arrive according to a Poisson process of rate $\gamma > 0$ (the offered load to the system) to be broadcast to all the N receivers. Each user has the option of either accepting or refusing the services provided by the base station. The decision is based on the utility derived by the user on accepting the service and the delay it will experience before the completion of the download. In particular, assume that each user will derive a utility $U(D)$ from transferring a single file of size K to all the N receivers, given a queueing delay of D. Assume further that demand for the file transfer is perfectly inelastic, i.e., users are only willing to join the system if the queueing delay is below a certain level, say, d_{max}. In order to model this behavior, let $U(D)$ be uniformly distributed over the interval $[0, d_{max}]$. Hence, the utility of each user as a function of the queueing delay D takes the following form:

$$U(D) = \begin{cases} \frac{1}{d_{max}} & \text{if } D < d_{max} \\ 0 & \text{if } D \geq d_{max} \end{cases} \tag{3.9}$$

This form for the utility function signifies that users gain a fixed amount of utility from files that meet the delay constraint, i.e., files that take less than d_{max} time slots to be transferred do not provide additional utility to the user. This is motivated by

[2]or files; we will use the terms *user* and *file* interchangeably

real-time applications where it is often the case that users are not concerned about how long it takes for the file to be transferred as long as the transfer is completed within a certain time. In Section 3.3.3, we analyze the performance of this system and compare the performance of network coding and scheduling under various scaling laws

3.3.3 Analysis

The system can be modeled in general as a $G/G/1$ queue. In order to derive expressions that represent the general case, we focus on the scenario in which the arrival process is Poisson and model the system as an $M/G/1$ queue. For more general arrival processes (such as a deterministic arrival process), various bounds such as Kingman's bound [26] can be employed to characterize the system delay.

We now focus on the $M/G/1$ case. Each user will experience a delay $D(\gamma, N, K)$ depending on the transmission strategy used by the base station (network coding or scheduling) and the number of users waiting in the queue (dictated by the arrival rate γ). The expression for the expected delay is given by the *Pollaczek-Khinchin* formula:

$$\mathbb{E}[Delay] = \frac{\lambda \mathbb{E}[Z^2]}{2(1 - \lambda \mathbb{E}[Z])}, \tag{3.10}$$

where Z is the service time of a single file broadcast. The distribution of Z will depend on the transmission strategy employed at the base station. Henceforth, we will use m_1 and m_2 to denote $\mathbb{E}[Z]$ and $\mathbb{E}[Z^2]$ respectively. We characterize the user admission rate λ in the following proposition.

Proposition 3. *Assuming Poisson arrivals with rate γ and utility function given by 3.9, the user admission rate λ takes the following form:*

$$\lambda = \frac{2\gamma d_{max} m_1 + \gamma m_2 + 2d_{max} - \sqrt{(2\gamma d_{max} m_1 + \gamma m_2 + 2d_{max})^2 - 16(d_{max})^2 \gamma m_1}}{4d_{max} m_1}$$

Proof. Let the queuing delay be μ at any given instant. Each user will decide to enter if and only if its net utility from the file transfer is non-negative, which will be the case only when $\mu < d_{max}$. This implies that the *effective input rate* or *accepted load* λ is given by

$$\lambda \;=\; \gamma \mathbb{P}\left(\mu \leq d_{max}\right) = \gamma \int_{\mu}^{d_{max}} \frac{1}{d_{max}}\, dD.$$

We therefore have

$$\lambda = \begin{cases} \gamma \frac{d_{max}-\mu}{d_{max}} & \text{if } \mu < d_{max} \\ 0 & \text{if } \mu \geq d_{max} \end{cases} \tag{3.11}$$

For the case where $\mu < d_{max}$, we have

$$\lambda \;=\; \gamma\left(1 - \frac{\mu}{d_{max}}\right) = \gamma\left(1 - \frac{\lambda m_2}{2d_{max}\left(1 - \lambda m_1\right)}\right)$$

using the *Pollaczek-Khinchin* formula (3.10). With a little algebra, we obtain a quadratic equation in λ,

$$2d_{max}m_1\lambda^2 - (2\gamma d_{max}m_1 + \gamma m_2 + 2d_{max})\lambda + 2\gamma d_{max} = 0, \tag{3.12}$$

with roots

$$\lambda_1 \;=\; \frac{2\gamma d_{max}m_1 + \gamma m_2 + 2d_{max} + \sqrt{(2\gamma d_{max}m_1 + \gamma m_2 + 2d_{max})^2 - 16(d_{max})^2\gamma m_1}}{4d_{max}m_1}$$
$$\lambda_2 \;=\; \frac{2\gamma d_{max}m_1 + \gamma m_2 + 2d_{max} - \sqrt{(2\gamma d_{max}m_1 + \gamma m_2 + 2d_{max})^2 - 16(d_{max})^2\gamma m_1}}{4d_{max}m_1}$$

In order to satisfy the constraint that the delay μ is finite and positive, we must have $\lambda < \frac{1}{m_1}$ (so that the term $(1 - \lambda m_1)$ in the *Pollaczek-Khinchin* formula (3.10) is positive). Out of λ_1 and λ_2, only λ_2 satisfies this constraint, and therefore

$$\lambda = \frac{2\gamma d_{max} m_1 + \gamma m_2 + 2d_{max} - \sqrt{(2\gamma d_{max} m_1 + \gamma m_2 + 2d_{max})^2 - 16(d_{max})^2 \gamma m_1}}{4d_{max} m_1}$$

This completes the proof.

$\square$

The characterization of the user admission rate λ given by Proposition 3 involves m_1 and m_2, the first and second moments of the service time process. Both m_1 and m_2 are functions of N and K. Therefore, in order to observe the dependence of N on d_{max} using (3), K and λ must be held constant and d_{max} must be varied. We now derive explicit expressions in order to efficiently compute m_1 and m_2. Since our goal is to compare the performance of network coding with that of scheduling, we will need to compute m_1 and m_2 for both network coding and scheduling. Let us introduce the subscripts NC for network coding and RR for round robin scheduling. Consequently, m_{1-NC} and m_{2-NC} will denote the first and second moments of the service time distribution using network coding, and m_{1-RR} and m_{2-RR} will denote the first and second moments of the service time distribution using round robin scheduling. Recall that m_{1-NC} and m_{1-RR} are the same as $\mathbb{E}[T_{NC}]$ and $\mathbb{E}[T_{RR}]$ and are given by equations (3.3) and (3.4) respectively. In order to characterize m_{2-NC} and m_{2-RR}, we will need the distributions of T_{NC} and T_{RR}. We first provide a characterization of m_{2-NC} by means of the following proposition.

