Robust Transportation Network Design Under User Equilibrium

by

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Abstract

We address the problem of designing a transportation network in the presence of demand uncertainty, multiple origin-destination pairs and a budget constraint for the overall construction cost, under the behavioral assumption that travelers optimize their own travel costs (i.e., the "user-equilibrium" condition). Under deterministic demand, we propose an exact integer optimization approach that leads to a quadratic objective, linear constraints optimization problem. As a result, the problem is efficiently solvable via commercial software, when the costs are linear functions of traffic flows. We then use an iterative algorithm to address the case of nonlinear cost functions. While the problem is intractable under probabilistic assumptions on demand uncertainty, we extend the previous model and propose an iterative algorithm using a robust optimization approach that models demand uncertainty. We finally report extensive numerical results to illustrate that our approach leads to tractable solutions for large scale networks.

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Chapter 1

Introduction

The network design problem (NDP) arises in many applications as diverse as telecommunication networks, logistics, transportation and production planning. It has both theoretical and practical implications and has therefore drawn considerable attention over the past fifty years. For a comprehensive discussion of the NDP and its applications, we refer the reader to the survey paper by Magnanti and Wong [31]. In this thesis, we consider the NDP in the area of transportation, which is an important sector for most economies. According to a report by the U.S. Department of Transportation, all levels of government spent $147.5 billion for highways in 2004 [50]. This reveals that advancement of research in this area will help improve the strategic planning and investment of transportation networks as well as benefiting the public as a whole.

We consider an uncapacitated transportation network with multiple origin-destination (OD) pairs and a budget constraint for the overall construction cost. While previous work studies the system-optimal routing of traffic flows and assumes that travelers act in a cooperative manner, we focus on the setting where travelers behave in a selfish, noncooperative fashion, routing flow so as to minimize their own travel cost and disregarding the effect of their decisions on others. This concept is defined by Wardrop [52] as the "user equilibrium" condition. Under user equilibrium, any path carrying a strictly positive amount of flow between a given OD pair must be a minimum-cost path for that OD pair. Indeed, in urban traffic systems, observed flows

are likely to be closer to a user than a system optimum [18], as a result of lacking a central directing authority. Not surprisingly, the total travel cost is not minimized by the user equilibrium [44, 26]. Therefore, in this research we are interested in designing a network under the user-equilibrium behavioral assumption for travelers, while keeping the total travel cost as small as possible. In addition, this work addresses demand uncertainty, which is a reasonable modeling assumption in real-world applications, as it is difficult to estimate the exact value of the current and future demand.

The NDP is a difficult problem to solve as its general form is NP-hard (for example, see [31]). Even simplified, special cases of the problem are difficult to solve. Johnson et al. [25] proved that the uncapacitated budget design problem, the version of the NDP we will focus on, is also NP-hard. Moreover, the NDP under user equilibrium is an instance of a mathematical programming problem with equilibrium constraints (MPEC). This formulation is in turn a generalization of the bilevel optimization problem. This increased difficulty is due, in part, to the phenomenon illustrated by Braess's Paradox [12], introduced by Murchland [36] to the transportation community. In Braess's Paradox, the removal of an edge from a network results in an equilibrium flow solution whose total system cost is strictly less than that for the original network. It demonstrates that providing greater choice to selfish users can actually have a deleterious effect on the total system cost.

1.1 Literature Review

1.1.1 The User Equilibrium Problem

The objective and constraints of an NDP are defined, in part, by the criterion of how to measure the network performance. In a traffic network with congestion effects, it is more realistic to study the user equilibrium problem rather than the system-optimal routing problem.

For a thorough review of the user equilibrium problem and the corresponding solution methods, we refer the user to the books by Nagurney [38] and Patriksson [43],

as well as the survey paper by Florian and Hearn [21], Magnanti [30], and to the references therein. Recently, Correa et al. [15] extended the study to capacitated networks and gave the definition of the so-called "capacitated user equilibrium". Ordóñez and Stier-Moses [39] incorporated uncertainty in the utility functions and studied a robust user equilibrium problem.

In addition, since the user equilibrium problem is an instance of the variational inequality (VI) problem, VI solution methods can be applied to the user equilibrium problem [16]. The book by Facchinei and Pang [20], the monograph by Patriksson [42], the review article by Harker and Pang [24] and the Ph.D thesis of Hammond [23] provide insightful reviews of the VI problem and associated algorithms. Marcotte and Wynter [34] showed that the multiclass network equilibrium problem may satisfy a weaker property induced by the hierarchical nature of the travel cost interactions and proposed a decomposition algorithm.

Recently, Aghassi et al. [1] used duality and reformulated the asymmetric variational inequality (VI) problem over a polyhedron as a single-level (and many-times differentiable) optimization problem. They provided sufficient conditions for the convexity of this reformulation and therefore suggested a possible approach to reformulate the user equilibrium problem as an equivalent optimization problem.

1.1.2 The NDP Under User Equilibrium and Demand Uncertainty

While the NDP is a mature area in integer optimization, the analysis of instances under user equilibrium or demand uncertainty, though very relevant in practice, is not fully developed. The NDP community has generally focused on the NDP involving centrally controlled, system-optimal routing. It has furthermore focused on problem instances involving known, deterministic demands.

The following literature provides a representative, though not comprehensive, sample of research in NDP under user equilibrium. LeBlanc and Boyce [27] and Marcotte [32, 33], have employed solution algorithms specialized to solving bilevel

optimization problems. Roughgarden [46] drew on ideas from the literature of price of anarchy and gave optimal inapproximability results for the NDP under user equilibrium, arc construction cost of zero and separable and nondecreasing arc usage costs. He considered the objective of minimizing the total system arc usage cost at equilibrium. Essentially, Roughgarden showed that unless $P = NP$, in this setting, there is no better approximation algorithm for this NDP than the "algorithm" of building the entire network.

Rather than addressing user equilibrium conditions, others have assumed system-optimal routing but have accounted for demand uncertainty in the network. Riis and Andersen [45], Lisser et al. [29], Andrade et al. [2], and Waller and Ziliaskopoulos [51], to name a few, have modeled demand uncertainty via probability distributions and have proposed solution approaches based on stochastic programming. In contrast, others have addressed demand uncertainty deterministically using the robust optimization approach. For instance, Chekuri et al. [13], Ordóñez and Zhao [40] and Atamturk and Zhang [3] presented robust optimization models of a version of the NDP, under demand uncertainty, and in which the network planner seeks to determine edge capacities for a given set of edges. Chekuri et al. [13] considered the minimum-cost capacity allocation problem that can accommodate any realization from the uncertain set of demands. Ordóñez and Zhao [40] treated the capacity allocation costs via a design budget and sought to minimize linear routing costs. Atamturk and Zhang [3] considered an objective arising from the sum of capacity allocation and linear routing costs. Finally, Mudchanatongsuk et al. [35] addressed a classic multi-commodity network design problem under transportation costs and demand uncertainty.

However, research in this area is far from complete. For instance, in the literature, robust optimization and other deterministic treatments of the problem have modeled the cost per unit flow on an arc as constant and consequently the total system routing cost as linear functions of the flow variables. In this case, they have ignored the possibility of congestion effects, which more realistically captures conditions inherent in real-world transportation systems. Furthermore, the robust optimization analysis

of the NDP under demand uncertainty has focused on the capacity allocation problem, rather than the binary choice, arc construction problem. Finally, to the best of our knowledge, no one has yet addressed the network design problem under both demand uncertainty and user equilibrium conditions, from a robust optimization standpoint.

1.2 Contributions of the Thesis

The main contributions of the thesis include:

1. We first propose an exact integer optimization formulation for the NDP under user equilibrium and deterministic demand. In the case of linear cost functions, our model leads to a quadratic objective, linear constraints, mixed-integer optimization problem and therefore can be efficiently solved by commercial optimization software. We further propose an iterative algorithm for the case of nonlinear cost functions.

2. We consider a robust optimization approach to the deterministic model, in order to address demand uncertainty in the network. We propose a robust model for the NDP under user equilibrium and polyhedral demand uncertainty. We then consider an iterative algorithm for solving the robust NDP.

1.3 Structure of the Thesis

This thesis is structured as follows: In Chapter 2, we develop an exact integer optimization reformulation for the NDP with deterministic demand, based on the optimization reformulation of the variational inequality problem. We show that when the costs are linear functions of traffic flows, our model leads to a quadratic objective, linear constraints, mixed-integer optimization problem. In Section 2.4, we extend the work to the case of nonlinear cost functions. In Chapter 3, we study the NDP with polyhedral demand uncertainty. We apply robust optimization ideas to extend the deterministic model and propose an iterative algorithm to solve the case with demand uncertainty. In Chapter 4, we investigate several examples of different sizes and report numerical results for both deterministic and uncertain demand cases. Computational

results show that our formulations are efficiently solvable via commercial software. Finally we conclude with a summary in Chapter 5.

Chapter 2

Network Design under Deterministic Demand

2.1 Notation

We consider a network represented by a directed graph $G(V, A)$, where V and A are the sets of nodes and arcs of the network, respectively. In addition, we denote OD pair d_w as the amount of flow to be routed from the origin node s_w to the destination node t_w and the set of all OD pairs as W. For each OD pair $w \in W$, we define a vector $\mathbf{d}^w \in \mathbb{R}^{|V|}$ as

$$d_v^w = \begin{cases} d_w & \text{if } v = s_w \\ -d_w & \text{if } v = t_w \\ 0 & \text{otherwise,} \end{cases} \qquad \forall v \in V, \forall w \in W. \tag{2.1}$$

We denote $\mathbf{f} \in \mathbb{R}^{|A|}$ as the vector of flows on the arcs and f_a^w as the amount of flow on arc a serving OD pair w. For each OD pair w, we group all f_a^w to form the vector $\mathbf{f}^w \in \mathbb{R}^{|V|}$. It follows that

$$f_a = \sum_{w \in W} f_a^w, \qquad \forall a \in A. \tag{2.2}$$

We also define P_w as the set of paths connecting OD pair $w \in W$ and P as the set of all paths. Then $\mathbf{F}$, the vector of path flows, satisfies the following relation with the vector of arc flows $\mathbf{f}$:

$$f_a = \sum_{p \in P | a \in p} F_p, \qquad \forall a \in A. \tag{2.3}$$

We then denote $\mathbf{c}(\mathbf{f})$ as the vector function mapping a vector of arc flows $\mathbf{f}$ to the vector of arc costs per unit flow. We also define

$$c_a^w(f_a) = c_a(f_a). \tag{2.4}$$

Similarly, we denote $\mathbf{C}(\mathbf{F})$ as the vector function mapping a vector of path flows $\mathbf{F}$ to the vector of path costs per unit flow. Then the quantity $\mathbf{c}(\mathbf{f})'\mathbf{f}$ is defined as the *total system cost* of the arc flow vector $\mathbf{f}$ and $\mathbf{C}(\mathbf{F})'\mathbf{F}$ as the *total system cost* of the path flow vector $\mathbf{F}$. For any pair of consistent arc and path flow vectors $\mathbf{f}$ and $\mathbf{F}$ satisfying equation (2.3),

$$C_p(\mathbf{F}) = \sum_{a \in p} c_a(\mathbf{f}), \qquad \forall p \in P. \tag{2.5}$$

Therefore we can show that

$$\mathbf{c}(\mathbf{f})'\mathbf{f} = \mathbf{C}(\mathbf{F})'\mathbf{F}. \tag{2.6}$$

We finally introduce $\mathbf{y} \in \{0,1\}^{|A|}$, a vector of binary decision variables, for the arcs and denote $y_a = 1$ if arc a is chosen to be built, and $y_a = 0$ otherwise.

2.2 User Equilibrium via Optimization

In a given network, user equilibrium is reached when any path carrying strictly positive amount of flow between a given OD pair is a minimum-cost path for that OD pair. We first present the definitions of the set of feasible arc flows K_A and the set of

feasible path flows K_P.

