Dirac Operators and Monopoles with Singularities

by

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Abstract

This thesis consists of two parts. In the first part of the thesis, we prove an index theorem for Dirac operators of conic singularities with codimension 2. One immediate corollary is the generalized Rohklin congruence formula. The eta function for a twisted spin Dirac operator on a circle bundle over a even dimensional spin manifold is also derived along the way.

In the second part, we study the moduli space of monopoles with singularities along an embedded surface. We prove that when the base manifold is Kahler, there is a holomorphic description of the singular monopoles. The compactness for this case is also proved.

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Chapter 1

Introduction.

The work in this thesis is part of a project, in which we want to define Seiberg Witten Knot Floer Homology. More explicitly, this thesis studies the four dimensional analog of this question: given a triple \((X, \Sigma, S)\), where \(X\) is a closed oriented four manifold, \(\Sigma\) is an embedded surface, and \(S\) is a \(Spin^c\) structure on \(X\setminus \Sigma\), restricted to the circle bundle \(P\) of \(\Sigma\) is torsion, can we define an appropriate monopole moduli space for this pair? The natural strategy is to study Seiberg Witten theory on the complement. Since here the base manifold is open, the asymptotic behavior of the monopoles might depend on the metric. And we choose to use the incomplete metric which is the restriction of a smooth metric on \(X\). For the \(Spin^c\) connection, we look at the ones which have nontrivial holonomy around the small circles linking the surface. In this thesis we compute the formal dimension of the moduli space defined under an appropriate Sobolev space, and prove that when the base manifold \(X\) is Kahler, the moduli space is compact with a holomorphic description.

The first part of the thesis is to find a proper space so that the Dirac operator is Fredholm and to compute the index. On an open manifold, the Fredholm property and index for Dirac operators certainly depends on the metric at the end of the manifold. If the metric is complete, it has been studied by [9], [15]. For incomplete metric, the situation is more sophisticated. For instance, the Dirac operators are no longer essentially selfadjoint and particular closed extensions have to be chosen. Some special cases of incomplete metrics have been carried out in the literature. If the metric has
isolated conical point, i.e., $M = M_1 \cup_N U$, with $\partial M_1 = N$, $U = (0,1) \times N$, and
$g|_U = dr^2 + r^2 g_N$ for some smooth metric $g_N$ on $N$, the analysis has been initiated by
Cheeger[7] who treated the case for signature operator. And for the Dirac operator,
the index is calculated by Chou in [8]. Their analysis was extended to the more
general case by Bruning and Seeley in [5], [4], [3]. To construct closed extensions
such that the operator $D$ is Fredholm, boundary conditions have to be imposed along
the singular point. The following facts have been shown by the above authors. Let
$\overline{D}$ denote the $L^2$ closure of $\{ \phi \in L^2, D\phi \in L^2, \phi \in C^\infty \}$, then $\overline{D}$ is selfadjoint if
and only if there is no eigenvalue of $D_N$ such that $|\nu| < \frac{1}{2}$, ($D_N$ is the associated
self adjoint Dirac operator on $N$), which is also equivalent to that $\overline{D}$ is equal to the
usual $L^2_1$ space. Thus the existence of the low eigenvalues makes the sections in $\overline{D}$
do not decay enough(or even have some singularities along the singular point) such
that they do not sit inside $L^2_1$. The $L^2$ closed extensions of $D$ are parametrized by
the subspaces of $\oplus_{|s|<\frac{1}{2}} \ker(D_N - s)$. For any subspace $S \subset \oplus_{|s|<\frac{1}{2}} \ker(D_N - s)$ and
its orthogonal complement $S^\perp$, the corresponding closed extensions $D_S$ and $D_{S^\perp}$ are
adjoint to each other. All these extensions are Fredholm and their indices can be
calculated explicitly.

In this thesis, we consider another special case of incomplete metrics, for which
the Dirac operator is singular along a codimension 2 submanifold. More specifically,
let $B$ be a codimension 2 submanifold of a closed even dimensional manifold $M$. $S$
is a spinor bundle on $M \setminus B$, whose positive part $S^+$, when restricted to the tubular
neighborhood of $B$, is the pull back of some spin bundle on $B$. (we will see later that
this is equivalent to that $S$ can not be extended to a spin bundle over $M$.) Denote the
normal bundle of $B$ by $N$, and the associated circle bundle by $P$. $\omega$ is a connection
form of $P$. We are interested in the metric which is the restriction of a smooth metric
on the entire $M$. Thus the Dirac operator $D^+$ looks like

$$\partial_r + \frac{1}{2r} + \frac{F}{r} + H,$$

where $F$ is a family of first order differential operator on the fiber $S^1$ over $B$, $H$ is
essentially a family of first order differential operators along $B$. Because $S$ is the pull back spin structure from $B$, it makes sense (which will be proved in Chapter 2) that $F$ has eigenvalue 0 with the eigenspace being infinite dimensional, thus some APS type boundary conditions must be imposed along $B$ to make $D^+$ Fredholm.

Define $\text{Dom}(D^{\text{max}}_{+}) = \{ \phi \in L^2(S^+), D^+\phi \in L^2(S^-) \}$. Notice that $\text{Dom}(D^{\text{max}}_{+})$ is not the usual $L^2(S^+, M \setminus B)$, intuitively it is because that the Dirac operator $D^+$ has holonomy $-1$ along the small circles linking $B$, so it has some singularities along $\Sigma$. An important aspect of the behavior of the spinors in $\text{Dom}(D^{\text{max}}_{+})$ is given by the following lemma, which will be proved in Chapter 2.

**Lemma 1.0.1.** There is a bounded linear operator

$$R : r^\frac{1}{2} \text{Dom}(D^{\text{max}}_{+}) \to L^2_{-\frac{1}{2}}(B; S_B)$$

The range is $(L^2_{\frac{1}{2}} \cap H^+) \oplus (L^2_{\frac{1}{2}} \cap H^-)$. Here $r$ is the distance function to $B$, which is extended to the entire $M$ by 1, $H^\pm$ are the spectral subspaces associated to $D_B$.

Just as the manifold with boundary case, we need to put some boundary conditions along $B$. Write $P_0$ for the projection

$$P_0 : L^2_{-\frac{1}{2}}(S_B) \to L^2_{-\frac{1}{2}}(S_B)$$

with image $H^-$ and kernel $H^+$. This is just the APS boundary condition. We can also consider a slightly different boundary condition, given by a commensurate projection $P$, which is defined as in [12]:

**Definition 1.0.2.** Two projections $P_1, P_2 : L^2_{-\frac{1}{2}}(S_B) \to L^2_{-\frac{1}{2}}(S_B)$ are called commensurate if the difference

$$P_1 - P_2 : L^2_{-\frac{1}{2}}(S_B) \to L^2_{-\frac{1}{2}}(S_B)$$

is compact.

The following index theorem in Chapter 2.
Theorem 1.0.3. Let $M$ be a $2n$-dimensional closed manifold, $B$ be a submanifold with codimension 2. Let $S^+$ be a spinor bundle on $M\setminus B$, which can not be extended to $M$ as a spinor bundle, $D^+$ be the Dirac operator on $S^+$. $M\setminus B$ has the induced metric from $M$, $r$ is the distance function near $B$ as above. Let $P$ be a projection commensurate with $P_0$, $W^+$ be the kernel of the composition map

$$r^{\frac{1}{2}} \text{Dom}(D^+_{\text{max}}) \xrightarrow{R} L^2_{-\frac{1}{2}}(S_B) \xrightarrow{P} L^2_{-\frac{1}{2}}(S_B),$$

then on $W^+$, $D^+$ is Fredholm. Furthermore, if we take the projection to be $P_0$, the index of $D^+$ is

$$\text{Ind}(D^+) = \int_{M\setminus B} \hat{A}(M) + \int_B \hat{A}(B) \frac{1 - \cosh \frac{e}{2}}{2 \sinh \frac{e}{2}} - \frac{1}{2} \dim \ker (D_B).$$

If we replace the above projection $P$ by any projection $P_1$ such that $\ker(P_1) \oplus \text{Im}(P_2) = L^2_{-\frac{1}{2}}(S_B)$ for some projection $P_2$ commensurate with $P_0$, $D^+$ is again Fredholm. In particular, if we project to the negative spinors, we have that $D^+$ is Fredholm and with index

$$d(D^+) = \int_{M\setminus B} \hat{A}(M) + \int_B \hat{A}(B) \frac{1 - \cosh \frac{e}{2}}{2 \sinh \frac{e}{2}} + \frac{1}{2} \text{Ind} \ker (D_B^1).$$

In the second part of this thesis we define a singular moduli space for the Seiberg Witten equations over $X\setminus \Sigma$ and a $Spin^c$ structure $S$, whose restriction to the circle bundle $P$(associated to the normal bundle $N$ over $\Sigma$) is determined by a torsion line bundle. The $Spin^c$ extension of $S$ to $X$ is not unique, however, when coupled with a special $Spin^c$ connection $A_k + i\frac{1}{2}\omega$(which has trivial holonomy), $(S, A_k + i\frac{1}{2}\omega)$ extends uniquely to a $Spin^c$ pair $(S_X, A_X)$ on $X$, such that $c_1(S_X^+)(\Sigma) = 2k + n, k \in \mathbb{Z}$ and $n = \Sigma \cdot \Sigma$ is the intersection number of $\Sigma$. Fix a reference connection $A_k + ic\omega$, $A_k$ has holonomy $\exp(i2(c - \frac{1}{2})\pi)$ along the fibers. We consider the configuration space $\mathcal{C}_k$(detailed definition in Chapter 3.) with spinors singular along $\Sigma$ and the connections having nontrivial holonomy. For the kahler case, we see that there is a holomorphic description of the singular monopoles. We will also show the compact-
ness for Kahler case using a variational argument. More specifically, the following properties are proved in Chapter 3.

**Theorem 1.0.4.** Let $X$ be a closed Kahler surface, $\Sigma$ be a holomorphically embedded curve. $S$ is a Spin$^c$ structure on $X$, which restricted to the circle bundle of $\Sigma$ is torsion. Then for any sequence $(A_i, \alpha_i) \in C_k$ which are solutions to the Seiberg-Witten equations, there exists a subsequence, still denoted by $(A_i, \alpha_i)$, and a sequence of gauge transformation $g_i \in G_2^0$, such that $g_i \cdot (A_i, \alpha_i)$ converges in $C_k$. 
Chapter 2

An Index Theorem for Dirac Operators with Conic Singularities.

In this chapter we will prove the index theorem stated in the introduction. Since local analysis on $M$ is well understood, we first focus our attention to the tubular neighborhood of $B$. For the isolated conic situation, where the metric $g = dr^2 + r^2 g_P$ near the singular point, with $P$ being the cross section, the eigenspaces for the Dirac operator $D_P$ on $P$ defined with respect to metrics $r^2 g_P$ remain the same for different $r$. For the metric we are working with, it only collapses along the fibers, i.e., the metric is dominated by the form $g = dr^2 + r^2 \omega + \pi^* g_B$. Thus a priori the eigenspaces for $D_P$ vary with $r$. Fortunately, as analyzed in Section 2.1.3., we can find another subspaces decomposition of $L^2(S_P)$, $S_P$ is the induced spin bundle on $P$, which is independent of $r$, and allows us to use separation of variables to reduce the local analysis near $B$ to the global analysis on $P$.

2.1 Dirac Operators on Circle bundles.

2.1.1 Review of Spin Geometry.

In this section, we collect some basic facts on spin manifolds and Dirac operators. For details the reader may refer to [14] or [2].
An orientable manifold $X$ is called a spin manifold if its second Stiefel-Whitney class $w_2(X)$ is zero. Suppose $X$ is equipped with a Riemannian metric, and let $P_{SO_n}(X)$ be the bundle of oriented orthonormal tangent frames. Let $Spin_n$ denote the spin group, which is the connected two-fold covering of $SO_n$ for $n \geq 2$. A spin structure on $X$ is a principal $Spin_n$ bundle $P_{Spin_n}(X)$ together with a $Spin_n$ equivariant map $\rho : P_{Spin_n} \to P_{SO_n}(X)$ which commutes with the projection onto $X$. The condition $w_2(X) = 0$ is equivalent to the existence of a spin structure. In fact using the Cech cohomology, we can easily see that the topological obstruction cocycle to the globalization of the local two-fold covering map

$$P_{Spin_n}(X) = Spin_n \times U \to P_{SO_n}(X)|_U = SO_n \times U,$$

where $U$ is a small neighborhood, is exactly $w_2(X) \in H^2(X, \mathbb{Z}_2)$.

Let $Cl_n$ denote the Clifford algebra of $R^n$ with its standard inner product, and also denote the complexified Clifford algebra $Cl_n = Cl_n \otimes_R C$. The Spin representation of the Spin group $Spin_n$ is, by definition, the restriction of the algebra representation $\rho$ of $Cl_n$ to $Spin_n \subset Cl_n$. The Spin group $Spin_n$ has only one irreducible representation if $n$ is odd and two irreducible representations $\Delta^\pm$ if $n$ is even. These two irreducible representation $\Delta$ of $Cl_{2k}$:

$$\Delta : Cl_{2k} \to \text{End}(C^{2k}),$$

$\text{End}(C^{2k})$ is the group of endomorphisms of $C^{2k}$.

When restricted to $Spin_{2k}$, $\Delta$ breaks into two irreducible representations $\Delta^\pm$ corresponding to the $(\pm)$ eigenspace of multiplication by the complex volume form $w = i^k e_1 e_2, \cdots, e_{2k}$, where $\{e_1, \cdots, e_{2k}\}$ is the orthonormal basis of $R^{2k}$.

Suppose now that $X$ is a spin manifold of dimension $n$ and $P_{Spin_n}(X) \to P_{SO_n}(X)$ is a spin structure on $X$. Then from the spin representation $\rho$ of $Spin_n$, we can form the associated complex vector bundle

$$S(X) = P_{Spin_n}(X) \times_{\rho} V,$$
where $V$ is the representation space of $\rho$. This is called the bundle of spinors. If $n = 2k$, then the above equation breaks into two pieces:

$$S(X) = P_{Spin_n}(X) \times_\Delta V$$

$$= P_{Spin_n}(X) \times_\Delta+ V^+ \oplus P_{Spin_n}(X) \times_\Delta- V^- = S^+ \oplus S^-$$

The section of $S(X)$ are called spinors, and the sections of $S^+(S^-)$ are called positive(negative) spinors. A local section $e = \{e_1, \cdots, e_n\}$ of $P_{SO_n}(X)$ can be lifted up to $P_{Spin_n}(X)$ and then embedded into $P_{Spin_n}$ as a local section $\phi = \{\phi_1, \cdots, \phi_N\}$. This section $\phi$ is a local orthonormal basis of the bundle $S(X)$.

Let $Cl(X)$ denote the associated bundle of Clifford algebras. This is the bundle over $X$ whose fiber at each point $x$ is the Clifford algebra of the tangent space $T^*_x(X)$ with its given metric. This bundle carries a natural unitary connection $\nabla$, induced from the principal $SO_n$ bundle, and characterized by the condition that $\nabla$ acts as a derivation on the algebra of sections $\Gamma(Cl(X))$, i.e., $\nabla(\alpha \cdot \beta) = (\nabla \alpha) \cdot \beta + \alpha \cdot (\nabla \beta)$ for all $\alpha, \beta \in \Gamma(Cl(X))$, where $\cdot$ is the Clifford multiplication.

We can easily see that $S(X)$ is a bundle of modules over $Cl(X)$.

Lifting the Riemannian connection on $P_{SO_n}(X)$ to $P_{Spin_n}(X)$ via the Lie algebra isomorphism, we have an associated connection $\nabla$ on $S(X)$ whose action on the spinor basis $\phi = \{\phi_1, \cdots, \phi_N\}$ can be described as follows. Let $e = \{e_1, \cdots, e_n\}$ be a local section of $P_{SO_n}(X)$ and $\nabla^T$ the Riemannian connection on the tangent bundle $T(X)$. Suppose that $\{\omega_{ij}\}$ are the one forms defined by

$$\nabla^T e_i = \sum_{j=1}^n \omega_{ij} e_j,$$

Then

$$\nabla^s \phi_i = \frac{1}{2} \sum_{i<j} \omega_{ij} e_i e_j \cdot \phi$$

It can be also be shown that $\nabla$ acts as a derivation with respect to module multiplication, i.e.,

$$\nabla^s(\alpha \cdot \phi) = (\nabla \alpha) \cdot \phi + \alpha \cdot (\nabla \phi)$$
for all $\alpha \in \Gamma(Cl)$ and all $\phi \in \Gamma(s)$.

The Dirac operator $D : C^\infty(S) \to C^\infty(S)$ is defined by

$$D\phi = \sum_{i=1}^{n} e_i \cdot \nabla_{e_i} \phi,$$

where $\{e_1, \cdots, e_n\}$ is a local orthonormal basis on $X$ and $\phi \in C^\infty(S)$. This is a first order elliptic differential operator with symbol $\sigma_\eta(D) = \eta \cdot \eta$ for $\eta \in T^*X$.

We also include the famous Weitzenbock formula here:

**Theorem 2.1.1. (Lichnerowicz- Bochner-Weitzenbock Formula)**

$$D^2\phi = \nabla^*\nabla\phi + \frac{1}{4} s\phi,$$

where $s$ is the scalar curvature of $X$.

The Dirac operator is a formally self adjoint elliptic differential operator of first order. If the manifold $M$ is closed, then $D$ has discrete real spectrum.

### 2.1.2 Spin structures on circle bundles.

In this section, we look at spin structures on a circle bundle $P$ over a closed manifold $B$, $\dim B = 2m - 2$. A spin structure is called *extendable* if it can be extended to a spin structure on the disk bundle bounded by $P$, otherwise called *nonextendable*. Let $\zeta \in TP$ be the infinitesimal generator of the $S^1$ action on $P$, $i\omega \in i\Omega^1(P)$ is a connection 1-form on $P$ such that $d\omega \in \pi^*\Omega^2(B)$. Let $P_{SO(2m-2)}(P)$ consists those frames on $P$ having $\zeta$ as the first vector.

We will see that $S$ is *nonextendable* is equivalent to that $S$ is the pull back of a spin bundle over $B$. More precisely, pull back means the follows: let $\phi : P_{\text{Spin}}(B) \to P_{SO}(B)$ be a spin structure over $B$, then $\pi^*\phi : \pi^*P_{\text{Spin}}(N) \to \pi^*P_{SO}(N) = P_{SO}(B)$ is a $\Theta(2m-2) : \text{Spin}(2m-2) \to \text{SO}(2m-2)$ equivariant map. Extending the structure group to $\text{Spin}(2m-1)$ by

$$\tilde{\phi} : \pi^*P_{\text{Spin}}(B) \times_{\text{Spin}(2m-2)} \text{Spin}(2m-1) \to P_{SO(2m-2)} \times_{SO(2m-2)} SO(2m-1)$$
gives a spin structure on $P$, here $\tilde{\phi} := \pi^* \phi \times \Theta_{2m-2} \Theta(2m-1)$.

The $S^1$ action on $P$ induces an $S^1$ action on $P_{SO(P)}$, the orthogonal bundle on $P$. For any spin structure $\pi : P_{Spin(P)} \to P_{SO(P)}$, it is called projectable if the action lifts. Otherwise it is called nonprojectable. See [1]. The relationship between projectable spin structures and spin structures on $B$ is shown in [1], and we include it below.