Proposition 4. *Let $F(y, K, p)$ denote the cumulative distribution function of a Pascal random variable of order K and success probability p. Then*

$$m_{2-NC} = \sum_{i=1}^{\infty} i^2 \left[\left(\sum_{\tau=K}^{i} \binom{\tau-1}{K-1} (1-c)^{(\tau-K)} c^K \right)^N - \left(\sum_{\tau=K}^{i-1} \binom{\tau-1}{K-1} (1-c)^{(\tau-K)} c^K \right)^N \right]$$

Proof. We know from (3.1) that $T_{NC} = \max_{1 \le i \le N} Y_i$, where each Y_i is a Pascal random variable of order K and success probability c.

$$\mathbb{P}(Y_i \leq y) = F(y, K, c),$$

and

$$\begin{aligned}
\mathbb{P}(T_{NC} \leq y) &= \mathbb{P}(\max_{1 \leq i \leq N} Y_i \leq y) = \mathbb{P}(\cap_{i=1}^{N} Y_i \leq y) \\
&= (\mathbb{P}(Y_i \leq y))^N \text{ since the } Y_i\text{'s are } \textit{i.i.d.} \\
&= F(y, K, c)^N.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\mathbb{P}(T_{NC} = y) &= \mathbb{P}(T_{NC} \leq y) - \mathbb{P}(T_{NC} \leq y - 1) \\
&= F(y, K, c)^N - F(y - 1, K, c)^N
\end{aligned}$$

and so

$$m_{2-NC} = \mathbb{E}[T_{NC}^2] = \sum_{i=1}^{\infty} i^2 \left[F(i, K, c)^N - F(i - 1, K, c)^N\right]$$

$$= \sum_{i=1}^{\infty} i^2 \left[\left(\sum_{\tau=K}^{i} \binom{\tau - 1}{K - 1} (1 - c)^{(\tau - K)} c^K \right)^N - \left(\sum_{\tau=K}^{i-1} \binom{\tau - 1}{K - 1} (1 - c)^{(\tau - K)} c^K \right)^N \right]$$

This completes the proof. □

The characterization of m_{2-NC} given in Proposition 4, although accurate, is cumbersome and inefficient to compute, requiring the computation of combinations and sums of large orders. In order to make m_{2-NC} computationally more efficient, we use the Pascal-to-Gamma approximation suggested by Guenther [17]. More specifically, we approximate the Pascal distribution function $F(y, K, p)$ by the Gamma distribution function as follows:

$$\begin{aligned} F(y,K,p) &\approx \mathbb{P}(M,X) \\ &= \int_0^X \frac{t^{(N-1)}e^{-t/2}}{2^N\Gamma(N)}, \end{aligned}$$

where $\mathbb{P}(M, X)$ is the Gamma distribution function with parameters α and β, and $M = \alpha = k(1-p)$ and $X = (2y+1)p$.

This approximation of the Pascal distribution by a Gamma distribution is not only accurate but also significantly easier to evaluate. Consequently, we can replace $F(y, K, p)$ by $\mathbb{P}(k(1-p), (2y+1)p)$ in Proposition 4 to obtain

$$m_{2-NC} = \sum_{i=1}^{\infty} i^2 \left[\mathbb{P}(K(1-c), (2y+1)p)^N - \mathbb{P}(K(1-c), (2y-1)p)^N\right] \quad (3.13)$$

Since our goal is to characterize the delay gains from network coding as compared to scheduling, it is sufficient to obtain lower bounds for the first and second moments of the service time distribution using round robin scheduling. The use of lower bounds for scheduling (if attainable) corresponds to the worst-case scenario in some sense, because in practice, the queueing delay for scheduling will always be larger than the queueing delay obtained using lower bounds for m_{1-RR} and m_{2-RR}, and delay gains from network coding will be better.

A lower bound on m_{1-RR} can be trivially obtained from (3.4):

$$m_{1-RR} \geq \frac{K}{2} + K \sum_{t=1}^{\infty} \left[1 - (1-(1-c)^t)^{KN}\right]. \quad (3.14)$$

In order to obtain a lower bound on m_{2-RR}, we first define a lower bound on T_{RR}

in the following proposition.

Proposition 5. *Let* $\hat{T}_{RR} = \max_{\substack{1\leq i\leq N\\1\leq k\leq K}} KW_i^k$*, where* W_i^k *is a geometric random variable with parameter c, and* T_{RR} *be given by (3.2). Then*

$$\hat{T}_{RR} < T_{RR}\ \forall K, N \in \mathbb{N}.$$

Proof. Suppose Proposition 5 is not true. Then there exist K and N such that

$$\hat{T}_{RR} = \max_{i,k} KW_i^k \geq \max_{i,k} KW_i^k + k = T_{RR}.$$

Let u denote the W_i^k which maximizes $\hat{T}_{RR}$. Then, by a cut-and-paste argument, $Ku \geq Ku + k$, which leads to a contradiction since $k > 0$ by construction. Therefore, $\hat{T}_{RR} < T_{RR}$ by contradiction. This completes the proof. □

Let $\hat{m}_{2-RR}$ be the second moment of $\hat{T}_{RR}$. Since $\hat{T}_{RR}$ is a lower bound on T_{RR}, $\hat{m}_{2-RR}$ constitutes a lower bound on m_{2-RR}. We characterize $\hat{m}_{2-RR}$ in the following proposition.

Proposition 6. *Let* $F_G(y,p)$ *denote the distribution function of a geometric random variable with parameter p. Then,*

$$\hat{m}_{2-RR} = K^2 \sum_{i=1}^{\infty} i^2 \left[F_G(i,c)^{KN} - F_G(i-1,c)^{KN}\right],$$

where c is the probability of a successful packet transmission on a channel in a given time slot.

