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DYNAMICS OF A THREE DEGREE OF FREEDOM KINEMATIC CHAIN

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ABSTRACT. In order to be able to design a control system for high-speed control of mechanical manipulators, it is necessary to understand properly their dynamics. Here we present an analysis of a detailed model of a three-link device which may be viewed as either a "leg" in a locomotory system, or the first three degrees of freedom of an "arm" providing for its gross motions. The equations of motion are shown to be non-trivial, yet manageable.

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INTRODUCTION.

In this paper, we analyze the relationship between actuator torques and joint angular accelerations for a device with three rotational degrees of freedom, such as a "leg" on a locomotory system or the first three joints of a manipulator or "arm". The kind of analysis we present here leads to a clear understanding of the effects of varying inertia, joint-interactions and coriolis forces and forms the basis for simulations of such systems and, most importantly, design of control systems. It is likely that high speed control of articulated kinematic chains is not possible without this kind of detailed understanding; conversely, we show that the computations required of such a control system are manageable.

Initially, we restrict our attention to arrangements with no offsets between links, as shown in figure 1. Later, we consider a more realistic case, the MIT-Scheinman electric arm, which has offsets, as do many practical devices. Further, we model links as thin rods and, finally, consider more complicated mass distributions at the end of the paper.

We use rather primitive techniques in order to avoid possible complications due to the potential difficulty of visualizing angular rotation vectors and components of inertia matrices. The same results, however, could be obtained using such advanced notions, with little savings in effort and considerable loss of insight.

NOTATIONAL CONVENTIONS.

The links, modelled as thin rods (see figures 2 and 3), are numbered starting with the base. The base, link 0, is rigidly attached to a fixed Cartesian coordinate system with the z-axis pointing up through the column, link 1. Joints are numbered systematically, with joint i connecting link $(i - 1)$ to link i . Thus the "hip" or "shoulder" is joint 2, with the "knee" or "elbow" being joint 3.

The lengths of the links will be l_1 , l_2 and l_3 , with masses m_1 , m_2 and m_3 . The joint-angles will be called θ_1 , θ_2 and θ_3 and the angular velocities represented as $\dot{\theta}_1$, $\dot{\theta}_2$, and $\dot{\theta}_3$. At times it is convenient to use vector notation with

$$\underline{\theta} = (\theta_1, \theta_2, \theta_3)$$

$$\underline{\dot{\theta}} = (\dot{\theta}_1, \dot{\theta}_2, \dot{\theta}_3)$$

Clearly, $\underline{\theta}$ together with $\underline{\dot{\theta}}$ specify the state of the device completely.

PLAN OF ACTION.

We treat each link in turn: first, we calculate the velocity of each point in the link as a function of the joint-angle rate; second, we calculate the total kinetic energy in each link; third, we calculate the torques required to support the motion of that link; and, finally, we add up the torques required to move all links to obtain the total torque that must be applied by each actuator. We calculate gravity components of torque at the very end.

REVIEW OF BASIC MECHANICS.

It is convenient to calculate the total kinetic energy of each link by dividing it into infinitesimal parts and integrating along the length of each link. The kinetic energy of a particle of mass m moving with velocity v is, of course, $(1/2)mv^2$. Thus we can obtain the total kinetic energy of a link from

$$K = \frac{m}{2\ell} \int_0^{\ell} v^2(s) ds$$

where s is distance along the link and $v(s)$ is the velocity at a point located a distance s from one end. Here we have assumed that all the mass is concentrated along a line and is distributed uniformly from one end to the other with linear density m/ℓ . More complicated models require more difficult analysis and are warranted only if measurements can be made of the actual mass-distribution in particular limbs.

We find that, in general, $v(s)$ is of the form

$$v^2(s) = a + bs + cs^2$$

Then clearly

$$K = \frac{m}{2\ell} \left[a\ell + b \frac{\ell^2}{2} + c \frac{\ell^3}{3} \right]$$

CALCULATION OF ACTUATOR TORQUES.

The easiest technique is based on the Euler-Lagrange formula,

$$T_i = \frac{d}{dt} \left(\frac{\partial K}{\partial \dot{\theta}_i} \right) - \frac{\partial K}{\partial \theta_i}$$

where $\dot{\theta}_i$ is the angular velocity

$$\dot{\theta}_i = \frac{d}{dt} \theta_i$$

and T_i is the torque required at joint i to support the motion. K is the kinetic energy. This may look complicated, but, in fact, is very convenient. In general, if the potential energy term is added in, this calculation leads to n equations for a device with n degrees of freedom.

$$T_i = G_i(\theta) + \sum_{j=1}^n I_{ij}(\theta) \ddot{\theta}_j + \sum_{j=1}^n \sum_{k=1}^n C_{ijk}(\theta) \dot{\theta}_j \dot{\theta}_k$$

Here T_i , the actuator torque required at joint i , is made up of three components. The first is the gravitational term obtained from the potential energy P ,

$$G_i = \frac{\partial P}{\partial \theta_i}$$

The second term is a sum of products of inertias and angular accelerations $\ddot{\theta}_j$, where

$$\ddot{\theta}_j = \frac{d^2}{dt^2} \theta_j$$

This term is thus composed of the inertial forces needed to accelerate the links along the desired trajectory.

The third term is a double sum of velocity product terms and constitutes the torque required to balance Coriolis forces; these include the centrifugal forces. Note that all three kinds of coefficients G_i , I_{ij} and C_{ijk} are functions of the configuration, $\underline{\theta}$, only, where

$$\underline{\theta} = (\theta_1, \theta_2, \theta_3 \dots \theta_n)$$

That is, they do not depend on joint angle velocities (or accelerations). In fact, we find that these terms are polynomials in link lengths and sines and cosines of the joint-angles.

It is convenient to think of the total kinetic energy of the device as a sum of the kinetic energies of the individual links,

$$K = \sum_{j=1}^n K_j$$

and to calculate the total torque required of a particular actuator as a sum of components, each obtained by applying the Euler-Lagrange equation to a component of the kinetic energy,

$$T_i = \sum_{j=1}^n T_{ij}$$

where T_{ij} is the actuator torque required at joint i to support the motion of link j .

$$T_{ij} = \frac{d}{dt} \left(\frac{\partial K_j}{\partial \dot{\theta}_i} \right) - \frac{\partial K_j}{\partial \theta_i}$$

THE UPRIGHT COLUMN.

If we model this link as a thin rod, it will have no inertia and not impede accelerations about its axis at all. It is more realistic to model it as a cylinder of uniform mass distribution. If it has height H , radius R and mass m_1 , it has inertia

$$I_1 = \frac{m_1 R^2}{2}$$

To introduce the techniques used later for the other links, we calculate this from first principles. The volume of the cylinder is

$$V = \pi R^2 H$$

Consequently, its mass density is $m_1/(\pi R^2 H)$.

Now consider a cylindrical shell of thickness d_s at distance s from the axis as in figure 4. It has mass

$$dm = m_1 \frac{2\pi s H}{\pi R^2 H} ds$$

Particles in this shell move with velocity $s\dot{\theta}_1$ when the column rotates at angular rate $\dot{\theta}_1$. The kinetic energy of the cylindrical shell is then

$$\frac{1}{2} (dm) [s\dot{\theta}_1]^2 = \frac{m_1}{R^2} s^3 \dot{\theta}_1^2 ds$$

Integrating over the whole cylinder we find the total kinetic energy is

$$K_1 = \frac{m}{R^2} \dot{\theta}_1^2 \int_0^R s^3 ds = \frac{1}{2} \left(\frac{mR^2}{2} \right) \dot{\theta}_1^2$$

The term $I_1 = mR^2/2$ is the inertia of the upright column, link 1, about its axis of rotation.

ACTUATOR TORQUES REQUIRED TO SUPPORT MOTION OF LINK 1.

We are now ready to apply the Euler-Lagrange equations to find the required actuator torques.

$$\tau_{11} = \frac{d}{dt} \left(\frac{\partial K_1}{\partial \dot{\theta}_1} \right) - \frac{\partial K_1}{\partial \theta_1}$$

where

$$\frac{\partial K_1}{\partial \theta_1} = 0 \quad \text{and} \quad \frac{\partial K_1}{\partial \dot{\theta}_1} = I_1 \dot{\theta}_1$$

so

$$\tau_{11} = \frac{d}{dt} (I_1 \dot{\theta}_1)$$

or

$$\tau_{11} = I_1 \ddot{\theta}_1$$

This rather obvious conclusion shows that we need only an inertial torque to support this motion. No other joints are affected.

THE SECOND LINK.

Link 2 is modelled as a thin rod of mass m_2 and length l_2 (see figures 5 and 6). To compress long expressions we adopt a convention for trigonometric terms:

$$c_i = \cos(\theta_i) \quad s_i = \sin(\theta_i)$$

and

$$c_{ij} = \cos(\theta_i + \theta_j) \quad s_{ij} = \sin(\theta_i + \theta_j)$$

The infinitesimal particle ℓ of length ds has mass

$$dm = \frac{m_2}{l_2} ds$$

First we determine the velocity of this particle; it can be found by differentiating its position with respect to time. Let $\underline{r} = (x, y, z)$ be the particle's position in reference to the rectangular coordinate system introduced earlier. If \underline{i} , \underline{j} , \underline{k} are unit vectors in the directions x , y , z respectively, then,

$$\underline{r} = s[(c_1 s_2) \underline{i} + (s_1 s_2) \underline{j} + (c_2) \underline{k}]$$

Taking the derivative we get the velocity

$$\underline{v} = s[(-s_1 \dot{\theta}_1 s_2 + c_1 \dot{c}_2) \underline{i} + (c_1 \dot{\theta}_1 s_2 + s_1 \dot{c}_2) \underline{j} + (-s_2 \dot{\theta}_2) \underline{k}]$$

To calculate the kinetic energy we need only the square of the absolute value of the velocity,

$$v^2 = \underline{v} \cdot \underline{v} = s^2 [(-s_1 \dot{\theta}_1 s_2 + c_1 c_2 \dot{\theta}_2)^2 + (c_1 \dot{\theta}_1 s_2 + s_1 c_2 \dot{\theta}_2)^2 + (s_2 \dot{\theta}_2)^2]$$

$$v^2 = s^2 [s_2^2 \dot{\theta}_1^2 + \dot{\theta}_2^2]$$

Finally, the kinetic energy is

$$K_2 = \frac{1}{2} \frac{m_2}{l_2} [s_2^2 \dot{\theta}_1^2 + \dot{\theta}_2^2] \int_0^{l_2} s^2 ds$$

That is,

$$K_2 = \frac{1}{6} m_2 l_2^2 [s_2^2 \dot{\theta}_1^2 + \dot{\theta}_2^2]$$

ACTUATOR TORQUES REQUIRED TO SUPPORT MOTION OF LINK 2:

First, we find the required derivatives:

$$\frac{\partial K_2}{\partial \theta_1} = 0 \quad \text{and} \quad \frac{\partial K_2}{\partial \theta_2} = \frac{1}{3} m_2 l_2^2 [s_2 c_2 \dot{\theta}_1^2]$$

$$\frac{\partial K_2}{\partial \dot{\theta}_1} = \frac{1}{3} m_2 l_2^2 [s_2^2 \dot{\theta}_1] \quad \text{and} \quad \frac{\partial K_2}{\partial \dot{\theta}_2} = \frac{1}{3} m_2 l_2^2 [\dot{\theta}_2]$$

$$\frac{d}{dt} \left(\frac{\partial K_2}{\partial \dot{\theta}_1} \right) = \frac{1}{3} m_2 l_2^2 [2s_2 c_2 \dot{\theta}_2 \dot{\theta}_1 + s_2^2 \dot{\theta}_1]$$

$$\frac{d}{dt} \left(\frac{\partial K_2}{\partial \dot{\theta}_2} \right) = \frac{1}{3} m_2 l_2^2 [\ddot{\theta}_2]$$

Finally, using the Euler-Lagrange equation,

$$T_{12} = \frac{d}{dt} \left(\frac{\partial K_2}{\partial \dot{\theta}_1} \right) - \frac{\partial K_2}{\partial \theta_1} = \frac{1}{3} m_2 l_2^2 [s_2^2 \dot{\theta}_1 + 2s_2 c_2 \dot{\theta}_1 \dot{\theta}_2]$$

$$T_{22} = \frac{d}{dt} \left(\frac{\partial K_2}{\partial \dot{\theta}_2} \right) - \frac{\partial K_2}{\partial \theta_2} = \frac{1}{3} m_2 l_2^2 [\ddot{\theta}_2 - s_2 c_2 \dot{\theta}_1^2]$$

ANALYSIS OF TORQUE COMPONENTS.

The two components of T_2 represent inertial and coriolis factors. First, note that $(1/3)m_2\ell_2^2s_2^2$ is the inertia of link 2 about the vertical axis. Multiplying this by the angular acceleration of joint 1 gives us the torque required to produce that angular acceleration. The second term, containing a product of angular velocities is a coriolis force factor which vanishes when $\dot{\theta}_1 = 0$, $\dot{\theta}_2 = 0$ or θ_2 is an integer multiple of 90° since

$$2s_2c_2 = \sin(2\theta_2)$$

This torque term has to do with the change in kinetic energy when the inertia about the vertical axis is changed -- it is the term which speeds up a spinning ice skater when (s)he pulls in his(her) arms and slows him(her) down as the arms are stretched out. It shows one of many interactions between motions for which it is hard to get an intuitive grasp.