Definition 1. *In a given network $G(V, A)$, an arc flow $\mathbf{f}$ resides in the set K_A of feasible flows, if $\mathbf{f}$ satisfies the following conditions:*

$$\sum_{i \in V} f_{vi}^w - \sum_{j \in V} f_{jv}^w = d_v^w, \qquad \forall v \in V, \forall w \in W, \tag{2.7}$$
$$f_{ij}^w \geq 0, \qquad \forall (i,j) \in A, \forall w \in W.$$

Similarly, a path flow $\mathbf{F}$ resides in the set K_P of feasible flows, if $\mathbf{F}$ satisfies

$$\sum_{p \in P_w} F_p = d_w, \qquad \forall w \in W, \tag{2.8}$$
$$F_p \geq 0, \qquad \forall p \in P.$$

Definition 2. *In a given network $G(V, A)$ with path cost vector $\mathbf{C}$, a path flow vector $\mathbf{F} \in K_P$ is a user equilibrium flow vector if for each OD pair $w \in W$ and any two paths p_1, p_2 connecting the origin s_w and the destination t_w of w,*

$$F_{p_1} \geq 0 \qquad \Longrightarrow \qquad C_{p_1}(\mathbf{F}) \leq C_{p_2}(\mathbf{F}). \tag{2.9}$$

Similarly, an arc flow vector $\mathbf{f}$ is a user equilibrium for $G(V, A)$ with arc cost vector $\mathbf{c}$, if its corresponding path flow vector satisfies equation (2.3) and (2.9).

In what follows we state a general result for the existence and uniqueness of user equilibrium.

Definition 3. *A vector function $\mathbf{c} : X \to \mathbb{R}^n$ is said to be **monotone** on $X \subseteq \mathbb{R}^n$ if*

$$[\mathbf{c}(\mathbf{x_1}) - \mathbf{c}(\mathbf{x_2})]'(\mathbf{x_1} - \mathbf{x_2}) \geq 0$$

*holds for $\forall \mathbf{x_1}, \mathbf{x_2} \in X$ and $\mathbf{x_1} \neq \mathbf{x_2}$. Furthermore, if the inequality is strict, then the vector function $\mathbf{c}$ is **strictly monotone**.*

Theorem 1 (see, e.g., [21]). *Consider a network $G(V, A)$ with continuous cost func-*

tions $\mathbf{c}$. *There exists a user equilibrium on* $G(V, A)$. *Moreover, under strict monotonicity of the vector cost function* $\mathbf{c}$, *the user equilibrium is unique.*

In the literature, it has been shown that the user equilibrium condition is equivalent to a variational inequality (VI) problem [16].

Definition 4. *Given a set* $K \in \mathbb{R}^n$ *and a mapping* $\mathscr{F} : K \to \mathbb{R}^n$, *the variational inequality (VI) problem, denoted as* $VI(K, \mathscr{F})$, *seeks an* $\mathbf{x}^* \in K$ *such that*

$$\mathscr{F}(\mathbf{x}^*)'(\mathbf{x} - \mathbf{x}^*) \geq 0, \forall \mathbf{x} \in K. \tag{2.10}$$

Theorem 2 ([48, 16]). *Consider a network* $G(V, A)$ *with arc cost vector* $\mathbf{c}$. *Denote the polyhedron set for the feasible arc flow vectors as* K_A; *then the user equilibrium problem is equivalent to the VI problem that seeks an optimal vector* $\mathbf{f}^* \in K_A$ *such that* $\mathbf{c}(\mathbf{f}^*)'(\mathbf{f} - \mathbf{f}^*) \geq 0, \forall \mathbf{f} \in K_A$.

Aghassi et al. [1] applied a duality-based proof method and reformulated the VI problem over a polyhedron as a single-level optimization problem. They also stated that this formulation applies even if the associated cost function has a asymmetric Jacobian matrix.

Theorem 3 ([1]). *Given a non-empty polyhedron* K *in standard form:*

$$K = \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{A}\mathbf{x} = \mathbf{b}, \mathbf{x} \geq 0\} \neq \emptyset, \tag{2.11}$$

where $\mathbf{A} \in \mathbb{R}^{m \times n}$ *and* $\mathbf{b} \in \mathbb{R}^m$. *Then* $\mathbf{x}^*$ *solves* $VI(K, \mathscr{F})$ *if and only if the following problem has an optimal value of zero, and there exists* $\boldsymbol{\lambda}^* \in R^m$ *such that* $(\mathbf{x}^*, \boldsymbol{\lambda}^*)$ *is an optimal solution:*

$$\begin{aligned} \min_{\mathbf{x}, \boldsymbol{\lambda}} \quad & \mathscr{F}(\mathbf{x})'\mathbf{x} - \mathbf{b}'\boldsymbol{\lambda}, \\ s.t. \quad & \mathbf{A}\mathbf{x} = \mathbf{b}, \\ & \mathbf{x} \geq 0, \\ & \mathbf{A}'\boldsymbol{\lambda} \leq \mathscr{F}(\mathbf{x}). \end{aligned} \tag{2.12}$$

Also note that $\mathscr{F}(\mathbf{x})'\mathbf{x} - \mathbf{b}'\boldsymbol{\lambda} \geq 0$ always holds due to weak duality.

Aghassi et al. [1] further gave a sufficient condition for the convexity of this reformulation in Theorem 4 and the next Corollary 1 discusses the convexity for the affine case.

Theorem 4 ([1]). *Suppose that K is the non-empty polyhedron given by (2.11). If $\mathscr{F}_j(\mathbf{x})$ is a concave function over K, $\forall j \in \{1, \ldots, n\}$, and $\mathscr{F}(\mathbf{x})'\mathbf{x}$ is a convex function over K, then the problem (2.12) defines a convex optimization problem.*

Corollary 1 ([1]). *Suppose that K is the non-empty polyhedron given by (2.11) and that $\mathscr{F}(\mathbf{x}) = \mathbf{G}\mathbf{x} + \mathbf{h}$, with $\mathbf{G} \succeq 0$, but not necessarily symmetric. Then the problem (2.12) is a convex linear constraints, quadratic programming (LCQP) problem.*

Based on Theorem 2 and Theorem 3, we prove a corollary that reformulates the user equilibrium problem as an equivalent optimization problem. We use the VI in the arc formulation rather than the path formulation, because the number of paths will grow exponentially when the size of the network increases.

Corollary 2. *Consider a network $G(V, A)$ with arc cost vector $\mathbf{c}$, then an arc flow vector $\mathbf{f}$ is said to be a user equilibrium if and only if there exits a vector $\boldsymbol{\lambda} \in \mathbb{R}^{|w||V|\times 1}$ such that the following optimization problem has an optimal value of zero:*

$$
\begin{aligned}
\min_{\mathbf{f^w},\boldsymbol{\lambda}} \quad & \sum_{(i,j)\in A}\sum_{w\in W} c_{ij}^w(f_{ij}^w)f_{ij}^w - \sum_{v\in V}\sum_{w\in W} d_v^w \lambda_v^w, \\
s.t. \quad & \sum_{i\in V} f_{vi}^w - \sum_{j\in V} f_{jv}^w = d_v^w \qquad \forall v \in V, \forall w \in W \\
& \lambda_i^w - \lambda_j^w \leq c_{ij}^w\left(f_{ij}^w\right), \qquad \forall (i,j) \in A, \forall w \in W \\
& f_{ij}^w \geq 0. \qquad \forall (i,j) \in A, \forall w \in W
\end{aligned}
\tag{2.13}
$$

Proof. Based on Theorem 2, the VI formulation of the user equilibrium problem in a network with multiple OD pairs can be represented as the following VI in the arc

formulation:

$$\mathbf{c}(\mathbf{f}^*)'(\mathbf{f}-\mathbf{f}^*) \geq 0, \qquad \forall \mathbf{f} \in K_A \tag{2.14}$$

$$\sum_{(i,j)\in A} c_{ij}(f_{ij}^*)(f_{ij}-f_{ij}^*) \geq 0, \qquad \forall \mathbf{f} \in K_A \tag{2.15}$$

$$\sum_{(i,j)\in A} c_{ij}(f_{ij}^*) \sum_{w\in W} (f_{ij}^w - f_{ij}^{w*}) \geq 0. \qquad \forall \mathbf{f} \in K_A \tag{2.16}$$

Using equation (2.4), it follows that

$$\sum_{(i,j)\in A} \sum_{w\in W} c_{ij}^w(f_{ij}^{w*})(f_{ij}^w - f_{ij}^{w*}) \geq 0. \qquad \forall \mathbf{f}^{\mathbf{w}} \in K_A \tag{2.17}$$

Directly applying Theorem 2, we reach Corollary 2. □

2.3 A Mixed Integer Optimization Model

The NDP under user equilibrium is an instance of an MPEC, which is a generalization of a bilevel programming problem. The objective of the problem is to minimize the total system cost $\mathbf{c}(\mathbf{f})'\mathbf{f}$ while following the condition that the arc flow vector $\mathbf{f}$ is a user equilibrium. Based on the single-level optimization reformulation of the user equilibrium condition from Corollary 2, we propose the following mixed-integer optimization model to address this problem.

Theorem 5. *Given a network $G(V, A)$ with arc cost vector $\mathbf{c}$, the NDP under user equilibrium is equivalent to find the optimal arc flow vector $\mathbf{f}$ (and its corresponding $\mathbf{f}^{\mathbf{w}}$) and the optimal binary decision vector $\mathbf{y}$ to the following optimization problem*

with a large enough θ:

$$
\begin{aligned}
\min_{\mathbf{f^w},\boldsymbol{\lambda},\mathbf{y}} \quad & \sum_{(i,j)\in A}\sum_{w\in W} c_{ij}^w(f_{ij}^w)f_{ij}^w + \theta\left(\sum_{(i,j)\in A}\sum_{w\in W} c_{ij}^w(f_{ij}^w)f_{ij}^w - \sum_{v\in V}\sum_{w\in W} d_v^w\lambda_v^w\right), \\
s.t. \quad & \sum_{i\in V} f_{vi}^w - \sum_{j\in V} f_{jv}^w = d_v^w, \qquad \forall v\in V, \forall w\in W \\
& \lambda_i^w - \lambda_j^w \le c_{ij}^w\left(f_{ij}^w\right) + M(1-y_{ij}), \qquad \forall (i,j)\in A, \forall w\in W \\
& \sum_{w\in W} f_{ij}^w \le K y_{ij}, \qquad \forall (i,j)\in A, \\
& f_{ij}^w \ge 0, \qquad \forall (i,j)\in A, \forall w\in W \\
& \sum_{(i,j)\in A} b_{ij}y_{ij} \le B, \\
& y_{ij}\in\{0,1\}. \qquad \forall (i,j)\in A.
\end{aligned}
\tag{2.18}
$$

Note that b_{ij} is the arc construction cost for arc (i,j), B is the budget for the overall construction cost and both M and K are large enough numbers.

We can further rewrite Model (2.18) in a vector form as follows:

$$
\begin{aligned}
\min_{\mathbf{f^w},\boldsymbol{\lambda},\mathbf{y}} \quad & \sum_{w\in W} \mathbf{c}^w(\mathbf{f}^w)'\mathbf{f}^w + \theta\left(\sum_{w\in W} \mathbf{c}^w(\mathbf{f}^w)'\mathbf{f}^w - \sum_{w\in W} \mathbf{d}^{w\prime}\boldsymbol{\lambda}_w\right) \\
s.t. \quad & \mathbf{N}\mathbf{f}^w = \mathbf{d}^w, \qquad \forall w\in W \\
& \mathbf{N}'\boldsymbol{\lambda}_w \le \mathbf{c}^w(\mathbf{f}^w) + M(\mathbf{1}-\mathbf{y}), \qquad \forall w\in W \\
& \sum_{w\in W} \mathbf{f}^w \le K\mathbf{y}, \\
& \mathbf{f}^w \ge 0, \qquad \forall w\in W \\
& \mathbf{b}'\mathbf{y} \le B, \\
& \mathbf{y}\in\{0,1\}^{|A|},
\end{aligned}
\tag{2.19}
$$

where $\mathbf{N}$ is the network's arc incidence matrix.