Proof. The proof is very similar to the proof of Proposition 4. Let $\mu_{max} = \max_{i,k} W_i^k$. Then

$$\hat{m}_{2-RR} \;=\; \mathbb{E}[(\hat{T}_{RR})^2] = \mathbb{E}[(K\mu_{max})^2] = K^2\mathbb{E}[(\mu_{max})^2].$$

Now, since $\mu_{max} = \max_{i,k} W_i^k$,

$$\mathbb{P}(\mu) max \leq r) = \mathbb{P}(W_i^k \leq r)^{KN} = F_G(r,p)^{KN},$$

and so,

$$\begin{aligned}\mathbb{P}(\mu_{max} = r) &= \mathbb{P}(\mu_{max} \leq r) - \mathbb{P}(\mu_{max}) \leq r - 1) \\ &= F_G(r,p)^{KN} - F_G(r-1,p)^{KN}\end{aligned}$$

Consequently, we have

$$\begin{aligned}\mathbb{E}[(\mu_{max})^2] &= \sum_{i=1}^{\infty} i^2 \mathbb{P}(\mu_{max} = i) \\ &= \sum_{i=1}^{\infty} i^2 \left[F_G(i,c)^{KN} - F_G(i-1,c)^{KN}\right].\end{aligned}$$

Hence,

$$\begin{aligned}\hat{m}_{2-RR} &= K^2 \mathbb{E}[(\mu_{max}^2)] \\ &= K^2 \sum_{i=1}^{\infty} i^2 \left[F_G(i,c)^{KN} - F_G(i-1,c)^{KN}\right].\end{aligned}$$

This completes the proof. □

Since the geometric cumulative distribution function $F_G(i,p) = (1-(1-p)^i)$, we obtain the following expression for $\hat{m}_{2-RR}$:

$$\hat{m}_{2-RR} = K^2 \sum_{i=1}^{\infty} i^2 \left[(1-(1-c)^i)^{KN} - (1-(1-c)^{i-1})^{KN}\right] \tag{3.15}$$

3.3.4 Performance of Network Coding and Scheduling for Inelastic Traffic

We use equation 3 along with the first and second moments of the service times process to observe the effect of the maximum allowable delay d_{max} on the system capacity, i.e., the number of receivers the system is able to support. The coding window K and the user admission rate λ are held constant.

Figures 3-2 and 3-3 show plots of the number of receivers N the system can support against the maximum allowable delay d_{max} for network coding and scheduling respectively. The coding window was held constant at $K = 20$.

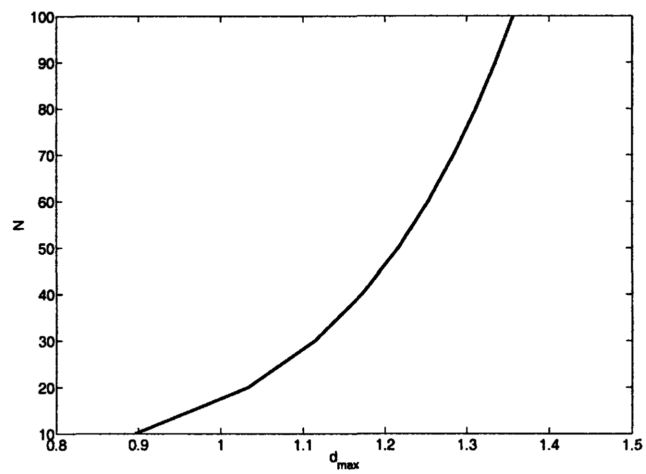

Figure 3-2: Number of receivers N against d_{max} for network coding, $K = 20$

We observe that with network coding, the system can support a large number of receivers ($N = 100$) for a significantly smaller delay than scheduling. Moreover, the user-to-delay curve for network coding is much steeper than that for scheduling. The number of receivers rises from about $N = 10$ to $N = 100$ as d_{max} increases from 0.9 to 1.3, a change of only 0.4. In the case of scheduling, on the other hand, the number of receivers rises from $N = 10$ to $N = 100$ as d_{max} increases from 1.69×10^7 to 1.74×10^7, which is an much larger change of 5×10^6. In other words, the delay constraint on the system must be considerably relaxed before scheduling can accommodate a larger number of receivers.

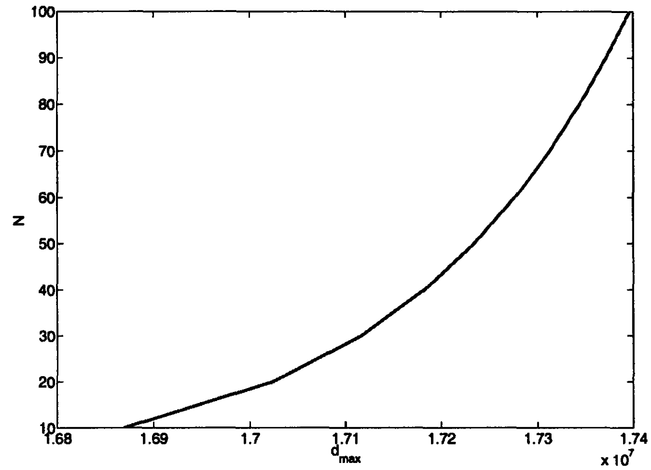

Figure 3-3: Number of receivers N against d_{max} for scheduling, $K = 20$

Figure 3-4 shows plots of the user admission rate λ against the maximum allowable delay d_{max} for network coding and scheduling. The number of receivers and the coding window was held constant at $N = 50$ and $K = 20$ respectively.

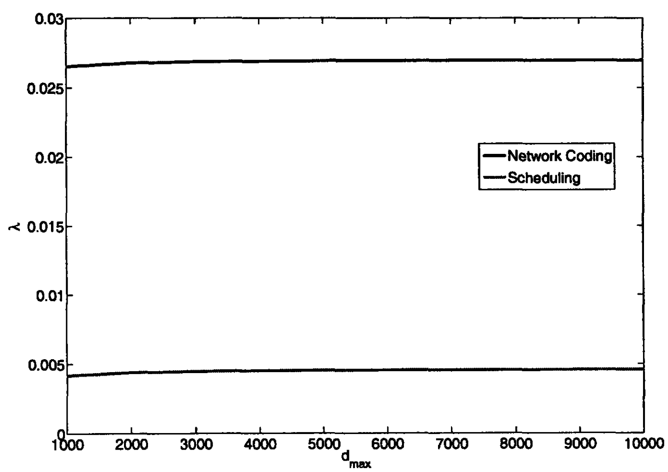

Figure 3-4: User admission rate λ against d_{max}, $N = 50$, $K = 20$

In this case, it is again evident that for a given number of receivers and a given coding window, network coding admits a much higher rate of user admission than scheduling for a wide range of values of the maximum allowable delay. For $K = 20$, the ratio of the user admission rate for network coding to that of scheduling is about six. In other words, over a long duration of time, network coding can on average support six times the number of users that scheduling can support. Figure 3-5 shows how this ratio varies as the coding window changes.

