Dynamic Rate-Control and Scheduling Algorithms for Quality-of-Service in Wireless Networks

by

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Submitted to the Department of Electrical Engineering and Computer Science
in partial fulfillment of the requirements for the degree of

Doctor of Philosophy in Electrical Engineering and Computer Science

at the

MASSACHUSETTS INSTITUTE OF TECHNOLOGY

September 2007

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Abstract

Rapid growth of the Internet and multimedia applications, combined with an increasingly ubiquitous deployment of wireless systems, has created a huge demand for providing enhanced data services over wireless networks. Invariably, meeting the quality-of-service requirements for such services translates into stricter packet-delay and throughput constraints on communication. In addition, wireless systems have stringent limitations on resources which necessitates that these must be utilized in the most efficient manner. In this thesis, we develop dynamic rate-control and scheduling algorithms to meet quality-of-service requirements on data while making efficient utilization of resources. Ideas from *Network Calculus* theory, *Continuous-time Stochastic Optimal Control* and *Convex Optimization* are utilized to obtain a theoretical understanding of the problems considered, and to develop various insights from the analysis.

We, first, address energy-efficient transmission of deadline-constrained data over wireless fading channels. In this setup, a transmitter with controllable transmission rate is considered, and the objective is to obtain a rate-control policy for transmitting deadline-constrained data with minimum total energy expenditure. Towards this end, a deterministic model is first considered and the optimal policy is obtained graphically using a novel *cumulative curves* methodology. We, then, consider stochastic channel fading and introduce the canonical problem of transmitting B units of data by deadline T over a Markov fading channel. This problem is referred to as the "BT-problem" and its optimal solution is obtained using techniques from stochastic control theory. Among various extensions, specific setups involving variable deadlines on the data packets, known arrivals and a Poisson arrival process are considered. Using a graphical approach, transmission policies for these cases are obtained through a natural extension of the results obtained earlier.

In the latter part of the thesis, a multi-user downlink model is considered which consists of a single transmitter serving multiple mobile users. Here, the quality-of-service requirement is to provide guaranteed average throughput to a certain class of users, and the objective is to obtain a multi-user scheduling policy that achieves this using the minimum number of time-slots. Based on a geometric approach we obtain the optimal policy for a general fading scenario, and, further specialize it to the case of symmetric Rayleigh fading to obtain closed-form relationships among the various performance metrics.

Thesis Supervisor: Eytan Modiano
Title: Associate Professor

Acknowledgments

As I look back on the years spent at MIT, I recall the moments of fun, play and learning, that have all together made this an immensely memorable endeavor. I realize the value of this opportunity and the change it has brought in my life, both in terms of professional and intellectual development. It is an honor and pleasure to have participated in this process, and I would like to express gratitude towards everyone and everything that made this happen.

First and foremost, I would like to profoundly thank my advisor, Prof. Eytan Modiano, for his invaluable guidance in research and career, and the fun we had as a research group. I relish the relationship that we have developed over these years and look forward to continue to seek his advice in the future; I thank him in advance for it. I would also like to express gratitude to my thesis committee members, Prof. Sanjoy Mitter, Prof. Asuman Ozdaglar and Prof. Devavrat Shah, for valuable comments and suggestions on the research work. In the Spring of 2006, it was a pleasure to work with Prof. John Wyatt as a teaching assistant and I would like to thank him for giving me this opportunity. I really appreciate his belief in intuition and the wonders it can bring in understanding and solving problems.

The other important part of my experience at MIT was the diverse set of people that I met and the friendships that I made over these years. I would like to thank all my friends and colleagues for their support and encouragement. Thanks to Anand Srinivas for adding 'unlimited' humour at and outside of work; I would always cherish his friendship and the times we spent together. Thanks to my other wonderful office mates, Andrew Brzezinski and Jun Sun, who were equal partners in the 'crimes' (read 'fun'), that included tennis, T.T. sessions, lunches, tea/coffee breaks and long 'enlightening' discussions. Thanks to Zeeshan Syed, Ebad Ahmed, Shashibhushan Borade and Siddhartha Jain for making S&P inhabitable and a fun place to live in. Thanks also to Sonia Jain, Masha Ishutkina, Dr. Gil Zussman, Prakash Pottukuchi, S. Krishnakumar, Krishna Jagannathan, Guner Celik, Ajay Deshpande, Saif Khan, Raj Rao, Siddharth Ray, Jay-kumar Sundararajan, Amit Deshpande, Rehan Tahir, Saad Zaheer, Danish Maqbool and others, for the enjoyable and memorable times spent together.

My deepest gratitude is for my parents, Abbas and Mehjabeen Zafer, for their unwavering support, love and encouragement, and above all, the sacrifices they have made for me; words will never be able to describe my appreciation for them. I would also like to extend my deepest thanks to my brother, Ali Zafer, for all his advices that have and will continue to shape my life. Thanks to my sister, Zainab, for her love and support, and my sister-in-law, Sadaf. My gratitude for my grand-parents and others in the family (especially my aunt Khudammi) who have been an integral part of the support and encouragement that I have received. My deepest thanks to my loving fiancee, Samreen, for her love and support during the latter years at MIT, as I look forward to spending my life with her.

I am grateful to the various funding agencies for their financial support for this research work. In particular, this work was supported by NSF ITR grant CCR-0325401, by DARPA/AFOSR through the University of Illinois grant no. F49620-02-1-0325, by NASA Space Communication Project grant number NAG3-2835, by ONR grant number N000140610064 and by a grant from Draper laboratory.

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Chapter 1

Introduction

Communication technology has advanced rapidly over the last few decades, from point-to-point telegraphic services to modern wired-telephone and computer networks, and now expanding to wireless systems. While the earlier telephone systems were designed primarily for voice communication, present day communication networks handle a large volume of data traffic which is expected to further grow exponentially, fuelled by the rapid growth of the Internet and multimedia applications. Data services are expected to expand beyond email and web-data transfers to more enhanced services such as video and real-time multimedia streaming, delay-constrained file transfers and Voice-over-IP (VoIP) [1]. To deliver these services there are various wireless data systems under development that include, for example, 1xEV-DO/HDR [3], 3G/4G and WiMAX systems. Applications involving delay constraints also arise in other communication systems such as sensor and mobile ad-hoc networks. For example, in real-time monitoring scenarios using sensor networks, the data collected by the sensor devices must be transmitted back to a central processing node within a certain fixed time-interval. Invariably, providing such enhanced Quality-of-Service (QoS) translates into stricter delay and throughput requirements on communication, thus, introducing new problems and challenges in addressing these concerns.

As compared to the wire-line networks, communication over wireless channels inherently involves dealing with time-varying and stochastic channel conditions and scarcity of resources. Time-varying channel conditions arise due to a variety of reasons, most common being multi-path fading, shadowing and weather conditions in case of satellite communication [4, 5]. Due to the time-varying nature of the channel gain and interference from other sources, the signal-to-noise power at the receiver fluctuates over time which translates into a time-varying rate at which data can be reliably received for a certain bit-error probability. In addition to the channel variability, wireless systems also have more stringent limitations on resources such as battery energy, bandwidth etc., and therefore it necessitates that these must be utilized in the most efficient manner.

In this thesis, we develop dynamic rate-control and scheduling algorithms to meet quality-of-service requirements on data while making efficient utilization of resources. We adopt a theoretical viewpoint and obtain optimal solutions under various setups, utilizing techniques from *Network Calculus* [50–53], *Continuous-time Stochastic Control* theory [63–65] and *Convex Optimization* [66]. In Chapters 2, 3 and 4, we consider a point-to-point wireless link model and treat various formulations in which the objective is to minimize the total transmission energy expenditure when packets have strict deadline constraints. In Chapter 5, we consider a wireless down-link model where there is a single transmitter serving multiple mobile users and the objective is to obtain a multi-user scheduling policy that minimizes the total time-slot utilization while providing throughput-rate guarantees.

For the remainder of this chapter, we delve into a more detailed overview of the problems addressed in the thesis, outline the related work in the literature and describe our contributions. Finally, we conclude the chapter with an outline of the thesis.

1.1 Deadline-Constrained Energy-Efficient Rate Control

1.1.1 Problem Overview

Energy consumption is an important concern in wireless system design [2, 8–13, 21, 22, 26–28, 41, 48] and minimizing the total energy expenditure has numerous advantages in terms of efficient battery utilization for mobile devices, increased lifetime of sensor devices and mobile ad-hoc networks, and better utilization of limited energy sources in satellites. Since in most scenarios the energy spent for transmission constitutes the bulk of the total energy expenditure, it is imperative to minimize this cost to achieve significant energy savings. The work presented in Chapters 2, 3 and 4, addresses energy-efficient transmission of data over a wireless channel with deadline constraints. Broadly speaking, we consider a point-to-point wireless link model with strict deadline constraints on data transmission and utilize dynamic rate-control to minimize the total transmission energy cost. A schematic diagram

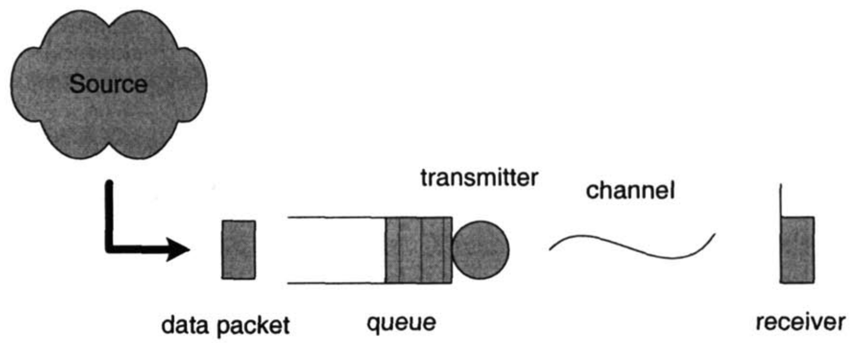


Figure 1-1: A schematic diagram of the system model for the deadline-constrained, energy-efficient, data transmission problem

of the setup is shown in Figure 1-1.

To understand how transmission energy expenditure can be minimized using rate control, we need to look at the power-rate function. The power-rate function defines the relationship that specifies the amount of transmission power required to reliably transmit at a certain rate. Two fundamental aspects of this function, which are exhibited by most encoding/communication schemes and hence are common assumptions in the literature are as follows [4,8–13,21,22,27,29,32,39,40]. First, for a fixed bit-error probability and channel state, the required transmission power is a convex function of the data rate, as shown in Figure 1-2(a). This implies, from Jensen's inequality, that transmitting data at low rates, over a longer duration, is more energy efficient as compared to high rate transmission. Second, the wireless channel is time-varying which shifts the convex power-rate curve as a function of the channel state as shown in Figure 1-2(b). As good channel conditions require less transmission power, one can exploit this variability over time by adapting the rate in response to the channel conditions. Thus, we see that by adapting the transmission rate intelligently over time, energy cost can be reduced.

Modern wireless devices are equipped with channel measurement and rate adaptation capabilities [3, 4, 6, 7]. Channel measurement allows the transmitter-receiver pair to measure the fade state using a pre-determined pilot signal while rate control capability allows the transmitter to adjust the transmission rate over time. Such a control can be achieved in various ways that include adjusting the power level, symbol rate, coding scheme, constellation size and any combination of these approaches; furthermore, in some technologies the receiver can detect these changes directly from the received data without the need for an explicit rate change control information [7]. In present systems, the transmission rate

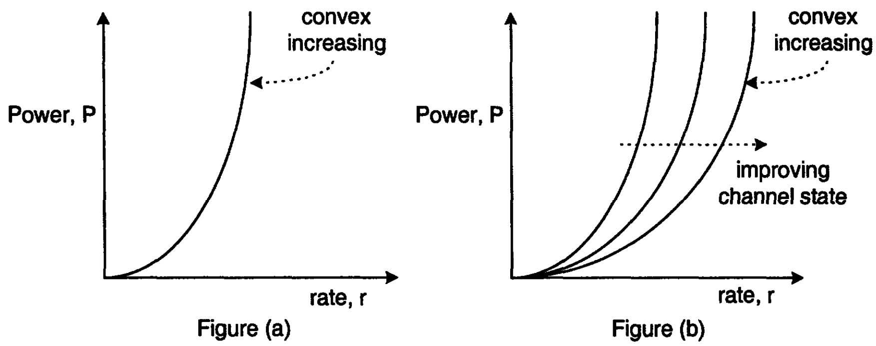


Figure 1-2: Transmission power as a function of the rate and the channel state; (a) fixed channel state, (b) variable channel state.

can be adapted very rapidly over millisecond duration time-slots [3,4,6], thereby, providing ample opportunity to utilize rate adaptation to optimize system performance.

Summarizing, for a given transmitter-receiver pair with rate-adaptation capabilities and the above mentioned power-rate function characteristics, the goal of this research work is to seek the transmission policy that minimizes the energy expenditure while ensuring that the strict QoS constraints are met. Throughout the thesis, the terms *transmission policy* and *rate-control policy* will be used interchangeably and they refer to the transmission rate to be selected for data transmission at a certain time.