Proof. Based on Corollary 2, we can formulate the NDP under user equilibrium as a bilevel optimization problem:

$$
\begin{aligned}
\min_{\mathbf{f^w},\boldsymbol{\lambda},\mathbf{y}} \quad & \sum_{(i,j)\in A}\sum_{w\in W} c_{ij}^w(f_{ij}^w)f_{ij}^w \\
& \text{s.t.} \\
& \min_{\mathbf{f^w},\boldsymbol{\lambda}} \sum_{(i,j)\in A}\sum_{w\in W} c_{ij}^w(f_{ij}^w)f_{ij}^w - \sum_{v\in V}\sum_{w\in W} d_v^w\lambda_v^w, \qquad (2.20) \\
& \text{s.t.} \quad \sum_{i\in V} f_{vi}^w - \sum_{j\in V} f_{jv}^w = d_v^w \qquad \forall v\in V, \forall w\in W \\
& \qquad \lambda_i^w - \lambda_j^w \le c_{ij}^w\left(f_{ij}^w\right), \qquad \forall (i,j)\in A, \forall w\in W \\
& \qquad f_{ij}^w \ge 0, \qquad \forall (i,j)\in A, \forall w\in W \\
& \sum_{w\in W} f_{ij}^w \le K y_{ij}, \qquad \forall (i,j)\in A, \\
& \sum_{(i,j)\in A} b_{ij}y_{ij} \le B, \\
& y_{ij}\in\{0,1\} \qquad \forall (i,j)\in A.
\end{aligned}
$$

From Theorem 3, it follows that the objective in the lower level optimization problem is nonnegative, i.e.,

$$
\sum_{(i,j)\in A}\sum_{w\in W} c_{ij}^w(f_{ij}^w)f_{ij}^w - \sum_{v\in V}\sum_{w\in W} d_v^w\lambda_v^w \ge 0. \qquad (2.21)
$$

Therefore we can introduce a penalty parameter $\theta > 0$ and reformulate model (2.20) as a single-level optimization problem (2.18) with the objective

$$
\sum_{(i,j)\in A}\sum_{w\in W} c_{ij}^w(f_{ij}^w)f_{ij}^w + \theta\left(\sum_{(i,j)\in A}\sum_{w\in W} c_{ij}^w(f_{ij}^w)f_{ij}^w - \sum_{v\in V}\sum_{w\in W} d_v^w\lambda_v^w\right). \qquad (2.22)
$$

We introduce the term $M(1-y_{ij})$ in the second constraint in order to ensure that this constraint only applies when $y_{ij}=1$ and only then that arc (i,j) is built.

We now prove that as $\theta_k \to \infty$, model (2.18) correctly solves the NDP under user

equilibrium. Let

$$\mathbf{x} = (\mathbf{f^w}, \boldsymbol{\lambda}, \mathbf{y}) \tag{2.23}$$

and

$$g(\mathbf{x}) = \sum_{(i,j)\in A} \sum_{w\in W} c_{ij}^w(f_{ij}^w) f_{ij}^w, \tag{2.24}$$

$$h(\mathbf{x}) = \sum_{(i,j)\in A} \sum_{w\in W} c_{ij}^w(f_{ij}^w) f_{ij}^w - \sum_{v\in V} \sum_{w\in W} d_v^w \lambda_v^w, \tag{2.25}$$

$$p(\mathbf{x}, \theta) = g(\mathbf{x}) + \theta h(\mathbf{x}). \tag{2.26}$$

We also denote the feasible set of model (2.18) as K. We then consider the following penalty problem:

$$\begin{aligned} P(\theta) : \min_{\mathbf{x}} \quad & p(\mathbf{x}, \theta) \\ \text{s.t.} \quad & \mathbf{x} \in K \end{aligned} \tag{2.27}$$

for an increasing sequence of constants θ as $\theta_k \to \infty$.

Let $\theta_k > 0, k = 1, \ldots, \infty$, be a sequence of penalty parameters that satisfies $\theta_{k+1} > \theta_k$ for all k and $\lim_{k\to\infty} \theta_k = +\infty$. Let $\mathbf{x}^k$ be an optimal solution to the problem $P(\theta_k)$.

We have

$$g(\mathbf{x}^k) + \theta_k h(\mathbf{x}^k) \leq g(\mathbf{x}^{k+1}) + \theta_k h(\mathbf{x}^{k+1}) \tag{2.28}$$

and

$$g(\mathbf{x}^{k+1}) + \theta_{k+1} h(\mathbf{x}^{k+1}) \leq g(\mathbf{x}^k) + \theta_{k+1} h(\mathbf{x}^k). \tag{2.29}$$

Thus

$$(\theta^{k+1} - \theta^k) h(\mathbf{x}^k) \geq (\theta^{k+1} - \theta^k) h(\mathbf{x}^{k+1}), \tag{2.30}$$

whereby

$$h(\mathbf{x}^k) \geq h(\mathbf{x}^{k+1}). \tag{2.31}$$

Since $\{h(\mathbf{x}^k)\}$ is a nonincreasing sequence and $h(\mathbf{x}^k) \geq 0$ from (2.21), we prove that $\lim_{k\to\infty} h(\mathbf{x}^k) = 0$. We use a contradiction argument. If $\lim_{k\to\infty} h(\mathbf{x}^k) > 0$, then $\lim_{k\to\infty} p(\mathbf{x}^k, \theta^k) = +\infty$, as $\lim_{k\to\infty} g(\mathbf{x}^k) \geq 0, \lim_{k\to\infty} h(\mathbf{x}^k) > 0$ and $\lim_{k\to\infty} \theta_k = +\infty$. On the other hand, for any k, we can get a solution $\tilde{\mathbf{x}}^k = (\tilde{\mathbf{f}}^{\mathbf{w}k}, \tilde{\boldsymbol{\lambda}}^k, \tilde{\mathbf{y}}^k)$ to problem (2.20) by first specifying a decision vector $\tilde{\mathbf{y}}^k$ that satisfies the budget constraint $\sum_{(i,j)\in A} b_{ij}\tilde{y}^k_{ij} \leq B$ and then solving the user equilibrium problem (2.13) on network $G(V, A(\tilde{\mathbf{y}}^k))$, where $A(\tilde{\mathbf{y}}^k)$ indicates the network constructed by vector $\tilde{\mathbf{y}}^k$,. According to Theorem 1, on the network $G(V, A(\tilde{\mathbf{y}}^k))$, a user equilibrium exists. By Corollary 2, it follows that $\lim_{k\to\infty} h(\tilde{\mathbf{x}}^k) = 0$. Consequently the objective in problem (2.18) satisfies $\lim_{k\to\infty} p(\tilde{\mathbf{x}}^k, \theta^k) = \lim_{k\to\infty} g(\tilde{\mathbf{x}}^k) < +\infty$. So $\tilde{\mathbf{x}}^k$ yields lower objective and improves the solution to problem $P(\theta^k)$ from solution $\mathbf{x}^k$. We reach a contradiction. Therefore, $\lim_{k\to\infty} h(\mathbf{x}^k) = 0$.

As parameter θ takes a large enough positive value, term $h(\mathbf{x})$ will be forced to become zero and the user equilibrium condition is reached in Corollary 2. Therefore the objective function in problem (2.18) will become $g(\mathbf{x})$, which is exactly the total system cost. As a result, we have illustrated that the mixed integer programming formulation we proposed minimizes the total system cost and solves the NDP under user equilibrium. □

By Theorem 4, if the cost function $\mathbf{c}(\mathbf{f})$ is a concave function and $\mathbf{c}(\mathbf{f})'\mathbf{f}$ is a convex function, then problem (2.18) is a convex programming problem with integer variables. In the model in this thesis, we consider congestion effects in the network. If we model the cost function $\mathbf{c}(\mathbf{f})$ as an affine function of the arc flow vector $\mathbf{f}$:

$$\mathbf{c}(\mathbf{f}) = \mathbf{G}\mathbf{f} + \mathbf{H}, \tag{2.32}$$

where $\mathbf{G} \in \mathbb{R}^{|A|\times|A|}$ and $\mathbf{H} \in \mathbb{R}^{|A|\times 1}$, then the objective in (2.18) becomes a quadratic

function of vector $\mathbf{f}$. Therefore, the model we propose has a quadratic objective with linear constraints and is a mixed-integer optimization problem. If $\mathbf{G}$ is further a positive semidefinite matirx, i.e., $\mathbf{G} \succeq 0$, then the problem is also convex by Corollary 1.

We next make some remarks here on the choices of the parameters θ, K and M in formulation (2.18).

Remarks:

(1) The best bound of the scalar θ that guarantees the term (2.22) to converge can be found by performing a binary search method.

(2) For the parameter K, if we have n OD pairs with demand $d_{w_1}, \ldots, d_{w_n}$, then the flow f_{ij} is bounded by

$$f_{ij} \leq \sum_{m=1}^{n} d_{w_m}. \tag{2.33}$$

Therefore, a reasonable bound for K is

$$K = \sum_{m=1}^{n} d_{w_m}. \tag{2.34}$$

(3) If the cost function $\mathbf{c}(\mathbf{f}) = \mathbf{G}\mathbf{f} + \mathbf{H}$ is monotone (i.e., $\mathbf{G} \succeq 0$), for each arc (i, j), we can set

$$M_{ij} = c_{ij}\left(\sum_{m=1}^{n} d_{w_m}\right) - c_{ij}(0). \tag{2.35}$$

2.4 Nonlinear Cost Functions

Arc cost functions used in practice are usually modeled as smooth functions [14, 47]. In general, they have the nonlinear form instead of the linear model we used in Section 2.3. In this section, we first discuss a general iterative scheme for VI problem. We then propose an iterative algorithm to address the case of nonlinear cost functions.

2.4.1 A General Iterative Scheme for the Solution of VI Problem

We consider the VI defined in (2.10) and assume that $\mathscr{F}$ is a continuous function and K is a closed, convex set. We consider a smooth function $g(\mathbf{x}, \mathbf{x}') : K \times K \to \mathbb{R}^n$ that satisfies

$$g(\mathbf{x}, \mathbf{x}) = \mathscr{F}(\mathbf{x}), \qquad \forall \mathbf{x} \in K. \tag{2.36}$$

We also assume that for any fixed point $\mathbf{x}'$, the VI

$$g(\mathbf{x}^*, \mathbf{x}')^T(\mathbf{x} - \mathbf{x}^*) \geq 0, \qquad \forall \mathbf{x} \in K, \tag{2.37}$$

has a unique solution $\mathbf{x}^*$.

We then construct a sequence $\{\mathbf{x}^i\}$ by the following procedure. Let $\mathbf{x}^0$ be an arbitrary point in the set K. At the i-th step, find the optimal solution $\mathbf{x}^i$ of (2.37) with $\mathbf{x}' = \mathbf{x}^{i-1}$. We then show under appropriate assumptions, (2.37) admits a unique solution, which is the limit point of the sequence $\{\mathbf{x}^i\}$ as $i \to +\infty$.

Theorem 6 ([17]). *Assume that the function g satisfies the strong monotonicity property:*

$$[g(\mathbf{x}_1, \mathbf{x}') - g(\mathbf{x}_2, \mathbf{x}')]^T(\mathbf{x}_1 - \mathbf{x}_2) \geq \alpha|\mathbf{x}_1 - \mathbf{x}_2|^2, \qquad \forall \mathbf{x}^1, \mathbf{x}^2, \mathbf{x}' \in K \tag{2.38}$$

where $\alpha > 0$ is a constant. Further, there is $0 < k < 1$ such that

$$sup\|\frac{\partial g(\mathbf{x}, \mathbf{x}')}{\partial \mathbf{x}'}\| \leq k\alpha, \qquad \forall \mathbf{x}, \mathbf{x}' \in K \tag{2.39}$$

Then there is a unique solution $\mathbf{x}^*$ to (2.37) and for any choice of the initial point $\mathbf{x}^0$, the sequence $\{\mathbf{x}^i\}$ converges to the limit point $\mathbf{x}^*$, as $i \to +\infty$.

