Extension of the Hodge Theorem to Certain Non-Compact Manifolds

by

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Bachelor of Arts, Boston University, June 2003

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Abstract

We prove an analogue of the Hodge cohomology theorem for a certain class of non-compact manifolds. Specifically, let $M$ be a compact manifold with boundary $\partial M$, and let $g$ be a metric on $\text{Int}(M)$. Assume that there exists a collar neighborhood of the boundary, $U \cong (0, 1)_x \times \partial M$, in which $g$ has the form

$$g = \frac{dx^2}{x^{2a+2}} + \frac{h}{x^{2b}}$$

where $h$ is a fixed metric on $\partial M$, and $a \geq b \geq 0$ are two real numbers. We show that the $k$-th Hodge cohomology group of $\text{Int}(M)$ is isomorphic to $H^k(M, \partial M)$ if $k < \frac{n+1-a/b}{2}$, to $H^k(M)$ if $k > \frac{n-1+a/b}{2}$, and to the image of the inclusion of $H^k(M, \partial M)$ into $H^k(M)$ if $\frac{n+1-a/b}{2} \leq k \leq \frac{n-1+a/b}{2}$ or if $b = 0$. In the proof we write the restriction of an arbitrary $k$-form on $M$ to $U$ as $\omega_k \frac{dx}{x^{2k}} + \frac{dx}{x^{a+1}} \wedge \frac{\omega_2}{x^{b(k-1)}}$. We then describe doubly weighted Sobolev spaces on $M$. For elements of these spaces the harmonic parts of $\omega_1$ and $\omega_2$ lie in one Sobolev space, while the non-harmonic parts of $\omega_1$ and $\omega_2$ lie in a differently defined Sobolev space. We prove that $d + \delta$ is Fredholm on almost all of these doubly weighted spaces, except for a finite number of values of $w$. This gives us an analogue of the Hodge decomposition theorem and leads to the result. This work generalizes earlier theorems of Atiyah, Patodi and Singer for $b$-metrics (case $a = b = 0$) and of Melrose for scattering metrics (case $a = b = 1$).

Thesis Supervisor: Richard B. Melrose
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Chapter 1

Introduction

For a compact Riemannian manifold $M$ without boundary, the Hodge theorem states that the $k$-th Hodge cohomology group of $M$, $\mathcal{H}^k(M)$, is isomorphic to the $k$-th DeRham cohomology of $M$, $H^k(M)$. Here $\mathcal{H}^k(M)$ is the space of forms of order $k$ in the kernel of the Hodge Laplacian $D = d\delta + \delta d$, where $d$ is the deRham operator and $\delta$ is its adjoint.

There is no general analogue of the Hodge theorem for non-compact Riemannian manifolds. It has been extended to manifolds with cylindrical and conical ends, as follows. By definition, a non-compact Riemannian manifold $M$ with metric $g$ has a cylindrical end if there is an open set $U \subset M$ (a neighborhood of infinity) which can be written as $U = (1, \infty) \times N$, where $N$ is a compact Riemannian manifold, $M - U$ is compact in $M$, and the metric $g$ on $U$ has the form

$$g = dt^2 + h(y, dy)$$

where $h$ is a metric on $N$. (We write $y$ and $dy$ as the arguments of $h$ to emphasize that in any local coordinates $y_i$ on $N$, $h$ has the form $\sum_{i,j} h_{ij}(y_1, \ldots y_{n-1})dy_idy_j$.)

So, intuitively speaking, a manifold with a cylindrical end is just a manifold in which a neighborhood of infinity looks like a product of a half-line with a compact manifold $N$. The definition of a manifold with a conical end is very similar: $M$ is defined to have a conical end if it has an open subset $U$ that can be represented as
\( U = (1, \infty) \times N \), where \( N \) is compact, \( M - U \) is compact, and \( g \) on \( U \) has the form

\[
g = e^{2t}dt^2 + e^{2t}h(y, dy)
\]

Here, as before, \( h(y, dy) \) is an Riemannian metric on \( N \). So, on a manifold with a conical end, the \( t \)- and \( y \)- components of the metric both grow at the same rate (here, exponentially) near infinity.

To simplify computations it is often a good idea to compactify such manifolds, turning them into compact manifolds with boundary and with a metric that grows to infinity near the boundary. In both definitions above, let \( x = e^{-t} \). Then the first metric becomes

\[
g = \frac{dx^2}{x^2} + h(y, dy)
\]

on \((0, \frac{1}{e}) \times N\), and the second one,

\[
g = \frac{dx^2}{x^4} + \frac{h(y, dy)}{x^2}
\]

in a neighborhood of the boundary.

These two classes of metrics on Riemannian manifolds with boundary are called the \( b \)-metrics and scattering metrics, respectively. They encompass many examples of non-compact manifolds that come up in practice. The analogues of the Hodge theorem for \( b \)- and scattering metrics have been known for some time. Specifically, Atiyah, Patodi and Singer proved the following result in [2]:

**Theorem 1.1** Let \( M \) be a compact manifold with boundary \( \partial M \), and \( g \) be a \( b \)-metric on \( \text{Int}(M) \). Then

\[
\mathcal{H}^k(M) \cong \text{Im}(i : H^k(M, \partial M) \to H^k(M))
\]

where \( i \) is the natural inclusion of the DeRham cohomology of \( M \) relative to its boundary into the DeRham cohomology of \( M \).

An analogous theorem for scattering metrics is found in Melrose’s work [6]
**Theorem 1.2** Let $M$ be a compact $n$-dimensional manifold with boundary $\partial M$, and $g$ a scattering metric on $\text{Int}(M)$. Then

$$\mathcal{H}^k(M) \cong \begin{cases} H^k(M, \partial M) & \text{if } k < \frac{n}{2} \\ \text{Im}(i : H^k(M, \partial M) \to H^k(M)) & \text{if } k = \frac{n}{2} \text{ and } n \text{ is even} \\ H^k(M) & \text{if } k > \frac{n}{2} \end{cases}$$

The goal of this paper is to generalize these results for another class of metrics on compact manifolds with boundary. Let $a \geq b \geq 0$ be two non-negative real numbers, and $M$ a compact $n$-dimensional manifold with boundary $\partial M$. A metric $g$ on $\text{Int}(M)$ is called a type $(a, b)$-metric if there is a collar neighborhood $U$ of $\partial M$ (diffeomorphic to $[0, 1)_x \times \partial M$) in which $g$ has the form

$$g = \frac{dx^2}{x^{2a+2}} + \frac{h(y, dy)}{x^{2b}}$$

for a metric $h$ on $\partial M$. Thus a type $(0, 0)$-metric is a $b$-metric, and a type $(1, 1)$-metric is a scattering metric. The Hodge theorem for type $(a, b)$-metrics is as follows:

**Theorem 1.3** Let $M$ be a compact $n$-dimensional manifold with boundary $\partial M$, $a \geq b \geq 0$ be two real numbers, and $g$ be a type $(a, b)$ metric on $\text{Int}(M)$. If $b = 0$, then

$$\mathcal{H}^k(M) \cong \text{Im}(i : H^k(M, \partial M) \to H^k(M))$$

If $b > 0$, then

$$\mathcal{H}^k(M) \cong \begin{cases} H^k(M, \partial M) & \text{if } k < \frac{n+1-\frac{a}{2}}{2} \\ \text{Im}(i : H^k(M, \partial M) \to H^k(M)) & \text{if } \frac{n+1-\frac{a}{2}}{2} \leq k \leq \frac{n+1-\frac{a}{2}}{2} \\ H^k(M) & \text{if } k > \frac{n+1-\frac{a}{2}}{2} \end{cases}$$

Of course, when $b > 0$ and $\frac{a}{b} \geq n + 1$, only the second of the three cases listed above will occur.

The case $b = 0$ follows trivially from the result of Atiyah, Patodi, and Singer: a change of variables $y = \exp(-\frac{1}{ax^2})$ reduces the problem to the $b$-metrics. If $b > 0$, the
case $a = b$ is easy to reduce to Melrose’s result: another change of variables, $y = ax^a$, transforms our metric into a scattering metric. (A similar change of coordinates can reduce any $(a, b)$-metric to an $(\frac{a}{b}, 1)$-metric. But when $a$ is greater than $b$, the assumption that $b = 1$ does not simplify the proof in any way.) From now on we assume that $a > b > 0$. In this case the structure of the argument is this:

For any differential form $\omega \in \Omega^k(M)$, the restriction of $\Omega$ to $U$ can be written as $\frac{\omega_1}{x^b} + \frac{d\sigma}{x^{a+1}} \wedge \frac{\omega_2}{x^{(k-1)b}}$, where $\omega_1$ and $\omega_2$ are $x$-dependent differential forms on $\partial M$ of order $k$ and $k - 1$, respectively. Thus the form $\omega|_U$ can be written as a $2 \times 1$ matrix with entries $\omega_1$ and $\omega_2$. We call $\omega_1$ the tangential part of $\omega$ in $U$, and $\omega_2$ the conormal part of $\omega$ in $U$. We say that an $x$-dependent form $\omega_1$ on $\partial M$ is boundary harmonic (or boundary non-harmonic) if $\omega_1(x)$ lies in the kernel of $d_{\partial M} + \delta_{\partial M}$ (or image of $d_{\partial M} + \delta_{\partial M}$). We will abbreviate ”boundary harmonic” and ”boundary non-harmonic” to BH and BNH.

In chapter 2 we compute the operator $D = d + \delta$ in matrix notation. In the collar neighborhood $U$ it has the form

$$D = \begin{pmatrix} x^b D' & -x^{a+1} \frac{\partial}{\partial x} + (n - A - 1)bx^a \\ x^{a+1} \frac{\partial}{\partial x} - Abx^a & -x^b D' \end{pmatrix}$$

where $D' = d_{\partial M} + \delta_{\partial M}$ is the Hodge operator on the boundary of $M$. It is easy to see that $D$ preserves the BH and BNH part of a form in $U$. This leads us to define the doubly weighted Sobolev spaces $H_{m,w,w'}^m$ of forms of order $k$ on $\text{Int}(M)$. Let $\rho$ be a smooth function such that $\text{supp}(\rho) \subset U$ and $\rho$ is identically equal to 1 on the subset of $U$ diffeomorphic to $(0, c) \times \partial M$ for a small $c \in (0, 1)$. By definition, $\omega$ belongs to $H_{m,w,w'}^m$ if three conditions hold. First, $(1 - \rho)\omega$ must lie in $H_{c}^m(M)$. Second, the BNH part of $\rho \omega$ must lie in $x^w L^2(U)$ along with its derivatives of order up to $m$ with respect to any $m$ vector fields that can be written as sums of $x^{a-b+1} \frac{\partial}{\partial x}$ with a smooth tangential vector field. Third, the BH part of $\rho_1 \omega$ must lie in $x^w L^2$ along with its derivatives of order up to $m$ with respect to $x \frac{\partial}{\partial x}$. The definition and properties of these spaces are discussed in detail in chapter 3.

In spite of their complicated definition, the doubly weighted Sobolev spaces are
natural objects to work with when trying to invert the $D$ operator. In fact, $D : \mathcal{H}^{m,w,w'} \to \mathcal{H}^{m-1,w+a,w'+b}$ will be Fredholm almost always, unless $w$ belongs to a finite set of exceptional values, which depend on $a$ and $b$. The Fredholm properties of $D$ are discussed in chapter 4. We first quote two equivalent definitions of a Fredholm operator: for two Banach spaces $B_1$ and $B_2$, an operator $P : B_1 \to B_2$ is Fredholm if it induces an invertible map $P' : B_1/V_1 \to B_2/V_2$ for some choice of finite-dimensional subspaces $V_1 \subset B_1$ and $V_2 \subset B_2$. Equivalently, $P$ is Fredholm if its kernel and cokernel are finite-dimensional and its range is closed. Equivalently, if $i : B_1 \to B_3$ is a compact embedding of Banach spaces, then $P : B_1 \to B_2$ is Fredholm if and only if the inequality

$$\|v\|_{B_1} \leq C (\|P(v)\|_{B_2} + \|i(v)\|_{B_3})$$

(1.1)

(where $C$ is a constant) holds for any $v \in B_1$, and a similar inequality is true for the adjoint $P^* : B_2^* \to B_1^*$.

We use the second definition to prove that $D : \mathcal{H}^{m,w,w'} \to \mathcal{H}^{m-1,w+a,w'+b}$ is Fredholm unless $w = \frac{b(n-2k-1)-a}{2}$ for some $k \in \{0, 1, \ldots, n-1\}$. In the proof we will work with $B_1 = \mathcal{H}^{m,w,w'}$, $B_2 = \mathcal{H}^{m-1,w+a,w'+b}$, and $B_3 = \mathcal{H}^{m-1,w-1,w'+b-a}$, and use the fact that $D$ is adjoint to itself. The inequality is proved by breaking an arbitrary $k$-form $\omega \in \mathcal{H}^{m,w,w'}$ into three parts as in the definition of $\mathcal{H}^{m,w,w'}$, proving the corresponding inequality for each part, and then bringing the three resulting estimates together by using the triangle inequality.

In section 4.1 we establish the Fredholm properties of a model operator which is closely related to the restriction of $D$ to boundary harmonic forms on $U$. The entries of $D$ on the main diagonal vanish in this case, and the entries off the main diagonal are multiples of $x \frac{\partial}{\partial x} + C_k$, where $C_k$ is a constant that depends on the degree $k$ of $\omega$. Using a change of variables $x = e^t$, we derive the Fredholm properties of $x \frac{\partial}{\partial x} + C_k$ on $U$ from the properties of $\frac{\partial}{\partial t} + C_k$ on the real line, with an exponentially weighted measure. This is the part from which the condition on $w$ arises.

In section 4.2 we prove that a model operator for $D$ on boundary non-harmonic forms on $U$ is Fredholm regardless of $m$ or the weight $w'$. To see this, we break $D$
into a sum of two parts

$$x^{-b}D = \begin{pmatrix} D' & -x^{a-b+1} \frac{\partial}{\partial x} \\ x^{a-b+1} \frac{\partial}{\partial x} & -D' \end{pmatrix} + \begin{pmatrix} 0 & (n - A - 1)bx^{a-b} \\ -(n - A - 1)bx^{a-b} & 0 \end{pmatrix}$$

By setting $t = \frac{x^{b-a}}{b-a}$ we can replace $x^{a-b+1} \frac{\partial}{\partial x}$ with $\frac{\partial}{\partial t}$. A Fourier transform in $t$ then allows us to go from $\frac{\partial}{\partial t}$ to $i \xi$. After these transformations, the first summand becomes elliptic, because its determinant is $-(D')^2 - \xi^2$, and $D'$ is elliptic and invertible on non-harmonic forms (this is the reason we treat the harmonic part separately). The first summand is therefore invertible, hence it must satisfy a Fredholm-type estimate, by the closed graph theorem. From there it is easy to see (by the triangle inequality) that the entire operator $D$ on non-harmonic forms supported in $U$ also satisfies a Fredholm-type inequality, because the second summand can be estimated by the norm of $\omega$ in a bigger doubly weighted space. We then derive the Fredholm properties of $D$ on forms supported in the interior of $M$. Because the forms we are working with are compactly supported this time, inequality (1.1) for these forms follows from the standard elliptic regularity results. We finally use the triangle inequality to bring together the three Fredholm-type inequalities for $D$. We show that, unless $\omega$ is one of the exceptional values, $D$ is Fredholm on the entire space $H^{m, w, w'}$.

In chapter 5 we use the Fredholm properties of $D$ on doubly weighted Sobolev spaces to give a proof of Theorem 1.3. It follows the same lines as the proof of the Hodge theorem for compact manifolds. We will frequently need to use the identity $\langle d\alpha, \beta \rangle = \langle \alpha, \delta\beta \rangle$ for various forms $\alpha$ and $\beta$. This transition will only be justified as long as $\alpha$ and $\beta$ decay sufficiently fast near the boundary. In section 5.1 we give several conditions on how fast $\alpha$ and $\beta$ have to decay for this integration by parts to be legitimate. In section 5.2. we prove Theorem 1.3 for the case $k < \frac{n+1-a/b}{2}$. Poincare duality enables us to immediately derive a similar result for $k > \frac{n-1+a/b}{2}$. We prove the existence and injectivity of a map from $\mathcal{H}^k(M)$ to $H^k(M, \partial M)$ similarly to the way it is done for a compact manifold, although we need to rely on the results of section 5.1 on multiple occasions. For surjectivity we have to use the Fredholm
properties of $D$, and the proof is slightly different depending on whether or not $D$ is Fredholm as a map into $L^2$. The key result in the proof of surjectivity is Lemma 5.14, which also explains where the distinction between $k < \frac{n+1-a/b}{2}$ and $k \geq \frac{n+1-a/b}{2}$ comes from. Finally, in section 5.3 we prove Theorem 1.3 for $\frac{n+1-a/b}{2} \leq k < \frac{n-1+a/b}{2}$; the result extends by Poincare duality to $k = \frac{n-1+a/b}{2}$. This is comparatively easy, because we are able to use Lemma 5.14 and other results of Section 5.2.

In conclusion, we note that our line of proof relies heavily on the fact that $a$ is no smaller than $b$. The general statement of the Hodge theorem for type $(a, b)$-metrics when $a$ is less than $b$ remains an open question. A result along these lines was obtained by Mazzeo in his paper [5], where he proved that for $a = 0$ and $b = 1$ $\mathcal{H}^k(M)$ will be isomorphic to $H^k(M, \partial M)$ for $k < \frac{n-1}{2}$, to $H^k(M)$ for $k > \frac{n+1}{2}$, and for $k = \frac{n}{2}$ the space of harmonic forms is infinite-dimensional.
Chapter 2

Matrix notation.

Let $M$ be a compact $n$-dimensional oriented manifold with boundary $\partial M$ and a boundary-defining function $x$. Then there is a collar neighborhood $U$ of $\partial M$ and a diffeomorphism $\phi: U \to [0,1)_x \times \partial M_y$. Our assumption is that $M$ has a Riemannian metric $g$, and that in the collar neighborhood $U$,

$$g(x, y) = \frac{dx^2}{x^{2a+2}} + \frac{h(y, dy)}{x^{2b}}$$

where $a > b$ are two positive real numbers and $h(y, dy)$ defines a metric on $\partial M$.

Consider a point $p = (x_0, y_0) \in (0,1)_x \times \partial M_y$. Let $e_1, \ldots, e_{n-1}$ be an orthonormal basis of $T_{y_0}(\partial M)$ in the metric $h$. Let $e_1^*, \ldots, e_{n-1}^*$ be a dual basis of $T_{y_0}^*(\partial M)$. Then $e_1, \ldots e_{n-1}$ are no longer orthonormal in the metric $g$, because $g$ divides their norm by $x_0^b$. Instead, an orthonormal basis of $T_pM$ consists of the vectors

$$x_0^b e_1, x_0^b e_2, \ldots, x_0^b e_{n-1} \quad \text{and} \quad x_0^{a+1} \frac{\partial}{\partial x}$$

and the corresponding dual basis of $T_p^*M$ contains

$$\frac{e_1^*}{x_0^a}, \frac{e_2^*}{x_0^a}, \ldots, \frac{e_{n-1}^*}{x_0^a} \quad \text{and} \quad \frac{dx}{x_0^{a+1}}$$

An orthonormal basis of $\bigwedge^k T_p^*M$ contains every possible wedge product of any $k$
of these $n$ covectors. In other words, it is the union of 

$$\left\{ \frac{1}{x^{k}b} e_{i_1}^* \wedge e_{i_2}^* \wedge \ldots e_{i_k}^* : 1 \leq i_1 < i_2 < \ldots < i_k \leq n \right\}$$

with

$$\left\{ \frac{1}{x^{a+1+(k-1)b}} dx \wedge e_{i_1}^* \wedge e_{i_2}^* \wedge \ldots e_{i_{k-1}}^* : 1 \leq i_1 < i_2 < \ldots < i_{k-1} \leq n \right\}$$

(To repeat again, all these covectors have norm 1 in the metric $g$.)

Let now $\alpha(y) \in \Omega^k(\partial M)$ be a differential form on $\partial M$ of order $k$. Then for any function $f(x, y)$ on $U$, $f(x, y)\alpha(y)$ is a differential form of order $k$ on $U$. Because of what we just said, the identity

$$||f(x_0, y_0)\alpha(y_0)||_{p,g} = x^{k}b ||f(x_0, y_0)\alpha(y_0)||_{y_0,h}$$

holds at every $p = (x_0, y_0) \in U$. Here $||\alpha||_{p,g}$ means "the norm of the restriction of $\alpha$ to $T^*_p M$ in the metric $g". Likewise if $\beta \in \Omega^{k-1}(\partial M)$ is a differential form of order $k-1$ on $\partial M$, then $f(x, y)dx \wedge \beta(y)$ is an element of $\Omega^k(U)$ for any function $f(x, y)$ on $U$. For that form, the identity

$$||f(x_0, y_0)dx \wedge \beta(y_0)||_{p,g} = x^{(k-1)b+a+1} ||f(x_0, y_0)\beta(y_0)||_{y_0,h}$$

will be true for every $p = (x_0, y_0) \in U$.

Finally, assume $\omega \in \Omega^k(U)$ is a form of order $k$ on $U$. Then there is a unique way to write $\omega = \alpha + dx \wedge \beta$, where both $\alpha$ and $\beta$ are forms on $\partial M$ which depend on $x$. Informally speaking, that means neither $\alpha$ nor $\beta$ has a $dx$ in any of its terms. In other words, both $\alpha$ and $\beta$ can be written as finite sums of terms of the form $f(x, y)\gamma_i(y)$, with each $\gamma_i$ a differential form on $\partial M$. We will call $\alpha$ the tangential part of $\omega$ in $U$, and $\beta$ the conormal part of $\omega$ in $U$. Clearly $\alpha$ has order $k$, and $\beta$ has order $k-1$. We are going to use the matrix notation: we will think of $\omega$ as consisting of two parts, $\alpha$
and $\beta$, and write

$$\omega = \alpha + dx \wedge \beta = \begin{pmatrix} x^{kb} \alpha \\ x^{(k-1)b+a+1} \beta \end{pmatrix}$$  \hfill (2.1)$$

This clever notation lets us recover the norm of $\omega$ from the norms of its components without juggling the powers of $x$. Namely, at each $p = (x_0, y_0) \in U$

$$\left\| \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix} \right\|_{p,g}^2 = \left\| \omega_1 \right\|_{y_0,h}^2 + \left\| \omega_2 \right\|_{y_0,h}^2$$

because if $\omega_1 = x^{kb} \alpha$, $\omega_2 = x^{(k-1)b+a+1} \beta$, then

$$\left\| \begin{pmatrix} x^{kb} \alpha \\ x^{(k-1)b+a+1} \beta \end{pmatrix} \right\|_{p,g}^2 = \left\| \alpha + dx \wedge \beta \right\|_{p,g}^2 = \left\| \alpha \right\|_{p,g}^2 + \left\| dx \wedge \beta \right\|_{p,g}^2 =$$

$$= \left\| x_0^{kb} \alpha \right\|_{y_0,h}^2 + \left\| x_0^{(k-1)b+a+1} \beta \right\|_{y_0,h}^2$$

It is convenient to rewrite (2.1) as follows:

$$\begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix} = \frac{\omega_1}{x^{kb}} + \frac{dx}{x^{a+1}} \wedge \frac{\omega_2}{x^{(k-1)b}}$$ \hfill (2.2)$$

where $\omega_1$ and $\omega_2$ are $x$-dependent differential forms on $\partial M$ of order $k$ and $k - 1$, respectively.

### 2.1 The $d + \delta$ operator in matrix notation.

Let $d : \Omega(M) \rightarrow \Omega(M)$ denote the deRham operator on the manifold $M$. Let $\delta \Omega(M) \rightarrow \Omega(M)$ be its adjoint. An important property of $d$ and $\delta$ is that they are both local. That is, if $\alpha$ and $\beta$ coincide on an open set $V \subset M$, then $d\alpha$ coincides with $d\beta$, and $\delta\alpha$ with $\delta\beta$, on $V$. So it makes sense to speak of $d$ and $\delta$ on the collar neighborhood $U$.

Differential forms on $U$ can be written as $(2 \times 1)$-matrices with entries in $\Omega(\partial M)$.
So the linear operator \( d + \delta \) on \( U \) can be written, in that notation, as a \((2 \times 2)\) matrix, and its entries will be first order differential operators on \( \Omega(\partial M) \). Our goal in this section is to compute that matrix.

Let \( d' \) denote the deRham operator on the boundary \( \partial M \) of \( M \). If \( \alpha \) is an \( x \)-dependent differential form on \( \partial M \), then \( d'\alpha \) is well-defined by

\[
(d'\alpha)(x_0) = d'(\alpha(x_0)) \in \Omega(\partial M)
\]

That is, \( d' \) differentiates \( x \)-dependent forms on \( \partial M \) in the directions tangent to \( \partial M \), treating \( x \) as a parameter. If \((W, y_1, y_2, \ldots, y_{n-1})\) is a coordinate patch on \( \partial M \), then \( d' \) has this form on \((0, 1) \times W_y\):

\[
d' \left( f(x, y_1, \ldots, y_{n-1}) dy_{i_1} \ldots \wedge dy_{i_k} \right) = \sum_{j=1}^{n-1} \frac{\partial f(x, y_1, \ldots, y_{n-1})}{\partial y_j} dy_j \wedge dy_{i_1} \ldots \wedge dy_{i_k}
\]

This, of course, extends linearly to \( \Omega((0, 1) \times W) \).

On the other hand, by the definition of \( d \),

\[
d \left( f(x, y_1, \ldots, y_{n-1}) dy_{i_1} \ldots \wedge dy_{i_k} \right) =
\]

\[
\frac{\partial f(x, y_1, \ldots, y_{n-1})}{\partial x} dx \wedge dy_{i_1} \ldots \wedge dy_{i_k} + \sum_{j=1}^{n-1} \frac{\partial f(x, y_1, \ldots, y_{n-1})}{\partial y_j} dy_j \wedge dy_{i_1} \ldots \wedge dy_{i_k}
\]

Therefore, if we denote \( f(x, y_1, \ldots, y_{n-1}) dy_{i_1} \ldots \wedge dy_{i_k} \) by \( \gamma \) for brevity, then

\[
d \gamma = dx \wedge \frac{\partial \gamma}{\partial x} + d'\gamma \tag{2.3}
\]

By linearity, (2.3) holds for any \( \gamma \in \Omega((0, 1) \times W) \). Because the left-hand and right-hand sides of (2.3) do not involve coordinates on \( \partial M \), (2.3) must be true on all of \( U \). (2.3) easily implies

\[
d(\omega_1 + dx \wedge \omega_2) = d\omega_1 - dx \wedge d\omega_2 = d'(\omega_1) + dx \wedge (\frac{\partial \omega_1}{\partial x} - d'\omega_2) \tag{2.4}
\]
where $\omega_1$ and $\omega_2$ are $x$-dependent differential forms on $\partial M$.

We need to perform two more steps. First, we want to put the powers of $x$ in this expression, and second, we want to do the same calculation for $\delta$. Let’s start with the powers of $x$. Consider an arbitrary element of $\Omega^k(U)$:

$$\omega = \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix} = \frac{\omega_1}{x^{k+b}} + \frac{dx}{x^{a+1}} \wedge \frac{\omega_2}{x^{(k-1)b}} \in \Omega^k(U)$$

Then (2.4) clearly implies that $d\omega$ is equal to

$$d\left( \frac{\omega_1}{x^{k+b}} + \frac{dx}{x^{a+1}} \wedge \frac{\omega_2}{x^{(k-1)b}} \right) = \frac{d'\omega_1}{x^{k+b}} + \frac{dx}{x^{a+1}} \wedge \left( \frac{\partial}{\partial x} \left( \frac{\omega_1}{x^{k+b}} \right) - d' \left( \frac{\omega_2}{x^{(k-1)b+a+1}} \right) \right)$$

$$= \frac{x^bd'\omega_1}{x^{(k+1)b}} + \frac{dx}{x^{a+1}} \wedge \left( \frac{-kb \omega_1}{x^{kb+1}} + \frac{1}{x^{kb}} \frac{\partial \omega_1}{\partial x} - \frac{1}{x^{a+1}} \frac{x^b d' \omega_2}{x^{kb}} \right)$$

$$= \frac{x^b d' \omega_1}{x^{(k+1)b}} + \frac{dx}{x^{a+1}} \wedge \left( \frac{-kb x^a \omega_1}{x^{kb}} + \frac{x^a+1}{x^{kb}} \frac{\partial \omega_1}{\partial x} - \frac{x^b d' \omega_2}{x^{kb}} \right)$$

Here we are using the fact that $d'$ commutes with powers of $x$, since it does not involve differentiation in $x$. The last expression is easy to write in the matrix form, because $d' \omega_1$ is a form of order $k+1$; and $\omega_1$, $\frac{\partial \omega_1}{\partial x}$, and $d' \omega_2$ are all forms of order $k$. The last expression is therefore equal to

$$\begin{pmatrix} x^bd' \omega_1 \\ -kb x^a \omega_1 + x^a+1 \frac{\partial \omega_1}{\partial x} - x^b d' \omega_2 \end{pmatrix} = d \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix}$$

Consequently,

$$d = \begin{pmatrix} x^bd' & 0 \\ -kb x^a + x^a+1 \frac{\partial}{\partial x} & -x^b d' \end{pmatrix}$$

is the expression in matrix form of $d : \Omega^k(U) \rightarrow \Omega^{k+1}(U)$.

To compute $D$ in matrix form we also need a similar expression for $\delta$. Since $\delta$ is loca, we can assume without loss of generality that $\omega$ is compactly supported inside
\[ U. \text{ It is well known that in this case} \]

\[ \delta \omega = (-1)^{n(k+1)+1} * d * \omega \]

where \(*\) is the Hodge star on \( U \). To get a matrix form expression for \( \delta \), we first need a relation between \(*\) and \(*'\), the Hodge star on \( \partial M \). Choose an orientation on \( \partial M \) which is compatible with the orientation on \( M \) in the following way: if \( p = (x_0, y_0) \in U \), then a basis \((e_1, \ldots, e_{n-1})\) of \( T_{y_0}(\partial M) \) is positively oriented if and only if \((\frac{\partial}{\partial x}, e_1, \ldots, e_{n-1})\) is a positively oriented basis of \( T_p M \). Equivalently, a basis \((e_1^*, \ldots, e_{n-1}^*)\) of \( T_{y_0}^*(\partial M) \) is positively oriented if and only if \((dx, e_1, \ldots, e_{n-1})\) is a positively oriented basis of \( T_p^* M \).

In the same notation, let \((e_1^*, \ldots, e_{n-1}^*)\) be a positively oriented basis of \( T_{y_0}^*(\partial M) \), orthonormal in the \( h \) metric. Then \((\frac{dx}{x^{n+1}}, \frac{e_1^*}{x^b}, \ldots, \frac{e_{n-1}^*}{x^b})\) is a positively oriented basis of \( T_p^* M \), orthonormal in the \( g \) metric. Consequently, for any \( k \leq n \)

\[ *(\frac{dx}{x^{a+1}} \wedge \frac{e_1^*}{x^b} \wedge \frac{e_2^*}{x^b} \wedge \ldots \wedge \frac{e_k^*}{x^b}) = \frac{e_{k+1}^*}{x^b} \wedge \frac{e_{k+2}^*}{x^b} \wedge \ldots \wedge \frac{e_{n-1}^*}{x^b} = \frac{*'(e_1 \wedge e_2 \wedge \ldots \wedge e_k)}{x^{(n-k-1)b}} \]

And similarly,

\[ *(\frac{dx}{x^{a+1}} \wedge \frac{e_1^*}{x^b} \wedge \ldots \wedge \frac{e_k^*}{x^b}) = (-1)^k \frac{dx}{x^{a+1}} \wedge \frac{e_{k+1}^*}{x^b} \wedge \frac{e_{k+2}^*}{x^b} \wedge \ldots \wedge \frac{e_{n-1}^*}{x^b} = \frac{*'(e_1 \wedge e_2 \wedge \ldots \wedge e_k)}{x^{(n-k-1)b}} \]

Consequently, for any \( x \)-dependent \( k \)-th order differential form \( \frac{\gamma}{x^{kb}} \) on \( \partial M \),

\[ *(\frac{\gamma}{x^{kb}}) = (-1)^k \frac{dx}{x^{a+1}} \wedge \frac{*'\gamma}{x^{(n-k-1)b}} \text{ and } *(\frac{dx}{x^{a+1}} \wedge \frac{\gamma}{x^{kb}}) = \frac{*'\gamma}{x^{(n-k-1)b}} \] (2.6)

It is also clear from the definition of \(*'\) that it commutes with \( \frac{\partial}{\partial x} \). Indeed, any \( x \)-dependent differential form \( \omega \) on \( \partial M \) can be written as a sum of terms of the form \( f(x, y)\omega_i(y) \), with each \( \omega_i(y) \) in \( \Omega(\partial M) \). Since

\[ \frac{\partial}{\partial x} *'(f(x, y)\omega_i(y)) = \frac{\partial}{\partial x} (f(x, y) *' \omega_i(y)) = \frac{\partial f(x, y)}{\partial x} *' \omega_i(y) = *' \frac{\partial}{\partial x} (f(x, y)\omega_i(y)) \]


\* must commute with \( \frac{\partial}{\partial x} \).