1.1.2 Related Work

Transmission power and rate control are an active area of research in communication networks in various contexts. Power control in cellular CDMA networks has been studied extensively, but with the primary motivation of mitigating interference and addressing the "near-far" problem [4,33–35]. Adaptive algorithms for network control have been studied in the context of network stability [36–40,42–45], wherein, various notions of stability are addressed, but the primary goal is to ensure that the queue sizes do not grow to infinity. Scheduling and power control have also been considered in the context of average throughput [46,49,75–77], average delay [8,9,27–29] and packet/call drop probability [30–32]. However, this body of literature considers "average metrics" that are measured over an infinite time horizon and hence do not directly apply for delay constrained/real-time data. Furthermore, with strict deadline constraints, adapting the transmission rate simply based on steady state distributions does not suffice and one needs to take into account the system dynamics over time, thereby, introducing new challenges and complexity into the problem.

Recent work in this direction includes [10–13,20–22,26]. The works in [10–12,26] studied various offline formulations for energy-efficient data transmission by assuming complete knowledge of the future arrivals and the channel states, and then devised heuristic online policies using the offline optimal solution. Thus, in this body of work, the sample path was assumed known for the optimization problem therein. The authors in [13] studied several data transmission problems using discrete-time Dynamic Programming (DP) [61], however, the problems that we consider in this work become intractable using this methodology due to the large state space in the DP-formulation or the well-known "curse of dimensionality". In [21], the authors considered packet deadlines and transmission over a time-invariant (non-fading) channel and used filtering techniques to obtain the energy efficient policy, while, the formulation in [22] allowed energy recovery when the transmitter is in the idle state. As we see later, the generalized formulation that we consider in Chapter 2 recovers back the results in [10, 21] as special cases.

Job scheduling with deadlines has also been considered in the Operations Research literature. Recent relevant work here includes [23–25] which deal with scheduling of jobs (or packets) with hard deadlines, where the service rate is fixed and the goal is to maximize the number of packets that get served. However, the difference in the system model for our case is that the service rate (transmission rate) is controllable and there is a power cost associated with using a particular service rate; furthermore, there is no dropping of packets and the goal is to minimize the total energy cost of transmission.

1.1.3 Contributions

We consider two different setups for the rate control problem – the *Deterministic Setup* and the *Stochastic Setup*. In Chapter 2, we consider the deterministic setup in which all time-variability in the system is known in advance and the goal is to seek the optimal off-line solution. Here, we describe the data flows in and out of the transmitter queue using cumulative curves, namely, the *Arrival Curve* and the *Departure Curve*; and model the quality-of-service (QoS) constraints using a new notion of a *Minimum Departure Curve*. Using this framework, we obtain the optimal policy for a general formulation that incorporates a wide set of QoS constraints in the problem, hence, some of the earlier results in the literature can be recovered as special cases from our general formulation. We also present a graphical visualization of the problem that provides an intuitive and easy way to understand the optimal minimum-energy transmission policy.

In Chapters 3 and 4, we consider the stochastic setup and begin in Chapter 3 with the following canonical problem (which is referred to as the "BT-problem") – the transmitter has B bits of data in the queue which must be transmitted by deadline T over a time-varying and stochastic channel. The channel state is modelled as a Markov process. And, the objective is to obtain the optimal transmission policy that minimizes the expected total energy expenditure. We consider two different formulations here – first in which there is no maximum power limit and the deadline constraint is a hard constraint, and second in which there is an average short-term power limit and the data left in the queue at time T incurs a penalty cost. Using a continuous-time formulation and techniques from *Stochastic Optimal Control* [63–65] theory and *Lagrangian Duality* [66,68], we obtain the optimal transmission policy for both these setups. From the closed-form structure of the optimal policy, various useful insights into the data transmission problem and results under special scenarios are also obtained. These are further discussed in detail in that chapter.

Finally in Chapter 4, we extend the above results to more generalized scenarios. First, we consider the variable deadlines setup where the packets in the transmitter queue have distinct individual deadlines and the goal is to serve these packets over a stochastic channel with minimum energy. Second, we consider the arrivals with a single deadline case, where there is a stream of known packet arrivals and a single deadline by which all the data must depart. Using the cumulative curves framework as discussed in Chapter 2 and a decomposition approach, we obtain a transmission policy through an intuitive and a natural extension of the previous results. This policy is shown to be optimal under a specific class of channel models. Using the above results, we also obtain an online energy-efficient policy for the case of arbitrary and unknown packet arrivals to the queue with individual packet deadlines. Lastly, we consider a stream of Poisson packet arrivals to the queue and a single deadline by which they must all be transmitted. In this setup, we obtain an energy-efficient transmission policy in closed-form, and also highlight the various insights that can be drawn from it regarding the effect of statistical knowledge of the packet arrivals on energy expenditure.

We have presented part of the results from Chapter 2 in [14,15], from Chapter 3 in [18,19] and from Chapter 4 in [14,16,17].

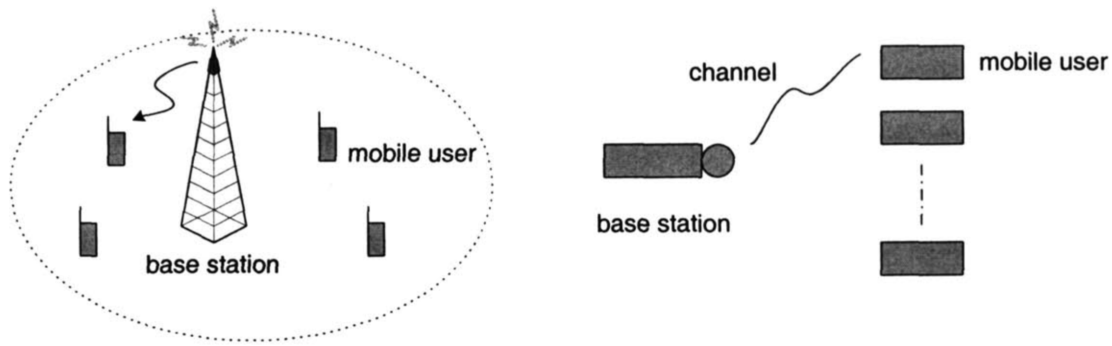


Figure 1-3: A schematic diagram of the system model for the multi-user scheduling problem.

1.2 Multi-user Scheduling with Throughput-rate Guarantees

1.2.1 Problem Overview

As mentioned earlier, wireless communication inherently involves dealing with time-varying channel conditions. To mitigate the effects of channel fading, much of the early research focus in cellular networks was to use a variety of diversity techniques such as time interleaving of data, frequency hopping and using power-control in CDMA systems [4]. However, with a single base-station serving multiple mobile users one can take advantage of channel fading by utilizing another form of diversity, which is referred to as *Multi-user Diversity* [4, 80] or *Opportunistic Scheduling* [75–79]. The main idea behind this technique is that with multiple mobile-users experiencing independent fading, at any given time there will some users with good channel conditions, and the base-station can then select the "best user" for transmission based on achieving certain required objectives.

In this part of the thesis, presented in Chapter 5, we address multi-user scheduling for Quality-of-Service (QoS) traffic that require a certain guaranteed throughput-rate. We consider a single server that represents the base station transmitting to multiple users that represent the mobile handsets. The system operates in a time-slotted manner and in each time-slot the base station can serve only one user. This setup is referred to in the literature as the *Wireless Downlink Scenario*, where "downlink" refers to the communication link from the base-station to the mobile user. A schematic diagram of the setup is shown in Figure 1-3. We further assume that the set of users are divided into two classes: (i) throughput-rate guaranteed, QoS users and (ii) "best effort" (BE) users. The QoS users in the system represent session applications such as FTP, high data-rate web-browsing, throughput-constrained data transfers etc., which require the base station to provide a certain long-term data rate on the downlink. In contrast, the BE users represent on-the-fly applications such as email transfers, low priority and latency tolerant data transfers etc., which do not have rate requirements and are short-lived. The goal of this work is to design a scheduling policy that provides the required throughput rates to the QoS users with the least time-slot utilization and maximizes the remaining fraction of time-slots assigned for the BE class.

1.2.2 Related Work

Downlink scheduling is an active area of research in wireless systems and has been studied in different contexts. The work relevant for our study includes [37–39, 75–79]. In [37–39], the authors studied the problem within the context of queue stability, wherein, the goal was to ensure that the queue sizes do not grow to infinity. The work in [75] studied opportunistic scheduling under a utility maximization framework and presented various formulations with different objective functions. In [76], the authors considered the objective of maximizing the minimum throughput-rate among a set of users and obtained the optimal policy for that setup, while in [77] the framework was extended to include a dynamic user population. In [78], the authors assumed multiple simultaneous transmissions employing spread spectrum and considered fairness constraints while in [79] the authors presented algorithms for scheduling users with average delay considerations.

1.2.3 Contributions

As mentioned earlier, we consider a setup where the set of users are divided into two classes – the QoS users which are guaranteed certain throughput-rates and the BE users which form the low-priority service. The goal is to obtain a multi-user scheduling policy that serves the QoS users with the least time-slot utilization and maximizes the remaining fraction of slots allocated to the BE class. To solve the problem, we adopt a geometric approach and show that the optimal policy satisfies a special structure. The geometric analysis is valid for a general fading model, and hence, is applicable for a wide set of scenarios. Specializing the results to case of Rayleigh fading, we obtain closed-form formulas that relate the achievable throughput-rate guarantee of the QoS users as a function of other system parameters, thus, providing closed-from relationships to understand the various system tradeoffs. Analytical comparison between the optimal policy and the random-scheduling policy also shows that gains on the order of $\ln(N)$ can be achieved, where N is the number of QoS users. We have presented part of the results from Chapter 5 in [73, 74].

1.3 Thesis Organization

The rest of the thesis is organized as follows. In Chapters 2, 3 and 4, we consider in detail the various setups for the energy-efficient transmission rate control problem, as described briefly in Section 1.1. The deterministic case for this problem is treated in Chapter 2 while the stochastic setup is presented in Chapters 3 and 4. In Chapter 5, we consider in detail the multi-user scheduling problem with throughput-rate guarantees as discussed briefly in Section 1.2. Finally in Chapter 6, we conclude the thesis.

Chapter 2

Deadline-Constrained Data Transmission – Deterministic Setup

2.1 Introduction

Delay constraints and energy-efficiency are important concerns in wireless data transmission, and as discussed in Chapter 1, these concerns arise frequently in real-time data communication. In principle, without energy concerns, strict deadline constraints can always be met by transmitting at high rates, albeit, incurring high transmission energy expenditure. When the transmitter has energy limitations, then as discussed in Chapter 1, one can utilize transmission rate-control to minimize the energy cost. More specifically, since transmission power is a convex function of the rate, data should be transmitted at low rates but ensuring that the deadline constraint is met. And furthermore, as transmission power also depends on the underlying channel state, the rate should be adapted in response to the channel variations.

In this part of the research work, presented in Chapters 2, 3 and 4, we address the question of optimal rate control to serve deadline-constrained data with minimum energy expenditure. We begin in this chapter by considering a deterministic setup, where the time-variability in the system is assumed known in advance. The problem is formulated over a finite-time horizon using a cumulative curves approach and its optimal policy is obtained.

As will be evident later, such an approach provides an appealing graphical visualization of the problem and the optimal solution. The formulation also generalizes the problems considered in [10, 21] which can be obtained as special cases, as further discussed later.

The rest of the chapter is organized as follows. In the next section, Section 2.2, we present the data flow and the transmission model. In Section 2.3, we consider the time-invariant power-rate function while in Section 2.4 the results are generalized to incorporate the time-varying power-rate function. Finally, in Section 2.5, we conclude the chapter and summarize the results.

2.2 System Model

We consider a continuous-time setup and assume that the rate can be varied continuously over time. Clearly, such a model is an approximation of a communication system which operates in discrete time-slots. However, the assumption is still justified since in practice the time-slot durations are very short on the order of 1 msec [3], and much smaller than packet delay requirements which are usually on the order of 100's of msec. An advantage of such a model is that it makes the problem mathematically tractable and also provides a simple and intuitive graphical visualization of the optimal solution. In fact, the results obtained here can be applied to a discrete-time system in a straightforward manner by simply evaluating the solution at the slot boundaries.

2.2.1 Data Flow Model

To describe the flow of data into the system, we utilize a cumulative curves methodology [50, 51, 53]. This model applies to a general setting where data could arrive in packets (packetized model) or in a continuum of bits (fluid model). Let $A(t)$, $D(t)$ and $D_{min}(t)$ denote the arrival curve, departure curve and the minimum departure curve respectively. These curves are assumed right-continuous functions and are defined as follows.