Proof. We have

$$g(\mathbf{x}^{i+1}, \mathbf{x}^i)^T(\mathbf{x} - \mathbf{x}^{i+1}) \geq 0, \qquad \forall \mathbf{x} \in K \tag{2.40}$$

$$g(\mathbf{x}^*, \mathbf{x}^*)^T(\mathbf{x} - \mathbf{x}^*) \geq 0, \qquad \forall \mathbf{x} \in K. \tag{2.41}$$

Let $\mathbf{x} = \mathbf{x}^*$ in (2.40) and $\mathbf{x} = \mathbf{x}^{i+1}$ in (2.41). Therefore,

$$g(\mathbf{x}^{i+1}, \mathbf{x}^i)^T(\mathbf{x}^* - \mathbf{x}^{i+1}) \geq 0, \tag{2.42}$$

$$g(\mathbf{x}^*, \mathbf{x}^*)^T(\mathbf{x}^{i+1} - \mathbf{x}^*) \geq 0. \tag{2.43}$$

(2.42)+(2.43):

$$\left(g(\mathbf{x}^{i+1}, \mathbf{x}^i) - g(\mathbf{x}^*, \mathbf{x}^*)\right)^T (\mathbf{x}^* - \mathbf{x}^{i+1}) \geq 0, \tag{2.44}$$

$$\left(g(\mathbf{x}^{i+1}, \mathbf{x}^i) - g(\mathbf{x}^*, \mathbf{x}^i) + g(\mathbf{x}^*, \mathbf{x}^i) - g(\mathbf{x}^*, \mathbf{x}^*)\right)^T (\mathbf{x}^* - \mathbf{x}^{i+1}) \geq 0, \tag{2.45}$$

$$\begin{aligned} &\left(g(\mathbf{x}^*, \mathbf{x}^i) - g(\mathbf{x}^{i+1}, \mathbf{x}^i)\right)^T (\mathbf{x}^* - \mathbf{x}^{i+1}) \leq \\ &\qquad \left(g(\mathbf{x}^*, \mathbf{x}^i) - g(\mathbf{x}^*, \mathbf{x}^*)\right)^T (\mathbf{x}^* - \mathbf{x}^{i+1}) \end{aligned} \tag{2.46}$$

By strong monotonicity, we have

$$\left(g(\mathbf{x}^*, \mathbf{x}^i) - g(\mathbf{x}^{i+1}, \mathbf{x}^i)\right)^T (\mathbf{x}^* - \mathbf{x}^{i+1}) \geq \alpha \|\mathbf{x}^* - \mathbf{x}^{i+1}\|^2. \tag{2.47}$$

By mean value theorem, there exists $\bar{\mathbf{x}}$ such that

$$g(\mathbf{x}^*, \mathbf{x}^i) - g(\mathbf{x}^*, \mathbf{x}^*) = \frac{\partial g(\mathbf{x}^*, \bar{\mathbf{x}})}{\partial \bar{\mathbf{x}}}(\mathbf{x}^i - \mathbf{x}^*). \tag{2.48}$$

Further applying Cauchy-Schwarz inequality, we have

$$\begin{aligned} & \left(g(\mathbf{x}^*, \mathbf{x}^i) - g(\mathbf{x}^*, \mathbf{x}^*)\right)^T (\mathbf{x}^* - \mathbf{x}^{i+1}) \\ \leq \quad & \|g(\mathbf{x}^*, \mathbf{x}^i) - g(\mathbf{x}^*, \mathbf{x}^*)\| \cdot \|\mathbf{x}^* - \mathbf{x}^{i+1}\| \\ \leq \quad & \|\frac{\partial g(\mathbf{x}^*, \bar{\mathbf{x}})}{\partial \bar{\mathbf{x}}}\| \cdot \|\mathbf{x}^* - \mathbf{x}^i\| \cdot \|\mathbf{x}^* - \mathbf{x}^{i+1}\| \\ \leq \quad & k\alpha \|\mathbf{x}^* - \mathbf{x}^i\| \cdot \|\mathbf{x}^* - \mathbf{x}^{i+1}\| \end{aligned} \tag{2.49}$$

Combining (2.46), (2.47) and (2.49), we have

$$\|\mathbf{x}^* - \mathbf{x}^{i+1}\| \leq k\|\mathbf{x}^* - \mathbf{x}^i\| \tag{2.50}$$

$$\frac{\|\mathbf{x}^* - \mathbf{x}^{i+1}\|}{\|\mathbf{x}^* - \mathbf{x}^i\|} \leq k < 1 \tag{2.51}$$

Therefore, the sequence $\{\mathbf{x}^i\}$ converges to $\mathbf{x}^*$, as $i \to +\infty$. □

2.4.2 An Iterative Algorithm for the Case of Nonlinear Cost Functions

We rewrite the objective function in model (2.19) as

$$(1+\theta)\mathbf{c}(\mathbf{f}^{\mathbf{w}})'\mathbf{f}^{\mathbf{w}} - \theta\mathbf{d}^{\mathbf{w}\prime}\boldsymbol{\lambda}^{\mathbf{w}}. \tag{2.52}$$

Let

$$\mathbf{x} = (\mathbf{f}^{\mathbf{w}}, \boldsymbol{\lambda}^{\boldsymbol{w}}, \mathbf{y}) \tag{2.53}$$

$$F(\mathbf{x}) = \nabla_{(\mathbf{f}^{\mathbf{w}},\boldsymbol{\lambda}^{\boldsymbol{w}},\mathbf{y})}\left((1+\theta)\mathbf{c}(\mathbf{f}^{\mathbf{w}})'\mathbf{f}^{\mathbf{w}} - \theta\mathbf{d}'\boldsymbol{\lambda}^{\boldsymbol{w}}\right). \tag{2.54}$$

We also denote the feasible set of $\mathbf{x}$ as K, where K is defined by the constraints in model (2.19). We assume that $\mathbf{c}(\mathbf{f})$ is a concave function and $\mathbf{c}(\mathbf{f})'\mathbf{f}$ is a convex function. Therefore, the superset induced by relaxing the integer variables $\mathbf{y}$ in K is a convex set by Corollary 1. Since every feasible solution $\mathbf{x} = (\mathbf{f}^{\mathbf{w}}, \boldsymbol{\lambda}^{\boldsymbol{w}}, \mathbf{y})$ in K belongs to the superset, we use the first order optimality condition and state that this problem is equivalent to find the optimal solution $\mathbf{x}^*$ to the following VI problem:

$$F(\mathbf{x}^*)^T(\mathbf{x} - \mathbf{x}^*) \geq 0, \qquad \forall \mathbf{x} \in K. \tag{2.55}$$

We consider a projection method with the corresponding choice of g as

$$g(\mathbf{x}, \mathbf{x}_0) = F(\mathbf{x}_0) + \frac{1}{\rho}G(\mathbf{x} - \mathbf{x}_0), \tag{2.56}$$

where $\rho > 0$ and G is a fixed symmetric positive definite matrix.

Moreover, at a given flow value $\mathbf{f_0^w}$, the cost function $\mathbf{c(f^w)}$ can be linearized by performing a first-order Taylor expansion and discarding the high-order terms:

$$\mathbf{c(f^w)} = \mathbf{c(f_0^w)} + [\nabla_{\mathbf{f^w}}\mathbf{c(f_0^w)}]'(\mathbf{f^w} - \mathbf{f_0^w}). \tag{2.57}$$

Consequently, the nonlinear constraint in model (2.19) can be replaced by

$$\mathbf{N}'\boldsymbol{\lambda}_w \leq \mathbf{c}(\mathbf{f}_0^w) + [\nabla_{\mathbf{f}^w}\mathbf{c}(\mathbf{f}_0^w)]'(\mathbf{f}^w - \mathbf{f}_0^w) + M(\mathbf{1} - \mathbf{y}), \qquad \forall w \in W. \tag{2.58}$$

We then propose an iterative algorithm to solve the NDP under nonlinear cost functions.

Algorithm 1:

Step 1: Compute a feasible solution vector $\mathbf{x_0} = (\mathbf{f_0^w}, \boldsymbol{\lambda_0}^{\mathbf{w}}, \mathbf{y_0})$ as the initial value.

Step 2: In the k-th iteration, use $\mathbf{x}_{k-1}$ as the given value and replace the objective function in model (2.19) by the quadratic approximation and the nonlinear constraint by (2.58).

Step 3: Solve the new model to get $\mathbf{x_k} = (\mathbf{f_k^w}, \boldsymbol{\lambda_k}^{\mathbf{w}}, \mathbf{y_k})$. If $\|\mathbf{y}_k - \mathbf{y}_{k-1}\|_2 = 0$ and $\|\mathbf{f}_k^w - \mathbf{f}_{k-1}^w\|_2 \leq \epsilon$, where ϵ is the tolerance parameter, stop. Otherwise go to **Step 2**.

We show that under certain conditions, the sequence $\{\mathbf{x}_k\}$ generated by the algorithm converges to the optimal solution, as $k \to \infty$.

Chapter 3

Network Design under Polyhedral Demand Uncertainty

3.1 Review of Robust Optimization

Robust optimization addresses the problem of data uncertainty by guaranteeing feasibility and optimality of the solution for the worst instances of parameters. It does not treat the uncertain parameters as random variables with known distributions. Instead it takes a deterministic view and characterizes the parameter uncertainty by an uncertainty set in which all possible realizations of data reside.

Soyster [49] was the first to propose a model to handle columnwise uncertainty in the context of linear optimization, where every uncertain parameter is equal to its worst-case value in the set. This method protects each constraint against its worst case. Nevertheless, it produces very conservative results. Subsequent research efforts, led by Ben-Tal and Nemirovski [6, 7, 8], and El Ghaoui et al. [19], to address over-conservativeness, have applied robust optimization ideas to linear programming (LP) problems with ellipsoidal uncertainty sets, thus obtaining the robust counterpart of the nominal problem in the form of conic quadratic programs. Later Bertsimas and Sim [11] introduced an approach based on the so-called interval uncertainty set, that yields linear robust counterparts of linear programming problems. Their approach is appealing because of the tractability of their linear formulation and the possibility

of controlling the degree of conservativeness. However, it only applies to row-wise uncertainty. Bertsimas et al. [10] extend this framework by considering the problem:

$$\begin{aligned} \min_{\mathbf{x}} \quad & \mathbf{c}'\mathbf{x}, \\ \text{s.t.} \quad & \mathbf{A}\mathbf{x} \leq \mathbf{b}, \\ & \mathbf{x} \in P^x, \\ & \forall \mathbf{A} \in P^A, \end{aligned} \tag{3.1}$$

where

$$P^A = \{\mathbf{A} | \|\mathbf{\Sigma}^{-\frac{1}{2}}(vec(\mathbf{A}) - vec(\bar{\mathbf{A}}))\|_1 \leq \Gamma\}, \tag{3.2}$$

and $vec(\mathbf{A}) \in \mathbb{R}^{(m.n\times 1)}$ denotes the vector equivalent to the uncertain coefficient matrix $\mathbf{A} \in \mathbb{R}^{m\times n}$, obtained by stacking the rows of $\mathbf{A}$ on top of one another, and $\mathbf{\Sigma}$ representing the covariance matrix of the uncertain coefficient vector $\mathbf{A}$. $\bar{\mathbf{A}}$ is the nominal value of the uncertain coefficient vector, and Γ represents the budget of uncertainty for the whole uncertain coefficient matrix $\mathbf{A}$.

Polyhedral uncertainty sets allow the modeling of dependencies among uncertain coefficients across constraints. Bertsimas et al. [10] show that it is equivalent to the following linear programming problem:

$$\begin{aligned} \min_{\mathbf{x}} \quad & \mathbf{c}'\mathbf{x}, \\ \text{s.t.} \quad & (\mathbf{x_i})'vec(\mathbf{A}) + \Gamma\|\mathbf{\Sigma}^{-\frac{1}{2}}\mathbf{x_i}\|_\infty \leq b_i, \qquad \forall i \\ & \mathbf{x} \in P^x, \end{aligned} \tag{3.3}$$

where $\mathbf{x_i} \in \mathbb{R}^{(m.n\times 1)}$ contains $\mathbf{x}$ in entries $(i-1)n+1$ through in and zero everywhere else.

In addition to the regular robust optimization methodology, Ben-Tal et al. [5] also introduced the concept of adjustable robust optimization. They consider linear programs with uncertain parameters lying in some prescribed uncertainty set, where

part of the variables must be determined before the realization of the uncertain parameters ("non-adjustable" variables), while the other part are variables that can be chosen after the realization ("adjustable variables"). In particular, they consider an uncertain LP problem:

$$\left\{\min_{\mathbf{u},\mathbf{v}} \mathbf{c}'\mathbf{u} : \mathbf{U}\mathbf{u} + \mathbf{V}\mathbf{v} \leq \mathbf{b}\right\}_{\zeta=[\mathbf{U},\mathbf{V},\mathbf{b}]\in\mathcal{Z}}. \tag{3.4}$$

where $\mathbf{u}$ is the non-adjustable part of the solution, $\mathbf{v}$ is adjustable and $\mathcal{Z} \in \mathbb{R}^n \times \mathbb{R}^{m\times n} \times \mathbb{R}^m$ is an uncertainty set. Then the Adjustable Robust Counterpart (ARC) of this uncertain LP is defined as

$$\min_{\mathbf{u}} \left\{\mathbf{c}'\mathbf{u} : \forall(\zeta = [\mathbf{U},\mathbf{V},\mathbf{b}] \in \mathcal{Z}), \exists \mathbf{v} : \mathbf{U}\mathbf{u} + \mathbf{V}\mathbf{v} \leq \mathbf{b}\right\}. \tag{3.5}$$

Often the ARC approach is significantly less conservative than the usual Robust Counterpart approach. However, ARC is usually computationally intractable, as Gusliter [22] showed that the ARC of an LP with polyhedral uncertainty set is NP-hard. Ben-Tal et al. [5] proposed an approximate solution of the ARC by introducing the Affine Adjustable Robust Counterpart (AARC) and limiting the adjustable variables as affine functions of the uncertainty. For the LP problem (3.4), they assume that for $\mathbf{u}$ given, $\mathbf{v}$ is forced to be an affine function of the data:

$$\mathbf{v} = \mathbf{w} + \mathbf{W}\zeta. \tag{3.6}$$

Therefore, the AARC of the LP is defined as:

$$\min_{\mathbf{u},\mathbf{w},\mathbf{W}} \left\{\mathbf{c}'\mathbf{u} : \mathbf{U}\mathbf{u} + \mathbf{V}(\mathbf{w} + \mathbf{W}\zeta) \leq \mathbf{b}, \forall(\zeta = [\mathbf{U},\mathbf{V},\mathbf{b}] \in \mathcal{Z})\right\}. \tag{3.7}$$

In this case, this approximate problem is potentially tractable as it is in the form of the regular robust counterpart.