The components of T_{22} are even easier to understand. The first term is just the inertial force needed to accelerate link 2, since the inertia of link 2 about joint 2 is simply $(1/3)m_2\ell_2^2$. The last term is a centrifugal force term, which again is zero when θ_2 is an integer multiple of 90° . It represents the tendency for the second link to become horizontal as a result of rotation about the vertical axis.

THE THIRD AND LAST LINK.

Link 3 is modelled as a thin rod of mass m_3 and length ℓ_3 (see figures 7 and 8). The location \underline{r} of the infinitesimal particle of length ds and mass dm is found first, where

$$dm = \frac{m_3}{\ell_3} ds$$

We find that,

$$\underline{r} = c_1(\ell_2 s_2 + s s_{23}) \underline{i} + s_1(\ell_2 s_2 + s s_{23}) \underline{j} + (\ell_2 c_2 + s c_{23}) \underline{k}$$

Differentiation with respect to time,

$$\begin{aligned} \underline{v} = & [-s_1 \dot{\theta}_1 (\ell_2 s_2 + s s_{23}) + c_1 (\ell_2 c_2 \dot{\theta}_2 + s c_{23} (\dot{\theta}_2 + \dot{\theta}_3))] \underline{i} + \\ & [c_1 \dot{\theta}_1 (\ell_2 s_2 + s s_{23}) + s_1 (\ell_2 c_2 \dot{\theta}_2 + s c_{23} (\dot{\theta}_2 + \dot{\theta}_3))] \underline{j} + \\ & [-\ell_2 s_2 \dot{\theta}_2 - s s_{23} (\dot{\theta}_2 + \dot{\theta}_3)] \underline{k} \end{aligned}$$

Then,

$$\begin{aligned} v^2 = \underline{v} \cdot \underline{v} = & \dot{\theta}_1^2 (\ell_2 s_2 + s s_{23})^2 + (\ell_2 c_2 \dot{\theta}_2 + s c_{23} (\dot{\theta}_2 + \dot{\theta}_3))^2 + \\ & [-\ell_2 s_2 \dot{\theta}_2 - s s_{23} (\dot{\theta}_2 + \dot{\theta}_3)]^2 \end{aligned}$$

$$v^2 = (\ell_2^2 s_2^2 + 2\ell_2 s_2 s_{23} s + s_{23}^2 s^2) \dot{\theta}_1^2 + \ell_2^2 \dot{\theta}_2^2 + 2\ell_2 c_3 \dot{\theta}_2 (\dot{\theta}_2 + \dot{\theta}_3) s + (\dot{\theta}_2 + \dot{\theta}_3)^2 s^2$$

Since $c_2 c_{23} + s_2 s_{23} = \cos(\theta_2 - (\theta_2 + \theta_3)) = c_3$.

So,

$$K_3 = \frac{1}{2} \int_0^{\ell_3} v^2 \frac{m_3}{\ell_3} ds$$

becomes

$$K_3 = \frac{m_3}{2} [(\ell_2^2 s_2^2 + \ell_2 \ell_3 s_2 s_{23} + \frac{\ell_3^2}{3} s_{23}^2) \dot{\theta}_1^2 + \ell_2^2 \dot{\theta}_2^2 + \ell_2 \ell_3 c_3 \dot{\theta}_2 (\dot{\theta}_2 + \dot{\theta}_3) + \frac{\ell_3^2}{3} (\dot{\theta}_2 + \dot{\theta}_3)^2]$$

That is,

$$K_3 = \frac{m_3}{2} [(\ell_2^2 s_2^2 + \ell_2 \ell_3 s_2 s_{23} + \frac{\ell_3^2}{3} s_{23}^2) \dot{\theta}_1^2 + (\ell_2^2 + \ell_2 \ell_3 c_3 + \frac{\ell_3^2}{3}) \dot{\theta}_2^2 + (\ell_2 \ell_3 c_3 + \frac{2\ell_3^2}{3}) \dot{\theta}_2 \dot{\theta}_3 + (\frac{\ell_3^2}{3}) \dot{\theta}_3^2]$$

This finally is the kinetic energy link 3!

PARTIAL DERIVATIVES NEEDED.

$$\frac{\partial K_3}{\partial \theta_1} = 0$$

$$\frac{\partial K_3}{\partial \theta_2} = \frac{m_3}{2} (2l_2^2 s_2 c_2 + l_2 l_3 (c_2 s_{23} + s_2 c_{23}) + \frac{2l_3^2}{3} s_{23} c_{23}) \dot{\theta}_1^2$$

$$\frac{\partial K_3}{\partial \theta_3} = \frac{m_3}{2} [(l_2 l_3 s_2 c_{23} + \frac{2l_3^2}{3} s_{23} c_{23}) \dot{\theta}_1^2 + (-l_2 l_3 s_3) \dot{\theta}_2^2 + (-l_2 l_3 s_3) \dot{\theta}_2 \dot{\theta}_3]$$

$$\frac{\partial K_3}{\partial \dot{\theta}_1} = \frac{m_3}{2} [2(l_2^2 s_2^2 + l_2 l_3 s_2 s_{23} + \frac{l_3^2}{3} s_{23}^2) \dot{\theta}_1]$$

$$\frac{\partial K_3}{\partial \dot{\theta}_2} = \frac{m_3}{2} [2(l_2^2 + l_2 l_3 c_3 + \frac{l_3^2}{3}) \dot{\theta}_2 + (l_2 l_3 c_3 + \frac{2l_3^2}{3}) \dot{\theta}_3]$$

$$\frac{\partial K_3}{\partial \dot{\theta}_3} = \frac{m_3}{2} [(l_2 l_3 c_3 + \frac{2l_3^2}{3}) \dot{\theta}_2 + \frac{2l_3^2}{3} \dot{\theta}_3]$$

$$\frac{d}{dt}\left(\frac{\partial K_3}{\partial \dot{\theta}_1}\right) = \frac{m_3}{2} \left[2(\ell_2^2 s_2^2 + \ell_2 \ell_3 s_2 s_{23} + \frac{\ell_3^2}{3} s_{23}^2) \dot{\theta}_1 + \right. \\ \left. 2(2\ell_2^2 s_2 c_2 \dot{\theta}_2 + \ell_2 \ell_3 (c_2 \dot{\theta}_2 s_{23} + s_2 c_{23} (\dot{\theta}_2 + \dot{\theta}_3))) + \frac{2\ell_3^2}{3} s_{23} c_{23} (\dot{\theta}_2 + \dot{\theta}_3) \dot{\theta}_1 \right]$$

$$\frac{d}{dt}\left(\frac{\partial K_3}{\partial \dot{\theta}_2}\right) = \frac{m_3}{2} \left[2(\ell_2^2 + \ell_2 \ell_3 c_3 + \frac{\ell_3^2}{3}) \dot{\theta}_2 + (\ell_2 \ell_3 c_3 + \frac{2\ell_3^2}{3}) \dot{\theta}_3 + \right. \\ \left. 2(-\ell_2 \ell_3 s_3 \dot{\theta}_3) \dot{\theta}_2 + (-\ell_2 \ell_3 s_3 \dot{\theta}_3) \dot{\theta}_3 \right]$$

$$\frac{d}{dt}\left(\frac{\partial K_3}{\partial \dot{\theta}_3}\right) = \frac{m_3}{2} \left[(\ell_2 \ell_3 c_3 + \frac{2\ell_3^2}{3}) \dot{\theta}_2 + \frac{2\ell_3^2}{3} \dot{\theta}_3 + (-\ell_2 \ell_3 s_3 \dot{\theta}_3) \dot{\theta}_2 \right]$$

ACTUATOR TORQUES REQUIRED TO SUPPORT MOTION OF LINK 3.

$$T_{13} = \frac{d}{dt} \left(\frac{\partial K_3}{\partial \dot{\theta}_1} \right) - \frac{\partial K_3}{\partial \theta_1}$$

$$T_{13} = \frac{m_3}{2} [2(l_2^2 s_2^2 + l_2 l_3 s_2 s_{23} + \frac{l_3^2}{3} s_{23}^2) \ddot{\theta}_1 +$$

$$2(2l_2^2 s_2 c_2 + l_2 l_3 (c_2 s_{23} + s_2 c_{23}) + \frac{2l_3^2}{3} s_{23} c_{23}) \dot{\theta}_1 \dot{\theta}_2 +$$

$$2(l_2 l_3 s_2 c_{23} + \frac{2l_3^2}{3} s_{23} c_{23}) \dot{\theta}_3 \dot{\theta}_1]$$

$$T_{23} = \frac{d}{dt} \left(\frac{\partial K_3}{\partial \dot{\theta}_2} \right) - \frac{\partial K_3}{\partial \theta_2}$$

$$T_{23} = \frac{m_3}{2} [2(l_2^2 + l_2 l_3 c_3 + \frac{l_3^2}{3}) \ddot{\theta}_2 + (l_2 l_3 c_3 + \frac{2l_3^2}{3}) \ddot{\theta}_3 -$$

$$-(2l_2^2 s_2 c_2 + l_2 l_3 (c_2 s_{23} + s_2 c_{23}) + \frac{2l_3^2}{3} s_{23} c_{23}) \dot{\theta}_1^2 -$$

$$-2l_2 l_3 s_3 \dot{\theta}_2 \dot{\theta}_3 - l_2 l_3 s_3 \dot{\theta}_3^2]$$

$$T_{33} = \frac{d}{dt} \left(\frac{\partial K_3}{\partial \dot{\theta}_3} \right) - \frac{\partial K_3}{\partial \theta_3}$$

$$T_{33} = \frac{m_3}{2} [(l_2 l_3 c_3 + \frac{2l_3^2}{3}) \ddot{\theta}_2 + \frac{2l_3^2}{3} \ddot{\theta}_3 -$$

$$(l_2 l_3 s_2 c_{23} + \frac{2l_3^2}{3} s_{23} c_{23}) \dot{\theta}_1^2 +$$

$$(l_2 l_3 s_3) \dot{\theta}_2^2]$$

GRAVITATIONAL TORQUE COMPONENTS IN SIMPLE CASE.

If gravity acts along the z-axis and has magnitude g , then the potential energy of the device can be found easily from the vertical positions of the centers of mass of the two links.

$$z_2 = \frac{l_2}{2} c_2 \quad \text{and} \quad z_3 = (l_2 c_2 + \frac{l_3}{2} c_{23})$$

So the total potential energy is

$$P = g[m_2 \frac{l_2}{2} c_2 + m_3(l_2 c_2 + \frac{l_3}{2} c_{23})]$$

So the torque components are simply

$$T_{2g} = - \frac{\partial P}{\partial \theta_2} = g[m_2 \frac{l_2}{2} s_2 + m_3(l_2 s_2 + \frac{l_3}{2} s_{23})]$$

and

$$T_{3g} = - \frac{\partial P}{\partial \theta_3} = g [m_3 \frac{l_3}{2} s_{23}]$$

These components may simply be added to the components already found for inertial and coriolis torques.

GRAVITY COMPENSATION IN GENERAL CASE.

The position of the center of mass of link 2 is

$$\underline{r}_2 = \frac{l_2}{2} [(c_1 s_2) \underline{i} + (s_1 s_2) \underline{j} + (c_2) \underline{k}]$$

If $\underline{g} = (g_1, g_2, g_3)$ is the vector of gravitational acceleration, then the potential energy is

$$P_2 = -(\underline{r}_2 \cdot \underline{g})m_2 = -m_2 \frac{l_2}{2} [c_1 s_2 g_1 + s_1 s_2 g_2 + c_2 g_3]$$

Similarly, the position of the center of mass of link 3 is

$$\underline{r}_3 = c_1 (l_2 s_2 + \frac{l_3}{2} s_{23}) \underline{i} + s_1 (l_2 s_2 + \frac{l_3}{2} s_{23}) \underline{j} + (l_2 c_2 + \frac{l_3}{2} c_{23}) \underline{k}$$

$$P_3 = -(\underline{r}_3 \cdot \underline{g})m_3 =$$

$$-m_3 [c_1 (l_2 s_2 + \frac{l_3}{2} s_{23}) g_1 + s_1 (l_2 s_2 + \frac{l_3}{2} s_{23}) g_2 + (l_2 c_2 + \frac{l_3}{2} c_{23}) g_3]$$

The total potential energy is the sum of P_2 and P_3 . The gravity compensation torques can be found from P by differentiation.

$$T_{1g} = -\frac{\partial P}{\partial \theta_1} = s_1 \left(-m_2 \frac{\ell_2}{2} s_2 - m_3 \left(\ell_2 s_2 + \frac{\ell_3}{2} s_{23} \right) \right) g_1 + \\ c_1 \left(m_2 \frac{\ell_2}{2} s_2 + m_3 \left(\ell_2 s_2 + \frac{\ell_3}{2} s_{23} \right) \right) g_2$$

$$T_{2g} = -\frac{\partial P}{\partial \theta_2} = c_1 \left(m_2 \frac{\ell_2}{2} c_2 + m_3 \left(\ell_2 c_2 + \frac{\ell_3}{2} c_{23} \right) \right) g_1 + \\ s_1 \left(m_2 \frac{\ell_2}{2} c_2 + m_3 \left(\ell_2 c_2 + \frac{\ell_3}{2} c_{23} \right) \right) g_2 + \\ \left(-m_2 \frac{\ell_2}{2} s_2 - m_3 \left(\ell_2 s_2 + \frac{\ell_3}{2} s_{23} \right) \right) g_3$$

$$T_{3g} = -\frac{\partial P}{\partial \theta_3} = m_3 \left[c_1 \left(\frac{\ell_3}{2} c_{23} \right) g_1 + s_1 \left(\frac{\ell_3}{2} c_{23} \right) g_2 - \frac{\ell_3}{2} s_{23} g_3 \right]$$

MASS CONCENTRATION AT THE END OF LINK 2.