***Definition** 1 (Arrival Curve)* *An arrival curve $A(t), t \geq 0, t \in \mathbb{R}$, is the total number of bits that have arrived in time interval $[0, t]$.*

***Definition** 2 (Departure Curve)* *A departure curve $D(t)$, $t \geq 0, t \in \mathbb{R}$, is the total number of bits that have departed (served) in time $[0, t]$.*

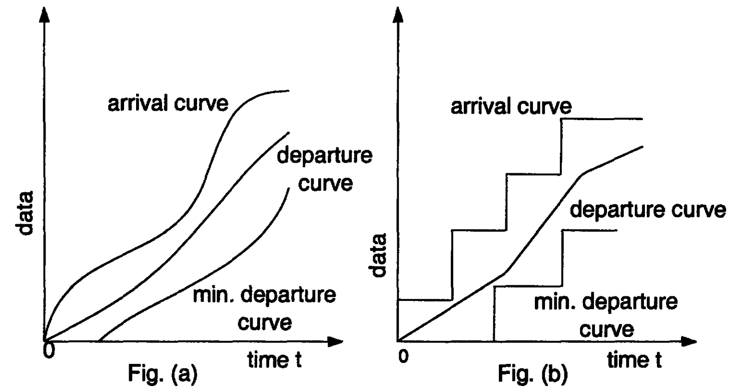


Figure 2-1: Data flow model: (a) Fluid arrival model, (b) Packetized arrival model

In case of a fluid arrival model, $A(t)$ is a continuous function, whereas, for a packet arrival model it is a piecewise-constant function as depicted in Figure 2-1. To ensure that the transmitter does not transmit more than the data that has arrived to the queue, we require that $D(t) \leq A(t)$. We refer to this as the *causality constraint.* Now, to model the quality-of-service constraints we introduce a new notion of a "*minimum departure curve*" which is defined as follows.

Definition **3 (Minimum Departure Curve)** *Given an arrival curve $A(t)$, a minimum departure curve $D_{min}(t)$ is a function such that $D_{min}(t) \leq A(t), \forall t \geq 0$, and is defined as the minimum cumulative number of bits that if departed by time t would satisfy the quality-of-service requirements.*

The function $D_{min}(t)$ can be viewed as the constraint function, so that in order to satisfy the QoS requirements the departure curve $D(t)$ must satisfy $D(t) \geq D_{min}(t)$. Thus, in a compact way the QoS and the causality constraints can be expressed as, $D_{min}(t) \leq D(t) \leq A(t),\ \forall t$. Note that the definition of $D_{min}(t)$ hides the implicitly assumed service discipline (the order in which data is served), as the above model looks at the data flow in a cumulative sense. Through a few illustrative examples, we show next that a number of commonly used QoS constraints with an appropriate service discipline can be modelled within this framework.

Delay Constraint: Consider an arrival curve $A(t)$ and a constant deadline constraint d on all the data. It is clear that by setting, $D_{min}(t) = 0, t \in [0, d)$ and $D_{min}(t) = A(t-d), t \geq d$, and following an earliest-deadline-first service discipline, the deadline constraints will be satisfied. Thus, here, $D_{min}(t)$ is simply a time-shifted version of $A(t)$ as shown in Figure 2-

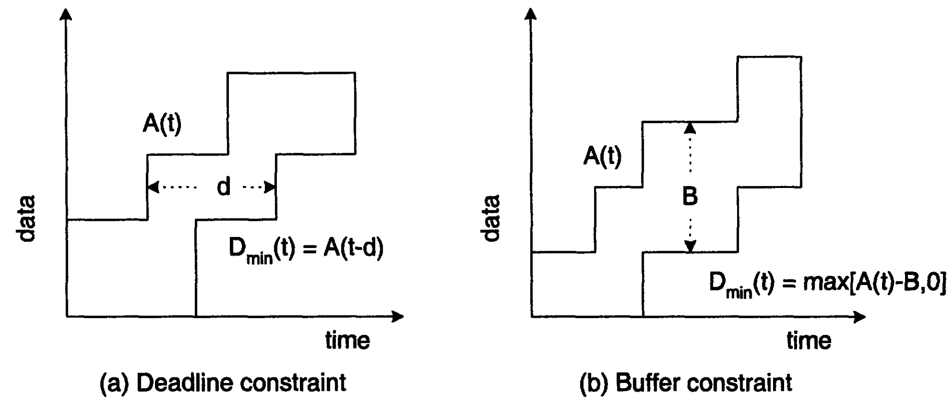


Figure 2-2: QoS Examples: (a) Packet deadline constraint of d, (b) Buffer constraint of B.

2(a). Generalizing this, suppose now that the data has variable deadlines and these deadlines are in the increasing order in which the bits arrive. Consider first a packet arrival model and let $\{t_i^A\}$ denote the arrival epochs, $\{d_i^A\}$ the deadlines and $\{b_i^A\}$ the sizes of the packets. Then, $D_{min}(t)$ is a piecewise constant function with jumps at times $\{t_i^D = t_i^A + d_i^A\}$ and the sizes of the jumps being $\{b_i^A\}$. Similarly, for a continuous data arrival model, let $d(t)$ be the general deadline function, where $d(t)$ is the deadline for data arriving at time t. Assuming that $h(s) \triangleq s + d(s)$ is a monotonically increasing function, the minimum departure curve is $D_{min}(t) = 0, t \in [0, d(0))$ and $D_{min}(t) = A(h^{-1}(t)), t \geq d(0)$.

Buffer Constraint: Consider a buffer constraint of B, i.e. the queue size must not exceed $B, \forall t \geq 0$. For an arrival curve $A(t)$ and a departure curve $D(t)$ the buffer size at any time t is given by $b(t) = A(t) - D(t)$. Since $b(t) \leq B$, we have $D(t) \geq max[A(t) - B, 0]$. Following a first-come-first-serve service discipline, it is easy to see that the minimum departure curve must be $D_{min}(t) = \max[A(t) - B, 0]$ as shown in Figure 2-2(b). It is easy to incorporate a time varying buffer constraint $B(t)$ as well.

Service-Curve Constraint: The notion of service curves forms an integral part of network calculus theory [53]. Given a service curve $\beta(t)$ and an arrival curve $A(t)$, the minimum departure curve can be obtained as $D_{min}(t) = A(t) \otimes \beta(t)$, where $\otimes$ is convolution in the min-plus algebra.

Thus, we see that a wide variety of QoS constraints can be abstracted by constructing the appropriate minimum departure curve.

2.2.2 Transmission Model

Let $P(t)$ denote the required transmission power to reliably transmit at rate $r(t)$ at time t. We assume the following power-rate relationship,

$$P(t) = g(r(t), t) \tag{2.1}$$

where the function $g(r, t)$ is a convex, increasing function with respect to the first argument (rate) and $g(r, t) \geq 0$ for $r \geq 0, \forall t$. The relationship in (2.1) is a general transmission model for most encoding schemes and has been widely studied in the literature in various forms [9–13,21,22,27,32]. As a well-known example, the Shannon formula for the power per bit gives the following relationship, $P = N_0 W(2^{r/W} - 1)$; in case of other coding schemes the Shannon formula gives a lower bound on the power per bit.

Given the relationship in (2.1), the transmission energy expenditure of a departure curve $D(t)$ over time interval $[0, T]$ is given by,

$$\mathcal{E}(D(t)) = \int_0^T g(D'(t), t)dt \tag{2.2}$$

where $D'(t)$ is the derivative at time t; it gives the transmission rate at that instant[1] and the term $g(D'(t), t)$ gives the instantaneous transmission power.

Throughout the paper, our focus will be on the time interval $[0, T]$ for some finite T, and with finite deadline constraints. Thus, we deal with energy minimization over a finite time interval rather than considering an infinite time horizon, as done in much of the literature on power-rate adaptation which studies average performance metrics. Since a departure curve specifies the transmission rate and vice-versa, we will use the terms departure curve and transmission policy interchangeably.

2.3 Time-Invariant Power-Rate Function

We first consider the case of a time-invariant power-rate function where $P(t)$ is only a function of $r(t)$, i.e. $P(t) = g(r(t))$. Such an assumption models a static or a slow fading wireless channel where over $[0, T]$ the channel gain does not change appreciably over time. This is a good model for wireless LAN settings and fixed wireless network scenarios.

[1] At points of non-differentiability $D'(t)$ is taken as the right-derivative.

2.3.1 Problem Formulation

Consider an arrival curve $A(t)$ and assume that this curve is known over the interval $[0, T]$. Based on the QoS requirements, one can construct the minimum departure curve $D_{min}(t)$ as discussed in Section 2.2. Now given $A(t)$ and $D_{min}(t)$ curves, a departure curve $D(t)$ is said to be *admissible* if it satisfies both the causality and the QoS constraints; i.e. $D_{min}(t) \leq D(t) \leq A(t),\ t \in [0, T]$. The energy minimization problem is to obtain the admissible departure curve with the least energy expenditure. Mathematically, this can be stated as follows,

$$\begin{aligned} \min_{D(t)} \quad & \mathcal{E}(D(t)) = \int_0^T g(D'(t))dt && (2.3) \\ \text{subject to} \quad & D_{min}(t) \leq D(t) \leq A(t),\ t \in [0, T] && \\ & D(t) \in \Gamma && (2.4) \end{aligned}$$

Without loss of generality, we take $D_{min}(0) = 0, D_{min}(T) = A(T)$, where the last equality simply states that all the data must depart by T. For admissibility, we also need the technical requirement that $D(t)$ belongs to the set Γ, where Γ consists of all non-decreasing, continuous functions with bounded right-derivative for all $t \in [0, T]$ and with $D(0) = 0$. For set Γ, the non-decreasing assumption follows from the cumulative nature of the departure curves, the continuity assumption is natural as any discontinuity would imply instantaneous transmission of non-zero amount of data which is practically infeasible and finally, the bounded right-derivative assumption ensures that the rate and the energy cost in (2.3) are finite[2]. Furthermore, if one makes the natural assumption that there is no data that arrives and needs to be transmitted instantaneously, then, admissible departure curves exist.

2.3.2 Optimality Properties

Consider first the following simple example – the transmitter has B units of data that must be transmitted by a deadline T. We refer to this as the "BT-problem". This example sheds important insights into the problem and will also serve as a building block for the general problem.

[2]Thus, we assume that $D'(t) < \mathcal{M}, \forall t \in [0, T], \forall D(t) \in \Gamma$, where $\mathcal{M}$ is chosen large enough such that finite-energy practical policies are all included. The curves $A(t)$ and $D_{min}(t)$ are also assumed right-continuous with bounded right-derivative for all $t \in [0, T]$.

BT-problem: The two curves $A(t)$ and $D_{min}(t)$ for this problem are as follows. Since there are no new arrivals and the queue has B units of data to begin with, the arrival curve is $A(t) = B, \ \forall t \in [0, T]$. Further, there is no minimum data transmission requirement until the deadline T, at which point all the data must have been sent; hence, we get $D_{min}(t) = 0, t \in [0, T)$ and $D_{min}(T) = B$. The admissibility criterion specialized to this case thus becomes $0 \leq D(t) \leq B$ and $D(T) = B$. We claim that the optimal policy is constant rate transmission at rate B/T, i.e. $(D^{opt})'(t) = \frac{B}{T}$ and $D^{opt}(t) = \frac{Bt}{T}$, $t \in [0, T]$, where $D^{opt}(t)$ denotes the optimal departure curve. To see why this is true consider the following integral version of Jensen's inequality.

***Lemma** 1 Let $f(t), p(t)$ be two functions defined for $a \leq t \leq b$ such that $\alpha \leq f(t) \leq \beta$ and $p(t) > 0$, with $p(t) \not\equiv 0$. Let $\phi(u)$ be a convex function defined on the interval $\alpha \leq u \leq \beta$; then*

$$\phi\left(\frac{\int_a^b f(t)p(t)dt}{\int_a^b p(t)dt}\right) \leq \frac{\int_a^b \phi(f)p(t)dt}{\int_a^b p(t)dt} \tag{2.5}$$

with strict inequality if $\phi()$ is strictly convex and $a \neq b, \alpha \neq \beta$.

Proof: See [84]. ■

Now, consider an admissible departure curve $D(t)$ and make the following substitution in the above lemma, $p(t) = 1$, $\phi() = g()$, $f() = D'()$, $a = 0$ and $b = T$. This gives,

$$g\left(\frac{\int_0^T D'(t)dt}{\int_0^T dt}\right) \leq \frac{\int_0^T g(D'(t))dt}{\int_0^T dt} \tag{2.6}$$

$$g\left(\frac{D(T) - D(0)}{T}\right)T \leq \int_0^T g(D'(t))dt \tag{2.7}$$

$$g(B/T)T \leq \int_0^T g(D'(t))dt \tag{2.8}$$

The left hand side in (2.8) is the total energy cost of the constant rate transmission policy at rate B/T, while, the right hand side is the total cost of any other admissible departure curve. The inequality in (2.8) thus proves the optimality claim.