3.2 Robust Formulation

In this section, we give the robust formulation of the NDP under demand uncertainty. We consider the following polyhedral uncertainty set:

$$\mathcal{U} = \{\mathbf{d^w} |\ \|\mathbf{d^w} - \bar{\mathbf{d}}^{\mathbf{w}}\|_1 \leq \Gamma\}, \tag{3.8}$$

where $\bar{\mathbf{d}}^{\mathbf{w}}$ is the nominal value of the demand vector and Γ is the budget of uncertainty.

We adopt the concept of AARC introduced by Ben-Tal et al. [5] and consider the flow vector $\mathbf{f^w}$ as the vector of adjustable variables. Following equation (3.6), we force $\mathbf{f^w}$ to be an affine function of the uncertainty $\mathbf{d^w}$:

$$\mathbf{f}^w = \mathbf{Q}^w\mathbf{d}^w + \mathbf{p}^w, \qquad \forall w \in W \tag{3.9}$$

where $\mathbf{Q}^w \in \mathbb{R}^{|V|\times|V|}$ and $\mathbf{p}^w \in \mathbb{R}^{|V|\times 1}, \forall w \in W$.

If we consider the affine cost function as denoted by equation (2.32) and apply equation (3.9), then the objective in model (2.19) can be represented as a function $F(\mathbf{Q^w}, \mathbf{p^w}, \mathbf{d^w})$ as follows:

$$\begin{aligned} F(\mathbf{Q^w}, \mathbf{p^w}, \mathbf{d^w}) = \quad & (1+\theta)\sum_{w\in W}\left((\sum_{w\in W}\mathbf{G}(\mathbf{Q}^w\mathbf{d}^w + \mathbf{p}^w) + \mathbf{H})'(\mathbf{Q}^w\mathbf{d}^w + \mathbf{p}^w)\right) \\ & -\theta\sum_{w\in W}\mathbf{d}^{w\prime}\boldsymbol{\lambda}_w. \end{aligned} \tag{3.10}$$

So the NDP under user equilibrium and polyhedral demand uncertainty can be

formulated as:

$$
\begin{aligned}
\min_{\mathbf{Q^w},\mathbf{p^w},\boldsymbol{\lambda},\mathbf{y}} \quad & \max_{\mathbf{d^w}\in\mathcal{U}} \quad F(\mathbf{Q^w},\mathbf{p^w},\mathbf{d^w}) \\
\text{s.t.} \quad & (\mathbf{N}\mathbf{Q}^w - \mathbf{I})\mathbf{d}^w + \mathbf{p}^w = 0, \qquad \forall \mathbf{d}^w \in \mathcal{U}, \forall w \in W && (3.11)\\
& \mathbf{N}'\boldsymbol{\lambda}_w \leq \min_{\mathbf{d}^w\in\mathcal{U}} \left(\sum_{w\in W} \mathbf{G}(\mathbf{Q}^w\mathbf{d}^w + \mathbf{p}^w) + \mathbf{H} \right) + M(\mathbf{1}-\mathbf{y}), \\
& \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \forall w \in W && (3.12)\\
& \max_{\mathbf{d}^w\in\mathcal{U}} \sum_{w\in W} (\mathbf{Q}^w\mathbf{d}^w + \mathbf{p}^w) \leq K\mathbf{y}, && (3.13)\\
& \min_{\mathbf{d}^w\in\mathcal{U}} (\mathbf{Q}^w\mathbf{d}^w + \mathbf{p}^w) \geq 0, \qquad \forall w \in W && (3.14)\\
& \mathbf{b}'\mathbf{y} \leq B, && (3.15)\\
& \mathbf{y} \in \{0,1\}^{|A|}. && (3.16)
\end{aligned}
$$

In order for equation (3.11) to be satisfied for every $\mathbf{d^w}$ residing in the uncertainty set $\mathcal{U}$, we force

$$(\mathbf{N}\mathbf{Q}^w)_i = \mathbf{I}_i, \qquad \forall i = 1\ldots(|A|-1), \quad \forall w \in W, \tag{3.17}$$

and

$$\mathbf{N}\mathbf{p}^w = \mathbf{0}, \quad \forall w \in W. \tag{3.18}$$

In equation (3.17), i is listed from 1 to $(|A|-1)$ instead of 1 to $|A|$, because of the dependence of the $|A|$ rows of the matrix $\mathbf{N}\mathbf{Q}^w - \mathbf{I}$.

The robust counterpart of constraints (3.12), (3.13) and (3.14) can be obtained respectively, by following the robust optimization framework by Bertsimas et al. [10] (see (3.1), (3.2) and (3.3)). Therefore, the robust counterpart of model (2.19) can be

reformulated as follows:

$$
\begin{aligned}
\min_{\mathbf{Q^w},\mathbf{p^w},\boldsymbol{\lambda},\mathbf{y}} \quad & \max_{\mathbf{d^w}\in\mathcal{U}} \quad F(\mathbf{Q^w},\mathbf{p^w},\mathbf{d^w}) \\
\text{s.t.} \quad & (\mathbf{N}\mathbf{Q}^w)_i = \mathbf{I}_i, \qquad \forall i = 1\ldots(|A|-1), \quad \forall w \in W \\
& \mathbf{N}\mathbf{p}^w = \mathbf{0}, \quad \forall w \in W \\
& (\mathbf{N}'\boldsymbol{\lambda}_w)_i \leq \sum_{w\in W}\Big((\mathbf{G}\mathbf{Q}^w)_i vec(\bar{\mathbf{d}}^w) - \Gamma\|(\mathbf{G}\mathbf{Q}^w)_i\Sigma^{-\frac{1}{2}}\|_\infty + (\mathbf{G}\mathbf{p}^w)_i\Big) \\
& \qquad\qquad +\mathbf{H}_i + M(\mathbf{1}-y_i), \qquad\qquad \forall i = 1\ldots|A|, \ \forall w \in W \\
& \sum_{w\in W}\Big((\mathbf{Q}^w)_i vec(\bar{\mathbf{d}}) + \Gamma\|(\mathbf{Q}^w)_i\Sigma^{-\frac{1}{2}}\|_\infty + (\mathbf{p}^w)_i\Big) \leq Ky_i, \qquad \forall i = 1\ldots|A| \\
& (\mathbf{Q}^w)_i vec(\bar{\mathbf{d}}) - \Gamma\|(\mathbf{Q}^w)_i\Sigma^{-\frac{1}{2}}\|_\infty + (\mathbf{p}^w)_i \geq 0, \qquad \forall i = 1\ldots|A|, \quad \forall w \in W \\
& \mathbf{b}'\mathbf{y} \leq B, \\
& \mathbf{y} \in \{0,1\}^{|A|}.
\end{aligned}
\tag{3.19}
$$

3.3 An Iterative Algorithm for the Robust NDP

In this section, we propose an iterative algorithm to solve robust formulation (3.19). We first define the *master problem* and the *subproblem* that will be used in our algorithm.

The master problem is defined as:

$$
\begin{aligned}
\min_{z,\mathbf{Q^w},\mathbf{p^w},\boldsymbol{\lambda},\mathbf{y}} \quad & z \\
\text{s.t.} \quad & z \geq F(\mathbf{Q^w},\mathbf{p^w},\mathbf{d^w}), \\
& (\mathbf{N}\mathbf{Q}^w)_i = \mathbf{I}_i, \quad \forall i = 1\ldots(|A|-1), \quad \forall w \in W \\
& \mathbf{N}\mathbf{p}^w = \mathbf{0}, \quad \forall w \in W \\
& (\mathbf{N}'\boldsymbol{\lambda}_w)_i \leq \sum_{w\in W} \Big((\mathbf{G}\mathbf{Q}^w)_i vec(\bar{\mathbf{d}}^w) - \Gamma \|(\mathbf{G}\mathbf{Q}^w)_i \Sigma^{-\frac{1}{2}}\|_\infty + (\mathbf{G}\mathbf{p}^w)_i \Big) \\
& \qquad + \mathbf{H}_i + M(\mathbf{1} - y_i), \qquad \forall i = 1\ldots|A|, \ \forall w \in W \\
& \sum_{w\in W} \Big((\mathbf{Q}^w)_i vec(\bar{\mathbf{d}}) + \Gamma \|(\mathbf{Q}^w)_i \Sigma^{-\frac{1}{2}}\|_\infty + (\mathbf{p}^w)_i \Big) \leq K y_i, \qquad \forall i = 1\ldots|A| \\
& (\mathbf{Q}^w)_i vec(\bar{\mathbf{d}}) - \Gamma \|(\mathbf{Q}^w)_i \Sigma^{-\frac{1}{2}}\|_\infty + (\mathbf{p}^w)_i \geq 0, \qquad \forall i = 1\ldots|A|, \ \forall w \in W \\
& \mathbf{b}'\mathbf{y} \leq B, \\
& \mathbf{y} \in \{0,1\}^{|A|},
\end{aligned}
\tag{3.20}
$$

and the subproblem is defined as:

$$
\begin{aligned}
\max_{\mathbf{d^w}} \quad & F(\mathbf{Q^w},\mathbf{p^w},\mathbf{d^w}) \\
\text{s.t.} \quad & \|\mathbf{d^w} - \bar{\mathbf{d}}^{\mathbf{w}}\|_1 \leq \Gamma,
\end{aligned}
\tag{3.21}
$$

The iterative algorithm is described as follows:

Algorithm 2:

Step 1: Initialization. Let $\mathbf{d}_0^{\mathbf{w}} := \bar{\mathbf{d}}^{\mathbf{w}}$.

Step 2: In the i-th iteration, let $F(\mathbf{Q^w},\mathbf{p^w},\mathbf{d^w}) := F(\mathbf{Q^w},\mathbf{p^w},\mathbf{d}_{i-1}^{\mathbf{w}})$ and solve the master problem (3.20) to get the values of $w_i, \mathbf{Q}_i^{\mathbf{w}}, \mathbf{p}_i^{\mathbf{w}}, \boldsymbol{\lambda}_i, \mathbf{y}_i$.

Step 3: Let $F(\mathbf{Q^w},\mathbf{p^w},\mathbf{d^w}) := F(\mathbf{Q}_i^{\mathbf{w}},\mathbf{p}_i^{\mathbf{w}},\mathbf{d^w})$ and solve the subproblem (3.21) to get the values of $\mathbf{d}_i^{\mathbf{w}}$;

Step 4: If $F(\mathbf{Q}_i^{\mathbf{w}},\mathbf{p}_i^{\mathbf{w}},\mathbf{d}_i^{\mathbf{w}}) \leq z_i$, stop and $\mathbf{Q}_i^{\mathbf{w}}, \mathbf{p}_i^{\mathbf{w}}, \boldsymbol{\lambda}_i, \mathbf{y}_i$ is the optimal solution to the robust problem. Otherwise, add a constraint $z \geq F(\mathbf{Q^w},\mathbf{p^w},\mathbf{d}_i^{\mathbf{w}})$ to the master problem (3.20). Go to **Step 2**.

Our algorithm can be viewed as a variant of Bender's decomposition [9]. Bender's

decomposition is a popular technique in solving certain classes of difficult problems such as stochastic programming problems and mixed integer programming problems. It can be regarded as special cases of cutting-plane methods. Empirical results have shown that Bender's decomposition converge in few iterations. In the language of our algorithm, part of the explanation is that when we add a new constraint to the master problem, we quickly cut the "abundant" candidate set and move toward the optimal solution.

We then discuss the complexity of the subproblem If we choose the matrix $\mathbf{G}$ to be positive semidefinite, i.e., $\mathbf{G} \succeq 0$, the objective function in the subproblem (3.21) is a convex function. Therefore, we are maximizing a convex function over a convex feasible set. The maximum will be reached at the extreme points of the feasible set. Moreover, since the feasible set has a finite number of extreme points, we can simply do an enumeration over all the extreme points and then find the maximum value.