Let there be an additional mass M_2 attached at the end of link 2. Its velocity squared is

$$v^2 = l_2^2 [s_2^2 \dot{\theta}_1^2 + \dot{\theta}_2^2]$$

So

$$K_2' = \frac{1}{2} M_2 l_2^2 [s_2^2 \dot{\theta}_1^2 + \dot{\theta}_2^2]$$

So the torque components would come to

$$T_1' = M_2 l_2^2 [s_2^2 \ddot{\theta}_1 + 2s_2 c_2 \dot{\theta}_1 \dot{\theta}_2]$$

$$T_2' = M_2 l_2^2 [\ddot{\theta}_2 - s_2 c_2 \dot{\theta}_1^2]$$

MASS CONCENTRATION AT THE END OF LINK 3 -- LOAD CARRIED.

Let there be an additional mass M_3 attached at the end of link 3. Its velocity squared is

$$v^2 = (\ell_2 s_2 + \ell_3 s_{23})^2 \dot{\theta}_1^2 + \ell_2^2 \dot{\theta}_2^2 + 2\ell_2 \ell_3 c_3 \dot{\theta}_2 (\dot{\theta}_2 + \dot{\theta}_3) + \ell_3^2 (\dot{\theta}_2 + \dot{\theta}_3)^2$$

So,

$$K'_3 = \frac{1}{2} M_3 [(\ell_2 s_2 + \ell_3 s_{23})^2 \dot{\theta}_1^2 + (\ell_2^2 + 2\ell_2 \ell_3 c_3 + \ell_3^2) \dot{\theta}_2^2 + 2\ell_3 (\ell_2 c_3 + \ell_3) \dot{\theta}_2 \dot{\theta}_3 + \ell_3^2 \dot{\theta}_3^2]$$

$$\frac{\partial K'_3}{\partial \theta_1} = 0$$

$$\frac{\partial K'_3}{\partial \theta_2} = M_3 [(\ell_2 s_2 + \ell_3 s_{23})(\ell_2 c_2 + \ell_3 c_{23}) \dot{\theta}_1^2]$$

$$\frac{\partial K'_3}{\partial \theta_3} = M_3 [(\ell_2 s_2 + \ell_3 s_{23}) \ell_3 c_{23} \dot{\theta}_1^2 - \ell_2 \ell_3 s_3 \dot{\theta}_2^2 - \ell_2 \ell_3 s_3 \dot{\theta}_2 \dot{\theta}_3]$$

$$\frac{\partial K'_3}{\partial \theta_1} = M_3 [(\ell_2 s_2 + \ell_3 s_{23})^2 \dot{\theta}_1]$$

$$\frac{\partial K'_3}{\partial \theta_2} = M_3 [(\ell_2 + 2\ell_2 \ell_3 c_3 + \ell_3^2) \dot{\theta}_2 + \ell_3 (\ell_2 c_3 + \ell_3) \dot{\theta}_3]$$

$$\frac{\partial K'_3}{\partial \dot{\theta}_3} = M_3[l_3(l_2c_3 + l_3) \dot{\theta}_2 + l_3^2 \dot{\theta}_3]$$

$$\frac{d}{dt} \left(\frac{\partial K'_3}{\partial \dot{\theta}_1} \right) = M_3[(l_2s_2 + l_3s_{23})^2 \ddot{\theta}_1 + 2(l_2s_2 + l_3s_{23})(l_2c_2 \dot{\theta}_2 + l_3c_{23}(\dot{\theta}_2 + \dot{\theta}_3)) \dot{\theta}_1]$$

$$\frac{d}{dt} \left(\frac{\partial K'_3}{\partial \dot{\theta}_2} \right) = M_3[(l_2^2 + 2l_2l_3c_3 + l_3^2) \ddot{\theta}_2 + l_3(l_2c_3 + l_3) \ddot{\theta}_3 -$$

$$2l_2l_3s_3 \dot{\theta}_2 \dot{\theta}_3 - l_2l_3s_3 \dot{\theta}_3^2]$$

$$\frac{d}{dt} \left(\frac{\partial K'_3}{\partial \dot{\theta}_3} \right) = M_3[l_3(l_2c_3 + l_3) \ddot{\theta}_2 + l_3^2 \ddot{\theta}_3 - l_2l_3s_3 \dot{\theta}_2 \dot{\theta}_3]$$

$$T'_1 = M_3[(l_2s_2 + l_3s_{23})^2 \ddot{\theta}_1 + 2(l_2s_2 + l_3s_{23})(l_2c_2 + l_3c_{23}) \dot{\theta}_1 \dot{\theta}_2 +$$

$$2l_3c_{23}(l_2s_2 + l_3s_{23}) \dot{\theta}_3 \dot{\theta}_1]$$

$$T'_2 = M_3 [(l_2^2 + 2l_2l_3c_3 + l_3^2) \ddot{\theta}_2 + l_3(l_2c_3 + l_3) \ddot{\theta}_3 -$$

$$(l_2s_2 + l_3s_{23})(l_2c_2 + l_3c_{23}) \dot{\theta}_1^2 - 2l_2l_3s_3 \dot{\theta}_2 \dot{\theta}_3 - l_2l_3s_3 \dot{\theta}_3^2]$$

$$T'_3 = M_3[l_3(l_2c_3 + l_3) \ddot{\theta}_2 + l_3^2 \ddot{\theta}_3 - l_3c_{23}(l_2s_2 + l_3s_{23}) \dot{\theta}_1^2 + l_2l_3s_3 \dot{\theta}_2^2]$$

These expressions can be used to calculate additional torques required to support the movements of a load carried by the third link.

GRAVITY COMPENSATION FOR THE MASS CONCENTRATIONS AT THE ENDS OF LINK 2 AND LINK 3.

The position of the mass concentration at the end of link 2 is

$$\underline{R}_2 = l_2[(c_1 s_2)\underline{i} + (s_1 s_2)\underline{j} + c_2 \underline{k}]$$

and its potential energy is

$$P'_2 = -(\underline{R}_2 \cdot \underline{g})M_2 = -M_2 l_2 [c_1 s_2 g_1 + s_1 s_2 g_2 + c_2 g_3]$$

Similarly, the position of the mass concentration at the end of link 3 is

$$\underline{R}_3 = c_1(l_2 s_2 + l_3 s_{23})\underline{i} + s_1(l_2 s_2 + l_3 s_{23})\underline{j} + (l_2 c_2 + l_3 c_{23})\underline{k}$$

and its potential energy is

$$P'_3 = -(\underline{R}_3 \cdot \underline{g})M_3 = -M_3 [c_1(l_2 s_2 + l_3 s_{23})g_1 + s_1(l_2 s_2 + l_3 s_{23})g_2 + (l_2 c_2 + l_3 c_{23})g_3]$$

The net potential energy of the mass concentrations is

$$P' = P'_2 + P'_3$$

and the gravity compensation torques for the three links, on account of the mass concentrations, are

$$T'_{1g} = -\frac{\partial P'}{\partial \theta_1} = s_1(-M_2 l_2 s_2 - M_3(l_2 s_2 + l_3 s_{23}))g_1 + c_1(M_2 l_2 s_2 + M_3(l_2 s_2 + l_3 s_{23}))g_2$$

$$T'_{2g} = - \frac{\partial P'}{\partial \theta_2} = c_1 (M_2 l_2 c_2 + M_3 (l_2 c_2 + l_3 c_{23})) g_1 +$$

$$s_1 (M_2 l_2 c_2 + M_3 (l_2 c_2 + l_3 c_{23})) g_2 -$$

$$(M_2 l_2 s_2 + M_3 (l_2 s_2 + l_3 s_{23})) g_3$$

$$T'_{3g} = - \frac{\partial P'}{\partial \theta_3} = M_3 (l_3 c_1 c_{23} g_1 + l_3 s_1 c_{23} g_2 - l_3 s_{23} g_3)$$

THREE DEGREE OF FREEDOM DEVICE WITH OFFSETS.

In some manipulators and legs, trade-offs in the mechanical design dictate a geometry with offsets between links as seen in figure 9, for example. Smaller packaging, better strength and larger range of motion can be achieved this way in return for a small increase in complexity of control. The MIT-Scheinman electric manipulator is an example of a device with offsets. It will be found that only a few extra terms appear in the expressions for the torques required of the actuators. Obviously the torques required to support the motions of link 1, the upright column, do not change, so we start with link 2.

THE SECOND LINK IN A DEVICE WITH OFFSETS.

The position r of a particle on the second link can be found by considering figures 10 and 11.

$$r = [sc_1s_2 - \delta_2s_1]i + [ss_1s_2 + \delta_2c_1]j + [sc_2]k$$

Differentiating,

$$v = [-ss_1\dot{\theta}_1s_2 + sc_1c_2\dot{\theta}_2 - \delta_2c_1\dot{\theta}_1]i + [sc_1\dot{\theta}_1s_2 + ss_1c_2\dot{\theta}_2 - \delta_2s_1\dot{\theta}_1]j + [-ss_2\dot{\theta}_2]k$$

So,

$$v^2 = s^2s_2^2\dot{\theta}_1^2 + (sc_2\dot{\theta}_2 - \delta_2\dot{\theta}_1)^2 + s^2s_2^2\dot{\theta}_2^2$$

$$v^2 = (\delta_2^2 + s^2s_2^2)\dot{\theta}_1^2 - 2s\delta_2c_2\dot{\theta}_1\dot{\theta}_2 + s^2\dot{\theta}_2^2$$

The kinetic energy can be found by integration

$$K_2 = \int_0^{l_2} \frac{1}{2} \frac{m_2}{l_2} v^2 ds$$

$$K_2 = \frac{1}{2} m_2 [(\delta_2^2 + \frac{l_2^2}{3} s_2^2)\dot{\theta}_1^2 - l_2\delta_2c_2\dot{\theta}_1\dot{\theta}_2 + \frac{l_2^2}{3}\dot{\theta}_2^2]$$

ACTUATOR TORQUES REQUIRED TO SUPPORT MOTION OF LINK 2.

$$\frac{\partial K_2}{\partial \theta_1} = 0 \quad \text{and} \quad \frac{\partial K_2}{\partial \theta_2} = \frac{m_2}{2} \left[\frac{2l_2^2}{3} s_2 c_2 \dot{\theta}_1^2 + l_2 \delta_2 s_2 \dot{\theta}_1 \dot{\theta}_2 \right]$$

$$\frac{\partial K_2}{\partial \dot{\theta}_1} = \frac{m_2}{2} \left[2(\delta_2^2 + \frac{l_2^2}{3} s_2^2) \dot{\theta}_1 - l_2 \delta_2 c_2 \dot{\theta}_2 \right]$$

$$\frac{\partial K_2}{\partial \dot{\theta}_2} = \frac{m_2}{2} \left[-l_2 \delta_2 c_2 \dot{\theta}_1 + \frac{2l_2^2}{3} \dot{\theta}_2 \right]$$

$$\frac{d}{dt} \left(\frac{\partial K_2}{\partial \dot{\theta}_1} \right) = \frac{m_2}{2} \left[2(\delta_2^2 + \frac{l_2^2}{3} s_2^2) \ddot{\theta}_1 - l_2 \delta_2 c_2 \ddot{\theta}_2 + \frac{4l_2^2}{3} s_2 c_2 \dot{\theta}_1 \dot{\theta}_2 + l_2 \delta_2 s_2 \dot{\theta}_2^2 \right]$$

$$\frac{d}{dt} \left(\frac{\partial K_2}{\partial \dot{\theta}_2} \right) = \frac{m_2}{2} \left[-l_2 \delta_2 c_2 \ddot{\theta}_1 + \frac{2l_2^2}{3} \ddot{\theta}_2 + l_2 \delta_2 s_2 \dot{\theta}_1 \dot{\theta}_2 \right]$$

Finally,

$$T_{12} = \frac{d}{dt} \left(\frac{\partial K_2}{\partial \dot{\theta}_1} \right) - \frac{\partial K_2}{\partial \theta_1}$$

$$T_{12} = \frac{m_2}{2} \left[2(\delta_2^2 + \frac{l_2^2}{3} s_2^2) \ddot{\theta}_1 - l_2 \delta_2 c_2 \ddot{\theta}_2 + \frac{4l_2^2}{3} s_2 c_2 \dot{\theta}_1 \dot{\theta}_2 + l_2 \delta_2 s_2 \dot{\theta}_2^2 \right]$$

$$T_{22} = \frac{d}{dt} \left(\frac{\partial K_2}{\partial \dot{\theta}_2} \right) - \frac{\partial K_2}{\partial \theta_2}$$

$$T_{22} = \frac{m_2}{2} \left[-l_2 \delta_2 c_2 \ddot{\theta}_1 + \frac{2l_2^2}{3} \ddot{\theta}_2 - \frac{2l_2^2}{3} s_2 c_2 \dot{\theta}_1^2 \right]$$

THE THIRD LINK IN A DEVICE WITH OFFSETS.