***Remark** 1* : The result for the BT-problem is fairly intuitive given the convexity property of the power-rate function. Its practical implication is interesting as it says that for the time-invariant case there is no gain achieved by a complex variable-rate policy; in fact, a constant rate policy suffices. Another observation is that when $g(\cdot)$ is strictly convex the inequality in (2.8) is strict and the constant rate policy is the unique optimal policy. Whereas, for the case of a linear power-rate there is equality in (2.8) and all policies have the same cost.

We now consider the general setup and assume without loss of generality that $A(t) > D_{min}(t), 0 < t < T$. Otherwise, if at some time t_e there is equality, the problem can be divided into two sub-problems over time intervals $[0, t_e]$ and $[t_e, T]$ and each can be solved independently. The first result, Theorem I, is a generalization of the result for the BT-problem and it gives a criterion for the optimality of a departure curve.

<u>Theorem</u> I (Optimality Criterion) *Let $D(t)$ be an admissible departure curve and $L(t)$ be a straight line segment over $[a, b]$ that joins points $D(a)$ and $D(b)$, $0 \leq a < b \leq T$. If $L(t)$ satisfies $D_{min}(t) \leq L(t) \leq A(t)$, and, $L(t) \not\equiv D(t)$, the new departure curve $D^{new}(t)$ constructed as,*

$$\begin{aligned} D^{new}(t) &= D(t),\ t \in [0, a) \\ &= L(t),\ t \in [a, b] \\ &= D(t),\ t \in (b, T] \end{aligned}$$

satisfies, $\mathcal{E}(D^{new}(t)) \leq \mathcal{E}(D(t))$, where the inequality is strict if $g(.)$ is strictly convex.

The above theorem states that if there exists any two points on the curve $D(t)$ that can be joined by a straight line without violating the admissibility constraints, replacing that part of $D(t)$ with the straight line can only lower the energy cost. The implication of this is that whenever admissible, it is optimal to transmit at a constant rate. A schematic diagram depicting this is given in Figure 2-3. Henceforth, the criterion that along a departure curve there does not exist any two points that can be joined by a distinct admissible straight line will be referred to as the *"Optimality Criterion"*.

Proof: First note that since $L(t)$ is admissible, the new curve $D^{new}(t)$ is also admissible. Consider,

$$\mathcal{E}(D^{new}(t)) - \mathcal{E}(D(t)) = \mathcal{E}(L(t)) - \int_a^b g(D'(t), t)dt \tag{2.9}$$

Over the interval $[a, b]$, we know from the result for the BT-problem that $L(t)$ has the least energy cost among all departure curves that would transmit $(D(b) - D(a))$ amount of data in time $(b - a)$. Hence, from (2.6)-(2.8), we get, $\mathcal{E}(L(t)) - \int_a^b g(D'(t), t)dt \leq 0$ and the

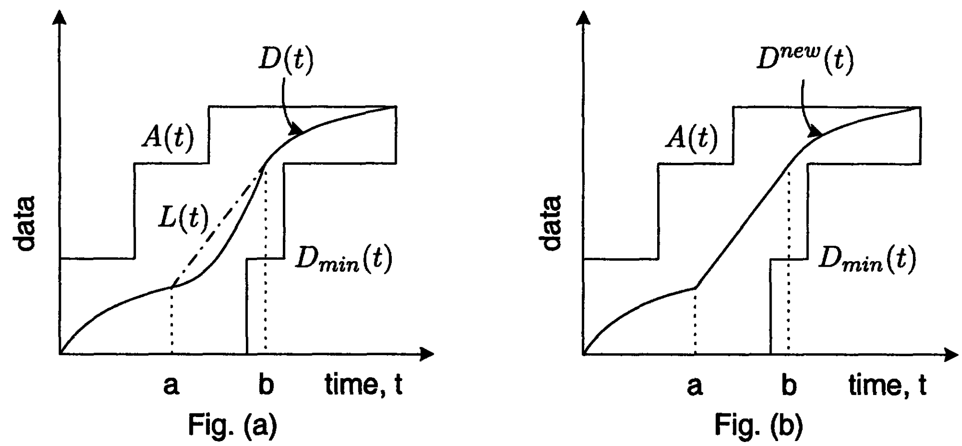


Figure 2-3: Figure for Theorem I: (a) an admissible departure curve $D(t)$ and (b) the new curve $D^{new}(t)$.

result follows. ∎

***Remark** 2 :(Linear power-rate function)* An interesting special case arises when the power-rate relationship is linear, i.e. $P = \kappa r$ where $\kappa > 0$ is a constant. In this case, the inequality in Lemma 1 becomes an equality from which it follows that all departure curves have the same energy cost. Thus, with a linear power-rate curve it does not matter, in terms of energy cost, how the data is transmitted as long as the QoS constraints are met. However, even in the special case of linear power-rate function, we will see next that the departure curve that satisfies the optimality criterion has appealing properties that make it a right candidate for the optimal transmission policy.

Henceforth, we consider the more interesting case of strictly convex $g(\cdot)$ function. The next result shows that the optimal departure satisfying the optimality criterion is unique.

***Theorem** II (Uniqueness) Consider the optimization problem in (2.3) with $g(\cdot)$ being strictly convex. Let $\tilde{D}(t)$ be an admissible departure curve that satisfies the optimality criterion, then, $\tilde{D}(t)$ is unique and it minimizes the energy cost in (2.3).*

Proof: Appendix A.1. ∎

Throughout now, we will denote the admissible departure curve satisfying the optimality criterion as $D^{opt}(t)$ and later in Section 2.3.3 give an algorithm for constructing $D^{opt}(t)$. We now present the various properties of $D^{opt}(t)$ and start by characterizing the points in time at which the optimal rate changes, i.e. points at which the slope/right-derivative of

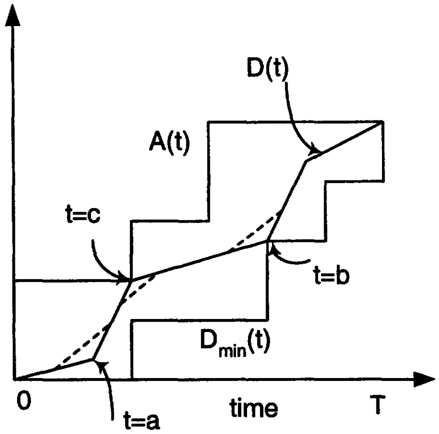


Figure 2-4: Example showing violation of Lemmas 2-4. The dotted line shows that $D(t)$ does not meet the optimality criterion.

$D^{opt}(t)$ changes, either continuously or in a discrete step. Denoting any such point as t_0, the following results follow[3].

***Lemma* 2** *At t_0, $D^{opt}(t)$ either intersects $A(t)$ or it intersects $D_{min}(t)$; i.e. we have $D^{opt}(t_0) = A(t_0)$ or $D^{opt}(t_0) = D_{min}(t_0)$. Note, if there is a discontinuity in $A(t)$ at t_0 (jump point for packetized data) then $D^{opt}(t_0) = A(t_0^-)$.*

***Lemma* 3** *Suppose that at t_0 we have $D^{opt}(t_0) = D_{min}(t_0)$, then, the slope change must be negative.*

***Lemma* 4** *Suppose that at t_0 we have $D^{opt}(t_0) = A(t_0)$ $\left(\text{or } A(t_0^-)\right)$ then the change in slope must be positive.*

The observations in the above lemmas are straightforward and can be easily understood from Figure 2-4. Point $t = a$ corresponds to a point of rate change and it violates Lemma 2. It is easy to see that around $t = a$ the optimality criterion is violated since an admissible straight line segment exists (the dotted segment around $t = a$ in the figure). Similarly, points $t = b$ and $t = c$ correspond to a violation of Lemmas 3 and 4 respectively.

Among other properties, the optimal departure curve $D^{opt}(t)$ has the least maximum transmission-power requirement and the shortest length metric. We first discuss the minimal maximum-power requirement of $D^{opt}(t)$ which states that among all admissible departure

[3]The notation $f(x^+)$ means $\lim_{n\to\infty} f(x+\epsilon_n)$ and $f(x^-)$ means $\lim_{n\to\infty} f(x-\epsilon_n)$ with $\epsilon_n > 0, \epsilon_n \to 0$.

curves, if we look at the maximum instantaneous power requirement over time, then, $D^{opt}(t)$ has the least such requirement.

Theorem ***III (Minimal Maximum Power)*** *Given any admissible departure curve $D(t)$, the optimal departure curve $D^{opt}(t)$ satisfies,*

$$\max_{t\in[0,T)} (D^{opt})'(t) \leq \max_{t\in[0,T)} D'(t) \tag{2.10}$$

Equivalently, $\max_{t\in[0,T)} P^{opt}(t) \leq \max_{t\in[0,T)} P(t)$, *where $P(.)$ denotes the power expenditure over time.*

Proof: See Appendix A.2 ■

Remark *3* : The above theorem is very significant if we impose an additional maximum power constraint in (2.4). In this case, the problem is first solved without the power constraint. If the optimal solution satisfies the maximum power constraint, we are done; otherwise from Theorem III it follows that there does not exist any other admissible departure curve that can satisfy the power constraint and the constrained optimization problem has no solution. Thus, we see that $D^{opt}(t)$ is the unique curve that satisfies the QoS constraints with both the least total energy cost and the least maximum power requirement.

Theorem ***IV (Shortest Length)*** *The optimal departure curve $D^{opt}(t)$ has the shortest length among admissible departure curves. Specifically, it minimizes the metric,*

$$len(D(t)) \triangleq \int_0^T \sqrt{(1 + (D'(t))^2)}dt \tag{2.11}$$

Proof: Since $D^{opt}(t)$ minimizes the integral in (2.3) for a convex increasing function $g(\cdot)$, the result follows by replacing $g(r)$ with $g(r) = \sqrt{(1 + r^2)}$. ■

2.3.3 Optimal Policy

In the last section, we presented the optimality criterion and the various properties of the optimal curve. We now construct the optimal departure curve $D^{opt}(t)$. However, before giving the algorithmic description, it is instructive to consider a very insightful visualization. This graphical picture provides a simple and intuitive way to understand $D^{opt}(t)$ and is described next.

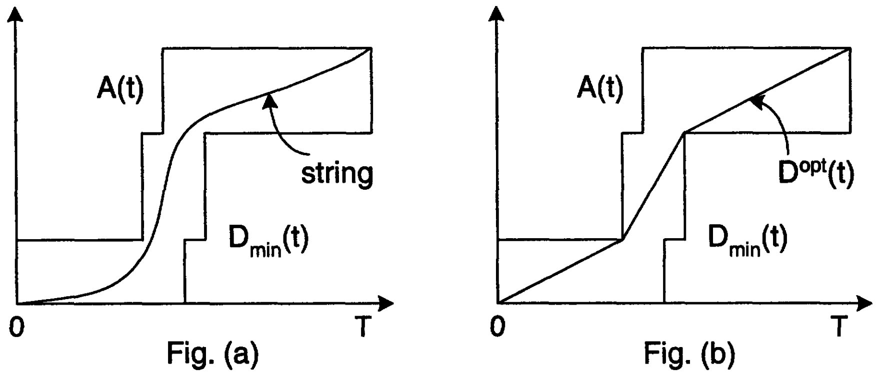


Figure 2-5: String visualization for the optimal curve, (a) string lying between $A(t)$ and $D_{min}(t)$; (b) $D^{opt}(t)$ as taut string.

String Visualization: *Consider a string restricted to lie between $A(t)$ and $D_{min}(t)$ (i.e. visualize $A(t), D_{min}(t)$ curves as hard boundaries for the string). Tie one end of the string at the origin and pass the other end through $D_{min}(T)$. If we now make the string tight, its trajectory gives the optimal departure curve*[4].

Intuitively, when the string is in the tight condition it cannot be made tighter between any two points along its curve. This means that the optimality criterion must be satisfied, because otherwise, the construction in Theorem I would make the string tighter thereby leading to a contradiction. By the uniqueness result, it then follows that this must be the optimal curve. Figure 2-5 is an illustration showing a general $A(t), D_{min}(t)$ curve and the corresponding $D^{opt}(t)$ visualized as a tight string. Note that depending on the shape of $A(t)$ and $D_{min}(t)$ curves, the curve $D^{opt}(t)$ could consist of segments of constant-rate transmission and/or segments where the rate is varying continuously over time; see for example Figure 2-7(b), where over time $[a, b]$ and $[c, d]$ the curve $D^{opt}(t)$ has a continuous rate change.

Examples: Using the above string visualization, we now present a few illustrative examples for which the optimal solution can be obtained in closed-form. Among these, the first two examples have been studied earlier in the literature [10, 21] and their solutions were obtained using a discrete-optimization approach which was mathematically tedious. By re-formulating the problems within our framework, the solutions can be obtained easily from the graphical picture.