Any projectable spin structure on $P$ is the pull back of some spin structure on $B$: let $\tilde{\phi} : P_{Spin(P)} \to P_{SO(P)}$ be projectable. We can identify $P_{SO(B)}$ with $P_{SO(2m-2)}(P)/S^1$. Now $\tilde{\phi}^{-1}(P_{SO(2m-2)}/S^1)$ is a $Spin$ bundle over $SO(B)$, and $\pi$ induces a corresponding $Spin$ structure on $B$.

Conversely, any spin structure on $B$ induces a projectable spin structure on $P$ via pull back as above. The corresponding spinor bundle is just the pull back of the spinor bundle on $B$. Let $S_B = S_B^+ \oplus S_B^-$ be the spinor bundle on $B$, $S_P = \pi^* S_B$ be the corresponding spin bundle on $P$.

We conclude the discussion on projectable spin structures by remarking that projectable spin structures are exactly the nonextendable ones. First let us take a look at spin structures on the circle $S^1$. There are two different structures up to isomorphism, the trivial one $C_1 = S^1 \times Spin(1)$, and the nontrivial one $C_2 = ([0, 2\pi] \times Spin(1))/\sim$, where $\sim$ identifies $0$ and $2\pi$ while interchanges the two elements of $Spin(1)$. Let $D$ be the disk with $S^1$ as its boundary. Since the disk is simply connected it can only have one spin structure. Thus only one spin structure on $S^1$ extends to the disk. The tangent vector to the boundary $S^1$ together with the outer unit normal vector forms a orthogonal frame which is a loop in the frame bundle on the disk, whose lift to the spin bundle does not close up. Thus the induced spin structure on the boundary $S^1$ is the nontrivial one. So the nontrivial spin structure on $S^1$ extends to the disk, and the trivial one does not extend.

Now given a projectable spin structure on $P$, from the previous discussion, it is the pull back of some spin structure. Hence when restricted to any fiber $S^1$, it induces a trivial spin structure, which does not extend to the corresponding disk. Thus the projectable spin structures do not extend to the disk bundles. On the other hand, if a spin structure $S$ on $P$ does not extend, then when restricted to the fiber, it does
not extend neither. Thus restricted to any chart \( U \times S^1 \subset P \), with \( U \subset B \) being a small ball, \( S \) is isomorphic to the trivial spin structure, and the \( S^1 \) action lifts, so \( S \) is projectable.

### 2.1.3 \( \eta \) invariants of the perturbed Dirac operators on circle bundles.

In this section we consider the Dirac operator for the nonextendable spin bundle \( S \) on a circle bundle \( P \) over a spin manifold \( B \). From the last section we know that \( S \) is the pull back of some spin bundle \( S_B \) over \( B \), i.e., \( S = \pi^* S_B \). We will calculate the eta function of the perturbed Dirac operator, which is going to be needed later for the index calculation. Recall that \( L \) is the line bundle associated to \( P \).

Define the metric \( g_r \) on \( P \) by \( g_r := r^2 \omega \otimes \omega + \pi^* g_B \), \( g_B \) is a Riemannian metric on \( B \). Let \( A_r \) be the Dirac operator on \( S_P \) defined with respect to the metric \( g_r \). The action of \( S^1 \) on \( P_{\text{sol}(2m-1)}(P) \) can be lifted to \( P_{\text{spin}(2m-1)}(P) \), and it induces an isometric action on \( L^2(S_P) \), where the inner product on \( L^2(S_P) \) is defined with respect to the metric \( g_1 = \omega \otimes \omega + \pi^* g_B \). Let \( L_\xi \) be the differential of the representation the Lie group \( S^1 \) on \( L^2(S_P) \), we have the following decomposition:

\[
L^2(S_P) = \bigoplus_{k \in \mathbb{Z}} V_k,
\]

\( V_k \) is the eigenspace of \( L_\xi \) with eigenvalue \( ik \), \( k \in \mathbb{Z} \).

In section 4 of [1], it is proved that there is an isometry \( Q_k \) (which preserves the splitting of the bundle) from \( L^2(S_B \otimes L^{-k}) \) to \( V_k \), such that

\[
A_r = \frac{1}{r} A_v + A_h - \frac{r}{4} \gamma \left( \frac{\xi}{r} \right) \gamma(\pi^* d\omega).
\]

with \( A_v = \gamma \left( \frac{\xi}{r} \right) L_\xi \), \( A_h \mid_{V_k} = Q_k \circ A_k \circ Q_k^{-1} \), \( A_k \) being the twisted Dirac operator on \( L^2(S_B \otimes L^{-k}) \), and \( \gamma \) is the Clifford multiplication.

In the rest of this section we will consider the perturbed Dirac operator defined by \( A_r = \frac{1}{r} A_v + A_h \), whose spectrum is much easier to analyze, and we will calculate
the eta function in this section. This perturbed operator also plays an essential role in the next section when we are trying to analyze the Dirac operator on $M \setminus B$, the main reason is that the perturbed term is of order $r$, which is not going to affect the index we want, but make the analysis much easier.

Now with respect to the bundle splitting $S_P = \pi^* S_B^+ \oplus \pi^* S_B^-$, the Clifford multiplication $\gamma(\frac{1}{r} \zeta)$ is given by

$$
\begin{pmatrix}
-i & 0 \\
0 & i
\end{pmatrix},
$$

and the horizontal part $A_h$ is

$$
A_h = \begin{pmatrix}
0 & A_h^- \\
A_h^+ & 0
\end{pmatrix}
$$

So $\gamma(\frac{1}{r})$ anticommutes with $A_h$. Let $\phi_{k,a} = \alpha \oplus \beta \in L^2(S_P)$ be a unit norm common eigenspinor of $L_\zeta$ and $A_h$ for eigenvalues $ik$ and $a$ respectively.

If $a \neq 0$, $k \neq 0$, then $\alpha \neq 0, \beta \neq 0$, and $\alpha \oplus 0, 0 \oplus \beta$ are linearly independent and using them as a basis for the two dimensional space they span, $A_r$ is represented by the matrix

$$
A_{k,a} = \begin{pmatrix}
k/r & a \\
a & -k/r
\end{pmatrix}
$$

If $a = 0$, let $V_{k,0}$ be the space spanned by the common eigenspinors of $L_\zeta$ and $A_h$ for eigenvalues $ik$ and $0$ respectively, then for $\phi = \alpha \oplus \beta \in V_{k,0}$, $\alpha$ is the eigenvector of $A_r$ with eigenvalue $k$ which has multiplicity $\text{dim Ker}(A_k|_{S_B^+ \otimes L^{-k}})$, while $\beta$ is eigenvector of $A_r$ with eigenvalue $-k$ of multiplicity $\text{dim Ker}(A_k|_{S_B^- \otimes L^{-k}})$.

If $k = 0$, $V_0$ is isometric to $L^2(S_B)$, and $A_r = A_v + A_h = D_B$ on $V_0$, $V_0 = \oplus V_{0,a}$, $a$ runs through all the eigenvalues of $D_B$.

To sum up, $L^2(S_P)$ can be decomposed into direct sum of
$$L^2(S_P) = \bigoplus_{k \neq 0, a \neq 0} (V_{k,a}) \oplus V_{k,0} \oplus V_0$$  \hspace{1cm} (2.1.1)$$

$$= H_1 \oplus H_2$$

$H_1$ is the sum of the first two blocks, and $H_2$ is the third. Accordingly, $A_r$ can be decomposed into

$$A_r = \bigoplus A_{k,a} \oplus \frac{k}{r} \oplus D_B$$  \hspace{1cm} (2.1.2)$$

and $\frac{k}{r}$ has multiplicity $\dim(Ker(A_k|_{S_B^+ \otimes L^k})) + \dim(Ker(A_{-k}|_{S_B^- \otimes L^{-k}}))$.

Remark: The Hilbert spaces $H_1, H_2$ are independent of $r$.

Due to the decomposition above, we can calculate the eta function $\eta_{A_r}(s)$ of $A_r$. By definition, $\eta_{A_r}(s) = \sum_{\lambda > 0} \frac{m_{\lambda} - m_{-\lambda}}{\lambda^s}$, $m_{\lambda}$ is the multiplicity of $\lambda$. On $V_{k,a}$, $A_r$ has $\pm \sqrt{k^2 + a^2}$ as eigenvalues with same multiplicities, so they have no contributions to $\eta_{A_r}(s)$. On $V_0$, since $B$ is even dimensional, $\pm a$ has same multiplicity, so no contributions to the eta function. On $V_{k,0}$, we have

$$m_k = \dim(Ker(A_k|_{S_B^+ \otimes L^k})) + \dim(Ker(A_{-k}|_{S_B^- \otimes L^{-k}})).$$

$$m_{-k} = \dim Ker(A_k|_{S_B^- \otimes L^{-k}}) + \dim Ker(A_{-k}|_{S_B^+ \otimes L^k}),$$

so

$$\eta_{A_r}(s) = \sum \frac{m_k - m_{-k}}{k^s}$$  \hspace{1cm} (2.1.3)$$

By Atiyah-Singer index theorem, $m_k - m_{-k} = - \int_B \hat{A}(B)(ch(L^k) - ch(L^{-k}))$. Define

$$f_s(x) = - \sum_{k \geq 1} \frac{e^{kx} - e^{-kx}}{k^s} = -2 \sum_{i \geq 1} \frac{\zeta(s - (2i - 1))x^{2i-1}}{(2i - 1)!}.$$ 

Here $\zeta(s)$ is the Riemann-Zeta function. Noticing that $\zeta(-(2n - 1)) = \frac{(-1)^{n-1}B_n}{2n}$, with $B_n$ being the $n$-th Bernoulli number, and the Taylor expansion

$$\frac{1}{2} \coth \frac{x}{2} = \sum_{i \geq 0} \frac{(-1)^i B_i}{(2i)!} x^{2i-1},$$
we have that
\[ f_s(x) \big|_{s=0} = -2(\frac{1}{2} \coth \frac{x}{2} - \frac{1}{x}). \] (2.1.4)

Hence
\[ \eta_A(s) = \int_B \hat{A}(B) f_s(e). \]

and the eta invariant of \( A \) is
\[ \eta_A(0) = -2 \int_B \hat{A}(B)(\frac{1}{2} \coth \frac{e}{2} - \frac{1}{e}). \]

Here \( e \) is again the Euler class of the normal bundle on \( B \). Since \( \zeta(s) \) has the only simple pole at 1 with residue 1, \( \eta_A(s) \) has simple poles only at 2, 4, 6,.. with residue 1. When \( B \) is a Riemann surface, this eta function has been calculated in [18].

### 2.2 An Index Theorem for Dirac Operators on Complements of the Submanifolds.

In this chapter, we will prove our index theorem for the Dirac operators on the complements of the codimension 2 submanifolds.

#### 2.2.1 The Dirac operator on the cone.

Let \( C(P) = (0,1) \times P \) be the finite cone on \( P \) with metric \( g = dr^2 + g_r \), \( g_r \) is the metric on \( P \) defined in the last section. \( \partial_r \) denote the unit radial tangent vector field. Our orientation convention is that if \( e_1,..,e_{2m-1} \) is an oriented orthonormal frame on \( P \), then \( \partial_r, e_1,..,e_{2m-1} \) will give an orientation on \( C(P) \). Let \( S^\pm_{C(P)} \) be the positive(negative) spinor bundle on \( C(P) \), and let \( D^+ \) denote the Dirac operator on \( S^+_{C(P)} \). \( \gamma(\partial_r) : S^+ \rightarrow S^- \) gives an identification of \( S^+ \) and \( S^- \) as spinor bundles when restricted to \( P \). In particular \( S^+ \) gives a spin bundle on \( P \) with Clifford multiplication defined as
\[ \gamma(e_i) = \gamma(\partial_r)^{-1} \gamma_{C(P)}(e_i), i = 1,..,2m-2. \]
The Dirac operator $D^+$ and $A_r$ is related in the following way (cf [12]):

$$D^+\phi = \gamma(\partial_r)(\partial_r\phi - \frac{H}{2}\phi + A_r\phi)$$  \quad (2.2.1)

$H$ is the mean curvature of $\{r\} \times P$ in $C(P)$, and can be computed as follows: let $\zeta_r, v_1, \cdots, v_{2m-1}$ be an oriented orthonormal frame on $P$, such that $v_i = \pi^*w_i$, $\{w_i\}$ is an oriented orthonormal frame on $B$. Obviously $[\partial_r, v_i] = 0$, and $[\partial_r, \zeta_r] = -\frac{\zeta_r}{r}$. Thus $<\nabla_{\zeta_r} \partial_r, \zeta_r> = -\frac{\zeta_r}{r^2} + \nabla_{\partial_r} \zeta_r, \zeta_r > = -\frac{1}{r}$, and $<\nabla_{v_i} \partial_r, v_i> = <\nabla_{\partial_r} v_i, v_i> = 0$, we have $H = -\frac{1}{r}$.

So (2.2.1) becomes

$$D^+\phi = \gamma(\partial_r)(\partial_r\phi + \frac{1}{2r}\phi + A_r\phi)$$  \quad (2.2.2)

For any cross section $\phi$ in $S^{+}_{C(P)} |_{\{r\} \times P}$, we can extend it to a global section on $S^{+}_{C(P)}$ by parallel transport along radial geodesics, and still denote it by $\phi$. So $H_1, H_2$ can be extended to $L^2(S^{+}_{C(P)})$, and

$$L^2(S^{+}_{C(P)}) = (H_1 \oplus H_2) \otimes L^2((0, 1), r dr).$$  \quad (2.2.3)

Define $\text{Dom}(D_{\text{max}}) = \{\phi \in L^2, D^+\phi \in L^2\}$.

In this section, we will prove the following:

**Lemma 2.2.1.** There exists a bounded linear operator $R : r^{\frac{1}{2}}\text{Dom}(D_{\text{max}}) \rightarrow \pi^*L^2_{-\frac{1}{2}}(B; S_B)$. The range is $\pi^*(L^2_{-\frac{1}{2}} \cap \pi^+H) \oplus \pi^*(L^2_{\frac{1}{2}} \cap \pi^-H)$. Here $r$ is the distance function to $B$, which is extended to the entire $M$ by 1, $H^\pm$ are the spectral subspaces associated to $D_B$.

We first focus our attention to APS boundary condition on $B$. The show that the general case follows easily from this special one. We still use $W^+$ to denote the kernel of the following composition map:

$$r^{\frac{1}{2}}\text{Dom}(D_{\text{max}}) \xrightarrow{R} L^2_{-\frac{1}{2}}(S_B) \xrightarrow{P_B} L^2_{-\frac{1}{2}}(S_B).$$
Recall that $P_0$ is the projection map to the nonnegative eigenspaces of $D_B$. In this section, we prove the following:

**Theorem 2.2.2.** With domain $W^+$, $D^+$ is Fredholm.

Define $D_0^+ \phi = \gamma(\partial_r)(\partial_r \phi + \frac{1}{2r} \phi + A_r \phi)$, which is just a perturbation of $D^+$ of order $O(r)$ near $B$. We will show later in this chapter that the Fredholm property of $D_0^+$ will guarantee the Fredholm property of $D^+$, and the two has the same index.

### 2.2.2 Construction of a boundary parametrix.

On $C(P) = (0, 1) \times P$, any spin section can be written as the direct sum of three different parts as in (2.1.1),

$$\psi = \otimes \psi_{k,a} \oplus \psi_{k,0} \oplus \psi_{0,a} = \psi_1 \oplus \psi_2$$

we will construct the boundary parametrix according to the decomposition.

First, we want to solve $D_0^+ \phi = \psi$, for $\phi, \psi \in L^2(S^+_C(P))$ then construct the boundary parametrix of $D_0^+$ based on the three components of $\psi$.

**Case I:**

Given $k \neq 0, a \neq 0$, $D_{k,a}^+ = \partial_r + \frac{1}{2r} + A_{k,a}$. It has two fundamental solutions

$$\phi_1(r) = |a|^{\frac{1}{2}} \left( \begin{array}{c} K_{\nu_+}(|a| r) \\ K_{\nu_-}(|a| r) \end{array} \right), \quad \phi_2(r) = |a|^{\frac{1}{2}} \left( \begin{array}{c} -I_{\nu_+}(|a| r) \\ I_{\nu_-}(|a| r) \end{array} \right)$$

with $\nu_{\pm} = |k \pm \frac{1}{2}|$, and $K_{\nu}$, $I_{\nu}$ are generalized Bessel functions with order $\nu$, and their asymptotic behavior has been studied extensively in the literature. The following two specific formulas for $K_{\nu}$ and $I_{\nu}$

$$K_{\nu}(z) = \frac{\Gamma\left(\frac{1}{2} + \nu\right)\left(\frac{1}{2}z\right)^{\nu}}{\sqrt{\pi}} \int_0^\infty \frac{\cos dt}{(t^2 + z^2)^{\nu + \frac{1}{2}}}$$

$$I_{\nu}(z) = \left(\frac{1}{2}z\right)^\nu \sum_{k=0}^\infty \frac{(1/4z)^k}{k\Gamma(\nu + k + 1)}.$$
give the estimates for $K_{\nu}(r)$ and $I_{\nu}(r)$ when $r$ is small:

\[ K_{\nu}(r) \leq C\frac{1}{2} \Gamma(\nu)\left(\frac{1}{2}r\right)^{-\nu} \leq Cr^{-\nu} \]

\[ I_{\nu}(r) \leq C\left(\frac{1}{2}r\right)^{\nu} / \Gamma(\nu + 1) \leq Cr^\nu \]

for some constant $C$ independent of $\nu$. The Wronskian for the fundamental system is

\[
W(\phi_1(r), \phi_2(r)) = \det \begin{pmatrix} \phi_1 & \phi_2 \\ \phi_2 & \phi_1 \end{pmatrix} = |a|(K_{\nu+}\nu_+ + K_{\nu-}\nu_-)
\]

\[
= \frac{1}{r}
\]

With the given fundamental system $\phi_1(r), \phi_2(r)$, the solution of the inhomogeneous equation $D_{k,a}\phi = \psi$ can be calculated by the method of variation of constants. Since $\phi \in L^2((0,1), rdr \otimes \mathbb{C}^2)$,

\[
\phi_{k,a} = c\phi_2(r) + c_1(r)\phi_1(r) + c_2(r)\phi_2(r)
\]

where $c$ is constant, and

\[
c_1(r) = \int_0^r W(\phi_1(s), \phi_2(s))^{-1} W(\psi(s), \phi_2(s)) ds
\]

\[
= \int_0^r W(\psi(s), \phi_2(s)) s ds
\]

\[
c_2(r) = -\int_r^1 W(\phi_1(s), \phi_2(s))^{-1} W(\phi_1(s), \psi(s)) ds
\]

\[
= -\int_r^1 W(\phi_1, \psi) s ds
\]

Define

\[
Q_{k,a}\psi = c_1(r)\phi_1(r) + c_2(r)\phi_2(r).
\]

**Case II:**

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For $k \geq 1$, $D^+_{k,0} = \partial_r + \frac{1}{2r} + \frac{k}{r}$, define
\[
Q_{k,0} \psi(r) = r^{-(k+\frac{1}{2})} \int_0^r s^{k+\frac{1}{2}} \psi(s) ds, \quad k \geq 1 \quad (2.2.8)
\]
For $k \leq -1$, define
\[
Q_{k,0} \psi(r) = -r^{-(k+\frac{1}{2})} \int_r^1 s^{k+\frac{1}{2}} \psi(s) ds, \quad k \leq -1 \quad (2.2.9)
\]

**Case III:** When $k = 0$, $T_{0,a}^+ = \partial_r + \frac{1}{2r} + a$,
\[
Q_{0,a} f = r^{-\frac{1}{2}} \int_0^r e^{a(s-r)} s^{\frac{1}{2}} f(s) ds \quad a \geq 0 \quad (2.2.10)
\]
\[
Q_{0,a} f = -r^{-\frac{1}{2}} \int_r^1 e^{a(s-r)} s^{\frac{1}{2}} f(s) ds \quad a < 0 \quad (2.2.11)
\]

Formally, we write our boundary parametrix to be
\[
Q = \oplus Q_{k,a} \oplus Q_{k,0} \oplus Q_{0,a}.
\]

The following lemma makes sure that $Q$ is well defined:

**Lemma 2.2.3.** For $\psi \in L^2(S_C(P))$, there exists a constant $C$ independent of $k$, $a$, such that

1. $\| Q_{k,0} \psi_{k,0} \| (r) \leq (2k + 2)^{-\frac{1}{2}} \| \psi_{k,0} \|_{L^2((0,r),rdr)}$, $k \geq 1 \quad \text{or} \quad k \leq -1$.
2. $\| Q_{0,a} \psi_{0,a} \| (r) \leq C \| \psi_{0,a} \|_{L^2((0,1),rdr)}$.
3. $\| Q_{k,a} \psi_{k,a} \| (r) \leq C \| \psi_{k,a} \|_{L^2}$.