The position \underline{r} of a particle in the third link can be found from figures 12 and 13.

$$\begin{aligned}\underline{r} = & [(l_2 s_2 + s s_{23})c_1 - \delta_3 s_1] \underline{i} + \\ & [(l_2 s_2 + s s_{23})s_1 + \delta_3 c_1] \underline{j} + \\ & [l_2 c_2 + s c_{23}] \underline{k}\end{aligned}$$

Differentiating,

$$\begin{aligned}\underline{v} = & [-(l_2 s_2 + s s_{23})s_1 \dot{\theta}_1 - \delta_3 c_1 \dot{\theta}_1 + (l_2 c_2 \dot{\theta}_2 + s c_{23}(\dot{\theta}_2 + \dot{\theta}_3))c_1] \underline{i} + \\ & [(l_2 s_2 + s s_{23})c_1 \dot{\theta}_1 - \delta_3 s_1 \dot{\theta}_1 + (l_2 c_2 \dot{\theta}_2 + s c_{23}(\dot{\theta}_2 + \dot{\theta}_3))s_1] \underline{j} + \\ & [-l_2 s_2 \dot{\theta}_2 - s s_{23}(\dot{\theta}_2 + \dot{\theta}_3)] \underline{k}\end{aligned}$$

So,

$$\begin{aligned}v^2 = \underline{v} \cdot \underline{v} = & (l_2 s_2 + s s_{23})^2 \dot{\theta}_1^2 + [-\delta_3 \dot{\theta}_1 + l_2 c_2 \dot{\theta}_2 + s c_{23}(\dot{\theta}_2 + \dot{\theta}_3)]^2 + \\ & [-l_2 s_2 \dot{\theta}_2 - s s_{23}(\dot{\theta}_2 + \dot{\theta}_3)]^2\end{aligned}$$

That is,

$$v^2 = (\ell_2 s_2 + s s_{23})^2 \dot{\theta}_1^2 + \delta_3^2 \dot{\theta}_1^2 + \ell_2^2 \dot{\theta}_2^2 + s^2 (\dot{\theta}_2 + \dot{\theta}_3)^2 - 2\delta_3 \ell_2 c_2 \dot{\theta}_1 \dot{\theta}_2 +$$

$$2s \ell_2 c_3 \dot{\theta}_2 (\dot{\theta}_2 + \dot{\theta}_3) - 2\delta_3 s c_{23} \dot{\theta}_1 (\dot{\theta}_2 + \dot{\theta}_3)$$

$$v^2 = (\ell_2^2 s_2^2 + 2s \ell_2 s_2 s_{23} + s^2 s_{23}^2 + \delta_3^2) \dot{\theta}_1^2 - 2\delta_3 (\ell_2 c_2 + s c_{23}) \dot{\theta}_1 \dot{\theta}_2 +$$

$$(\ell_2^2 + 2s \ell_2 c_3 + s^2) \dot{\theta}_2^2 + (2s \ell_2 c_3 + 2s^2) \dot{\theta}_2 \dot{\theta}_3 + (s^2) \dot{\theta}_3^2 - 2s \delta_3 c_{23} \dot{\theta}_3 \dot{\theta}_1$$

$$K_3 = \int_0^{\ell_3} \frac{1}{2} \frac{m_3}{\ell_3} v^2 ds$$

$$K_3 = \frac{m_3}{2} [(\ell_2^2 s_2^2 + \ell_2 \ell_3 s_2 s_{23} + \frac{\ell_3^2}{3} s_{23}^2 + \delta_3^2) \dot{\theta}_1^2 - \delta_3 (2\ell_2 c_2 + \ell_3 c_{23}) \dot{\theta}_1 \dot{\theta}_2 +$$

$$(\ell_2^2 + \ell_2 \ell_3 c_3 + \frac{\ell_3^2}{3}) \dot{\theta}_2^2 + (\ell_2 \ell_3 c_3 + \frac{2\ell_3^2}{3}) \dot{\theta}_2 \dot{\theta}_3 + (\frac{\ell_3^2}{3}) \dot{\theta}_3^2 - \delta_3 (\ell_3 c_{23}) \dot{\theta}_3 \dot{\theta}_1]$$

PARTIAL DERIVATIVES NEEDED.

$$\frac{\partial K_3}{\partial \theta_1} = 0$$

$$\frac{\partial K_3}{\partial \theta_2} = \frac{m_3}{2} [(2l_2^2 s_2 c_2 + l_2 l_3 (c_2 s_{23} + s_2 c_{23})) + \frac{2l_3^2}{3} s_{23} c_{23}] \dot{\theta}_1^2 + \delta_3 (2l_2 s_2 + l_3 s_{23}) \dot{\theta}_1 \dot{\theta}_2 + \delta_3 l_3 s_{23} \dot{\theta}_3 \dot{\theta}_1]$$

$$\frac{\partial K_3}{\partial \theta_3} = \frac{m_3}{2} [(l_2 l_3 s_2 c_{23} + \frac{2l_3^2}{3} s_{23} c_{23}) \dot{\theta}_1^2 + \delta_3 (l_3 s_{23}) \dot{\theta}_1 \dot{\theta}_2 - (l_2 l_3 s_3) \dot{\theta}_2^2 - (l_2 l_3 s_3) \dot{\theta}_2 \dot{\theta}_3 + \delta_3 (l_3 s_{23}) \dot{\theta}_3 \dot{\theta}_1]$$

$$\frac{\partial K_3}{\partial \theta_1} = \frac{m_3}{2} [2(l_2^2 s_2^2 + l_2 l_3 s_2 s_{23} + \frac{l_3^2}{3} s_{23}^2 + \delta_3^2) \dot{\theta}_1 - \delta_3 (2l_2 c_2 + l_3 c_{23}) \dot{\theta}_2 - \delta_3 (l_3 c_{23}) \dot{\theta}_3]$$

$$\frac{\partial K_3}{\partial \theta_2} = \frac{m_3}{2} [-\delta_3 (2l_2 c_2 + l_3 c_{23}) \dot{\theta}_1 + 2(l_2^2 + l_2 l_3 c_3 + \frac{l_3^2}{3}) \dot{\theta}_2 + (l_2 l_3 c_3 + \frac{2l_3^2}{3}) \dot{\theta}_3]$$

$$\frac{\partial K_3}{\partial \theta_3} = \frac{m_3}{2} [(l_2 l_3 c_3 + \frac{2l_3^2}{3}) \dot{\theta}_2 + \frac{2l_3^2}{3} \dot{\theta}_3 - \delta_3 (l_3 c_{23}) \dot{\theta}_1]$$

$$\begin{aligned} \frac{d}{dt} \left(\frac{\partial K_3}{\partial \dot{\theta}_1} \right) &= \frac{m_3}{2} [2(\ell_2^2 s_2^2 + \ell_2 \ell_3 s_2 s_{23} + \frac{\ell_3^2}{3} s_{23}^2 + \delta_3^2) \ddot{\theta}_1 - \\ &\quad \delta_3(2\ell_2 c_2 + \ell_3 c_{23}) \ddot{\theta}_2 - \delta_3(\ell_3 c_{23}) \ddot{\theta}_3 + \\ &\quad 2\{2\ell_2^2 s_2 c_2 \dot{\theta}_2 + \ell_2 \ell_3 (c_2 \dot{\theta}_2 s_{23} + s_2 c_{23} (\dot{\theta}_2 + \dot{\theta}_3)) + \\ &\quad \frac{2\ell_3^2}{3} s_{23} c_{23} (\dot{\theta}_2 + \dot{\theta}_3)\} \dot{\theta}_1 + \delta_3(2\ell_2 s_2 \dot{\theta}_2 + \ell_3 s_{23} (\dot{\theta}_2 + \dot{\theta}_3)) \dot{\theta}_2 + \\ &\quad \delta_3(\ell_3 s_{23} (\dot{\theta}_2 + \dot{\theta}_3)) \dot{\theta}_3] \end{aligned}$$

$$\begin{aligned} \frac{d}{dt} \left(\frac{\partial K_3}{\partial \dot{\theta}_2} \right) &= \frac{m_3}{2} [-\delta_3(2\ell_2 c_2 + \ell_3 c_{23}) \ddot{\theta}_1 + 2(\ell_2^2 + \ell_2 \ell_3 c_3 + \frac{\ell_3^2}{3}) \ddot{\theta}_2 + \\ &\quad (\ell_2 \ell_3 c_3 + \frac{2\ell_3^2}{3}) \ddot{\theta}_3 + \delta_3(2\ell_2 s_2 \dot{\theta}_2 + \ell_3 s_{23} (\dot{\theta}_2 + \dot{\theta}_3)) \dot{\theta}_1 - \\ &\quad 2(\ell_2 \ell_3 s_3 \dot{\theta}_3) \dot{\theta}_2 - (\ell_2 \ell_3 s_3 \dot{\theta}_3) \dot{\theta}_3] \end{aligned}$$

$$\begin{aligned} \frac{d}{dt} \left(\frac{\partial K_3}{\partial \dot{\theta}_3} \right) &= \frac{m_3}{2} [(\ell_2 \ell_3 c_3 + \frac{2\ell_3^2}{3}) \ddot{\theta}_2 + \frac{2\ell_3^2}{3} \ddot{\theta}_3 - \delta_3(\ell_3 c_{23}) \ddot{\theta}_1 - \\ &\quad (\ell_2 \ell_3 s_3 \dot{\theta}_3) \dot{\theta}_2 + \delta_3(\ell_3 s_{23} (\dot{\theta}_2 + \dot{\theta}_3)) \dot{\theta}_1] \end{aligned}$$

ACTUATOR TORQUES REQUIRED TO SUPPORT MOTION OF LINK 3.

$$T_{13} = \frac{d}{dt} \left(\frac{\partial K_3}{\partial \dot{\theta}_1} \right) - \frac{\partial K_3}{\partial \theta_1}$$

$$\begin{aligned} T_{13} = & \frac{m_3}{2} [2(l_2^2 s_2^2 + l_2 l_3 s_2 s_{23} + \frac{l_3^2}{3} s_{23}^2 + \delta_3^2) \ddot{\theta}_1 - \\ & \delta_3 (2l_2 c_2 + l_3 c_{23}) \ddot{\theta}_2 - \delta_3 (l_3 c_{23}) \ddot{\theta}_3 + \\ & 2(2l_2^2 s_2 c_2 + l_2 l_3 (c_2 s_{23} + s_2 c_{23}) + \frac{2l_3^2}{3} s_{23} c_{23}) \dot{\theta}_1 \dot{\theta}_2 + \\ & \delta_3 (2l_2 s_2 + l_3 s_{23}) \dot{\theta}_2^2 + 2\delta_3 l_3 s_{23} \dot{\theta}_2 \dot{\theta}_3 + \delta_3 l_3 s_{23} \dot{\theta}_3^2 + \\ & 2l_3 c_{23} (l_2 s_2 + \frac{2l_3}{3} s_{23}) \dot{\theta}_3 \dot{\theta}_1] \end{aligned}$$

$$T_{23} = \frac{d}{dt} \left(\frac{\partial K_3}{\partial \dot{\theta}_2} \right) - \frac{\partial K_3}{\partial \theta_2}$$

$$\begin{aligned} T_{23} = & \frac{m_3}{2} [-\delta_3 (2l_2 c_2 + l_3 c_{23}) \ddot{\theta}_1 + 2(l_2^2 + l_2 l_3 c_3 + \frac{l_3^2}{3}) \ddot{\theta}_2 + \\ & (l_2 l_3 c_3 + \frac{2l_3^2}{3}) \ddot{\theta}_3 - (2l_2^2 s_2 c_2 + l_2 l_3 (c_2 s_{23} + s_2 c_{23}) + \frac{2l_3^2}{3} s_{23} c_{23}) \dot{\theta}_1^2 - \\ & 2(l_2 l_3 s_3) \dot{\theta}_2 \dot{\theta}_3 - (l_2 l_3 s_3) \dot{\theta}_3^2] \end{aligned}$$

Note the cancellation of terms in $\dot{\theta}_1 \dot{\theta}_2$ and $\dot{\theta}_3 \dot{\theta}_1$.

$$T_{33} = \frac{d}{dt} \left(\frac{\partial K_3}{\partial \dot{\theta}_3} \right) - \frac{\partial K_3}{\partial \theta_3}$$

$$T_{33} = \frac{m_3}{2} \left[-\delta_3 (\ell_3 c_{23}) \ddot{\theta}_1 + (\ell_2 \ell_3 c_3 + \frac{2\ell_3^2}{3}) \ddot{\theta}_2 + \frac{2\ell_3^2}{3} \ddot{\theta}_3 - \ell_3 c_{23} (\ell_2 s_2 + \frac{2\ell_3}{3} s_{23}) \dot{\theta}_1^2 + (\ell_2 \ell_3 s_3) \dot{\theta}_2^2 \right]$$

Note cancellation of terms in $\dot{\theta}_1 \dot{\theta}_2$, $\dot{\theta}_2 \dot{\theta}_3$ and $\dot{\theta}_3 \dot{\theta}_1$.