[4]This observation was pointed out by Rene L. Cruz

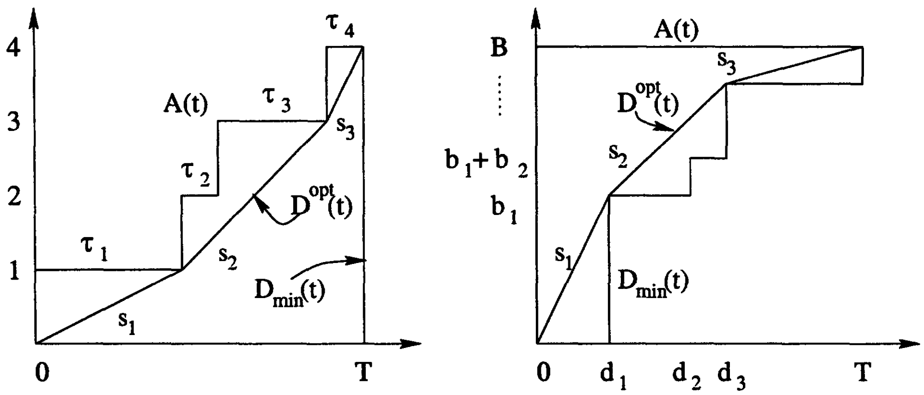
Example 1 *[10]*: Consider N packets of unit size arriving in time $[0, T)$ with known inter-arrival times $\tau_1, .., \tau_{N-1}$ and the first packet arrival at time 0. The deadline constraint is that all the packets must depart by time T (common deadline), where $T > (\tau_1 + .. + \tau_{N-1})$. Let $\tau_N = T - \sum_1^{N-1} \tau_i$. The curves $A(t)$ and $D_{min}(t)$ for this problem are depicted in Figure 2-6(a). From the string visualization it is easy to see that the optimal curve consists of piecewise linear segments with increasing slopes and the points at which the slope changes, the optimal policy just empties the buffer. The optimal curve $D^{opt}(t)$ can be constructed as follows. Let T_i be the jump points of the $A(t)$ curve then, $T_i = \sum_{l=1}^{i} \tau_l,\ i = 1, \ldots, N-1$ and let $T_N = T$. Denote A_i as the cumulative amount of data arrived to the queue just before time T_i (the total data in the first i packets). Now, starting at time 0, consider the straight line segments that join the points $(0,0)$ (origin) and (T_i, A_i) (jump points of $A(t)$). From among these, choose the segment with the minimum slope, i.e. the segment having slope equal to the minimum over i of $\left(\frac{A_i}{T_i}\right)$. Denoting the minimizing index as π, the first segment of $D^{opt}(t)$ is constant-rate transmission with rate $\frac{A_\pi}{T_\pi}$ from $t = 0$ until $t = T_\pi$. Starting at T_π, the procedure is repeated by shifting the origin to this point. Thus, the slopes of the linear segments denoted as $\{s_1, .., s_q\}$ can be computed recursively as follows. Take $l_1 = 1, T_0 = 0, A_0 = 0$ and initialize $m = 1$, we then have,

$$s_m = \min_{i \in \{l_m, .., N\}} \left(\frac{A_i - A_{(l_m - 1)}}{T_i - T_{(l_m - 1)}} \right) \tag{2.12}$$

$$l_{m+1} = 1 + \arg \min_{i \in \{l_m, .., N\}} \left(\frac{A_i - A_{(l_m - 1)}}{T_i - T_{(l_m - 1)}} \right) \tag{2.13}$$

The above iteration stops when $l_{m+1} = N + 1$. Intuitively, the optimal policy follows a constant rate transmission until points where the future arrivals are such that relative to the deadline constraint, the transmission rate must be higher.

Example 2 *[21]*: Consider M data packets in the buffer at time 0 and no new arrivals. Let the i^{th} packet have b_i units of data and a deadline d_i, $i = 1, .., M$. Let $d_M = T$ and $B = \sum_{i=1}^{M} b_i$. The packets in the queue are served in the earliest-deadline-first order and for this case, the $A(t)$ and $D_{min}(t)$ curves are shown in Figure 2-6(b). Note that the structure of this problem is the reverse of Example 1 and in some loose sense one can regard these problems as "*duals*" of each other. The string interpretation gives the optimal policy but now the piecewise linear segments have decreasing slopes. Let $T_j = d_j$ and $B_j = \sum_{l=1}^{j} b_l$,



(a)- Example 1 (b)- Example 2

Figure 2-6: Curves $A(t), D_{min}(t)$ and $D^{opt}(t)$ for Examples 1 and 2.

then, T_j denotes the deadlines of the packets and B_j denotes the cumulative data in the first j packets. Starting at time 0, consider the straight line segments that join the points $(0,0)$ (origin) and (T_j, B_j) (jump points of $D_{min}(t)$). From among these choose the segment with the maximum slope, i.e. the segment having slope equal to the maximum over j of $\left(\frac{B_j}{T_j}\right)$. Denoting the maximizing index as π, the first segment of $D^{opt}(t)$ is constant-rate transmission with rate $\frac{B_\pi}{T_\pi}$ from $t = 0$ until $t = T_\pi$. Starting at T_π, the procedure is repeated by shifting the origin to this point. Algebraically, these slopes $\{s_1, .., s_q\}$ are obtained as follows. Take $l_1 = 1, T_0 = 0, B_0 = 0$ and initialize $m = 1$, we then have,

$$s_m = \max_{j \in \{l_m,..,M\}} \left(\frac{B_j - B_{(l_m-1)}}{T_j - T_{(l_m-1)}} \right) \tag{2.14}$$

$$l_{m+1} = 1 + \arg \max_{j \in \{l_m,..,M\}} \left(\frac{B_j - B_{(l_m-1)}}{T_j - T_{(l_m-1)}} \right) \tag{2.15}$$

The above iteration stops when $l_{m+1} = N + 1$.

***Example* 3** : Consider a stream of N packet arrivals of size B with a constant inter-arrival time τ. Each packet has a deadline d before which it must depart (Figure 2-7(a)). Such an arrival stream is a good model for applications which generate packets at regular times (or with a small variance), e.g. voice data. The solution is obvious from the figure and is given as follows. If $d < \tau$, the solution is trivial and the packet must be transmitted before the next arrival. If $d \geq \tau$, the optimal curve is a straight line with slope $NB/(d + (N-1)\tau)$.

We now proceed to present an algorithm for constructing the optimal departure curve for the general case.

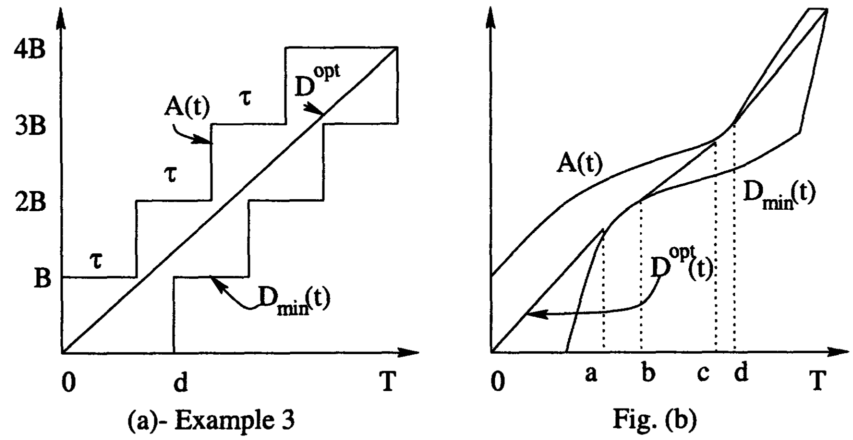


Figure 2-7: Curves $A(t), D_{min}(t)$ and $D^{opt}(t)$ for (a) Example 3 and (b) Continuous data flow.

Construction of the Optimal Departure Curve: The main idea behind constructing the optimal curve, $D^{opt}(t)$, is to obtain its segments in a recursive fashion. To proceed, we first present definitions of the various terms used in the algorithm later. The first definition concerns a tangent and is defined as follows.

Definition *4 A tangent to $D_{min}(t)$ at $t = t_0$ is a line passing through $(t_0, D_{min}(t_0))$ and slope $D'_{min}(t_0)$.*

In the above, $D'_{min}(t_0)$ is taken as the right-derivative. A similar definition holds for a tangent to $A(t)$ as well.

Next, we need the notion of intersection of curves. Since the data model includes piecewise constant functions (packet arrival model) that have discontinuities, we need to define what it means for such curves to intersect. Consider a line $L(t)$ of non-negative slope starting from an admissible point (t_0, α); where, admissibility of a point means that $D_{min}(t_0) \leq \alpha \leq A(t_0)$ and $0 \leq t_0 < T$.

Definition **5** *Starting at t_0, $L(t)$ intersects $D_{min}(t)$ if for some point $\tilde{t} > t_0$, called the point of intersection, one of the following holds: (a) either $L(\tilde{t}) = D_{min}(\tilde{t})$ or, (b) the function $L(t) - D_{min}(t)$ changes sign at $\tilde{t}$ (here $\tilde{t}$ is a discontinuity point).*

Intuitively, the above definition means that the straight line $L(t)$ crosses the curve $D_{min}(t)$ at $\tilde{t}$. A similar definition holds for intersection with $A(t)$. We now define what it means for the straight line $L(t)$ to intersect a curve first.

***Definition* 6** *We say that $L(t)$ intersects $D_{min}(t)$ first, if $L(t)$ intersects $D_{min}(t)$ curve at $\tilde{t}$ $(> t_0)$ and $L(t) < A(t), t \in (t_0, \tilde{t})$ (that is, $L(t)$ does not intersect $A(t)$ in $(t_0, \tilde{t})$).*

Similarly, we say that $L(t)$ *intersects* $A(t)$ *first* if $L(t)$ intersects $A(t)$ at $\tilde{t}$ and $L(t) > D_{min}(t), t \in (t_0, \tilde{t})$.

Given the above definitions, we now obtain the slope of the optimal segment of $D^{opt}(t)$ starting at an admissible point. To proceed, consider an admissible point (t_0, α) and consider straight lines with non-negative slopes starting at this point. Among these, choose those lines that starting at (t_0, α) remain admissible for some finite duration. In other words, consider straight lines $L(t)$ for which there exists an $\epsilon > 0$ (ϵ could depend on the chosen $L(t)$) such that $L(t)$ is admissible for $t \in [t_0, t_0 + \epsilon)$, i.e. $D_{min}(t) \leq L(t) \leq A(t)$, for $t \in [t_0, t_0 + \epsilon)$. Denote this set as $\mathcal{F}$. Intuitively, the slopes of the lines in $\mathcal{F}$ are the possible admissible slopes that $D^{opt}(t)$ can have. Note that the set $\mathcal{F}$ depends on the starting point (t_0, α) but to make the notations simple we drop the explicit dependence. The following lemmas summarize the properties of the set $\mathcal{F}$.

***Lemma* 5** *The slopes of the lines in $\mathcal{F}$ lie in a continuous interval.*

Proof: See Appendix A.3. ■

For $A(t_0) > D_{min}(t_0)$, the set $\mathcal{F}$ has the following three possibilities: (i) If $D_{min}(t_0) < \alpha < A(t_0)$, due to right-continuity of the curves all points in a small region around α are admissible. Hence, all lines with slopes lying in $[0, \infty)$ belong to the set $\mathcal{F}$. (ii) If $\alpha = D_{min}(t_0)$, all lines with slope less than the tangent at $D_{min}(t_0)$ (say slope c) are not admissible while lines with slope greater than the tangent are admissible. If the tangent itself is admissible, the slopes of $\mathcal{F}$ lie in $[c, \infty)$; else they lie in (c, ∞). (iii) If $\alpha = A(t_0)$, lines with slopes less than the tangent at $A(t_0)$ belong to $\mathcal{F}$. If the tangent is admissible, the slopes of $\mathcal{F}$ lie in $[0, l]$; else the slopes belong to $[0, l)$. Finally, at $t_0 = 0$ if we have $A(0) = D_{min}(0)$, the set $\mathcal{F}$ consists of lines with slopes lying between the tangents to each curve.

***Lemma* 6** *The lines in $\mathcal{F}$ must intersect $A(t)$ first or intersect $D_{min}(t)$ first.*

The above lemma is straightforward since $L(t)$ will eventually at some time cross either the $A(t)$ or the $D_{min}(t)$ curve. Now, let the set $\mathcal{F}$ be partitioned into a set of lines that intersect $A(t)$ first and those that intersect $D_{min}(t)$ first. Denote these sets as $\mathcal{F}_A$ and $\mathcal{F}_{D_m}$

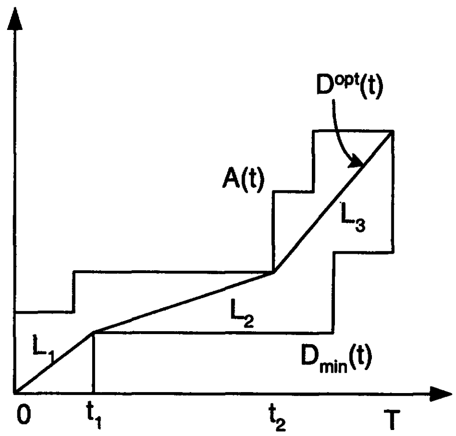


Figure 2-8: Example depicting $A(t)$, $D_{min}(t)$ and the constructed $D(t)$.

respectively. The following result states that the slopes of the lines in $\mathcal{F}_A$ and $\mathcal{F}_{D_m}$ lie in non-overlapping continuous intervals.