The way is now open for us to compute \( \delta \) in the matrix form. Because this computation is harder, we will work with \( \omega_1 \) and \( \frac{dx}{x^{a+1}} \wedge \frac{\omega_2}{x^{(k-1)b}} \) separately. Using the identities (2.6), we get

\[
\delta \left( \frac{\omega_1}{x^{kb}} \right) = (-1)^{n(n+1)+1} \ast d \ast \left( \frac{\omega_1}{x^{kb}} \right) \\
= (-1)^{n(n+1)+1} d \left( (-1)^k \frac{dx}{x^{a+1}} \wedge \frac{\delta' \omega_1}{x^{(n-k-1)b}} \right) \\
= (-1)^{n(n+1)+1} (-1)^k \left( \frac{dx}{x^{a+1}} \wedge \frac{x^b \delta' \delta' \omega_1}{x^{(n-k)b}} \right) \\
= (-1)^{n(n+1)+k} \frac{x^b \delta' \omega_1}{x^{(k-1)b}}
\]

The last transition is just the second identity in (2.6) applied to a form of order \( n-k+1 \) instead of order \( k+1 \). Observe that \( \omega_1 \) is an \( (x\text{-dependent}) \) differential form of order \( k \) on an \( (n-1)\)-dimensional manifold \( \partial M \). Therefore \( \delta' \omega_1 = (-1)^{(n-1)(k+1)+1} \ast d' \ast \omega_1 \), where \( \delta' \) is the adjoint of \( d' \) on \( \Omega(\partial M) \). Hence the above expression is equal to

\[
(-1)^{n(n+1)+k} (-1)^{(n-1)(k+1)+1} \frac{x^b \delta' \omega_1}{x^{(k-1)b}}
\]

The combined power of \(-1\) in this expression is

\[
nk + n + k + nk + n - k - 1 + 1 = 2(nk + n)
\]

so it’s just 1, and therefore

\[
\delta \frac{\omega_1}{x^{kb}} = \frac{x^b \delta' \omega_1}{x^{(k-1)b}}
\]

or, using matrix notation,

\[
\delta \left( \begin{array}{c} \omega_1 \\ 0 \end{array} \right) = \left( \begin{array}{c} x^b \delta' \omega_1 \\ 0 \end{array} \right) \quad (2.7)
\]

The calculation will be harder for \( \delta \left( \frac{dx}{x^{a+1}} \wedge \frac{\omega_2}{x^{(k-1)b}} \right) \), because it will involve the
product rule. Nonetheless, clenching our teeth and using (2.6), we write

\[
\delta \left( \begin{array}{c} 0 \\ \omega_2 \end{array} \right) = \delta \left( \frac{dx}{x^{a+1}} \wedge \frac{\omega_2}{x^{(k-1)b}} \right)
\]

\[
= (-1)^{n(k+1)+1} \delta_\ast \left( \frac{dx}{x^{a+1}} \wedge \frac{\omega_2}{x^{(k-1)b}} \right)
\]

\[
= (-1)^{n(k+1)+1} \delta_\ast \left( \frac{\omega_2}{x^{(n-k)b}} \right)
\]

\[
= (-1)^{n(k+1)+1} \left( dx \wedge \frac{\partial}{\partial x} \left( \frac{\omega_2}{x^{(n-k)b}} \right) + d' \frac{\omega_2}{x^{(n-k)b}} \right)
\]

\[
= (-1)^{n(k+1)+1} \left( dx \wedge \frac{\partial}{\partial x} \left( \frac{\omega_2}{x^{(n-k)b}} \right) + d' \frac{\omega_2}{x^{(n-k)b+1}} \right)
\]

\[
= (-1)^{n(k+1)+1} \left( dx \wedge \frac{\partial}{\partial x} \left( \frac{\omega_2}{x^{(n-k)b}} \right) + d' \frac{\omega_2}{x^{(n-k)b+1}} \right)
\]

\[
= (-1)^{n(k+1)+1} \left( \frac{dx}{x^{a+1}} \wedge \frac{x^{a+1} \ast \frac{\partial}{\partial x} \omega_2}{x^{(n-k)b}} + \frac{x^{a} (n - k)b \ast \omega_2}{x^{(n-k)b+1}} \right)
\]

Recall that, for a \( k \)-th order form \( \alpha \) on an \( n \)-dimensional manifold \( M \), \( \ast \ast \alpha = (-1)^{k(n-k)} \alpha \). In our case \( \omega_2 \) and \( \frac{\partial \omega_2}{\partial x} \) are \( (x \text{-dependent}) \) forms of order \( k - 1 \) on an \( (n - 1) \)-dimensional manifold \( \partial M \). It follows that

\[
\ast' \ast' \omega_2 = (-1)^{(k-1)(n-k)} \omega_2 \quad \text{and} \quad \ast' \ast' \frac{\partial \omega_2}{\partial x} = (-1)^{(k-1)(n-k)} \frac{\partial \omega_2}{\partial x}
\]

We also know that \( \delta' \omega_2 = (-1)^{(n-1)k+1} \ast' d' \ast' \omega_2 \). These three identities allow us to further simplify the last expression above. It is equal to

\[
= (-1)^{n(k+1)+1} \left( (-1)^{(k-1)(n-k)} \frac{x^{a+1} \frac{\partial \omega_2}{\partial x}}{x^{(k-1)b}} - (-1)^{(k-1)(n-k)} \frac{(n - k)b x^a \omega_2}{x^{(k-1)b}} \right)
\]

\[
+ (-1)^{n-k+1} (-1)^{(n-1)k+1} \frac{dx}{x^{a+1}} \wedge \frac{x^b \delta' \omega_2}{x^{(k-2)b}} \right)
\]

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And now the time has come to count the powers of $-1$. We are counting them modulo 2, of course. So we can use identities like $n = -n$, $k = k^2$, etc., which hold modulo 2 for integer numbers.

\[ n(k + 1) + 1 + (k - 1)(n - k) = nk + n + 1 + nk + n + k^2 + k = 1 \]

\[ n(k + 1) + 1 + n - k + 1 + (n - 1)k + 1 = nk + n + 1 + n + k + 1 + nk + k + 1 = 1 \]

So our expression is finally equal to

\[
\delta \left( \frac{dx}{x^{a+1}} \wedge \frac{\omega_2}{x(k-1)b} \right) = -\frac{x^{a+1} \partial \omega_2}{x(k-1)b} + \frac{(n - k)b a \omega_2}{x(k-1)b} - \frac{dx}{x^{a+1}} \wedge \frac{x b \delta' \omega_2}{x(k-2)b}
\]

Rewriting the same equation in matrix notation, we get

\[
\delta \begin{pmatrix} 0 \\ \omega_2 \end{pmatrix} = \begin{pmatrix} -x^{a+1} \partial \omega_2 + (n - k)b a \omega_2 \\ -x b \delta' \omega_2 \end{pmatrix}
\]

It is now easy to add (2.7) to this identity and obtain

\[
\delta \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix} = \begin{pmatrix} x b \delta' \omega_1 - x^{a+1} \partial \omega_2 + (n - k)b a \omega_2 \\ -x b \delta' \omega_2 \end{pmatrix}
\]

This means the matrix form of $\delta$ is

\[
\delta = \begin{pmatrix} x b \delta' & -x^{a+1} \partial + (n - k)b a \\ 0 & -x b \delta' \end{pmatrix}
\]

(2.8)

as an operator on $\Omega^k(U)$.

Adding (2.5) to (2.8), we get an expression for $d + \delta$ in the matrix form. Let $D$ denote the operator $d + \delta$, and $D'$ denote $d' + \delta'$. Then

\[
D = \begin{pmatrix} x b D' & -x^{a+1} \partial + (n - k)b a \\ x^{a+1} \partial - k b a & -x b D' \end{pmatrix}
\]
as an operator on $\Omega^k(U)$.

We would like to extend this notation to the entire $\Omega(U)$. For that we need to replace multiplication by $k$ in this matrix with the number operator. The number operator is just that - multiplication by the order of the form.

**Definition 2.1** For a manifold $M$, the number operator $A : \Omega(M) \rightarrow \Omega(M)$ is the linear operator uniquely defined by

$$A\omega = k\omega \quad \text{if } \omega \in \Omega^k(M)$$

In our computations, $\omega_1$ has order $k$, so $A\omega_1 = k\omega_1$. However, $\omega_2$ has order $k - 1$, so $A\omega_2 = (k - 1)\omega_2$. This explains why the final expression for $D = d + \delta$ in the matrix form is

$$D = \begin{pmatrix}
  x^bD' & -x^{a+1}\frac{\partial}{\partial x} + (n - A - 1)bx^a \\
x^{a+1}\frac{\partial}{\partial x} - Abx^a & -x^bD'
\end{pmatrix} \quad (2.9)$$

and this expression stays the same on all of $\Omega(U)$. This matrix form of $D$ will motivate our definition of the doubly weighted Sobolev spaces, on which $D$ will (almost always) be Fredholm.
Chapter 3

Doubly weighted Sobolev spaces.

As before, let \( D' = d' + \delta' \) be the Hodge-deRham operator on \( \partial M \), where \( \delta' \) is the adjoint of \( d' \) with respect to the metric \( h \). According to the ordinary Hodge theorem for the compact manifold \( \partial M \), for each \( k \in \{0, 1, \ldots, n - 1\} \) there is an orthogonal decomposition

\[
\Omega^k(\partial M) = d'\Omega^{k-1}(\partial M) \oplus \delta'\Omega^{k+1}(\partial M) \oplus \Omega_h^k(\partial M)
\]

where \( \Omega_h^k(\partial M) = \text{Ker}(D') \cap \Omega^k(\partial M) \) is the space of harmonic forms of order \( k \) on \( \partial M \). Simplifying notation, we will denote \( d'\Omega^{k-1}(\partial M) \oplus \delta'\Omega^{k+1}(\partial M) \) by \( \Omega_{nh}^k(\partial M) \) (the letters \( nh \) stand for non-harmonic). Let \( \pi \) denote the orthogonal projection from \( \Omega^k(\partial M) \) onto \( \Omega_h^k(\partial M) \). Then any element \( \omega_1 \in \Omega^k(\partial M) \) can be expressed as

\[
\omega_1 = \pi \omega_1 + (1 - \pi) \omega_1,
\]

where \( \pi \omega_1 \in \Omega_h^k(\partial M) \) and \( (1 - \pi) \omega_1 \in \Omega_{nh}^k(\partial M) \).

If \( \omega_1 \) is an \( x \)-dependent form of order \( k \) on \( \partial M \), then it is likewise possible to write

\[
\omega_1 = \pi \omega_1 + (1 - \pi) \omega_1,
\]

where, by definition, \( (\pi \omega_1)(x) = \pi(\omega_1(x)) \). Because \( \pi \) is an orthogonal projection, the \( x \)-dependent forms \( \pi \omega_1 \) and \( (1 - \pi) \omega_1 \) will have the same regularity as \( \omega \).

We have seen that the restriction to \( U \) of any \( \omega \in \Omega^k(M) \) can be written as

\[
\omega|_U = \omega_1 \frac{dx}{x^{k+b}} + \frac{dx}{x^{a+1}} \wedge \frac{-\omega_2}{x^{(k-1)b}} = \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix}
\]
where $\omega_1$ and $\omega_2$ are differential forms on $\partial M$ which depend on $x$. Therefore, $\omega|_U$ can also be decomposed into the following sum:

$$\omega = \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix} = \begin{pmatrix} \pi \omega_1 \\ \pi \omega_2 \end{pmatrix} + \begin{pmatrix} (1 - \pi) \omega_1 \\ (1 - \pi) \omega_2 \end{pmatrix}$$

where, for each $x \in (0, 1)$, $\pi \omega_1(x) \in \Omega^k_x(\partial M)$, and $\pi \omega_2(x) \in \Omega^{k-1}_x(\partial M)$, and likewise $(1 - \pi) \omega_1 \in \Omega^k_{nx}(\partial M)$ and $(1 - \pi) \omega_2 \in \Omega^{k-1}_{nx}(\partial M)$. We will denote the form $\begin{pmatrix} \pi \omega_1 \\ \pi \omega_2 \end{pmatrix}$ by $\pi \omega$, and call it the boundary harmonic part of $\omega$ in $U$. Similarly, $(1 - \pi) \omega$ will be called the boundary non-harmonic part of $\omega$ in $U$. Through the rest of this exposition we will abbreviate "boundary harmonic" and "boundary non-harmonic" to BH and BNH, respectively. The definition of the BH and BNH part of a form extends by linearity to the space $\Omega(M)$.

If $\omega$ has a zero BH part or a zero BNH part, it is possible to simplify the matrix by which $D$ acts on $\omega$ in the collar neighborhood $U$. Assume first that the BNH part of $\omega$ in $U$ is zero. Then $\omega_1(x)$ and $\omega_2(x)$ are harmonic for each $x$, i.e. they satisfy

$$D\omega_1(x) = D\omega_2(x) = 0.$$ According to (2.9), $D$ acts on $\omega|_U$ by

$$D = x^a \begin{pmatrix} 0 & -x \frac{\partial}{\partial x} + (n - A - 1)b \\ x \frac{\partial}{\partial x} - Ab & 0 \end{pmatrix} \quad (3.1)$$

As we will see later, the matrix in the right-hand side will define a Fredholm operator on all but a finite number of weighted Sobolev spaces. We remark in passing that $d$ and $\delta$ act on $\omega|_U$ by

$$d = x^a \begin{pmatrix} 0 & 0 \\ x \frac{\partial}{\partial x} - Ab & 0 \end{pmatrix} \quad \text{and} \quad \delta = x^a \begin{pmatrix} 0 & -x \frac{\partial}{\partial x} + (n - A - 1)b \\ 0 & 0 \end{pmatrix} \quad (3.2)$$

respectively. This follows from (2.5) and (2.8) and the fact that $d'$ and $\delta'$ restrict to 0 on $\Omega_h(\partial M)$. (A harmonic form on $\partial M$ is $d'$-closed and $\delta'$-closed).
On the other hand, if $\omega_1$ and $\omega_2$ have zero BH parts, then $D$ acts on $\omega|_U$ via

\[
x^b \left( \begin{pmatrix} D' & -x^{a-b+1}\frac{\partial}{\partial x} \\ x^{a-b+1}\frac{\partial}{\partial x} & -D' \end{pmatrix} + \begin{pmatrix} 0 & (n - A - 1)bx^{a-b} \\ -Abx^{a-b} & 0 \end{pmatrix} \right)
\]

In section 4 we will see that this operator satisfies a Fredholm-type estimate on an appropriately defined space of forms with no BH part. The first matrix will induce an isomorphism, because $D'$ is an isomorphism on $\Omega_{mh}(\partial M)$. The second matrix will be too small compared to the first to stop the entire operator from being Fredholm.

Expressions (3.1) and (3.3) point us to the fact that $D$ shifts the weight (intuitively thought of as "power of $x$") by $a$ on the BH part of a form in $U$, and by $b$ on the BNH part of that form in $U$. So, in order to define a family of spaces on which $D$ will be Fredholm, we will need to keep track of the weights of the BH and the BNH part of a form separately. This leads to the definition of doubly weighted Sobolev spaces $H^{m,w,w'}$, which we will now give.

For any $c \in (0, 1)$, let $U_c$ denote the subset of the collar neighborhood $U$ that is diffeomorphic to $(0, c) \times \partial M$. A vector field on $U$ is said to be tangential if its natural projection onto $(0, 1)$ is zero. For $m \in \mathbb{Z}_+$, we say that a differential form $\omega$ on $U$ lies in $H_{nz}^m$ if three conditions hold. The first is that $\pi \omega = 0$, i.e. $\omega$ has no BH part. The second is that $\text{supp}(\omega) \subset U_c$ for some $c \in (0, 1)$, i.e. $\omega$ is compactly supported inside $[0, 1) \times \partial M$. The third condition is that for any $l \leq m$, $i \leq m - l$, and any $l$ vector fields $v_1, \ldots, v_l$ on $U$ which are pullbacks to $U$ of smooth vector fields on $\partial M$,

\[
\nabla_{v_1} \nabla_{v_2} \cdots \nabla_{v_l} (\nabla_{x^{a-b+1}\frac{\partial}{\partial x}})^i \omega \in L^2\Omega^k(U)
\]

Here $\nabla$ is the standard Levi-Civita connection associated with the metric $h$. Informally speaking, the third condition states that the elements of $H_{nz}^m$ must stay in $L^2(U)$ if we hit them with $\frac{\partial}{\partial y_i}$ or $x^{a-b+1}\frac{\partial}{\partial x}$ up to $m$ times. Here the $y_i$-s are the local coordinates on $\partial M$.

We remark that, since the elements of $H_{nz}^m$ are compactly supported inside $[0, 1) \times \partial M$, it may have been more appropriate to denote that space $H_{nz,c}^m$. We leave the
index $c$ out to avoid overloading our notation with indices when we proceed to define the space $H_{nz}^{m,w}$.

The definition of $H_{nz}^{-m}$ for a positive integer $m$ will be similar to that of an ordinary negative Sobolev space. Instead of requiring that the derivatives of a form up to $m$-th order lie in $L^2$, we will require that it is a sum of derivatives of forms in $H_{nz}^0$ of order up to $m$. The derivatives here are taken in the sense of distributions. So, strictly speaking, a form $\omega$ with distributional coefficients lies in $H_{nz}^{-m}$ if it can be written as a finite sum

$$\omega = \sum_{j=1}^N \nabla_{v_{1,j}} \nabla_{v_{2,j}} \cdots \nabla_{v_{l,j}} (\nabla_{x^a-b^+1, \partial_x})^{i_j} \omega_j$$

where, for each $j$, each $v_{p,j}$ is a pullback to $U$ of a smooth vector field on $\partial M$, and $l_j + i_j \leq m$, and each $\omega_j$ lies in $H_{nz}^0(U)$. This definition implies that $\text{supp}(\omega) \subset U_c$ for some $c < 1$; if $\text{supp}(\omega_j) \subset U_{c_j}$ for each $j$, then $c$ can be taken to be the biggest of the numbers $c_1, \ldots, c_N$.

We will now show that a suitable change of variables can identify space $H_{nz}^m$ to a subspace of the ordinary weighted Sobolev space on $\mathbb{R} \times \partial M$. Let $\Phi : (-\infty, -\frac{1}{a-b}) \times \partial M \to (0, 1) \times \partial M$ be defined by $\Phi(t, p) = ((a-b)^{-\frac{t}{b-a}}(-t)^{\frac{1}{b-a}}, p)$. It is clear that $\Phi$ is a diffeomorphism with inverse $\Phi^{-1}(x, p) = \left(\frac{(-1)}{(a-b)^{\frac{x}{b-a}}}, p\right)$. Moreover, $\Phi$ commutes with projection onto $\partial M$, so it maps vector fields which are pullbacks of smooth vector fields on $\partial M$ to other vector fields which have the same property. We finally compute

$$dx = (a - b)^{\frac{1}{b-a}}(-t)^{\frac{1}{b-a}}$$
and hence $\Phi_* \left(\frac{\partial}{\partial t}\right) = x^{a-b+1} \frac{\partial}{\partial x}$

(Of course, this is the reason why we defined $t$ this way in the first place.) The volume form on $U$ is pulled back by $\Phi$ to

$$\Phi^*(dVol_g) = \Phi^* \left(\frac{dx \wedge dVol_h}{x^{a+1+b(n-1)}}\right) = ((a - b)(-t))^{\frac{a+1+b(n-1)}{a-b}} dt \wedge dVol_h$$

Consequently, a differential form $\omega$ will be an element of $H_{nz}^m$ iff $\text{supp}(\omega) \subset U_c$ for some $c \in (0, 1)$, and $\pi \omega = 0$, and $\Phi^* \omega \in H^m \left(\mathbb{R} \times \partial M, (-t)^{\frac{a+1+b(n-1)}{a-b}} dt \wedge dVol_h\right)$, the ordinary Sobolev space on $(-\infty, \frac{1}{a-b} \times \partial M)$. (We remove the constant from the
measure, because that does not affect the definition of the Sobolev space.) This is true since $\Phi^*$ maps $x^{a-b+1} \frac{\partial}{\partial x}$ to $\frac{\partial}{\partial t}$. Note that the power of $t$ in the measure can be transformed into a weight, by the following lemma:

**Lemma 3.1** For any real $W \in \mathbb{R}$ and $K > 0$ and any compact Riemannian manifold $\partial M$ with metric $h$,

$$H^m((\infty, -K) \times \partial M, (-t)^{2W} dt \wedge dVol_h) = t^{-W} H^m((\infty, -K) \times \partial M)$$

Here $H^m((\infty, -K) \times \partial M)$ denotes the Sobolev space for the usual measure $dt \wedge dVol_h$. The proof of this lemma involves induction on $m$ and repeated uses of the product rule, and it is almost exactly the same as the argument in the proof of Lemma 4.13.

To simplify notation, let us temporarily denote the quantity $\frac{a+1+b(n-1)}{2(a-b)}$ by $W$. Recall that $t^{-W} H^m_c((\infty, -\frac{1}{a-b}))$ carries a natural Sobolev inner product: for any two forms $\alpha$ and $\beta$ in that space, the inner product of $\alpha$ and $\beta$ is defined to be the inner product of $t^W \alpha$ and $t^W \beta$ in $H^m(\mathbb{R} \times \partial M)$. While $t^{-W} H^m_c((\infty, -\frac{1}{a-b}) \times \partial M)$ is not closed under that inner product, each of its subspaces of the form $t^{-W} H^m((\infty, -K] \times \partial M)$ is closed and hence is a Hilbert space (for each $K > \frac{1}{a-b}$).

The map $\Phi^*$ enables us to carry over the same property to $H^m_{nz}$. That is, there is an intrinsic inner product on $H^m_{nz}$ defined by

$$\langle \alpha, \beta \rangle_{H^m_{nz}} = \langle \Phi^* \alpha, \Phi^* \beta \rangle_{t^{-W} H^m_c((\infty, -\frac{1}{a-b}) \times \partial M)}$$

and while $H^m_{nz}$ is not a Hilbert space under that inner product, its subspaces of the form $\{ \omega \in H^m_{nz} : \text{supp}(\omega) \subset [0, K] \times \partial M \}$ will be Hilbert for any $K \in (0, 1)$. This is the case because $\pi$ is a continuous projection, and its kernel $\Omega_{nz}(\partial M)$ is a closed subspace.

Let $w'$ be any real number. We will define $H^{m,w'}_{nz}$ as $x^{-w'} H^m_{nz}$: by definition, a form $\omega \in \Omega(U)$ lies in $H^{m,w'}_{nz}$ if and only if $x^{-w'} \omega \in H^m_{nz}$. This, of course, implies that $\omega$ is compactly supported inside $[0, 1) \times \partial M$. The natural isomorphism $\omega \to x^{-w'} \omega$ from
$H_{nz}^{m,w}$ to $H_{nz}^{m}$ induces an inner product on $H_{nz}^{m,w}$ in the usual way: for any two forms $\alpha, \beta \in H_{nz}^{m,w}$,

$$\langle \alpha, \beta \rangle_{H_{nz}^{m,w}} = \left\langle x^{-w} \alpha, x^{-w} \beta \right\rangle_{H_{nz}^{m}}$$

As with $H_{nz}^{m}$, any subspace $\{\omega \in H_{nz}^{m,w} : \text{supp}(\omega) \subset [0, K] \times \partial M\}$ for any $K \in (0, 1)$ will become a Hilbert space when equipped with this inner product. We finally define $H_{nz}^{m,w} \Omega^k(U)$ to be the space of elements of $H_{nz}^{m,w}$ of degree $k$.

Our definition of $H_{nz}^{m,w}$ was motivated by the fact that the entries in the matrix (3.3) of the operator $D$ on boundary non-harmonic forms contain $D'$ and $x^{a-b+1} \frac{\partial}{\partial x}$. The entries in the matrix (3.1) of $D$ on boundary harmonic forms only contain $x \frac{\partial}{\partial x}$, and so our definition of the corresponding space $H_{z}^{m,w}$ will be that much simpler. As before, let $m \geq 0$ be an integer.

We will say that an element $\omega \in \Omega(U)$ belongs to $H_{z}^{m}$ if three conditions hold. The first is that $\pi \omega = \omega$, i.e. $\omega$ has a zero BNH part. (This automatically implies that $\omega$ is smooth in the tangential variables.) The second is that $\text{supp}(\omega) \subset U_c$ for some $c \in (0, 1)$, in other words, $\omega$ is compactly supported inside $[0, 1) \times \partial M$. The third condition is that for any $l \leq m$,

$$\left( x \frac{\partial}{\partial x} \right)^l \omega \in L^2 \Omega(U)$$

In effect, this says that $\omega$ lies in $L^2$ and remains there if it is differentiated up to $m$ times with respect to $x \frac{\partial}{\partial x}$.

By the Hodge theorem for compact manifolds, $\Omega_h(\partial M)$ must be finite-dimensional. So forms on $U$ that have no BNH part can be thought of as functions from $(0, 1)$ to a finite-dimensional vector space $\Omega_h(\partial M) \oplus \Omega_h(\partial M)$. This direct sum of two copies of $\Omega_h(\partial M)$ arises because forms on $U$ have a tangential and a conormal part. So $H_{z}^{m}$ can be thought of as the Sobolev space of functions from $(0, 1)$ to a finite-dimensional vector space which are supported away from 1, which are square integrable with respect to the measure $x^{a+1+b(n-1)}\frac{dx}{x^{a+1+b(n-1)}}$, and which stay $L^2$ with respect to that measure if they are differentiated with $x \frac{\partial}{\partial x}$ up to $m$ times.

The definition of $H_{z}^{-m}$ is similar to that of $H_{nz}^{-m}$: we say that a form $\omega$ on $U$ with
distributional coefficients lies in $H^m_{x}$ if $\omega$ can be written as a finite sum

$$\omega = \sum_{j=0}^{m} \left( x \frac{\partial}{\partial x} \right)^j \omega_j$$

where each $\omega_j$ belongs to $H^0$. As with $H^m_{n\Omega}$, this immediately implies that the support of $\omega$ is compact inside $[0, 1) \times \partial M$.

As with $H^m_{n\Omega}$, the definition of $H^m_{x}$ can be made to coincide with that of an ordinary Sobolev space by an appropriate change of coordinates. Define $\Psi : (-\infty, 0) \times \partial M \to (0, 1) \times \partial M$ by the formula $\Psi(t, p) = (e^t, p)$. Then $\Psi$ clearly is a diffeomorphism whose inverse $\Psi^{-1}$ maps $(0, 1) \times \partial M$ to $(-\infty, 0) \times \partial M$ by $\Psi^{-1}(x, p) = (\ln(x), p)$. A trivial calculation shows that $\Psi^* \varphi = x \frac{\partial}{\partial x} \varphi$. It is also clear that

$$\Psi^* \left( \frac{dx}{x^{a+1+b(n-1)}} \right) = e^{(-a-b(n-1))t} dt$$

Therefore, a form $\omega$ that has no BNH part and is supported away from 1 will lie in $H^m_{x}$ if and only if $\Psi^* \omega$ belongs to $H^m(\Psi((0, 1) \times \partial M), e^{(-a-b(n-1))t} dt)$, the Sobolev space with respect to $e^{(-a-b(n-1))t} dt$ of forms on the negative real line. The exponential in the measure can be absorbed into the weight of the space, by the following lemma:

**Lemma 3.2** For any $\alpha \in \mathbb{R}$ and any finite-dimensional normed space $U$,

$$H^m((-\infty, 0), V, e^{2\alpha t} dt) = e^{-\alpha t} H^m((-\infty, 0), V, dt)$$

where the index $c$ means that both Sobolev spaces contain only functions which are supported away from zero.

This lemma can again be proved by iterated use of the product rule. (We will give the proof of a more general result for Sobolev spaces on the entire line, with different exponential weights at $+\infty$ and $-\infty$, in Lemma 4.13.) It follows that $\Psi^*$ maps $H^m_{x}$ to the exponentially weighted Sobolev space $e^{\frac{1}{2}(a+b(n-1))t} H^m_{e}(\Psi((-\infty, 0), \Omega_h(\partial M)^2)$. The latter space has a natural inner product, which induces a Hilbert space structure on
each subspace

\[ \{ \omega \in e^{\frac{1}{2}(a+b(n-1)t)}H_c^m((-\infty, 0), \Omega_h(\partial M)^2) : \text{supp}(\omega) \subset (-\infty, -K] \} \]

where \( K \) is any positive real number. Hence \( \Psi^* \) enables us to define an inner product on \( H_c^m \) via \( \langle \alpha, \beta \rangle = \langle \Psi^*\alpha, \Psi^*\beta \rangle \), where the inner product in the right-hand side is taken in the sense of the space \( e^{\frac{1}{2}(a+b(n-1)t)}H_c^m((-\infty, 0), \Omega_h(\partial M)^2) \). Since \( \Psi \) maps \((-\infty, -K] \times \partial M \) diffeomorphically onto \((0, e^{-K}] \times \partial M \) for all \( K > 0 \), this inner product on \( H_c^m \) induces a Hilbert space structure on the subspace \{\omega \in H_c^m : \text{supp}(\omega) \subset (0, K] \times \partial M \} \) for any \( K \in (0, 1) \).

Our next step is to describe \( H_z^{m,w} = x^w H_z^m \). By definition, \( \omega \in \Omega(U) \) belongs to \( H_z^{m,w} \) if \( x^{-w}\omega \in H_z^m \). This, of course, means \( \omega \) is supported away from \( \{1\} \times \partial M \). The isomorphism \( \omega \rightarrow x^{-w}\omega \) between \( H_z^{m,w} \) and \( H_z^m \) induces an inner product on \( H_z^{m,w} \) by

\[ \langle \alpha, \beta \rangle_{H_z^{m,w}} = \langle x^{-w}\alpha, x^{-w}\beta \rangle_{H_z^m} \]

Since multiplication by a power of \( x \) preserves support, any subspace of \( H_z^{m,w} \) of the type \{\omega \in H_z^{m,w} : \text{supp}(\omega) \subset (0, K] \times \partial M \} \) will be a Hilbert space under this inner product (for any \( K \in (0, 1) \)). Finally, \( H_z^{m,w}\Omega^k(U) \) will denote the space of elements of \( H_z^{m,w} \) of form degree \( k \).

Having defined \( H_z^{m,w} \) and \( H_z^{m,w'} \), we are now in a position to define the doubly weighted spaces \( H^{m,w,w'} \). Let \( \epsilon \in (0, \frac{1}{2}) \). Let \( \rho \in C^\infty(M) \) be a smooth function with the properties that

\[ \rho(p) \in [0, 1] \forall p \in M; \quad \rho|_{U_\epsilon} \equiv 1; \quad \rho|_{M-U_{2\epsilon}} \equiv 0 \]

That is, \( \rho \) is a smooth cutoff function near \( \partial M \).

**Definition 3.3** Let \( m \in \mathbb{Z} \) and \( w, w' \in \mathbb{R} \). By definition, the doubly weighted Sobolev space \( H^{m,w,w'} \) contains all forms \( \omega \in \Omega(M) \) such that \( \pi(\rho \omega) \in H_z^{m,w}, (1 - \pi)(\rho \omega) \in H^{m,w'}_{nz}, \) and \( (1 - \rho)\omega \in H^m_c(\text{Int}(M)) \). Here \( H^m_c(M) \), is the usual Sobolev space of compactly supported forms on \((M,g)\).
The first fact we need to check is that the above definition of $H^{m,w,w'}$ is independent of the choice of $\epsilon$ and $\rho$.

**Lemma 3.4** Let $\epsilon, \epsilon' \in (0, \frac{1}{2})$. Assume $\rho, \rho' \in C^\infty(M)$ satisfy $\rho|_{U_\epsilon} \equiv 1$, $\rho|_{M - U_{2\epsilon}} \equiv 0$, $\rho(M) \subset [0,1]$, and likewise $\rho'|_{U_{\epsilon'}} \equiv 1$, $\rho'|_{M - U_{2\epsilon'}} \equiv 0$, and $\rho'(M) \subset [0,1]$. Let $\omega \in \Omega(M)$, $m \in \mathbb{Z}$, $w, w' \in \mathbb{R}$. If $\pi(\rho \omega) \in H^{m,w}_z$, $(1 - \pi)(\rho \omega) \in H^{m,w}_{nz}$, and $(1 - \rho)\omega \in H^m_c(M)$, then $\pi(\rho' \omega) \in H^{m,w}_z$, $(1 - \pi)(\rho' \omega) \in H^{m,w}_{nz}$, and $(1 - \rho')\omega \in H^m_c(M)$.