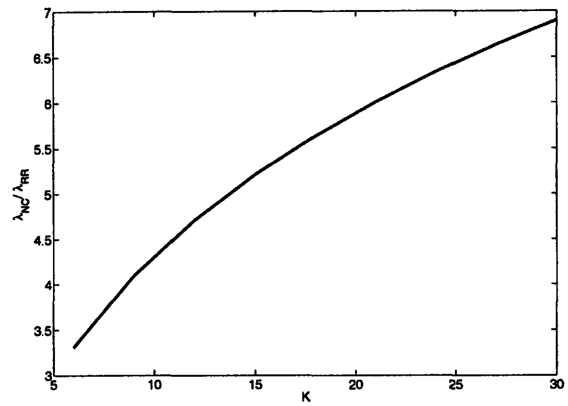

Figure 3-5: Ratio of the two admission rates for network coding and scheduling against the coding window K, $N = 50$

We observe that for $K = 5$, this ratio is about 3.5, increasing monotonically to approximately 7 at $K = 30$. This result serves to illustrate that network coding can accommodate significantly more real-time traffic than scheduling. from Figure 3-5, it may seem that $K \rightarrow \infty$ will yield the best "relative" gain for network coding as compared to scheduling. It is worth pointing out that although the ratio of the two throughputs may increase, the absolute values of both throughputs will decay to zero as the queueing delay becomes large on account of the increase in K. Therefore, in this case, a comparison of the throughput ratios for large K does not make much sense.

Chapter 4

Network Coding in Multi-hop Networks

So far we have considered wireless networks consisting only of a single hop. Packets are sent directly from the transmitter to the receivers over time-varying stochastic channels. In other words, packet transmission from the source to the receivers is accomplished over a single link, without any intermediate transmission or routing mechanism. Although such a model is quite helpful for gaining an insight into the workings of wireless systems, it provides a very simplistic view of a wireless network. Most real-world wireless systems are multi-hop: packet transmission between two end-to-end nodes is accomplished by relaying packets between intermediate nodes over multiple links. This is primarily due to the widespread deployment of the 802.11 protocol, and the fact that from relaying packets from node to node results in energy and capacity gains. Wireless mesh architectures for the provision of broadband internet, for instance, constitute an important and widely-used class of multi-hop wireless networks.

In light of the significance and prevalence of multi-hop wireless networks, it seems worthwhile to extend our model to include multi-hop wireless networks and quantify the performance gains to be had from network coding in this more general case. A secondary motivation for doing so is that recently there has been an appreciable interest in multi-hop wireless applications such as message sharing and file transfer over peer-to-peer networks, and there has been considerable ongoing research into optimizing packet transmission in these systems. Such systems can be readily modeled as a tree-based multicast topology, which is a classic model for multi-hop networks.

In this chapter, we extend the single-hop cellular downlink model to a multi-hop case. The multi-hop link can be conceived of as a chain of single-hop links placed in series. The utility of this approach is that it enables us to analyze multicast settings in general network topologies using the results derived for the single-hop case. We take the following approach while analyzing general network topologies:

- Rearrange the general topology into a layered topology from which multi-hop paths between the source and the terminals can be readily identified.
- Analyze the multi-hop paths as a series of single-hop paths, using the previously derived results for single-hop networks.

4.1 An Example of the Multi-hop Case

Consider as an example the multicast setting shown in Figure 4-1. The multicast set consists of two terminal nodes, and there is a single source node and a number of intermediate nodes. Such a network topology can be representative of a peer-to-peer network with the source as a central server and the terminals as peers requesting information from the server.

The first step is to modify the network and rearrange it in a layered topology as shown in Figure 4-2. Each layer in the layered network topology consists of all nodes which are located at a given number of hops from the source. For instance, Layer 1 consists of all nodes that are a single hop away from the source, Layer 2 consists of all nodes that are two hops away from the source, and so on.

While rearranging the network into layers, we make the assumption that there is no communication among nodes within the same layer. In other words, we drop all links (that may have existed in the original network) between two nodes within the same layer. For instance, consider node A in Figure 4-1. It is a single-hop away from