***Lemma* 7** *(a) Let $L_D(t) \in \mathcal{F}_{D_m}$ then any $L(t) \in \mathcal{F}$ that has slope less than L'_D intersects $D_{min}(t)$ first. (b) Let $L_A(t) \in \mathcal{F}_A$ then any $L(t) \in \mathcal{F}$ that has slope greater than L'_A intersects $A(t)$ first.*

Proof: See Appendix A.4. ■

The above lemma has the following implications. First, the slopes of the lines in $\mathcal{F}_A$ and $\mathcal{F}_{D_m}$ lie in non-overlapping continuous intervals which we denote as $\mathcal{S}_A$ and $\mathcal{S}_{D_m}$ respectively. Second, the slopes in $\mathcal{F}_A$ are greater than in $\mathcal{F}_{D_m}$. The line with slope β_o at the boundary of the two intervals[5] is given as,

$$\beta_o = \inf \mathcal{S}_A = \sup \mathcal{S}_{D_m} \tag{2.16}$$

The equality of inf and sup above follows from the continuity property in Lemma 5. If either $\mathcal{S}_A$ or $\mathcal{S}_{D_m}$ is empty, it is neglected. We call β_o the *optimal slope* and the line with slope β_o the *optimal line.* It is denoted as L_o. Simply stated, L_o is the least slope line that intersects $A(t)$ first, or the maximum slope line that intersects $D_{min}(t)$ first. Using this line L_o, we can now construct the optimal departure curve as illustrated next in the following algorithm.

[5]A case of singularity occurs at $t_0 = 0$ if $A(0) = D_{min}(0)$, $A'(0) = D'_{min}(0)$, then, both the sets $\mathcal{S}_A, \mathcal{S}_{D_m}$ are empty. Here, simply define $\beta_o = A'(0)$.

To begin with, we have $D^{opt}(0) = 0$. The segments of $D^{opt}(t)$ are now constructed in a recursive fashion starting at $(0,0)$. Let t_0 denote a generic time instant, where $t_0 = 0$ in the first iteration.

1. Obtain β_o as in (2.16) and the optimal line L_o.

2. If L_o is not tangent to $D_{min}(t)$ (or $A(t)$) at t_0, obtain the first instant t_1 such that, (a) $L_o(t_1) = D_{min}(t_1)$ (QoS constraint is just met); or (b) $L_o(t_1) = A(t_1)$ or $L_o(t_1) = A(t_1^-)$ (buffer is just empty). Let $D^{opt}(t) = L_o(t)$, $t \in (t_0, t_1]$.

3. If L_o is tangent to $D_{min}(t)$ (or $A(t)$) at t_0 then let $t_1 = \min\{\tilde{t}, T\}$ where $\tilde{t}$ is the first instant[6] at which the corresponding tangent is no more the optimal line. Let $D^{opt}(t) = D_{min}(t)$ (or $A(t)$), $t \in (t_0, t_1]$.

If $t_1 = T$ terminate; else repeat the above steps with the new starting point as $(t_1, D^{opt}(t_1))$. The correctness and optimality of the above algorithm is shown in Appendix A.5.

As an example, consider $A(t)$ and $D_{min}(t)$ shown in Figure 2-8 for which the algorithm executes as follows. Start at the origin $(0,0)$ and note that L_1 is the optimal line as defined above and t_1 is the first instant at which it equals $D_{min}(t)$. Thus, segment L_1 from $t = [0, t_1]$ is the first part of the optimal curve. Note that lines with slope greater than L_1' intersect $A(t)$ first and lines with slope less than L_1' intersect $D_{min}(t)$ first. The line L_1 is the one with slope at the boundary (as defined in (2.16)). Next, starting from the new point $(t_1, D_{min}(t_1))$, L_2 is the optimal line and t_2 is the first instant such that $L_2(t_2) = A(t_2^-)$. The segment L_2 from $t = [t_1, t_2]$ forms part of the optimal curve. The segment L_3 is also obtained in a similar fashion and it is the last segment as $t = T$ is reached. Finally, for the case when $A(t)$ and $D_{min}(t)$ are piece-wise constant functions, the optimal departure curve $D^{opt}(t)$ is piecewise linear and the algorithm presented above can be further specialized to obtain the slopes directly by only looking at the jump points of the two curves. The algorithm for this is presented in Appendix A.6.

2.4 Time-varying Power-Rate Function

In the previous section, we considered the time-invariant power-rate function case and utilized a cumulative curves methodology to obtain the optimal solution. The framework

[6]If $\tilde{t} = t_0$, the interval $[t_0, \tilde{t}]$ is just the point t_0; in this case, obtain t_1 as in step 2.

provided a graphical visualization of the problem and the optimal solution. In this section, we generalize those results and consider a time-varying power-rate function, i.e. the function $P(t)$ is given as $P(t) = g(r(t), t)$. Thus, for a fixed time t_0, the amount of power required to transmit at a certain rate r is governed by the convex function $g(\cdot, t_0)$, but now, this convex function could be different at different times.

2.4.1 Problem Formulation

The problem formulation remains the same as given in Section 2.3.1 with the data flows being described using cumulative curves and the objective is to obtain the minimum energy departure curve. Mathematically, the optimization problem is given as,

$$\begin{aligned} \min_{D(t)} \quad & \mathcal{E}(D(t)) = \int_0^T g(D'(t), t)dt \\ \text{subject to} \quad & D_{min}(t) \leq D(t) \leq A(t), \;\; t \in [0, T] \\ & D(t) \in \Gamma \end{aligned} \tag{2.17}$$

In the above formulation, we assume that $g(r, t)$ as a function of r is a strictly convex, increasing and continuously differentiable function for all t. We also assume that $g(r, t)$ is a deterministic function of time $t \in [0, T]$ and piecewise continuous in t.

The above formulation provides a general framework to model various scenarios involving time-variability in the system. It generalizes the problem in Section 2.3.1 to include time-dependent parameters in transmission arising due to phenomena such as beam-forming, antenna patterns etc. Since it models a more general power-rate cost function, one can also introduce an artificial cost for control purposes; for example, by imposing a high cost over certain intervals one can control the times over which data should be transmitted. Finally, it also models scenarios where we have a time-varying channel but the channel gain is predictable or known over time.

2.4.2 Optimality Properties

We proceed as in Section 2.3 by first considering the BT-problem and then extending the results to general $A(t)$ and $D_{min}(t)$ curves. As in the time-invariant case, the BT-problem provides useful insights into the problem and also plays an important role as a building block.

BT-problem: Consider the BT-problem where the transmitter has B units of data in the queue and a deadline T by which this data must be transmitted using minimum energy. The following lemma gives the optimal solution for this problem; its proof is based on results from the theory of *Calculus of Variations.*

***Lemma** 8 The optimal transmission rate $r^{opt}(t)$ for the BT-problem is given by,*

$$r^{opt}(t) = \max(0, r^*) \tag{2.18}$$

where r^* is the unique positive value that satisfies $\frac{\partial}{\partial r}g(r,t)|_{r=r^*} = k$ and k is a positive constant such that $\int_0^T r^{opt}(t)dt = B$.

Proof: See Appendix A.7. ■

As examples to understand the above solution, we first specialize (2.18) to two specific forms of $g(r,t)$, namely, the *Monomial* class and the *Exponential* class of functions. The general solution is then explained later.

***Example** 4 : (Monomial Class)* Let $g(r,t) = \frac{r^n}{c(t)}$, $n > 1$, $c(t) > 0$, be the class of positive monomial functions with $c(t)$ representing the channel gain or the time-dependent parameter. For any positive constant k, $\frac{\partial}{\partial r}\frac{r^n}{c(t)}|_{r=r^*} = k$, gives, $r^* = \left(\frac{kc(t)}{n}\right)^{\frac{1}{n-1}}$. Since k and $c(t)$ are positive, we have $r^* > 0, \forall t$, and from (2.18) we get $r^{opt}(t) = \left(\frac{kc(t)}{n}\right)^{\frac{1}{n-1}}$. The value of k such that the deadline constraint is met is obtained from, $\int_0^T r^{opt}(t)dt = B$, which gives, $k^{\frac{1}{n-1}} = \frac{B}{\gamma}$, where $\gamma = \int_0^T (c(t)/n)^{\frac{1}{n-1}}dt$. Substituting back in $r^{opt}(t)$ thus gives,

$$r^{opt}(t) = \frac{B}{\gamma}\left(\frac{c(t)}{n}\right)^{\frac{1}{n-1}} \tag{2.19}$$

***Example** 5 :(Exponential Class)* Let $g(r,t) = \frac{\alpha^r - 1}{c(t)}$, $\alpha > 1$, $c(t) > 0$, be the class of exponential functions with $c(t)$ being the time-dependent parameter. Note that taking $\alpha = 2$ and $c(t) = |h(t)|^2$ gives the Shannon formula for the power per bit. For the exponential case, $\frac{\partial g(r^*,t)}{\partial r} = \frac{\alpha^{r^*}\ln(\alpha)}{c(t)} = k$, gives,

$$r^{opt}(t) = \max\left(0, \frac{\ln(k) - \ln(\ln(\alpha)/c(t))}{\ln(\alpha)}\right) \tag{2.20}$$

This is the well-known "water-filling" solution over time [49] but now with deadline constraints. Lastly, the value of k such that the deadline constraint is met is obtained from $\int_0^T \max\left(0, \frac{\ln(k)-\ln(\ln(\alpha)/c(t))}{\ln(\alpha)}\right) dt = B$.

Re-examining (2.18) we see that the optimal rate is such that the partial derivative of $g(r,t)$ with respect to r at the positive value r^* equals a constant k, for all t. The value of this constant is chosen such that the deadline constraint at T is met. We refer to the constant k as the *"marginal cost"* for the BT-problem. For positive rate, since the marginal cost (or the first-derivative of $g(r,t)$ with respect to r) is the same for all t, it implies that for the optimal solution infinitesimal changes in the rate would not change the total energy cost. This observation is intuitive since otherwise, we could decrease the rate over the intervals when the marginal cost is high and correspondingly increase the rate over the intervals when the marginal cost is low, thereby, reducing the total energy cost and violating the optimality claim. Now, for all t such that $r^{opt}(t) = 0$ we must have $\frac{\partial}{\partial r} g(r,t)|_{r=0} \geq k$. This means that at all such times, the marginal cost is high and it is relatively costly to transmit the data, hence, the optimal policy chooses a zero rate.

As compared to the time-invariant power-rate function case, clearly, the optimal rate now is not constant over time. However, interestingly, the marginal cost though is constant. Thus, the constant slope property translates here into a constant marginal cost property. As a check, if we remove the time-dependence in $g(r,t)$, then r^* is the same for all t. This gives $r^{opt}(t) = r^*$ and from $\int_0^T r^{opt}(t)dt = B$, we get $r^* = \frac{B}{T}$. Thus, the optimal solution is constant-rate transmission in conformity with the result in Section 2.3.2. The solution in (2.18) also has an interesting monotonicity property with respect to the marginal cost k. This is presented in the lemma below.

Lemma *9 Let $r^{opt}(t)$ be given by (2.18) for some $k \geq 0$ and $D^{opt}(t) = \int_0^t r^{opt}(s)ds$. Then, $D^{opt}(t)$ is monotonically non-decreasing in k, unique for a given value of k and zero throughout for $k = 0$. Furthermore, for $D^{opt}(T) = B > 0$, there is a unique positive value of k that achieves it.*

Proof: Let k_1, k_2 be two positive values such that $0 < k_1 < k_2$. Let $r^{opt}_{k_1}(t), r^{opt}_{k_2}(t)$ be the corresponding optimal rate functions as given in (2.18). Suppose at time t, we have $r^{opt}_{k_1}(t) > 0$, then, due to strict convexity $\frac{\partial}{\partial r} g(r,t)$ is an increasing function of r and since $k_2 > k_1$ the unique r^* value for k_2 must be greater than for k_1. This gives, $r^{opt}_{k_2}(t) > r^{opt}_{k_1}(t)$.