Chapter 4

Computational Results

The computational tests in this thesis are divided into two parts: Section 4.1 studies the case of deterministic demand while Section 4.2 studies the case of polyhedral demand uncertainty. Some of the examples are downloaded from the "Transportation Network Test Problems" website by Bar-Gera [4] and the "Test networks for the Asymmetric Network Equilibrium Problem" website by Passacantando [41]. The tested examples are coded in AMPL and use ILOG CPLEX 10.1 and LOQO as solvers.

4.1 Deterministic Demand

In this section, we present the computational results for different networks with deterministic demand. We choose problem with different size and cost functions and also investigate the effect of the parameters θ and B on both the solution and the computational time.

4.1.1 Example 1

The first example we present has 2 nodes and 5 arcs, as shown in Figure 4.1.

There is a single OD pair $w_1(1,2)$ and the demand $d_{w_1} = 10$. The construction cost b_a for each arc a is 1 and the overall budget B is 3. We consider both the cases

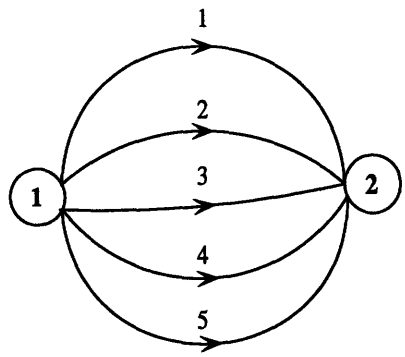

Figure 4.1: Example 1

of linear and nonlinear arc cost functions, as shown in Table 4.1.

Table 4.1: Arc cost functions (linear and nonlinear cases), Example 1

Cost	Linear	Nonlinear
$c_1(f_1) =$	$f_1 + 30$	$f_1^4 + 30$
$c_2(f_2) =$	$2f_2 + 30$	$2f_2^4 + 30$
$c_3(f_3) =$	$3f_3 + 30$	$3f_3^4 + 30$
$c_4(f_4) =$	$4f_4 + 20$	$4f_4^4 + 20$
$c_5(f_5) =$	$5f_5 + 20$	$5f_5^4 + 20$

For the linear case, we increase the value of θ from 0.1 to 100 and summarize the optimal objectives under different values of θ in Table 4.2. We notice that as the value of θ becomes large enough, the optimal cost converges to a fixed value and the NDP under user equilibrium is therefore solved. In order to find a best lower bound for θ, we perform a binary search method and find that $\theta = 0.805$ is the smallest value to guarantee convergence. For the nonlinear case, we set $\theta = 10$ and solve the NDP using the iterative algorithm described in Section 2.4.

For both the linear and nonlinear cost cases, the computation takes less than 1 second. The two cases reaches the same arc construction result: arcs 1, 4 and 5. The optimal flows for both cases are presented in Table 4.3. Since each arc can be viewed as a single path in this example, we can verify the user equilibrium condition by calculating the arc flow cost on each arc. For the linear case, the costs on arc 1, 4 and 5 are all 33.7931, while for the nonlinear case, the costs on these 3 arcs are all

338.1. By Definition 2, in the network we construct, each path is a minimum cost path connecting node 1 and node 2. Therefore, we have correctly solved the user equilibrium problem in the constructed network.

Table 4.2: Optimal objective with different values of θ (linear cost, Example 1)

θ	0.01	0.1	1	10	100
Objective	315.361	317.761	324.088	324.088	324.088

Table 4.3: Optimal flows for Example 1

flow	Linear Case	Nonlinear Case
f_1	3.7931	4.18957
f_2	0	0
f_3	0	0
f_4	3.44828	2.98623
f_5	2.75862	2.8242

4.1.2 Example 2

The second example we present is from [37], as shown in Figure 4.2. This network has 20 nodes, 28 arcs and 8 OD pairs. The demands for the OD pairs are summarized in Table 4.4. The cost functions are affine and non-separable, as shown in Table 4.5.

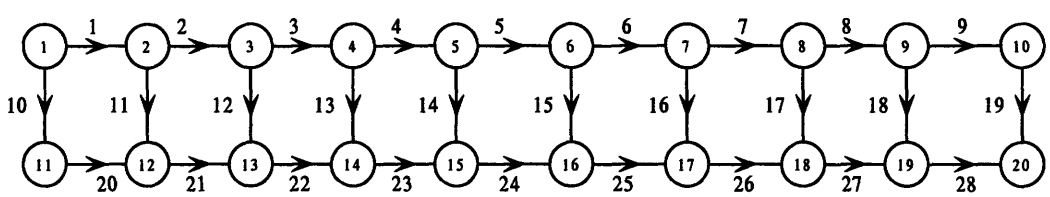

Figure 4.2: Example 2

The construction cost b_a for each arc a is 1. We let $B = 24, 20, 15, 12$ and increase the value of θ from 0.1 to 20 for each B. Figure 4.3 shows the constructed networks for different choices of the construction budget B and Table 4.6 presents the optimal flow values for each of the four constructed network. We also notice that if we take

Table 4.4: OD pairs and demands, Example 2

No.	OD Pair	Demand
w_1	(1,19)	60
w_2	(1,20)	50
w_3	(2,13)	100
w_4	(2,17)	100
w_5	(2,20)	100
w_6	(3,14)	50
w_7	(4,20)	100
w_8	(6,19)	100

Table 4.5: Arc cost functions, Example 2

$c_1(f_1) = 5f_1 + 2f_2 + 50$	$c_{15}(f_{15}) = 9f_{15} + 2f_{14} + 20$
$c_2(f_2) = 4f_2 + f_1 + 20$	$c_{16}(f_{16}) = 8f_{16} + 5f_{12} + 30$
$c_3(f_3) = 3f_3 + f_4 + 35$	$c_{17}(f_{17}) = 7f_{17} + 2f_{15} + 45$
$c_4(f_4) = 6f_4 + 3f_5 + 40$	$c_{18}(f_{18}) = 5f_{18} + f_{16} + 30$
$c_5(f_5) = 6f_5 + 4f_6 + 60$	$c_{19}(f_{19}) = 8f_{19} + 3f_{17} + 60$
$c_6(f_6) = 7f_6 + 3f_7 + 50$	$c_{20}(f_{20}) = 6f_{20} + f_{21} + 30$
$c_7(f_7) = 8f_7 + 2f_8 + 40$	$c_{21}(f_{21}) = 4f_{21} + f_{22} + 40$
$c_8(f_8) = 5f_8 + 2f_9 + 65$	$c_{22}(f_{22}) = 6f_{22} + f_{23} + 50$
$c_9(f_9) = 6f_9 + 2f_{10} + 70$	$c_{23}(f_{23}) = 9f_{23} + 2f_{24} + 35$
$c_{10}(f_{10}) = 4f_{10} + f_{12} + 80$	$c_{24}(f_{24}) = 8f_{24} + f_{25} + 40$
$c_{11}(f_{11}) = 7f_{11} + 4f_{12} + 65$	$c_{25}(f_{25}) = 9f_{25} + 3f_{26} + 45$
$c_{12}(f_{12}) = 8f_{12} + 2f_{13} + 70$	$c_{26}(f_{26}) = 7f_{26} + 8f_{27} + 30$
$c_{13}(f_{13}) = 7f_{13} + 3f_{18} + 60$	$c_{27}(f_{27}) = 8f_{27} + 3f_{28} + 50$
$c_{14}(f_{14}) = 8f_{14} + 3f_{15} + 50$	$c_{28}(f_{28}) = 7f_{28} + 3f_{27} + 65$

even smaller B's, i.e., $B < 12$, there will be no solution for the NDP problem, because in this case, we cannot even build a network that connects all the OD pairs, within small budget B.

We then investigate the effect of varying θ on the optimal objective and the computation time. Table 4.7 shows that as the value of θ becomes large enough, the optimal value converges. We also observe that though in theory the objective converges as $\theta \to \infty$, practical θ does not need to be very large. In this example, $\theta \geq 5$ is already large enough for all different B's. In the case $B = 20$ and $B = 12$, even $\theta = 0.1$ suffices. The computation time plot in Figure 4.4 further illustrates the benefit of a smaller θ. As the value of θ increases, the computation time increases in general. Under $B = 24$, the time increase significantly, as $\theta = 20$ takes 10 more times computation time than $\theta = 1$. Therefore, a good choice of θ is an important issue to take into account, as the choice of θ plays an important role not only in the convergence of the objective, but also in the computation time.

4.1.3 Example 3 - The Sioux Falls Network

The third example we present is the classic Sioux Falls network that is well known in the literature [28]. The Sioux Falls network has 24 nodes, 76 arcs, as shown in Figure 4.5. We consider a simplified version with 6 OD pairs and separable, linear arc cost functions. The OD pairs are summarized in Table 4.8 and the arc cost functions are shown in Table 4.9. We also set the construction cost $b_{ij} = 1$ for each arc (i, j).

For the Sioux Falls example, we first solve the user equilibrium problem in the whole network. We use the optimization reformulation in (2.13). We then solve the NDP under user equilibrium with the overall construction budget $B = 70$. The user equilibrium problem takes less than 1 second to solve and the NDP takes 15 minutes. This is because the NDP under user equilibrium is a mixed integer problem, which is more difficult to solve than the user equilibrium problem involving no integer variables. The optimal flows to the user equilibrium problem are shown in Table 4.10, while the optimal flows to the NDP under user equilibrium are shown in Table 4.11. In the NDP problem, the arcs that are not selected are arcs 30, 50, 51, 55, 70 and 72.