**Proof.** This can be deduced from the product rule and the fact that the vector fields used in the definitions of $H^{m,w}_z$ and $H^{m,w}_{nz}$ are bounded away from zero on sets of the form $M - U_\epsilon$. □

We are next going to list the properties of doubly weighted Sobolev spaces. All of them follow from the definition of $H^{m,w,w'}$ and from the corresponding properties of ordinary Sobolev spaces, as described in [1]. We first observe that each $H^{m,w,w'}$ is a Hilbert space.

**Lemma 3.5** The projections $\omega \rightarrow \pi(\rho \omega)$, $\omega \rightarrow (1 - \pi)(\rho \omega)$, and $\omega \rightarrow (1 - \rho)\omega$ of $H^{m,w,w'}$ induce an inner product on $H^{m,w,w'}$ by pulling back the inner products on those standard Sobolev spaces.

**Proof** The norm $\|\omega\|_{H^{m,w,w'}} = \|\pi(\rho \omega)\|_{H^{m,w}_z} + \|(1 - \pi)(\rho \omega)\|_{H^{m,w}_{nz}} + \|(1 - \rho)\omega\|_{H^m_c}$ is clearly linear and positive-definite, and it satisfies the triangle inequality. The fact that $H^{m,w,w'}$ is complete under this norm follows from the fact that $H^{m,w}_z$, $H^{m,w}_{nz}$, and $H^m_c$ are all complete with respect to theirs. □

We note two trivial properties of $H^{m,w,w'}$:

**Lemma 3.6** If $m_1 < m_2$, $w_1 < w_2$, and $w'_1 < w'_2$, then $H^{m_2,w_2,w'_2}_{nz} \subset H^{m_1,w_1,w'_1}_{nz}$. Also, $D$ is a continuous map from $H^{m,w,w'}$ to $H^{m-1,w+a,w'+b}$ for all $m$, $w$, and $w'$. 

**Proof.** The first fact follows directly from the definition of $H^{m,w,w'}$; the second, from the restrictions (3.1) and (3.3) of $D$ to boundary harmonic and boundary non-harmonic forms, respectively. □
We will also need the fact that smooth, compactly supported forms on $M$ are dense in each $H^{m,w,w'}$, and that any form that belongs to every $H^{m,w,w'}$ must be smooth and rapidly decaying near $\partial M$.

**Lemma 3.7** For each $m$, $w$, and $w'$, $C_c^\infty(M)$ is a dense subset of $H^{m,w,w'}$. Also,

$$\bigcap_{m,w,w'} H^{m,w,w} = C^\infty_c(M)$$

**Proof.** Both facts follow from the corresponding properties of $H^{m,w}$, $H^{m,w'}$, and $H^c_c(M)$. □

We next note that the dual of $H^{m,w,w'}$ under the $L^2$ pairing is the doubly weighted space $H^{-m,-w,-w'}$.

**Lemma 3.8** The standard $L^2$ pairing on $C_c^\infty(M)$ extends by continuity to a non-degenerate sesquilinear pairing between $H^{m,w,w'}$ and $H^{-m,-w,-w'}$.

**Proof.** This again follows from the same property of $H_c^m(M)$ and the standard weighted Sobolev spaces on $\mathbb{R} \times \partial M$. □

**Corollary 3.9** The adjoint of the operator $D : H^{m,w,w'} \to H^{m-1,w+a,w'+b}$ under the $L^2$ pairing is $D : H^{-m+1,-w-a,-w'-b} \to H^{-m,-w,-w'}$.

**Proof.** This follows at once from the fact that $D$ is self-adjoint on smooth forms that are compactly supported inside $M$, that the space of those forms is dense in any doubly weighted Sobolev space, and from the previous lemma. □

The last property of doubly weighted Sobolev spaces that we will need is this:

**Lemma 3.10** If $m_1 < m_2$, $w_1 < w_2$, and $w_1' < w_2'$, then the inclusion of $H^{m_2,w_2,w_2'}$ into $H^{m_1,w_1,w_1'}$ is a compact map.

**Proof.** Like most other properties of doubly weighted Sobolev spaces, this one follows from the same properties of $H^{m,w}$, $H^{m,w'}$, and $H^c_c(M)$, as well as the definition of the norm on $H^{m,w,w'}$. □
Chapter 4

Fredholm properties of $D$.

The motivation for our definition of doubly weighted Sobolev spaces is that the operator $D$ is Fredholm as a map from $H^{m,w,w'}$ to $H^{m-1,w+a,w'+b}$, unless $w$ belongs to a finite set of exceptional weights. In this section we describe the Fredholm properties of $D$. Our key result will be this:

**Lemma 4.1** Let $m \in \mathbb{Z}$, and $w, w' \in \mathbb{R}$. Assume that $w \notin \left\{ \frac{1}{2}(-a - b(2k - n + 1)) : k \in \mathbb{Z}, 0 \leq k \leq n - 1 \right\}$. Then $D : H^{m,w,w'} \to H^{m-1,w+a,w'+b}$ is a Fredholm operator.

Plugging in $w = -a$ in the above formula, we immediately deduce the conditions on $a$ and $b$ for when $D : H^{1,-a,-b} \to H^{0,0,0} = L^2$ is Fredholm.

**Corollary 4.2**. As long as $\frac{n-1+a/b}{2} \notin \{0, 1, \ldots, n-1\}$, $D : H^{1,-a,-b} \to L^2$ is a Fredholm operator.

**Proof.** Plugging $w = -a$ into the statement of Lemma 4.1 yields the condition $k \neq \frac{n-1+a/b}{2}$ for all $k \in \mathbb{Z}$ between 0 and $n - 1$. □

In this section we will also give a description of harmonic forms in $H^{m,w,w'}$. As in the case of a compact manifold, harmonic forms will have far more smoothness (and decay) than the definition of $H^{m,w,w'}$ requires.

**Lemma 4.3** Assume $\omega \in H^{m,w,w'}$ satisfies $D\omega = 0$. Then $\omega \in H^{\infty,w,\infty}$. In particular, $\omega$ is smooth and its boundary non-harmonic part in $U$ belongs to $H^{\infty,\infty}$, i.e. is
rapidly decaying near $\partial M$. The boundary harmonic part of $\omega$ in $U$ is equal to

$$\pi(\omega|_U) = \sum_{k=0}^{n-1} \begin{pmatrix} x^{kb}\alpha_k \\ x^{(n-k-1)b}\beta_k \end{pmatrix}$$

where $\alpha_k, \beta_k \in \mathcal{H}^k(\partial M)$ are $x$-independent harmonic forms on $\partial M$ for each $k$.

Recall that there are three equivalent definitions of a Fredholm operator. (We refer the reader to [4] for a detailed description of Fredholm operators and their properties.) An operator $P : V_1 \rightarrow V_2$, where $V_1$ and $V_2$ are Hilbert spaces, is said to be Fredholm if $\dim(\text{Ker}(D)) < \infty$, $\dim(\text{Im}(D)^\perp) < \infty$, and the image of $D$ is a closed subspace of $V_2$. Equivalently, $P : V_1 \rightarrow V_2$ is a Fredholm operator if there exist finite-dimensional subspaces $U_1 \subset V_1$, $U_2 \subset V_2$ such that $P$ induces an isomorphism from $V_1/U_1$ to $V_2/U_2$. If $V_1$ is compactly embedded into a Banach space, $V_3$, and the dual $V_2^*$ of $V_2$ is compactly embedded into $V_4$, then there is another equivalent definition of $P$ being Fredholm. It is this:

**Definition 4.4** If $P : V_1 \rightarrow V_2$ is a continuous operator and $i : V_1 \rightarrow V_3$, $j : V_2^* \rightarrow V_4$ are two compact embeddings (where $V_1$, $V_2$, $V_3$, and $V_4$ are Hilbert spaces, and $V_2^*$ is the dual of $V_2$), $P$ is called a Fredholm operator if there exists a constant $C \in \mathbb{R}$ such that for any $v \in V_1$,

$$\|v\|_{V_1} \leq C \left( \|Pv\|_{V_2} + \|v\|_{V_3} \right) \quad (4.1)$$

and for any $w \in V_2^*$,

$$\|w\|_{V_2^*} \leq C \left( \|P^*w\|_{V_1^*} + \|w\|_{V_4} \right)$$

In Section 4.1 we will prove inequality (4.1) for $D : H^{m,w}_{m,w} \rightarrow H^{m-1,w+a}_{m-1,w}$. This will only be possible as long as $w \notin \left\{ \frac{1}{2}(-a - b(2k - n + 1)) : k \in \{0, 1, \ldots, n - 1\} \right\}$. In Section 4.2 we will establish a similar inequality for $D : H^{m,w}_{m,w'} \rightarrow H^{m-1,w'+b}_{m-1,w'}$. This time the space $V_3$ will be $H^{m-1,w'+b-a}_{m-1,w'}$. Finally, in Section 4.3 we will prove (or, rather, quote) a similar inequality for $D$ on forms which are compactly supported away from $\partial M$. We will then use the triangle inequality to bring the three estimates together and prove (4.1) for $D : H^{m,w,w'} \rightarrow$
$H^{m-1,w+a,w'+b}$, with $V_3$ being $H^{m-1,w-1,w'+b-a}$, as long as $w$ is not an element of the finite set of exceptional weights. Because $D : H^m,w,w' \to H^{m-1,w+a,w'+b}$ is the adjoint of $D : H^{1-m,-w-a,-w'-b}$, and because $w$ will lie in the set $\{ \frac{1}{2}(-a - b(2k - n + 1)) : k \in \{0,1,\ldots,n-1\} \}$ if and only if $-w - a$ lies in that set, this inequality will show that $D : H^m,w,w' \to H^{m-1,w+a,w'+b}$ is indeed Fredholm as long as $w$ is not an exceptional weight.

### 4.1 Fredholm properties of $D$ on $H^m_z,w$.

Our goal in this section is to prove the following two lemmas.

**Lemma 4.5** Let $w \in \mathbb{R} - \{ \frac{1}{2}(-a - b(2k - n + 1)) : k \in \{0,1,\ldots,n-1\} \}$, and $m \in \mathbb{Z}$. Then there exists a constant $C_1$ such that for any $\omega \in H^m_z,w$,

$$||\omega||_{H^m_z,w} \leq C_1 (||D\omega||_{H^{m-1,w+a}} + ||\omega||_{H^{m-1,w-1}})$$

**Lemma 4.6** If $\omega \in H^m,w,w'$ satisfies $D\omega = 0$, the BH part of $\omega$ in $U$ must be

$$\pi(\omega|_U) = \sum_{k=0}^{n-1} \left( \begin{array}{c} x^{kb} \alpha^k \\ x^{(n-k-1)b} \beta^k \end{array} \right)$$

where $\alpha^k, \beta^k \in \mathcal{H}^k(\partial M)$ for all $k$.

**Proof of Lemma 4.6.** This instantly follows from the fact that $D$ acts on the BH part of a form in $U$ by (3.1). Assume $\omega$ is a harmonic form on $M$, and let the BH part of $\omega$ in $U$ be $\left( \begin{array}{c} \omega_T \\ \omega_C \end{array} \right)$. Assume further that $\omega_T = \sum_{k=0}^{n-1} \omega^T_k$ and $\omega_C = \sum_{k=0}^{n-1} \omega^C_k$, where each $\omega^T_k$ and $\omega^C_k$ is an $x$-dependent form on $\partial M$ of form degree $k$. Since $D$ acts on the BH part of $\omega$ in $U$ by (3.1) and the BH part of $D\omega$ in $U$ is zero, we have

$$0 = x^a \left( \begin{array}{cc} 0 & -(n-A-1)b \\ x^{\frac{\partial}{\partial x}} - Ab & 0 \end{array} \right) \left( \sum_{k=0}^{n-1} \left( \begin{array}{c} \omega^T_k \\ \omega^C_k \end{array} \right) \right)$$

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\[\begin{aligned}
&= x^n \sum_{k=0}^{n-1} \left( -x^k \frac{\partial \omega^k_T}{\partial x} + (n - k - 1)k \omega^k_T \right) \\
&\quad \times \left( x^k \frac{\partial \omega^k_C}{\partial x} - k \omega^k_T \right)
\end{aligned}\]

We see that the top entries of any two summands will have different form degrees, and the same is true of the bottom entries. So this sum can be equal to zero if and only if each summand is equal to zero. Solving the ordinary differential equation for each \(\omega^k_T\) and \(\omega^k_C\), we get \(\omega^k_T = x^k \alpha^k\) and \(\omega^k_C = x^{(n-k-1)b} \beta^k\), where the constants of integration \(\alpha^k\) and \(\beta^k\) are \(x\)-independent harmonic forms on \(\partial M\).

The proof of Lemma 4.5 is more sophisticated. Since \(x \frac{\partial}{\partial x}\) is conjugate to \(\frac{\partial}{\partial t}\) under the change of variables \(x = e^t\), we will be able to deduce the Fredholm properties of \(D\) from the Fredholm properties of \(\frac{\partial}{\partial x}\) on exponentially weighted Sobolev spaces on the real line. Those spaces are defined as follows.

Fix a form degree \(k\). Define the space \(L^2_{0,0}(\mathbb{R})\) (also denoted \(L^2_{0,0}\)) by

\[L^2_{0,0}(\mathbb{R}) = \{\omega \in L^1_{\text{loc}}(\mathbb{R}, L^2 \Omega^k_z(\partial M)) : \int_{-\infty}^{\infty} ||\omega(t)||^2_h dt < \infty\}\]

Here || \cdot ||_h is the norm on \(L^2 \Omega^k(\partial M)\) induced by \(h\). In other words, \(L^2_{0,0}\) is the space of \(k\)-th order differential forms on \(\mathbb{R} \times \partial M\) which do not have \(dt\) in them, and which are \(L^2\) with respect to the metric \(dt^2 + h\) and harmonic when restricted to each cross-section \(\{x_0\} \times \Omega^k(\partial M)\).

Let \(a\) and \(b\) be two real numbers. The weight function \(W_{a,b}\) is, by definition,

\[W_{a,b}(t) = \begin{cases} 
  e^{-at} & \text{if } t < 0 \\
  e^{-bt} & \text{if } t \geq 0
\end{cases}\]

The space \(L^2_{a,b}(\mathbb{R})\) (or simply \(L^2_{a,b}\)) is defined as

\[L^2_{a,b}(\mathbb{R}) = \{\omega : \mathbb{R} \to L^2 \Omega^k_z(\partial M) : W_{a,b}(t)\omega \in L^2_{0,0}\}\]

Intuitively speaking, if \(H\) is the heavyside function (equal to 0 on \((-\infty, 0)\) and 1 on \([0, \infty)\)), then \(L^2_{a,b}\) contains all forms \(\omega\) whose "left half" \((1 - H(t))\omega(t)\) lies in...
$e^{at}L^2_{0,0}$, and whose "right half" $H(t)\omega(t)$ is in $e^{bt}L^2_{0,0}$. It is a space with different weights, $a$ and $b$, at the positive and negative infinity.

We define $H^1_{a,b}(\mathbb{R})$ to be the space

$$H^1_{a,b}(\mathbb{R}) = \{ \omega \in L^2_{a,b} : \frac{\partial \omega}{\partial t} \in L^2_{a,b} \}$$

Here the derivative is taken in the sense of distributions.

We begin with this key lemma:

**Lemma 4.7** The operator $\frac{\partial}{\partial x} : H^1_{a,b} \to L^2_{a,b}$ is Fredholm with index 0 if $a > 0$ and $b < 0$, or if $a < 0$ and $b > 0$. It is Fredholm with index 1 if $a > 0$ and $b > 0$, and Fredholm with index $-1$ if $a < 0$ and $b < 0$.

Our proof will follow [7], where the same result is proved for functions on a line. The basic idea of the proof, of course, is that differentiation is the inverse of integration: to solve $-\frac{\partial \omega}{\partial t} = \alpha$, you have to integrate $\alpha$. The only question is whether or not the result will lie in the same space $L^2_{a,b}$ that $\alpha$ was in. The answer is given by the following lemma.

**Lemma 4.8** Let $\alpha \in L^2_{a,b}$ satisfy $\alpha(t) = 0$ for each $t < 0$. If $b > 0$, then $\int_0^t \alpha(s)ds \in L^2_{a,b}$. If $b < 0$, then $H(t)\int_0^t \alpha(s)ds \in L^2_{a,b}$, where $H$ is the Heavyside function.

**Proof.** Since $\alpha(t) = 0$ whenever $t < 0$, we have

$$\int_{-\infty}^{\infty} ||W_{a,b}(t)\alpha(t)||^2dt = \int_0^\infty ||W_{a,b}(t)\alpha(t)||^2dt = \int_0^\infty ||e^{-bt}\alpha(t)||^2dt$$

and the condition that $\alpha \in L^2_{a,b}$ means that the above quantity is finite. So $e^{-bt}\alpha$ is an element of $L^2_{0,0}$. In other words, $\alpha(t) = e^{bt}\gamma(t)$ for some $L^2$ form $\gamma$.

Assume first that $b > 0$. Let $\omega(t) = \int_0^t \alpha(s)ds$. Obviously, $\omega$ is zero for $t < 0$, because $\alpha(s) = 0$ for each $s < 0$. So the condition $\omega \in L^2_{a,b}$, which we have to prove, also translates to

$$\int_0^\infty ||e^{-bt}\omega(t)||^2dt < \infty$$
Observe that

\[
\int_0^\infty |e^{-bt}\omega(t)|^2 dt = \int_0^\infty e^{-2bt} \left( \int_0^t |\alpha(s)| ds \right)^2 dt \\
\leq \int_0^\infty e^{-2bt} \left( \int_0^t |e^{bs}\gamma(s)| ds \right)^2 dt \\
= \int_0^\infty e^{-2bt} \int_0^t e^{bs} |\gamma(s)| ds \int_0^t e^{by} |\gamma(y)| dy dt \\
= \int_{t=0}^\infty \int_{s=0}^t \int_{y=0}^t e^{b(s-t)} e^{b(y-t)} |\gamma(s)| |\gamma(y)| dy ds dt \\
\leq \left( \int_{t=0}^\infty \int_{s=0}^t \int_{y=0}^t e^{b(s-t)} e^{b(y-t)} |\gamma(s)| |\gamma(y)| dy ds dt \right)^{\frac{1}{2}} \times \\
\left( \int_{t=0}^\infty \int_{s=0}^t \int_{y=0}^t e^{b(s-t)} e^{b(y-t)} |\gamma(s)|^2 dy ds dt \right)^{\frac{1}{2}}
\]

In the last transition, we use the Hölder inequality for \( L^2 \), rewriting the integrand \( e^{b(s-t)} e^{b(y-t)} |\gamma(s)| |\gamma(y)| \) as

\[
\left( \sqrt{e^{b(s-t)} e^{b(y-t)} |\gamma(s)|} \right) \cdot \left( \sqrt{e^{b(s-t)} e^{b(y-t)} |\gamma(y)|} \right)
\]

To prove that the product of the square roots of these two integrals is finite, it suffices to prove that they themselves are finite. But the two integrals are identical: if we rename \( s \) to \( y \) and \( y \) to \( s \) in one of them, we get the other. Hence all we need to prove is that the first integral is finite. First notice that

\[
\int_{t=0}^\infty \int_{s=0}^t \int_{y=0}^t e^{b(s-t)} e^{b(y-t)} |\gamma(s)|^2 dy ds dt = \\
= \int_{t=0}^\infty \int_{s=0}^t e^{b(s-t)} |\gamma(s)|^2 \int_{y=0}^t e^{b(y-t)} dy ds dt \\
\]
as only one term in the product depends on \( y \). The inner integral evaluates to

\[
\int_{y=0}^t e^{b(y-t)} dy = \frac{1}{b} e^{b(y-t)} \bigg|_{y=0}^{t} = \frac{1}{b} \left( 1 - e^{-bt} \right)
\]
Plugging this in the original integral, we get

\[
\int_{t=0}^{\infty} \int_{s=0}^{t} e^{b(s-t)} ||\gamma(s)||^2 \frac{1}{b} (1 - e^{-bt}) dsdt \\
= \frac{1}{b} \int_{s=0}^{\infty} \int_{t=s}^{\infty} (e^{b(s-t)} - e^{b(s-2t)}) ||\gamma(s)||^2 dt ds \\
= \frac{1}{b} \int_{s=0}^{\infty} ||\gamma(s)||^2 \int_{t=s}^{\infty} e^{b(s-t)} - e^{b(s-2t)} dt ds
\]

Once again, the inner integral is easy to evaluate:

\[
\int_{t=s}^{\infty} e^{b(s-t)} - e^{b(s-2t)} dt = \left( \frac{1}{b} e^{b(s-t)} + \frac{1}{2b} e^{b(s-2t)} \right) \bigg|_{t=s}^{\infty} = \frac{1}{b} - \frac{1}{2b} e^{-bs}
\]

because the crucial fact that \( b \) is positive implies

\[
\lim_{t \to \infty} e^{b(s-t)} = \lim_{t \to \infty} e^{b(s-2t)} = 0
\]

for any fixed \( s \). Plugging this in the original integral, we obtain

\[
\frac{1}{b^2} \int_{s=0}^{\infty} \left( 1 - \frac{e^{-bs}}{2} \right) ||\gamma(s)||^2 ds \leq \frac{1}{b^2} \int_{s=0}^{\infty} \left( 1 - \frac{e^{-bs}}{2} \right) ||\gamma(s)||^2 ds
\]

again by Hölder’s inequality. This last quantity is clearly finite, because \( b > 0 \) implies \( 0 < e^{-bs} < 1 \) for all positive \( s \), and the fact that \( \gamma \) is \( L^2 \) means that \( ||\gamma(s)||^2 \) has a bounded \( L^1 \) norm. This finally proves that \( \omega(t) = \int_0^t \alpha(t) dt \) lies in \( L^2_{a,b} \), as long as \( b > 0 \).

Next, assume that \( b < 0 \). We now set \( \omega(t) = H(t) \int_\infty^t \alpha(s) ds \). Note first that the integral converges, because

\[
\left| \int_{\infty}^{t} \alpha(s) ds \right| \leq \int_{\infty}^{t} e^{bs} ||\gamma(s)|| ds \leq \left( \int_{\infty}^{\infty} e^{2bs} ds \right)^{\frac{1}{2}} \left( \int_{\infty}^{t} ||\gamma(s)||^2 ds \right)^{\frac{1}{2}}
\]

and both quantities on the right are finite, because \( b \) is negative and \( \gamma \) is \( L^2 \). Since \( \omega(t) = 0 \) for any negative \( t \), we only have to prove that \( \int_{0}^{\infty} ||e^{-bt}\omega(t)||^2 ds \) is finite.

The argument is very similar to the case \( b < 0 \), because for any positive \( t \) the Heavyside function in the definition of \( \omega(t) \) disappears (it is 1). Just as above, we
rewrite the integral in terms of γ. Since the inner integral is squared, we can write it as an integral from t to ∞ instead of ∞ to t. We then replace the square of the inner integral with the product of two integrals in two different variables. The only difference is that the inner integrals now go from t to ∞, instead of zero to t. Hölder’s inequality again allows us to estimate that expression by the product of square roots of two identical integrals. So now we have to prove the finiteness of

\[ \int_{t=0}^{\infty} \int_{s=t}^{\infty} e^{b(s-t)} \gamma(s)^2 dy ds dt = \int_{t=0}^{\infty} \int_{s=t}^{\infty} e^{b(s-t)} \gamma(s)^2 dy ds dt \]

Since b is now negative, the inner integral evaluates to \(-\frac{1}{b}\). The rest of the expression then becomes

\[ -\frac{1}{b} \int_{t=0}^{\infty} \int_{s=t}^{\infty} e^{b(s-t)} \gamma(s)^2 ds dt = -\frac{1}{b} \int_{s=0}^{\infty} \int_{t=0}^{s} e^{b(s-t)} \gamma(s)^2 ds dt = -\frac{1}{b} \int_{s=0}^{\infty} \gamma(s)^2 \int_{t=0}^{s} e^{b(s-t)} ds dt \]

Computing the inner integral, we get \(-\frac{1}{b} (1 - e^{bs})\). So the outer integral equals

\[ \frac{1}{b^2} \int_{0}^{\infty} (1 - e^{bs}) \gamma(s)^2 ds \leq \frac{1}{b^2} \|1 - e^{bs}\|_{L^\infty(\mathbb{R}^+)} \|\gamma(s)^2\|_{L^1(\mathbb{R}^+)} \]

and the first norm is bounded because b is negative (hence \(0 < 1 - e^{bs} < 1\) for all positive s), and the second one, because γ is an element of \(L^2\). This shows that \(\omega\) lies in \(L^2_{a,b}\), and concludes the proof of Lemma 4.8.

**Corollary 4.9** Suppose that \(\alpha \in L^2_{a,b}\) satisfies \(\alpha(t) = 0\) for all \(t > 0\). If \(a < 0\), then \(\int_{0}^{t} \alpha(s) ds \in L^2_{a,b}\). If \(a > 0\), then \(H(-t) \int_{-t}^{t} \alpha(s) ds \in L^2_{a,b}\).

**Proof.** This follows immediately from Lemma 4.8 by a symmetry argument. Namely, for any form \(\beta \in L^2_{a,b}\), define \(\mathcal{F}\beta\) by \(\mathcal{F}\beta(t) = \beta(-t)\) for each \(t\). It is clear that \(\beta \in L^2_{a,b}\) if and only if \(\mathcal{F}\beta \in L^2_{-b, -a}\), because

\[ \int_{-\infty}^{\infty} \|W_{a,b}(t)\beta(t)\|^2 dt = \int_{-\infty}^{\infty} \|W_{a,b}(-t)\beta(-t)\|^2 dt = \int_{-\infty}^{\infty} \|W_{-b, -a}(t)\mathcal{F}\beta(t)\|^2 dt \]
In our case, since \( \alpha \in L^2_{a,b} \) satisfies \( \alpha(t) = 0 \) for each \( t > 0 \), \( \mathcal{F} \alpha \) lies in \( L^2_{-b,-a} \) and satisfies \( \mathcal{F} \alpha(t) = 0 \) for any \( t < 0 \). By Lemma 4.8, if \(-a > 0\), then \( \int_0^t \mathcal{F} \alpha(y)dy \in L^2_{-b,-a} \). If \(-a < 0\), then \( H(t) \int_{-\infty}^t \mathcal{F} \alpha(y)dy \in L^2_{-b,-a} \). So if \( a < 0 \), then

\[
\int_0^t \alpha(s)ds = \mathcal{F} \left( \int_0^{-t} \alpha(s)ds \right) = \mathcal{F} \left( \int_0^t \alpha(-y)d(-y) \right) = -\mathcal{F} \left( \int_0^t \mathcal{F} \alpha(y)dy \right)
\]

which lies in \( L^2_{a,b} \), because \( \int_0^t \mathcal{F} \alpha(y)dy \) is in \( L^2_{-b,-a} \).

The following conditions are clearly equivalent:

\[
\int_{-\infty}^0 \alpha(s)ds = 0 \quad \Leftrightarrow \quad \int_{0}^\infty \alpha(s)d(-s) = 0 \quad \Leftrightarrow \quad \int_0^\infty \mathcal{F} \alpha(s)ds = 0
\]

So if \( a > 0 \) and \( \int_{-\infty}^0 \alpha(s)ds = 0 \), then Lemma 4.8 implies that

\[
H(-t) \int_{-\infty}^t \alpha(s)ds = \mathcal{F} \left( H(t) \int_{-\infty}^{-t} \alpha(s)ds \right) = \mathcal{F} \left( H(t) \int_{0}^t \alpha(-y)d(-y) \right) = -\mathcal{F} \left( H(t) \int_0^t \mathcal{F} \alpha(y)dy \right) \in L^2_{a,b}
\]

because \( H(t) \int_0^t \mathcal{F} \alpha(y)dy \) is in \( L^2_{-b,-a} \).

Now that we know how to invert \( \frac{\partial}{\partial x} \) on the left and right half-lines, we can spell out the rules for inverting it on the entire line.

**Lemma 4.10** Let \( \alpha \in L^2_{a,b}(\mathbb{R}) \).

1. If \( a < 0 \), \( b > 0 \), set \( \omega(t) = \int_0^t \alpha(s)ds \)
2. If \( a < 0 \), \( b < 0 \), set \( \omega(t) = \int_0^t \alpha(s)ds \)
3. If \( a > 0 \), \( b > 0 \), set \( \omega(t) = \int_{-\infty}^t \alpha(s)ds \)
4. If \( a > 0 \), \( b < 0 \), and \( \int_{-\infty}^\infty \alpha(s)ds = 0 \), set \( \omega(t) = \int_{-\infty}^t \alpha(s)ds \).

Then, in each of these cases, \( \omega(t) \) is a well-defined element of \( H^1_{a,b} \) which satisfies \( \frac{\partial \omega(t)}{\partial t} = \alpha(t) \).
Proof. First let us explain why \( \omega \) is well defined in each case. Since \( \alpha \in L_{loc}^1 \), any finite integral \( \int_a^b \alpha(s)ds \) makes sense, and we only need to show that the infinite integrals in cases 2, 3, and 4 converge for each \( t \). In cases 3 and 4 it is equivalent to saying that \( \int_0^\infty \alpha(s)ds \) converges, because it differs from any \( \omega(t) \) by the finite quantity \( \int_0^t \alpha(s)ds \). We estimate \( \int_0^t \alpha(s)ds \) as

\[
\int_{-\infty}^t \alpha(s)e^{-as}e^{as}ds \leq \left( \int_{-\infty}^0 \|\alpha(s)e^{-as}\|^2 ds \right)^{\frac{1}{2}} \left( \int_{-\infty}^0 e^{2as}ds \right)^{\frac{1}{2}}
\]

\[
= \left( \int_{-\infty}^0 \|W_{a,b}(s)\alpha(s)\|^2 ds \right)^{\frac{1}{2}} \cdot \frac{1}{\sqrt{2a}} \leq \frac{1}{\sqrt{2a}} \left( \int_{-\infty}^\infty \|W_{a,b}(s)\alpha(s)\|^2 ds \right)^{\frac{1}{2}}
\]

and the last expression is finite, because \( W_{a,b} \alpha \) is an element of \( L^2 \). We’re using the condition \( a > 0 \) to conclude that \( \int_0^0 e^{2as}ds = \frac{1}{2a} \). In case 2 the fact that \( \omega \) is well-defined follows from an identical argument, except that we use \( b \) instead of \( a \) and \( +\infty \) instead of \( -\infty \).

We next notice that, by the definition of \( \omega \), it must be continuous and hence necessarily \( L_{loc}^1 \). The fundamental theorem of calculus implies that \( \frac{d\omega(t)}{dt} = \alpha(t) \) (almost everywhere, but that’s what equality means for our forms). So now we only need to prove that \( \omega \) lies in \( L_{a,b}^2 \).

Let \( H \) be the Heavyside function: \( H(t) = 1 \) if \( t \geq 0 \), and \( H(t) = 0 \) otherwise. To prove that a given form \( \omega \) lies in \( L_{a,b}^2 \), it suffices to prove that its right half \( H(t)\omega(t) \) and its left half \( (1 - H(t))\omega(t) \) both lie in \( L_{a,b}^2 \) (as \( L_{a,b}^2 \) is closed under addition). When \( a < 0 \) and \( b > 0 \), it is clear that

\[
H(t)\omega(t) = \int_0^t H(s)\alpha(s)ds \quad \text{and} \quad (1 - H(t))\omega(t) = \int_0^t (1 - H(s))\alpha(s)ds
\]

by the definitions of \( H \) and \( \omega \). (In other words, to get the right half of \( \omega \), we integrate the right half of \( \alpha \) from zero, and the same is true for the left half.) Since \( H(s)\alpha(s) \) and \( (1 - H(s))\alpha(s) \) satisfy the conditions of Lemma 4.8 and Corollary 4.9, the left and right halves of \( \omega \) are both in \( L_{a,b}^2 \).
Cases 2 and 3 are almost identical: conjugation by $F$ turns one into the other, so we will write the proof only for case 2. The basic idea is that the right half of $\omega$ is the same as it was in Lemma 4.8, so of course, it will be in $L^2_{a,b}$. And the left half differs from the one in Corollary 4.9 only by the constant $\int_0^0 \alpha(s)ds$, but any function that’s constant on $\mathbb{R}_-$ and zero on $\mathbb{R}_+$ is also element of $L^2_{a,b}$ for any negative $a$.