If instead at time t, we have $r_{k_1}^{opt}(t) = 0$, then, $r_{k_2}^{opt}(t)$ can be either 0 or positive. Thus, we see that $r_{k_2}^{opt}(t) \geq r_{k_1}^{opt}(t)$, $\forall t$, with equality only if both are zero. This shows that $D^{opt}(t)$ is non-decreasing in k. For a given k value, the uniqueness of $D^{opt}(t)$ follows since r^* is unique. Now suppose $k = 0$, then, since $g(r,t)$ is increasing in r we have $\frac{\partial}{\partial r}g(r,t) \geq 0, \forall r$. Also, as before $\frac{\partial}{\partial r}g(r,t)$ is an increasing function in r, thus, there is no positive r^* such that $\frac{\partial}{\partial r}g(r,t)|_{r=r^*} = 0$ ($= k$ as taken). This gives $r^{opt}(t) = 0$ and $D^{opt}(t) = 0$, $\forall t$. Lastly, suppose $D^{opt}(T) = B > 0$ and let k_1, k_2 be two distinct k values such that $\int_0^T r_{k_1}^{opt}(s)ds = \int_0^T r_{k_2}^{opt}(s)ds = B$. Without loss of generality assume $k_2 > k_1$. From the earlier arguments we know that whenever $r_{k_1}^{opt}(t) > 0$, we have $r_{k_2}^{opt}(t) > r_{k_1}^{opt}(t)$. Since $B > 0$, an interval exists over which $r_{k_1}^{opt}(t) > 0$. Thus, we see that $\int_0^T r_{k_1}^{opt}(s)ds < \int_0^T r_{k_2}^{opt}(s)ds$, which leads to a contradiction, hence there is a unique k value that achieves $D^{opt}(T) = B$.

■

Consider now the setup with general $A(t)$ and $D_{min}(t)$ curves. Theorem V below gives the optimality criterion for this case and is a generalization of Theorem I presented earlier. It states that if there exists any two points on an admissible departure curve that can be replaced with a constant marginal-cost solution without violating the admissibility constraints, the new departure curve obtained will have a lower energy cost. The notation, *"constant marginal-cost curve over interval* $[a,b]$ *between* $[B_1, B_2]$*"* will refer to the departure curve, $L(t)$, obtained using the solution in (2.18) as follows: $L(a) = B_1$, $L(t) = L(a) + \int_a^t r(s)ds$, $t \in [a,b]$, where $r(s) = \max(0, r^*)$ and marginal-cost k chosen such that $L(b) = B_2$. From Lemma 9, this value of k and the corresponding $L(t)$ are unique.

***<u>Theorem</u>* V (Optimality Criterion)** *Let $D(t)$ be an admissible departure curve and $L(t)$ be the constant marginal-cost curve over $[a,b]$ between $[D(a), D(b)]$, $0 \leq a < b \leq T$. If $L(t)$ is admissible, i.e. $D_{min}(t) \leq L(t) \leq A(t)$, and, $L(t) \not\equiv D(t)$, the new departure curve $\tilde{D}(t)$ constructed as,*

$$\begin{aligned}\tilde{D}(t) &= D(t),\ t \in [0,a) \\ &= L(t),\ t \in [a,b] \\ &= D(t),\ t \in (b,T]\end{aligned}$$

satisfies $\mathcal{E}(\tilde{D}(t)) \leq \mathcal{E}(D(t))$, where $\mathcal{E}(\cdot)$ is as given in (2.17).

Proof: First note that since $L(t)$ is admissible, the new curve $\tilde{D}(t)$ is also admissible. Consider,

$$\mathcal{E}(\tilde{D}(t)) - \mathcal{E}(D(t)) = \mathcal{E}(L(t)) - \int_a^b g(D'(t), t)dt \qquad (2.21)$$

From Lemmas 8 and 9, we know that $L(t)$ is the unique curve, that has the least energy cost among all departure curves that would transmit $(D(b) - D(a))$ units of data over time interval $[a, b]$. Thus, $\mathcal{E}(L(t)) \leq \int_a^b g(D'(t), t)dt$, which completes the proof. ■

From the above theorem, we see that whenever admissible segments of the optimal departure curve follow the constant marginal cost curve. This property translated into constant rate (straight line) segments in the time-invariant power-rate function case, as outlined earlier in Theorem I. Thus, we see that the pictorial representation and the properties from the time-invariant case apply here in terms of constant marginal costs. Lastly, as illustrative examples for the time-varying case, we re-visit Examples 1 and 2 in Section 2.3.3 and obtain the departure curve that satisfies the optimality criterion. The algorithms presented are obtained by translating the respective ones from the time-invariant case, where instead of constant-slope segments we will be constructing constant marginal-cost segments.

Example 6 : Consider the setup in Example 1 where there is a stream of N packet arrivals and a deadline T by which all the data must depart. The curves $A(t)$ and $D_{min}(t)$ for this problem are depicted in Figure 2-6(a) with the same notations as used in that example. To obtain the departure curve satisfying the optimality criterion proceed as follows. Start at time 0; let $\{k_i\}$, $i = 1, \ldots, N$, be the marginal costs to meet each of the (T_i, A_i) points individually, i.e. k_i is the marginal cost associated with optimally transmitting A_i bits over time $[0, T_i]$. Let k_{min} be the minimum among $\{k_i\}$ and i_{min} the corresponding index of the minimizing jump point. The first segment of $D^{opt}(t)$ is then the constant marginal cost solution between $[0, T_{i_{min}}]$ with marginal cost k_{min}. Now, starting at $(T_{i_{min}}, A_{i_{min}})$ repeat the algorithm by shifting the origin to this point and considering the jump points beyond $T_{i_{min}}$, i.e. considering all i such that $T_i > T_{i_{min}}$. Finally, the algorithm stops when $T_{i_{min}} = T$.

Example 7 : Consider the setup in Example 2 where the queue has M data packets with the i^{th} packet having b_i bits and a deadline d_i, $i = 1, .., M$. For this problem, the curves $A(t)$ and $D_{min}(t)$ are shown in Figure 2-6(b) with the same notations as used in that example. As in the previous example, the departure curve satisfying the optimality criterion is constructed as follows. At time 0, let $\{k_j\}$, $j = 1, \ldots, M$ be the marginal costs to meet the (T_j, B_j) points, i.e. k_j is the marginal cost associated with optimally transmitting B_j bits over time $[0, T_j]$. Let k_{max} be the maximum among $\{k_j\}$ and j_{max} be the corresponding index of the maximizing jump point. The first segment of $D^{opt}(t)$ is then the constant marginal-cost solution between $[0, T_{j_{max}}]$ with marginal cost k_{max}. Now, starting at $(T_{j_{max}}, B_{j_{max}})$ repeat the algorithm by shifting the origin to this point and considering the jump points beyond $T_{j_{max}}$. The algorithm finally stops at the step when $T_{j_{max}} = T$.

2.5 Chapter Summary

In this chapter, we considered the deterministic setup for the deadline-constrained energy-efficient data transmission problem. We presented a finite-time horizon formulation where the objective was to obtain the minimum-energy transmission policy with hard deadline or other QoS constraints on the data. The data flow model was setup using three cumulative curves – the arrival curve $A(t)$, the departure curve $D(t)$, and the minimum departure curve $D_{min}(t)$. The arrival curve modelled the cumulative amount of data arrived to the queue, the departure curve modelled the cumulative amount of data departed from the queue, and the minimum departure curve modelled the cumulative minimum amount of data that must depart to satisfy the quality-of-service constraints. Under this framework, the optimization problem reduced to obtaining the minimum energy departure curve that satisfied the arrival and the QoS constraints.

We considered both the time-invariant and the time-varying power-rate function. In the time-invariant case, the graphical visualization of the problem provided an intuitive understanding of the optimality properties and the optimal departure curve, $D^{opt}(t)$. As outlined in Section 2.3.3, the optimal departure curve can be visualized as a taut string lying between the $A(t)$ and $D_{min}(t)$ curves. It was also shown that $D^{opt}(t)$ not only minimizes the total energy expenditure but it also has the least maximum instantaneous power requirement and the shortest length among admissible departure curves. In the time-varying power-rate function case, the corresponding optimality criterion was obtained and it was shown to be based on a constant marginal-cost property.

The formulation considered in this chapter generalizes the work presented in [10, 21]. Hence, the problems considered in those works can be re-formulated within the cumulative curves framework of this chapter, and the solutions can be obtained in a simpler way through the graphical picture. Having considered the deterministic case, we next consider a stochastic setup in Chapters 3 and 4 which involves a stochastic and time-varying fading channel.

Chapter 3

Stochastic Setup – “BT-problem”

3.1 Introduction

Communication over a wireless channel inherently involves dealing with channel variations that arise due to multi-path fading and other similar effects; however, the effect of channel fading on data transmission depends on the time-scale of this variability [4, 5]. In slow mobility environments, such as fixed wireless LAN's with non-mobile or slowly moving users, the time-scale of channel variations is much larger as compared to the time-scale of packet deadlines. Thus, in this case, data communication takes place over a virtually time-invariant channel. In scenarios of higher mobility, the time-scale of channel variations is much shorter and a network-layer data packet would then need to be transmitted over multiple fade states of the channel. As discussed in Chapter 1, since the power-rate function depends on the underlying channel fade state, clearly, to achieve energy efficiency the transmission rate must be adapted dynamically over time in response to the channel variations.

In this chapter and the next, i.e. Chapters 3 and 4, we focus on such scenarios where the channel fluctuates on a time-scale shorter than the packet deadlines. Our main focus in this chapter will be on the canonical “BT-problem”, where the transmitter queue has B bits of data and a single deadline T by which this data must be transmitted. The channel state varies stochastically over time and is modelled as a Markov process. The transmitter can control the transmission rate over time and the expended power depends on both the chosen rate and the present channel state. The objective is to obtain a transmission policy that minimizes the expected energy expenditure while meeting the deadline constraint. As we see later in Chapter 4, the solution to the BT-problem helps build towards many important generalizations involving variable deadlines and packet arrivals.

In the previous chapter, we observed that the optimal solution under a deterministic setup has a very elegant description using a graphical visualization. We also observed that under the time-invariant channel setup the optimal policy to transmit B bits of data by deadline T was to transmit at a constant rate B/T (see Section 2.3.2). With a stochastic channel, clearly, a constant transmission rate does not suffice and the rate must be further adapted in response to the channel variations. Intuitively, when the channel is in a "good" state the transmission rate must be increased while in a "bad" state it must be decreased. In this chapter, we obtain the optimal rate-control policy and discuss the various interesting insights that can be drawn from its functional form.

The rest of this chapter is organized as follows. In Section 3.2, we describe the transmission and the channel model. In Section 3.3, we address the BT-problem when there is no limit on the instantaneous maximum transmission power, while, in Section 3.4, we consider the setup when there is a short-term average power limit. Finally, in Section 3.5, we conclude the chapter and summarize the results. The proofs for the various results in this chapter are presented in Appendix B.

3.2 System Model

As assumed in Chapter 2, we consider a continuous-time model of the system where the rate can be varied continuously in time. Clearly, such a model is an approximation of a communication system that operates in discrete time-slots; however, the assumption is justified since, in practice, transmission-rate can be adapted over time-slots of 1 msec duration, while, in comparison packet deadlines are usually on the order of 100's of msec [3, 4, 6]. A significant advantage of such a model is that it makes the problem mathematically tractable and yields simple solutions, whereas, the alternative discrete-time optimization setup (eg. discrete-time Dynamic Programming) is intractable, computationally intensive and would only yield numerical solutions without much insights. In fact, the results obtained here using the continuous-time model can be applied to the discrete-time system in a very straightforward manner, by simply evaluating the solution at the time-slot boundaries, as done for the simulation results. We now describe the transmission model followed by the Markov model for the channel state process.

modulation	bits/symbol	SNR/symbol
2 PAM	1	$0.25\ d^2$
4 QAM	2	$0.50\ d^2$
16 QAM	4	$1.25\ d^2$
64 QAM	6	$5.25\ d^2$

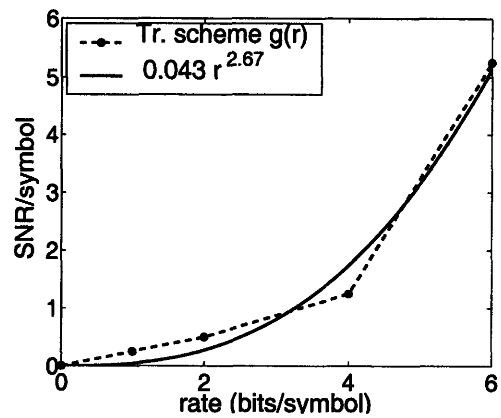


Figure 3-1: Modulation scheme considered in [40] as given in the table. The corresponding plot shows the least squares monomial fit, $0.043r^{2.67}$, to the scaled piecewise linear power-rate curve.