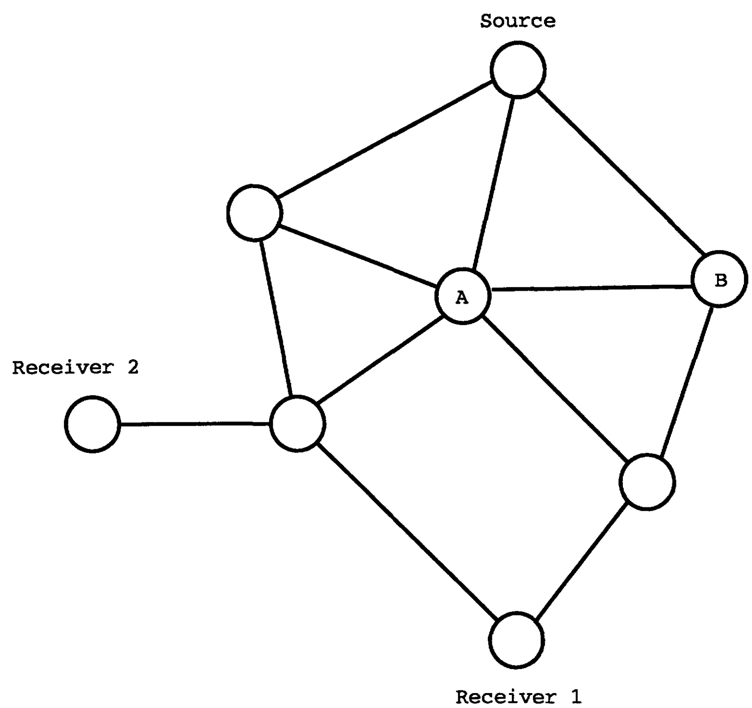

Figure 4-1: A multicast setting in a general network topology

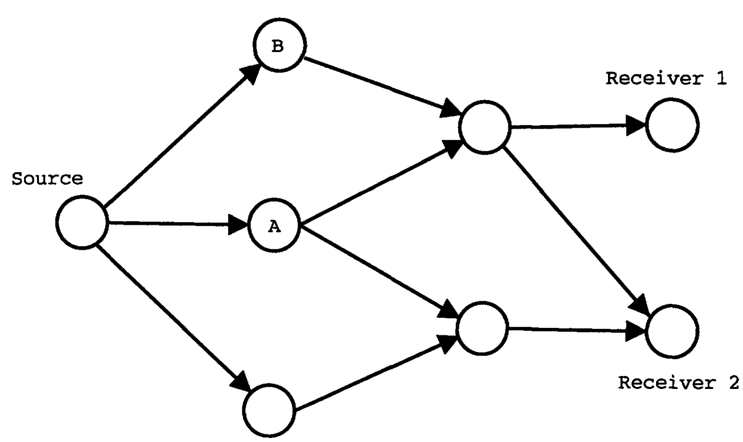

Figure 4-2: A layered network

the source node, and therefore should be placed in Layer 1. However, node A can also be reached from the source in two hops via node B. In this case, we take the distance of a node from the source to be the minimum number of hops required to reach that node starting at the source node. In order to resolve any ambiguity in case there are multiple paths from the source to a given node (as is the case for node A), we drop all links between any two nodes within the same layer. This translates to the aforementioned assumption that nodes within the same layer cannot communicate with one another.

It is easy to identify the layer in which each node is located with respect to a given multicast by simply flooding the network or by using more sophisticated shortest path algorithms. We assume that all nodes which are members of the same layer compete for the same network resources and therefore can interfere with one another. On the other hand, nodes which are members of different layers are assumed to be operating in orthogonal channels (for instance, disjoint frequency bands or time slots), and consequently are assumed not to interfere with one another. Note that such a separation of channels can be achieved through a simple scheduling mechanism. This separation allows us to focus exclusively on transmissions between adjacent layers.

Once the layered network topology has been created, the next step is to analyze the layered network as a series of single-hop networks. More specifically, the layered network in our model operates in the following manner. The source transmits the file to the first layer, the first layer then transmits the file to the second layer, and so on. The file is thus transmitted from layer to layer until it reaches the receivers. Receivers can be in different layers, in which case the layer-to-layer transmission will continue until the file is successfully received by the receiver in the last layer. We thus define the depth of the network as the number of hops required to reach the last layer, starting at the source. The depth of the network in the example given above is 3.

Notice that packet transmission from one layer to the next (adjacent) layer is the same as the single-hop case we have previously looked at, with two important differences; first, both the transmitting layer and the receiving layer may have more than one node (i.e., there can be multiple transmitters and multiple receivers), and second, the presence of multiple transmitting nodes in the same layer may lead to collisions at the receivers. We assume that a receiver successfully receives a packet only if one of the transmitters transmits a packet. Otherwise, the transmission fails. Therefore, before we can apply our results from the single-hop scenario to the layered network topology, we must extend our analysis of the single-hop case from single-source multiple-receivers to multiple-sources multiple-receivers. This extension is described in the following section.

4.2 The Multiple-source Multiple-receiver Model

Consider a system with N_s sources and N_r receivers. Transmissions take place in regularly arranged time slots with each slot long enough to accommodate a single packet transmission. Assume for simplicity that each receiver is linked to a randomly chosen subset of the sources, and that the cardinality of the subset is the same for each receiver, i.e., all receivers are connected to an equal number of sources. This is the symmetric case. In the asymmetric case, each receiver will be connected to a different number of sources. The channel conditions on each link are identical to the channel conditions for the single-source multiple-receiver case. Figure 4-3 illustrates the system topology for $N_s = 3$ and $N_r = 4$. Each receiver is connected to two sources.

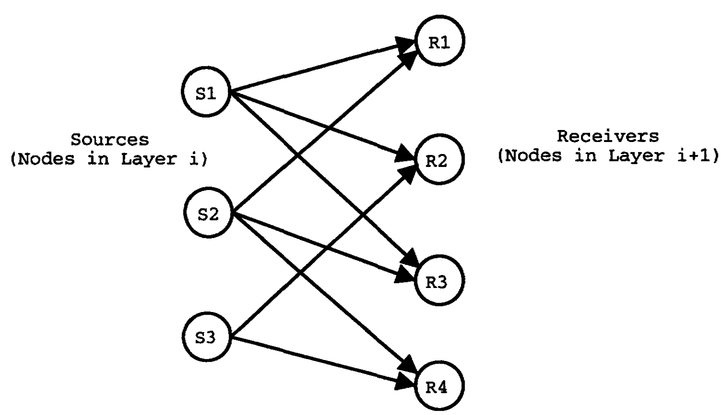

Figure 4-3: Multiple-source multiple-receiver system with three sources and four receivers

Initially, all the sources within the layer possess a single file consisting of K packets. Our goal is to minimize the amount of time necessary for the file to be transmitted to all the receivers. In particular, we are interested in finding the mean file completion times for the cases of network coding and scheduling, and evaluate the delay performance of network coding as compared to scheduling in the more general scenario of multiple sources and multiple receivers.

As we mentioned, we assume that when a receiver receives packets from more than one source in a single time slot, the packets collide and are erased. Therefore, a successful transmission occurs only when a receiver receives one packet in a single time slot. In such a scenario, it clearly does not make sense for each source to attempt transmission in every time slot. Assuming that the sources are not able to communicate with one another, a better transmission strategy is for source S_i to attempt transmission with probability p_i in every time slot. In order to keep the analysis simple, we restrict our attention to the symmetric case in which $p_i = p$ for all sources S_i. Consider the scenario in which source S_i attempts to transmit a packet with probability p. The channel between S_i and, say, receiver R_i is ON with probability c. The probability that R_i successfully receives a packet from S_i is therefore pc. Suppose each receiver is connected to L nodes ($L < N$). Then, the number of packets a receiver receives in one time-slot, X, is given by a binomial distribution with parameters (L, pc). Then,

$$
\begin{aligned}
\mathbb{P}(\text{Receiver } R_i \text{ successfully receives a packet}) &= \mathbb{P}(\text{Receiver } R_i \text{ receives one packet}) \\
&= \mathbb{P}(X = 1) \\
&= Lpc(1 - pc)^{L-1}
\end{aligned}
$$

In order to minimize the file download completion time, we must maximize the probability of a successful transmission/reception given by the expression above. This expression is identical to the probability of a successful capture in the Aloha system. Assuming that the number of sources L is reasonably large, say $L \geq 10$, we can use our knowledge of the Aloha system to conclude that the above probability is maximized when $p = 1/L$, attaining a maximum value of $1/e$. Therefore, the number of packets a receiver receives in one time slot is Bernoulli distributed with a success probability of $1/e$.