**Proof.** The first part follows from the Cauchy Schwartz inequality, so does the second part when $a = 0$. For part 3, when $|a| \leq 1$, by using the asymptotic behavior of $K_\nu(r)$ and $I_\nu(r)$ near $r$, Cauchy Schwartz inequality gives that $\| Q_{k,a} \psi_{k,a} \| (r) \leq |a|(2k + 2)^{-\frac{1}{2}} \| \psi_{k,a} \|_{L^2((0,1),rdr)}$

When $|a| \geq 1$, we can use the Hankel transform to get the desired bound:
Let \( \psi = \begin{pmatrix} h \\ l \end{pmatrix} \in C_0^\infty((0, 1), r dr \otimes C^2) \), \( \phi = Q_{k,a} \psi = \begin{pmatrix} f \\ g \end{pmatrix} \). Since

\[
(\partial_r + \frac{1}{2r} + \begin{pmatrix} k/r & a \\ a & -k/r \end{pmatrix}) \phi = \psi
\]  
(2.2.12)

We have

\[
(-\partial_r^2 + \frac{k^2 - k + \frac{1}{4}}{r^2} - \frac{1}{r} \partial_r)g + a^2g = -\left(\partial_r - \frac{k + \frac{1}{2}}{r}\right)h + al.
\]  
(2.2.13)

Now recall that for \( g \in C_0^\infty(R_+) \), Hankel transform is defined by

\[
H_\nu(g)(\lambda) = \int_0^\infty J_\nu(\lambda r)g(r)r dr,
\]
where \( J_\nu \) is the Bessel function of order \( \nu \) and \( \nu > -1 \). Also we have the Plancherel equality:

\[
\int_0^\infty |g(r)|^2 r dr = \int_0^\infty |H_\nu(g)(\lambda)|^2 \lambda d\lambda.
\]

If we denote \( \Delta_{\nu_\pm} = -\partial_r^2 + \frac{k^2 \pm k + \frac{1}{4}}{r^2} - \frac{1}{r} \partial_r \), then

\[
H_{\nu_\pm}(\Delta_{\nu_\pm}g) = \lambda^2 H_{\nu_\pm}(g).
\]

\[
H_\nu(\partial_r - \frac{\nu + 1}{r}) = \lambda H_{\nu + 1}
\]

From the above facts, and applying \( H_{\nu_-} \) to (2.2.14), we get

\[
\lambda^2 H_{\nu_-}(g) + a^2 H_{\nu_-}(g) = \lambda H_{\nu_+}(h) + a H_{\nu_-}(l)
\]

\[
(\lambda + ia)H_{\nu_-}(g) = \frac{\lambda H_{\nu_-}(h)}{\lambda + ia} + \frac{a H_{\nu_-}(l)}{\lambda + ia}.
\]

So

\[
(\lambda^2 + a^2) \|g\|_{L^2(R_+, r dr)}^2 \leq \|h\|_{L^2(R_+, r dr)}^2 + \|l\|_{L^2(R_+, r dr)}^2.
\]
The same inequality holds for \( f \). So

\[
\| a \phi \|_{L^2} \leq \| \psi \|_{L^2(R^+ \times (r, dr))}.
\]

and \( \| \phi \|_{L^2} \leq \frac{1}{2c} \| \psi \|_{L^2} \).

The above argument also applies to the second part when \( a \neq 0 \), by letting \( k = 0 \) above and noting that \( |a| \) has a lower bound now since \( a \) is the eigenvalue for the Dirac operator \( D_B \), which is a self-adjoint elliptic operator on closed manifold \( B \).

\[ \square \]

With all these estimates at hand, we are ready to prove Theorem 2.2.2.

**Proof of lemma 2.2.1:** In the tubular neighborhood, write \( \phi = \phi_{k} \oplus \phi_{0} = \phi_{1} \oplus \phi_{2} \) as usual. From Lemma 2.2.3 and Lemma 2.2.4, we see that \( |r^{\frac{1}{2}} \phi_{1}|_{L^2(S_{p, g_1})} \) goes to zero as \( r \) goes to 0. As for \( \phi_{2} \), we have \( \psi(r) = r^{\frac{1}{2}} \phi_{2} \in L^2((0, 1), V_0), \partial_{r} \psi(r) + D_{B} \psi(r) = r^{\frac{1}{2}} D_{0}^{+} \phi_{2} \in L^2((0, 1), V_0) \), so up to the isometry \( Q \) in Section 2.1.2, \( \psi \in L^2_{1}(S_{B \times (0, 1)}) \), here \( S_{B \times (0, 1)} = \pi^{*}S_{B} \) is the pull back bundle over \( B \times (0, 1) \) from \( S_{B} \), with \( \pi : B \times (0, 1) \to B \), and \( \partial_{r} + D_{B} \) is an elliptic operator on \( S_{B \times (0, 1)} \). Now from the following lemma, we have the restriction map \( R \) with the corresponding range as claimed. Q. E. D.

Let \( X \) be a compact manifold with boundary \( \partial X \). Let \( D : C^\infty(E) \to C^\infty(E) \) be a first order elliptic operator. Near the boundary, \( D \) has the form

\[
D = \partial_{t} + B,
\]

here \( B \) is a first order elliptic self-adjoint operator on \( E | \partial X \). And we write a collar neighborhood of \( \partial X \) as \((-1, 0] \times \partial X \). Define \( D_{\text{max}} = \{ \phi \in L^2 \mid D\phi \in L^2 \} \), and let \( D_{\text{min}} \) denote the closure of smooth section which vanish on \( \partial X \) with respect to the \( L^2_{1} \) norm. Let \( H^\pm \subset L^2 \) be the positive and nonnegative spectrum of \( B \). Then we have the following lemma, which is due to Furutani and Booss.

**Lemma 2.2.4.** The restriction of smooth sections extends to define a bounded linear
operator

\[ R : D_{\text{max}} \to L_{-\frac{1}{2}}^2(Y; E |_{\gamma}). \]

The kernel of this map is \( D_{\text{min}} \) and the range is \( L_{\frac{1}{2}}^2 \cap H^- \oplus L_{-\frac{1}{2}}^2 \cap H^+ \).

Proof. Let \( D^* \) denote the adjoint of \( D \). Then integration by part gives

\[ \int_X < D\phi, \psi > - < \phi, D^* \psi > = \int_{\partial X} < \phi, \psi > . \]

Thus for any \( \hat{\psi} \in L_{\frac{1}{2}}^2(\partial X, E \mid \partial X) \), by Sobolev restriction we have that there is a \( \hat{\psi} \in L_1^2(X, E) \) extending \( \hat{\psi} \) and the extension can be chosen so as to define a bounded linear operator. Applying the equality to \( \phi \in D_{\text{max}} \) and \( \hat{\psi} \) we see that the left hand side of the equality is well defined and thus defines the right hand side. Thus the restriction of \( \phi \in D_{\text{max}} \) lies in \( L_{-\frac{1}{2}}^2(\partial X, E \mid \partial X) \).

From APS we know that if \( \phi \in D_{\text{max}} \) and \( R(\phi) \in H_{-\frac{1}{2}}^- \), then \( R(\phi) \in H_{\frac{1}{2}}^- \).

If \( \phi \in H_{-\frac{1}{2}}^+ \) then

\[ \beta(t)e^{Bt} \hat{\phi} = \beta(t) \sum_{\lambda > 0} e^{\lambda t} \hat{\phi}_\lambda \in D_{\text{max}}, \]

here we view the above a section on \((-1, 0] \times \partial X \) and \( \beta \) is a cutoff function which is 1 near \( t = 0 \) and zero near \( t = -1 \).

Remark 2.2.5. For any \( \phi \in W^+ \), \( \phi = \phi_0 + r^{-\frac{1}{2}} e(\psi_B) \), such that \( \phi_0 \in L_1^2(S^+, M\setminus B) \), \( \psi_B \in L_1^2(S_B) \), and \( e(\psi_B) \) is the \( L_1^2 \) extension of \( \psi_B \) to \( M\setminus B \). This fact can be seen easily from the proof of Lemma 2.2.1: since we are projecting to the nonnegative eigenspaces, we know that \( R(r^\frac{1}{2} \phi) \in H_{\frac{1}{2}}^- \) for any \( \phi \in W^+ \), so the extension map is well defined.

The estimates also give us the following

Lemma 2.2.6.

\[ Q : C_0^\infty((0, 1), H) \to C^\infty((0, 1), H; P) \]

is a linear operator such that

\[ D_0^+ Q = Id \]
\[ QD_0^+ \phi = \phi, \text{ for } \phi \in C^\infty(R_+, H; P), \phi(r) = 0, \quad r \geq 1, \]

and there exists constant \( C \) such that

\[ \| Q \psi \|_{L^2((0,1), rdr \otimes H)} \leq C \| \psi \|_{L^2((0,1), rdr \otimes H)}. \]

**Proof.** \( D_0^+ Q = Id \) follows from the construction of \( Q \). Let \( \phi = \oplus \phi_{k,a} \oplus \phi_{k,0} \oplus \phi_{0,a} \), \( \psi = D_0^+ \phi \). Then \( \psi_{k,0} = (\partial_r + \frac{1}{2r} + \frac{k}{r})\phi_{k,0} \). If \( k \leq -1 \), from the construction, \( (Q\psi_{k,0})(1) = 0 \), on the other hand, \( \phi_{k,0}(1) = 0 \) from the assumption, thus \( Q\psi_{k,0} = \phi_{k,0} \) since they both solve the same ODE. If \( k \geq 1 \),

\[ Q\psi_{k,0} = r^{-(k+\frac{1}{2})} \int_0^r s^{k+\frac{1}{2}} \psi_{k,0}(s) ds - r^{-(k+\frac{1}{2})} \int_1^r s^{k+\frac{1}{2}} \psi_{k,0}(s) ds. \]

But \( Q\psi_{k,0} \in L^2((0,1), rdr) \), so \( c = 0 \), thus \( Q\psi_{k,0}(1) = 0 \), and \( Q\psi_{k,0} = \phi_{k,0} \). The same argument applies to \( \phi_{k,a} \) and \( \phi_{0,a} \), thus if \( \phi(1) = 0 \), \( QD_0^+ \phi = \phi \).

\[ \square \]

Let \( 0 < \epsilon \leq 1 \), then for any bump function \( 0 \leq \chi \leq 1 \), such that \( \chi = 1 \) for \( 0 < r < \frac{\epsilon}{2} \), and \( \chi = 0 \) for \( r > \epsilon \), we have the following:

**Lemma 2.2.7.** Let \( Q \) acting in \( L^2((0,1), rdr) \), then there exists a constant such that

\[ \| \chi Q \| + \| Q \chi \| \leq C \epsilon. \]

Here \( \epsilon, \chi \) are as above.

**Proof.** From the lemma above, \( \| Q \| \leq C \). Using Cauchy Schwartz inequality, \( \| \chi Q \| \leq \| \chi \|_{L^2((0,1))} \| Q \| \leq C \epsilon. \) Similarly for \( Q \chi \). Here the norm means the \( L^2 \) operator norm.

\[ \square \]

**Proof of Theorem 2.2.2** To show that \( D^+ \) is Fredholm, we will use our boundary parametrix to construct a global right and left parametrices. Choose \( \phi, \bar{\phi} \in C_0^\infty(-\epsilon, \epsilon) \) such that \( \phi = 1 \) near 0 and \( \bar{\phi} = 1 \) near \( \text{supp} \phi \), choose \( \psi, \bar{\psi} \in C_0^\infty(\mathbb{X}) \) such that \( \phi + \psi = 1 \) and \( \bar{\psi} = 1 \) in a neighborhood of \( \text{supp} \psi \). Let \( Q_i \) be an interior parametrix
for $D^+$ with

$$D\bar{\psi}Q_i\psi = \psi + R_i \quad (2.2.14)$$

$$\bar{\psi}Q_i\psi D = \psi + L_i \quad (2.2.15)$$

with $R_i, L_i$ compact smoothing operators in $L^2(S^+), L^2(S^-)$, respectively. Define

$$R := \bar{\phi}Q\phi + \bar{\psi}Q_i\psi \quad (2.2.16)$$

From Lemma 3.6, we know that

$$R : L^2(S^-) \rightarrow Dom(D^+) \quad (2.2.17)$$

Now

$$DR = I + \bar{\phi}Q\phi + \frac{n}{2}\bar{\phi}rQ\phi + R_i \quad (2.2.18)$$

When the support of $\phi$ is sufficiently small, from Lemma 2.2.4, we have

$$\| S = \bar{\phi}Q\phi + \frac{n}{2}\bar{\phi}rQ\phi \| < \frac{1}{2}$$

and we can write

$$DR = I + S_1 + R_i$$

since $R_i$ is compact and

$$\| S_1 \| < \frac{1}{2}.$$ 

This implies

$$DR(I + S_1)^{-1} = I + R_i(I + S_1)^{-1},$$

Next we can see that with Lemma 2.2.7.,

$$RD = \bar{\phi}Q\phi T + \psi + \frac{n}{2}\bar{\phi}rQ\phi + L_i \quad (2.2.19)$$

$$= I + \bar{\phi}Q\phi^{'} T + \bar{\phi}Q\phi + L_i$$
and again, we have
\[ \| S_2 = \bar{\phi}Q_2^r \phi + \bar{\phi}Q\phi' \| < \frac{1}{2}, \]
and
\[ RD = I + S_2 + L_i \]
\[ (I + S_2)^{-1}RD = I + (I + S_2)^{-1}L_i. \]

Thus we find operators \( A_R \) and \( A_L \) such that \( DA_R = I + D_R \) and \( A_L D = I + D_L \), \( D_R \) and \( D_L \) are compact operators. Notice that \( A_L DA_R = A_R + D_L A_R \), on the other hand \( A_L DA_R = A_L + A_L D_L \), we have \( C = A_R - A_L = A_L D_R - A_R D_L \) is compact, \( A_R D = (A_L + C)D = I + CD + D_R \). So we find a bounded operator \( A_R : L^2 \to W^+ \) such that \( A_R D - I \) and \( DA_R - I \) are compact respectively, which says \( D \) is Fredholm.

Q. E. D.

Lemma 2.2.8. Let \( g \) be a bounded smooth function on \( M \setminus B \), then
\[ \bar{D}^+ := D^+ + g \]
is a Fredholm operator on \( W^+ \), and
\[ \text{Ind}(\bar{D}^+) = \text{Ind}(D^+). \]

Proof. Define \( D_t^+ = D^+ + tg, t \in [0, 1] \). Let \( \chi \) be the bump function in Lemma 2.2.5., then \( \| t\chi gQ \| + \| Qt\chi g \| \leq C\epsilon \) from the Cauchy Schwartz inequality. Thus we can still use \( Q \) to construct a boundary parametrix of \( D_t^+ \) to prove that \( D_t^+ \) is Fredholm in \( W^+ \), just as in Theorem 2.2.2. And \( \text{Ind}(\bar{D}^+) = \text{Ind}(D_1^+) = \text{Ind}(D_0^+) = \text{Ind}(D^+) \) follows immediately.

Remark: we conclude this section by remarking that instead of \( L^2 \), \( D^+ \) is also Fredholm with the same index on the following domain: \( \{ \phi \in L^p, D^+\phi \in L^p, P(\phi) = 0 \} \), for some \( p > 2 \). To see this, first observe that on this domain, \( D^+ \) has finite dimensional kernel since \( p > 2 \) and \( D^+ \) has finite kernel on \( W^+ \). Next, for a sequence
$D^+ \phi \rightarrow \psi$ in $L^p$, we have $D^+ \phi$ converges in $L^2$, thus there exists $\phi \in W^+$, such that $D^+ \phi = \psi$. Since $\psi \in L^p$, we know that $\phi \in L^p_{\text{loc}}$. Moreover $\phi \in L^p(U)$, here $U$ is the tubular neighborhood of $\Sigma$, from Remark 2.2.5., we can write $\phi = \phi_0 + r^{-\frac{1}{2}}e(\psi_B)$, $\phi_0 \in L^p$ since $\phi_0 \in L^2_\Sigma$, also $e(\psi_B) \in L^2_\Sigma(S^+, M\setminus B)$, from Sobolev multiplication we have $r^{-\frac{1}{2}}e(\psi_B) \in L^p$, thus $\phi \in L^p$, and $D^+$ has closed image. Finally, $D^+$ has finite dimensional cokernel: let $\psi \in L^q_\Sigma$ lie in the cokernel of $D^+$, with $\frac{1}{p} + \frac{1}{q} = 1$. From local regularity we have $\psi \in C^\infty(S^\Sigma)$, and $D^-\psi = 0$. By solving $D^-\psi = 0$ in $U$ and from all the estimates above, we see that $\psi \in L^2(U)$, thus $\psi \in L^2$, so $\psi$ lies in the cokernel of $W^+$. Thus $D^+$ is Fredholm on the above domain.

2.3 Index of $D^+_{M\setminus B}$.

The index of $D^+$ will be calculated in this section. We first consider the case when the metric near $B$ is of the form $g_0 = dr^2 + r^2 \omega \otimes \omega + \pi^*g_B$, then show that the Dirac operator associated with any general metric is just a bounded perturbation of the one with $g_0$, and the index remains equal from lemma 2.2.8. Also we will first focus on the APS boundary conditions, the general case follows immediately.

Let

$$\Delta^+ = D^-D^+, \Delta^- = D^+D^-.$$ 

Since in the tubular neighborhood the operator is a combination of singular regular operator and product operator, from [5], we know that $(\Delta^\pm + \lambda)^{-m}$ are trace class for large $m$. And the nonzero eigenvalues of $\Delta^+$ and $\Delta^-$ coincide. Thus

$$tr(\Delta^+ + \lambda)^{-m} - tr(\Delta^- + \lambda)^{-m} = \lambda^{-m}indD^+.$$ 

To get the expansion of $(\Delta^\pm + \lambda)^{-m}$, we will construct a parametrix for $(\Delta^\pm + \lambda)^{-m}$. As usual, we construct a boundary parametrix, and then patch it with the canonical interior one. Let $P_{i}^\pm$ be the interior pseudodifferential parametrix for $(\Delta^\pm + \lambda)^{-m}$. 