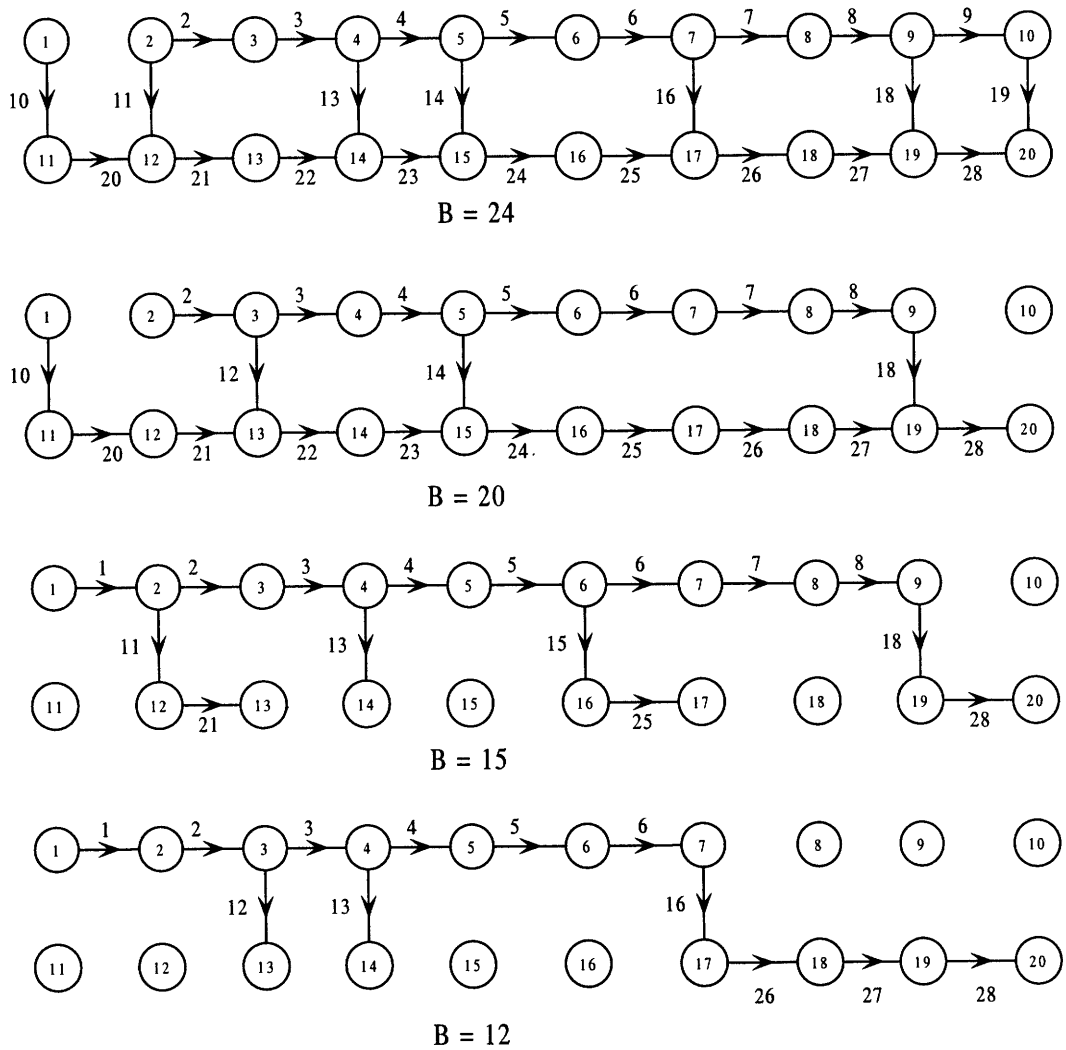

Figure 4.3: Constructed network for $B = 24, 20, 15, 12$, Example 2

Table 4.6: Optimal flow values for $B = 24, 20, 15, 12$, Example 2

flow	$B = 24$	$B = 20$	$B = 15$	$B = 12$
f_1	0	0	110	110
f_2	190.847	300	310	410
f_3	240.847	188.125	360	360
f_4	259.964	288.125	410	410
f_5	181.761	163.75	410	410
f_6	281.761	263.75	410	510
f_7	242.547	263.75	410	0
f_8	242.547	263.75	410	0
f_9	124.973	0	0	0
f_{10}	110	110	0	0
f_{11}	109.153	0	100	0
f_{12}	0	161.875	0	100
f_{13}	80.8826	0	50	50
f_{14}	78.2034	124.375	0	0
f_{15}	0	0	100	0
f_{16}	39.2144	0	0	510
f_{17}	0	0	0	0
f_{18}	117.573	263.75	410	0
f_{19}	124.973	0	0	0
f_{20}	110	110	0	0
f_{21}	219.153	110	100	0
f_{22}	119.153	171.875	0	0
f_{23}	150.036	121.875	0	0
f_{24}	228.239	246.25	0	0
f_{25}	228.239	246.25	100	0
f_{26}	167.453	146.25	0	410
f_{27}	167.453	146.25	0	410
f_{28}	125.027	250	250	250

Table 4.7: The effect of different values of θ on optimal objectives

θ	$B = 24$	$B = 20$	$B = 15$	$B = 12$
0.1	6425451.9	7050956.25	9635630	14112450
0.5	6427513.946	7050956.25	9709550	14112450
1	6427970.994	7050956.25	9801950	14112450
2	6427970.994	7050956.25	9986750	14112450
5	6427970.994	7050956.25	10304400	14112450
10	6427970.994	7050956.25	10304400	14112450
15	6427970.994	7050956.25	10304400	14112450
20	6427970.994	7050956.25	10304400	14112450

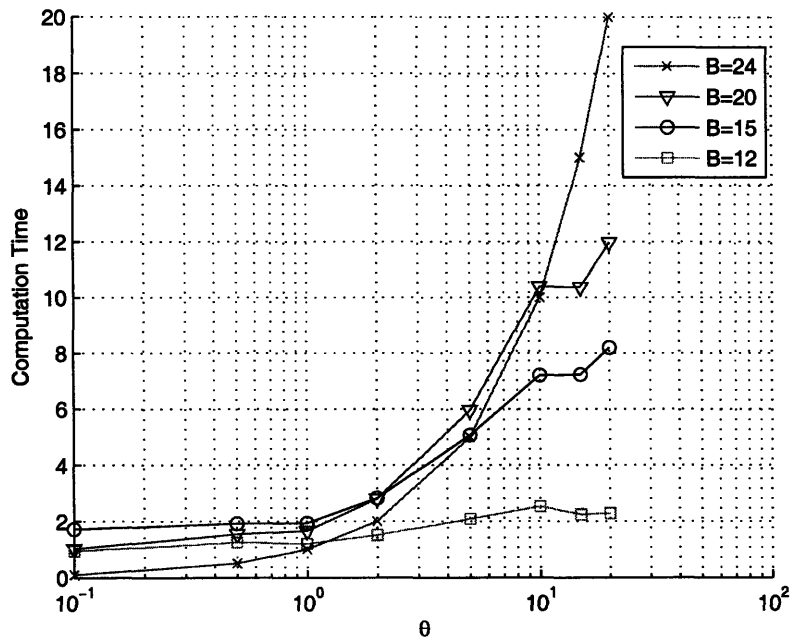

Figure 4.4: The effect of different values of θ on computation time

Table 4.8: OD pairs and demands for the Sioux Falls network

No.	OD Pair	Demand
w_1	(1,20)	25
w_2	(20,1)	25
w_3	(1,24)	20
w_4	(24,1)	20
w_5	(7,20)	15
w_6	(20,7)	15

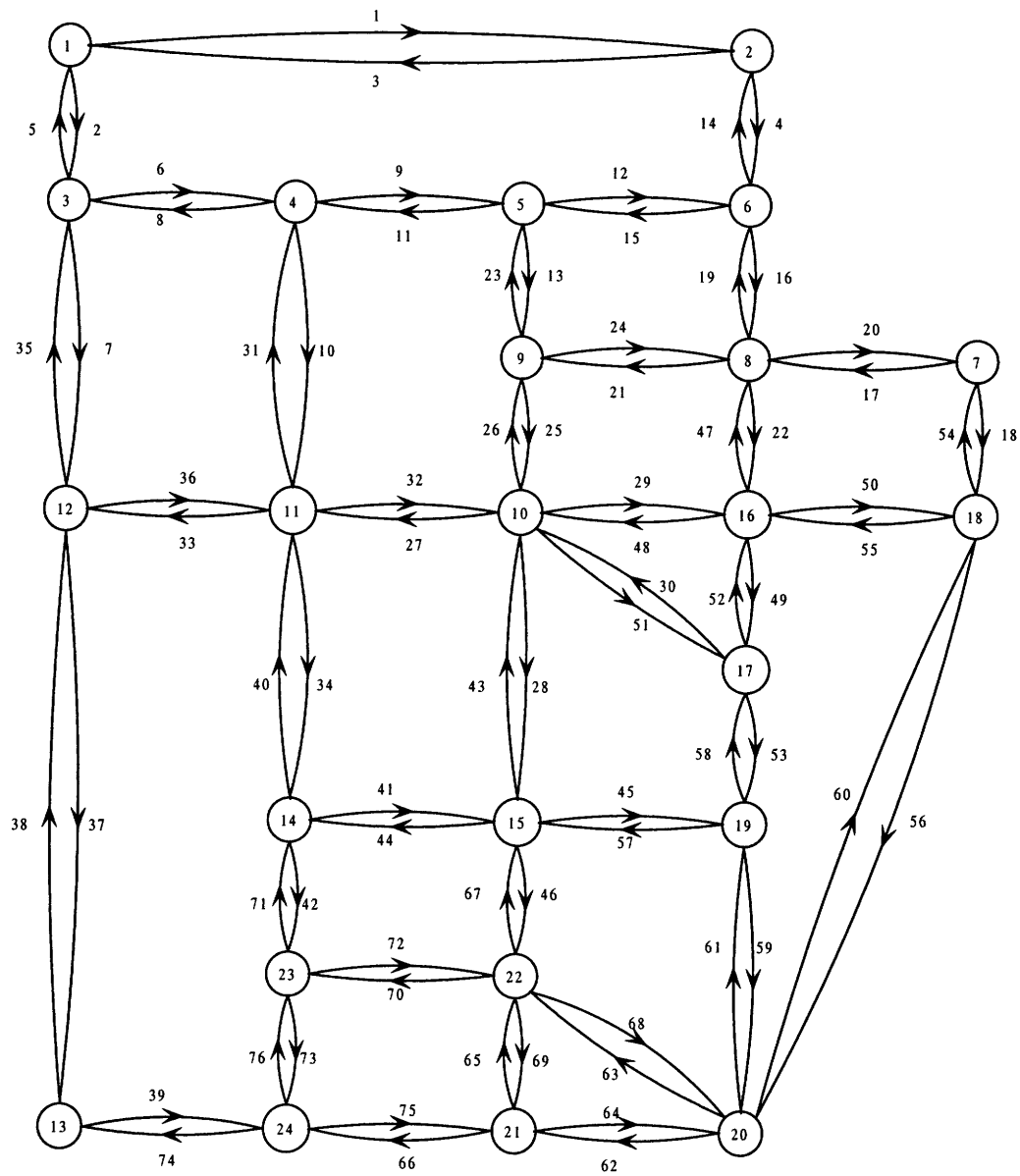

Figure 4.5: The Sioux Falls network example

Table 4.9: Cost functions for the Sioux Falls network

$c_1(f_1) = 6f_1 + 6$	$c_{27}(f_{27}) = 3f_{27} + 5$	$c_{53}(f_{53}) = 6f_{53} + 2$
$c_2(f_2) = 2f_2 + 4$	$c_{28}(f_{28}) = 4f_{28} + 6$	$c_{54}(f_{54}) = 4f_{54} + 2$
$c_3(f_3) = 6f_3 + 6$	$c_{29}(f_{29}) = 2f_{29} + 4$	$c_{55}(f_{55}) = 2f_{55} + 3$
$c_4(f_4) = 2f_4 + 5$	$c_{30}(f_{30}) = 6f_{30} + 8$	$c_{56}(f_{56}) = 16f_{56} + 4$
$c_5(f_5) = 2f_5 + 4$	$c_{31}(f_{31}) = 3f_{31} + 6$	$c_{57}(f_{57}) = 2f_{57} + 3$
$c_6(f_6) = 2f_6 + 4$	$c_{32}(f_{32}) = 3f_{32} + 5$	$c_{58}(f_{58}) = 6f_{58} + 2$
$c_7(f_7) = 4f_7 + 4$	$c_{33}(f_{33}) = 4f_{33} + 6$	$c_{59}(f_{59}) = 4f_{59} + 4$
$c_8(f_8) = 2f_8 + 4$	$c_{34}(f_{34}) = 3f_{34} + 4$	$c_{60}(f_{60}) = 16f_{60} + 4$
$c_9(f_9) = 2f_9 + 2$	$c_{35}(f_{35}) = 4f_{35} + 4$	$c_{61}(f_{61}) = 4f_{61} + 4$
$c_{10}(f_{10}) = 3f_{10} + 6$	$c_{36}(f_{36}) = 4f_{36} + 6$	$c_{62}(f_{62}) = 4f_{62} + 6$
$c_{11}(f_{11}) = 2f_{11} + 2$	$c_{37}(f_{37}) = 14f_{37} + 3$	$c_{63}(f_{63}) = 6f_{63} + 5$
$c_{12}(f_{12}) = 10f_{12} + 4$	$c_{38}(f_{38}) = 14f_{38} + 3$	$c_{64}(f_{64}) = 4f_{64} + 6$
$c_{13}(f_{13}) = 2f_{13} + 5$	$c_{39}(f_{39}) = 4f_{39} + 4$	$c_{65}(f_{65}) = 8f_{65} + 2$
$c_{14}(f_{14}) = 2f_{14} + 5$	$c_{40}(f_{40}) = 3f_{40} + 4$	$c_{66}(f_{66}) = 4f_{66} + 3$
$c_{15}(f_{15}) = 10f_{15} + 4$	$c_{41}(f_{41}) = 4f_{41} + 5$	$c_{67}(f_{67}) = 2f_{67} + 3$
$c_{16}(f_{16}) = 2f_{16} + 2$	$c_{42}(f_{42}) = 2f_{42} + 4$	$c_{68}(f_{68}) = 6f_{68} + 5$
$c_{17}(f_{17}) = 2f_{17} + 3$	$c_{43}(f_{43}) = 4f_{43} + 6$	$c_{69}(f_{69}) = 8f_{69} + 2$
$c_{18}(f_{18}) = 4f_{18} + 2$	$c_{44}(f_{44}) = 4f_{44} + 5$	$c_{70}(f_{70}) = 2f_{70} + 4$
$c_{19}(f_{19}) = 2f_{19} + 2$	$c_{45}(f_{45}) = 2f_{45} + 3$	$c_{71}(f_{71}) = 2f_{71} + 4$
$c_{20}(f_{20}) = 2f_{20} + 3$	$c_{46}(f_{46}) = 2f_{46} + 3$	$c_{72}(f_{72}) = 2f_{72} + 4$
$c_{21}(f_{21}) = 4f_{21} + 10$	$c_{47}(f_{47}) = 2f_{47} + 5$	$c_{73}(f_{73}) = 2f_{73} + 2$
$c_{22}(f_{22}) = 2f_{22} + 5$	$c_{48}(f_{48}) = 2f_{48} + 4$	$c_{74}(f_{74}) = 4f_{74} + 4$
$c_{23}(f_{23}) = 2f_{23} + 5$	$c_{49}(f_{49}) = 2f_{49} + 2$	$c_{75}(f_{75}) = 4f_{75} + 3$
$c_{24}(f_{24}) = 4f_{24} + 10$	$c_{50}(f_{50}) = 2f_{50} + 3$	$c_{76}(f_{76}) = 2f_{76} + 2$
$c_{25}(f_{25}) = 2f_{25} + 3$	$c_{51}(f_{51}) = 6f_{51} + 8$	
$c_{26}(f_{26}) = 2f_{26} + 3$	$c_{52}(f_{52}) = 2f_{52} + 2$	

We also notice that if we choose smaller values for B, it takes much longer solving time because there will be more combinations of B arcs chosen from the total 76 arcs and this causes the solver to perform more branch-and-bound iterations.