4.3 Performance Analysis

4.3.1 Network Coding

Let T_{NC} denote the file download completion time for all receivers of the optimal network coding policy. Since the number of packets a receiver receives in one time slot is Bernoulli distributed with a success probability of approximately $1/e$, the expected file completion time is simply the mean of the maximum of N_r Pascal random variables according to the analysis in [10], and is given by

$$\mathbb{E}[T_{NC}] \approx K + \sum_{t=K}^{\infty}\left[1 - \prod_{i=1}^{N_r}\left(\sum_{\tau=K}^{t}\begin{pmatrix}\tau-1\\K-1\end{pmatrix}\left(1-\frac{1}{e}\right)^{(\tau-K)}\left(\frac{1}{e}\right)^{K}\right)\right],$$

where $\begin{pmatrix}n\\m\end{pmatrix}$ gives the number of combinations of size m of n elements.

4.3.2 Scheduling

Based on our previous assumption that there is no communication among the sources, a source cannot know which packet other sources will attempt to transmit in a particular time slot unless there has been a previous agreement among the sources to transmit packets in a particular order in successive time slots. It has been shown in [10] that in scheduling without CSI, one of the optimal scheduling policies is *Round Robin* (RR), where Packet-k is transmitted in time slots $(mK + k)$ for $m = 0, 1, \cdots$ until all the receivers get the file. Consequently, one of the simplest strategies to consider for multiple sources is the case in which all sources attempt to transmit the same packet in a particular time slot, i.e., all sources transmit Packet-k in time slots $(mK+k)$ for $m = 0, 1, \cdots$ with probability p. Again, throughput is maximized when $p = 1/L$, and the maximum throughput is $1/e$.

Let T_{RR} denote the the file completion time for this strategy. Then, using the analysis in [10],

$$\frac{\mathbb{E}[T_{RR}]}{K} \approx \gamma + \sum_{t=1}^{\infty} \left[1 - \left(1 - (1 - \frac{1}{e})^t \right)^{KN_r} \right],$$

for some $\gamma \in (1/2, 1)$.

4.3.3 Comparison

Figure 4-4 illustrates the delay performance of network coding versus scheduling for $K = 30$. There is approximately a four-fold gain in delay from network coding for $N_r = 10$, which increases to five-fold as N_r increases to 40.

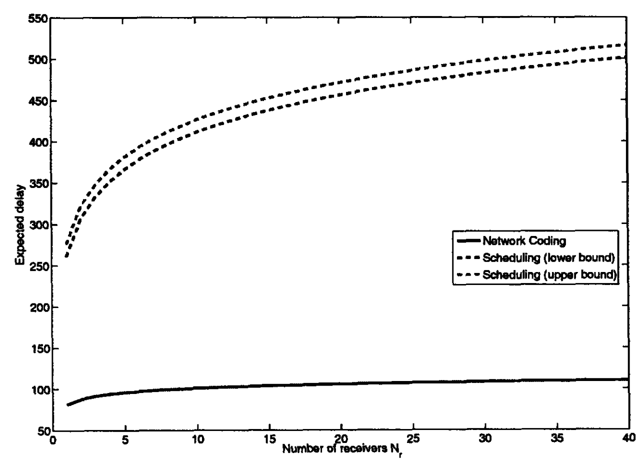

Figure 4-4: Delay Performance of Network Coding versus Scheduling (Round robin) for $K = 30$

The gains from network coding as shown by Figure 4-4 seem very promising. Indeed, during the course of packet transmission from one layer to the next, we would expect the gains from network coding to accumulate from layer to layer. If the depth of the network is sufficiently large, the cumulative gains from network coding will be significantly higher.

Chapter 5

Conclusion

Network coding indeed appears to be a promising approach as far as efficient utilization of resources in communication networks is concerned, and has provided significant gains in various aspects of networking theory - chiefly energy, throughput, and capacity. However, one area in which network coding has previously not been analyzed is delay minimization. Delay is an important practical consideration in communication networks and in particular wireless networks: networks with large delays become infeasible to operate in practice. Analyzing network coding with the goal of optimizing delay is made especially challenging by the fact that network coding requires packets to be encoded and decoded in bulk. This feature of network coding, as we have mentioned earlier, necessitates the use of a rateless transmission scenario, in which the delay performance of a scheme is quantified only after transmission has been successfully completed.

The delay performance of network coding in such a rateless transmission scenario had previously been studied in [10]. However, [10] constituted the only work that presented a quantification of the delay gains from network coding. Our goal in this thesis was to further our understanding of the delay characteristics of network coding. We proposed a number of extensions to the model in [10] and analyzed the performance of network coding in each of those settings. We now summarize our efforts and results.

In Chapter 2, we looked at a simple file transfer system with a single base station and multiple receivers. We investigated the system from an economic viewpoint and obtained characterizations of the optimal user admission rate and its dependence on the moments of the service time at the base station. We also approximated the optimal price and maximum revenue of the base station, and showed that the revenue was a unimodal function of the file size with a single maximum. Furthermore, we showed that the optimum file size (i.e., the file size at which the revenue is maximized), is highly insensitive to changes in the number of receivers, suggesting that pricing decisions can be decoupled from coding decisions. We also compared the revenue, price and delay obtained from network coding to those obtained from scheduling, and observed that network coding yielded significant gains in revenue while allowing for a higher user admission rate at the same time. We also discussed the effect of channel conditions on the maximum revenue generated by the base station and the optimum file size.

In Chapter 3, we studied the asymptotic performance of network coding and scheduling under the scenarios of elastic and inelastic traffic. For the case of elastic traffic, we showed that the ratio of the the file completion time of network coding to that of scheduling asymptotically approached $1/K$ (i.e., the reciprocal of the file size K) as the number of receivers increased to a sufficiently large value, signifying that file downloads took K times longer to complete in the case of scheduling. Arbitrarily large gains could be realized by appropriately choosing the file size. In the case of inelastic traffic, we analyzed the behavior of the number of receivers and throughput as functions of the delay constraint under various scaling laws, and observed that network coding allowed significantly more receivers to be supported and permitted an appreciably higher throughput as compared to scheduling for the same delay constraint.