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In the tubular neighborhood $C_{0,1}(P)$, $\text{tr}P^\pm_i$ has expansion

$$\text{tr}P^\pm_i(r, r; x, x; \lambda) r d\lambda = \sum_j p_j^\pm(r, x) \lambda^{-\frac{j}{2}} r d\lambda, \quad r \in (0, 1), \quad x \in P.$$  \hspace{1cm} (2.3.1)

Recall that we have the decomposition (2.1.1):

$$L^2(S_P) = \bigoplus (V_{k,a}) \oplus_{k \in \mathbb{Z}} V_{k,0} \oplus_{a \neq 0} (V_{0,a}),$$

and $\dim\ker(A_\tau) = \dim V_{0,0} = \dim\ker(D_B)$.

On $V_{k,a} \otimes L^2((0, 1), r d\tau)$,

$$\Delta^+ = -\partial^2_r - \frac{1}{r} \partial_r + \frac{1}{4r^2} + \frac{1}{2r^2} \begin{pmatrix} k^2 & k \\ k & k^2 \end{pmatrix} + a^2$$ \hspace{1cm} (2.3.2)

After changing a basis, obviously we can write it as

$$\Delta^+ = -\partial^2_r - \frac{1}{r} \partial_r + \frac{1}{4r^2} + \frac{1}{2r^2} \begin{pmatrix} k^2 + k & 0 \\ 0 & k^2 - k \end{pmatrix} + a^2$$ \hspace{1cm} (2.3.3)

Under the same basis,

$$\Delta^- = -\partial^2_r - \frac{1}{r} \partial_r + \frac{1}{4r^2} + \frac{1}{2r^2} \begin{pmatrix} k^2 - k & 0 \\ 0 & k^2 + k \end{pmatrix} + a^2$$ \hspace{1cm} (2.3.4)

Both are with boundary condition $(r^\frac{1}{2} \phi)(r) = 0$. So on $V_{k,a}$, $\text{tr}(\Delta^+ + \lambda)^{-m} - \text{tr}(\Delta^- + \lambda)^{-m} = 0$.

Given $a \neq 0$, on $V_{0,\pm a} \otimes L^2((0, 1), r d\tau)$, $\Delta^\pm = -\partial^2_r - \frac{1}{r} \partial_r + \frac{1}{4r^2} + a^2$, with the same boundary condition

$$(r^\frac{1}{2} \phi_{0,a})(0) = 0, \quad a \geq 0$$ \hspace{1cm} (2.3.5)

$$(r^\frac{1}{2}(\partial_r + \frac{1}{2r} + a) \phi_{0,a})(0) = 0, \quad a < 0.$$ \hspace{1cm} (2.3.6)

Thus we get $\text{tr}(\Delta^+ + \lambda)^{-m} - \text{tr}(\Delta^- + \lambda)^{-m} = 0$
On $\oplus_{k \in \mathbb{Z}\setminus\{0\}} V_{k,0}$, 
\[ \Delta^\pm = -\partial_r^2 + \frac{k^2 \pm k + \frac{1}{4}}{r^2} - \frac{1}{r} \partial_r. \]

with boundary conditions $(r^{1/2} \phi)(0) = 0$, this is just the conical operator considered in [5], where the following facts have been proved.

1. The kernels for $(\Delta^\pm + z^2)^{-1}$ with the above boundary conditions are 

\[ \oplus I_{\nu_\pm}(zr) K_{\nu_\pm}(zr) \]

Noting that 
\[ (\Delta^\pm + z^2)^{-m} = \frac{1}{(m - 1)!} (-\frac{1}{2z} \partial_z)^{m-1} (\Delta^\pm + z^2)^{-1}, \]

let $rz = \zeta$, define 

\[ \sigma^+(r, \zeta) = \frac{r^{2m-2}}{(m - 1)!} (-\frac{1}{2\zeta} \partial_\zeta)^{m-1} \sum I_{\nu_+}(\zeta) K_{\nu_+}(\zeta) \quad (2.3.7) \]

\[ \sigma^-(r, \zeta) = \frac{r^{m-2}}{m - 1} (-\frac{1}{2\zeta} \partial_\zeta)^{m-1} \sum I_{\nu_-}(\zeta) K_{\nu_-}(\zeta) \quad (2.3.8) \]

2. 
\[ \text{tr}(\psi(\Delta^\pm + z^2)^{-m}) = \int_0^\infty \psi(r) \sigma^\pm(r, rz) rdr. \]

$\psi, \phi$ are smooth functions, vanishing near $r = 0$, $\psi \equiv 1$ near supp$\phi$.

Let $\sigma^\pm(r, \zeta) \sim \sum_{j=1}^\infty \sigma_j^\pm(r) \zeta^{-j}$, $\zeta \to +\infty$. From the pointwise expansion of boundary parametrix on $H_3$ and the interior parametrix, we have 

\[ \sigma^\pm(1, \zeta) \sim \sum_j \int p_j^\pm(1, x) dx \zeta^{-j}. \]

Due to (4.6), (4.7), (4.8), 

\[ \sigma^\pm(r, \zeta) = r^{2m-2} \sigma^\pm(1, \zeta) \quad (2.3.9) \]

\[ \sigma_j^\pm = r^{2m-2} \int p_j^\pm(1, x) dx. \quad (2.3.10) \]
\[ tr \phi(\Delta^{\pm} + z^2)^{-m} \sim \sum_j \int_0^\infty \phi(r) \sigma_j^{\pm}(r)(rz)^{-j} rdr + \sum_j (\int_X \phi_j p_j) z^{-j} \]
\[ + \sum_{k=0}^\infty z^{-k-1} \int_0^\infty \frac{1}{k!} \zeta^k \sigma^{(k)}(0, \zeta) d\zeta \]
\[ + \sum_{k=0}^\infty z^{-k-1} a \log z \sigma_{k+1}^{(k)}(0)/k! \]

To construct a parametrix for \((\Delta^{\pm} + \lambda)^{-m}\), we patch the interior parametrix \(P_i^\pm\) with the boundary parametrix above. From [5], the remainder term does not affect the asymptotics of the trace. Thus we can use this parametrix to compute the expansion.

\[ z^{-2m} \text{ind}(D^+) = \sum_j (\int_X \phi_j(p_j^+ - p_j^-)) z^{-j} \]
\[ + \sum_j \int_0^\infty \phi(r)(\sigma_j^+(r)(rz)^{-j} - \sigma_j^-(r)(rz)^{-j}) rdr \]
\[ + \sum_{k=0}^\infty z^{-k-1} \int_0^\infty \frac{1}{k!} \zeta^k (\sigma_0^{(k)} - \sigma_0^{(k)}(0, \zeta)) d\zeta \]
\[ + \sum_{k=0}^\infty z^{-k-1} a \log z (\sigma_{k+1}^{(k)}(0) - \sigma_{-k+1}^{(k)}(0))/k! \]  \hspace{1cm} (2.3.11)

So the terms in \(z^{-2m} \log z\) coming from \(\Delta^+\) and \(\Delta^-\) must cancel:

\[ \int_N p_{2m}^+ dx = \int_N p_{2m}^- dx \]

thus \(\sigma_{2m}^+ = \sigma_{2m}^-\). Hence (4.11) becomes:

\[ \text{ind}(D^+) = \int_X \phi_i(p_i^{2m} - p_i^{-2m})) \]
\[ + \int_0^\infty \frac{\zeta^{2m-2}}{(2m-2)!} (\sigma_0^{(2m-2)}(0, \zeta) - \sigma_0^{(2m-2)}(0, \zeta)) d\zeta \]  \hspace{1cm} (2.3.12)

Since \(p_{2m}^+ - p_{2m}^-\) vanishes in the tubular neighborhood, we can drop \(\phi_i\) in the first
integral. Now it remains to compute the second integral above. Define

$$h_\pm(w) = \int_0^\infty \frac{\zeta^w}{(2m-1)!} \sigma_\pm^{(2m-1)}(0, \zeta) d\zeta.$$  

It has been shown in [5] that the integral is just the analytic continuation of $h_+(w) - h_-(w)$, i.e., it is equal to $Res_0(h_+ - h_-)(2m - 1)$, where $Res_k h(w_0)$ is the coefficient of $(w - w_0)^{-1}$ in the Laurent expansion of the meromorphic function $h(w)$.

Thus

$$\text{Ind}(D^+) = \int_M \phi_i(p_+^{2m} - p_-^{2m}) + Res_0(h_+ - h_-)(2m - 1).$$  

(2.3.13)

From [7],

$$h_\pm(w) = \frac{\Gamma(w+1/2)\Gamma(m - 1 - w/2)}{4\sqrt{\pi}\Gamma(m)} \sum_{b \neq 0} \frac{\Gamma(\nu_\pm + w/2 - m)}{\Gamma(1 + \nu_\pm - w/2 + m)}.$$  

Define $\zeta$ function of $A_r$ as follows:

$$\zeta_\pm(z) = \sum_{b \neq 0} b \pm \frac{1}{2} \left| z \right|. $$

Let $z = (w + 1 - 2m)/2$, then $h_\pm(w) = \frac{\Gamma(z+m)\Gamma(-z-\frac{1}{2})}{4\sqrt{\pi}(m-1)!} \sum_{b \neq 0} \frac{\Gamma(\nu_\pm + z + 1)}{\Gamma(\nu_\pm - z)}$. This expression is analyzed in [7]:

$$\sum_{b \neq 0} \frac{\Gamma(\nu_\pm + z + 1)}{\Gamma(\nu_\pm - z)} = \sum_{j=0}^{N} Q_j(z) \zeta_\pm(j - 1 - 2z) + R(z),$$

$R(z)$ is analytic in $\text{Re}(z) < 1$, $R(0) = 0$. So if $\zeta_+(s) - \zeta_-(s)$ only has simple poles, we only need the linear parts of $Q_j$ to get $Res_0(h_+ - h_-)(2m - 1)$. From [7], we have

$$Q_j(z) = O(z^2), j \quad \text{odd}$$

$$Q_j(z) = -z^2 \frac{2}{j} B_j, j \quad \text{even} > 0$$

$B_j$ is the $j$-th Bernoulli number, we have $B_2 = \frac{1}{6}$ which we are going to use in our
So $Res_0(h_+ - h_-)(2m - 1)$ is the 0–th residue at 0 of the difference function of the following two functions:

$$\frac{\Gamma(z + m)\Gamma(-z - \frac{1}{2})}{4\sqrt{\pi}(m - 1)!}(\zeta_+(-1 - 2z) - \sum_{k=1}^{N/2} k^{-1}B_k z \zeta_+(2k - 1 - 2z)),$$

$$\frac{\Gamma(z + m)\Gamma(-z - \frac{1}{2})}{4\sqrt{\pi}(m - 1)!}(\zeta_+(-1 - 2z) - \sum_{k=1}^{N/2} k^{-1}B_k z \zeta_-(2k - 1 - 2z)).$$

If $f$ is analytic and $g$ has a simple pole at 0, then $Res_0(fg)(0) = f(0)Res_0g(0) + f'(0)Res_1g(0)$, from this and noting that $\frac{f'(m)}{\Gamma(m)} = \sum_{j=1}^{m-1} \frac{j}{\gamma} - \frac{1}{\gamma}$, $\gamma$ is Euler’s constant, and $\Gamma'(-\frac{1}{2}) = -2\sqrt{\pi}$,

$$Res_0(h_+ - h_-)(2m - 1) = -\frac{1}{2}Res_0(\zeta_+ - \zeta_-)(-1)$$

$$-\frac{1}{4} \sum_{k \geq 1} (-1)^k \frac{1}{k} B_k Res_1(\zeta_+ - \zeta_-)(2k - 1)$$

$$+ \frac{1}{2} \left( \frac{\Gamma'(-\frac{1}{2})}{4\sqrt{\pi}} + \frac{1}{2} \sum_{j=1}^{m-1} \frac{1}{j} - \gamma \right) Res_1(\zeta_+ - \zeta_-)(-1)$$

$$- \frac{1}{2} dimV_{0,0}$$

(2.3.14)
while \((\zeta_+ - \zeta_-)(z)\) is given by the following:

\[
(\zeta_+ - \zeta_-)(z) = \sum_{b \neq 0} \left( \frac{1}{|b + \frac{1}{2}|} z - \sum_{b \neq 0} \frac{1}{|b - \frac{1}{2}|} z \right)
\]

\[
= \sum_{b \neq 0} \left( \frac{1}{|b|} z \right) \left( \frac{1}{2b} - \frac{1}{2b} \right) - \left( \frac{1}{2b} - \frac{1}{2b} \right)
\]

\[
= 2 \sum_{b \neq 0} \left( \frac{1}{|b|} z \right) \sum_{k \geq 0} \left( \frac{2k+1}{z} (2b)^{-2k-1} \right)
\]

\[
= \sum_{k \geq 0} 2^{-2k} \left( \frac{2k+1}{z} \right) \sum_{b \neq 0} \left( \frac{1}{|b|} z \right) \text{sign} b
\]

\[
= \sum_{k \geq 0} 2^{-2k} \left( \frac{1}{z} \right) \left( z - 1 \right) \cdots \left( z - 2k \right) \frac{1}{(2k+1)!} \eta_{A_r}(z + 2k + 1)
\]

\[
= \eta_{A_r}(z + 1) + \sum_{k \geq 1} 2^{-2k} \left( \frac{1}{z} \right) \left( z - 1 \right) \cdots \left( z - 2k \right) \frac{1}{(2k+1)!} \eta_{A_r}(z + 2k + 1)
\]

here \(\eta_{A_r}(z)\) is the \(\eta\) function of the operator \(A_r\) on \(N_r\).

From (4.13), we have

\[
Res_0(\zeta_+ - \zeta_-)(-1) = \eta_{A_r}(0) + \sum_{k \geq 1} 2^{-2k} \frac{1}{2k(2k+1)} Res_1 \eta_{A}(2k)
\] (2.3.15)

For \(k \geq 1\),

\[
Res_1(\zeta_+ - \zeta_-)(2k - 1) = - \sum_{j \geq 0} 2^{-2j} \left( \frac{2k + 2j - 1}{2j + 1} \right) Res_1 \eta_{A}(2j + 2k)
\] (2.3.16)

Lastly we can get the explicit index formula by using the calculation for \(\eta_{A}(s)\) in Section 2:

\[
Res_0(h_+ - h_-)(2m - 1) = \frac{1}{2} \eta_{A}(s) - \frac{1}{2} \sum_{l \geq 1} 2^{-2l} \frac{Res_1 \eta_{A}(2l)}{2l(2l+1)}
\]

\[
- \frac{1}{4} \sum_{l \geq 1} \sum_{i \geq 1, i+j=l} (-1)^i \frac{B_i}{j} (-1)^{2j} \left( \frac{2l-1}{2j+1} \right) Res_1 \eta_{A}(2l)
\]

\[
- \frac{1}{2} \text{dim } V_{0,0}
\] (2.3.17)
And we can write the first three terms explicitly:

\[-\frac{1}{2} \eta_A = -\frac{1}{2} (-2) \sum_{l \geq 1} \int_B \hat{A}(B) e^{2l-1} \frac{\zeta(-(2l - 1))}{(2l - 1)!} \]

\[= \sum_{l \geq 1} \frac{(-1)^{l-1} B_l \int_B \hat{A}(B) e^{2l-1}}{(2l)!} \]

\[= \int_B \hat{A}(B) \left( \frac{1}{2} \coth \frac{e}{2} - \frac{1}{e} \right) \]

In the third equality above we used equation (2.1.4).

For the sum of the middle two terms in \(Res_0(h_+ - h_-)(2m - 1)\), we claim it is equal to \(\int_B \left(-\frac{\cosh \frac{e}{2}}{2 \sinh \frac{e}{2}} + \frac{1}{e}\right)\). To see this, note that

\[-\frac{1}{4} \sum_{l \geq 1} \sum_{i+j=l} \frac{(-1)^i B_i}{i} (-1)^{2j} \left(\frac{2l - 1}{2j + 1}\right) Res_1 \eta_A(2l) \]

\[= -\frac{1}{4} \sum_{l \geq 1} \sum_{i+j=l} \frac{(-1)^{i-1} B_i (-2)^{-2j} \int_B \hat{A}(B) e^{2l-1}}{(2i - 2)!(2j + 1)!} \]

\[= 2 \sum_{l \geq 1} \sum_{i+j=l} \frac{(-1)^{i-1} B_i (2i - 1) 2^{-2j-1} \int_B \hat{A}(B) e^{2l-1}}{(2i)! (2j + 1)!} \]

thus the sum of the middle two is

\[2 \int_B \hat{A}(B) \frac{d}{de} \left( \frac{1}{2} \coth \frac{e}{2} \right) (\sinh \frac{e}{2}) = \int_B \left(-\frac{\cosh \frac{e}{2}}{2 \sinh \frac{e}{2}} + \frac{1}{e}\right).\]

Note \(dim V_{0,0} = dim Ker(D_B)\), we have

\(Res_0(h_+ - h_-)(2m - 1) = \int_B \hat{A}(B) \left( \frac{1}{2} \coth \frac{e}{2} \right) - \frac{1}{2} \dim Ker(D_B)\).

Combining this with (4.12), we have

\( \text{Ind}(D^+) = \int_X \omega_{D^+} + \int_B \hat{A}(B) \left( \frac{1}{2} \coth \frac{e}{2} \right) - \frac{1}{2} \dim Ker(D_B) \)

(2.3.18)

Since \(D^+\) is the Dirac operator on a spin bundle, we know that \(\omega_{D^+} = \hat{A}(X)\).
Thus

\[ \text{Ind}(D^+) = \int_M \hat{A}(M) + \int_B \hat{A}(B) \frac{1 - \cosh \frac{\epsilon}{2}}{2 \sinh \frac{\epsilon}{2}} - \frac{1}{2} \dim \text{Ker}(D_B) \quad (2.3.19) \]

Now suppose \((M, g)\) is an arbitrary smooth Riemann metric on \(M\), \((p, rn)\) be a point near \(B\), \(p \in B\), \(n\) is a unit normal vector. Denote by \(A_n\) the shape operator of the submanifold \(B\) in the direction \(n\). Then the metric \(g\) is of the following form near \(B\), which is proved in [13]:

**Lemma 2.3.1.** Let \(R\) denote the Riemann curvature \((0, 4)\) tensor of \((M, g)\). Then for \(u_1, u_2 \in T_{(p, rn)}M\), we have

\[
\exp^* g(u_1, u_2) = g_0(u_1, u_2) - 2g(A_n \pi_* u_1, \pi_* u_2) r
\]
\[
+ \{ g(A_n \pi_* u_1, A_n \pi_* u_2) + R(\pi_* u_1, n, \pi_* u_2, n) + \frac{2}{3} R(\pi_* u_1, n, Ku_2, n) \}
\]
\[
+ \frac{2}{3} R(\pi_* u_2, n, Ku_1, n) + \frac{1}{3} R(Ku_1, n, Ku_2, n) \} r^2 + O(r^3)
\]

Here \(\exp\) is the exponential map, \(g_0\) is the model metric we used in the previous sections, and \(K\) is the projection to the vertical tangent space.