More formally, we can easily see that

$$H(t)\omega(t) = H(t) \int_\infty^t \alpha(s)ds = H(t) \int_\infty^t H(s)\alpha(s)ds$$

because all these are zero when $t$ is negative, and when $t$ is non-negative all the $H$-s disappear. So $H(t)\omega(t) \in L^2_{a,b}$, because $H(s)\alpha(s)$ satisfies the assumptions of Lemma 4.8 and hence $H(t) \int_\infty^t H(s)\alpha(s)ds$ lies in $L^2_{a,b}$.

On the other hand,

$$(1 - H(t))\omega(t) = (1 - H(t)) \left( \int_0^t \alpha(s)ds + \int_\infty^0 \alpha(s)ds \right)$$

$$= \int_0^t (1 - H(s))\alpha(s)ds + (1 - H(t)) \int_\infty^0 \alpha(s)ds$$

because the first summand is zero when $t$ is positive and $\int_0^t \alpha(s)ds$ when $t$ is negative. Since $a < 0$ and $(1 - H(s))\alpha(s)$ is zero on the positive half-line, Corollary 4.9 tells us that the first summand is in $L^2_{a,b}$. And the second summand is a constant multiple of $1 - H(t)$, so to prove that it lies in $L^2_{a,b}$, it suffices to show that $W_{a,b}(t)(1 - H(t))$ lies in $L^2$ for $a < 0$. This follows directly from the definition of $W_{a,b}$:

$$\int_{-\infty}^\infty (W_{a,b}(t)(1 - H(t)))^2dt = \int_{-\infty}^0 W_{a,b}^2(t)dt = \int_0^0 e^{-2at}dt = -\frac{1}{2a}$$

because $a$ is negative.

Finally, in case 4 it is clear that

$$(1 - H(t))\omega(t) = (1 - H(t)) \int_{-\infty}^t (1 - H(s))\alpha(s)ds \in L^2_{a,b}$$
because \((1 - H(s))\alpha(s)\) satisfies the assumptions of Corollary 4.9, and therefore \(H(-t) \int_{-\infty}^{t} (1 - H(s))\alpha(s)ds\) lies in \(L_{a,b}^2\). Since the right-hand side differs from an element of \(L_{a,b}^2\) only at a single point (zero), it must itself lie in \(L_{a,b}^2\). As for the right half of \(\omega\), it equals

\[
H(t)\omega(t) = H(t) \int_{-\infty}^{t} \alpha(s)ds = H(t) \int_{\infty}^{t} \alpha(s)ds = H(t) \int_{\infty}^{t} H(s)\alpha(s)ds
\]

because \(\int_{-\infty}^{\infty} \alpha(s)ds = 0\) implies that \(\int_{-\infty}^{t} \alpha(s)ds = \int_{t}^{\infty} \alpha(s)ds\) for any \(t\). Since \(H(s)\alpha(s)\) vanishes on \(\mathbb{R}_-\) and \(b < 0\), \(H(t) \int_{\infty}^{t} H(s)\alpha(s)ds\) must lie in \(L_{a,b}^2\) by Lemma 4.8. Since the left and right halves of \(\omega\) lie in \(L_{a,b}^2\), so does \(\omega\) itself. \(\square\)

**Corollary 4.11** The operator \(\frac{\partial}{\partial t} : H_{a,b}^1 \to L_{a,b}^2\) is Fredholm as long as \(a \neq 0\) and \(b \neq 0\).

**Proof.** By definition, \(\frac{\partial}{\partial t}\) will be Fredholm if its kernel is finite-dimensional, its range is closed, and the complement of its range is finite-dimensional. The kernel of \(\frac{\partial}{\partial t}\) clearly consists of only the constant maps from \(\mathbb{R}\) to \(L^2 \Omega^k(\partial M)\) when \(a < 0 < b\), because then these constant maps lie in \(L_{a,b}^2\). If \(a > 0\) or \(b < 0\), then the kernel contains only zero, because the constants do not lie in \(L_{a,b}^2\). Either way, the kernel is finite-dimensional, because \(\dim (\Omega^k(\partial M)) < \infty\) (By the ordinary Hodge theorem, the space of harmonic forms on the compact Riemannian manifold \(\partial M\) must be finite-dimensional.)

By Lemma 4.10, if \(a < 0\) or \(b > 0\), then the range of \(\frac{\partial}{\partial t} : H_{a,b}^1 \to L_{a,b}^2\) is all of \(L_{a,b}^2\), so, of course, the range is closed and has a finite-dimensional (even zero-dimensional) complement. In the remaining case \(b < 0 < a\), let \(\omega\) be any element of \(L_{a,b}^2\). By definition, this means \(\omega(t) = W_{-a,-b}(t)\eta(t)\), where \(\eta \in L_{0,0}^2\). This implies that \(\int_{-\infty}^{0} \omega(t)dt\) converges, because

\[
\left| \int_{-\infty}^{0} \omega(t)dt \right| = \left| \int_{-\infty}^{0} W_{-a,-b}(t)\eta(t)dt \right| \leq \int_{-\infty}^{0} e^{at}||\eta(t)||dt \leq \left( \int_{-\infty}^{0} e^{2at}dt \int_{-\infty}^{0} ||\eta(t)||^2dt \right)^{\frac{1}{2}}
\]

(4.3)
by the Hölder inequality. The last quantity is finite, because \( a > 0 \) and \( \eta \in L^2_{0,0} \). An identical argument

\[
\left| \int_0^\infty \omega(t)dt \right| = \left| \int_0^\infty W_{-a,-b}(t) \eta(t)dt \right| \\
\leq \int_0^\infty e^{bt} ||\eta(t)|| dt \leq \left( \int_0^\infty e^{2bt} dt \int_0^\infty ||\eta(t)||^2 dt \right)^{\frac{1}{2}} < \infty
\]

shows that \( \int_0^\infty \omega(t)dt \) is also finite, because \( b < 0 \) and \( \eta \in L^2_{0,0} \).

Let \( \int_{-\infty}^\infty \omega(t)dt = \beta \). We see that \( \beta \) is a well-defined element of \( \Omega^k_2(\partial M) \). By the Hodge theorem for \( \partial M \), \( \Omega^k_2(\partial M) \) is finite-dimensional. Let \( e_1, \ldots, e_N \) be an orthonormal basis for \( \Omega^k_2(\partial M) \). Let \( f \) be an infinitely smooth, compactly supported function from \( \mathbb{R} \) to \( \mathbb{R} \) satisfying \( \int_{-\infty}^\infty f(t)dt = 1 \). Then it is clear that for each \( i \leq N \), \( f(t)e_i \) is an infinitely smooth, compactly supported map from \( \mathbb{R} \) to \( \Omega^k_2(\partial M) \) which satisfies

\[
\int_{-\infty}^\infty f(t)e_idt = e_i \int_{-\infty}^\infty f(t)dt = e_i
\]

and because \( f(t)e_i \) is compactly supported, it lies in \( L^2_{a,b} \) for any \( a \) and \( b \).

Finally, let \( \beta = \sum_{i=1}^N c_i e_i \). Then clearly \( \omega(t) - \sum_{i=1}^N c_i f(t)e_i \) is an element of \( L^2_{a,b} \) which satisfies

\[
\int_{-\infty}^\infty \omega(t)dt - \sum_{i=1}^n c_if(t)e_idt = \int_{-\infty}^\infty \omega(t)dt - \sum_{i=1}^n c_i \int_{-\infty}^\infty f(t)e_idt = \beta - \sum_{i=1}^n c_i e_i = 0
\]

and hence, by Lemma 4.10, \( \omega - \sum_{i=1}^N c_i f(t)e_i \) lies in the image of \( \frac{\partial}{\partial t} : H^1_{a,b} \to L^2_{a,b} \). So any form \( \omega \in L^2_{a,b} \) can be written as a sum of an element of \( \text{Im}(\frac{\partial}{\partial t}) \) and an element of the finite-dimensional subspace spanned by \( \{f(t)e_i\} \). Therefore, the orthonormal complement of \( \text{Im}(\frac{\partial}{\partial t}) \) is finite-dimensional; its dimension cannot be greater than \( N \). (It is possible to see that it is actually equal to \( N \), because \( f(t)e_i \) does not integrate to zero over \( \mathbb{R} \) and hence cannot lie in the image of \( \frac{\partial}{\partial t} \), by the converse to Lemma 4.10.) Also, because \( L^2_{a,b} \) can be written as a direct sum of \( \text{Im}(\frac{\partial}{\partial t}) \) with a finite-dimensional subspace, \( \text{Im}(\frac{\partial}{\partial t}) \) must be closed. So, by definition, \( \frac{\partial}{\partial t} : H^1_{a,b} \to L^2_{a,b} \) is Fredholm when \( b < 0 < a \). \( \square \)
Our next step is to extend this result to maps from $H^{m}_{a,b}$ to $H^{m-1}_{a,b}$ for any integer $m$. Let us first define these spaces. Let $m \in \mathbb{Z}_{+}$. Then, by definition,

$$H^{m}_{a,b}(\mathbb{R}) = \left\{ \omega \in L^{2}_{a,b}(\mathbb{R}) : \frac{\partial^{i}\omega}{\partial t^{i}} \in L^{2}_{a,b}(\mathbb{R}) \quad \forall i \leq m \right\}$$

If $m$ is a negative integer, define $H^{m}_{a,b}(\mathbb{R})$ to be the space of all formal sums

$$H^{m}_{a,b}(\mathbb{R}) = \left\{ \sum_{i=0}^{-m} \frac{\partial^{i}\omega_{i}}{\partial t^{i}} : \omega_{i} \in L^{2}_{a,b}(\mathbb{R}) \quad \forall i \leq m \right\}$$

When $m$ is negative, some elements of $H^{m}_{a,b}$ are distributions with values in $\Omega^{k}_{\mathbb{Z}}(\partial M)$, but when $m$ is non-negative, they are all functions from $\mathbb{R}$ to $\Omega^{k}_{\mathbb{Z}}(\partial M)$. It is clear from this definition that $H^{m}_{a,b} \subset H^{m'}_{a,b}$ for any $m > m'$, and that $\frac{\partial}{\partial t} : H^{m}_{a,b} \to H^{m-1}_{a,b}$ is a well-defined map.

**Lemma 4.12** The operator $\frac{\partial}{\partial t} : H^{m}_{a,b} \to H^{m-1}_{a,b}$ is Fredholm for any $m$ and any $a \neq 0$ and $b \neq 0$.

**Proof.** For $m = 1$ this is the statement of Lemma 4.11. Assume that $m > 1$. Let $\omega$ be any element of $H^{m-1}_{a,b}$. If $a < 0$ or $b > 0$, then Lemma 4.10 shows that there is a form $\alpha \in L^{2}_{a,b}$ satisfying $\frac{\partial \alpha}{\partial t} = \omega$. If $a > 0$ and $b < 0$, one can quickly see that $\int_{-\infty}^{\infty} \omega(t)dt$ converges. (We proved this for any element of $L^{2}_{a,b}$ in the case $b < 0 < a$ during the proof of Lemma 4.11.) Lemma 4.10 then states that there is an element $\alpha \in L^{2}_{a,b}$ satisfying $\frac{\partial \alpha}{\partial t} = \omega$ if $\int_{-\infty}^{\infty} \omega(t)dt = 0$.

Regardless of the sign of $a$ and $b$, $\omega$ will definitely be an element of the image of $\frac{\partial}{\partial t} : H^{m}_{a,b} \to H^{m-1}_{a,b}$ if $\omega = \frac{\partial \alpha}{\partial t}$ for some $\alpha \in L^{2}_{a,b}$, because

$$\forall i \in \{1,2,\ldots,m\} \quad \frac{\partial^{i+1}\alpha}{\partial t^{i}} = \frac{\partial^{i+1-1}\omega}{\partial t^{i-1}} \in L^{2}_{a,b}$$

so $\alpha \in H^{m}_{a,b}$, because $\omega \in H^{m-1}_{a,b}$.

We now use the same argument as we did for $m = 1$. The kernel of $\frac{\partial}{\partial t}$ on $H^{m}_{a,b}$ is a subset of the kernel of $\frac{\partial}{\partial t}$ on $H^{1}_{a,b}$, and the former must be finite-dimensional, because the latter is. (It is also easy to see that the kernel contains only constants.
when \( a < 0 < b \), and only zero otherwise.) If \( b > 0 \) or \( a < 0 \), then the above reasoning shows that \( \frac{\partial}{\partial t} : H_{a,b}^m \to H_{a,b}^{m-1} \) is surjective, because for any form \( \omega \in H_{a,b}^{m-1} \) there is an \( \alpha \in H_{a,b}^m \) satisfying \( \frac{\partial \alpha}{\partial t} = \omega \). Hence the range of \( \frac{\partial}{\partial t} \) is closed and has a zero-dimensional complement (it is all of \( H_{a,b}^{m-1} \)), so \( \frac{\partial}{\partial t} \) is Fredholm.

If \( b < 0 < a \), then, as we saw in the proof of Lemma 4.11, there exist \( N \) infinitely smooth and compactly supported maps \( f(t)e_i \) from \( \mathbb{R} \) to \( \Omega_z^k(\partial M) \) such that any \( \omega \in L_{a,b}^2 \) can be written as \( \omega(t) = \gamma(t) + \sum_{i=1}^N c_i f(t)e_i \), where \( \gamma \) satisfies \( \int_{-\infty}^{\infty} \gamma(t)dt = 0 \). Because the \( f(t)e_i \)'s are \( C^\infty \) with compact support, they lie in \( H_{a,b}^m \) for any \( m, a, \) and \( b \). So if \( \omega(t) \) is an element of \( H_{a,b}^{m-1} \), then \( \gamma(t) \) also lies in the same space. We know that \( \int_{-\infty}^{\infty} \gamma(t)dt = 0 \) implies that \( \gamma(t) = \frac{\partial \alpha}{\partial t} \) for some \( \alpha \in H_{a,b}^m \). Thus \( \gamma \) lies in the image of \( \frac{\partial}{\partial t} : H_{a,b}^m \to H_{a,b}^{m-1} \). We again see that any element \( \omega \) of \( H_{a,b}^{m-1} \) can be written as a sum of a form \( \gamma \) in the image of \( \frac{\partial}{\partial t} \) and an element of the finite-dimensional subspace spanned by \( \{f(t)e_i\} \). Hence the image of \( \frac{\partial}{\partial t} : H_{a,b}^m \to H_{a,b}^{m-1} \) must be closed, and its orthogonal complement finite-dimensional, so the operator is Fredholm.

Now assume that \( m < 1 \). Because \( \Omega_z^k(\partial M) \) is finite-dimensional, a choice of orthonormal basis in \( \Omega_z^k(\partial M) \) enables us to identify maps from \( \mathbb{R} \) to \( \Omega_z^k(\partial M) \) with maps from \( \mathbb{R} \) to \( \mathbb{R}^N \) for some integer \( N \). It is now easy to see that any formal sum

\[
\omega = \sum_{i=0}^{-m} \frac{\partial \omega_i}{\partial t^i} \quad \text{where } \omega_i \in L_{a,b}^2
\]

that satisfies \( \frac{\partial \omega}{\partial t} = 0 \) must be equal to a constant. This is true because each of the coordinates of \( \omega \) will be a constant, by the same result for distributions on \( \mathbb{R} \) (a distribution whose derivative is zero is a constant). Hence the kernel of \( \frac{\partial}{\partial t} \) on \( H_{a,b}^m \) is a subspace of the finite-dimensional space of constant maps from \( \mathbb{R} \) to \( \Omega_z^k(\partial M) \), so it is finite-dimensional itself.

Let now \( \omega = \sum_{i=0}^{-m+1} \frac{\partial \omega_i}{\partial t^i} \) (here \( \omega_i \in L_{a,b}^2 \)) be any element of \( H_{a,b}^{m-1} \). If \( a < 0 \) or \( b > 0 \), then Lemma 4.10 shows that \( \omega_0 = \frac{\partial \alpha}{\partial t} \) for some \( \alpha \in L_{a,b}^2 \). In this case

\[
\omega = \frac{\partial}{\partial t}\left(\alpha + \omega_1 + \sum_{i=1}^{-m} \frac{\partial \omega_{i+1}}{\partial t^i}\right) \in \text{Im}\left(\frac{\partial}{\partial t} : H_{a,b}^m \to H_{a,b}^{m-1}\right)
\]
because each $\omega_i$ and $\alpha$ lies in $L^2_{a,b}$. So the image of $\frac{\partial}{\partial t} : H_{a,b}^{m+1} \to H_{a,b}^m$ in this case is all of $H_{a,b}^m$ ($\omega$ was arbitrary), and hence the operator is Fredholm.

As before, the case $b < 0 < a$ is slightly different. We saw during the proof of Lemma 4.11 that in this case any element $\omega_0 \in L^2_{a,b}$ can be written as $\omega(t) = \gamma(t) + \sum_{i=1}^N c_i f(t) e_i$, where the $e_i$'s are certain fixed elements of $\Omega^k_z(\partial M)$, $f$ is smooth and compactly supported, and $\gamma(t) = \frac{\partial \alpha}{\partial t}$ for some $\alpha \in H^1_{a,b}$. Therefore, we can write $\omega$ as

$$
\omega = \sum_{i=1}^N c_i f e_i + \frac{\partial}{\partial t} \left( \alpha + \omega_1 + \sum_{i=1}^{m} \frac{\partial^i \omega_{i+1}}{\partial t^i} \right)
$$

so any $\omega \in H_{a,b}^{m-1}$ can be written as a sum of an element in the image of $\frac{\partial}{\partial t} : H_{a,b}^m \to H_{a,b}^{m-1}$ and an element of the finite-dimensional subspace of $H_{a,b}^{m-1}$ spanned by the $f(t)e_i$'s. Hence the image of $\frac{\partial}{\partial t} : H_{a,b}^m \to H_{a,b}^{m-1}$ must be closed and have a finite-dimensional complement. So in the case $b < 0 < a$ the operator $\frac{\partial}{\partial t} : H_{a,b}^m \to H_{a,b}^{m-1}$ is also Fredholm.

It is now easy to describe the Fredholm properties of the operator $\frac{\partial}{\partial t} c : H_{a,b}^m \to H_{a,b}^{m-1}$. They will follow immediately once we realize that $\frac{\partial}{\partial t} + c$ on $H_{a,b}^m$ is conjugate to $\frac{\partial}{\partial t}$ on $H_{a+c,b+c}^m$. For any $c \in \mathbb{R}$ and $\omega \in H_{a,b}^m$, let $\phi_c(\omega) = e^{ct}\omega$.

**Lemma 4.13** For any $a, b, c \in \mathbb{R}$ and $m \in \mathbb{Z}$, $\phi_c$ is a linear isomorphism from $H_{a,b}^m$ to $H_{a+c,b+c}^m$. It also satisfies $\frac{\partial}{\partial t} \circ \phi_c = \phi_c \circ (\frac{\partial}{\partial t} + c)$.

**Proof.** The definition of $\phi_c$ makes it clear that it is linear, and $(\phi_c)^{-1} = \phi_{-c}$. So once we show that $\phi_c$ maps $H_{a,b}^m$ to $H_{a+c,b+c}^m$, it will also imply that its inverse $\phi_{-c}$ maps $H_{a+c,b+c}^m$ to $H_{a,b}^m$, and hence that $\phi_c$ is a linear isomorphism.

When $m = 0$, we need to show that $\phi_c$ maps $L^2_{a,b}$ to $L^2_{a+c,b+c}$. Obviously

$$
\phi_c \left( L^2_{a,b} \right) = e^{ct} W_{-a,-b}(t)L^2_{0,0} = W_{-a-c,-b-c}(t)L^2_{0,0} = L^2_{a+c,b+c}
$$

as $e^{ct}W_{a,b}(t) = W_{a-c,b-c}(t)$ by the definition of $W_{a,b}$.
Assume that $m > 0$ and $\omega \in H^m_{a,b}$. Then $\phi_c \omega \in L^2_{a+c,b+c}$, and for any $i \in \{1,2,\ldots,m\}$ we have

$$\frac{\partial^i \phi_c \omega(t)}{\partial t^i} = \frac{\partial^i e^{ct} \omega(t)}{\partial t^i} = \sum_{j=0}^{i} \frac{i!}{j!(i-j)!} e^{ct} \frac{\partial^i \omega(t)}{\partial t^j}$$

which lies in $L^2_{a+c,b+c}$, because each term in the above sum is a constant times $\phi_c \frac{\partial^i \omega}{\partial t^i}$, and it lies in $L^2_{a+c,b+c}$ because $\frac{\partial^i \omega}{\partial t^i}$ lies in $L^2_{a,b}$ (as $j \leq m$ and $\omega \in H^m_{a,b}$). Since any derivative of $\phi_c \omega(t)$ of order up to $m$ lies in $L^2_{a+c,b+c}$, $\phi_c \omega(t)$ lies in $H^m_{a+c,b+c}$ by definition.

To prove the statement for $m < 0$ we use induction on $-m$. We have already established the fact for $-m = 0$. Assume that $\phi_c : H^m_{a,b} \to H^{m+1}_{a+c,b+c}$ is an isomorphism. Let $\omega$ be any element of $H^m_{a,b}$. By definition, this means $\omega = \sum_{i=0}^{-m} \frac{\partial^i \omega}{\partial t^i}$, where each $\omega_i$ lies in $L^2_{a,b}$. The inductive assumption tells us that $\phi_c \left(\sum_{i=0}^{-m} \frac{\partial^i \omega}{\partial t^i}\right)$ lies in $H^{m+1}_{a+c,b+c}$ and hence in $H^m_{a+c,b+c}$. All we need to check is that $\phi_c \left(\sum_{i=0}^{-m} \frac{\partial^i \omega}{\partial t^i}\right) \in H^m_{a+c,b+c}$.

But that is also clear, because

$$\phi_c \left(\frac{\partial^{-m} \omega_{-m}(t)}{\partial t^{-m}}\right) = e^{ct} \frac{\partial^{-m} \omega_{-m}(t)}{\partial t^{-m}} \tag{4.6}$$

$$= \frac{\partial}{\partial t} \left(e^{ct} \frac{\partial^{-m-1} \omega_{-m-1}(t)}{\partial t^{-m-1}}\right) - ce^{ct} \frac{\partial^{-m-1} \omega_{-m-1}(t)}{\partial t^{-m-1}} \tag{4.7}$$

and since $e^{ct} \frac{\partial^{-m-1} \omega_{-m-1}(t)}{\partial t^{-m-1}}$ lies in $H^{m+1}_{a+c,b+c}$ by the inductive assumption, its $t$-derivative and its constant multiple must both be in $H^{-m}_{a+c,b+c}$. By mathematical induction, this means $\phi_c$ maps $H^{-m}_{a,b}$ into $H^{-m}_{a+c,b+c}$ for any $m \in \mathbb{Z}_+$.

The formula $\frac{\partial}{\partial t} \circ \phi_c = \phi_c \circ (\frac{\partial}{\partial t} + c)$ is just a restatement of the product rule:

$$\frac{\partial}{\partial t} (\phi_c \omega(t)) = \frac{\partial}{\partial t} (e^{ct} \omega(t)) = ce^{ct} \omega(t) + e^{ct} \frac{\partial \omega(t)}{\partial t} = \phi_c \left(\frac{\partial}{\partial t} + c\right) \omega(t)$$

for any $\omega \in H^m_{a,b}$. \hfill \Box

**Corollary 4.14** The operator $\frac{\partial}{\partial t} + c : H^m_{a,b} \to H^{m-1}_{a,b}$ is Fredholm for any $m$, $a$ and $b$ and any $c$ satisfying $c \neq -a$, $c \neq -b$. 

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Proof. We have just seen that the operator $\frac{\partial}{\partial t} + c : H_{a,b}^m \rightarrow H_{a,b}^{m-1}$ is conjugate to $\frac{\partial}{\partial t} : H_{a,b}^{m-1} \rightarrow H_{a,b}^{m-1}$ via the conjugacy map $\phi_c$. Consequently, the first operator is Fredholm if and only if the second one is. By Lemma 4.12, $\frac{\partial}{\partial t} : H_{a,c,b+c}^m \rightarrow H_{a,c,b+c}^{m-1}$ is Fredholm when $a + c \neq 0$ and $b + c \neq 0$. Hence $\frac{\partial}{\partial t} + c : H_{a,b}^m \rightarrow H_{a,b}^{m-1}$ is Fredholm when $c \neq -a$ and $c \neq -b$.

We now want to extend these results to spaces of functions from $\mathbb{R}$ to $\Omega^k_z(\partial M)$ which are $L^2$ with respect to an exponential measure. Thankfully, these spaces will be exactly the same as the ones we have been working with, except that the exponential measure will cause a shift in $a$ and $b$.

Let $v \in \mathbb{R}$ be a constant. By definition,

$$L^2_{0,0}(e^{vt}dt) = \left\{ \omega \in L^1_{\text{loc}}(\mathbb{R}, \Omega^k_z(\partial M)) : \int_{-\infty}^{\infty} \|\omega(t)\|^2 e^{vt}dt < \infty \right\}$$

$$L^2_{a,b}(e^{vt}dt) = \left\{ \omega \in L^1_{\text{loc}}(\mathbb{R}, \Omega^k_z(\partial M)) : W_{a,b}(t)\omega(t) \in L^2_{0,0}(e^{vt}dt) \right\}$$

$$H^m_{a,b}(e^{vt}dt) = \left\{ \omega \in L^2_{a,b}(e^{vt}dt) : \frac{\partial^i \omega}{\partial t^i} \in L^2_{a,b}(e^{vt}dt) \forall i \leq m \right\} \text{ if } m \geq 0$$

$$H^m_{a,b}(e^{vt}dt) = \left\{ \sum_{i=0}^{-m} \frac{\partial^i \omega_i}{\partial t^i} : \omega_i \in L^2_{a,b}(e^{vt}dt) \forall i \leq -m \right\} \text{ if } m < 0$$

These are exactly the same as the definitions of the usual $L^2_{a,b}$ and $H^m_{a,b}$, the only difference is that the measure on the real line is now $e^{vt}dt$ instead of $dt$.

Lemma 4.15 For any $a$, $b$, $m$, and $v$,

$$L^2_{a,b}(e^{vt}dt) = L^2_{a-\frac{v}{2}, b-\frac{v}{2}} \quad \text{and} \quad H^m_{a,b}(e^{vt}dt) = H^m_{a-\frac{v}{2}, b-\frac{v}{2}}$$

Proof. The definition of $W_{a,b}$ implies that $W(a, b)(t)e^{\frac{vt}{2}} = W_{a-\frac{v}{2}, b-\frac{v}{2}}(t)$. So

$$\int_{-\infty}^{\infty} \|W_{a,b}(t)\omega(t)\|^2 e^{vt}dt = \int_{-\infty}^{\infty} \left\|W_{a,b}(t)e^{\frac{vt}{2}}\omega(t)\right\|^2 dt$$

$$= \int_{-\infty}^{\infty} \|W_{a-\frac{v}{2},b-\frac{v}{2}}(t)\omega(t)\|^2 dt$$

so the first integral converges if and only if the third one converges. Consequently, $\omega \in L^2_{a,b}(e^{vt}dt)$ if and only if $\omega \in L^2_{a-\frac{v}{2},b-\frac{v}{2}}$. This also implies that $\omega \in H^m_{a,b}(e^{vt}dt)$ if
and only if \( \omega \in H^m_{a-\frac{1}{2},b-\frac{1}{2}} \). For if \( m \geq 0 \), then all derivatives of \( \omega \) of order up to \( m \) lie in \( L^2_{a,b}(e^{ut}dt) \) if and only if all of those derivatives lie in \( L^2_{a-\frac{1}{2},b-\frac{1}{2}} \). And if \( m \leq 0 \), then \( \omega \) is a sum of derivatives of elements of \( L^2_{a,b}(e^{ut}dt) \) of order up to \( -m \) if and only if it is a sum of derivatives of elements of \( L^2_{a-\frac{1}{2},b-\frac{1}{2}} \) of order up to \( -m \).

Lemma 4.15 immediately lets us derive the Fredholm properties of \( \frac{\partial}{\partial t} + c \) on \( H^m_{a,b}(e^{ut}dt) \) from Lemma 4.14.

**Lemma 4.16** The operator \( \frac{\partial}{\partial t} + c : H^m_{a,b}(e^{ut}dt) \to H^{-m-1}_{a,b}(e^{ut}dt) \) is Fredholm if \( c \neq \frac{v}{2} - a \) and \( c \neq \frac{v}{2} - b \).

**Proof.** According to lemma 4.15, \( H^m_{a,b}(e^{ut}dt) = H^m_{a-\frac{1}{2},b-\frac{1}{2}} \), and \( H^{-m-1}_{a,b}(e^{ut}dt) = H^{-m-1}_{a-\frac{1}{2},b-\frac{1}{2}} \). By lemma 4.12, \( \frac{\partial}{\partial t} + c : H^m_{a-\frac{1}{2},b-\frac{1}{2}} \to H^{-m-1}_{a-\frac{1}{2},b-\frac{1}{2}} \) will be Fredholm if \( c \neq -(a-\frac{1}{2}) \) and \( c \neq -(b-\frac{1}{2}) \).

Of course, the reason we started to describe these spaces in the first place is that they are isomorphic to the weighted \( L^2 \) spaces of functions from \( \mathbb{R}_+ \) to \( \Omega^k(\partial M) \) via a logarithmic transformation. Let \( x = e^t \). For any \( \omega \in H^m_{a,b} \), let \( \omega(x) = \omega(\log(x)) = \omega(t) \). It is then clear that

\[
\frac{\partial \omega(t)}{\partial t} = \frac{\partial \omega(x)}{\partial t} = \frac{\partial \omega(x)}{\partial x} \frac{\partial x}{\partial t} = e^t \frac{\partial \omega(x)}{\partial x} = x \frac{\partial \omega(x)}{\partial x}
\]

so the logarithmic transformation conjugates \( \frac{\partial}{\partial t} \) to \( x \frac{\partial}{\partial x} \). It is also clear that

\[
W_{a,b}(t) = \begin{cases} 
eq a t & \text{if } t < 0 \\ e^{bt} & \text{if } t \geq 0 \end{cases}
= \begin{cases} x^{-a} & \text{if } x < 1 \\ x^{-b} & \text{if } x \geq 1 \end{cases}
\]

We will define \( W_{a,b}(x) \) to be \( W_{a,b}(\log(x)) \). Finally, for any \( f \in L^1(\mathbb{R}) \)

\[
\int_{-\infty}^{\infty} f(t)e^{ut}dt = \int_{0}^{\infty} \overline{f}(x)x^u dx
\]

where \( \overline{f}(x) = f(\log x) \). These three formulas allow us to directly translate our results from exponentially weighted \( L^2 \) spaces on the line to polynomially weighted \( L^2 \) spaces on the half-line.
By definition, for any real \( a, b, v, \)

\[
\frac{L_{a,b}^2}{x} \left( x^v \frac{dx}{x} \right) = \left\{ \omega \in L_{\text{loc}}^1(\mathbb{R}^+, \Omega_x^p(\partial M)) : \int_0^\infty \| W_{a,b}(x) \omega(x) \|^2 x^{v-1} dx < \infty \right\}
\]

and, for \( m \geq 0, \)

\[
H_{a,b}^m \left( x^v \frac{dx}{x} \right) = \left\{ \omega \in L_{a,b}^2 \left( x^v \frac{dx}{x} \right) : \left( x \frac{\partial}{\partial x} \right)^i \omega \in L_{a,b}^2 \left( x^v \frac{dx}{x} \right) \forall i \leq m \right\}
\]

whereas for \( m < 0 \)

\[
H_{a,b}^{-m} \left( x^v \frac{dx}{x} \right) = \left\{ \sum_{i=0}^{-m} \left( x \frac{\partial}{\partial x} \right)^i \omega_i : \omega_i \in L_{a,b}^2 \left( x^v \frac{dx}{x} \right) \forall i \leq -m \right\}
\]

We have just seen that the logarithmic transformation \( t = \log x \) turns \( x \frac{\partial}{\partial x}, x^v \frac{dx}{x} \) into \( e^{vt} dt, \) and \( W_{a,b}(x) \) into \( W_{a,b}(t). \) This proves that \( H_{a,b}^m \) corresponds to \( \overline{H}_{a,b}^m \) under the logarithmic transformation.

**Lemma 4.17** For any \( m, a, \) and \( b, \omega \in H_{a,b}^m \) if and only if \( \overline{\omega} \in \overline{H}_{a,b}^m. \) \( \square \)

Since \( x \frac{\partial}{\partial x} + c : \overline{H}_{a,b}^m \left( x^v \frac{dx}{x} \right) \to \overline{H}_{a,b}^{m-1} \left( \frac{dx}{x} \right) \) and \( \frac{\partial}{\partial t} + c : H_{a,b}^m \left( e^{vt} dt \right) \to H_{a,b}^{m-1} \left( e^{vt} dt \right) \) are conjugate under the map \( \overline{\omega} \to \omega, \) the first operator must be Fredholm whenever the second one is. We restate this fact in another lemma.