TORQUES REQUIRED TO SUPPORT GRAVITATIONAL FORCES.

The center of gravity of link 2 is

$$\underline{r}_2 = \left(\frac{l_2}{2} c_1 s_2 - \delta_2 s_1\right) \underline{i} + \left(\frac{l_2}{2} s_1 s_2 + \delta_2 c_1\right) \underline{j} + \left(\frac{l_2}{2} c_2\right) \underline{k}$$

Its potential energy is,

$$P_2 = -(\underline{r}_2 \cdot \underline{g})m_2 = -m_2 \left[\left(\frac{l_2}{2} c_1 s_2 - \delta_2 s_1\right) g_1 + \left(\frac{l_2}{2} s_1 s_2 + \delta_2 c_1\right) g_2 + \left(\frac{l_2}{2} c_2\right) g_3 \right]$$

Similarly, the position of the center of gravity of link 3 is

$$\underline{r}_3 = \left[(l_2 s_2 + \frac{l_3}{2} s_{23}) c_1 - \delta_3 s_1 \right] \underline{i} + \left[(l_2 s_2 + \frac{l_3}{2} s_{23}) s_1 + \delta_3 c_1 \right] \underline{j} + \left[l_2 c_2 + \frac{l_3}{2} c_{23} \right] \underline{k}$$

Its potential energy is

$$P_3 = -(\underline{r}_3 \cdot \underline{g})m_3$$

$$P_3 = -m_3 \left[\left[(l_2 s_2 + \frac{l_3}{2} s_{23}) c_1 - \delta_3 s_1 \right] g_1 + \left[(l_2 s_2 + \frac{l_3}{2} s_{23}) s_1 + \delta_3 c_1 \right] g_2 + \left[l_2 c_2 + \frac{l_3}{2} c_{23} \right] g_3 \right]$$

The total potential energy is the sum of P_2 and P_3 .

Torques required then are:

$$\begin{aligned} T_{1g} = -\frac{\partial P}{\partial \theta_1} = & g_1[-m_2(\frac{\ell_2}{2} s_1 s_2 + \delta_2 c_1) - m_3(\ell_2 s_2 + \frac{\ell_3}{2} s_{23})s_1 - \\ & m_3 \delta_3 c_1] + g_2[m_2(\frac{\ell_2}{2} c_1 s_2 - \delta_2 s_1) + m_3(\ell_2 s_2 + \frac{\ell_3}{2} s_{23}) c_1 - \\ & m_3 \delta_3 s_1] \end{aligned}$$

$$\begin{aligned} T_{2g} = -\frac{\partial P}{\partial \theta_2} = & g_1[m_2 \frac{\ell_2}{2} c_1 c_2 + m_3(\ell_2 c_2 + \frac{\ell_3}{2} c_{23})c_1] + \\ & g_2[m_2 \frac{\ell_2}{2} s_1 c_2 + m_3(\ell_2 c_2 + \frac{\ell_3}{2} c_{23})s_1] + \\ & g_3[-m_2 \frac{\ell_2}{2} s_2 - m_3(\ell_2 s_2 + \frac{\ell_3}{2} s_{23})] \end{aligned}$$

$$T_{3g} = -\frac{\partial P}{\partial \theta_3} = g_1[m_3 \frac{\ell_3}{2} c_{23} c_1] + g_2[m_3 \frac{\ell_3}{2} c_{23} s_1] + g_3[-m_3 \frac{\ell_3}{2} s_{23}]$$

MASS CONCENTRATION AT THE END OF LINK 3 -- LOAD CARRIED.

$$v^2 = (l_2^2 s_2^2 + 2l_2 l_3 s_2 s_{23} + l_3^2 s_{23}^2 + \delta_3^2) \dot{\theta}_1^2 -$$

$$2\delta_3 (l_2 c_2 + l_3 c_{23}) \dot{\theta}_1 \dot{\theta}_2 + (l_2^2 + 2l_2 l_3 c_3 + l_3^2) \dot{\theta}_2^2 +$$

$$(2l_2 l_3 c_3 + 2l_3^2) \dot{\theta}_2 \dot{\theta}_3 + l_3^2 \dot{\theta}_3^2 - 2l_3 \delta_3 c_{23} \dot{\theta}_3 \dot{\theta}_1$$

$$K'_3 = \frac{1}{2} M_3 v^2 \quad \frac{\partial K'_3}{\partial \theta_1} = 0$$

$$\frac{\partial K'_3}{\partial \theta_2} = M_3 [(l_2 s_2 + l_3 s_{23})(l_2 c_2 + l_3 c_{23}) \dot{\theta}_1^2 + \delta_3 (l_2 s_2 + l_3 s_{23}) \dot{\theta}_1 \dot{\theta}_2 +$$

$$l_3 \delta_3 s_{23} \dot{\theta}_3 \dot{\theta}_1]$$

$$\frac{\partial K'_3}{\partial \theta_3} = M_3 [(l_2 s_2 + l_3 s_{23}) l_3 c_{23} \dot{\theta}_1^2 + \delta_3 l_3 s_{23} \dot{\theta}_1 \dot{\theta}_2 - l_2 l_3 s_3 (\dot{\theta}_2 + \dot{\theta}_3) \dot{\theta}_2 +$$

$$l_3 \delta_3 s_{23} \dot{\theta}_3 \dot{\theta}_1]$$

$$\frac{\partial K'_3}{\partial \theta_1} = M_3 [(l_2^2 s_2^2 + 2l_2 l_3 s_2 s_{23} + l_3^2 s_{23}^2 + \delta_3^2) \dot{\theta}_1 - \delta_3 (\delta_2 c_2 + l_3 c_{23}) \dot{\theta}_2 -$$

$$\delta_3 l_3 c_{23} \dot{\theta}_3]$$

$$\frac{\partial K'_3}{\partial \theta_2} = M_3 [-\delta_3 (l_2 c_2 + l_3 c_{23}) \dot{\theta}_1 + (l_2^2 + 2l_2 l_3 c_3 + l_3^2) \dot{\theta}_2 +$$

$$(l_2 l_3 c_3 + l_3^2) \dot{\theta}_3]$$

$$\frac{\partial K'_3}{\partial \dot{\theta}_3} = M_3 [\dot{\theta}_3 \delta_3 l_3 c_{23} \dot{\theta}_1 + (l_2 l_3 c_3 + l_3^2) \dot{\theta}_2 + l_3^2 \dot{\theta}_3]$$

$$\begin{aligned} \frac{d}{dt} \left(\frac{\partial K'_3}{\partial \dot{\theta}_1} \right) &= M_3 [(l_2^2 s_2^2 + 2l_2 l_3 s_2 s_{23} + l_3^2 s_{23}^2 + \delta_3^2) \ddot{\theta}_1 - \\ &\quad \delta_3 (l_2 c_2 + l_3 c_{23}) \ddot{\theta}_2 - \delta_3 l_3 c_{23} \ddot{\theta}_3 + \\ &\quad 2(l_2 s_2 + l_3 s_{23})(l_2 c_2 \dot{\theta}_2 + l_3 c_{23}(\dot{\theta}_2 + \dot{\theta}_3)) \dot{\theta}_1 + \\ &\quad \delta_3 (l_2 s_2 \dot{\theta}_2 + l_3 s_{23}(\dot{\theta}_2 + \dot{\theta}_3)) \dot{\theta}_2 + \\ &\quad \delta_3 l_3 s_{23}(\dot{\theta}_2 + \dot{\theta}_3) \dot{\theta}_3] \\ &= M_3 [(l_2^2 s_2 + 2l_2 l_3 s_2 s_{23} + l_3^2 s_{23}^2 + \delta_3^2) \ddot{\theta}_1 - \\ &\quad \delta_3 (l_2 s_2 + l_3 c_{23}) \ddot{\theta}_2 - \delta_3 l_3 c_{23} \ddot{\theta}_3 + \\ &\quad 2(l_2 s_2 + l_3 s_{23})(l_2 c_2 + l_3 c_{23}) \dot{\theta}_1 \dot{\theta}_2 \\ &\quad (l_2 s_2 + l_3 s_{23}) \delta_3 \dot{\theta}_2^2 + 2\delta_3 l_2 s_{23} \dot{\theta}_2 \dot{\theta}_3 + \\ &\quad \delta_3 l_3 s_{23} \dot{\theta}_3^2 + 2(l_2 s_2 + l_3 s_{23}) l_3 c_{23} \dot{\theta}_3 \dot{\theta}_1] \end{aligned}$$

$$\begin{aligned}
 \frac{d}{dt} \left(\frac{\partial K'_3}{\partial \dot{\theta}_2} \right) &= M_3 [-c_3(\ell_2 c_2 + \ell_3 c_{23}) \ddot{\theta}_1 + (\ell_2^2 + 2\ell_2 \ell_3 c_3 + \ell_3^2) \ddot{\theta}_2 + \\
 &\quad (\ell_2 \ell_3 c_3 + \ell_3^2) \ddot{\theta}_3 + \delta_3 (\ell_2 s_2 \dot{\theta}_2 + \ell_3 s_{23} (\dot{\theta}_2 + \dot{\theta}_3)) \dot{\theta}_1 - \\
 &\quad - 2\ell_2 \ell_3 s_3 \dot{\theta}_2 \dot{\theta}_3 - \ell_2 \ell_3 s_3 \dot{\theta}_3^2] \\
 &= M_3 [-c_3(\ell_2 c_2 + \ell_3 c_{23}) \ddot{\theta}_1 + (\ell_2^2 + 2\ell_2 \ell_3 c_3 + \ell_3^2) \ddot{\theta}_2 + \\
 &\quad (\ell_2 \ell_3 c_3 + \ell_3^2) \ddot{\theta}_3 + \delta_3 (\ell_2 s_2 + \ell_3 s_{23}) \dot{\theta}_1 \dot{\theta}_2 - \\
 &\quad 2\ell_2 \ell_3 s_3 \dot{\theta}_2 \dot{\theta}_3 - \ell_2 \ell_3 \dot{\theta}_3^2 + \delta_3 \ell_3 s_{23} \dot{\theta}_3 \dot{\theta}_1]
 \end{aligned}$$

$$\begin{aligned}
 \frac{d}{dt} \left(\frac{\partial K'_3}{\partial \dot{\theta}_3} \right) &= M_3 [-\delta_3 \ell_3 c_{23} \dot{\theta}_1 + (\ell_2 \ell_3 c_3 + \ell_3^2) \ddot{\theta}_2 + \ell_3^2 \ddot{\theta}_3 + \\
 &\quad \delta_3 \ell_3 s_{23} \dot{\theta}_1 \dot{\theta}_2 - \ell_2 \ell_3 s_3 \dot{\theta}_2 \dot{\theta}_3 + \delta_3 \ell_3 s_{23} \dot{\theta}_3 \dot{\theta}_1]
 \end{aligned}$$

$$\begin{aligned}
 T'_1 &= M_3 [\ell_2^2 s_2^2 + 2\ell_2 \ell_3 s_2 s_{23} + \ell_3^2 s_{23}^2 + \delta_3^2] \ddot{\theta}_1 - \\
 &\quad \delta_3 (\ell_2 c_2 + \ell_3 c_{23}) \ddot{\theta}_2 - \delta_3 \ell_3 c_{23} \ddot{\theta}_3 + 2(\ell_2 s_2 + \ell_3 s_{23}) \\
 &\quad (\ell_2 c_2 + \ell_3 c_{23}) \dot{\theta}_1 \dot{\theta}_2 + (\ell_2 s_2 + \ell_3 s_{23}) \delta_3 \dot{\theta}_2^2 + \\
 &\quad 2\delta_3 \ell_2 s_{23} \dot{\theta}_2 \dot{\theta}_3 + \delta_3 \ell_3 s_{23} \dot{\theta}_3^2 + 2\ell_3 c_{23} (\ell_2 s_2 + \ell_3 s_{23}) \dot{\theta}_3 \dot{\theta}_1
 \end{aligned}$$

$$\begin{aligned} T_2' = & M_3[-\delta_3(l_2c_2 + l_3c_{23})\ddot{\theta}_1 + (l_2^2 + 2l_2l_3c_3 + l_3^2)\ddot{\theta}_2 + \\ & l_3(l_2c_3 + l_3)\ddot{\theta}_3 - (l_2s_2 + l_3s_{23})(l_2c_2 + l_3c_{23})\dot{\theta}_1^2 - \\ & 2l_2l_3s_3\dot{\theta}_2\dot{\theta}_3 - l_2l_3s_3\dot{\theta}_3^2 \end{aligned}$$

$$\begin{aligned} T_3' = & M_3[-\delta_3l_3c_{23}\ddot{\theta}_1 + l_3(l_2c_3 + l_3)\ddot{\theta}_2 + l_3^2\ddot{\theta}_3 - l_3c_{23} - \\ & (l_2s_2 + l_3s_{23})\dot{\theta}_1^2 + l_2l_3s_3\dot{\theta}_2^2] \end{aligned}$$

GRAVITY COMPENSATION FOR LOAD CARRIED.