Given a Riemannain metric \(g\), the following Koszul formula is helpful if one wants to calculate the Levi-Civita connection:

**Lemma 2.3.2.** Koszul Formula (cf [19])

Let \((M, g)\) be a Riemannian manifold, \(\nabla\) is the Levi-Civita connection, \(X, Y, Z\) are tangent vector fields, then

\[
2g(\nabla_X Y, Z) = X \cdot g(Y, Z) + Y \cdot g(Z, X) - Z \cdot g(X, Y)
\]
\[
+ g([X, Y], Z) - g([Y, Z], X) + g([Z, X], Y)
\]

The Dirac operator associated with \(g\) is related to the Dirac operator \(D_0\) associated with \(g_0\) in the following way:

**Theorem 2.3.3.** Let \((M, g)\) be a smooth Riemannian manifold of dimension \(2m\), \(B\) is an embedded submanifold with codimension 2. \(S\) is a Spin bundle over \(M \setminus B\) which
can not be extended to a spin bundle over $M$. Then near $B$, the Dirac operator $D^+$ can be written as

$$D^+ = \partial_r + \frac{1}{2r} + \frac{1}{2} H + A_r + \eta(r).$$

$H$ is the mean curvature of $B$, which we can view as a function on the unit circle bundle of $B$, and $A_r$ is the perturbed operator introduced in Chapter 2. $\eta(r)$ is compactly supported in a small neighborhood of $B$, equal to $O(r)$ near $B$. Thus equivalently, we have that near $B$

$$D^+ = D_0^+ + \frac{1}{2} H + O(r),$$

$D_0^+$ is equal to $\partial_r + \frac{1}{2} r + A_r$.

**Proof.** Let $e_1, \ldots, e_{2m-2}$ be an oriented orthonormal frame for $TB$, and $\zeta$ is the infinitesimal of the $s^1$ action on $N$. First from the Koszul formula, it is straightforward to see that the contribution to the Levi-Civita connection $\nabla_M$ from the third and fourth part in the metric formula in Lemma 2.3.1 is $O(r)$, hence the same is true for $D^+$. Thus it suffices to consider metric $g_1 = g_0 - 2g(A_n \pi_*, \pi_*) r$. As shown in [13], at $(p, \theta, n)$, the Dirac operators $D^+$ and $D_N$ are related by

$$D_1^+ = \partial_r + D_N - \frac{H_N}{2}.$$

Here $D_N$ is the Dirac operator on $N$ with metric induced from $g_1$, i.e., $g_N = r^2 \omega \otimes \omega + \pi^* g_B - 2g(A_n \pi_*, \pi_*)$. $H_N$ is the mean curvature of $N$ at $(p, \theta, n)$.

For any fixed $r$, $g_N$ is a small perturbation of the model metric we considered before, again from the Koszul formula, the Levi-Civita connection is perturbed by an $O(r)$ term, so is true for the Dirac operator: $D_N^+ = A_r + O(r)$.

At $(p, \theta, n)$, if we still use $e_1, \ldots, e_{2m-2}$ denote the pull back vectors from $B$, then under the metric $g_1$, $|e_i| = 1 + O(r)$, $|\zeta| = r$, and $\frac{\xi}{|\xi|}$, $\xi$, $\partial_r$ is an orthonormal frame for $TM$. On the other hand, $H_N = \sum_{i=1}^{2m-2} g(\nabla_{e_i/|e_i|} n, e_i/|e_i|) + g(\nabla_{\zeta/\xi} n, \zeta/\xi)$. Since $g_1(\nabla_{\xi} n, \zeta) = g_1(\nabla_{\xi} n, \xi) = -\frac{1}{r}$, the last term in the above is $-\frac{1}{r}$. From the Koszul formula, $2g_1(\nabla_{e_i} n, e_i) = n \cdot g_1(e_i, e_i) = -2g_1(A_n e_i, e_i)$, the last equality follows
from the definition of our metric $g_1$. So $H_N(p, r_n) = -\frac{1}{r} - 2H(p, r_n)$, thus $D_1^+ = \partial_r + \frac{1}{2r} + \frac{1}{2}H + D_N^1 = \partial_r + \frac{1}{2r} + A_r + \frac{1}{2}H = D_0 + \frac{1}{2}H$. \qed

Now for any smooth metric $g$ on $M$, the corresponding Dirac operator $D$ is just a bounded perturbation of $D_0$: $D = D_0 + \frac{H}{2} + \eta(r)$, from Lemma 2.2.8., we know that the index of $D$ will remain equal.

**Proof of theorem 1.0.2.** To prove that $D^+$ is Fredholm, notice that it suffices to prove

$$D^+ \oplus (P \circ R) : \text{Dom}(D_{\text{max}}^+) \to L^2(S^+) \oplus (\text{Im}(P) \cap L^2_{-\frac{1}{2}}(S_B))$$

is Fredholm, for which the proof of proposition 17.2.5. in [12] applies here: first $PP_0 : \text{Im}(P_0) \cap L^2_{-\frac{1}{2}} \to \text{Im}(P) \cap L^2_{-\frac{1}{2}}$ is Fredholm, since $(P_0P)(PP_0) = P_0 + P_0(P - P_0)P_0$, so is a compact perturbation of the identity operator on $\text{Im}(P_0)$, so $PP_0$ has a right inverse operator moduli compact operators, similarly it also has a left inverse. Thus $(D^+, PP_0)$ is Fredholm. At last, we can write $(D^+, P) = (D^+, PP_0) + (0, P(P - P_0))$, the first is Fredholm and the second is compact, so $(D^+, P)$ is Fredholm. If we replace $P$ by $P_1$ as in the theorem, following from the proof of proposition 17.2.6 in [12], $D^+$ is Fredholm.

As for the index, the first index formula has just been proved. For the second index formula, since the only difference in the boundary contribution is from $\text{ker}(D_B)$, which now is projecting to $\text{ker}(D_B^1)$ instead of to $\text{ker}(D_B)$, thus the index formula follows immediately.

### 2.3.1 Weitzenbock formula.

In this section, we look at the Weitzenbock formula for spinors in the domains we considered before. For simplicity, we assume the metric $g$ equal to $dr^2 + r^2\omega \otimes \omega + \pi^*g_B$ near $B$.  

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Let $M_\epsilon = \{p \in M \mid d(p, B) \geq \epsilon\}$, $\epsilon > 0$. For $\phi \in L^2(S^+), D^+ \phi \in L^2(S^-)$, we have

$$\int_{M_\epsilon} \| D^+ \phi \|^2 = \int_{M_\epsilon} \| \nabla \phi \|^2 + \int_{M_\epsilon} \frac{s}{4} \| \phi \|^2 + \int_{N_\epsilon} < \phi, D_\epsilon \phi > d\sigma - \frac{1}{2\epsilon} \int_{N_\epsilon} \| \phi \|^2 d\sigma$$

(2.3.21)

The last term is because with our metric, the boundary circle bundle has mean curvature $\frac{1}{\epsilon}$. As $\epsilon \to 0$, we will see that both $\int_{M_\epsilon} \| \nabla \phi \|^2$ and $\frac{1}{2\epsilon} \int_{N_\epsilon} \| \phi \|^2 d\sigma$ diverge, but the divergent terms of this two cancel out, so the left hand side of the above equation remains finite.

For any $\phi \in L^2(S^+)$ such that $D^+ \phi \in L^2(S^-)$, from the last section, $\phi = \phi_0 + r^{-\frac{1}{2}} \psi$, $\phi_0 \in L^2(S^+, M\setminus B)$, $\psi$ is a section which in the tubular neighborhood is in $\pi^* L^2(S_B)$, and vanish in the interior. Thus $\int_{M_\epsilon} \| \nabla r^{-\frac{1}{2}} \psi \|^2 (\frac{1}{2\epsilon} \int_{N_\epsilon} \| \phi \|^2 d\sigma$ respectively) is the term that renders $\int_{M_\epsilon} \| \nabla \phi \|^2 (\frac{1}{2\epsilon} \int_{N_\epsilon} \| \phi \|^2 d\sigma$ respectively) divergent.

In the tubular neighborhood, the spin connection $\nabla$ can be written as

$$\nabla = \nabla^1 + \nabla^2.$$

$$\nabla^1 = \partial_r \otimes dr + \frac{1}{2} \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} \otimes \omega$$

(2.3.22)

$$\nabla^2 = \partial_r \otimes \eta_r + \partial_1 \otimes \eta^1 + \partial_2 \otimes \eta^2 - \frac{1}{2} k \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} \otimes \eta^1$$

$$+ \frac{1}{2} (-rn) \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix} \otimes \eta^2 + \frac{1}{2} rn \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \otimes \eta^1$$

(2.3.23)

$$+ \frac{1}{2} rn \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} \otimes \eta_r$$
The divergent term of \(\int_{M_c} \| \nabla \phi \|^2\) comes from \(\int_{M_c} \| \nabla r^{-\frac{1}{2}} \psi \|^2\), which is

\[
\int_{M_c} \| \nabla r^{-\frac{1}{2}} \psi \|^2 = C + \int_{M_c \setminus M_a} \| \nabla^3 r^{-\frac{1}{2}} \psi \|^2 \\
= C + \int_{M_c \setminus M_a} \frac{1}{4} \psi^2 + \| \frac{1}{2} \psi - \frac{1}{2} r^{-\frac{3}{2}} \gamma \psi \otimes \omega \|^2 \\
= C + \int_{M_c \setminus M_a} \frac{1}{4} \psi^2 + \frac{1}{2} r^{-3} \psi^2 \\
= C + \frac{1}{2} \int_B \psi^2 + \frac{1}{2} \int_B \psi^2 \\
= C - \frac{1}{2} \int_B \psi^2 + \frac{1}{2} \int_B \psi^2 \\
= C - \frac{1}{2} \int_B \psi^2
\]  

(2.3.24)

Here \(C\) is a constant which depends on \(\psi\), and in the third equality above we used the fact that \(\| \omega \|_{N_r} = \frac{1}{r}\).

At the same time, the divergent term of \(\frac{1}{2\epsilon} \int_{N_\epsilon} \| \phi \|^2\) is

\[
\frac{1}{2\epsilon} \int_{N_\epsilon} \| \epsilon^{-\frac{1}{2}} \psi \| \, dg_\epsilon = \frac{1}{2\epsilon} \int_B \| \psi \|^2 \, dg_B
\]  

(2.3.25)

Thus if we let

\[
Q(\phi) = \lim_{\epsilon \to 0} (\int_{M_c} \| \nabla \phi \|^2 - \frac{1}{2\epsilon} \int_{N_\epsilon} \| \phi \|^2)
\]  

(2.3.26)

Also notice that \(\lim_{\epsilon \to 0} \int_{N_\epsilon} < D_\epsilon \phi, \phi > = \int_B < D_B \psi, \psi >\), Weitzenbock formula for \(\phi \in L^2(S^+), D^+ \phi \in L^2(S^-)\) becomes

\[
\int_{M \setminus B} \| \phi \|^2 = Q(\phi) + \int_{M \setminus B} \frac{1}{4} \| \phi \|^2 + \int_B < D_B \psi, \psi >
\]  

(2.3.27)

2.3.2 Index for \(\text{Spin}^c\) Case.

Let \(S \to M \setminus B\) be a \(\text{Spin}^c\) bundle, \(A_0\) is a \(\text{Spin}^c\) connection such that restricted to the circle bundle \(P\) of \(B\), it is \((S \mid_P, A_0 \mid_P) = \pi^*(S_B, A_B)\), where \(S_B\) is a \(\text{Spin}^c\) bundle over \(B\), and \(A_B\) is a \(\text{Spin}^c\) connection on \(S_B\). So \(A_0\) has holonomy \(-1\) along the circles linking \(B\). Let \(\omega\) be a connection form of \(P\). Thus \((S, A_0 + i\frac{1}{2} \omega)\) extends to a \(\text{Spin}^c\) pair \((S_M, A_M)\), \(\omega\) is a connection form on \(P\). And \(A_0 + i\frac{1}{2} \omega\) has holonomy zero along circles linking \(B\).
Use $A_0$ as a reference connection, then all the $Spin^c$ connection can be characterized by $i\alpha \in \Lambda^1(iT^*M \setminus B)$. Now suppose that when restricted to the tubular neighborhood, $i\alpha = ic\omega$, $c \in (0, \frac{1}{2})$ is constant. Let $D^\xi_A$ be the Dirac operator associated with connection $A_c = A_0 + ic\omega$. In the tubular neighborhood $U$, as before, we have $L^2(S^+_U) = L^2((0,1), L^2(S^+_N) \otimes rdr)$. Since the $Spin^c$ bundle is a pull back of a $Spin^c$ bundle over $B$, the $S^1$ action on $N$ again can be lifted to $S$, and $L^2(S^+_|N) = \oplus V_k$, where $V_k$ is the eigenspace of the differential of the $S^1$ action, $L_\zeta$.

The Dirac operator corresponding to $A_c$ and $g|_N = r^2 \omega \otimes \omega + \pi^*g_B$ is denoted by $A^c_r$, and as in Chapter 2, can be written as

$$A^c_r = \frac{1}{r}A_v + A_h - \frac{r}{4} \gamma(\frac{\zeta}{r})\gamma(\pi^*d\omega).$$

We also have the isometry $Q_k : V_k \to L^2(S_B \otimes L^{-k})$, such that when restricted to $V_k$, $A_h = Q_k^{-1} \circ A_k \circ Q_k$.

$A^c_r = \frac{1}{r}A_v + A_h$ is the twisted operator.

Let $\phi_{k,a} = \alpha \oplus \beta \in L^2(S_N)$ be a unit norm common eigenvector of $L_\zeta$ and $A_h$ with eigenvalues $ik$ and $a$ respectively.

If $a \neq 0$, $k \neq 0$, then $\alpha, \beta$ span a two dimensional space, under this basis, $A^c_r$ is represented by

$$A_{k,a} = \begin{pmatrix} -\frac{k+c}{r} & a \\ a & \frac{k+c}{r} \end{pmatrix}$$

If $a = 0$, $(\alpha, \beta) \in \ker(D_B)$, then $\alpha$ is the eigenvector of $A_r$, with eigenvalue $\frac{k+c}{r}$ and multiplicity $\dim \ker(A_k|_{S_B \otimes L^{-k}})$, and $\beta$ is the eigenvector of $A_k$ with eigenvalue $-\frac{k+c}{r}$ and multiplicity $\dim \ker(A_k|_{S_B^{-} \otimes L^{-k}})$.

If $k = 0$, up to the isometry, $A_r|_{V_0} = D_B$.

Then as in Chapter 2, the eta function $\eta^0_{A_0}$ of $A^0_1$ is again

$$\eta^{A_0}_s = \int_B \hat{A}(B)f_s(e),$$

where $f_s(x) = -\sum_{k \geq 1} \frac{e^{kx} - e^{-kx}}{k} = -2\sum_{i \geq 1} \frac{\zeta(s-(2i-1))x^{2i-1}}{(2i-1)!}$. 

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For $0 < c < \frac{1}{2}$, the eta function $\eta_{A^c}(s)$ of $A^c$ with holonomy $e^{2\pi(c+\frac{1}{2})}$ is

$$\eta_{A^c}(s) = \sum_{k \geq 0} \int_B \hat{A}(B) \text{ch}((\det(S_B)L^{-k})) \frac{k^s}{(k+c)^s} + \sum_{k < 0} \int_B \hat{A}(B) \text{ch}((\det(S_B)L^k)) \frac{(-k)^s}{(-k-c)^s}$$

$$= \int_B \hat{A}(B) f_s(e)$$

with

$$f_s(x) = \sum_{i \geq 0} \sum_{j \geq 0} c^{i-j} \zeta(s-j, c) \frac{x^i}{i!} + \sum_{i \geq 0} \sum_{j \geq 0} c^{i-j} \zeta(s-j, 1-c) \frac{(-x)^i}{i!}$$

Here $\zeta(s, a)$ is the Hurwitz Zeta function, given by

$$\zeta(s, a) = \sum_{k=0}^{+\infty} \frac{1}{(k+a)^s}.$$ 

$\zeta(s, a)$ defines a meromorphic function on the complex plane, with a simple pole at $s = 1$ of residue 1, and $\zeta(s, a)$ is continuous in the second variable $a$.

Let

$$C^\infty(M \setminus B, S^+; P) = \{ \phi \in C^\infty \cap L^2, D_A^c \phi \in L^2 \}. \quad (2.3.29)$$

Still let $D_{\text{max}}^c$ denote the closure operator defined by above. When $c = 0$, define $\text{Dom}(D_A^c)$ is the kernel of the composition:

$$r^\frac{1}{2} \text{Dom}(D_{\text{max}}^c) \xrightarrow{R} L^2_{-\frac{1}{2}} (S_B \otimes L_B \otimes L^k) \xrightarrow{P_B} L^2_{-\frac{1}{2}} (S_B \otimes L_B \otimes L^k).$$

Here $P_B$ is as usual the projection map to the negative spinors. When $c \neq 0$, for any $\phi \in D_{\text{max}}^c$, the leading order term of $\phi$ is $\psi = c_1(r^{-\frac{1}{2}}-c)\alpha \oplus r^{-\frac{1}{2}+c}\beta) + c_2(r^{\frac{1}{2}+c}\alpha' \oplus r^{\frac{1}{2}-c}\beta') + O(lnr)$, for some $\alpha, \beta, \alpha', \beta' \in L^2(S_P)$. Define

$$\text{Dom}(D^c) = \{ \phi | \lim_{r \to 0} |r^\frac{1}{2}\phi|_{L^2(S_{N, \eta})} = 0 \}.$$
Due to the behavior of $\phi \in D'_{\text{max}}$ for near $B$, we see that this boundary condition is exactly the same as the second boundary condition we considered before, i.e., projection to the negative spinors.

The main result in this section is the following theorem:

**Theorem 2.3.4.** Let $S \to M \setminus B$ be a $\text{Spin}^c$ bundle, $A_0$ is a $\text{Spin}^c$ connection such that when restricted to the circle bundle $P$ of $B$, $(S|_P, A_0|_P) = (\pi^*S_B, \pi^*A_B), S_B$ is a $\text{Spin}^c$ bundle over $B$, $A_B$ is a $\text{Spin}^c$ connection on $S_B$. If the connection in the tubular neighborhood is $A_0 + i\omega$, then with the domain defined above, $D_A'$ is Fredholm and

$$\text{Ind}(D_A') = \int_M \hat{A}(M) \wedge \exp\left(\frac{i}{4\pi}(F_A - i\omega) - \int_B \hat{A}(B)\frac{1 - \cosh \frac{x}{2}}{2 \sinh \frac{x}{2}} + \frac{1}{2} \text{Ind}(D_B^+),ight)$$

where $F_M$ is the curvature term of $A_M$. When $M$ is four dimensional,

$$\text{Ind}(D_A^c) = \frac{\sigma(M) - c_1(S_M^c)^2}{8}.$$

where $(S_M, A_M)$ is the extension of the pair $(S, A_0 + i\frac{1}{2}\omega)$.

**Proof.** Freehomness can be proved in exactly the same way as before.

When $c = 0$, the index calculation is almost the same as the $\text{Spin}$ case. The eta function of $A'$ is of the exact same form, which makes the boundary contribution the same. As for the interior form, notice that $F_X - i\omega$ is the curvature form of our $\text{Spin}^c$ connection.