**Lemma 4.18** The operator \( x \frac{\partial}{\partial x} + c : \overline{H}_{a,b}^m \left( x^v \frac{dx}{x} \right) \to \overline{H}_{a,b}^{m-1} \left( x^v \frac{dx}{x} \right) \) is Fredholm for any \( a, b, v, c \) and \( m \) such that \( c \neq \frac{v}{2} - a \) and \( c \neq \frac{v}{2} - b. \)

**Proof.** As already stated, this follows from Lemma 4.16 by conjugation. \( \square \)

It follows that \( x \frac{\partial}{\partial x} + c : \overline{H}_{a,b}^m \left( x^{v-1} dx \right) \to \overline{H}_{a,b}^{m-1} \left( x^{v-1} dx \right) \) satisfies an estimate (4.1) with \( V_3 \) being the space \( \overline{H}_{a-1,b-1}^{m-1} \left( x^{v-1} dx \right) \), into which \( \overline{H}_{a,b}^m \) is compactly included. But the operator \( x^{-a} D \) on \( H_z^{m,w} \), whose matrix form is (3.1), is a direct sum of \( 2n \) copies of \( x \frac{\partial}{\partial x} + c \) on the subspace of \( \overline{H}_{w,q}^m \left( x^{-a-1-b(n-1)} dx \right) \) that contains forms supported inside \([0, 1). \) Here \( q \) can be chosen to be any real number. Therefore, \( D : H_z^{m,w} \to H_z^{m-1,w+a} \) will satisfy the inequality

\[
\| \omega \|_{H_z^{m,w}} \leq \| D \omega \|_{H_z^{m-1,w+a}} + \| \omega \|_{H_z^{m-1,w-1}}
\]
as long as, for each \( k \in \{0, 1, \ldots, n - 1\} \), we have

\[
bk \neq \frac{-a - b(n - 1)}{2} - w \quad \text{and} \quad b(n - k - 1) \neq \frac{-a - b(n - 1)}{2} - w
\]

The first of these conditions turns into the second when replace \( k \) with \((n - 1) - k\), so they are equivalent to each other. And the second condition translates to \( w \neq \frac{-a - b(2k - n + 1)}{2} \). This finally concludes the proof of Lemma 4.5.

### 4.2 Fredholm properties of \( D \) on \( H^{m, w'}_{nz} \).

The purpose of this section is to establish two more crucial lemmas.

**Lemma 4.19** For any \( m \in \mathbb{Z} \), \( w' \in \mathbb{R} \) there exists a \( C_2 > 0 \) such that for all \( \omega \in H^{m, w'}_{nz} \),

\[
\|\omega\|_{H^{m, w'}_{nz}} \leq C_2 \left( \|D\omega\|_{H^{m-1, w'+b}_{nz}} + \|\omega\|_{H^{m-1, w'+b-a}_{nz}} \right)
\]

**Lemma 4.20** Assume that \( \omega \in H^{m, w, w'} \) satisfies \( D\omega = 0 \), and let \( \rho \in C^\infty(M) \) be a cutoff function (i.e. \( \rho|_{U_\epsilon} \equiv 1 \) and \( \rho|_{M - U_{2\epsilon}} \equiv 0 \) for some \( \epsilon \in (0, \frac{1}{2}) \), and \( \rho(M) \subset [0, 1] \)). Then \( (1 - \pi)(\rho\omega) \in H^{\infty, \infty}_{nz} \). In particular, the BNH part of \( \omega \) in \( U \) must be rapidly decaying near \( \partial M \).

The second lemma follows easily from the first. For if \( D\omega = 0 \), then \( D((1 - \rho)(\pi\omega)) \) will be smooth and compactly supported inside \( U \), and therefore lie in \( H^{m, w'}_{nz} \) for any \( m \) and \( w' \). Iterated use of Lemma 4.19 now enables us to conclude that if the \( H^{m, w'}_{nz} \)-norm of \( (1 - \rho)(\pi\omega) \) is finite, then so is its \( H^{m+1, w'+b-a}_{nz} \)-norm, and its \( H^{m+2, w'+2b-2a}_{nz} \)-norm, ad infinitum. So it must belong to \( H^{\infty, \infty}_{nz} \), as claimed.

To prove the Lemma 4.19, we recall that

\[
D = x^b \begin{pmatrix}
D' & -x^{a-b-1} \frac{\partial}{\partial x} \\
x^{a-b+1} \frac{\partial}{\partial x} & -D'
\end{pmatrix} + \begin{pmatrix}
0 & (n - A - 1)bx^a \\
-Abx^a & 0
\end{pmatrix}
\]

and denote the first and second summand by \( D_1 \) and \( D_2 \), respectively. The proof hinges on the following result:
Lemma 4.21 For any $m, w'$ there exists a constant $C$ such that for all $\omega \in H_{nz}^{m,w'}$,
$$\|\omega\|_{H_{nz}^{m,w'}} \leq \|D_1 \omega\|_{H_{nz}^{m-1,w'+b}}.$$ 

Lemma 4.19 follows from Lemma 4.21 by the following argument:

$$\|\omega\|_{H_{nz}^{m,w'}} \leq C \|D_1 \omega\|_{H_{nz}^{m-1,w'+b}} \leq C \|D \omega\|_{H_{nz}^{m-1,w'+b}} + C \|D_2 \omega\|_{H_{nz}^{m-1,w'+b}} = C'((\|D \omega\|_{H_{nz}^{m-1,w'+b}} + C \|\omega\|_{H_{nz}^{m-1,w'+b-a}})$$

for another constant $C'$, because $D_2 : H_{nz}^{m,w'+b-a} \to H_{nz}^{m,w'+b}$ is bounded from above by the constant $b(n-1)$.

Lemma 4.21 will in turn follow from the closed graph theorem once we show that $x^{-b}D_1$ is invertible on an appropriate weighted space. For that we change variables to $t = \frac{-(a-b)}{x+a-b}$, and then perform a Fourier transform in $t$. That turns $x^{-b}D_1$ first into

$$\left( \begin{array}{cc} D' & -\frac{\partial}{\partial t} \\ \frac{\partial}{\partial t} & -D' \end{array} \right)$$

and then into

$$\left( \begin{array}{cc} D' & -i\tau \\ i\tau & -D' \end{array} \right)$$

The latter operator is fully elliptic on a suitably defined space of $x$-dependent non-harmonic forms on $\partial M$, because its determinant is $-(D')^2 - \tau^2$ and $D'$ is invertible on boundary non-harmonic forms (which is why we consider the boundary harmonic case separately). This enables us to conclude that it is invertible, and so is $D_1$ on a certain weighted space containing $H_{nz}^{m,w'}$, so $D_1$ will have to satisfy the required estimate.

An estimate of the type of (4.1) for $D$ on forms supported inside $\partial M$ follows from the fact that $D$ is elliptic inside $\partial M$. For example, for forms supported outside $U_\epsilon$ for some fixed $\epsilon$ we can replace $M$ with a compact manifold by doubling it across $\{\epsilon\} \times \partial M$, and then use a similar result for an elliptic operator on a compact manifold.

Taken together, the three estimates for $D$ on the three components of a differential form on $M$ combine to give the statement of Lemma 4.1 by a relatively easy application of the triangle inequality.
Chapter 5

Proof of the Hodge theorem.

The fact that $D : H^{m,w,w'} \to H^{m-1,w+a,w'+b}$ is Fredholm unless $w = \frac{1}{2}(-a + b(n - 2k - 1))$ for some $k \in \{0, 1, \ldots, n - 1 \}$ enables us to prove Theorem 1.1. Let us first deal with the case when $\frac{n-1+a/b}{2} \notin \{0, 1, \ldots, n - 1 \}$. If that is so, $D : H^{1,-a,-b} \to L^2$ is Fredholm, because for each $k \in \{0, 1, \ldots, n - 1 \}$

$$k \neq \frac{n-1+a/b}{2} \iff -a \neq b(n - 2k - 1) \iff -a \neq \frac{1}{2}(-a + b(n - 2k - 1))$$

so $-a$ is not one of the exceptional weights. Consequently, any form $\omega \in L^2\Omega(M)$ can be written as $\omega = \alpha + D\beta$, where $\beta \in H^{1,-a,-b}$ and $\alpha \in \text{Ker}(D^*)$. Since the adjoint $D^*$ of $D : H^{1,-a,-b} \to L^2$ is $D : L^2 \to H^{-1,a,b}$, this means $\alpha$ is harmonic.

Let us first explain where the conditions on $k$ come from. We have seen that the BNH part of any harmonic form $\omega$ on $U$ rapidly decays near $\partial M$. The principal question is: does $\omega$ represent a relative cohomology class? To answer this, we will have to check whether its tangential part vanishes on $\partial M$. Since its BNH part vanishes to infinite order near $\partial M$, the only question is whether the tangential part of the BH part of $\omega$ vanishes on $\partial M$. But we have seen that the BH part of any harmonic $k$-form in $U$ is equal to $C_1\omega_1 + C_2x_1^{(n-2k+1)b-a-1}dx \wedge \omega_2$, where $C_1$ and $C_2$ are constants, $\omega_1$ is a harmonic $k$-form on $\partial M$, and $\omega_2$ is a harmonic $k-1$-form on $\partial M$. Clearly, the tangential part of such a form will vanish at $\partial M$ if and only if $C_1\omega_1$ is zero. The question then becomes: for which values of $k$ can a smooth form whose tangential
part in $U$ is $C_1\omega_1 \neq 0$ lie in $L^2\Omega^k(M)$? The answer is given by the following lemma.

**Lemma 5.1** Let $\omega_1 \in \mathcal{H}^k(\partial M)$, $\omega_2 \in \mathcal{H}^{k-1}(\partial M)$ be non-zero harmonic forms (they also define forms on $U$ that are independent of $x$). Let $w \in \mathbb{R}$.

$$\omega_1 \in x^w L^2 \Omega^k(U) \iff w < b \left( k - \frac{n-1+a/b}{2} \right)$$

$$x^{(n-2k+1)b-a-1} dx \wedge \omega_2 \in x^w L^2 \Omega^k(U) \iff w < b \left( \frac{n+1-a/b}{2} - k \right)$$

**Proof.** These statements follow from the relation $g = \frac{dx^2}{x^{a+b}} + \frac{h}{x^a}$ between $g$ and the metric $h$ on $\partial M$. We immediately see that

$$\|x^{-w}\omega_1\|_{L^2}^2 = \int_U x^{2kb-2w} |\omega_1|^2 h dvol_g = \int_0^1 x^{2kb-2w} \|\omega_1\|^2 h_{x^{a+1+b(n-1)}} \frac{dx}{x^{a+1+b(n-1)}}$$

The integral will converge if and only if $2kb - 2w - a - 1 - b(n-1) > -1$, which is equivalent to $w < b \left( k - \frac{n-1+a/b}{2} \right)$.

The second statement is likewise easy to prove:

$$\|x^{-w}x^{(n-2k+1)b-a-1} dx \wedge \omega_2\|_{L^2}^2 =$$

$$= \int_U x^{-2w} x^{2(n-2k+1)b-a-1+((k-1)b+a+1)} |\omega_2|^2 h dvol_g$$

$$= \int_0^1 x^{2b(n-k)-2w} \|\omega_1\|^2 h_{x^{a+1+b(n-1)}} \frac{dx}{x^{a+1+b(n-1)}}$$

This integral converges if and only if $2b(n-k) - a - 1 - b(n-1) - 2w > -1$, i.e. if and only if $w < b \left( \frac{n+1-a/b}{2} - k \right)$. \hfill $\Box$

Plugging $w = 0$ in the above statement, we immediately get

**Corollary 5.2** Let $\omega_1 \in \mathcal{H}^k(\partial M)$ and $\omega_2 \in \mathcal{H}^{k-1}(\partial M)$ be non-zero. Then

$$\omega_1 \in L^2\Omega^k(U) \iff k > \frac{n-1+a/b}{2}$$

$$x^{(n-2k+1)b-a-1} dx \wedge \omega_2 \in L^2\Omega^{k-1}(U) \iff k < \frac{n+1-a/b}{2} \quad \Box$$

We have already seen that the BH part in $U$ of any harmonic $k$-form $\omega$ on $M$ is
\[ w_1 + x^{(n-2k+1)b-a-1}dx \wedge \omega_2, \text{ where } \omega_1 \in \mathcal{H}^k(\partial M) \text{ and } \omega_2 \in \mathcal{H}^{k-1}(\partial M). \] For \( \omega \) to be in \( L^2 \), its BH part has to be \( L^2 \). By the last corollary, for \( k \leq \frac{n-1+a/b}{2} \) that is only possible when \( \omega_1 = 0 \). Likewise, for \( k \geq \frac{n+1-a/b}{2} \) that can only happen when \( \omega_2 = 0 \). So Corollary 5.2 immediately implies the following

**Corollary 5.3** Let \( \omega \in \Omega^k(M) \) be an \( L^2 \) form whose BH part in \( U \) is equal to \( w_1 + x^{(n-2k+1)b-a-1}dx \wedge \omega_2, \) where \( \omega_1 \in \mathcal{H}^k(\partial M) \) and \( \omega_2 \in \mathcal{H}^{k-1}(\partial M) \). (For example, any \( \omega \in \mathcal{H}^k(M) \) satisfies these assumptions.) If \( k \leq \frac{n-1+a/b}{2} \), then \( \omega_1 \) is zero, and if if \( k \geq \frac{n+1-a/b}{2} \), then \( \omega_2 \) is zero. \( \square \)

5.1 Justification for integration by parts.

We need a few more technical lemmas before we can proceed to deal with actual ranges of \( k \). Just as with the Hodge theorem for compact manifolds, we will frequently need to use integration by parts: that is, the relation \( \langle d\alpha, \beta \rangle = \langle \alpha, \delta \beta \rangle \). When \( \alpha \) and \( \beta \) are two smooth forms on a compact manifold with no boundary, this follows immediately from the Stokes' theorem. However, our manifold \( M \) does have a boundary. To use this relation, we must therefore make sure that \( \alpha \) and \( \beta \) decay near the boundary fast enough for the boundary term in Stokes' theorem to vanish. Exactly how fast do \( \alpha \) and \( \beta \) have to decay? Our next lemma gives one possible answer:

**Lemma 5.4** Let \( \alpha \in C^\infty \Omega^{k-1}(\text{Int}(M)) \), and \( \beta \in C^\infty \Omega^k(\text{Int}(M)) \). Assume that \( \alpha \in x^p L^2 \Omega^{k-1}(M) \), and \( \beta \in x^q L^2 \Omega^k(M) \), where \( p \) and \( q \) are two real numbers satisfying \( p + q > -a \). Assume further that \( d\alpha \in x^{-q+\epsilon} L^2 \Omega^k(M) \), and \( \delta \beta \in x^{-p+\epsilon} L^2 \Omega^{k-1}(M) \), where \( \epsilon > 0 \) is a fixed number. Then \( \langle d\alpha, \beta \rangle = \langle \alpha, \delta \beta \rangle \).

The statement of the lemma immediately implies that \( \langle d\alpha, \beta \rangle \) and \( \langle \alpha, \delta \beta \rangle \) are well-defined, because \( d\alpha \in x^{-q+\epsilon} L^2 \subset x^{-q} L^2 \) and \( \delta \beta \in x^{-p+\epsilon} L^2 \subset x^{-p} L^2 \), and the \( L^2 \) pairing between \( x^p L^2 \) and \( x^{-p} L^2 \) (or \( x^q L^2 \) and \( x^{-q} L^2 \)) is well-defined.

**Proof.** By the definitions of \( \delta \) and the scalar product on forms,

\[
\langle d\alpha, \beta \rangle - \langle \alpha, \delta \beta \rangle = \int_M d\alpha \wedge *\beta - \alpha \wedge *\delta \beta = \int_M d(\alpha \wedge *\beta)
\]
The lemma states that this difference is zero. To prove it, we will partition $M$ into two pieces. One of them will be a subset of the collar neighborhood $U$ diffeomorphic to $[0, l) \times \partial M$, where $l$ is small. The other one will be $M - [0, l) \times \partial M$. We will estimate the integral of $d(\alpha \wedge \ast \beta)$ over the first piece directly, and estimate the integral over the second piece by using the Stokes theorem.

The first calculation is the easier one. We claim that

$$\lim_{l \to 0} \int_{[0,l) \times \partial M} d\alpha \wedge \ast \beta - \alpha \wedge \ast \delta \beta = 0$$

By assumption, $d\alpha \in x^{-q+\epsilon}L^2\Omega^k(M)$, $\beta \in x^qL^2\Omega^k(M)$, $\alpha \in x^pL^2\Omega^{k-1}(M)$, and $\delta \beta \in x^{-p+\epsilon}L^2\Omega^{k-1}(M)$. Because the Hodge star preserves pointwise norm, this means $\ast \beta \in x^qL^2\Omega^{n-k}(M)$ and $\ast \delta \beta \in x^{-p+\epsilon}L^2\Omega^{n-k+1}(M)$. Consequently,

$$d\alpha \wedge \ast \beta - \alpha \wedge \ast \delta \beta \in x^\epsilon L^1\Omega^n(M), \text{ i.e. } d\alpha \wedge \ast \beta - \alpha \wedge \ast \delta \beta = x^\epsilon \sigma \text{ where } \sigma \in L^1\Omega^n(M)$$

Since $\sigma \in L^1\Omega^n(M)$, it follows that $\sigma |_{[0,l) \times \partial M} \in L^1\Omega^n([0, l) \times \partial M)$. By the Hölder inequality for $L^1$ and $L^\infty$, we have

$$|\int_{[0,l) \times \partial M} x^\epsilon \sigma| \leq \|x^\epsilon \sigma\|_{L^1\Omega^n([0,l) \times \partial M)} \leq \|x^\epsilon\|_{L^\infty([0,l) \times \partial M)} \|\sigma\|_{L^1\Omega^n([0,l) \times \partial M)} \leq l^\epsilon \|\sigma\|_{L^1\Omega^n(M)}$$

Here $\|\sigma\|_{L^1\Omega^n(M)}$ does not depend on $l$, and $\lim_{l \to 0} l^\epsilon = 0$, because $\epsilon$ is positive. Therefore,

$$\lim_{l \to 0} \int_{[0,l) \times \partial M} d\alpha \wedge \ast \beta - \alpha \wedge \ast \delta \beta = 0$$

which is what we had to prove.

The second estimate is somewhat trickier. We need to show that

$$\int_{M - [0, l) \times \partial M} d(\alpha \wedge \ast \beta) = \int_{(l) \times \partial M} \alpha \wedge \ast \beta$$

becomes arbitrarily small when $l$ is small. Let $\phi = p + q + a$. Since $p + q > -a$, $\phi$ is positive. What we will show is that for any $l > 0$, there is an $x \in (0, l)$ for which
\[ | \int_{\{x\} \times \partial M} \alpha \wedge *\beta | < x^\phi. \] Let us explain how the lemma follows from this fact. The lemma essentially says that for any \( c > 0, \) \( | < d\alpha, \beta > - < \alpha, \delta \beta | < c. \) Let us fix some \( c > 0. \) We have seen that for any \( x < l, \)

\[
< d\alpha, \beta > - < \alpha, \delta \beta > = \int_{[0,x) \times \partial M} d\alpha \wedge *\beta - \alpha \wedge *\delta \beta + \int_{\{x\} \times \partial M} \alpha \wedge *\beta
\]

We have just seen that the first summand tends to 0 as \( x \) goes to 0 from above. So there exists an \( l > 0 \) such that for each \( x < l \) the integral \( \int_{[0,x)} d\alpha \wedge *\beta - \alpha \wedge *\delta \beta \) will be less than \( c/2 \) in absolute value. Choose a positive number \( l_1 < l \) that satisfies \( l_1^\phi < c/2 \) (always possible since \( \phi > 0 \)). Then our estimate on the second summand (which we will prove in a moment) says there is an \( x < l_1 \) for which \( \int_{\{x\} \times \partial M} \alpha \wedge d\beta | < x^\phi. \) But because \( x < l_1, \) \( x^\phi < l_1^\phi < c/2. \) So for that \( x, \)

\[
\left| \int_{[0,x)} d\alpha \wedge *\beta - \alpha \wedge *\delta \beta + \int_{\{x\} \times \partial M} \alpha \wedge *\beta \right| < c/2 + c/2 = c
\]

(as \( x < l_1 < l \) implies \( x < l \)), and this is the desired inequality.

So all that remains is to prove that for each \( l > 0, \) there is an \( x < l \) for which \( \int_{\{x\} \times \partial M} \alpha \wedge *\beta | < x^\phi. \) Let us take a closer look at the behavior of \( \alpha \) and \( \beta \) in the collar neighborhood \( U. \) Since \( \alpha \in x^p L^2 \Omega^{k-1}(U) \) and \( \beta \in x^q L^2 \Omega^k(U), \) the restriction of \( \alpha \) to \( U \) must have the form

\[
\alpha|_U = x^p \left( \frac{\alpha_T}{x^{k-1}b} + \frac{dx}{x^{a+1}} \wedge \frac{\alpha_C}{x^{k-2}b} \right)
\]

where \( \alpha_T \in L^2_g((0,1), L^2 \Omega^{k-1}(\partial M, h)) \) and \( \alpha_C \in L^2_g((0,1), L^2 \Omega^{k-2}(\partial M, h)). \) \( L^2_g \) here means that the forms are \( L^2 \) with respect to the measure \( \frac{dx}{x^{\alpha+1} + \beta(a-1)}, \) which comes from the metric \( g. \) Similarly, the restriction of \( \beta \) to \( U \) equals

\[
\beta|_U = x^q \left( \frac{\beta_T}{x^{k^b}} + \frac{dx}{x^{a+1}} \wedge \frac{\beta_C}{x^{k-1}b} \right)
\]

so that \( \beta_T \in L^2_g((0,1), L^2 \Omega^k(\partial M, h)) \) and \( \beta_C \in L^2_g((0,1), L^2 \Omega^{k-1}(\partial M, h)). \) Moreover, since \( \alpha \) and \( \beta \) are infinitely smooth, \( \alpha_T, \beta_T, \alpha_C \) and \( \beta_C \) must all be smooth functions
from \((0, 1)\) to smooth forms on \(\partial M\). (The indices \(T\) and \(C\) stand for "tangential" and "conormal", of course.)

The condition \(\alpha_T \in L^2_g\) literally means that 

\[
\int_0^1 \|\alpha_T(x)\|^2 \frac{dx}{x^{a+1+b(n-1)}} < \infty \iff \alpha_T \in x^{\frac{a+1+b(n-1)}{2}} L^2((0, 1), L^2(\partial M, h))
\]

Here the \(L^2\) space in the right-hand side is with respect to the usual measure \(dx\). By the same token, \(\alpha_C, \beta_T,\) and \(\beta_C\) must all lie in the same weighted \(L^2\) space with respect to \(dx\).

Next, if \(\ast\) denotes the Hodge star on \(\partial M\) (with respect to the metric \(h\)), then the definition of \(\ast\) implies that in \(U\),

\[
\ast \beta = x^a \ast \left( \frac{\beta_T}{x^{k/b}} + \frac{dx}{x^{a+1}} \wedge \frac{\beta_C}{x^{(k-1)b}} \right) = x^a \left( \frac{\ast \beta_C}{x^{(n-k)b}} + (-1)^k \frac{dx}{x^{a+1}} \wedge \frac{\ast \beta_T}{x^{(n-k-1)b}} \right)
\]

When we compute \(\int_{\{x\} \times \partial M} \alpha \wedge \ast \beta\), only the tangential part of \(\alpha \wedge \ast \beta\) will be relevant, because the conormal part of any \((n-1)\)-form integrates to zero over \(\{x\} \times \partial M\) (as the restriction of \(dx\) to that submanifold is zero). But the tangential part of \(\alpha \wedge \ast \beta\) is equal to

\[
x^{p+q} \frac{\alpha_T \wedge \ast \beta_C}{x^{(n-1)b}}
\]

We have seen that \(\alpha_T \in x^{\frac{a+1+b(n-1)}{2}} L^2((0, 1), L^2 \Omega^{k-1}(\partial M, h))\). We also know that \(\beta_C \in x^{\frac{a+1+b(n-1)}{2}} L^2((0, 1), L^2 \Omega^{n-k}(\partial M, h))\). Since \(\ast\) preserves pointwise norm, this also means \(\ast \beta_C \in x^{\frac{a+1+b(n-1)}{2}} L^2((0, 1), L^1 \Omega^{n-k}(\partial M, h))\). Adding together all powers of \(x\), we get

\[
x^{p+q} \frac{\alpha_T \wedge \ast \beta_C}{x^{(n-1)b}} \in x^{p+q+a+1} L^1((0, 1), L^1 \Omega^{n-1}(\partial M, h))
\]

because a wedge product of two \(L^2\) forms is \(L^1\).

Since the tangential part of \(\alpha \wedge \ast \beta\) lies in \(x^{1+\phi} L^1((0, 1), L^1 \Omega^{n-1}(\partial M, h))\), it must be equal to \(x^{1+\phi} \gamma(x)\), where \(\gamma\) is an element of \(L^1((0, 1), L^1 \Omega^{n-1}(\partial M, h))\). So we see that

\[
\int_{\{x\} \times \partial M} \alpha \wedge \ast \beta = \int_{\{x\} \times \partial M} x^{p+q} \frac{\alpha_T \wedge \ast \beta_C}{x^{(n-1)b}} = \int_{\{x\} \times \partial M} x^{1+\phi} \gamma(x)
\]
and our goal is to show that for any \( l > 0 \), there is an \( x < l \) for which the absolute value of this quantity is less than \( x^\phi \). That is easy to see by contradiction. Assume that there is an \( l \) such that for each \( x < l \), the above quantity is greater than or equal to \( x^\phi \) in absolute value. Clearly

\[
\left| \int_{\{x\} \times \partial M} x^{1+\phi} \gamma(x) \right| \geq x^\phi \iff \left| \int_{\{x\} \times \partial M} \gamma(x) \right| \geq \frac{1}{x}
\]

and since this identity holds for each \( x < l \), \( \gamma \) cannot possibly be an element of \( L^1([0,1), L^1 \Omega^{n-1}(\partial M, h)) \): the integral \( \int_0^1 \left| \int_{\{x\} \times \partial M} \gamma(x) \right| \, dx \) will already diverge, by the dominated convergence theorem, and thus \( \int_0^1 \left| \int_{\{x\} \times \partial M} \|\gamma\|_h \, dvol_h \right| \, dx \) will not converge either. (By the triangle inequality, the second integral is no smaller than the first.) This is a contradiction, so our assumption was false and there is no positive \( l \) for which any \( x < l \) satisfies \( \left| \int_{\{x\} \times \partial M} x^{1+\phi} \gamma(x) \right| \geq x^\phi \). This completes the proof.

Lemma 5.4 can also be restated as follows.

**Lemma 5.5** Assume that \( \alpha \in C^\infty \Omega^{k-1}(\text{Int}(M)) \), and \( \beta \in C^\infty \Omega^k(\text{Int}(M)) \). Assume further that \( \alpha \in x^p L^2 \Omega^{k-1}(M) \), \( \beta \in x^q L^2 \Omega^k(M) \), \( d\alpha \in x^{p'} L^2 \Omega^k(M) \), and \( d\beta \in x^{q'} L^2 \Omega^{k-1}(M) \), where \( p, q, p', \) and \( q' \) are real numbers. If \( p + q + a > 0 \), \( p + q' > 0 \) and \( p' + q > 0 \), then \( \langle d\alpha, \beta \rangle = \langle \alpha, d\beta \rangle \).

**Proof.** Setting \( \epsilon = \min(p + q', q + p') \) reduces this to Lemma 5.4.

This lemma can be easily generalized to forms of mixed degree:

**Lemma 5.6** Assume that \( \alpha, \beta \in C^\infty \Omega(M) \) are smooth differential forms (possibly of mixed degree). If \( \alpha \in x^p L^2 \Omega(M) \), \( \beta \in x^q L^2 \Omega(M) \), \( d\alpha \in x^{p'} L^2 \Omega(M) \), and \( d\beta \in x^{q'} L^2 \Omega(M) \) for some real \( p, p', q, \) and \( q' \), and if \( p + q' > 0 \), \( q + p' > 0 \) and \( p + q + a > 0 \), then \( \langle d\alpha, \beta \rangle = \langle \alpha, d\beta \rangle \).

**Proof.** Let \( \alpha = \sum_{k=0}^n \alpha^k \) and \( \beta = \sum_{k=0}^n \beta^k \), where for each \( k \alpha^k \in C^\infty \Omega^k(M) \) and \( \beta^k \in C^\infty \Omega^k(M) \) are the \( k \)-th degree parts of \( \alpha \) and \( \beta \), respectively. Since any two forms of different degrees are \( L^2 \)-orthogonal, we have

\[
\langle d\alpha, \beta \rangle = \left\langle \sum_{k=0}^n d\alpha^k, \sum_{k=0}^n \beta^k \right\rangle = \sum_{k=1}^n \left\langle d\alpha^{k-1}, \beta^k \right\rangle \quad \text{and by the same token,}
\]
\[ \langle \alpha, \delta \beta \rangle = \left( \sum_{k=0}^{n} \alpha^k, \sum_{k=0}^{n} \delta \beta^k \right) = \sum_{k=1}^{n} \langle \alpha^{k-1}, \delta \beta^k \rangle \]

So it suffices to show that \( \langle d\alpha^{k-1}, \beta^k \rangle = \langle \alpha^{k-1}, \delta \beta^k \rangle \) for each \( k \) between 1 and \( n \).
But for each such \( k \) we have \( \alpha^{k-1} \in x^p L^2 \Omega^{k-1}(M), \beta^k \in x^q L^2 \Omega^k(M), d\alpha^{k-1} \in x^{p'} L^2 \Omega^{k}(M), \) and \( \delta \beta^k \in x^{q'} L^2 \Omega^{k-1}(M) \), because the norms of \( \alpha^{k-1}, \beta^k, d\alpha^{k-1} \) and \( \delta \beta^k \) in these spaces are bounded from above by the norms of \( \alpha, \beta, d\alpha \) and \( \delta \beta \), respectively. Since each \( \alpha^{k-1} \) and \( \beta^k \) is smooth (as \( \alpha \) and \( \beta \) are), Lemma 5.5 shows that \( \langle d\alpha^{k-1}, \beta^k \rangle = \langle \alpha^{k-1}, \delta \beta^k \rangle \) for all \( k \in \{1, \ldots, n\} \).

In the proof of Lemma 5.4 we only used the facts that \( d\alpha \wedge * \beta \) and \( \alpha \wedge * \delta \beta \) both lie in \( x^s L^1 \Omega^n(M) \), while the tangential part of the restriction of \( \alpha \wedge * \beta \) to the collar neighborhood \( U \) lies in \( x^{p+q} L^1 \Omega^{n-1}(U) \). The first fact enabled us to estimate \( \int_{[0,1] \times \partial M} d\alpha \wedge * \beta - \alpha \wedge * \delta \beta \), and the second fact gave us an estimate on \( \int_{[1] \times \partial M} \alpha \wedge * \beta \). So Lemma 5.4 can be also restated in a more general (though less elegant) way:

**Lemma 5.7** Let \( \alpha \in C^\infty k^{-1}(\text{Int}(M)) \) and \( \beta \in C^\infty k(\text{Int}(M)) \) be smooth differential forms. Assume that \( d\alpha \wedge * \beta \) and \( \alpha \wedge * \delta \beta \) lie in \( x^s L^1 \Omega^n(M) \), and the tangential part of \( (\alpha \wedge * \beta)|_U \) lies in \( x^{-a+\phi} L^1 \Omega^{n-1}(U) \), where \( \epsilon, \phi > 0 \) are real. Then \( \langle d\alpha, \beta \rangle = \langle \alpha, \delta \beta \rangle \).

This result can also be generalized to mixed form degree, but we will never need to apply it to forms of mixed degree. The lemmas on integration by parts will enable us to prove the Hodge cohomology theorem for type \((a, b)\)-metrics. We begin with the case of "small" \( k \).