The position of the load is,

$$\begin{aligned} r_3 = & [(l_2 s_2 + l_3 s_{23})c_1 - \delta_3 s_1]i + [(l_2 s_2 + l_3 s_{23})s_1 + \\ & \delta_3 c_1]j + [l_2 c_2 + l_3 c_{23}]k \end{aligned}$$

The potential energy is,

$$\begin{aligned} P_3' = & -(r_3 \cdot g)M_3 = -M_3 [g_1((l_2 s_2 + l_3 s_{23})c_1 - \delta_3 s_1) + \\ & g_2(l_2 s_2 + l_3 s_{23})s_1 + \delta_3 c_1] + g_3(l_2 c_2 + l_3 c_{23})] \end{aligned}$$

$$\begin{aligned} T_{1g}' = & M_3 \{ -g_1[(l_2 s_2 + l_3 s_{23})s_1 + \delta_3 c_1] + \\ & g_2[(l_2 s_2 + l_3 s_{23})c_1 - \delta_3 s_1] \} \end{aligned}$$

$$\begin{aligned} T_{2g}' = & M_3 \{ g_1(l_2 c_2 + l_3 c_{23})c_1 + g_2(l_2 c_2 + l_3 c_{23})s_1 - \\ & g_3(l_2 s_2 + l_3 s_{23}) \} \end{aligned}$$

$$T_{3g}' = M_3 \{ g_1 l_3 c_{23} c_1 + g_2 l_3 c_{23} s_1 - g_3 l_3 s_{23} \}$$

SUMMARY AND CONCLUSIONS.

We have shown that detailed analysis of the dynamics of mechanical manipulators is feasible and leads to complicated, but manageable equations. Such equations may be used in the simulation of such devices or directly in control systems based on open-loop computation of required joint torques. Simulations may be used in the design or analysis of traditional control systems, which can maintain stable control only for low speeds.

New kinds of control systems can be envisaged, where negative feedback is only used to account for small errors which come about because of differences between the actual device and the mathematical model used in deriving these equations.

We have derived the necessary equations for devices with rotational degrees of freedom and no offsets (pages 10, 13, 19). Compensation for gravitational forces have also been calculated for arbitrary orientation of the device (page 22) as have the torques required to move a load carried at the tip of link 3 (page 25). Finally, compensation for gravitational forces on this extra load were considered (page 26).

The same calculations were then repeated for a device with offsets (pages 10, 29, 34, 35).

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2. B. K. P. Horn, "Kinematics, Statics, and Dynamics of Two-D Manipulators," MIT AI Lab Working Paper No. 99, June 1975.
3. B. K. P. Horn, "The Fundamental Eel Equations," MIT AI Lab Working Paper No. 116, December 1975.

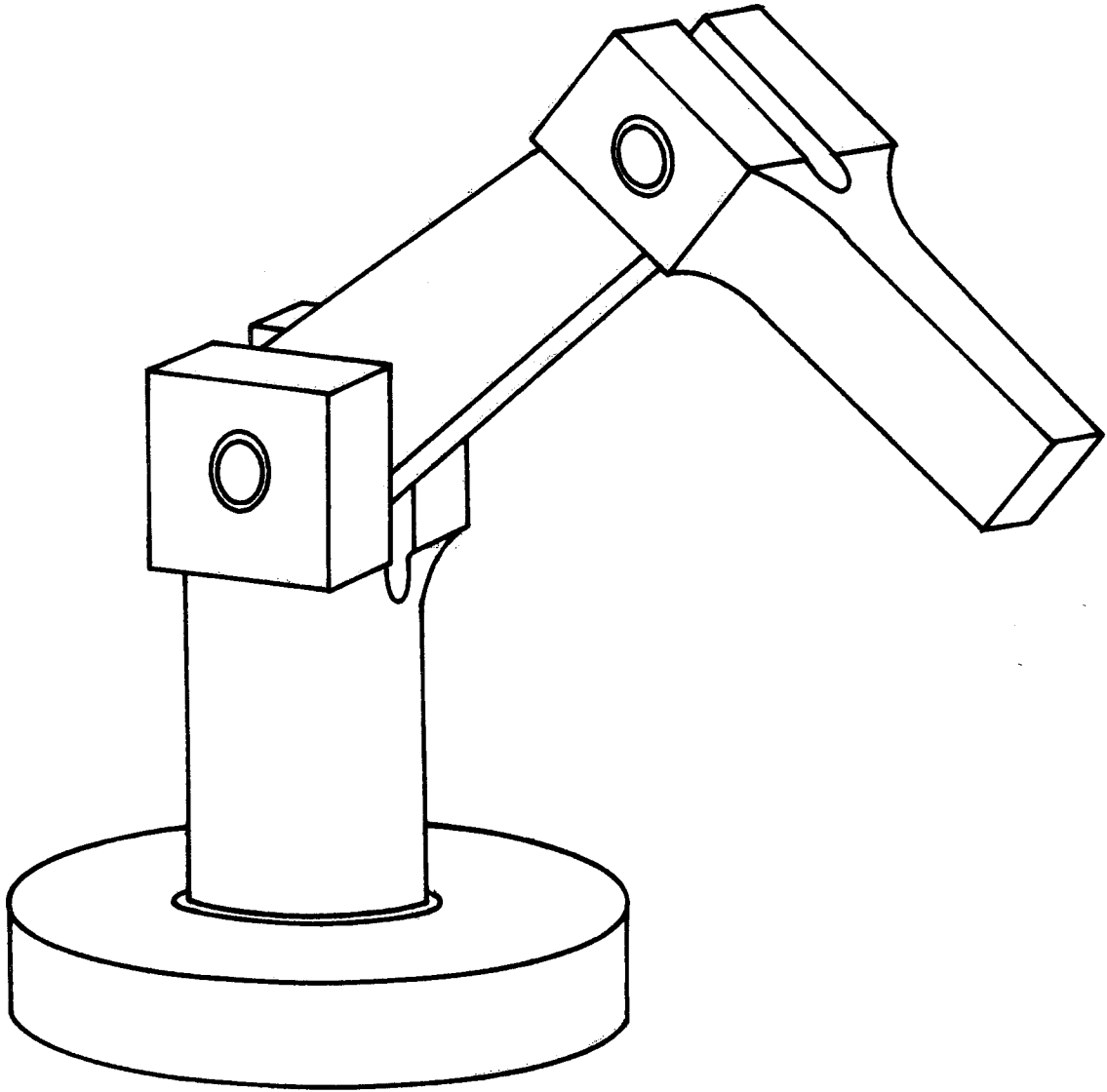


FIGURE 1

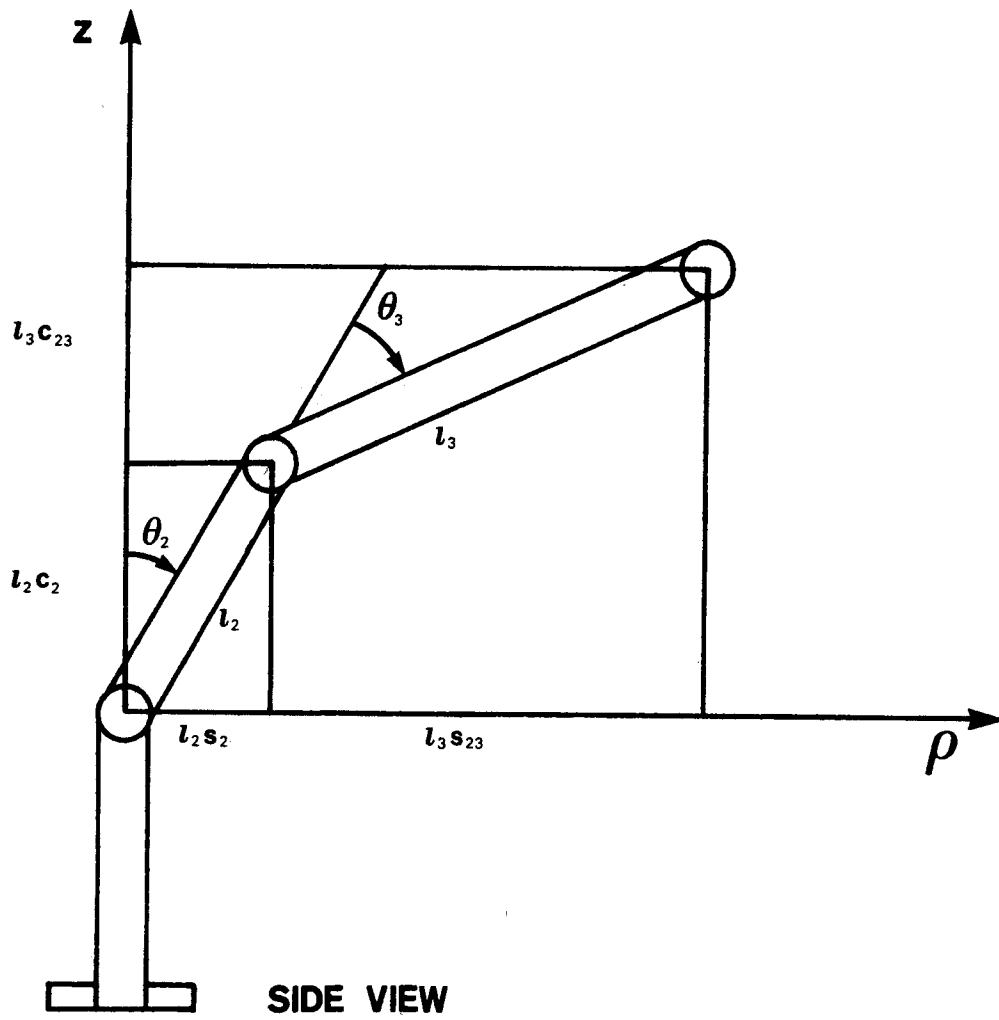
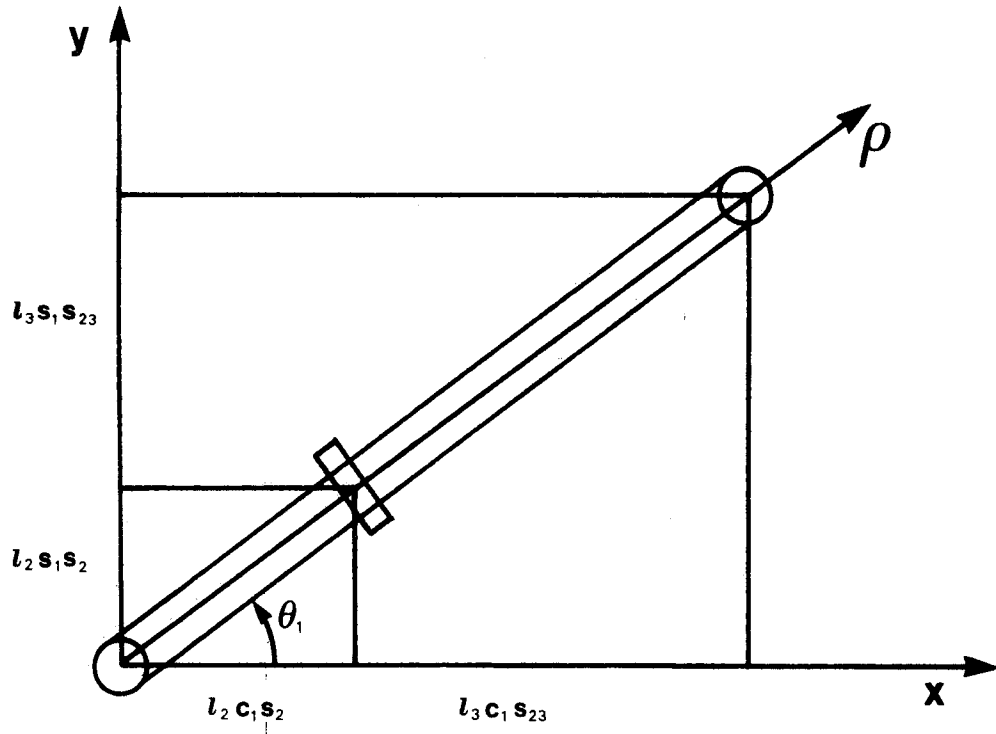