When $c \neq 0$, the eta function $\eta_{A^c}(s)$ is given by (4.26), and since the Hurwitz function $\zeta(s, q)$ has simple pole at $s = 1$ with residue 1 and $\zeta(s, a)$ is continuous in the second argument, we see that the boundary contribution to the index of $D^c$ is continuous in $c$. But on the other hand, the index form of $D^c$ also continuously depends on $c$, they are all equal to $\int_M \hat{A}(M) \wedge \exp\left(\frac{i}{4\pi}(F_{A_m} - i\omega + i2cd\omega)\right)$. Since $\text{Ind}(D^c)$ is integer valued, we get that $\text{Ind}(D^c) = \text{Ind}(D^0)$. When $c = 0$, the boundary contribution is the same as before, the index formula follows immediately.

The formula for four dimensional case follows as below: $\Sigma$ is $\text{Spin}$, we can write
$S|_P = (\pi^* K_{\Sigma}^{-\frac{1}{2}} \oplus \pi^* K_{\Sigma}^{\frac{1}{2}}) \otimes H$, such that $(H, B) = \pi^*(H\Sigma, B\Sigma, A\Sigma)$, $H\Sigma$ is a line bundle over $\Sigma$ with degree $k$. And $A_0$ comes from the connection of the pull back $Spin$ connection on $K_{\Sigma}^{-\frac{1}{2}} \oplus K_{\Sigma}^{\frac{1}{2}}$ and $B\Sigma$. Thus $(S, A_0 + i\frac{1}{2}\omega)$ extends to $(S_X, A_X)$ such that $c_1(S_X^+)(\Sigma) = 2k + n$. Now from the general formula, $\text{Ind}(D^c) = \frac{\sigma(X) - c_1(S_X^+) - d\omega^2}{8} - \frac{n}{8} - \frac{1}{2}k$, while $(c_1(S_X) - d\omega)^2 = c_1(S_X^+)^2 - 2c_1(S_X) \wedge d\omega + n$, and $c_1(S_X^+) \wedge d\omega = 2k + n$, we get

$$\text{Ind}(D^c) = \frac{\sigma(X) - c_1(S_X^+)}{8}.$$
Chapter 3

Moduli Space of Singular Monopoles.

3.1 Basic Seiberg Witten Theory.

Seiberg-Witten equations are first order elliptic equations. And the solutions are absolute minimum of a so called Seiberg-Witten functional. The Euler-Lagrange equations of the functional are second order equations, and Seiberg-Witten solutions also satisfy the Euler-Lagrange equations. One strategy to study the Seiberg-Witten solutions is to first solve the E-L equations, then to identify conditions under which certain solutions actually also solve the first order equations. The main source of reference is [10].

3.1.1 The monopole equations.

Let $X$ be a compact connected oriented 4-manifold and fix a $Spin^c$ structure $S$ on $X$. Such a $Spin^c$ structure always exists. As before, $\gamma : TX \to End(S)$ is the Clifford multiplication, and $\gamma$ extends to an isomorphism of algebra bundles $C(TX) \to End(S)$. There is a natural splitting of $S$:

$$S^+ \oplus S^-$$
into the $\pm$ eigenspaces of $\gamma(e_1 e_2 e_3 e_4)$ where $e_1, e_2, e_3, e_4$ is any positively oriented orthonormal frame of $TX$. Let $L_S$ be the determinant line bundle of $S^+$ and $S^-$:

$$L_S = det(S^+) = det(S^-)$$

Denote by $\mathcal{A}$ the space of $Spin^c$ connection on $S^+$. For any given $A \in \mathcal{A}$, $D_A$ is the associated Dirac operator.

The Seiberg-Witten monopole equations are a system of first order differential equations for a pair $(A, \Phi)$ where $A \in \mathcal{A}$ and $\Phi \in C^\infty(X, S^+)$. They read

$$D_A \Phi = 0, \quad F_A^+ = \sigma^+((\Phi \Phi^*)_0).$$

(3.1.1)

Here the endomorphism $\Phi \Phi^* \in C^\infty(X, End(S^+))$ is defined by

$$\Phi \Phi^* \tau = \langle \Phi, \tau \rangle \Phi$$

for $\tau \in C^\infty(X, S^+)$. Its traceless part is given by

$$(\Phi \Phi^*)_0 \tau = \langle \Phi, \tau \rangle \Phi - \frac{1}{2} \| \Phi \|^2 \tau.$$ 

Let $End_1(S^+)$ denote the bundle of traceless endomorphisms of $S^+$. The bundle isomorphism

$$\sigma^+ : End_0(S^+) \to \Omega^+ T^* X \otimes \mathbb{C}$$

is the inverse of the map $\gamma : \Omega^+ T^* X \to End_0(S^+)$ defined by Clifford multiplication. Recall that in the 4-dimensional case, $\gamma$ identifies the imaginary self-dual 2-forms on $X$ with the traceless Hermitian endomorphisms of $S^+$. Thus $\sigma^+((\Phi \Phi^*)_0)$ is an imaginary valued self-dual 2-form and so is $F_A^+$.

### 3.1.2 Space of Configurations.

In order to use these equations to produce a moduli space, out of which one can define Seiberg-Witten invariant of the $Spin^c$ structure, we need to put these equations in a
nonlinear elliptic framework. The space of configurations is the space on which the
equations define a function. It is the space of all pairs \((A, \Phi)\) where \(A\) is a \(\text{Spin}^c\)
connections and \(\Phi\) is a spinor. For technical reasons, we need to work with Banach
or Hilbert spaces. And usually people work with \(L_2^p\) or \(L_p^p\) for \(p > 4\). Here we choose
the working space to be \(L_2^p\), based on [10].

Let \(L_2^p(S^+)\) be the space of sections of \(W^+\) of \(L_2^p\) class, and \(\mathcal{A}_2^2\) be the connections
on \(S^+\) of \(L_2^p\) class.

The Seiberg-Witten functional for a pair \((A, \Phi)\) is defined by

\[
SW(A, \Phi) = \int_X (\| \nabla_A \Phi \|^2 + \| F_A^+ \|^2 + \frac{s}{4} \| \phi \|^2 + \frac{1}{8} \| \phi \|^4) dvol.
\]
and \(s\) is the scalar curvature of \((X, g)\).

Seiberg-Witten solutions are absolute minimum of this functional. The Euler-
Lagrange equations of the Seiberg-Witten functional are

\[
-\triangle_A \phi + \frac{s}{4} \phi + \frac{1}{4} \| \phi \|^2 \phi = 0, \tag{3.1.2}
\]

\[
d^* F_A^+ + \frac{1}{2} \text{Im} \langle \nabla_i \phi, \phi \rangle e^i = 0. \tag{3.1.3}
\]
Here \(\triangle_A\) is the Laplacian, and \(\nabla_i = \nabla_{e_i}\), \(\{e_i\}\) is an orthonormal basis of \(TX\).

The following maximum-principle property is proved by Kronheimer-Mrowka in
[11]:

**Theorem 3.1.1.** For a smooth solution \((A, \Phi)\) of the Seiberg-Witten equation,

\[
\| \phi(x) \| \leq \max\{-s, 0\}.
\]

It is not hard to see that the functional \(SW\) is well defined on \(\mathcal{A}_2^2 \times L_2^p(S^+)\) and
\(SW\) is smooth.

A suitable Lie group is required to serve as a gauge group. Let \(G_0 = \exp(iL_2^p(X, R))\).
Then \(G_0\) is a Lie group: obviously the quotient \(Y = L_2^p/\sim\) is a Lie group with the
usual addition of functions, \(\sim\) is the equivalence relation in \(L_2^p(X)\), \(\phi_1 \sim \phi_2\) if and
only if \( \phi_1(x) - \phi_2(x) = 2\pi n \), for almost all \( x \in X \), for some integer \( n \). And \( G_0 \) can be identified with \( Y \) by the exponential map. Hence \( G_0 \) is a Lie group with the multiplication of functions.

**Lemma 3.1.2.** Define \( G = \bigcup g \cdot G_0 \). \( G \) is a Lie group, here the union is over all components of \( C^\infty(X, s^1) \). \( G \) acts smoothly on \( A_2 \times L^2_1(S^+) \).

The proof of the following gauge fixing lemma can be found in [16].

**Lemma 3.1.3.** For any pair \((A, \phi), A_0\) a fixed smooth connection on \( S^+ \), there exists a gauge transformation \( g \), such that \( \alpha = g \cdot A - A_0 \) satisfies \( d^* \alpha = 0 \), and

\[
\| \alpha \|_{L^2_1} \leq c_1 \| F^+_A \|_{L^2} + c_2.
\]

where \( c_1, c_2 \) are constants.

### 3.1.3 Regularity of solutions.

The crucial point in the proof of both the regularity and compactness of weak solutions is the \( L^\infty \) bound of the spinor \( \phi \). As long as this is guaranteed, the regularity can then be obtained through the standard elliptic bootstrap method.

Since now the configuration space is \( L^2_1 \), the usual maximal-principle argument does not apply to prove the \( L^\infty \) bound. And the authors in [10] use the method in [20] to prove the following:

**Theorem 3.1.4.** Let \((A, \phi) \in A_2^2 \times L^2_1(S^+)\) be a weak solution of (5.1). Then

\[
\| \phi \|_{L^\infty} \leq \max\{-s(x), 0\}.
\]

**Proof.** Let \( s_0 = \min\{s(x) \mid x \in X\} \). If \( s_0 \geq 0 \), then since

\[
\int \| \nabla_A \phi \|^2 + \frac{s}{4} \| \phi \|^2 + \frac{1}{4} \| \phi \|^4 = 0,
\]

thus \( \phi = 0 \). Without loss of generality, assume \( s_0 = -1 \). Define test function \( \eta \) to be
\[ \eta = \begin{cases} (\| \phi \| - 1) \frac{\phi}{\| \phi \|}, & \text{for } \| \phi \| > 1, \\ 0, & \text{for } \| \phi \| \leq 1. \end{cases} \]  

(3.1.4)

and \( \Omega = \{ x \in X \mid \| \phi(x) \| > 1 \} \).

Let \( \nu = \frac{\phi}{\| \phi \|} \) for \( \| \phi \| \geq 1 \), \( \chi(\Omega) \) be the characteristic function of \( \Omega \). By noticing that \( \| \nu \| = 1 \) and \( \| \phi \| > 1 \), we conclude

\[ \nabla \eta = \chi(\Omega)((\| \phi \|)\nu + (\| \phi \| - 1)\nabla \nu) \in L^2, \]

since \( d(\| \phi \|)\nu = \frac{\nabla \phi \cdot \nu}{\| \phi \|} = \nabla \phi \cdot \nu = \nu \in L^2 \), and

\[ \chi(\Omega)((\| \phi \| - 1)\nabla \nu) = \chi(\Omega)((\| \phi \| - 1) \frac{\nabla \phi}{\| \phi \|} - \frac{1}{2}(\| \phi \| - 1) < \nabla \phi, \phi > \phi > \phi) > \]

\[ = \chi(\Omega)((1 - \frac{1}{\| \phi \|})\nabla \phi - \frac{1}{2}(1 - \frac{1}{\| \phi \|}) < \nabla \phi, \nu > \nu \in L^2). \]

Now that \((A, \phi)\) is a solution of (5.1), we have

\[ 0 = \int_\Omega < \nabla A \phi, \nabla A \eta > + \frac{s}{4} < \phi, \eta > + \frac{1}{4} \| \phi \|^2 < \phi, \eta > > \]

\[ = \int_\Omega < \nabla A \phi, \nabla A \eta > + \frac{1}{4}(\| \phi \|^2 + s)(\| \phi \| - 1) \| \phi \| \]

\[ \geq \int_\Omega < \nabla A \phi, \nabla A \eta > + \frac{1}{4}(\| \phi \|^2 - 1)(\| \phi \| - 1) \| \phi \|, \]

for \( s \geq -1 \). We show that the first term is also nonnegative: for

\[ \int_\Omega < \nabla A \phi, \nabla A \eta > = \int_\Omega < \nabla A \phi, d(\| \phi \|)\nu > + < \nabla A \phi, (\| \phi \| - 1)\nabla A \nu > \]

\[ = d(\| \phi \|)\nu, d(\| \phi \|)\nu > + < \nabla \phi, \nabla \nu, d(\| \phi \|) > \]

\[ + < d(\| \phi \|)\nu, (\| \phi \| - 1)\nabla A \nu > + < \| \phi \| \nabla A \nu, (\| \phi \| - 1)\nabla A \nu > \]

\[ = d(\| \phi \|)\nu, d(\| \phi \|)\nu > + \| \phi \| (\| \phi \| - 1) < \nabla A \nu, \nabla A \nu > \]

\[ \geq 0 \]

(3.1.5)
The last equality in the above is due to the fact that $\langle \nu, \nabla_A \nu \rangle = 0$ since $\nu$ is a unit normal vector. Thus $\Omega$ has measure zero, and $\| \phi \|_{L^\infty} \leq 1$.

With the $L^\infty$ bound at hand, it is straightforward to prove the regularity property and the compactness theorem.

**Theorem 3.1.5.** Let $(A, \phi) \in A_1^2 \times L_1^2(S^+)$ be a solution of the Seiberg-Witten equation. Then there exists a gauge transformation $g \in G$ such that $g(A, \phi) = (g(A), g^{-1} \phi)$ is smooth. Moreover, for any sequence $(A_i, \phi_i)$, there exists a subsequence (also denoted by $(A_i, \phi_i)$), and $g_i$, such that $g_i \cdot (A_i, \phi_i)$ converges in $C^\infty$ to a solution $(A, \phi)$.

**Proof.** As before, we choose a base connection $A_0$. From the gauge fixing lemma, there exists a gauge transformation $g, \alpha = g \cdot A - A_0$, such that $d^* \alpha = 0$. Thus the Seiberg-Witten equation is of the form

$$\begin{cases} (D_{A_0} \phi = i\alpha \cdot \phi, \\ (d\alpha)^+ = \sigma(\phi) - F_{A_0}^+, \\
 d^* \alpha = 0, \end{cases} \quad (3.1.6)$$

Since $\| \phi \|_{L^\infty} < C$, again from Lemma 3.1.3, $\alpha$ has uniformly bounded $L_1^p$ norm for any $p > 1$. In particular, take $p > 4$, we see that it is bounded in $C^0$ by the Sobolev Embedding Theorem, and hence $\alpha \cdot \phi$ is bounded in $L^p$ for any $p$. It follows that $\phi$ is bounded in $L_1^p$ for any $p > 4$. Since $L_1^p$ has Banach algebra structure, it follows from elliptic bootstrap that $(\alpha, \phi)$ is bounded in $L_1^p$ for any $k \geq 2$. Then Sobolev Embedding Theorem says that $(\alpha, \phi)$ is smooth.

For compactness property, notice that $\phi$ has a uniform $L^\infty$ bound, from the exact argument above, for any sequence $(A_i, \phi_i)$, we have a uniform $L_1^p$ bound for any integer $k \geq 1$ and $p > 4$, thus Rellich's Theorem produces a subsequence converges in $L_1^p$ for all $k$. This proves the compactness.

□

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3.2 Configuration space over the complement of the surface.

3.2.1 Spin$^c$ structures.

As before, $X$ is a smooth, connected, oriented closed four manifold, $\Sigma$ is a smooth, oriented, embedded surface in $X$. Denote by $P$ the circle bundle of the normal bundle of $\Sigma$ with degree $n$. $\omega$ is a connection form on $P$, such that $d\omega = \pi^* d\text{vol}_\Sigma$. The following lemma summarizes some relevant cohomological information about these spaces. The (co)homology groups are with integer coefficients.

**Lemma 3.2.1.** Let $X, P, \Sigma, n \neq 0$ are as above.

\[
H^1(P) \cong H^1(\Sigma), \quad H^2(P) \cong \mathbb{Z}_n \oplus H^1(\Sigma)
\]

\[
H^1(X\setminus \Sigma) \cong H^1(X), \quad H^2(X\setminus \Sigma) \cong H^2(X)/\xi \oplus F,
\]

where $\xi$ is the Poincare dual of $\Sigma$ and $F$ is a subgroup of $H^1(\Sigma)$.

**Proof.** The cohomology groups of $P$ comes from the Gysin exact sequence of $S^1 \rightarrow P \rightarrow \Sigma$. The Gysin exact sequence implies

\[
H^2(P; \mathbb{Z}) \cong \pi^* H^2(\Sigma; \mathbb{Z}) \oplus H^1(\Sigma; \mathbb{Z}) \cong \mathbb{Z}_n \oplus \mathbb{Z}^{2g}.
\]

As for the cohomology of $X\setminus \Sigma$, we can use the Poiccare duality and the excision principle: $H^2(X\setminus \Sigma) \cong H_2(X\setminus \Sigma, P) \cong H_2(X, U)$, $U$ is the tubular neighborhood of $\Sigma$. And $H_2(X, U)$ can be calculated from the exact sequence of the pair $H_2(X, U)$:

\[
H_3(X) \rightarrow H_3(X, N) \rightarrow H_2(U) \rightarrow H_2(X) \rightarrow H_2(X, U) \rightarrow H_1(U).
\]

Since $H_1(N)$ is a free abelian group, we have $H_2(X, U) \cong H_2(X)/\xi \oplus F$, $F$ is some subgroup of $H_1(U)$. Similarly $H^1(X\setminus \Sigma) \cong H_3(X, U) \cong H^1(X)$.

$\square$
Any line bundle \( H \) on \( P \) such that \( c_1(H) = k \in \mathbb{Z}_{[n]} \) can be obtained as a pull back of a line bundle \( H_{\Sigma} \) on \( \Sigma \), \( c_1(H_{\Sigma}) = k \in \mathbb{Z} \). Note that \( k \) is determined only modulo \( n \). So this does not give a faithful correspondence between the line bundles on \( \Sigma \) and the torsion line bundles over \( P \). However, if one equips the line bundle with a connection, one gets a faithful correspondence, as described in [17]:

**Lemma 3.2.2.** ([17]) There is a natural one-to-one correspondence between pairs bundles-with-connection over \( \Sigma \) and bundles-with-connection over \( P \), whose curvature forms pull up from \( \Sigma \) and whose fiberwise holonomy is trivial.

Let \( C \) be a flat connection on \( H \) with holonomy \( \exp(2ik\pi/n) \). Set \( C_k = C + (2ik\pi/n)\omega \), then \( C_k \) has trivial holonomy along the fibers and the curvature form of \( C_k \) is horizontal, \( F_{C_k} = 2ik\pi \omega \). Thus \( C_k \) is the pull back of a connection \( C_{\Sigma} \) on a line bundle \( H_{\Sigma} \) whose curvature satisfies

\[
\pi^*F_{C_{\Sigma}} = 2i\pi k/n\omega = 2i k \pi^*d\nuol_{\Sigma}.
\]

Thus \( c_1(H_{\Sigma}) = k \).

Now any \( Spin^c \) bundle \( S_P \) over \( P \) with torsion determinant line bundle can be written as \( S^+ = \pi^*K_{\Sigma}^{-\frac{1}{2}} \otimes H \oplus \pi^*K_{\Sigma}^{\frac{1}{2}} \otimes H \).