Table 4.10: Optimal flow solutions to the user equilibrium problem in the Sioux Falls network

f_1	11.8327	f_{17}	5.17813	f_{33}	5.91066	f_{49}	9.97023	f_{65}	3.67224
f_2	33.1673	f_{18}	9.82187	f_{34}	17.6629	f_{50}	0.105025	f_{66}	1.51466
f_3	11.8327	f_{19}	10.1274	f_{35}	13.844	f_{51}	0.588265	f_{67}	11.1048
f_4	11.8327	f_{20}	5.17813	f_{36}	5.91066	f_{52}	9.97023	f_{68}	8.85425
f_5	33.1673	f_{21}	2.23494	f_{37}	7.93334	f_{53}	10.5585	f_{69}	3.67224
f_6	19.3233	f_{22}	13.0706	f_{38}	7.93334	f_{54}	9.82187	f_{70}	1.42168
f_7	13.844	f_{23}	7.57125	f_{39}	7.93334	f_{55}	0.105025	f_{71}	15.003
f_8	19.3233	f_{24}	2.23494	f_{40}	17.6629	f_{56}	9.71685	f_{72}	1.42168
f_9	5.86604	f_{25}	9.80619	f_{41}	2.65994	f_{57}	5.68352	f_{73}	13.5813
f_{10}	13.4573	f_{26}	9.80619	f_{42}	15.003	f_{58}	10.5585	f_{74}	7.93334
f_{11}	5.86604	f_{27}	1.70502	f_{43}	14.1284	f_{59}	16.242	f_{75}	1.51466
f_{12}	1.70521	f_{28}	14.1284	f_{44}	2.65994	f_{60}	9.71685	f_{76}	13.5813
f_{13}	7.57125	f_{29}	3.20543	f_{45}	5.68352	f_{61}	16.242		
f_{14}	11.8327	f_{30}	0.588265	f_{46}	11.1048	f_{62}	5.18689		
f_{15}	1.70521	f_{31}	13.4573	f_{47}	13.0706	f_{63}	8.85425		
f_{16}	10.1274	f_{32}	1.70502	f_{48}	3.20543	f_{64}	5.18689		

4.2 Uncertain Demand

In this section, we present computational results of the robust approach. In our robust formulation, the flow $\mathbf{f}^{\mathbf{w}}$ is an affine function of the uncertain demand $\mathbf{d}^{\mathbf{w}}$ as in (3.9). We compare the robust and the deterministic solutions in the following ways:

(1) We solve the deterministic problem with the nominal value of demand $\bar{\mathbf{d}}^W$ and obtain the optimal binary decision vector $\mathbf{y}_D$.

(2) For a given budget of uncertainty Γ, we solve the robust problem and obtain the binary decision vector $\mathbf{y}_R$ and the corresponding optimal $\mathbf{Q}^{\mathbf{w}}$ and $\mathbf{p}^{\mathbf{w}}$.

(3) We randomly pick up a demand vector $\mathbf{d}^{\mathbf{w}}$ residing in the polyhedral uncertainty set $\mathcal{U}$ defined in (3.8).

(4) We solve the user equilibrium problem in the network $G(V, A(\mathbf{y}_D))$ with demand $\mathbf{d}^{\mathbf{w}}$ and obtain the deterministic solution of flow $\mathbf{f}_D$. We then calculate the total system cost $\mathbf{c}(\mathbf{f}_D)'\mathbf{f}_D$ for the deterministic case.

(5) We use $\mathbf{d}^{\mathbf{w}}$ and equation (3.9) to calculate the robust solution of flow $\mathbf{f}_R$ and the corresponding total system cost $\mathbf{c}(\mathbf{f}_R)'\mathbf{f}_R$.

(6) We repeat procedure (3), (4) and (5) for 1000 times and compare the deterministic and the robust total system costs.

We perform this comparison because we can view the robust NDP under user equilibrium in the following way. After the network has determined the subsets of arcs to build, an adversary may select from the uncertain demand set $\mathcal{U}$ a vector of demand $\mathbf{d}^{\mathbf{w}}$ and a corresponding equilibrium flow vector for $\mathbf{d}^{\mathbf{w}}$, such that these selections are maximally hostile with respect to the network designer's decisions.

Table 4.11: Optimal flow solutions to the NDP under user equilibrium in the Sioux Falls network with $B = 70$

f_1	11.8829	f_{17}	5.21405	f_{33}	5.91559	f_{49}	10.479	f_{65}	3.36278
f_2	33.1171	f_{18}	9.78595	f_{34}	17.4306	f_{50}	0	f_{66}	2.0391
f_3	11.8829	f_{19}	10.2083	f_{35}	13.7843	f_{51}	0	f_{67}	11.9477
f_4	11.8829	f_{20}	5.21405	f_{36}	5.91559	f_{52}	10.479	f_{68}	8.58496
f_5	33.1171	f_{21}	2.14722	f_{37}	7.86866	f_{53}	10.479	f_{69}	3.36278
f_6	19.3328	f_{22}	13.2751	f_{38}	7.86866	f_{54}	9.78595	f_{70}	0
f_7	13.7843	f_{23}	7.62956	f_{39}	7.86866	f_{55}	0	f_{71}	14.1704
f_8	19.3328	f_{24}	2.14722	f_{40}	17.4306	f_{56}	9.78595	f_{72}	0
f_9	5.95493	f_{25}	9.77678	f_{41}	3.26012	f_{57}	5.74821	f_{73}	14.1704
f_{10}	13.3779	f_{26}	9.77678	f_{42}	14.1704	f_{58}	10.479	f_{74}	7.86866
f_{11}	5.95493	f_{27}	1.86294	f_{43}	14.4358	f_{59}	16.2262	f_{75}	2.0391
f_{12}	1.67463	f_{28}	14.4358	f_{44}	3.26012	f_{60}	9.78595	f_{76}	14.1704
f_{13}	7.62956	f_{29}	2.79611	f_{45}	5.74821	f_{61}	16.2272		
f_{14}	11.8829	f_{30}	0	f_{46}	11.9477	f_{62}	5.40188		
f_{15}	1.67463	f_{31}	13.3779	f_{47}	13.2751	f_{63}	8.58496		
f_{16}	10.2083	f_{32}	1.86294	f_{48}	2.79611	f_{64}	5.40188		

4.2.1 Example 1

We use the Example 1 in Section 4.1.1, with the linear arc cost functions in Table 4.1. We increase the budget of uncertainty Γ from 0.5 to 5.0, with a stepsize of 0.5. We calculate the mean, standard deviation, the maximum and the minimum of the total costs for both the deterministic and the robust approaches and summarize the result in Table 4.12. We also plot a comparison of the mean of the total costs in Figure 4.6.

For $\Gamma = 1, \ldots, 3.5$, the robust solution $\mathbf{y}_R$ chooses to build the same network as the deterministic solution (arcs 1, 4 and 5). The statics also shows that the robust solution has a lower mean than the deterministic solution. This is expected as robust optimization offers more protection against randomness and therefore has a better performance. For $\Gamma = 4, 4.5$ and 5, the robust solution builds a different network (arcs 1, 2 and 4), and the robust solution has very close mean with the deterministic solution, as shown in Figure 4.6. Very interestingly, this indicates that in this case, the robust solution does not seem to have any benefit.

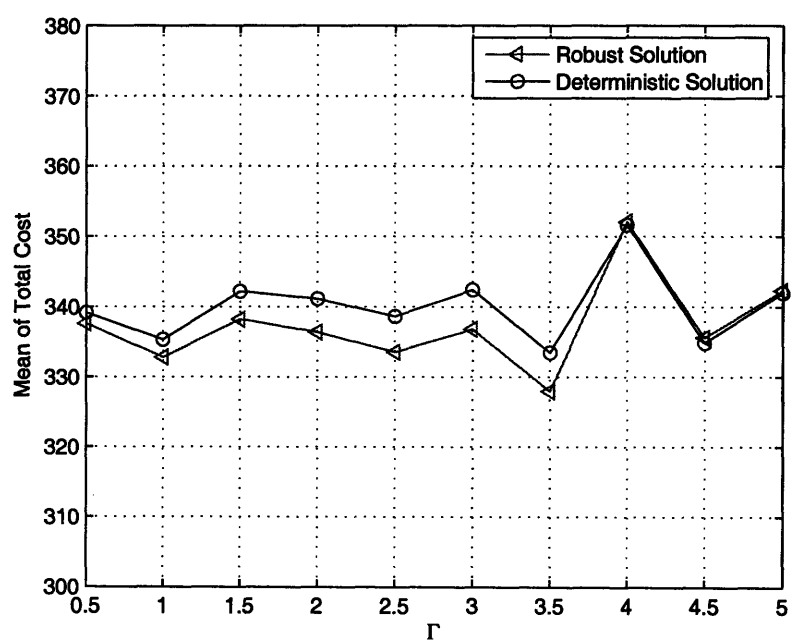

Figure 4.6: Comparison of the mean of the total costs for the deterministic and the robust approaches, Example 1.

Table 4.12: Comparison of robust and deterministic solutions

Γ	Approach	Mean	Standard Deviation	Maximum	Minimum
$\Gamma = 0.5$	Robust	337.61905	10.68305561	355.189	318.124
	Deterministic	339.14825	11.46832815	357.952	318.155
$\Gamma = 1$	Robust	332.79625	21.25062317	373.903	298.862
	Deterministic	335.31969	22.65689033	378.848	298.942
$\Gamma = 1.5$	Robust	338.24588	31.43916807	393.917	279.045
	Deterministic	342.19957	33.18724751	400.432	279.097
$\Gamma = 2$	Robust	336.3958	40.97565639	408.176	263.329
	Deterministic	341.13114	42.90642319	415.524	263.682
$\Gamma = 2.5$	Robust	333.55941	58.62437482	436.119	242.953
	Deterministic	338.6621	60.88903627	443.877	243.172
$\Gamma = 3$	Robust	336.80875	70.76134139	456.761	224.469
	Deterministic	342.4023	73.01493303	464.236	224.69
$\Gamma = 3.5$	Robust	327.95365	77.73963904	475.581	204.79
	Deterministic	333.48909	79.88122574	482.57	204.475
$\Gamma = 4$	Robust	352.08086	90.09075647	503.546	193.173
	Deterministic	351.59784	93.42413777	508.401	186.509
$\Gamma = 4.5$	Robust	335.62265	111.5811685	522.027	176.258
	Deterministic	334.8634	115.3821202	527.421	169.842
$\Gamma = 5$	Robust	342.30695	124.1844125	552.099	158.231
	Deterministic	341.99239	128.1481512	558.24	151.802

Chapter 5

Conclusion

In this thesis, we address the transportation network design problem under the user-equilibrium behavioral assumption of travelers. We study both the cases of deterministic demand and polyhedral demand uncertainty. The major contributions of our work are as follows:

(1) For the NDP with deterministic demand, we propose an exact optimization formulation. When we model the costs as linear functions of arc flows, our model turns out to be a quadratic objective, linear constraints, mixed-integer optimization problems. We further extend the work to nonlinear cost functions and offer an iterative algorithm to solve the nonlinear case. We give three examples to illustrate that our model can be efficiently solved by commercial software and is applicable to large scale networks.

(2) We also incorporate demand uncertainty into the previous model by applying the robust optimization approach. We propose an iterative algorithm to solve the robust formulation. Computational results show that our robust optimization formulation produces tractable solutions and have well protected against the data uncertainty.

We hope that our work with the NDP under user equilibrium and robust optimization will lead to better understanding and provide new insights into the network design problems.

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