### 5.2 Case \( k < \frac{n+1-a/b}{2} \).

The result that we obtain in this section will apply only to the case \( k < \frac{n+1-a/b}{2} \), \( n-1+a/b \not\in \{0, 1, \ldots, n-1\} \). However, along the way we will derive several lemmas that will directly extend either to the case \( \frac{n+1-a/b}{2} \leq k \leq \frac{n-1+a/b}{2} \), or to \( \frac{n+1-a/b}{2} \in \{0, 1, \ldots, n-1\} \), or both. So we will spell out our assumptions on \( k \) and on \( a/b \) in the statement of each lemma.
If \( k \) is smaller than \( \frac{n+1-a/b}{2} \), our goal is to prove that \( \mathcal{H}^k(M) \cong H^k(M, \partial M) \). We will construct a map from \( \mathcal{H}^k(M) \) to \( H^k(M, \partial M) \), and then prove that this map is both injective and surjective. The trick is to use an appropriate model of \( H^k(M, \partial M) \) for the proof of each of these statements. Its standard definition uses the chain complex of smooth and compactly supported forms on \( M \):

\[
C^\infty_c \Omega^k(M) = \{ \omega \in C^\infty \Omega^k(M) : \text{supp}(\omega) \text{ is compact in } \text{Int}(M) \}
\]

An equivalent definition of \( H^k(M, \partial M) \) is that it is the cohomology of the chain complex \( A^* \), where the space \( A^k \) is, by definition

\[
\{ \omega \in C^\infty \Omega^k(\text{Int}(M)) : \omega|_U = \omega_1 + dx \wedge \omega_2, \omega_1 \in C^\infty((0,1), C^\infty \Omega^k(\partial M)), \\
\omega_2 \in C^\infty((0,1), C^\infty \Omega^{k-1}(\partial M)), \lim_{x \to 0} ||\omega_1(x)||_{L^2} = \lim_{x \to 0} ||d'\omega_1(x)||_{L^2} = 0; \\
\exists L, C : \forall x, p \ |\omega_2(x,p)|_g \leq C x^L, |d'\omega_2(x,p)|_g \leq C x^L, \left| \frac{\partial \omega_1}{\partial x}(x,p) \right|_g \leq C x^L \}
\]

Here \( d' \) is the DeRham operator on \( \partial M \). In effect, \( A^k \) contains all smooth \( k \)-forms \( \omega \) on \( M \) whose tangential part in \( U \) is vanishing at the boundary, whose conormal part grows at most polynomially near the boundary, and such that \( d\omega \) satisfies the same conditions.

We will need yet another equivalent definition of \( H^k(M, \partial M) \). It is the cohomology of the chain complex \( B^* \), where the space \( B^k \) contains all smooth forms on \( M \) whose tangential part is rapidly decaying near \( \partial M \), and whose conormal part grows at most polynomially near \( \partial M \). The definition of \( B^k \) is

\[
\{ \omega_1 + dx \wedge \omega_2 \in A^k : \omega_1 \in \hat{C}^\infty((0,1), C^\infty \Omega(\partial M)), \omega_2 \in C^\infty((0,1), C^\infty \Omega(\partial M)) \}
\]

where \( \hat{C}^\infty((0,1), C^\infty \Omega^k(\partial M)) \) is the set

\[
\left\{ \omega \in C^\infty((0,1), \Omega^k(\partial M)) : \forall i, j, l \in \mathbb{Z}_+, \lim_{z \to 0} \left| x^{-i} \frac{\partial^j \omega(x)}{\partial x^j} \right|_{H^l(\partial M)} = 0 \right\}
\]
It is clear that the spaces $A^k$ form a chain complex, as do the $B^k$-s, because
\[
d(\omega_1 + dx \wedge \omega_2) = d'\omega_1 + dx \wedge \left( \frac{\partial \omega_1}{\partial x} - d'\omega_2 \right)
\]
so the conormal part of $\omega$ does not affect the rate of growth of the tangential part of $d\omega$. It is relatively easy to show (by integration from the boundary) that the inclusions $C^\infty \Omega^k(M) \subset B^k \subset A^k$ induce an isomorphism in cohomology.

We will use the complex $A^k$ to prove the existence and injectivity of a map from $\mathcal{H}^k(M)$ to $H^k(M, \partial M)$. To prove its surjectivity, we need to show that each relative cohomology class has a harmonic representative. Since the inclusion of $C^\infty_c \Omega^k(M)$ into $B^k$ induces an isomorphism in cohomology, every cohomology class in $H^k(M, \partial M)$ has a representative that lies in $C^\infty_c \Omega^k(M)$. So, in order to show that the natural map from $\mathcal{H}^k(M)$ to $H^k(M, \partial M)$ is surjective, it suffices to show that any closed element of $C^\infty_c \Omega^k(M)$ differs from a harmonic form by an exact element of $B^k$.

We first show that there is a natural map from $\mathcal{H}^k(M)$ to the third model of $H^k(M, \partial M)$. This argument extends to the case $\frac{n-1+a/b}{2} \leq k \leq \frac{n-1+a/b}{2}$.

**Lemma 5.8** Assume that $k \leq \frac{n-1+a/b}{2}$, and $\omega \in \mathcal{H}^k(M)$ is an $L^2$ harmonic form. Then $\omega \in B^k$.

**Proof.** We have seen that any harmonic form on $M$ must be $C^\infty$ in Int($M$), so the smoothness requirement is satisfied. The restriction of any smooth $k$-form $\omega$ to $U$ has the form
\[
\omega|_U = \omega_{nh} + \omega_h = \left( \begin{array}{c} \omega_{1,nh} \\ \omega_{2,nh} \end{array} \right) + \left( \begin{array}{c} \omega_{1,h} \\ \omega_{2,h} \end{array} \right)
\]
Here $\omega_{1,nh}, \omega_{2,nh} \in C^\infty((0,1), \Omega_{nh}(\partial M))$ and $\omega_{1,h}, \omega_{2,h} \in C^\infty((0,1), \Omega_h(\partial M))$. By the results of section 4.2, if $\omega$ is harmonic, then $\omega_{1,nh}$ and $\omega_{2,nh}$ are both rapidly decaying near $\partial M$, and $\omega_{1,h} = x^{kb}\alpha^k$ and $\omega_{2,h} = x^{(n-k)b}\beta^{k-1}$, where $\alpha^k \in \mathcal{H}^k(\partial M)$ and $\beta^{k-1} \in \mathcal{H}^{k-1}(\partial M)$. For $\omega$ to be square integrable, both $\left( \begin{array}{c} x^{kb}\alpha^k \\ 0 \end{array} \right)$ and $\left( \begin{array}{c} 0 \\ x^{(n-k)b}\beta^{k-1} \end{array} \right)$ must be square integrable on $U$. In ordinary notation (without matrices) this means $\alpha^k$ and $x^{(n-2k+1)b-a-1}dx \wedge \beta^{k-1}$ have to lie in $L^2\Omega^k(U)$. But Corollary 5.2 states that
this is impossible unless \( \alpha^k = 0 \), because in our case \( k \) is less than or equal to \( \frac{n+1-a/b}{2} \).

(By the same token, \( \beta^{k-1} \) will be zero if \( k \in \left[ \frac{n+1-a/b}{2}, \frac{n-1+a/b}{2} \right] \). But we will not use this fact.) So if \( \omega \) is an \( L^2 \) harmonic form on \( M \) of order \( k \), then it is smooth in \( \text{Int}(M) \), and its tangential part in \( U \) is

\[
\omega_1 = \begin{pmatrix} \omega_{1,h} \\ 0 \end{pmatrix} + \begin{pmatrix} \omega_{1,nh} \\ 0 \end{pmatrix} = \begin{pmatrix} x^{kb} \alpha^k \\ 0 \end{pmatrix} + \begin{pmatrix} \omega_{1,nh} \\ 0 \end{pmatrix} = \frac{\omega_{1,nh}}{x^{kb}}
\]

which is rapidly decaying near \( \partial M \), because \( \omega_{1,nh} \) is.

Let \( dx \wedge \omega_2 \) denote the conormal part of \( \omega \) in \( U \). It is equal to

\[
dx \wedge \omega_2 = \begin{pmatrix} 0 \\ \omega_{2,h} \end{pmatrix} + \begin{pmatrix} 0 \\ \omega_{2,nh} \end{pmatrix} = \begin{pmatrix} 0 \\ x^{(n-k)b}\beta^{k-1} \end{pmatrix} + \begin{pmatrix} 0 \\ \omega_{2,nh} \end{pmatrix}
\]

For each \( x \in (0,1) \) and \( p \in \partial M \), we have

\[
\|\omega_2(x,p)\|_g = \left\| \frac{dx}{x^{a+1}} \wedge \omega_2(x,p) \right\|_g = x^{-a-1} \|x^{(n-k)b}\beta^{k-1}(p) + \omega_{2,nh}(x,p)\|_h \\
\leq x^{-a-1} \left( x^{(n-k)b}\|\beta^{k-1}(p)\|_h + \|\omega_{2,nh}(x,p)\|_h \right)
\]

(by the triangle inequality). Since \( \beta^{k-1} \) is continuous on \( \partial M \), its \( h \)-norm on \( \partial M \) has a finite maximum \( C_1 = \max_{p \in \partial M} \|\beta^{k-1}(p)\|_h \). Because \( \omega_{2,nh} \) is rapidly decaying near \( \partial M \), there exists a constant \( C_2 \) such that for each \( x \in (0,1) \) and \( p \in \partial M \), \( \|\omega_{2,nh}(x,p)\|_h \leq C_2 x^{(n-k)b} \). So for every \( x \in (0,1) \) and every \( p \in \partial M \), \( \|\omega_2(x,p)\|_g \leq (C_1 + C_2) x^{(n-k)b-a-1} \).

At the same time

\[
dx \wedge d'\omega_2 = \begin{pmatrix} 0 \\ x^{(n-k)b}d'\beta^{k-1} + d'\omega_{2,nh} \end{pmatrix}
\]

because \( d' \) commutes with the powers of \( x \). As \( \beta^{k-1} \) is smooth, \( d'\beta^{k-1} \) is continuous, hence its \( h \)-norm on \( \partial M \) has a finite maximum \( C_3 = \max_{p \in \partial M} \|d'\beta^{k-1}(p)\| \). Just as before, \( d'\omega_{2,nh} \) is rapidly decaying near \( \partial M \) (because \( \omega_{2,nh} \) is), so there is a constant \( C_4 \)
for which \( \|d'\omega_{2,nh}(x)\|_{\sup} \leq C_4 x^{(n-k)b} \) for every \( x \in (0,1) \). Together these estimates imply that for each \( x \in (0,1) \) and \( p \in \partial M \), \( \|d'\omega_2(x,p)\|_g \leq (C_3 + C_4)x^{(n-k)b-a-1} \) (by the same triangle inequality as before). Finally, since \( \frac{\partial \omega_1}{\partial x} \) is rapidly decaying near \( \partial M \) (because \( \omega_1 \) is), there exists a constant \( C_5 \in \mathbb{R} \) such that \( \|\frac{\partial \omega_1}{\partial x}(x,p)\|_g \leq C_5 x^{(n-k)b-a-1} \) for any \( x \in (0,1), \ p \in \partial M \). So \( \omega \) satisfies the conditions of the definition of \( A^k \): the constant \( C \) from the definition of \( A^k \) is equal to the maximum of \( C_1 + C_2, C_3 + C_4, \) and \( C_5 \), and \( l \) can be any integer smaller than \( (n-k)b-a-1 \). Since \( \omega \) is smooth, and its tangential part in \( U \) is rapidly decaying near \( \partial M \), \( \omega \) lies in \( B^k \).

So \( \omega \) is an element of \( B^k \). To show that it represents a relative cohomology class, we need to prove that it is closed. Of course, for a harmonic \( k \)-form that is completely trivial: since \( d\omega \) is a \((k+1)\)-form and \( \delta \omega \) a \((k-1)\)-form, and \( D\omega = d\omega + \delta \omega = 0 \), \( d\omega \) and \( \delta \omega \) must each be zero. However, we will later need to know that a harmonic \( L^2 \) form is always closed, even if it has mixed degree. This is slightly harder to prove, and it is the subject of our next lemma.

**Lemma 5.9** Let \( \omega \in \mathcal{H}(M) \) be an \( L^2 \) harmonic form (possibly of mixed form degree). Then \( d\omega = \delta \omega = 0 \).

**Proof.** Since \( D\omega = (d + \delta)\omega = 0 \), we have \( 0 = \delta(d + \delta)\omega = \delta d\omega \). We now want to write

\[
\langle d\omega, d\omega \rangle = \langle \omega, \delta d\omega \rangle = 0
\]

but because \( M \) has a boundary, we will need to invoke Lemma 5.4 to prove that this integration by parts is allowed.

Let \( \omega = \sum_{k=0}^{n} \omega^k \), where \( \omega^k \in \Omega^k(M) \) for each \( k \). By the results of Section 4, \( \omega^k \) must be smooth for each \( k \), and its restriction to \( U \) must be equal to \( \alpha^k + x^{(n-2k+1)b-a-1} dx \wedge \beta^{k-1} + \omega^k_{nh} \). Here \( \alpha^k \in \mathcal{H}^k(\partial M) \) and \( \beta^k \in \mathcal{H}^{k-1}(\partial M) \), and \( \omega^k_{nh} \) is the BNH part of \( \omega^k \) in \( U \), smooth and rapidly decaying near \( \partial M \). Observe that

\[
\omega = \sum_{k=0}^{n} \omega^k \quad \Rightarrow \quad d\omega = \sum_{k=0}^{n} d\omega^k, \quad \delta \omega = \sum_{k=0}^{n} \delta \omega^k, \text{ hence}
\]

\[
\langle d\omega, d\omega \rangle = \sum_{k=0}^{n} \langle d\omega^k, d\omega^k \rangle, \quad \langle \omega, \delta d\omega \rangle = \sum_{k=0}^{n} \langle \omega^k, \delta d\omega^k \rangle
\]
because the scalar product of two forms of different degree is zero. So it suffices to prove that for each \( k \), \(< dw^k, dw^k > = < \omega^k, \delta dw^k >\). But according to Section 4, \( d\alpha^k = 0 \) and \( d(x^{(n-2k+1)b-a-1}dx \land \beta^{k-1}) = 0 \) in \( U \). So \((dw^k)|_U = dw^k_{nh}\) and \((\delta dw^k)|_U = \delta dw^k_{nh}\). These forms are rapidly decaying near \( \partial M \), because \( \omega^k_{nh} \) is. Thus \( dw^k \) and \( \delta dw^k \) lie in \( x^C L^2 \Omega(M) \) for any real \( C \). On the other hand, Lemma 5.1 states that \( \alpha^k \in x^w L^2 \Omega^k(U) \) for any \( w < b \left( k - \frac{n-1+a/b}{2} \right) \). By the same lemma, \( x^{(n-2k+1)b-a-1}dx \land \beta^{k-1} \in x^w L^2 \Omega^k(U) \) for any \( w < b \left( \frac{n+1-a/b}{2} - k \right) \). Since \( \omega^k \) is smooth inside \( M \) (because \( \omega \) is) and its restriction to \( U \) is \( \alpha^k + x^{(n-2k+1)b-a-1}dx \land \beta^{k-1} + \omega^k_{nh} \), and since \( \omega^k_{nh} \) lies in \( x^w L^2 \Omega^k(U) \) for all \( w \) by virtue of being smooth and rapidly decaying near \( \partial M \), it follows that \( \omega^k \in x^w L^2 \Omega^k(M) \) for each \( w < \min \left( b \left( k - \frac{n-1+a/b}{2} \right), b \left( \frac{n+1-a/b}{2} - k \right) \right) \).

It is now easy to see why Lemma 5.4 applies to the transition

\[
< dw^k, dw^k > = < \omega^k, \delta dw^k >
\]  

(5.1)

Fix any number \( p \) less than \( \min \left( b \left( k - \frac{n-1+a/b}{2} \right), b \left( \frac{n+1-a/b}{2} - k \right) \right) \). As explained above, \( \omega^k \) will then lie in \( x^p L^2 \Omega^k(M) \). Let \( q = -a - p + 1 \). Because \( dw \) and \( \delta dw \) lie in \( x^w L^2 \Omega(M) \) for any \( w \), we have \( dw \in x^q \Omega^{k+1}(M) \), \( dw \in x^{-q+1} \Omega^{k+1}(M) \) and \( \delta dw \in x^{-p+1} \Omega^k(M) \). Since all forms in question are smooth in the interior of \( M \), the assertions of Lemma 5.4 are satisfied (with \( \epsilon = 1 \)). This proves (5.1), which in turn implies \( < dw, dw > = < \omega, \delta dw > = 0 \) and \( dw = 0 \). As \( dw = 0 \), we also have \( \delta \omega = D \omega - dw = 0 \).

Every harmonic \( k \)-form thus lies in \( B^k \) and is closed. That means it represents a relative cohomology class, so there is a natural linear map from \( \mathcal{H}^k(M) \) to \( H^k(M, \partial M) \). Our next claim is that this map is injective.

**Lemma 5.10** Let \( k < \frac{n+1-a/b}{2} \), and assume that \( \omega \in \mathcal{H}^k(M) \) is an \( L^2 \) harmonic form that satisfies \( \omega = d\sigma \) for some \( \sigma \in A^{k-1} \). Then \( \omega = 0 \).

**Proof.** Since the inclusion of \( B^k \) into \( A^k \) induces an isomorphism in cohomology and \( \omega \) lies in \( B^k \), \( \omega \) will represent a zero cohomology class in the chain complex \( B^k \).
if and only if it represents a zero cohomology class in $A^k$. So if $\omega = d\sigma$ for some $\sigma \in A^{k-1}$, then there also exists a form $\sigma' \in B^{k-1}$ satisfying $\omega = d\sigma'$. Changing notation, we can assume without loss of generality that the form $\sigma$ itself lies in $B^{k-1}$. That is, the tangential part of $\sigma$ in $U$ can be assumed to rapidly decay near $\partial M$.

We have just seen that any $L^2$ harmonic form $\omega$ satisfies $\delta \omega = 0$. Our goal is to justify the integration by parts in

$$0 = \langle \delta \omega, \sigma \rangle = \langle \delta d\sigma, \sigma \rangle = \langle d\sigma, d\sigma \rangle \quad \Rightarrow \quad d\sigma = 0$$

This time we will use Lemma 5.7 to show that the last transition is legitimate. Observe first that $\sigma$ and $d\sigma$ are both smooth inside $M$, by the definition of the space $B^{k-1}$. So at least the smoothness conditions of Lemma 5.7 are fulfilled.

Let us fix a positive real $\epsilon$ which is less than $b \left( \frac{n+1-\alpha/b}{2} - k \right)$. According to Corollary 5.3, the BH part of $\omega$ in $U$ is equal to $x^{(n-2k+1)b-a-1}dx \wedge \beta^{k-1}$ for some $\beta^{k-1} \in \mathcal{H}^{k-1}(\partial M)$. By Lemma 5.1, the BH part of $\omega$ in $U$ lies in $x^\epsilon L^2\Omega^k(U)$ for our choice of $\epsilon$. The BNH part of $\omega$ lies in $x^w L^2\Omega^k(U)$ for all $w$ by virtue of being smooth and rapidly decaying near $\partial M$. Since $\omega$ is smooth inside $M$, that implies $\omega \in x^\epsilon L^2\Omega^k(M)$. Since $d\sigma = \omega \in x^\epsilon L^2\Omega^k(M)$ and the Hodge star preserves pointwise norm, we have $*d\sigma \in x^{\epsilon}L^2\Omega^{n-k}(M)$, and consequently $d\sigma \wedge *d\sigma \in x^{2\epsilon}L^1\Omega^n(M)$.

Further, $\delta d\sigma = \delta \omega = 0$, and hence $\sigma \wedge *\delta d\sigma$ trivially lies in $x^w L^1\Omega^n(M)$ for any real $w$. In particular, $\sigma \wedge *\delta d\sigma$ is an element of $x^{2\epsilon}L^1\Omega^n(M)$.

We need one more condition to invoke Lemma 5.7. That is, we need to know that the tangential part of $\sigma \wedge *d\sigma$ lies in $x^{1+\phi}L^1((0,1), L^1\Omega^{n-1}(\partial M))$ for some positive number $\phi$. But the tangential part of $\sigma \wedge *d\sigma$ is the wedge product of the tangential parts of $\sigma$ and $*d\sigma$. Since $d\sigma = \omega$, the conormal part of $d\sigma$ is $\frac{dx}{x^{\alpha+1}} \wedge \frac{x^{(n-k)b}\beta^{k-1}+\omega_{2,nh}}{x^{(k-1)b}}$. Hence the tangential part of $*d\sigma$ is $\frac{x^{(n-k)b}\beta^{k-1}+\omega_{2,nh}}{x^{(n-k)b}}$. Here $*'$ is the Hodge star on $\partial M$. Because $\beta^{k-1}$ is a harmonic $(k-1)$-form on $\partial M$, $*'\beta^{k-1}$ is a harmonic $(n-k)$-form on $\partial M$. By Lemma 5.1, this means $*'\beta^{k-1}$ lies in $x^w L^2\Omega^{n-k}(U)$ for any
w satisfying
\[ w < b \left( n - k - \frac{n - 1 + a/b}{2} \right) = b \left( \frac{n + 1 - a/b}{2} - k \right) \]
and, in particular, \(*' \beta \in x^a L^2 \Omega^{n-k}(U)\). Further, \(\omega_{2,nh}\) is rapidly decaying near \(\partial M\), by the results of Section 4.2. Hence \(\frac{*' \omega_{2,nh}}{x^{(n-k)b}}\) must lie in \(x^a L^2 \Omega^{n-k}(U)\), because \(\frac{\omega_{2,nh}}{x^{(n-k)b}}\) is smooth and rapidly decaying near \(\partial M\), and \(*'\) preserves pointwise norm. We conclude that the entire tangential part of \(*d\sigma\) lies in \(x^a L^2 \Omega^{n-k}(U)\).

On the other hand, the tangential part of \(\sigma\) in \(U\), which we will denote \(\sigma_T\), must be smooth and rapidly decaying near \(\partial M\), because \(\sigma \in B^{k-1}\). So \(\sigma_T\) lies in \(x^w L^2 \Omega^{k-1}(U)\) for any real \(w\). In particular, it belongs to \(x^{-a} L^2 \Omega^{k-1}(U)\).

We now see that the tangential part of \(\sigma \wedge *d\sigma\) is a wedge product of
\[
\sigma_T \in x^{-a} L^2 \Omega^{k-1}(U) \quad \text{and} \quad *' \beta^{k-1} + \frac{*' \omega_{2,nh}}{x^{(n-k)b}} \in x^a L^2 \Omega^{n-k}(U)
\]
So that tangential part lies in \(x^{-a+\epsilon} L^1 \Omega^{n-1}(U)\). The forms \(\sigma\) and \(d\sigma\) thus satisfy all assumptions of Lemma 5.7. So the transition \(\langle d\sigma, d\sigma \rangle = \langle \sigma, \delta d\sigma \rangle\) is justified. As \(\delta d\sigma = \delta \omega = 0\), this means \(\omega = d\sigma = 0\). \(\square\)

So the map from \(H^k(M)\) to \(H^k(M, \partial M)\) is injective. It remains to prove that it is surjective. We will now use the fact that the chain complexes \(C_c^\infty \Omega^k(M)\) and \(B^k\) define the same cohomology. That is, each relative cohomology class has a representative in \(C_c^\infty \Omega^k(M)\). Hence it suffices to prove that each closed element of \(C_c^\infty \Omega^k(M)\) is cohomologous to a harmonic form in \(B^k\).

Our proof of this fact will consist of several steps. First, we will show that an element \(\omega \in C_c^\infty \Omega^k(M)\) can always be written as \(\omega = \omega_0 + D\gamma\) with \(\omega_0 \in \mathcal{H}(M)\) and \(\gamma \in H^\infty, -a, \infty(M)\), as long as \(\frac{n-1+a/b}{2} \notin \{0, 1, \ldots, n-1\}\). In the exceptional case \(\frac{n-1+a/b}{2} \notin \{0, 1, \ldots, n-1\}\) we will show that there exists a similar decomposition \(\omega = \omega_0 + D\gamma\) for \(\omega_0 \in \mathcal{H}(M)\) and \(\gamma \in H^\infty, -a, -r, \infty\), where \(r\) is an arbitrary positive number less than a certain bound \(r_0\). To prove these lemmas we will need the Fredholm properties of \(D\). Next we will prove that if \(k < \frac{n-1+a/b}{2}\) and \(\omega \in C_c^\infty \Omega^k(M)\) is closed, then \(\omega = d\gamma^{k-1} + \omega_0^k\) for \(\gamma^{k-1} \in H^\infty, -a, \infty \Omega^{k-1}(M)\) (or \(\gamma^{k-1} \in H^\infty, -a, -r, \infty \Omega^{k-1}(M)\))
and \( \omega_0^k \in \mathcal{H}^k(M) \). That will involve an integration by parts. We will then prove that if \( k \) is less than \( \frac{n+1-a/b}{2} \), then the form \( \gamma'^{-1} \) above lies in the space \( B^{k-1} \). For this we will compute \( \gamma'^{-1} \) in a small tubular neighborhood of \( \partial M \) in which \( \omega \) is equal to 0 by solving an ordinary differential equation.

In the first step of our proof of surjectivity we will rely on the following very useful lemma.

**Lemma 5.11** Let \( w \in \mathbb{R} \) be such that \( D : H^m,w,w' \to H^{m-1,w+a,w'+b} \) is Fredholm for all \( m \in \mathbb{Z} \) and \( w' \in \mathbb{R} \). If, for some \( m_0 \in \mathbb{Z} \), \( w'_0 \in \mathbb{R} \), and \( \gamma \in H^{m_0,w,w'_0} \) we have \( D\gamma \in H^{-1,w+a,w++,} \), then \( \gamma \in H^{w,w} \).

**Proof.** For any \( m \in \mathbb{Z} \) and any \( w \in \mathbb{R} \), \( D : H^m,w,w' \to H^{m-1,w+a,w'+b} \) is, by assumption, Fredholm. That is, a form \( \omega \in H^{m-1,w+a,w'+b} \) will lie in the image of \( D : H^m,w,w' \to H^{m-1,w+a,w'+b} \) if and only if it is \( L^2 \)-orthogonal to the kernel of the adjoint of that operator. But the adjoint of \( D : H^m,w,w' \to H^{m-1,w+a,w'+b} \) is \( D : H^{1-m,-w-a,-w'-b} \to H^{-m,-w,-w'} \). Consequently,

\[
\forall \omega \in H^{m-1,w+a,w'+b}, \; \omega \in \text{Im}(D : H^m,w,w' \to H^{m-1,w+a,w'+b}) \iff \\
\omega \perp_{L^2} \text{Ker}(D : H^{1-m,-w-a,-w'-b} \to H^{-m,-w,-w'})
\]

(5.2)

Because \( \gamma \in H^{m_0,w,w'_0} \), this immediately implies

\[
D\gamma \perp_{L^2} \text{Ker}(D : H^{1-m_0,-w-a,-w'_0-b} \to H^{-m_0,-w,-w'})
\]

However, by Lemma 4.3, every harmonic form in each doubly weighted space \( H^{m,q,w'} \) must be smooth, and its BNH part rapidly decaying near \( \partial M \), i.e.

\[
\text{Ker}(D : H^m,q,w' \to H^{m-1,q+a,w'+b}) \subset H^{\infty,q,\infty} \quad \forall m, q, w'
\]

(5.3)

This fact enables us to conclude that for any \( m \) and \( w' \), \( D\gamma \) is \( L^2 \)-orthogonal to \( \text{Ker}(D : H^{1-m,-w-a,-w'-b} \to H^{-m,-w,-w'}) \) for all \( m \) and \( w' \). To prove it, observe that any element \( \sigma \in \text{Ker}(D : H^{1-m,-w-a,-w'-b} \to H^{-m,-w,-w'}) \) must satisfy \( \sigma \in \)
$H^\infty,-w_{-a,\infty}$ by (5.3). Hence $\sigma$ will also belong to $H^{1-m_0,-w_{-a,-w'_0-b}}$, and thus be an element of $\text{Ker} \left( D : H^{1-m_0,-w_{-a,-w'_0-b}} \to H^{-m_0,-w_{-w'_0}} \right)$, because $H^{1-m_0,-w_{-a,-w'_0-b}}$ contains $H^\infty,-w_{-a,\infty}$. This implies $\sigma \perp_{L^2} D\gamma$, and since $\sigma$ was arbitrary, we have $D\gamma \perp_{L^2} \text{Ker} \left( D : H^{1-m_0,-w_{-a,-w'_0-b}} \to H^{-m_0,-w_{-w'_0}} \right)$. It now follows from (5.4) that 

$$D\gamma \in \text{Im} \left( D : H^{m,w,w'} \to H^{m-1,w+a,w'+b} \right).$$

So for every $m$ and $w'$, there exists an element $\gamma_{m,w'} \in H^{m,w,w'}$ satisfying $D\gamma_{m,w'} = D\gamma$. Let now $m \in \mathbb{Z}$ and $w' \in \mathbb{R}$ be arbitrary numbers satisfying $m > m_0$ and $w' > w'_0$. These inequalities imply $H^{m,w,w'} \subset H^{m_0,w,w'_0}$ and hence $\gamma_{m,w'} \in H^{m_0,w,w'_0}$. So $\gamma - \gamma_{m,w'}$ lies in $H^{m_0,w,w'_0}$ and satisfies $D(\gamma - \gamma_{m,w'}) = 0$. By (5.3), this means $\gamma - \gamma_{m,w'} \in H^{\infty,w,\infty}$. Therefore $\gamma = (\gamma - \gamma_{m,w'}) + \gamma_{m,w'} \in H^{m,w,w'}$, as $H^{\infty,w,\infty} \subset H^{m,w,w'}$. But because $m$ and $w'$ can be arbitrarily large, we have

$$\gamma \in \bigcap_{m>m_0,w'>w'_0} H^{m,w,w'} = H^{\infty,w,\infty}$$

which was the statement of the lemma. \qed

This will be of great help in the proof of the next lemma, which lays the groundwork for our proof of surjectivity of the map from $\mathcal{H}^k(M)$ to $H^k(M,\partial M)$.

**Lemma 5.12** Let $\omega \in C^\infty_c \Omega^k(M)$, and assume $\frac{n-1+a/b}{2} \notin \{0,1,\ldots,n-1\}$. Then $\omega = \omega_0 + D\gamma$ for some $\omega_0 \in \mathcal{H}(M)$ and $\gamma \in H^{\infty,-a,\infty}(M)$.

**Proof.** We will again utilize the Fredholm properties of $D$. Recall that since $\frac{n-1+a/b}{2} \notin \{0,1,\ldots,n-1\}$, the operator $D : H^{m,-a,w'} \to H^{m-1,0,w'+b}$ is Fredholm for any integer $m$ and any real $w'$. So

$$\forall \omega \in H^{m-1,0,w'+b} \quad \omega \in \text{Im} \left( D : H^{m,-a,w'} \to H^{m-1,0,w'+b} \right) \iff$$

$$\iff \omega \perp_{L^2} \text{Ker} \left( D : H^{1-m,0,-w'-b} \to H^{-m,a,-w'} \right)$$

(5.4) as $D : H^{1-m,0,-w'-b} \to H^{-m,a,-w'}$ is adjoint to $D : H^{m,-a,w'} \to H^{m-1,0,w'+b}$ under the $L^2$ pairing.

Setting $m = 1$ and $w' = -b$, we see that $D : H^{1,-a,-b} \to L^2$ must be Fredholm
too. Thus there is a natural splitting

\[ L^2 = \text{Im}(D : H^{1,-a,-b} \to L^2) \oplus_{L^2} \text{Ker}(D : L^2 \to H^{-1,a,b}) \]

because the sesquilinear form on \( H^{0,0,0} = L^2 \) is the usual \( L^2 \) inner product, and \( L^2 \) is self-dual under that inner product (this makes it different from any other space \( H^{m,w,w'} \)). Since our original form \( \omega \) is smooth and compactly supported inside \( M \), it certainly lies in \( L^2 \) and can therefore be written as \( \omega = \omega_0 + D\gamma \), where \( \omega_0 \in \text{Ker}(D : L^2 \to H^{-1,a,b}) \) and \( \gamma \in H^{1,-a,-b} \).

According to (5.3), the form \( \omega_0 \) must belong to \( H^{\infty,0,\infty} \). Since \( \omega \) lies in \( C_c^\infty \Omega(M) \), it too will belong to \( H^{\infty,0,\infty} \). We see that \( D\gamma = \omega - \omega_0 \in H^{\infty,0,\infty} \). In addition, \( H^{m,-a,w'} \to H^{m-1,0,0'+b} \) is Fredholm for all \( m \) and \( w' \), because \( \frac{n-1+a/b}{2} \notin \{0,1,\ldots,n-1\} \). Hence Lemma 5.11 (with \( w = -a \)) applies to \( \gamma \). By that lemma, \( \gamma \) will belong to \( H^{\infty,-a,\infty} \). Since \( \omega_0 \in \mathcal{H}(M) \) by the definition of \( \mathcal{H}(M) \) and \( \omega = \omega_0 + d\gamma \), the proof is complete. \( \square \)

In the exceptional case \( \frac{n-1+a/b}{2} \in \{0,1,\ldots,n-1\} \), \( D \) will not be Fredholm as a map into \( L^2 \). Lemma 5.12 will then have to be replaced by the following analogue:

**Lemma 5.13** Assume that \( \frac{n-1+a/b}{2} \in \{0,1,\ldots,n-1\} \). Let \( \omega \in C_c^\infty \Omega^k(M) \). There exists a real \( r_0 > 0 \) such that for any positive \( r < r_0 \), \( \omega \) can be written as \( \omega = \omega_0 + D\gamma \) with \( \omega_0 \in \mathcal{H}(M) \) and \( \gamma \in H^{\infty,-a-r,\infty} \).