Suppose \( S \) is a \( Spin^c \) structure on \( X\setminus \Sigma \), whose restriction to \( P \) is determined by a torsion line bundle and it extends to a \( Spin^c \) structure on \( X \). The extension is not unique, and any two extensions differ by a power of the line bundle \( N \), the normal bundle of \( \Sigma \). (The pull back of \( N \) to \( P \) is trivial, so we can extend \( N \) to a line bundle over \( X \), still denoted by \( N \)). This fact can be seen as follows: first changing the \( Spin^c \) extension on \( X \) by a power of \( N \) does not change the induced \( Spin^c \) structure on \( X\setminus \Sigma \), since \( N \) is trivial on \( X\setminus \Sigma \). On the other hand, suppose two \( Spin^c \) structures on \( X \) differ by a line bundle \( L \). If restricted to \( X\setminus \Sigma \), they give the same \( Spin^c \) structure, then \( L \) is trivial on \( X\setminus \Sigma \), hence \( c_1(L) \) lies in the kernel of the restriction map \( H^2(X) \rightarrow H^2(X\setminus \Sigma) \), which is generated by the Poincare dual of \( \Sigma \), thus \( c_1(L) = m c_1(N) \), \( L = N^m \).

Although the extension of a given \( Spin^c \) structure \( S \) on \( X\setminus \Sigma \) is not unique, when
coupled with an extendable connection, from the above lemma, we do have a unique extension. More specifically, write $S^+|_P = (\pi^*K_{\Sigma}^{-\frac{1}{2}} \oplus \pi^*K_{\Sigma}^{\frac{1}{2}}) \otimes H$, let $A_k$ a $Spin^c$ connection on $S^+$, such that restricted to $P$, it is obtained by coupling the $Spin$ connection on $\pi^*K_{\Sigma}^{-\frac{1}{2}} \oplus \pi^*K_{\Sigma}^{\frac{1}{2}}$ and a connection $C_k$ on $H$. (Recall that $C_k = C + ik/n\omega$ is a connection with trivial holonomy on $H$.) And $A_k$ has $-1$ holonomy since the $Spin$ connection on $\pi^*K_{\Sigma}^{-\frac{1}{2}} \oplus \pi^*K_{\Sigma}^{\frac{1}{2}}$ does. From the above lemma, the pair $(H, C_k)$ is a pull back of a unique pair $(H_{\Sigma}, C_{\Sigma})$ on $\Sigma$. Thus $(S^+, A_k + \frac{1}{2}i\omega)$ extends uniquely to a pair $(S_X, A_X)$ on $X$. Let $\mathcal{L}$ denote the determinant line bundle of $S_X$, then $c_1(\mathcal{L})(\Sigma) = 2k + n$.

3.2.2 Configurations space of singular monopoles.

In order to be able to consider the Seiberg Witten equations over the complement of the surface, we need to choose an appropriate function space to study the equations. Recall that we chose to work with smooth metric $g$ which can be extended to $X$. And the $Spin^c$ structure $S$ is the one we mentioned above, whose determinant line bundle is torsion restricted to $P$. Let $A_k + ic\omega$ be the reference connection. Thus $A_k + ic\omega$ has holonomy $\exp(2i(c - \frac{1}{2})\pi)$ along the circles. Notice that if we change $c$ to $c + l$ for some integer $l$, then $A_k + il\omega + ic\omega$ has the same holonomy and $A_k + il\omega = A_{k+l}$.

Thus we only need to fix $c \in (-\frac{1}{2},0) \cup (0, \frac{1}{2})$. When $c = \frac{1}{2}$, the holonomy is trivial, and it reduces to the usual analysis on the compact manifold $X$.

Now for the spinors, we will use the space we studied in Chapter 2, i.e., $V^+ = \{\phi \in L^p(S^+), D_{A_k}\phi \in L^p(S^-), (r^{\frac{1}{2}}\phi)(0) \in S^+\}$ for $c = 0, V^+ = \{\phi \in L^p(S^+), D_{A_k}\phi \in L^p(S^-), (r^{\frac{1}{2}}\phi)(0) = 0\}$ for $c \in (-\frac{1}{2},0) \cup (0, \frac{1}{2})$.

For the connections, we use the following weighted sobolev spaces, with different weight of the spaces introduced in [11].

For some $p > 2$,

$$C^p_1 = \{\alpha \in \Omega^1(X\setminus \Sigma) \mid \alpha \in L^p(i\Omega^1(X\setminus \Sigma)), \nabla_X\alpha \in r^{-1}L^p(i\Omega^+(X\setminus \Sigma))\}, \quad (3.2.1)$$

$$C^p_2 = \{f \mid f \in rL^p, \nabla f \in L^p, \nabla^2 f \in r^{-1}L^p\}. \quad (3.2.2)$$

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And \( C_0^p = \{ f \mid f \in r^{-1} L^p \} \). Here \( \nabla_X \) is the induced Levi-Civita connection on 1-forms, and \( \alpha \in r^c L^p \) means that \( r^{-c}\alpha \in L^p \).

Let \( \mathcal{A}_k = \{ A_k + ic\omega + \alpha \mid \alpha \in C_1^p \} \). And the configuration space we choose to work with is \( \mathcal{C}_k = V^+ \times \mathcal{A}_k \).

The above spaces are isometric to the following weighted Sobolev spaces introduced in [11]:

For nonnegative integer \( k \) and \( p > 2 \), \( W_k^p \) is the completion of the space of compactly supported smooth functions on \( X \setminus \Sigma \) in the norm

\[
\| f \|_{W_k^p} = \| \frac{1}{r^k} f \|_p + \| \frac{1}{r^{k-1}} \nabla f \|_p + \cdots + \| \nabla^k f \|_p.
\]

Obviously \( W_k^p \subset L_k^p \), and \( W_0^p = L^p \). And the multiplication map \( T : C_k^p \rightarrow W_k^p \), \( T(f) = r^2 f \) is an isometry map.

In [11], the authors studied singular Yang-Mills equations along an embedded surface. There the connection near the surface is

\[ A = a^0 + b \]

with \( a^0 \) being the diagonal part and \( b \) being the off diagonal part such that \( a^0 \) is in the ordinary \( L^p_1 \) space and \( b \) is in \( W_1^p \) as above. The anti-self-dual equations can be written as

\[ d^a a^0 = -[b,b]^0 \]
\[ d^+ b = -[a^0,b] \]

Formally, the Seiberg Witten equations look similar as the above equations, with \( a^0 \) being replaced by the \( spin^c \) connection and \( b \) being replaced by the spinor. The configuration spaces we chose here are actually the analogue of the configurations spaces for the singular connections, at least up to an isometry: for any spinor \( \phi \in V^+ \), since \( \phi \sim O(r^{-\frac{1}{2}}) \), we can check that \( r\phi \in W_1^p(S^+, X \setminus \Sigma) \) for appropriate constant \( p > 2 \), also for any \( \alpha \in C_0^p \), we know that \( r\alpha \in L^p \), and \( \nabla(r\alpha) = dr \otimes \alpha + r\nabla\alpha \in L^p \), so \( r\alpha \in L_1^p \). Thus up to the weight \( r \), our configuration spaces are such that spinors lie in \( W_1^p \) and connections lie in \( L_1^p \).
It is shown in [11] that usual embedding theorem also holds for $W^p_k$, and so does the elliptic theory: let

$$w^p_k = k - p/4,$$

**Lemma 3.2.3.** ([11].) If $k \geq l$ and $w^p_k \geq w^q_l$, then there is an inclusion $W^p_k \hookrightarrow W^q_l$. If both inequalities are strict, then the inclusion is compact.

**Lemma 3.2.4.** The usual statements of elliptic regularity also holds for $S$:

$$S = (d^* + \frac{c_1}{r}) \oplus (d^* + \frac{c_2}{r}) : \Omega^1(X) \to \Omega^0(X) \oplus \Omega^+(X).$$

Due to the isometry $T$, we have

**Lemma 3.2.5.** The Sobolev embedding property also holds for $C^p_k$. And the elliptic regularity also holds for $d^* \oplus d^+$ acting on $C^p_k$.

Define $\mathcal{A} = A_k + C^p_1$. The configuration space we choose to work with is $\mathcal{C} = V^+ \times \mathcal{A}$.

A gauge transformation $\sigma \in L^p_{2,loc}(X \setminus \Sigma, S^1)$ belongs to the gauge group based at $\Sigma, \mathcal{G}_\Sigma$, if there exists $f \in C^p_2$, so that $\sigma = \exp(f)$. Define $\mathcal{G} = \bigcup_g g \cdot \mathcal{G}_\Sigma$, where the union is over all the harmonic representatives of $H^1(X)$.

### 3.3 Deformation Complex.

Seiberg-Witten equations on the complement of the surface give rise to a well defined map

$$SW : V^+ \times \mathcal{A} \to LP(S^-, X \setminus \Sigma) \times r^{-1}LP(i\Omega^+).$$

The deformation complex $\mathcal{D}_{(A, \Psi)}$ of the solution $(A, \Psi)$, taking into account of the action of $\mathcal{G}$, is

$$0 \to C^p_2 \xrightarrow{K_{(A, \Psi)}} V^+ \times \mathcal{A} \xrightarrow{T_{(A, \Psi)}SW} L^2(S^-) \times r^{-1}LP(i\Omega^+(X \setminus \Sigma)) \to 0, \quad (3.3.1)$$
where $K_{(A,\psi)}(f) = (2df, -f\psi)$ is the infinitesimal gauge group action and

$$T_{(A,\psi)}SW(\alpha, \psi) = (d^+\alpha - 2Q(\Psi, \psi), D_A\psi + \alpha \cdot \Psi)$$

is the linearization of the Seiberg-Witten map at $(A, \Psi)$. Here $Q$ is the bilinear map associated to the quadratic map $q$ in the Seiberg-Witten equations. The cohomology groups of this complex provide some local information about the based moduli space. The first observation is the following:

**Lemma 3.3.1.** The zeroth cohomology group of the deformation complex is trivial.

**Proof.** If $f \in C^2_2$ is in the kernel of $K_{(A,\psi)}$, then $df = 0$. Thus $f$ is constant and from our definition of $G^2_2$, $f$ converges to $0$ near $\Sigma$, it must be identically equal to zero. □

The first cohomology group of the deformation complex is called the *Zariski tangent space* of the moduli space and the second cohomology group is called the *obstruction space*. If $(A, \Psi)$ is a regular point for the Seiberg-Witten map, then the obstruction space vanishes and the first cohomology of the complex is isomorphic to the tangent space of the moduli at $[A, \Psi]$.

Let $(A, \Psi)$ be a configuration on $X \setminus \Sigma$. The adjoint $K^*_{(A,\psi)}$ of the infinitesimal gauge group action $K_{(A,\psi)}$ is defined with respect to the following inner products: for imaginary-valued forms $\alpha$ and $\beta$ let

$$< \alpha, \beta >_r = \int_X \alpha \wedge * \beta r^{-2},$$

where $*$ is the complex anti-linear extension of the Hodge star operator.

Then $K^*_{(A,\psi)}(\alpha, \psi) = 2r^2 d^* r^{-2} + 2i\text{Im} < \Psi, \psi >$; we can drop the factor $2$, thus obtaining the wrapped-up complex

$$F : A \times V^+ \to L^p(i\Omega(X) \oplus i\Omega^+(X)) \oplus L^p(S^-),$$

$$F(\alpha, \psi) = (d^*\alpha - \frac{2}{r} + i\text{Im} < \Psi, \psi >, d^+\alpha - 2Q(\Psi, \psi), D_A\psi + \alpha \cdot \Psi).$$
3.4 The equations over a Kahler manifold.

On a closed Kahler manifold \((X, \Omega)\), there is a canonical \(\text{Spin}^c\) structure given by

\[
S_0^+ = \Omega^0(X; \mathbb{C}) \oplus \Omega^{0,2}(X; \mathbb{C}),
\]

\[
S_0^- = \Omega^{0,1}(X; \mathbb{C}).
\]

Any other \(\text{Spin}^c\) structure \(S\) differs from \(S_0\) by tensoring with some complex line bundle \(L\), and is given by

\[
S^+ = \Omega^0(X; L) \oplus \Omega^{0,2}(X; L),
\]

\[
S^- = \Omega^{0,1}(X; L).
\]

Furthermore, the determinant of \(S^+\) is \(K_X^{-1} \otimes L^2\).

Recall that \((S, A_k)\) is a pair of \(\text{Spin}^c\) bundle and connection over \(X\setminus \Sigma\), such that \((S, A_k + i\frac{1}{2} \omega)\) extends uniquely to a pair of \((S_X, A_X)\) on \(X\). Write \(S_X^+ = \Omega^0(X; L) \oplus \Omega^{0,2}(X; L)\). And \(L\) is the determinant line bundle of \(S_X\), \(c_1(L)(\Sigma) = 2k + n\). For any holomorphic connection \(B\) on \(L\), let \(B_1\) be the induced holomorphic connection on \(L\).

Let \((A, \phi) \in \mathcal{C}_k\) be a solution to the Seiberg-Witten equation. From our definition of the singular configuration space, \(A = A_k + i \omega + i a\), for some \(a \in C_1^p\). Write \(\phi = (\alpha, \beta) \in S^+ \oplus S^-, \alpha\) has leading order \(r^{\frac{k}{2} - c}\) near \(\Sigma\) while \(\beta\) is of \(O(1)\) near \(\Sigma\). Due to the Kahler structure, we can write the Seiberg-Witten equations:

\[
\begin{align*}
\bar{\partial}_A \alpha + \partial_A \beta &= 0, \\
(F_A^+)^{1,1} &= \frac{1}{4} (|| \alpha ||^2 - || \beta ||^2) \omega, \\
F_A^{0,2} &= \frac{\bar{\alpha} \beta}{2}.
\end{align*}
\]

Now the configuration space \(\mathcal{C}_k\) is acted on by the complex gauge group which is defined to be \(\mathcal{G}^C = \text{Map}(X \setminus \Sigma, \mathbb{C}^*)\). A complex gauge transformation \(e^f\) acts on configurations by

\[
A \to A + \bar{\partial} f - \partial f
\]
\[\alpha \rightarrow e^f \alpha\]

\[\beta \rightarrow e^{-f} \beta\]

When \(f\) is pure imaginary this coincides with the usual action of \(G\). And it is easy to verify that the first and third equation in (3.4.1) are preserved by this action.

Choose \(e^f = r^{c-\frac{1}{2}}\), and \(\alpha_1 = r^{c-\frac{1}{2}} \alpha, \beta_1 = r^{\frac{1}{2} - c} \beta\). Under this change, \(B = A + (\frac{1}{2} - c) \partial \ln r - (\frac{1}{2} - c) \overline{\partial} \ln r\) is a nonsingular \(Spin^c\) connection which can be extended to a \(Spin^c\) connection on \(S_X\).

Now the standard proof that Seiberg-Witten solutions on Kahler manifold have vanishing \(F_0, 2\) directly generalizes to our case: we have the harmonic spinor equation

\[\overline{\partial}_B \alpha + \overline{\partial}_B^* \beta = 0,\]

Apply \(\overline{\partial}_B\) to this equation we get

\[\overline{\partial}_B \overline{\partial}_B \alpha_1 + \overline{\partial}_B \overline{\partial}_B^* \beta_1 = 0.\]

Of course, \(\overline{\partial}_B \overline{\partial}_B \alpha = F_B^{0,2} \cdot \alpha_1\), but \(F_A^{0,2} = F_B^{0,2} = \overline{\alpha} \beta = \overline{\alpha}_1 \beta_1\), we have

\[\| \alpha_1 \|^2 \beta_1 + \overline{\partial}_B \overline{\partial}_B^* \beta_1 = 0,\]

Due to the behavior of \(\alpha_1\) and \(\beta_1\) near \(\Sigma(\alpha_1\) is of order \(O(1)\), and \(\beta_1\) is of order \(O(r^{\frac{1}{2} - c})\), we have that \(\int_P \overline{\partial}_B^* \beta_1, \beta_1 > \rho \text{d}g_{N_1}\) goes to 0 when \(r\) goes to 0, here \(g_{N_1}\) is a fixed metric on \(N\). Thus we are justified to take the \(L^2\) inner product with \(\beta\) and get

\[\int_X \frac{1}{2} \| \alpha_1 \|^2 \| \beta_1 \|^2 \text{dvol} + \| \overline{\partial}_B^* \beta_1 \|^2_{L^2} = 0,\]

Since each of these terms is nonnegative, it follows that they both vanish. And \(\overline{\alpha}_1 \beta_1 = 0\). This means that \(F_A^{0,2} = 0\), and hence \(A\) is a holomorphic connection, \(\beta_1\) is an anti-holomorphic section, and \(\alpha_1\) is a holomorphic section. In particular, if \(X\) is connected, from unique continuation principle, either \(\alpha_1\) or \(\beta_1\) vanishes identically on \(X\).
Furthermore, since

\[(F_A^+)^{1,1} = (F_B^+)^{1,1} + i(c - \frac{1}{2})d\omega = \frac{i}{4}(\|\alpha\|^2 - \|\beta\|^2)\Omega.\]

Thus \[(2c_1(L) - c_1(K_X) + (c - \frac{1}{2})\xi) \cdot \Omega = \int_X (2c_1(L) - c_1(K_X) + (c - \frac{1}{2})\xi) \wedge \Omega = \frac{1}{8\pi} \int_X (\|\beta_1\|^2 - \|\alpha_1\|^2) d\text{vol},\] here \(\xi\) is the Poincare dual of \(\Sigma\).

Since at least one of \(\alpha_1\) and \(\beta_1\) is zero, we see that if \[(2c_1(L) - c_1(K_X) + (c - \frac{1}{2})\xi) \cdot \Omega\]
is nonnegative, then \(\alpha_1 = 0\) and if \[(2c_1(L) - c_1(K_X) + (c - \frac{1}{2})\xi) \cdot \Omega\]
is non-positive, then \(\beta_1 = 0\). Without loss of generality, we assume that \[(2c_1(L) - c_1(K_X) + (c - \frac{1}{2})\xi) \cdot \Omega\]
is nonpositive. Thus the Seiberg-Witten equations reduce to the following:

\[
\begin{cases}
\overline{\partial}_B \alpha_1 = 0, \\
(2F_{B_1}^+ - F_{K_X}^+)^{1,1} = \tau^{2c-1}\frac{i}{4} \|\alpha_1\|^2 \omega - cd\omega, \\
F_{B_1}^{0,2} = 0.
\end{cases}
\] (3.4.2)

To sum up, if \[(2c_1(L) - c_1(K_X) + (c - \frac{1}{2})\xi) \cdot \Omega \leq 0,\] the Seiberg Witten equations reduce to the above singular vortex equation, \((B_1, \alpha_1)\) is an effective divisor on \(L\).

### 3.5 Identification of two moduli spaces.

For compact Kahler manifold, there is a holomorphic description of the solutions to the Seiberg-Witten equations. More specifically, the following correspondence has been proved in [16]:

**Theorem 3.5.1.** Let \((X, \omega)\) be a closed Kahler surface, \(S\) be a \(\text{Spin}^c\) bundle over \(X\), \(S^+ = (\Omega^0(X; L) \oplus \Omega^{0,2}(X; L))\). Let \((A, \psi)\) be a solution to the Seiberg-Witten equations for \(S\), with \(\psi = (\alpha, \beta) \in \Omega^0(X; L) \oplus \Omega^{0,2}(X; L)\). Then if \(\int_X (2c_1(L) - c_1(K_X)) \wedge \Omega \leq 0,\) we have \(\beta = 0, A\) is a holomorphic connection on \(L\), \(\alpha\) is a holomorphic section of \(L\), with

\[
(F_A^+)^{1,1} = \frac{i}{4} |\alpha|^2 \omega.
\]

if the \(\int_X (2c_1(L) - c_1(K_X)) \wedge \Omega \geq 0,\) we have \(\alpha = 0, A\) is a holomorphic connection.
on $\mathcal{L}$, and $\beta$ is an anti holomorphic section on $L$, with

$$(F_A^+)_{1,1} = -\frac{i}{4}|\beta|^2\omega.$$ 

Conversely, suppose that $\int_X (2c_1(L) - c_1(K_X)) \wedge \Omega \leq 0$, $A$ is a hermitian, holomorphic connection on $\mathcal{L}$, $\alpha$ is a nonzero holomorphic section of $L$, then there exists another hermitian structure $h'$ on $L$ such that for the connection $A'$ which is hermitian with respect to $h'$ and which defines the same holomorphic structure on $L$ as $A$ does we have

$$F_{A'}^{1,1} = \frac{i}{4}(\alpha|_{h'})^2 \omega$$

where $(\alpha|_{h'})^2$ means the norm measured with respect to the hermitian structure on $L_0$ determined by $h'$.