In Chapter 4, we proposed a way to extend the single-hop model in [10] to general multi-hop network topologies. We described a method to structure a multi-hop network as a chain of single-hop networks. We further enhanced the single-hop model to include multiple base stations, and demonstrated that gains from network coding for each single-hop unit accumulated over the whole chain to yield significantly larger delay gains as compared to scheduling.

The possibilities for future work are vast. In the cellular downlink model that we have used, we have made the assumption that the user arrival process is Poisson. This assumption is reasonable and leads to a tractable analysis. However, some real-world networks cannot always be modeled by a Poisson arrival process. For instance, if arrivals are very rapid and the arrival rate does not change much over time, which can often be the case, it may be more realistic to model arrivals as a deterministic process. A potential future research direction, therefore, is to relax the assumption of Poisson arrivals and use more general processes to model arrivals to the system.

Much scope for future work exists in the generalization of single-hop networks to multi-hop networks that we have delineated in Chapter 4. As we mentioned, we assume in our analysis that each single-hop layer within our layered multi-hop topology is symmetric, i.e., each receiver is connected to an equal number of sources. This assumption is somewhat restrictive; many general network topologies cannot be modeled under the assumption of symmetry. A good example is networks with non-uniform density, i.e., networks which are densely connected in some regions and sparsely connected in other regions. Ideally, we do not want to impose any constraints on the topology of the network we wish to model. However, without the assumption of symmetry, the single-hop case will be tough to tackle. In the presence of erasures and collisions, the transmitters will need to optimize their individual probabilities of transmission so that total throughput is maximized. This leads to an interesting non-convex min-max problem.

Another open question related to the previous one pertains to finding the optimal scheduling strategy in our model of multi-hop networks. In Chapter 4, we have used a simplistic method based on round robin to schedule file transmissions in the presence of collisions; each transmitter sends the same packet in a give time slot. This scheme constitutes one way to schedule packet transmissions. It is most likely not the optimal way, and therefore, an potential research direction would be to evaluate different scheduling schemes and find the one which gives the highest throughput. In fact, solving the aforementioned min-max problem to optimize the transmission probabilities at each transmitter will provide sufficient insight into the nature of the optimal scheduling strategy in our case.

Yet another research direction can be identified with regard to the topology of the layered network. In our model, we have discarded all links present between any two nodes within the same layer under the assumption that nodes within the same layer are not able to communicate with one another. The removal of these links serves to decrease throughput in some sense. The presence of links between two nodes within the same layer results in more efficient transmission. This is certainly the case in peer-to-peer networks in which peers help improve download rates by forwarding packets to other peers. In order to improve the model in Chapter 4, it is worthwhile to relax the assumption that there is no communication among nodes in the same layer, and investigate how the presence of links between two nodes in the same layer affects the efficiency of the file download.

Another point that requires further consideration is the synchronization of layers in the layered network. In the model we have proposed, we have assumed that nodes in Layer i begin transmitting the file to Layer $(i+1)$ only after all the nodes in Layer i have successfully received the file. This may introduce an undesirable delay in the system, since file transmission can surely be achieved more quickly if nodes begin transmission as soon as they have received the file themselves, without having to wait for other nodes in the same layer to receive the file. The resulting asynchronous system will lead to smaller file download completion times, and an interesting future research direction would be to focus on quantifying the throughput gains from the lack of synchronization within layers.

Finally, a worthwhile extension to the model in Chapter 4 would be to model multiple multicast, i.e, to have multiple sources transmitting multiple files over the layered network. Such a model would involve multiple flows across one link. Since most routed packet networks operate in this fashion, analyzing such a model will deepen our understanding of the delay gains from network coding and will provide a solid basis for the comparison of the delay performance of network coding to that of traditional routing.

Bibliography

[1] D. Acemoglu, A. Ozdaglar, and R. Srikant. The marginal user principle for resource allocation in wireless networks. In *Proceedings of IEEE Conference on Decision and Control*, Paradise Island, Bahamas, December 2004.

[2] R. Ahlswede, Ning Cai, S. R. Li, and R. W. Yeung. Network information flow. *IEEE Transactions on Information Theory*, 46:1204–1216, July 2000.

[3] E. Ahmed, A. Eryilmaz, A. Ozdaglar, and M. Médard. Economic aspects of network coding. In *Proc. 44th Annual Allerton Conference on Communications, Control and Computing*, Monticello, IL, September 2006.

[4] T. Başar and G. J. Olsder. *Dynamic Noncooperative Game Theory.* SIAM Series in Classics in Applied Mathematics, Philadelphia, PA, 1999.

[5] K. Bharath-Kumar and J. M. Jaffe. Routing to multiple destinations in computer networks. *IEEE Transactions on Communications*, 31(3):343–351, March 1983.

[6] K. Bhattad and K. R. Narayanan. Weakly secure network coding. In *Proc. WINMEE, RAWNET and NETCOD 2005 Workshops*, April 2005.

[7] N. Cai and R. W. Yeung. Secure network coding. In *Proc. 2002 IEEE International Symposium on Information Theory (ISIT 2002)*, page 323, June/July 2002.

[8] H. A. David. *Order Statistics.* John Wiley, 2003. Second Edition.

[9] R. Dougherty, C. Freiling, and K. Zeger. Insufficiency of linear coding in network information flow. *IEEE Transactions on Information Theory*, 51(8):2745–2759, August 2005.

[10] A. Eryilmaz, A. Ozdaglar, and M. Médard. On delay performance gains from network coding. In *Proceedings of Conference on Information Sciences and Systems (CISS)*, Princeton, NJ, March 2006.

[11] J. Feldman, T. Malkin, R. A. Servedio, and C. Stein. On the capacity of secure network coding. In *Proc. 42nd Annual Allerton Conference on Communication, Control, and Computing*, September/October 2004.

[12] C. Fragouli and E. Soljanin. Decentralized network coding. In *Proc. 2004 IEEE Information Theory Workshop (ITW 2004)*, pages 310–314, October 2004.

[13] M. Ghaderi, D. Towsley, and Jim Kurose. Reliability benefit of network coding. *In preparation.*

[14] C. Gkantsidis and P. R. Rodriguez. Network coding for large scale content distribution. In *Proc. IEEE Infocom*, volume 4, pages 2235–2245, March 2005.