FIGURE 2



TOP VIEW

FIGURE 3

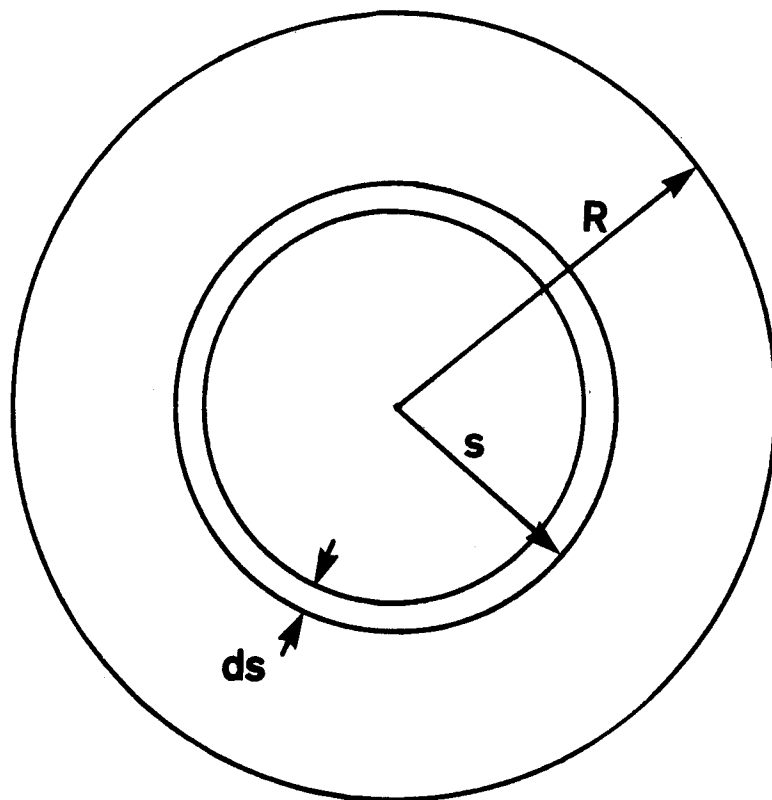
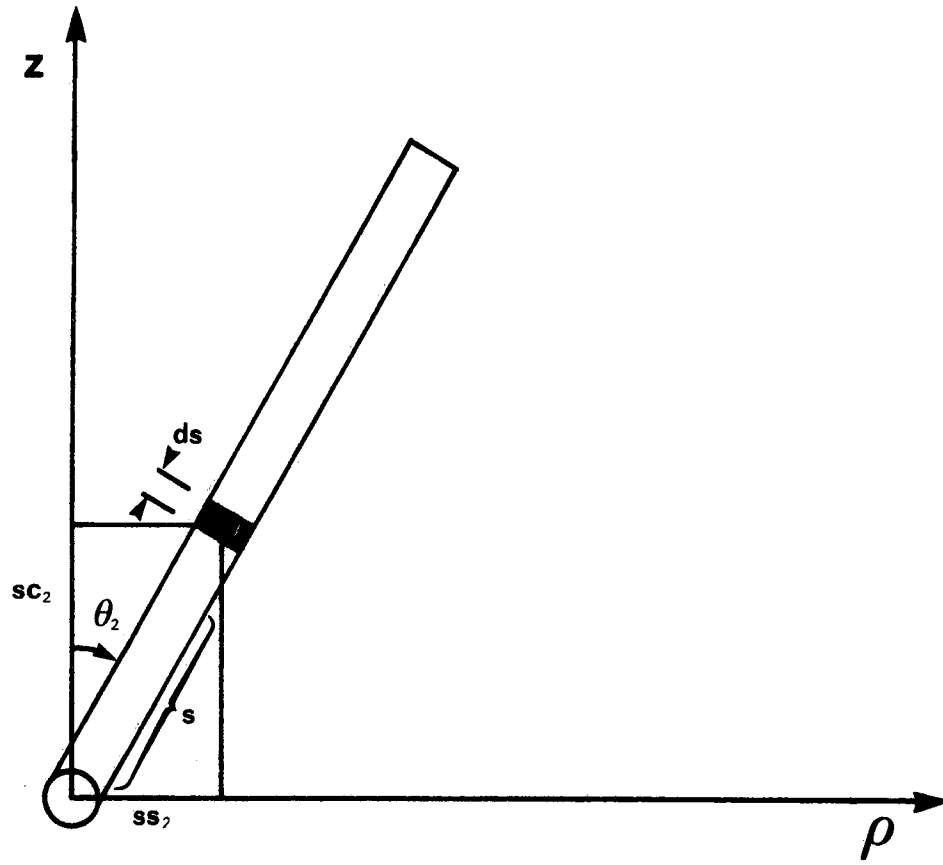


FIGURE 4



SIDE VIEW

FIGURE 5

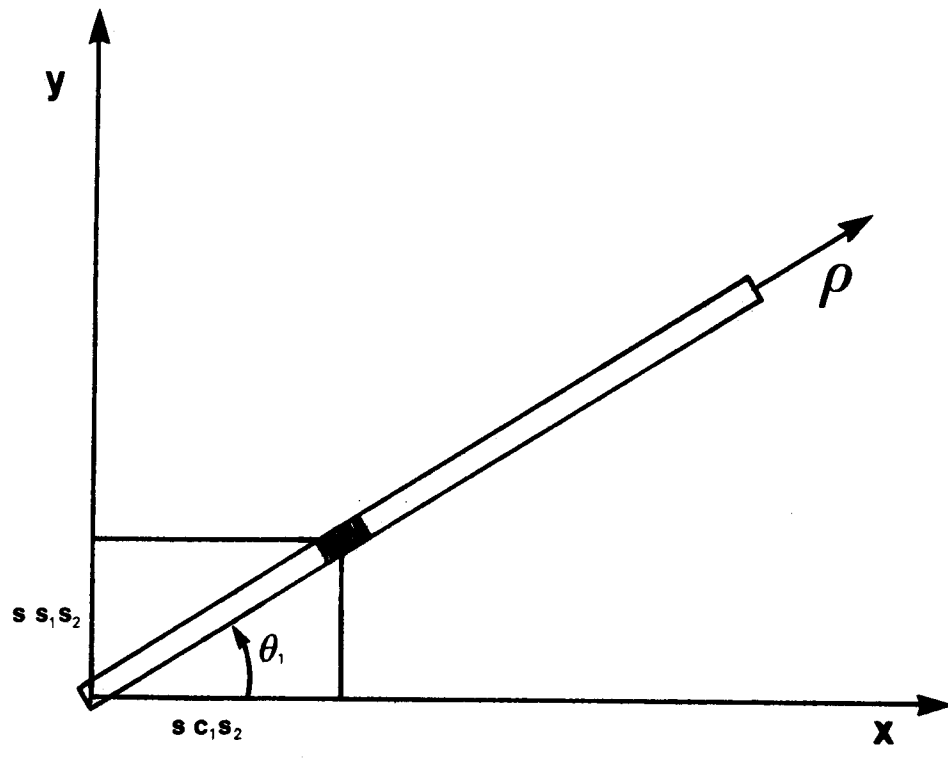
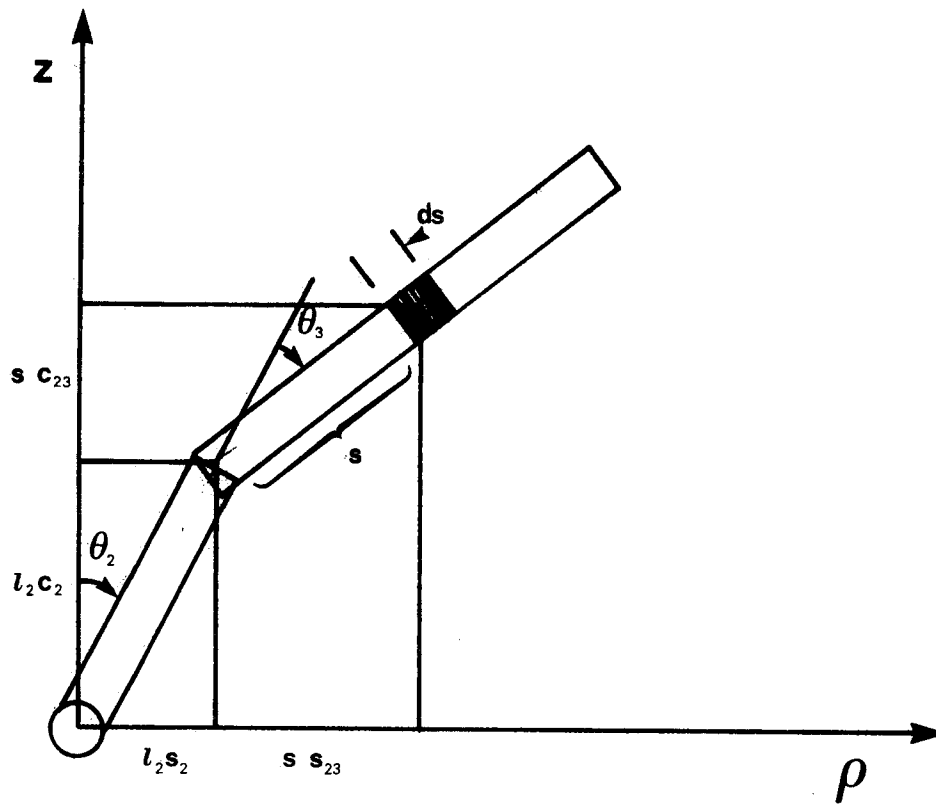
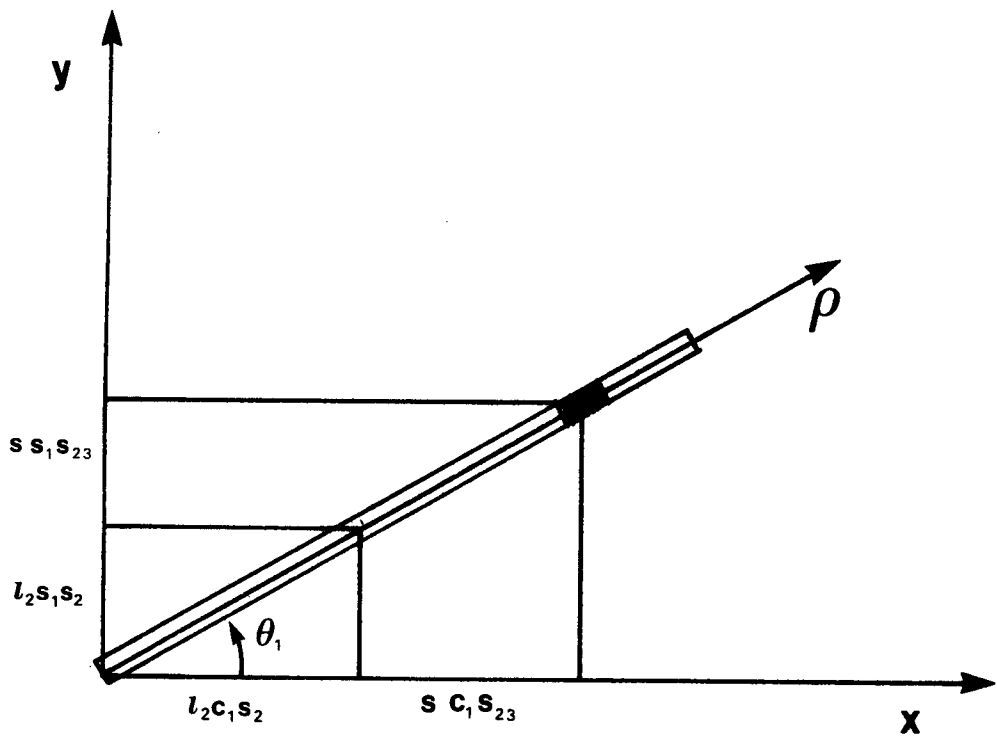