Similar results hold for $\beta$ when $\int_X (2c_1(L) - c_1(K_X)) \wedge \Omega \geq 0$.

In this section, we are going to prove an analogous identification theorem for the singular moduli space considered previously.

Let $\Sigma$ be a holomorphically embedded curved in a closed Kahler manifold $(X, \Omega)$. $\Sigma$ is a holomorphically embedded surface. $(L, h_0)$ is hermitian line bundle over $X$, such that $L|_P$ is torsion, $c_1(L)(\Sigma) = k + g - 1$. $B_k$ is the connection on $L$ such that when restricted to $P$, $B_k$ is a pull back connection from a connection on a line bundle $L_{\Sigma}$ over $\Sigma$.

For some $p > 2$, define

$$B^p = \overline{\partial}_B + \{a \in i\Omega^{0,1}(X), a \in C^p_1\},$$

$$\mathcal{G}^p = \{e^g | g \in C^p_2 \text{ is a complex valued function}\},$$

$$\mathcal{D}^p_1 = \{((\overline{\partial}_B, \alpha) \in B^p \times L^p_1(L) | F^0_{B} = 0, \overline{\partial}_B \alpha = 0\},$$

where $L^p_1(L)$ is the usual Sobolev space. Also recall that $\mathcal{M}_k = \{(A, \phi) \in C_1 | SW(A, \phi) = 0\}$.

So $\mathcal{G}^p_2$ is the complex gauge group, and $\mathcal{D}^p_1$ is the collection of effective divisors.
\( \mathcal{G}_2^{C_p} \) acts on \( \mathcal{D}_1^p \) as follows: for any \( e^g \in \mathcal{G}_2^{C_p} \), \( (\overline{\partial}_B, \alpha) \in \mathcal{D}_1^p \),

\[
\overline{\partial}_B \rightarrow e^g \overline{\partial}_B e^{-g},
\]

\[
\alpha \rightarrow e^g \alpha,
\]

\[
B \rightarrow B + \partial g - \overline{\partial} g.
\]

here \( B \) is the connection compatible with \( \overline{\partial}_B \) and \( h_0 \). From the definition the complex gauge group does form a group and the above action preserves \( \mathcal{D}_1^p \).

Now we are ready to state the following correspondence theorem:

**Theorem 3.5.2.** Let \( S_X = S_0 \otimes L \) be a Spin\(^c\) bundle on a Kahler manifold \((X, \Omega)\), \( \Sigma \) be a holomorphically embedded surface, \( n \) is the intersection number of \( \Sigma \). \( \mathcal{L} \) is the determinant line bundle such that \( c_1(\mathcal{L})(\Sigma) = 2k + n \). If \( (2c_1(L) - c_1(K_X) + (c - \frac{1}{2})\xi) \cdot \Omega \) is nonpositive, then we have a one to one correspondence:

\[
i : \mathcal{M}_k/G_2^p \rightarrow \mathcal{D}_1^{kp}/\mathcal{G}_2^{C_p}.
\]

**Proof.** First from the adjunction formula, \( c_1(\mathcal{L})(\Sigma) = 2k + n \) implies that \( c_1(L)(\Sigma) = k + g - 1 \). We have proved in Section 3.4 that if \( (2c_1(L) - c_1(K_X) + (c - \frac{1}{2})\xi) \cdot \Omega \) is nonpositive, then any for solution \((A, \phi) \in C_k, \phi = (\alpha, 0), \) after the complex gauge group transformation \( e^f = r^{\frac{1}{2} - c} \), the pair \((B, \alpha_1)(B = A + (\frac{1}{2} - c)\partial \overline{f} - (\frac{1}{2} - c)\overline{\partial} f, \alpha_1 = r^{\frac{1}{2} - c}\alpha) \) satisfy equation (3.4.2), i.e., \( B \) defines a holomorphic connection on the determinant line bundle \( \mathcal{L} \), let \( B \) also denote the induced holomorphic connection on \( L, B - B_k \in C_1^p, \) and \( \alpha_1 \) is a holomorphic section of \( L \), which give us the map \( i : (A, \phi) \rightarrow (B, \alpha_1). \)

The following theorem tells us that for any \( (\overline{\partial}_B, \alpha) \in \mathcal{D}_1^p \), there exists a unique \( e^f \in \mathcal{G}_1^{C_p} \) such that after the gauge transform, the effective divisor also satisfies the curvature equation in (3.4.2), thus \( i \) is sujective and injective.

\[\square\]

**Theorem 3.5.3.** Let \( S_X = S_0 \otimes L \) be a Spin\(^c\) bundle on \( X \), \( \Sigma \) is a holomorphically
embedded surface with genus $g$. $c_1(L)(\Sigma) = k + g - 1$. If $(2c_1(L) - c_1(K_X) + (c - \frac{1}{2})\xi) \cdot \Omega$ is nonpositive, then for any pair $(\overline{\partial} B, \alpha) \in \mathcal{D}_1^p$, there exists a $f \in C^2_\psi$ such that after the gauge transform $e^f$, $(\overline{\partial}_H', \alpha')$ satisfies

$$(2F_{B'} - F_{KX})^{1,1} = \frac{1}{4} r^{2c-1}|\alpha_1|^{2}_{h'} - ic(d\omega)^{1,1}.$$

Proof. In order to solve

$$(2F_{B'} - F_{KX})^{1,1} = \frac{1}{4} r^{2c-1}|\alpha_1|^{2}_{h'} - ic(d\omega)^{1,1}.$$  

First observe that for the gauge transformation $e^f$, we can write the above as an equation for $f$:

$$-i \wedge (2F_B - F_{K_X} + \partial f - \overline{\partial} f) = \frac{1}{4} r^{2c-1}|\alpha_1|^{2} e^{2f} - ic(d\omega)^{1,1}.$$  

Define $w = -4i \wedge (2F_B - F_{K_X} - 4icd\omega)$, $\int w \geq 0$ from the assumption. And using the Kahler identity $-4i \wedge (\partial \overline{\partial} f = \Delta f$, the above equation becomes

$$\Delta f + r^{2c-1}|\alpha_1|^{2} e^{2f} - w = 0,$$  

(3.5.1)

Now follow the exact procedure in [6], we can construct a solution $f$. And the elliptic regularity in Section 3.2 will make sure that $f \in C^2_\psi$. First using the method in [6] to construct a sub-solution $f_-$ and a super-solution $f_+$ for equation (3.5.1), i.e., functions $f_-$ and $f_+$ such that $f_+ > f_-$ everywhere and

$$\Delta f_- + r^{2c-1}|\alpha_1|^{2} e^{2f_-} - w < 0$$

$$\Delta f_+ + r^{2c-1}|\alpha_1|^{2} e^{2f_+} - w > 0.$$  

Then define a sequence of functions inductively by setting $f_0 = f_-$ and defining $f_{i+1}$ to be the unique solution to
\[ Lf_{i+1} = -r^{2c-1}|\alpha_1|^2 e^{2f_i} + w + kf_i \]

where \( L = \Delta + k \) for a suitably defined nonnegative function \( k \).

Then using \( f_i \) to construct a solution to equation (3.5.1).

Notice that \( \Delta v = u \) has a solution whenever \( \int u = 0 \). If \( u \in \mathcal{L}^p \), then \( v \in \mathcal{L}^p \).

To construct \( u_+ \), let \( v_1, v_2 \) be solutions to \( \Delta v_1 = w - \bar{w} \) and \( \Delta v_2 = c - r^{2c-1}|\alpha_1|^2 \) with \( c = \int r^{2c-1}|\alpha_1|^2 \) and \( \bar{w} = \int w \). Choose \( a \) to be constant large enough so that \( ac > \bar{w} \) and then choose a constant \( b \) large enough so that \( e^{v_1+av_2+b} - a > 0 \) and \( a - e^{-v_1-av_2-b} > 0 \). Let \( f_+ = v_1 + av_2 + b \) and it is easy to check that \( f_+ \) is a sup solution,

Simply define \( f_- = v_1 - n \) with \( n \) large enough constant so that \( f_- < f_+ \) and \( \bar{w} + r^{2c-1}|\alpha_1|^2 e^{v_1-n} < 0 \), then \( f_- \) will be a sub solution.

Obviously \( f_+, f_- \in \mathcal{L}^p \).

Let \( k = r^{2c-1}|\alpha_1|^2 e^{f_+} \).

Just as in [6], the sequence \( f_i \) defined recursively above satisfy \( f_{i+1} \geq f_i \). Thus we have uniform upper and lower bounds on \( f_i \), we get a uniform \( \mathcal{L}_2^p \) bounds on \( f_i \) from elliptic theory. Thus \( f_i \) has a subsequence converging uniformly to \( f \) in \( \mathcal{L}_1^r \) for some \( r > 4 \). In particular, \( f_i \) converge to \( f \) in \( C^0 \) norm.

From the definition of \( f_i \), it is not hard to see that \( f \) is a weak solution of 3.5.1.: for any smooth function \( h \),

\[
\int_X \nabla f_{i+1} \nabla h + kf_{i+1} h = \int_X (\Delta f_{i+1} + kf_{i+1}) h = \int_X (-r^{2c-1}|\alpha_1|^2 e^{2f_i} + w + kf_i) h
\]

let \( i \) go to infinity, we have

\[
\int_X \nabla f \nabla h + kf h = \int_X (r^{2c-1}|\alpha_1|^2 e^{2f} + w + kf) h.
\]

Since \( f \) is a weak solution of 3.5.1, \( f \in \mathcal{L}_2^p \) from the elliptic theory.

To prove that \( f \in C_2^p \), it suffices to prove \( g = rf \in W_2^p \) by recalling the isometry we mentioned:
Since \( f \) is the solution to (3.5.1), \( g \) is the solution to the following equation

\[
r \Delta \left( \frac{1}{r} g \right) + r r^{2c-1} |\alpha_1|^2 e^{2\frac{1}{r} g} - rw = 0
\]

(3.5.2)

Now that \( \Delta = -4i \wedge_\omega \partial \bar{\partial} \), we have

\[
-4i \wedge_\omega (\partial \bar{\partial} \left( \frac{1}{r} g \right) + r r^{2c-1} |\alpha_1|^2 e^{2\frac{1}{r} g} - rw = 0,
\]

alternatively, we can write

\[
-4i \wedge_\omega (\partial \bar{\partial} \left( \frac{1}{r} g \right) + r \partial r \wedge \bar{\partial} \left( \frac{1}{r} g \right)) = -rr^{2c-1} |\alpha_1|^2 e^{2\frac{1}{r} g} - rw = 0,
\]

since \( \frac{1}{r} g = f \) is continuous the right hand side of the above equation is in \( L^p \), and \( r \partial \bar{\partial} \left( \frac{1}{r} g \right) \) is also in \( L^p \) because \( f \in L^p \), thus \( r \partial \bar{\partial} \left( \frac{1}{r} g \right) \) is in \( W^p \), equivalently, \( r \partial \bar{\partial} \left( \frac{1}{r} g \right) = \bar{\partial} g - \frac{1}{r} g \) is in \( W^p \), which shows that \( g \in W^p \).

Uniqueness: if \( f_1, f_2 \) are two solutions, we have

\[
0 \leq |d(f_1 - f_2)|^2
= \int_X < \Delta (f_1 - f_2), f_1 - f_2 >
= \int_X -r^{-2(\frac{1}{r}-c)} |\alpha_1|^2 (e^{f_1} - e^{f_2})(f_1 - f_2)
\]

and the last integral is nonpositive, thus \( d(f_1 - f_2) = 0 \), \( f_1 - f_2 = c \) for some constant \( c \). But \( c \) has to be 0 otherwise the integral above is strictly negative. Thus \( f_1 = f_2 \).

### 3.6 Compactness of the moduli space for Kahler case.

In this section, we are going to prove that the moduli spaces we identified above is compact. The crucial point is to get a uniform \( L^\infty \) bound for the spinors. Notice that we are only working with \( L^p \) space here, so the maximal principle does not apply. Fortunately the argument in Section 3.1 works here. When the base manifold is not
Kahler, we can not reduce the equation, and we still do not know whether the moduli is compact or not.

By using the argument introduced in Section 3.1.3., we prove that $\alpha_1$ admits a uniform $L^\infty$ bound.

**Lemma 3.6.1.** Let $X$ be a closed Kahler surface, $\Sigma$ be a holomorphically embedded curve. $S$ is a $Spin^c$ structure on $X \setminus \Sigma$, which restricted to the tubular neighborhood of $\Sigma$ is torsion. If $(A, \phi) \in C_k$ is a solution to the Seiberg-Witten equation, then there exist a constant $C$ independent of $(A, \phi)$, such that $\| \frac{1}{2-c} \phi \|_{L^\infty} < C$.

**Proof.** First suppose that $(2c_1L) - c_1(K_X) + c\xi) \cdot \Omega$ is non-positive, thus $\beta = 0$, and $\phi = (\alpha, 0)$. From the above discussion, $(B, \alpha_1)$ satisfies equation (3.4.2), with $\alpha_1 = r^{1-c}\alpha$, and $B = A + \frac{1}{2} - c)\partial \ln r - (\frac{1}{2} - c)\bar{\partial} \ln r$ is a $Spin^c$ connection being able to be extended to $X$. Thus it suffices to prove that $\alpha_1$ has a uniform $L^\infty$ bound. Notice that $\alpha_1 \in L^p_1$. The same method for Theorem 3.1.1 applies here:

Without loss of generality, we can assume that $s(x) \geq -1$, $s(x)$ is the scalar curvature. Again define the test function $\eta \in L^2_1$ to be

$$\eta = \begin{cases} (\| \alpha_1 \| -1) \frac{\alpha_1}{\| \alpha_1 \|}, & \text{for } \| \alpha_1 \| > 1, \\ 0, & \text{for } \| \alpha_1 \| \leq 1. \end{cases} \quad (3.6.1)$$

and $\Omega = \{ x \in X \mid \| \phi(x) \| > 1 \}$.

Since $(A_1, \alpha)$ is a solution, we have

$$0 = \int_{\Omega} < \nabla_B \alpha_1, \nabla_B \eta > + \frac{s}{4} < \alpha_1, \eta > + \frac{1}{4} \| \phi \|^2 < r^{2c-1} \alpha_1, \eta >$$

$$= \int_{\Omega} < \nabla_B \alpha, \nabla_B \alpha_1 > + \frac{1}{4} (\| \alpha_1 \|^2 + s)(r^{2c-1} \| \alpha_1 \| \| -1 \| \| \alpha_1 \|$$

$$\geq \int_{\Omega} < \nabla_B \phi, \nabla_A \eta > + \frac{1}{4} (\| \phi \|^2 - 1)(r^{2c-1} \| \phi \| \| -1 \| \| \phi \|,$$

for $s \geq -1$. The first term is nonnegative from the proof of Theorem 3.1.1. Thus $\Omega$ has measure zero, and $\| \alpha_1 \|_{L^\infty} \leq 1$. \qed
Now we are ready to prove the following compactness theorem:

**Theorem 3.6.2.** Let $X$ be a closed Kahler surface, $\Sigma$ be a holomorphically embedded curve. $S$ is a Spin$^c$ structure on $X \setminus \Sigma$, which restricted to the tubular neighborhood of $\Sigma$ is torsion. Then for any sequence $(A^i, \alpha^i) \in C_k$ which are solutions to the Seiberg-Witten equations, there exists a subsequence, still denoted by $(A^i, \alpha^i)$, and a sequence of gauge transformation $g^i \in G$, such that $g^i \cdot (A^i, \alpha^i)$ converges in $C_k$.

**Proof.** Let $B_k = A_k + ic \omega + \overline{\partial f} - \overline{\partial f}$ be the reference connection, with $\omega' = r^{1 - c}$. After the complex gauge transformation of $e^i$ defined before, we have a sequence $(a^i, \alpha^i)$, such that $(B_k + a^i, \alpha^i)$ solve equation (3.4.2), with $a^i \in C^p_1$, and $\alpha^i \in L^2(S^+)$. And $(A^i, \alpha^i)$ converges in $C$ if and only if $(a^i, \alpha^i)$ converges in $C^p_1 \times L^2(S^+)$. From the usual gauge fixing theorem, for any sequence $(B^i, \alpha^i)$, we can find a sequence $g^i$ of gauge transformations, such that $g^i \cdot B^i$, still denoted by $B^i$, satisfy $d^* a^i = 0$, and $|a^i|_{L^p} < C$ for some constant $C$. For convenience, we rewrite equation 3.4.2. here:

$$
\begin{align*}
\left\{ \begin{array}{l}
\overline{\partial} B_k \alpha^i_1 + a^i_1 \cdot \alpha^i_1 = 0, \\
d^+ a^i = F_{B_k} + r^{2c-1} \sigma(a^i_1, \alpha^i_1), \\
d^* a^i = 0.
\end{array} \right.
\end{align*}
$$

Since $d^+ \oplus d^*$ has the usual elliptic regularity on $C^p_1$, $a^i$ has uniform $C^p_1$ bound, which in turn, through the first equation, gives a uniform $L^p_1$ bound on $\alpha^i_1$.

To get the bound for higher derivatives, we use the argument in [10]:

$\alpha_1$ is a weak solution to the following second order equation:

$$\begin{align*}
-\Delta_B \alpha_1 + \frac{s}{4} \alpha_1 + \frac{1}{4} r^{-2c} \| \alpha_1 \|^2 \alpha_1 = 0.
\end{align*}$$

From the above equation, we get for $1 < r \leq 2$,

$$\| \Delta \alpha_1 \|_{L^r} \leq c(\| \Delta_B \alpha_1 \|_{L^r} + \| A \|_{L^r} + \| B \| \| \nabla_B \alpha_1 \|_{L^r} + \| B \|^2 \|_{L^r}),$$

From the Holder inequality
Thus \( \| \alpha_1 \|_{L^2} \leq C. \)

From Rellich theorem, \( L^2_2 \rightarrow L^2_1 \) is compact, thus there exists a subsequence \( \alpha_i \) converging in \( L^2_1 \).

Now the right hand side of the second equation (3.4.4) has a uniform \( C^p \) bound: \( \sigma(\alpha_i, \alpha_i^j) \) has uniform \( L^s \) bound for any \( s < 4 \), thus a uniform \( C^s \) bound, and \( r^{2c-1} \in C^l_1 \) for \( l > 2 \), thus from Holder inequality, the whole term is bounded in \( C^p \). From Lemma 3.2.3, \( a^i \) has a uniform bound in \( C^p_2 \), and has a subsequence converging in \( C^p \). \( \square \)

Remark: Notice that due to the identification theorem in the previous section, the compactness argument here also proves the compactness for the moduli space of the effective divisors \( D^p_1 \).
Bibliography