[15] R. Gowaikar, A. F. Dana, R. Palanki, B. Hassibi, and M. Effros. On the capacity of wireless erasure networks. In *Proc. 2004 IEEE International Symposium on Information Theory (ISIT 2004)*, page 401, June/July 2004.

[16] P. Grabner and H. Prodinger. Maximum statistics of n random variables distributed by the negative binomial distribution. *Combinatorics, Probability and Computing*, 6(4):179–183, 1997.

[17] W. C. Guenther. A simple approximation to the negative binomial (and regular binomial). *Technometrics*, 14(2):385–389, 1972.

[18] T. Ho, B. Leong, R. Koetter, M. Médard, M. Effros, and D. R. Karger. Byzantine modification detection in multicast networks using randomized network coding. In *Proc. 2004 IEEE International Symposium on Information Theory (ISIT 2004)*, page 144, June/July 2004.

[19] T. Ho, M. Médard, and R. Koetter. An information-theoretic view of network management. *IEEE Transactions on Information Theory*, 51(4):1295–1312, April 2005.

[20] T. Ho, M. Médard, R. Koetter, D. R. Karger, M. Effros, J. Shi, and B. Leong. A random linear network coding approach to multicast. *IEEE Transactions on Information Theory*, 52:4413–4430, October 2006.

[21] S. Jaggi, M. Langberg, T. Ho, and M. Effros. Correction of adversarial errors in networks. In *Proc. 2005 IEEE International Symposium on Information Theory (ISIT 2005)*, pages 1455–1459, September 2005.

[22] S. Jaggi, P. Sanders, P. A. Chou, M. Effros, S. Egner, K. Jain, and L. M. G. M. Tolhuizen. Polynomial time algorithms for multicast network code construction. *IEEE Transactions on Information Theory*, 51(6):1973–1982, June 2005.

[23] K. Jain, L. Lovasz, , and P. A. Chou. Building scalable and robust peer-to-peer overlay networks for broadcasting using network coding. In *PODC '05: Proc. 24th Annual ACM SIGACT-SIGOPS Symposium on Principles of Distributed Computing*, pages 51–59, 2005.

[24] D. Katabi, S. Katti, W. Hu, H. Rahul, and M. Médard. The importance of being opportunistic: On practical network coding for wireless environments. In *Proceedings of Annual Allerton Conference on Communication, Control and Computing*, October 2005.

[25] R. Khalili and K. Salamatian. On the capacity of erasure relay channel: Multirelay case. In *Proc. 2005 IEEE Information Theory Workshop (ITW 2005)*, August 2005.

[26] J. F. C. Kingman. Some inequalities for the queue GI/G/1. *Biometrika*, 49:315–324, 1962.

[27] R. Koetter and M. Médard. Beyond routing: An algebraic approach to network coding. *IEEEACM Transactions on Networking*, 11:782–795, October 2003.

[28] A. Rasala Lehman and E. Lehman. Complexity classification of network information flow problems. In *Proc. 41st Annual Allerton Conference on Communication, Control, and Computing*, October 2003.

[29] S.-Y. R. Li, R. W. Yeung, and Ning Cai. Linear network coding. *IEEE Transactions on Information Theory*, 49:371–381, February 2003.

[30] D. S. Lun. Efficient operation of coded packet networks. PhD thesis, Massachusetts Institute of Technology, June 2006.

[31] D. S. Lun, N. Ratnakar, R. Koetter, M. Médard, E. Ahmed, and H. Lee. Achieving minimum cost multicast: A decentralized approach based on network coding. In *Proceedings of IEEE Infocom*, pages 1607–1617, 2005.

[32] D. S. Lun, N. Ratnakar, M. Médard, R. Koetter, D. R. Karger, T. Ho, E. Ahmed, and F. Zhao. Minimum-cost multicast over coded packet networks. *IEEE Transactions on Information Theory*, 51:2608–2623, 2005.

[33] C. Magalaras and A. Zeevi. Pricing and performance analysis for a system with differentiated services and customer choice. In *Proc. Allerton Conference on Control, Communications and Computing*, 2003.

[34] M. Médard, M. Effros, D. Karger, and T. Ho. On coding for non-multicast networks. In *Proc. 41st Annual Allerton Conference on Communication, Control, and Computing*, October 2003.

[35] I. Ch. Paschalidis and J. N. Tsitsiklis. Congestion-dependent pricing of network services. *IEEE/ACM Transactions on Networking*, 8(2):171–184, 2000.

[36] A. Ramamoorthy, J. Shi, and R. D. Wesel. On the capacity of network coding for random networks. *IEEE Transactions on Inofrmation Theory*, 51(8):2878–2885, August 2005.

[37] N. Ratnakar, D. Traskov, and R. Koetter. Approaches to network coding for multiple unicasts. In *Proc. 2006 International Zurich Seminar on Communications (IZS 2006)*, pages 70–73, February 2006. invited paper.

[38] S. Riis. Linear versus non-linear boolean functions in network flow. In *Proc. 2004 Conference on Information Sciences and Systems CISS 2004)*, March 2004.

[39] Y. E. Sagduyu and A. Ephremides. Crosslayer design and distributed mac and network coding in wireless ad hoc networks. In *Proc. 2005 IEEE International Symposium on Information Theory (ISIT 2005)*, pages 1863–1867, September 2005.

[40] J. Sundararajan, M. Médard, M. Kim, A. Eryilmaz, D. Shah, and R. Koetter. To appear in proc. ieee infocom. In *Network Coding in a multicast switch*, Alaska, May 2007.

[41] B. M. Waxman. Routing of multicast connections. *IEEE Journal on Selected Areas in Communications*, 6(9):1617–1622, December 1988.

[42] Y. Wu, P. A. Chou, and S.-Y. Kung. Information exchange in wireless networks with network coding and physical-layer broadcast. In *Proc. 2005 Conference on Information Sciences and Systems (CISS 2005)*, March 2005.

[43] Y. Wu, P. A. Chou, and S.-Y. Kung. Minimum-energy multicast in mobile ad hoc networks using network coding. *IEEE Transactions on Communications*, 53(11):1906–1918, November 2005.

[44] Y. Zhu, B. Li, and J. Guo. Multicast with network coding in application-layer overlay networks. *IEEE Journal on Selected Areas in Communications*, 22(1):107–120, January 2004.