**Proof.** The argument will be similar to the proof of Lemma 5.12. By Lemma 4.1, there is a finite set \( \{w_0,\ldots,w_n\} \) such that \( D : H^{m+1,w-a,w'-b} \to H^{m,w,w'} \) is Fredholm for all \( m \), all \( w' \) and all \( w \notin \{w_0,\ldots,w_n\} \). In addition, it follows from Lemma 4.3 that there exists a small \( \epsilon_h \) such that any \( L^2 \) harmonic form also lies in \( H^{\infty,\epsilon_h,\infty} \). Let \( r_0 \) be so small that \( r_0 < \epsilon_h \), and the interval \((-r_0,0)\) does not contain an element of \( \{w_0,\ldots,w_N\} \). Let \( r \) be any number between 0 and \( r_0 \).

Our first step will be to show that \( \omega \) can be written as \( \omega = \omega_0 + D\gamma \) with \( \omega_0 \in \mathcal{H}(M) \) and \( \gamma \in H^{1,-a-r,-b} \). By the construction, \(-r \) cannot be an element of \( \{w_0,\ldots,w_n\} \), and \( D : H^{1,-a-r,-b} \to H^{0,-r,0} \) must therefore be Fredholm. Its \( L^2 \)
dual is \( D : H^{0,r,0} \to H^{-1,a+r,b} \). An element \( \sigma \in H^{0,-r,0} \) will therefore lie in the image of \( D : H^{1,-a-r,-b} \to H^{0,-r,0} \) if and only if it is \( L^2 \)-orthogonal to the kernel of \( D : H^{0,r,0} \to H^{-1,a+r,b} \). Because \( D : H^{1,-a-r,-b} \to H^{0,-r,0} \) is Fredholm, its image will have a finite codimension. And since the sesquilinear \( L^2 \) pairing between \( H^{0,-r,0} \) and \( H^{0,r,0} \) is non-degenerate, we have

\[
\text{codim}(\text{Im}(D : H^{1,-a-r,-b} \to H^{0,-r,0})) = \dim(\text{Ker}(D : H^{0,r,0} \to H^{-1,a+r,b}))
\]

Since \( r \) is positive, \( H^{0,r,0} \) is a subset of \( H^{0,-r,0} \). Hence the kernel of \( D : H^{0,r,0} \to H^{-1,a+r,b} \) is also a subset of \( H^{0,-r,0} \). We now want to prove that its intersection with the image of \( D : H^{1,-a-r,-b} \to H^{0,-r,0} \) is zero. In effect, what we want to show is that any \( \sigma \in H^{0,r,0} \) that satisfies \( D\sigma = 0 \) and \( \sigma = D\tau \) for a form \( \tau \in H^{1,-a-r,-b} \) must satisfy \( \sigma = 0 \). (This, of course, is just another way of saying that the intersection of \( \text{Ker}(D : H^{0,r,0} \to H^{-1,a+r,b}) \) with \( \text{Im}(D : H^{1,-a-r,-b} \to H^{0,-r,0}) \) is zero.)

To prove that \( \sigma = 0 \) under these assumptions, we want to justify the integration by parts in \( \langle D\tau, D\tau \rangle = \langle \tau, D^2\tau \rangle \). First, since \( \sigma \in H^{0,r,0} \) satisfies \( D\sigma = 0 \), \( \sigma \) will also belong to \( H^{\infty,r,\infty} \), by (5.3). It will thus also lie in \( H^{\infty,-r,\infty} \), because \( r > 0 > -r \). Because \( -r \) is not one of the exceptional weights \( w_0, \ldots, w_n \), the operator \( D : H^{m,-a-r,w} \to H^{m-1,-r,w'+b} \) will be Fredholm for all \( m \) and \( w' \). Lemma 5.11 tells us that in this case \( \tau \) will be an element of \( H^{\infty,-a-r,\infty} \), as \( D\tau = \sigma \in H^{\infty,-r,\infty} \). In particular, \( \tau \) has to be smooth.

Because \( r > 0 \), we have \( \sigma \in H^{0,r,0} \subset H^{0,0,0} = L^2 \). So \( \sigma \) is an \( L^2 \) harmonic form, and therefore it will lie in \( H^{\infty,\epsilon h,\infty} \) by the definition of \( \epsilon_h \). Because \( d \) and \( \delta \) map each \( H^{\infty,w,\infty} \) to \( H^{\infty,w+a,\infty} \), we also have \( d\tau, \delta\tau \in H^{\infty,-r,\infty} \), and likewise \( d\sigma, \delta\sigma \in H^{\infty,\epsilon_h+a,\infty} \).

(We actually know that \( d\sigma \) and \( \delta\sigma \) are zero, by Lemma 5.9, but \( H^{\infty,\epsilon_h+a,\infty} \) already has enough decay for our purposes). We will now justify the transitions \( \langle d\tau, \sigma \rangle = \langle \tau, d\sigma \rangle \) and \( \langle \delta\tau, \sigma \rangle = \langle \tau, d\sigma \rangle \) using Lemma 5.6.

Since \( H^{\infty,w,\infty} \subset x^w L^2 \) for any \( w \), we have \( d\tau \in x^{-r} L^2 \), \( \sigma \in x^{\epsilon_h} L^2 \), \( \tau \in x^{-a-r} L^2 \).
and \( \delta \sigma \in x^{\epsilon_h +a}L^2 \). Because these forms are smooth and

\[-r + \epsilon_h = (-a - r) + (\epsilon_h + a) = (-a - r) + (\epsilon_h) + a > 0 \quad \text{as} \quad r < r_0 < \epsilon_h \quad (5.5)\]

the conditions of Lemma 5.6 are fulfilled. According to that lemma, \( \langle d \tau, \sigma \rangle = \langle \tau, \delta \sigma \rangle \). It is also clear that \( \delta \tau \in H^{\infty, -r, \infty} \subset x^{-r}L^2 \) and \( d \sigma \in H^{\infty, \epsilon_h +a, \infty} \subset x^{\epsilon_h +a}L^2 \). Because \( \tau \) and \( \sigma \) are smooth and because of (5.5), Lemma 5.6 shows that \( \langle \delta \tau, \sigma \rangle = \langle \tau, d \sigma \rangle \). Added to \( \langle d \tau, \sigma \rangle = \langle \tau, \delta \sigma \rangle \), the last equality yields \( \langle D \tau, \sigma \rangle = \langle \tau, D \sigma \rangle \). As \( \sigma = D \tau \), we have \( \langle D \tau, D \tau \rangle = \langle \tau, D^2 \tau \rangle = 0 \) because \( \sigma \) is harmonic. Hence \( \sigma = D \tau \) must be zero. This proves that \( \text{Ker}(D : H^{0,r,0} \to H^{-1,a+r,b}) \) and \( \text{Im}(D : H^{1,-a-r,-b} \to H^{0,-r,0}) \) have a zero intersection.

Since \( \text{Ker}(D : H^{0,r,0} \to H^{-1,a+r,b}) \) is a subspace of \( H^{0,-r,0} \) whose dimension is \( \text{codim}(\text{Im}(D : H^{1,-a-r,-b} \to H^{0,-r,0})) \), and since these subspaces, as we have just seen, are transverse, the space \( H^{0,-r,0} \) must be equal to their direct sum:

\[ H^{0,-r,0} = \text{Ker}(D : H^{0,r,0} \to H^{-1,a+r,b}) \oplus \text{Im}(D : H^{1,-a-r,-b} \to H^{0,-r,0}) \]

Since \( \omega \) lies in \( H^{0,-r,0} \) by virtue of being smooth and compactly supported inside \( M \), it can therefore be written as \( \omega = \omega_0 + D \gamma \), where \( \omega_0 \in \text{Ker}(D : H^{0,r,0} \to H^{-1,a+r,b}) \) and \( \gamma \in H^{1,-a-r,-b} \). But, by (5.3), the form \( \omega_0 \) with this property must also lie in \( H^{\infty,r,\infty} \), and hence in \( H^{\infty,-r,\infty} \). \( \omega \) belongs to \( H^{\infty,-r,\infty} \) too, because it is smooth and compactly supported inside \( M \). So \( D \gamma \in H^{\infty,-r,\infty} \). We also know that \( D : H^{m,-a-r,w'} \to H^{m-a,-r,w'+b} \) is Fredholm for all \( m \) and \( w' \), because \( -r \in (-r_0, 0) \) is not an exceptional weight. Hence Lemma 5.11 applies to \( \gamma \). According to that lemma, \( \gamma \) belongs to \( H^{\infty,-a-r,\infty} \). So \( \omega = \omega_0 + d \gamma \), where \( \omega_0 \in H^{\infty,r,\infty} \subset L^2 \) is a harmonic form and \( \gamma \) is an element of \( H^{\infty,-a-r,\infty} \). \( \square \)

For any \( \epsilon \in (0,1) \), let \( U_\epsilon \) denote the subset of the collar neighborhood \( U \) diffeomorphic to \( (0, \epsilon) \times \partial M \). Since \( \omega \) is compactly supported inside \( M \), there exists a small positive \( \epsilon \) such that \( \text{supp}(\omega) \subset M - U_\epsilon \).

Our next step is to replace \( D \gamma \) with \( d \gamma \), and then \( \omega_0 \) with its part of degree \( k \) and \( \gamma \) with its part of degree \( k-1 \), respectively. This will be the subject of our next
lemma. In the statement of that lemma we will only assume that $\omega$ can be expressed as $\omega_0 + d\gamma$ for $\omega_0 \in \mathcal{H}(M)$ and $\gamma \in H^{\infty,-a-s,\infty}$, where the real number $s$ can be made arbitrarily small. We add this complication to make the lemma below apply to the exceptional case $\frac{n-1+a/b}{2} \in \{0, 1, \ldots, n-1\}$, when instead of Lemma 5.12 we will need to rely on Lemma 5.13.

**Lemma 5.14** Let $k < \frac{n-1+a/b}{2}$, and let $\omega \in C^\infty \Omega^k(M)$ be a closed $k$-form. There exists a small real $s$ such that if $\omega$ can be written as $\omega_0 + d\gamma$ with $\omega_0 \in \mathcal{H}(M)$ and $\gamma \in H^{\infty,-a-s,\infty}$, then $\omega$ can also be represented as $\omega = \omega_0^k + d\gamma_1^k-1$ for some $\omega_0^k \in \mathcal{H}^k(M)$ and $\gamma_1^k-1 \in H^{\infty,-a-s,\infty} \Omega^{k-1}(M)$. In addition, if $\varepsilon$ is so small that $\text{supp}(\omega) \subset M - U_\varepsilon$, then $\gamma_1^k$ restricts on $U_\varepsilon \cong (0, \varepsilon) \times \partial M$ to

$$\gamma_1^{k-1}|_{U_\varepsilon} = \begin{cases} \left\{ \begin{array}{ll} x^{(n-k)b-a} \phi^{k-1} + \sigma_T & \text{if } k < \frac{n+1-a/b}{2} \\ x^{(n-k+1)b} \psi^{k-2} + \sigma_C & \text{if } \frac{n+1-a/b}{2} \leq k < \frac{n-1+a/b}{2} \end{array} \right. \end{cases}$$

where $\phi^{k-1} \in \mathcal{H}^{k-1}(\partial M)$, $\psi^{k-2} \in \mathcal{H}^{k-2}(\partial M)$, $\sigma_T, \sigma_C \in \dot{C}^\infty(\varepsilon, \mathcal{C}^\infty(\partial M))$.

**Proof.** If $M$ were compact, we could simply write

$$d\omega = 0 \Rightarrow d(\omega_0 + d\gamma + \delta\gamma) = 0 \Rightarrow d\delta\gamma = 0 \quad \text{and} \quad \langle \delta\gamma, \delta\gamma \rangle = \langle \gamma, d\delta\gamma \rangle = 0$$

which would imply $\delta\gamma = 0$. But because $M$ is not compact, we will need an intricate calculation to estimate the rate of growth of $\gamma$ inside $U_\varepsilon$ and make sure that the integration by parts in $\langle \delta\gamma, \delta\gamma \rangle = \langle \gamma, d\delta\gamma \rangle$ is justified.

Assume that $\omega_0 = \sum_{j=0}^n \omega_0^j$ and $\gamma = \sum_{j=0}^n \gamma^j$, where $\omega_0^j, \gamma^j \in \Omega^j(M)$ for each $j$. By Lemma 5.9, $D\omega_0 = 0$ implies that $d\omega_0 = \delta\omega_0 = 0$. That means $d\omega_0^j = \delta\omega_0^j = 0$ for each $j$, because $d\omega_0^j$ is the $(j+1)$-th degree part of $d\omega_0$, and $\delta\omega_0^j$ is the $(j-1)$-th degree part of $\delta\omega_0$. So we see that each $\omega_0^j$ must itself be harmonic. By Lemma 4.3, that implies that the BH part of $\omega_0^j$ in $U$ must be $\left( \begin{array}{c} x^j \alpha^j \\ x^{(n-j)b} \beta^{j-1} \end{array} \right)$, where $\alpha^j \in \mathcal{H}^j(\partial M)$ and $\beta^j \in \mathcal{H}^{j-1}(\partial M)$ are harmonic forms on $\partial M$ for each $j$. 

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Since $D$ preserves the BH and BNH parts of a form, the BH part of $D\gamma$ in $U_\epsilon$ must be the difference of the BH parts of $\omega$ and $\omega_0$. As $\omega_0$ vanishes inside $U_\epsilon$, its BH part is zero inside $U_\epsilon$. So the BH part of $D\gamma$ in $U_\epsilon$ is equal to 

$$\sum_{j=0}^{n} \begin{pmatrix} -x^{j}b\alpha^{j} \\ -x^{(n-j)}b\beta^{j-1} \end{pmatrix},$$

the difference of the BH parts of $\omega$ and $\omega_0$.

Let us denote the BH part of $\gamma^j$ in $U$ by \begin{pmatrix} \gamma^j_T \\ \gamma^j_C \end{pmatrix}. Because $D$ acts on the BH part of a form by (3.1), the BH part of $D\gamma$ is equal to

\[
\begin{pmatrix}
0 & -x^{a+1} \frac{\partial}{\partial x} + x^{a}(n - A - 1)b \\
x^{a+1} \frac{\partial}{\partial x} - x^{a}Ab & 0
\end{pmatrix}
\begin{pmatrix}
\gamma^j_T \\
\gamma^j_C
\end{pmatrix} = 
\sum_{j=0}^{n} \begin{pmatrix} -x^{a+1} \frac{\partial\gamma^j_{C}}{\partial x} + x^{a}(n - j)b\gamma^j_{C} \\ x^{a+1} \frac{\partial\gamma^j_{T}}{\partial x} - x^{a}jb\gamma^j_{T} \end{pmatrix}
\]

This is equal to the difference between the BH parts of $\omega$ and $\omega_0$, i.e.

$$\sum_{j=0}^{n} \left( -x^{a+1} \frac{\partial\gamma^j_{C}}{\partial x} + x^{a}(n - j)b\gamma^j_{C} \right) = -\sum_{j=0}^{n} x^{j}b\alpha^{j}$$
$$\sum_{j=0}^{n} \left( x^{a+1} \frac{\partial\gamma^j_{T}}{\partial x} - x^{a}jb\gamma^j_{T} \right) = -\sum_{j=0}^{n} x^{(n-j)}b\beta^{j-1}$$

We now shift indices, replacing each $j$ with a $j + 1$ in the left-hand side of the first equation and in the right-hand side of the second. Then we separate each side by form degree. We get

$$-x^{a} \left( x \frac{\partial\gamma^j_{C}}{\partial x} - (n - j - 1)b\gamma^j_{C} \right) = -x^{2b}\alpha^{j}$$
$$x^{a} \left( x \frac{\partial\gamma^j_{T}}{\partial x} - x^{a}jb\gamma^j_{T} \right) = -x^{(n-j-1)}b\beta^{j}$$

for all $j$ between 0 and $n-1$. According to the product rule, $x \frac{\partial f}{\partial x} + Cf = x^{-C}x \frac{\partial}{\partial x} (x^{C} f)$ for any smooth $f$ and any real $C$, so the above is equivalent to 

$$x^{a}x^{(n-j-1)}b \frac{\partial}{\partial x} (x^{-(n-j-1)}b\gamma^j_{C}) = x^{j}b\alpha^{j}$$

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To compute \( \gamma_T^j \) and \( \gamma_C^j \), we need to integrate the right-hand side of the following two ODEs:

\[
\frac{\partial}{\partial x} \left( x^{-(n-j-1)b} \gamma_T^j \right) = x^{(2j-n+1)b-a-1} \alpha^j \\
\frac{\partial}{\partial x} \left( x^{-j'b} \gamma_C^j \right) = -x^{(n-2j-1)b-a-1} \beta^j
\]

Observe that we cannot get a logarithm by integrating the right-hand side of the first (respectively, second) equation unless \( j = \frac{n-1+a/b}{2} \) (or \( j = \frac{n-1-a/b}{2} \)). Indeed, the power of \( x \) in the right-hand side of the first (respectively, second) equation is equal to \(-1\) if and only if

\[
(2j - n + 1)b - a - 1 = -1 \iff j = \frac{n - 1 + a/b}{2} \quad \text{or, respectively,} \\
(n - 2j - 1)b - a - 1 = -1 \iff j = \frac{n - 1 - a/b}{2}
\]

Let us denote the constant of integration \( \psi^j \) in the first ODE and \( \phi^j \) in the second. \( \phi^j \) and \( \psi^j \) are just some \( x \)-independent harmonic forms on \( \partial M \). Solving the two ODEs, we get

\[
\gamma_T^j = \begin{cases} 
\frac{-x^{(n-j-1)b-a}}{(2j-n+1)b-a} \beta^j + x^b \phi^j & \text{if } j \neq \frac{n-1-a/b}{2} \\
x^b \ln(x) \beta^j + x^b \phi^j & \text{if } j = \frac{n-1-a/b}{2}
\end{cases}
\quad \text{(5.6)}
\]

\[
\gamma_C^j = \begin{cases} 
\frac{-x^{(n-j-1)b-a}}{(2j-n+1)b-a} \alpha^j + x^{(n-j-1)b} \psi^j & \text{if } j \neq \frac{n-1+a/b}{2} \\
x^{(n-j-1)b} \ln(x) \alpha^j + x^{(n-j-1)b} \psi^j & \text{if } j = \frac{n-1+a/b}{2}
\end{cases}
\quad \text{(5.7)}
\]

where the denominators, as we have just seen, will be non-zero under these assumptions on \( j \).

The condition \( \gamma \in H^{\infty,-a-s,\infty} \) will imply that some of these terms vanish. First, it means that \( \gamma \) is smooth in \( M \), and its BNH part is rapidly decaying near \( \partial M \). Thus the only thing we need to test to see whether \( \gamma^j \) lies in \( x^w L^2 \) is whether its BH part
lies in $x^w L^2(U_\varepsilon)$. So, for any real weight $w$ we have

$$\gamma \in x^w L^2 \iff \gamma^j \in x^w L^2 \forall j \iff \left( \begin{array}{c} \gamma^j_T \\ \gamma^j_C \end{array} \right) \in x^w L^2(U_\varepsilon) \forall j \iff \left( \begin{array}{c} \gamma^j_T \\ 0 \end{array} \right) \in x^w L^2(U_\varepsilon) \text{ and } \left( \begin{array}{c} 0 \\ \gamma^j_C \end{array} \right) \in x^w L^2(U_\varepsilon) \forall j \in \{0, 1, \ldots, n-1\}$$

This condition is in turn equivalent to

$$\frac{x^{(n-j-1)b-a} \beta^j}{(n-2j-1)b-a} + x^{j b} \phi^j \in x^w L^2\left((0, \varepsilon), \Omega^j_h(\partial M), \frac{dx}{x^{a+1+b(n-1)}}\right)$$

(5.8)

$$\frac{x^{(j-1)b-a} \alpha^{j-1}}{(2j-n-1)b-a} + x^{(n-j)b} \psi^{j-1} \in x^w L^2\left((0, \varepsilon), \Omega^{j-1}_h(\partial M), \frac{dx}{x^{a+1+b(n-1)}}\right)$$

(5.9)

as long as $j$ is not equal to either $\frac{n-1-a/b}{2}$ or $\frac{n+1+a/b}{2}$. If $j = \frac{n-1-a/b}{2}$, then (5.8) has to be replaced by

$$x^{j b} \ln(x) \beta^j + x^{j b} \phi^j \in x^w L^2\left((0, \varepsilon), \Omega^k_h(\partial M), \frac{dx}{x^{a+1+b(n-1)}}\right)$$

Likewise, if $j = \frac{n+1+a/b}{2}$, (i.e. $j - 1 = \frac{n-1+a/b}{2}$), then instead of (5.9) we have

$$x^{(n-j)b} \ln(x) \alpha^{j-1} + x^{(n-j)b} \psi^{j-1} \in x^w L^2\left((0, \varepsilon), \Omega^{j-1}_h(\partial M), \frac{dx}{x^{a+1+b(n-1)}}\right)$$

Observe that, if $j \neq \frac{n-1-a/b}{2}$, then the powers of $x$ in the two summands of the left-hand side of (5.8) are distinct. Likewise, if $j \neq \frac{n+1+a/b}{2}$, then the two summands in the right-hand side of (5.9) have distinct powers of $x$. This is clear, since

$$(n-j-1)b-a = j b \Leftrightarrow j = \frac{n-1-a/b}{2}; \quad (j-1)b-a = (n-j)b \Leftrightarrow j = \frac{n+1+a/b}{2}$$

Thus, if $j \neq \frac{n-1-a/b}{2}$, then $\gamma^j_T$ can only lie in $x^w L^2((0, \varepsilon), \Omega^j_h(\partial M), \frac{dx}{x^{a+1+b(n-1)}})$ if $x^{(n-j-1)b-a} \beta^j$ and $x^{j b} \phi^j$ both lie in that space. Likewise, if $j \neq \frac{n+1+a/b}{2}$, then $\gamma^{j-1}_C$ will be an element of that space if and only if $x^{(j-1)b-a} \alpha^{j-1}$ and $x^{(n-j)b} \psi^{j-1}$ belong to it. This is true because the powers of $x$ in the summands of $\gamma^j_T$ and $\gamma^{j-1}_C$
are distinct under these assumptions on \( j \). Similarly, if \( j = \frac{n-1-a/b}{2} \), then \( \gamma_{j}^{+} \) will lie in \( x^{w}L^{2}((0,\epsilon),\Omega_{h}(\partial M),\frac{dx}{x+a+1+b(n-1)}) \) if and only if \( x^{j}\ln(x)\beta^{j} \) lies in that space. If \( j = \frac{n+1-a/b}{2} \), then \( \gamma_{j}^{-1} \) will lie in \( x^{w}L^{2}((0,\epsilon),\Omega_{h}(\partial M),\frac{dx}{x+a+1+b(n-1)}) \) if and only if \( x^{(n-j)b}\ln(x)\alpha^{j-1} \) lies in that space. This is the case because \( x^{j}\phi^{j} \) becomes small compared to \( x^{j}\ln(x)\beta^{j} \), and \( x^{(n-j)b}\phi^{j-1} \) becomes small compared to \( x^{(n-j)b}\ln(x)\alpha^{j-1} \), as \( x \) tends to zero. (We are using the fact that \( \alpha^{j}, \beta^{j}, \phi^{j} \) and \( \psi^{j} \) do not depend on \( x \).)

We now perform six calculations: for non-zero \( \alpha^{j}, \beta^{j}, \phi^{j} \) and \( \psi^{j} \) we have

\[
\int_{0}^{\epsilon} x^{2(n-j)b-2a-2b} \beta^{j} \| \beta^{j} \|^{2}_{L^{2}} \frac{dx}{x+a+1+b(n-1)} < \infty \Leftrightarrow w < -3a + (n - 2j - 1)b < 0 \tag{5.10}
\]

\[
\int_{0}^{\epsilon} x^{2j-2w} \phi^{j} \| \phi^{j} \|^{2}_{L^{2}} \frac{dx}{x+a+1+b(n-1)} < \infty \Leftrightarrow w < -a + (2j - n + 1)b < 0 \tag{5.11}
\]

\[
\int_{0}^{\epsilon} x^{2(n-j)b-2a-2b} \| \alpha^{j-1} \|^{2}_{L^{2}} \frac{dx}{x+a+1+b(n-1)} < \infty \Leftrightarrow w < -3a + (n - 2j - 1)b < 0 \tag{5.12}
\]

\[
\int_{0}^{\epsilon} x^{2j-2w} \| \phi^{j-1} \|^{2}_{L^{2}} \frac{dx}{x+a+1+b(n-1)} < \infty \Leftrightarrow w < -a + (2j - n + 1)b < 0 \tag{5.13}
\]

\[
\int_{0}^{\epsilon} x^{2(n-j)b-2a-2b} \ln^{2} x \| \beta^{j} \|^{2}_{L^{2}} \frac{dx}{x+a+1+b(n-1)} < \infty \Leftrightarrow w < -a + (2j - n + 1)b < 0 \tag{5.14}
\]

\[
\int_{0}^{\epsilon} x^{2(n-j)b-2a-2b} \ln^{2} x \| \alpha^{j-1} \|^{2}_{L^{2}} \frac{dx}{x+a+1+b(n-1)} < \infty \Leftrightarrow w < -a + (2j - n + 1)b < 0 \tag{5.15}
\]

Each of these inequalities comes from the condition that the power of \( x \) in the corresponding integral is greater than \( -1 \). In the last two inequalities we use the fact that \( \ln x \) is dominated by any negative power of \( x \) near \( x = 0 \).

According to our assumptions, \( \gamma \) lies in \( H^{\infty,-a-s,\infty} \), and thus it must be an element of \( x^{-a-s}L^{2} \). Plugging \(-a - s\) for \( w \), we see that the first four inequalities translate to

\[
\beta^{j} \neq 0 \Rightarrow j < \frac{n - 1 - a/b}{2} + \frac{s}{b} ; \quad \alpha^{j-1} \neq 0 \Rightarrow j > \frac{n + 1 + a/b}{2} - \frac{s}{b} ;
\]

\[
\phi^{j} \neq 0 \Rightarrow j > \frac{n - 1 - a/b}{2} - \frac{s}{b} ; \quad \psi^{j-1} \neq 0 \Rightarrow j < \frac{n + 1 + a/b}{2} + \frac{s}{b} .
\]

We do not need to mention the exceptional values of \( j \) here, because \( \frac{n-1-a/b}{2} \) satisfies the inequalities arising from \( \beta^{j} \) or \( \phi^{j} \) being non-zero, and \( \frac{n+1+a/b}{2} \) likewise satisfies the inequalities following from \( \alpha^{j-1} \) or \( \psi^{j-1} \) being non-zero.

We now claim that, if \( s \) is small enough, then the extra term \( \frac{s}{b} \) can be thrown away from the above inequalities. Indeed, if \( s_{1} > 0 \) is so small that there are no
integers in \( \left( \frac{n-1-a/b}{2}, \frac{n-1-a/b}{2} + \frac{s_1}{b} \right) \), then any integer \( j \) which satisfies \( j < \frac{n-1-a/b}{2} + \frac{s_1}{b} \) will also satisfy \( j \leq \frac{n-1-a/b}{2} \). A small positive number with this property must exist, since for any \( t \in \mathbb{R} \) there exists a small \( \chi \) such that there are no integers in \( (t, t + \chi) \). By the same token, let \( s_2, s_3 \) and \( s_4 \) be small positive numbers such that there are no integers in \( \left( \frac{n+1+a/b}{2} - \frac{s_2}{b}, \frac{n+1+a/b}{2} - \frac{s_3}{b}, \frac{n-1-a/b}{2} - \frac{s_4}{b} \right) \), or in \( \left( \frac{n+1+a/b}{2} - \frac{s_2}{b}, \frac{n+1+a/b}{2} - \frac{s_3}{b}, \frac{n-1-a/b}{2} + \frac{s_4}{b} \right) \). These conditions clearly remain true if \( s_1, s_2, s_3, \) or \( s_4 \) are decreased. Now any integer \( j \) such that \( j > \frac{n-1-a/b}{2} - \frac{s_3}{b} \) will also satisfy \( j \geq \frac{n-1-a/b}{2} \).

Likewise, any \( j \in \mathbb{Z} \) that satisfies \( j > \frac{n+1+a/b}{2} - \frac{s_2}{b} \) will also satisfy \( j \geq \frac{n+1+a/b}{2} \), and any \( j \in \mathbb{Z} \) such that \( j < \frac{n+1+a/b}{2} + \frac{s_4}{b} \) will automatically satisfy \( j \leq \frac{n+1+a/b}{2} \). We finally set \( s \) to be the smallest of the numbers \( s_1, s_2, s_3, \) and \( s_4 \). This value of \( s \) will remain fixed through the rest of the proof. With this choice of \( s \), the implications from \( \alpha^j, \beta^j, \phi^j, \) or \( \psi^j \) being non-zero can be rewritten as follows:

\[
\beta^j \neq 0 \Rightarrow j \leq \frac{n-1-a/b}{2} \quad \alpha^{j-1} \neq 0 \Rightarrow j \geq \frac{n+1+a/b}{2} \quad (5.16)
\]

\[
\phi^j \neq 0 \Rightarrow j \geq \frac{n-1-a/b}{2} \quad \psi^{j-1} \neq 0 \Rightarrow j \leq \frac{n+1+a/b}{2} \quad (5.17)
\]

Applying these conditions to (5.6) and (5.7), we see that if \( j < \frac{n-1-a/b}{2} \), then

\[
\gamma_T^j = \frac{x^{(n-j-1)b-a}}{(n-2j-1)b-a} \beta^j \quad \text{and} \quad \gamma_C^{j-1} = x^{(n-j)b} \psi^{j-1}.
\]

By (5.10) and (5.13),

\[
\text{For } j \leq \frac{n-1-a/b}{2}, \quad \gamma^j \in x^w L^2 \iff w < \frac{-3a}{2} + \frac{(n-2j-1)b}{2} \quad (5.18)
\]

\[
\left( \text{because } \frac{-3a}{2} + \frac{(n-2j-1)b}{2} < \frac{-a}{2} + \frac{(n-2j+1)b}{2} \right)
\]

Similarly, if \( \frac{n-1-a/b}{2} < j < \frac{n+1+a/b}{2} \), then \( \gamma_T^j = x^b \phi^j \) and \( \gamma_C^{j-1} = x^{(n-j)b} \psi^{j-1} \). This case has two subcases:

\[
\text{For } \frac{n-1-a/b}{2} < j \leq \frac{n}{2}, \quad \gamma^j \in x^w L^2 \iff w < \frac{-a + b(2j-n+1)}{2} \quad (5.19)
\]

\[
\text{For } \frac{n}{2} \leq j < \frac{n+1+a/b}{2}, \quad \gamma^j \in x^w L^2 \iff w < \frac{-a + b(n-2j+1)}{2} \quad (5.20)
\]
This follows from (5.11) and (5.13) and the fact that

$$\frac{-a + b(2j - n + 1)}{2} \leq \frac{-a + b(n - 2j + 1)}{2} \iff j \leq \frac{n}{2}$$

Finally, if $j > \frac{n+1+a/b}{2}$, then $\gamma_T^j = x^j \phi^j$ and $\gamma_C^{j-1} = \frac{x^{(j-1)b-a}}{(2j-n-1)b-a} \alpha^{j-1}$. In this case (5.11) and (5.12) imply

For $j > \frac{n + 1 + a/b}{2}$, $\gamma^j \in x^w L^2 \iff w < \frac{-3a}{2} + \frac{(2j - n - 1)b}{2}$ (5.21)

(because $\frac{-3a}{2} + \frac{(2j - n - 1)b}{2} < \frac{-a}{2} + \frac{(2j - n + 1)b}{2}$)

Similar inequalities can be deduced for the borderline values of $j$, i.e. for $j = \frac{n \pm (1 + a/b)}{2}$. If $j = \frac{n - 1 - a/b}{2}$, then (5.6), (5.7), and (5.16) imply that $\gamma_T^j = x^j \ln(x) \beta^j + x^j \psi^j$ and $\gamma_C^{j-1} = x^{n-j} \psi^j$. By (5.14) and (5.13), $\gamma^j$ will lie in $x^w L^2$ if $w < \frac{-a + (n-2j+1)b}{2}$ and $w < \frac{-a + (2j-n+1)b}{2}$. Plugging $j = \frac{n - 1 + a/b}{2}$ into these inequalities, we see that they are equivalent to $w < -a$ and $w < -a$. So $\gamma^j$ will belong to $x^w L^2$ for any $w < -a$, as $a$ and $b$ are positive.