FIGURE 6



SIDE VIEW

FIGURE 7



TOP VIEW

FIGURE 8

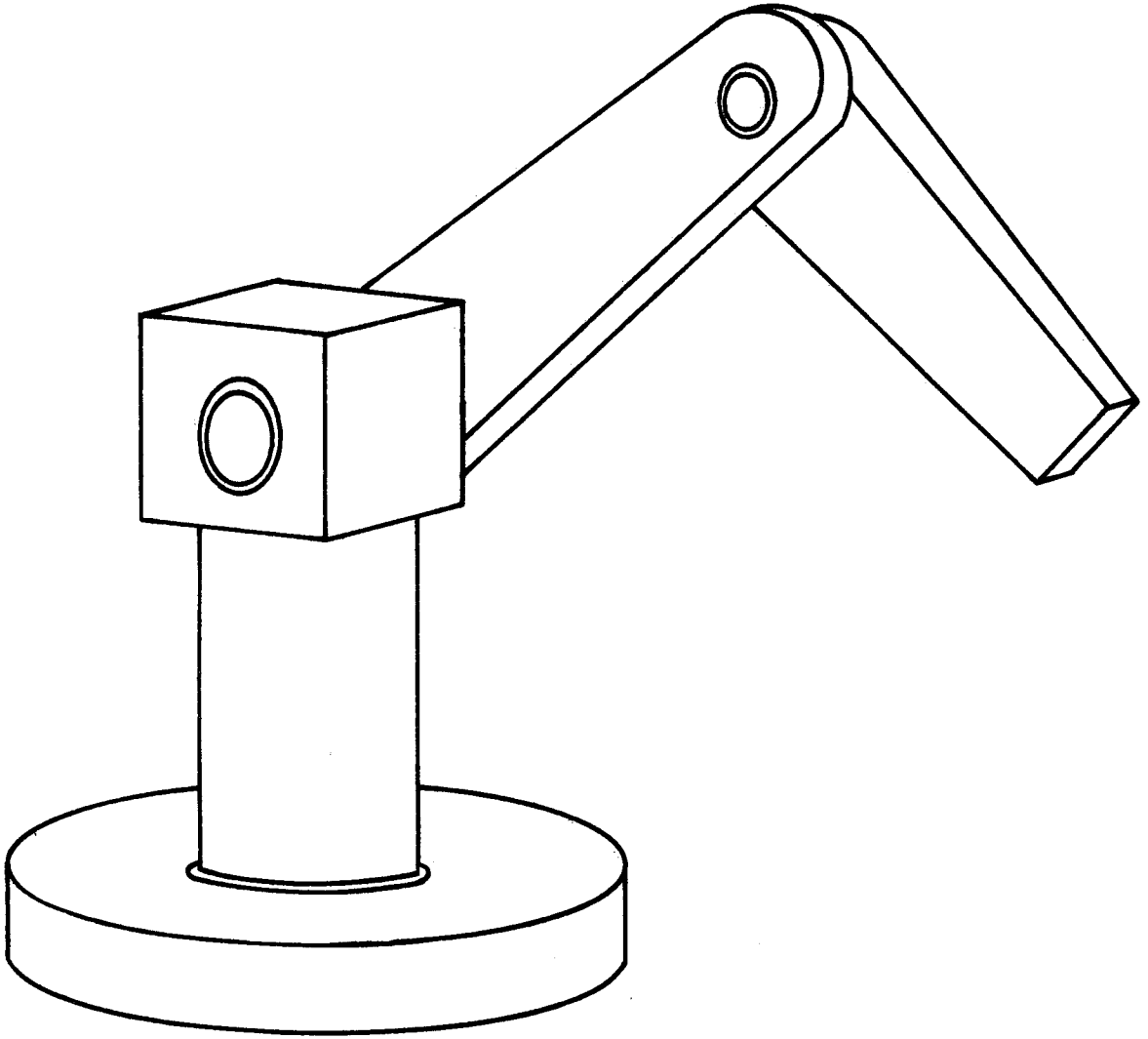


FIGURE 9

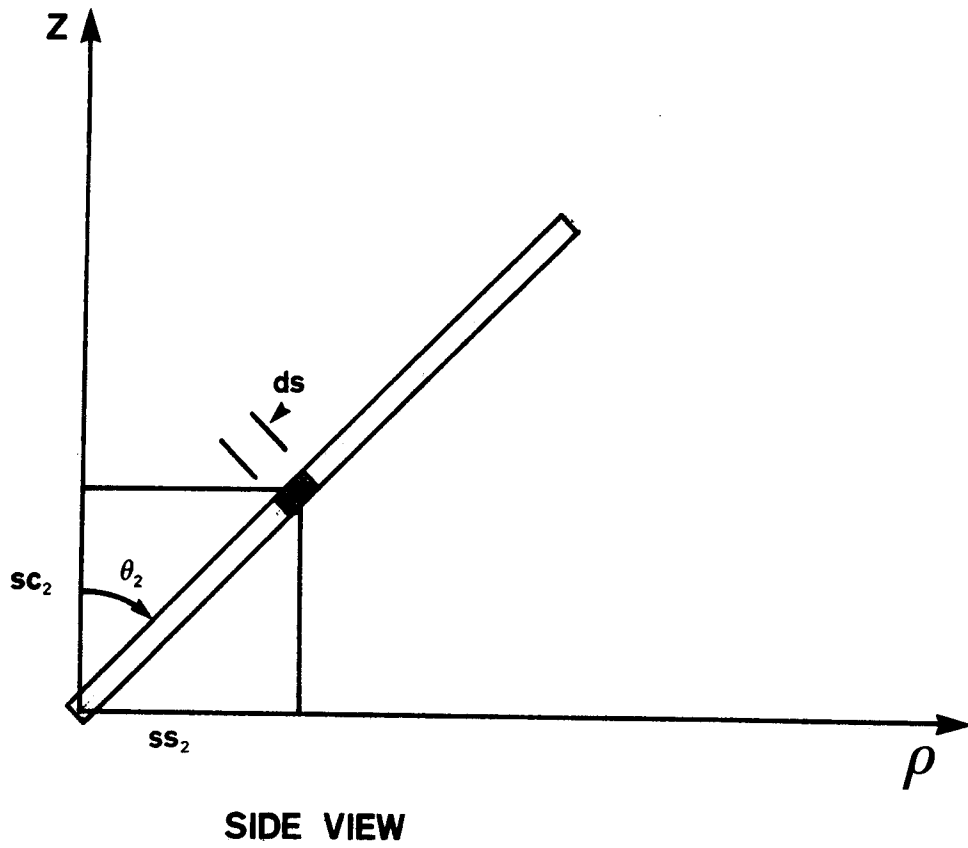
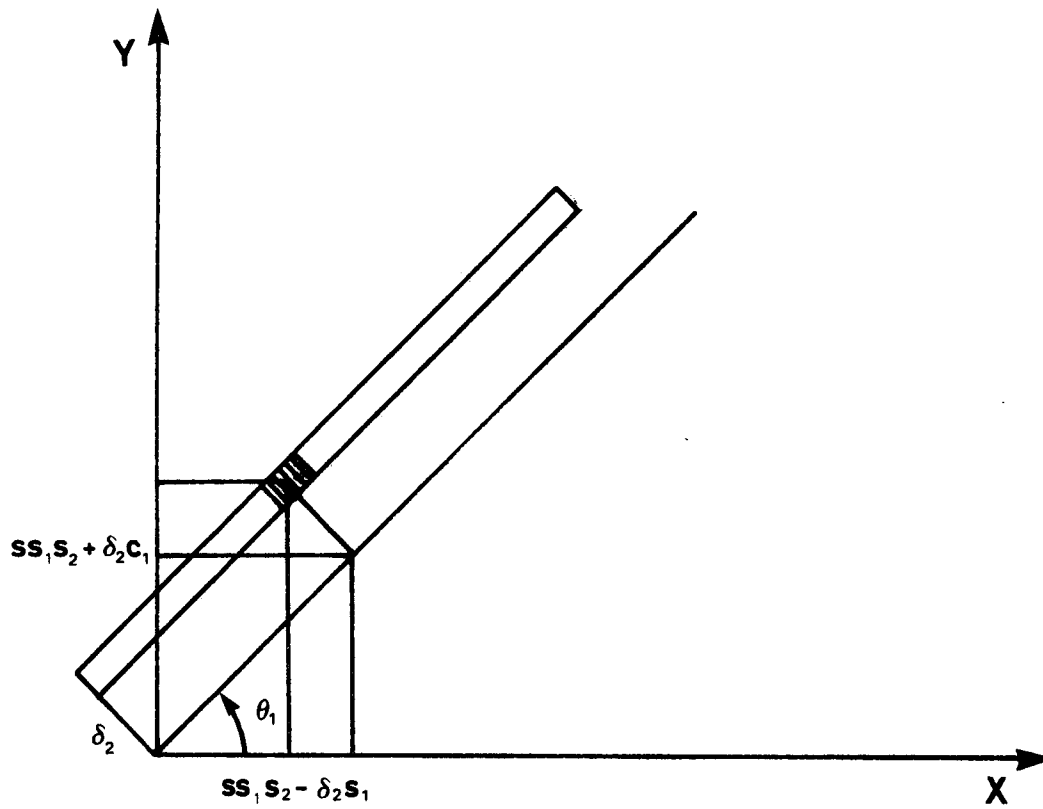
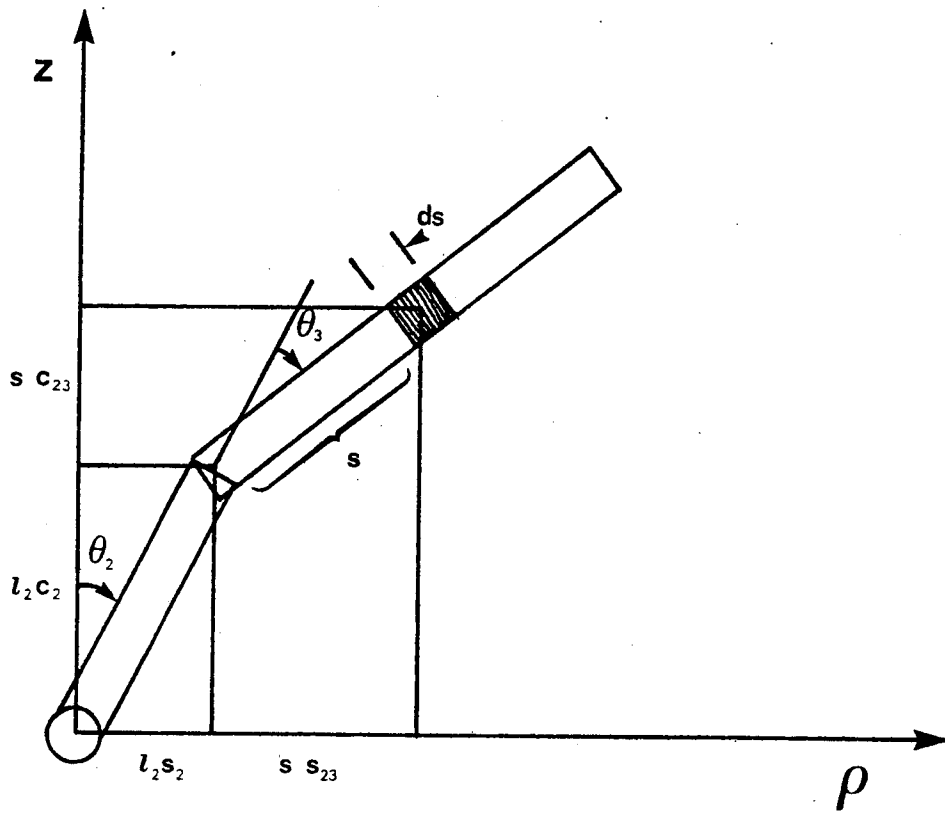


FIGURE 10



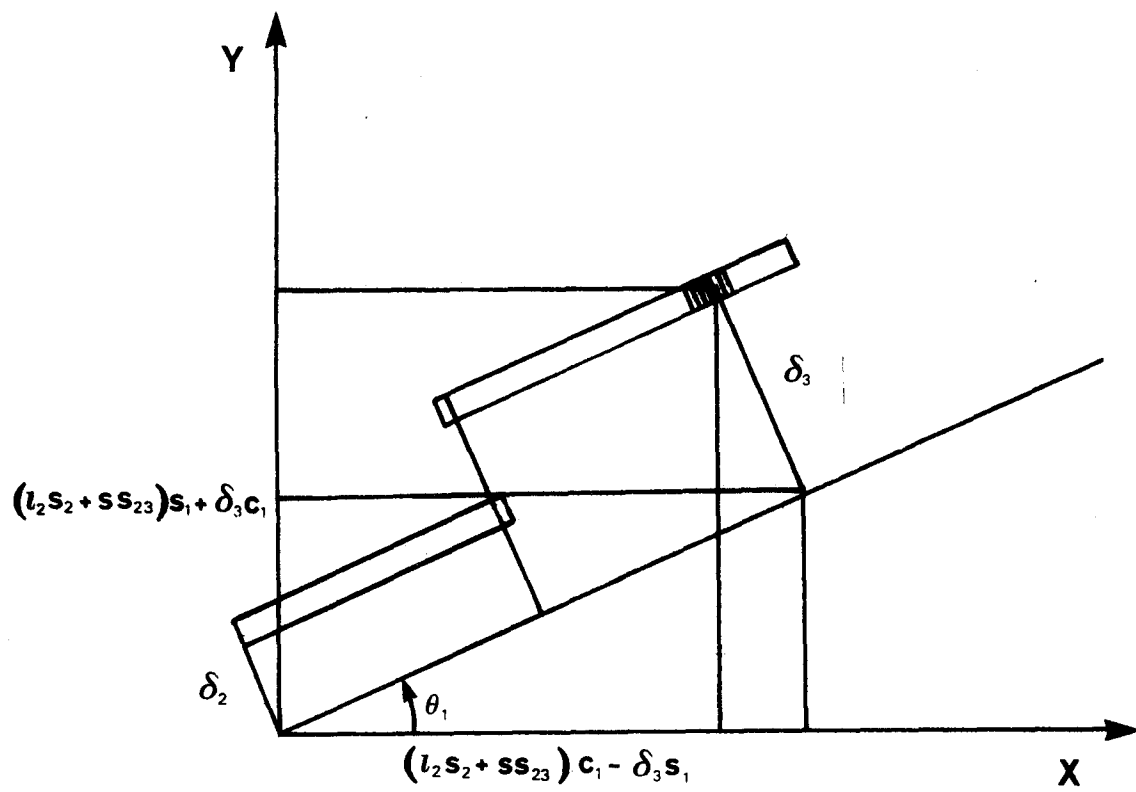
TOP VIEW

FIGURE 11



SIDE VIEW

FIGURE 12



TOP VIEW

FIGURE 13