If $j = \frac{n + 1 + a/b}{2}$, it follows from (5.6), (5.7), and (5.16) that $\gamma_T^j = x^j \phi^j$ and $\gamma_C^{j-1} = x^{(n-j)b} \ln(x) \alpha^{j-1} + x^{(n-j)b} \psi^{j-1}$. In this case, according to (5.15) and (5.11), $\gamma^j$ will belong to $x^w L^2$ if $w < \frac{-a + (n-2j+1)b}{2}$ and $w < \frac{-a + (2j-n+1)b}{2}$. Once we plug $j = \frac{n+1+a/b}{2}$ into these inequalities, they again become $w < -a$ and $w < -a$, so in this case $\gamma^j$ will again lie in $x^w L^2$ for each $w < -a$.

We need to use these estimates on $\gamma_T^j$ and $\gamma_C^{j-1}$ to justify the integration by parts in $\langle \delta \gamma, \delta \gamma \rangle = \langle \gamma, d \delta \gamma \rangle$. Observe first that

$$\langle \delta \gamma, \delta \gamma \rangle = \left( \sum_{j=0}^{n} \delta \gamma^j, \sum_{j=0}^{n} \delta \gamma^j \right) = \sum_{j=0}^{n} \langle \delta \gamma^j, \delta \gamma^j \rangle$$

and likewise

$$\langle \gamma, d \delta \gamma \rangle = \left( \sum_{j=0}^{n} \gamma^j, \sum_{j=0}^{n} d \delta \gamma^j \right) = \sum_{j=0}^{n} \langle \gamma^j, d \delta \gamma^j \rangle$$

because any two forms of different form degree are orthogonal. So our goal is to show that $\langle \delta \gamma^j, \delta \gamma^j \rangle = \langle \gamma^j, d \delta \gamma^j \rangle$ for each $j \in \{0, 1, \ldots n\}$. We will prove this separately for
each of the four cases \( j < \frac{n-1-a/b}{2} \), \( \frac{n-1-a/b}{2} \leq j \leq \frac{n}{2} \), \( \frac{n}{2} \leq j < \frac{n+1+a/b}{2} \), and \( j > \frac{n+1+a/b}{2} \), and then deal with the borderline cases \( j = \frac{n-1-a/b}{2} \) and \( j = \frac{n+1+a/b}{2} \). In each of those cases we will invoke Lemma 5.5. Observe first that \( \gamma^j \), and hence \( d\gamma^j \) and \( \delta\gamma^j \), are always smooth by the definition of \( H^{\infty,-a-s,\infty} \). So the smoothness assumptions of Lemma 5.5 are satisfied, and we only need to show that \( \gamma^j, \delta\gamma^j \) and \( d\delta\gamma^j \) lie in the appropriate weighted \( L^2 \) spaces. This is done differently for different ranges of \( j \).

Assume first that \( j < \frac{n-1-a/b}{2} \). Let \( \epsilon_1 \) be a positive real number. According to (5.18), in this case \( \gamma^j \in x^pL^2 \) for \( p = -a + (n-j-1)b - \epsilon_1 \). In conjunction with \( \gamma^j \in H^{\infty,-a-s,\infty} \), this implies \( \gamma^j \in H^{\infty,p,\infty} \). Since \( d \) and \( \delta \) map \( H^{\infty,w,\infty} \) to \( H^{\infty,w+a,\infty} \) for any real \( w \), it follows that \( \delta\gamma^j \in H^{\infty,q,\infty} \) for \( q = -a + (n-j-1)b - \epsilon_1 \). By the same token, \( d\delta\gamma^j \in H^{\infty,q',\infty} \) for \( q' = a + (n-j-1)b - \epsilon_2 \). It is also clear that \( \delta\gamma^j \in H^{\infty,p',\infty} \) for \( p' = q \). Observe that

\[
p + q + a = p' + q = p + q' = -a + (n-2j-1)b - 2\epsilon_1
\]

In order to justify the integration by parts \( \langle \gamma^j, d\delta\gamma^j \rangle = \langle \delta\gamma^j, \delta\gamma^j \rangle \) by using Lemma 5.5, we therefore only need to show that \( -a + (n-2j-1)b - 2\epsilon_1 \) will be positive for the right choice of \( \epsilon_1 \). (Here and below we are using the fact that \( H^{\infty,w,\infty} \) is a subset of \( x^wL^2 \), which follows immediately from the definition of these spaces.) In effect, that means we have to show that \( -a + (n-2j-1)b \) is greater than 0, because \( \epsilon_1 \) can then be chosen between zero and half of that number. But

\[
-a + (n-2j-1)b > 0 \Leftrightarrow j < \frac{n-1-a/b}{2}
\]

and that is exactly the range of \( j \) that we are dealing with. So the last inequality holds, and by Lemma 5.5 the equality \( \langle \gamma^j, d\delta\gamma^j \rangle = \langle \delta\gamma^j, \delta\gamma^j \rangle \) is true.

Consider now the case \( \frac{n-1-a/b}{2} < j \leq \frac{n}{2} \). For any positive real \( \epsilon_2 \), \( \gamma^j \) will belong to \( H^{\infty,p,\infty} \) for \( p = -a + b(2j-n+1) - \epsilon_2 \), according to (5.19) and because \( \gamma^j \in H^{\infty,-a-s,\infty} \). As \( d \) and \( \delta \) map \( H^{\infty,w,\infty} \) to \( H^{\infty,w+a,\infty} \) for all \( w \), it follows that \( \delta\gamma^j \in H^{\infty,q,\infty} \), \( d\delta\gamma^j \in H^{\infty,q',\infty} \) and \( \delta\gamma^j \in H^{\infty,p',\infty} \) for \( q = p' = a + b(2j-n+1) - \epsilon_2 \) and \( q' = 3a + b(2j-n+1) - \epsilon_2 \).
We easily compute

\[ p + q + a = p + q' = p' + q = a + b(2j - n + 1) - 2\epsilon_2 \]

To justify \( \langle \gamma^j, d\delta \gamma^j \rangle = \langle \delta \gamma^j, \delta \gamma^j \rangle \) by invoking Lemma 5.5, we only need to show that this quantity is positive for a small enough \( \epsilon_2 \). That will immediately follow if we can show that \( a + b(2j - n + 1) \) is positive. However,

\[ a + b(2j - n + 1) > 0 \iff j > \frac{n - 1 - a/b}{2} \]

and since we're dealing with the case \( \frac{n - 1 - a/b}{2} < j \leq \frac{n}{2} \), Lemma 5.5 applies and tells us that \( \langle \gamma^j, d\delta \gamma^j \rangle = \langle \delta \gamma^j, \delta \gamma^j \rangle \).

The other two ranges of \( j \) are almost identical. If \( \frac{n}{2} \leq j < \frac{n+1+a/b}{2} \), assume \( \epsilon_3 \) is real and positive. We know from (5.20) and the condition \( \gamma^j \in H^{\infty,-a-s,\infty} \) that in this case \( \gamma^j \in H^{\infty,p,\infty} \) for \( p = \frac{-a+b(n-2j+1)}{2} - \epsilon_3 \). Consequently, \( \delta \gamma^j \in H^{\infty,q,\infty} \), \( d\delta \gamma^j \in H^{\infty,q',\infty} \), and \( \delta \gamma^j \in H^{\infty,p',\infty} \) for \( q = p' = \frac{a+b(n-2j+1)}{2} - \epsilon_3 \) and \( q' = \frac{3a+b(n-2j+1)}{2} - \epsilon_3 \). Also,

\[ p + q + a = p + q' = p' + q = a + b(n - 2j + 1) - 2\epsilon_3 \]

so the assumptions of Lemma 5.5 will be satisfied for a small enough \( \epsilon_3 \) as long as \( a + b(n - 2j + 1) > 0 \). The last inequality holds because

\[ a + b(n - 2j + 1) > 0 \iff j < \frac{n + 1 + a/b}{2} \]

and we assumed \( j \in \left[ \frac{n}{2}, \frac{n+1+a/b}{2} \right) \). So \( \langle \gamma^j, d\delta \gamma^j \rangle = \langle \delta \gamma^j, \delta \gamma^j \rangle \), by Lemma 5.5.

If \( j > \frac{n+1+a/b}{2} \), let \( \epsilon_4 \) be a positive real number. According to (5.21), \( \gamma^j \in H^{\infty,p,\infty} \) for \( p = \frac{-3a+(2j-n-1)b}{2} - \epsilon_4 \) (because \( \gamma^j \in H^{\infty,-a-s,\infty} \)). So \( \delta \gamma^j \in H^{\infty,q,\infty} \), \( d\delta \gamma^j \in H^{\infty,q',\infty} \), and \( \delta \gamma^j \in H^{\infty,p',\infty} \) for \( q = p' = \frac{-a+(2j-n-1)b}{2} - \epsilon_4 \) and \( q' = \frac{a+(2j-n+1)b}{2} - \epsilon_4 \). Just as in the other cases, we see that

\[ p + q + a = p + q' = p' + q = -a + b(2j - n + 1) - 2\epsilon_4 \]
and this number will be positive if $\epsilon_4$ is sufficiently small, because

$$-a + b(2j - n - 1) > 0 \Leftrightarrow j > \frac{n + 1 + a/b}{2}$$

which is our assumption on $j$. So Lemma 5.5 implies $\langle \gamma^j, d\delta \gamma^j \rangle = \langle \delta \gamma^j, \delta \gamma^j \rangle$.

We now need to deal with the borderline values of $j$. We have seen that for $j = \frac{n+1+a/b}{2}$, $\gamma^j$ will lie in $x^wL^2$ for any $w < -a$. Assume first that $j = \frac{n-1-a/b}{2}$. Then $\gamma_T^j = x^b \ln(x) \beta^j + x^b \phi^j$ and $\gamma_{\gamma}^{j-1} = x^{n-j}b \psi^{j-1}$, by (5.6), (5.7), and (5.16). According to (3.2), the BH part of $\delta \gamma^j$ is equal to

$$\begin{pmatrix} 0 & -x^{a+1} \frac{\partial}{\partial x} + (n-j)b x^a \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x^b \ln(x) \beta^j + x^b \phi^j \\ x^{(n-j)}b \psi^{j-1} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

inside $U_\epsilon$. The BH part of $d\delta \gamma^j$ will thus also vanish inside $U_\epsilon$. Because the BNH part of $\gamma^j$ is rapidly decaying near $\partial M$ (since the same is true of $\gamma$), the BNH parts of $\delta \gamma^j$ and $d\delta \gamma^j$ are also rapidly decaying near $\partial M$. Hence $\delta \gamma^j$ and $d\delta \gamma^j$ are smooth and rapidly decaying near $\partial M$, and they therefore must lie in $x^wL^2$ for any $w$. So if $q = -a - 1$, $p = 2$, $q' = -1$, and $p' = a + 2$, then $\gamma^j \in x^qL^2$ (as $q < -a$), $\delta \gamma^j \in x^pL^2$, $d\delta \gamma^j \in x^{p'}L^2$, and $\delta \gamma^j \in x^{q'}L^2$. Since all forms involved are smooth, and $p + q + a = p + q' = p' + q = 1 > 0$, Lemma 5.5 justifies the transition $\langle d\delta \gamma^j, \gamma^j \rangle = \langle \delta \gamma^j, \delta \gamma^j \rangle$.

The remaining case is $j = \frac{n+1+a/b}{2}$. Equations (5.6) and (5.7) and the condition (5.16) then imply $\gamma_T^j = x^b \phi^j$ and $\gamma_{\gamma}^{j-1} = x^{(n-j)}b \ln(x) \alpha^{j-1} + x^{(n-j)}b \psi^{j-1}$. As $\delta$ acts on the BH part of a form by (3.2), the BH part of $\delta \gamma^j$ in $U_\epsilon$ will be

$$\begin{pmatrix} 0 & -x^{a+1} \frac{\partial}{\partial x} + (n-j)b x^a \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x^b \phi^j \\ x^{(n-j)}b \ln(x) \alpha^{j-1} + x^{(n-j)}b \psi^{j-1} \end{pmatrix} = \begin{pmatrix} -x^{a+(n-j)}b \alpha^{j-1} \\ 0 \end{pmatrix}$$

Since $j = \frac{n+1+a/b}{2}$, $a + (n-j)b = (j-1)b$. So the BH part of $d\gamma^j$ in $U_\epsilon$ is
This is exactly the tangential part of the BH part of $\omega_0^{j-1}$ in $U_\epsilon$. But since $\omega_0$ is square-integrable on $M$, $\omega_0^{j-1}$ must be $L^2$ on $U_\epsilon$. So the BH part of $\omega_0^{j-1}$, and hence also the tangential part of that BH part, must be square-integrable inside $U_\epsilon$. That is the same as saying that $-x^{(j-1)b}\alpha^{j-1}$ belongs to $L^2((0, \epsilon), \mathcal{H}^{k-1}(\partial M), \frac{dx}{x^{a+1+b(n-1)}})$. But this last condition is equivalent to

$$
\int_0^\epsilon x^{2\frac{(n+1+a/b-1)b}{2}}\|\alpha^{j-1}\|^2_{L^2} \frac{dx}{x^{a+1+b(n-1)}} < \infty \iff \int_0^\epsilon \|\alpha^{j-1}\|^2_{L^2} \frac{dx}{x} < \infty
$$

which is only possible as long as $\alpha^{j-1}$ is zero. We conclude that the BH part of $\gamma^j$ inside $U_\epsilon$ is zero. The rest of the argument is identical to the case $j = \frac{n-1-a/b}{2}$. The BNH parts of $\delta\gamma^j$ and $d\delta\gamma^j$ will be rapidly decaying near $\partial M$, because the same is true of the BNH part of $\gamma^j$. Since the BH parts of $\delta\gamma^j$ and $d\delta\gamma^j$ vanish inside $U_\epsilon$, these forms themselves are rapidly decaying near $\partial M$. They must therefore lie in $x^w L^2$ for any real $w$. We now set $q = -a - 1$, $p = 2$, $q' = -1$ and $p' = a + 2$ and observe that $\gamma^j \in x^q L^2$ (since $q < -a$), $\delta\gamma^j \in x^p L^2$, $d\delta\gamma^j \in x^{p'} L^2$, and $\delta\gamma^j \in x^{q'} L^2$.

Since all these forms are smooth and $p + q' = q + p' = p + q + a = 1 > 0$, the transition $\langle d\delta\gamma^j, \gamma^j \rangle = \langle \delta\gamma^j, \delta\gamma^j \rangle$ is justified by Lemma 5.5.

We have proved that $\langle \gamma^j, d\delta\gamma^j \rangle = \langle \delta\gamma^j, d\gamma^j \rangle$ for each $j$ between 0 and $n$. Consequently, $\langle \gamma, d\delta\gamma \rangle = \langle \delta\gamma, \delta\gamma \rangle$, and since $d\delta\gamma = d(\omega - \omega_0 - d\gamma) = 0$ this means $\delta\gamma = 0$. Thus $\omega = \omega_0 + d\gamma$. But since $\omega$ has degree $k$, it is equal to the $k$-th degree term of $\omega_0 + d\gamma$, hence $\omega = \omega_0^k + d\gamma^{k-1}$. We have seen at the beginning of this proof that $\omega_0^k$ is harmonic. We also know that $\gamma^{k-1} \in H^{\infty, -a-s, \infty}(\Omega^{k-1}(M))$, because $\gamma$ is an element of $H^{\infty, -a-s, \infty}(\Omega(M))$. Moreover, we have deduced the expression for $\gamma^{k-1}$.

Since $k < \frac{n-1+a/b}{2}$, it follows trivially that $k - 1 < \frac{n+1+a/b}{2}$. So the only cases we need to consider are $k - 1 < \frac{n-1-a/b}{2}$, $k - 1 = \frac{n-1-a/b}{2}$, and $\frac{n-1-a/b}{2} < k - 1 < \frac{n+1+a/b}{2}$. According to the expressions (5.7) and (5.6) for $\gamma^{k-1}$, and the conditions (5.16) and
(5.17) which make some of the terms vanish, we have

\[ \gamma^{k-1} = \begin{cases} 
-x^{(n-k)b-a}\beta^{k-1} + \frac{1}{x^{(n-2k+1)b-a}} + \sigma_T & \text{if } k < \frac{n+1-a/b}{2} \\
x^{(n-k+1)b}\psi^{k-2} + \sigma_C & \end{cases} \]

\[ \gamma^{k-1} = \begin{cases} 
x^{(k-1)b}\phi^{k-1} + \sigma_T & \text{if } \frac{n+1-a/b}{2} < k < \frac{n-1+a/b}{2} \\
x^{(n-k+1)b}\psi^{k-2} + \sigma_C & \end{cases} \]

in the neighborhood \( U_\epsilon \cong (0,\epsilon) \times \partial M \). Here \( \beta^{k-1}, \phi^{k-1} \) and \( \psi^{k-2} \) are harmonic forms on \( \partial M \), and \( \left( \begin{array}{c} \sigma_T \\ \sigma_C \end{array} \right) \) is the non-harmonic part of \( \gamma^{k-1} \) in \( U \), rapidly decaying near \( \partial M \) because \( \gamma^{k-1} \in H^{\infty,-a,\infty} \). All that is left to get the statement of the lemma for \( k \neq \frac{n+1-a/b}{2} \) is to rename \( \frac{-\beta^{k-1}}{x^{(n-2k+1)b-a}} \) to \( \phi^{k-1} \) in the first case.

If \( k = \frac{n+1-a/b}{2} \), we only need to show that \( \beta^{k-1} = 0 \). But \( \left( \begin{array}{c} 0 \\ x^{(n-k)b}\beta^{k-1} \end{array} \right) \) is the conormal part of the BH part of \( \omega_0^k \). Since \( \omega_0^k \) is square integrable on \( M \), \( \left( \begin{array}{c} 0 \\ x^{(n-k)b}\beta^{k-1} \end{array} \right) \) must be square integrable on \( U_\epsilon \). Hence \( x^{(n-k)b}\beta^{k-1} \) must be an element of \( L^2((0,\epsilon),\mathcal{H}^{k-1}(\partial M),\frac{dx}{x^{a+1+b(n-1)}}) \). This is only possible if

\[ \int_0^\epsilon x^{2(n+\frac{1+a/b}{2})b}||\beta^{k-1}||^2_{L^2} \frac{dx}{x^{a+1+b(n-1)}} < \infty \Leftrightarrow \int_0^\epsilon ||\beta^{k-1}||^2_{L^2} \frac{dx}{x} < \infty \]

(we are using the fact that \( k \) is equal to \( \frac{n+1-a/b}{2} \). Hence \( \beta^{k-1} \) must be equal to zero, and this proves the lemma for the exceptional case \( k = \frac{n+1-a/b}{2} \).

Lemma 5.14 makes it extremely easy to show that the map from \( \mathcal{H}^k(M) \) to the cohomology of the complex \( A^k \) is surjective.

**Lemma 5.15** If \( k < \frac{n+1-a/b}{2} \), then any element \( \omega \in A^k \) can be written as \( \omega = \omega_0^k + d\gamma^{k-1} \) for some \( \omega_0^k \in \mathcal{H}^k(M), \gamma^{k-1} \in A^{k-1} \).

**Proof.** Since the inclusion of \( C^\infty_c\Omega^k(M) \) into \( A^k \) induces an isomorphism in coho-
mology, there exists a form in the cohomology class of $\omega$ that is compactly supported inside $M$. By replacing $\omega$ with that form, we can assume without loss of generality that $\omega$ itself has compact support in $M$. If $\frac{n-1+a/b}{2} \notin \{0, 1, \ldots, n-1\}$, we can use Lemma 5.12 to express $\omega$ as $\omega_0 + D\gamma$, where $\omega_0 \in \mathcal{H}(M)$ and $\gamma$ is an element of $H^{\infty, -a, \infty}$. Since $H^{\infty, -a, \infty} \subset H^{\infty, -a-s, \infty}$ for any positive $s$, $\gamma$ will then lie in $H^{\infty, -a-s, \infty}$ for the number $s$ from the statement of Lemma 5.14. On the other hand, if $\frac{n-1+a/b}{2} \in \{0, 1, \ldots, n-1\}$, then let $w$ be a small positive number such that $w < r$ and $w < s$, where $r$ is the bound from Lemma 5.13 and $s$ is the constant from the statement of Lemma 5.14. Then, according to Lemma 5.13, $\omega$ can be written as $\omega = \omega_0 + D\gamma$ for $\omega_0 \in \mathcal{H}(M)$ and $\gamma \in H^{\infty, -a-w, \infty}$. But since $H^{\infty, -a-w, \infty} \subset H^{\infty, -a-s, \infty}$, in this case $\gamma$ will also belong to $H^{\infty, -a-s, \infty}$. In either case the decomposition $\omega = \omega_0 + d\gamma$ satisfies the assumptions of Lemma 5.14. According to that lemma, $\omega$ can also be written as $\omega = \omega^k + d\gamma^{k-1}$, where $\omega^k \in \mathcal{H}^k(M)$ and $\gamma^{k-1}$ is an element of $H^{\infty, -a-s, \infty}$ which satisfies

$$
\gamma = \left( \begin{array}{c}
x^{(n-k)b-a} \phi^{k-1} + \sigma_T \\
x^{(n-k+1)b} \psi^{k-2} + \sigma_C
\end{array} \right)
$$

in a small neighborhood $U_\varepsilon \cong (0, \varepsilon) \times \partial M$ of the boundary of $M$. In this expression $\phi^{k-1} \in \mathcal{H}^{k-1}(\partial M)$, $\psi^{k-2} \in \mathcal{H}^{k-2}(\partial M)$, $\sigma_T \in \dot{C}^\infty((0, \varepsilon), C^\infty \Omega^{k-1}(\partial M))$, and $\sigma_C \in \dot{C}^\infty((0, \varepsilon), C^\infty \Omega^{k-2}(\partial M))$. It remains to check that $\gamma$ is an element of $A^{k-1}$. It certainly is smooth inside $M$, because it belongs to $H^{\infty, -a-s, \infty}$. So we need to check that its tangential part vanishes inside $M$, and its conormal part grows at most polynomially near $\partial M$. Both conditions will be true if they are true of the restriction of $\gamma$ to $U_\varepsilon$. The conormal part of $\gamma$ clearly cannot grow faster than polynomially near $\partial M$, because

$$
\left( \begin{array}{c}
x^{(n-k)b-a} \phi^{k-1} \\
0
\end{array} \right) = \frac{x^{(n-k)b-a} \phi^{k-1}}{x^{(k-1)b}} = x^{(n-2k+1)b-a} \phi^{k-1} \to 0 \text{ as } x \to 0
$$

and the tangential part of $\gamma$ is vanishing at $\partial M$, because $\sigma_T$ is rapidly decaying near $\partial M$ and

$$
\left( \begin{array}{c}
x^{n-kb-a} \phi^{k-1} \\
0
\end{array} \right) = \frac{x^{(n-k)b-a} \phi^{k-1}}{x^{(k-1)b}} = x^{(n-2k+1)b-a} \phi^{k-1} \to 0 \text{ as } x \to 0
$$

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because $\phi^{k-1}$ does not depend on $x$ and

$$(n - 2k + 1)b - a > 0 \iff k < \frac{n + 1 - a/b}{2}$$

which is the case we are considering. So $\gamma$ is indeed an element of $A^{k-1}$.

Lemma 5.8 shows that there is a well-defined natural map from $\mathcal{H}^k(M)$ to the cohomology of the chain complex $A^k$ (because $B^k \subset A^k$). By Lemma 5.10, that natural map is injective. And Lemma 5.15 proves that the map is surjective. Taken together, those lemmas prove the following part of Theorem 1.1:

**Theorem 5.16** Assume that $a$ and $b$ are such that $\frac{n - 1 + a/b}{2} \notin \{0, 1, \ldots, n - 1\}$. Then, for every $k < \frac{n + 1 - a/b}{2}$, $\mathcal{H}^k(M) \cong H^k(M, \partial M)$.

This result immediately implies its counterpart by Poincare duality:

**Theorem 5.17** Let $a$ and $b$ satisfy $\frac{n - 1 + a/b}{2} \notin \{0, 1, \ldots, n - 1\}$. Then, for every $k > \frac{n + 1 - a/b}{2}$, $\mathcal{H}^k(M) \cong H^k(M)$.

We remark that it is also possible to prove Theorem 5.17 without using Poincare duality, by showing that the natural map from $\mathcal{H}^k(M)$ to the cohomology of $E^k$ is injective and surjective. That would again involve justifying the appropriate integrations by parts.

### 5.3 Case $\frac{n + 1 - a/b}{2} \leq k < \frac{n - 1 + a/b}{2}$.

We now aim to prove that for $\frac{n + 1 - a/b}{2} < k < \frac{n - 1 + a/b}{2}$, $\mathcal{H}^k(M)$ is isomorphic to the image of $H^k(M, \partial M)$ under its natural inclusion in $H^k(M)$. This will be relatively easy, because most of the work has already been done in the previous section. Just as before, the key to the proof is to choose the right models for relative and absolute cohomologies.

We will define $H^k(M)$ as the cohomology of the chain complex $E^*$, where $E^k$ is defined as

$$\{\omega \in C^\infty \Omega^k(\text{Int}(M)) : \omega|_U = \omega_1 + dx \wedge \omega_2; \ \omega_1, \omega_2 \in C^\infty((0, 1), \Omega(\partial M))\},$$

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∀x ∈ (0, 1), p ∈ ∂M \|ω_1(x, p)\|_g ≤ Cx^l, \\
\|d'ω_1(x, p)\|_g ≤ Cx^l, \|ω_2(x, p)\|_g ≤ Cx^l, \|d'ω_2(x, p)\|_g ≤ Cx^l\}
where $\omega_0^k \in \mathcal{H}^k(M)$ and $\gamma^{k-1}$ is an element of $H^{\infty,-a-s,\infty}$ which satisfies

$$\gamma = \left(\begin{array}{c} x^{(k-1)b}\phi^{k-1} + \sigma_T \\ x^{(n-k+1)b}\psi^{k-2} + \sigma_C \end{array}\right)$$

in a small neighborhood $U_\epsilon \cong (0, \epsilon) \times \partial M$ of the boundary of $M$. In the above expression $\phi^{k-1}, \psi^{k-2} \in \mathcal{H}(\partial M)$, and $\sigma_T, \sigma_C \in \dot{C}^{\infty}((0, \epsilon), C^\infty\Omega(\partial M))$. It is clear that $\gamma|_{U_\epsilon}$ will grow at most polynomially near $\partial M$ if the same is true of $\gamma|_{U_\epsilon}$. But $\gamma|_{U_\epsilon}$ definitely does not grow faster than polynomially near $\partial M$, since $\sigma_T$ and $\sigma_C$ are rapidly decaying near $\partial M$, and $\phi^{k-1}$ and $\psi^{k-2}$ do not depend on $x$. The same is true of $d'$ of the tangential and conormal part of $\gamma$, because powers of $x$ commute through $d'$. Since $\gamma$ is also smooth (being an element of $H^{\infty,-a,\infty}$), it lies in $E^{k-1}$ by definition, and this completes the proof. 

So any cohomology class of $E^k$ which lies in the image of the cohomology of $B^k$ must contain a harmonic element. Hence the map of $\mathcal{H}^k(M)$ to the image of the inclusion of $H^k(M, \partial M)$ into $H^k(M)$ is surjective. It remains to show that this map is injective. For this we cannot draw an analogy with the case $k < \frac{n+1-a/b}{2}$; we will need another argument that uses integration by parts.

**Lemma 5.19** Let $k \in \left[\frac{n+1-a/b}{2}, \frac{n-1+a/b}{2}\right)$, and let $\omega \in \mathcal{H}^k(M)$ be a harmonic form that satisfies $\omega = d\sigma$ for an element $\sigma \in E^{k-1}$. Then $\omega = 0$.

**Proof.** This is similar to Lemma 5.10, though not implied by it. We again seek to justify the transition $\langle d\sigma, \sigma \rangle = \langle d\sigma, d\sigma \rangle$. Once that is done, we will be able to write

$$\delta \omega = 0 \Rightarrow \delta d\sigma = 0 \Rightarrow \langle d\sigma, d\sigma \rangle = \langle \sigma, \delta d\sigma \rangle = 0 \Rightarrow \omega = d\sigma = 0$$

To justify the transition $\langle d\sigma, \sigma \rangle = \langle d\sigma, d\sigma \rangle$, we will use Lemma 5.5. Assume that $\sigma|_U = \sigma_1 + dx \wedge \sigma_2$, where $\sigma_1, \sigma_2 \in C^\infty((0, 1), C^\infty\Omega(\partial M))$. By the definition of $E^k$, $\sigma \in E^{k-1}$ implies that there exist an $l \in \mathbb{Z}$ and a $C \in \mathbb{R}$ such that $\|\sigma_1(x, p)\|_g \leq Cx^l$. 

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and \( \| \sigma_2(x, p) \|_g \leq C x^l \) for all \( x \in (0, 1), p \in \partial M \). Therefore

\[
\int_U \| x^{-w} \sigma_1 \|^2_g dVol_g \leq \int_{\partial M} \int_0^1 C^2 x^{-2w+2l} \frac{dx}{x^{a+1+b(n-1)}} dVol_h
\]

and this quantity will be finite if

\[
-2w + 2l - a - 1 - b(n - 1) > -1 \Leftrightarrow w < l - \frac{b(n - 1) + a}{2}
\]

It is likewise easy to see that

\[
\int_U \| x^{-w} dx \wedge \sigma_2 \|^2_g dVol_g = \int_U \left\| \frac{dx}{x^{a+1-w}} \wedge \sigma_2 \right\|^2_g dVol_g \leq \int_{\partial M} \int_0^1 C^2 x^{2a+2-2w+2l} \frac{dx}{x^{a+1+b(n-1)}} dVol_h
\]

where the last integral converges if

\[
a + 1 - 2w + 2l - b(n - 1) > -1 \Leftrightarrow w < l - \frac{b(n - 1) - a - 2}{2}
\]

Since \( 2l - b(n - 1) - a < 2l - b(n - 1) + a + 2 \), we see that both the tangential and the conormal part of \( x^{-w} \sigma|_U \) will be square integrable for any \( w < l - \frac{b(n - 1) + a}{2} \). Since \( \sigma \) is smooth on \( M \), this means \( \sigma \in x^p L^2 \Omega^{k-1}(M) \) for any \( p < l - \frac{b(n - 1) + a}{2} \). Choose any real \( p \) satisfying this inequality, and let \( q = -p - a + 1, q' = -p + 1 \) and \( p' = p + a \). Because \( \omega \in \mathcal{H}^k(M) \) and \( k \in \left[ \frac{n+1-a/b}{2}, \frac{n-1+a/b}{2} \right) \), \( \omega \) must be smooth and rapidly decaying near \( \partial M \) by Corollary 5.3. A rapidly decaying form lies in \( x^w L^2 \) for any weight \( w \), so we have \( d\sigma = \omega \in x^p L^2 \Omega^k(M) \) and \( d\sigma \in x^q L^2 \Omega^k(M) \). Finally, since \( \delta d\sigma = \delta \omega = 0 \) by Lemma 5.9, \( \delta d\sigma \) trivially lies in \( x^s L^2 \Omega^{k-1}(M) \). Because \( \sigma, d\sigma, \) and \( \delta d\sigma \) are all smooth (\( \sigma \) being an element of \( E^{k-1} \)), and because \( p + q' = p' + q = p + q + a = 1 > 0 \), Lemma 5.5 justifies the transition \( \langle d\sigma, d\sigma \rangle = \langle \sigma, \delta d\sigma \rangle \). As \( \delta d\sigma = 0 \), we have \( \omega = d\sigma = 0 \). □

So the natural map of \( \mathcal{H}^k(M) \) into the image of the cohomology of \( B^k \) in the cohomology of \( E^k \) is surjective by Lemma 5.18, and injective by Lemma 5.19. We have thus proved another part of Theorem 1.1:
Theorem 5.20 $H^k(M)$ is isomorphic to the image of $H^k(M, \partial M)$ under the inclusion in $H^k(M)$ for all $k \in \left[\frac{n+1-a/b}{2}, \frac{n-1+a/b}{2}\right]$.

If $\frac{n-1+a/b}{2}$ is an integer, it will satisfy $\frac{n-1+a/b}{2} = n - \frac{n+1-a/b}{2}$. Poincare duality then implies that the statement of last theorem extends to $k = \frac{n-1+a/b}{2}$. Combining it with Theorem 5.16 and Theorem 5.17, we get the statement of Theorem 1.3.
Bibliography