3.2.1 Transmission Model

Let h_t denote the channel gain, $P(t)$ the transmitted signal power and $P^{rcd}(t)$ the received signal power at time t. As discussed in Chapter 1 (see Section 1.1), the required received-signal-power for reliable communication, with a certain low bit-error probability, is convex in the rate [4, 8–13, 21, 22, 27, 29, 32, 39, 40]; i.e. $P^{rcd}(t) = g(r(t))$, where $g(r)$ is a non-negative, convex and increasing function for $r \geq 0$. Since the received signal power is given as $P^{rcd}(t) = |h_t|^2 P(t)$, the required transmission power to achieve rate $r(t)$ is given by,

$$P(t) = \frac{g(r(t))}{c(t)} \tag{3.1}$$

where $c(t) \triangleq |h_t|^2$. The quantity $c(t)$ is referred to as the *channel state* at time t. Its value at time t is assumed known either through one-step channel prediction or direct channel measurement, but it evolves stochastically in the future. As an example of (3.1), with optimal channel coding the well-known Shannon capacity formula gives the power per bit as, $P = \frac{N_0 W (2^{r/W} - 1)}{|h_t|^2}$ [67]; other examples can be found in [9, 10]. It is worth emphasizing that while we defined $c(t)$ as $|h_t|^2$ to motivate the relationship in (3.1), more generally, $c(t)$ could include other stochastic variations in the system and (uncontrollable) interference from other transmitter-receiver pairs, as long as the power-rate relationship obeys (3.1)[1].

[1]For example, let $I(t)$ be the interference power from other sources. Treating this as gaussian noise, the reliable communication rate, $r(t)$, for transmit power $P(t)$ is given by, $r(t) = W \log_2 \left(1 + \frac{P(t)|h_t|^2}{N_0 W + I(t)}\right)$, where N_0 is the thermal noise power per unit bandwidth and W is the total bandwidth. Re-arranging and defining $c(t) = \frac{|h_t|^2}{N_0 W + I(t)}$, we get, $P(t) = \frac{g(r(t))}{c(t)}$, where $g(r) = 2^{r/W} - 1$.

In this work, our primary focus will be on $g(r)$ belonging to the class of *Monomial functions*, namely, $g(r) = kr^n,\ n > 1, k > 0\ (n, k \in \mathbb{R})$. While this assumption restricts the generality of the problem, it serves several purposes. First, mathematically it leads to simple optimal solutions in explicit-form and insightful observations that can be applied in practice. Second, most importantly, for most practical transmission schemes, the function $g(\cdot)$ is described numerically and its exact analytical form is unknown. In such situations, one can obtain the best approximation of that function to the form kr^n by choosing the appropriate k, n and then applying the results thus obtained. For example, consider the QAM modulation scheme considered in [40] and reproduced here in Figure 3-1. The table gives the rate and the normalized signal power per symbol, where d represents the minimum distance between signal points and the scheme is designed for error probabilities less than 10^{-6}. The plot gives the least squares monomial fit to the transmission scheme and one can see from the plot that for this example the monomial approximation is fairly close. Third, monomials form the first step towards studying extensions to polynomial functions which would then apply to a general $g(\cdot)$ function using the polynomial expansion. Under a more restrictive setting in Section 3.3.3, we also study the class of *Exponential functions*, namely, $g(r) = k(\alpha^r - 1),\ \alpha > 1, k > 0\ (\alpha, k \in \mathbb{R})$. Finally, without loss of generality, throughout the chapter we take $k = 1$, since any other value of k simply scales the total energy cost without affecting the results on the optimal transmission policy.

3.2.2 Channel Model

We consider a general first-order, continuous-time, discrete state space Markov model for the channel state process. Markov processes constitute a large class of stochastic processes that exhaustively model a wide set of fading scenarios, and there is substantial literature on these models [54–59] and their applications to communication networks [27, 58–60]. In fact, in [54–57] and the body of literature referenced therein, first-order Markov models have been proposed for various commonly studied fading scenarios such as Rayleigh, Rician and Nakagami fading. In this work, we do not restrict attention to a specific fading scenario but rather assume a fairly general Markov model, as described in detail next.

Denote the channel stochastic process as $C(t)$ and the state space as $\mathcal{C}$. Let $c \in \mathcal{C}$ denote a particular channel state and $\{c(t), t \geq 0\}$ denote a sample path. Starting from state c, the channel can transition to a set of new states ($\neq c$) and this set is denoted as $\mathcal{J}_c$. Let

$\lambda_{c\tilde{c}}$ denote the channel transition rate from state c to $\tilde{c}$, then, the sum transition rate at which the channel jumps out of state c is, $\lambda_c = \sum_{\tilde{c} \in \mathcal{J}_c} \lambda_{c\tilde{c}}$. Clearly, the expected time that $C(t)$ spends in state c is $1/\lambda_c$ and one can view $1/\lambda_c$ as the coherence time of the channel in state c.

Now, define $\lambda \triangleq \sup_c \lambda_c$ and a random variable, $Z(c)$, as,

$$Z(c) \triangleq \begin{cases} \tilde{c}/c, & \text{with probability } \lambda_{c\tilde{c}}/\lambda, \ \tilde{c} \in \mathcal{J}_c \\ 1, & \text{with probability } 1 - \lambda_c/\lambda \end{cases} \tag{3.2}$$

With this definition, we obtain a compact and simple description of the process evolution as follows. *Given a channel state c, there is an Exponentially distributed time duration with rate λ after which the channel state changes. The new state is a random variable which is given as $C = Z(c)c$.* Clearly, from (3.2) the transition rate to state $\tilde{c} \in \mathcal{J}_c$ is unchanged at $\lambda_{c\tilde{c}}$, whereas with rate $\lambda - \lambda_c$ there are indistinguishable self-transitions. This is a standard Uniformization technique and there is no process generality lost with the new description as it yields a stochastically identical scenario [86].

Other technical assumptions in the channel model are as follows. The channel state space, $\mathcal{C}$, is assumed to be a countable space and $\mathcal{C} \subseteq \mathbb{R}^+$. The states $c = 0, \infty$ are excluded from $\mathcal{C}$ since each of this state leads to a singularity in (3.1). The set $\mathcal{J}_c, \forall c$, is a finite subset of $\mathcal{C}$. Transition rate $\lambda_c, \forall c$, is bounded which ensures that λ defined as the supremum is finite. For all c, the support of $Z(c)$ lies in $[z_l, z_h]$, where $0 < z_l \leq z_h < \infty$. This ensures that $C(t)$ does not converge to 0 or ∞, a.s. (almost surely), over a finite time interval.

Example: As an example, consider a two-state channel model with states b and g denoting the "bad" and the "good" channel conditions respectively. The two states correspond to a two level quantization of the channel gain square (i.e. $|h(t)|^2$). If the measured channel gain square is below some value, the channel is labelled as "bad" and $c(t)$ is assigned an average value c_b, otherwise $c(t) = c_g$ for the good condition. Let the transition rate from the good to the bad state be λ_{gb} and from the bad to the good state be λ_{bg}. Let $\gamma = c_b/c_g$, and using the earlier definition, $\lambda = \max(\lambda_{bg}, \lambda_{gb})$. For state c_g we have,

$$Z(c_g) = \begin{cases} \gamma, & \text{with probability } \lambda_{gb}/\lambda \\ 1, & \text{with probability } 1 - \lambda_{gb}/\lambda \end{cases} \tag{3.3}$$

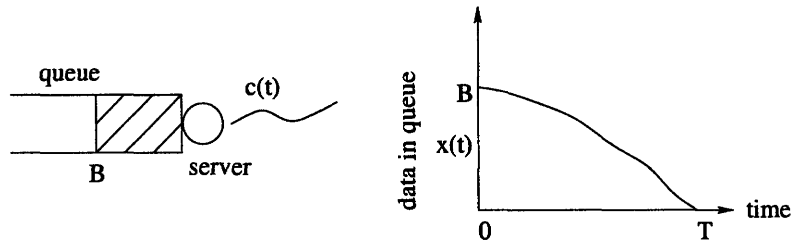


Figure 3-2: Schematic description of the system for the BT-problem.

To obtain $Z(c_b)$, replace γ with $1/\gamma$ and λ_{gb} with λ_{bg} in (3.3).

3.3 BT-problem

As a recapitulation, the BT-problem is to transmit B bits of data by deadline T. The channel state is stochastic and the objective is to obtain a rate-control policy that minimizes the expected energy expenditure while meeting the deadline constraint. We assume that there is no power limit and given a particular channel state the transmitter can transmit at any non-negative rate. While this assumption simplifies the mathematical exposition, the convexity of the power-rate curve does impose a "soft" power limit since higher rate transmissions incur a rapidly increasing power expenditure. Thus, the optimal policy would tend to avoid high-power transmissions; furthermore, in case of a practical implementation one can modify the optimal policy by simply restricting its rate to the maximum allowable. To further explore the issues arising out of power-limits, we consider a formulation involving short-term power limits which is presented later in Section 3.4.

We now discuss in detail the optimal control formulation for the BT-problem, obtain the optimal policy and discuss the insights that can be drawn from it.

3.3.1 Optimal Control Formulation

Let $x(t)$ denote the amount of data left in the queue at time t. The system state can then be described as (x, c, t), where this notation means that at the present time t, the amount of data left is $x(t) = x$, and the channel state is $c(t) = c$. Let $r(x, c, t)$ denote the chosen transmission rate for the corresponding system state (x, c, t). Since the underlying channel process is Markov, it is sufficient to restrict attention to transmission policies that depend only on the present system state [65]. Clearly then, (x, c, t) is a Markov process. A schematic diagram of the system is depicted in Figure 3-2.

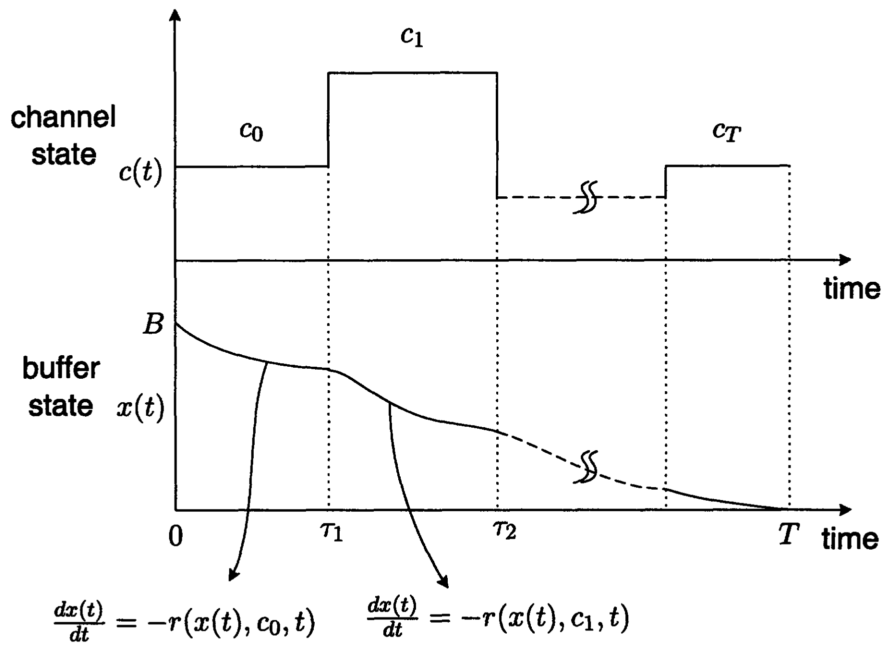


Figure 3-3: System evolution over time for the BT-problem.

Given a policy $r(x, c, t)$, the system evolves in time as a Piecewise-Deterministic-Process (PDP) [63] as follows. It starts with $x(0) = B$ and $c(0) = c_0$ ($c_0 \in \mathcal{C}$). Until τ_1, where τ_1 is the first time instant after $t = 0$ at which the channel changes, the buffer is reduced at the rate $r(x(t), c_0, t)$ (throughout the thesis, the terms "queue" and "buffer" are used interchangeably). Hence, over the interval $[0, \tau_1)$, $x(t)$ satisfies the ordinary differential equation,

$$\frac{dx(t)}{dt} = -r(x(t), c_0, t) \tag{3.4}$$

Equivalently in integral form, $x(t) = x(0) - \int_0^t r(x(s), c_0, s)ds,\ t \in [0, \tau_1]$. Then, starting from the new state $(x(\tau_1), c(\tau_1), \tau_1)$, until the next channel transition we have, $\frac{dx(t)}{dt} = -r(x(t), c(\tau_1), t)$, $t \in [\tau_1, \tau_2)$; and this procedure repeats until $t = T$ is reached. A schematic diagram of the process for a particular channel sample path is depicted in Figure 3-3.

A transmission policy, $r(x, c, t)$, is *admissible* for the BT-problem, if it satisfies the following,

(a) $0 \le r(x, c, t) < \infty$, (rate must be non-negative)

(b) $r(x, c, t) = 0$, if $x = 0$ (no data left to transmit) and,

(c) $x(T) = 0$, almost surely (a.s.), (deadline constraint)

Additionally, another technical requirement is that $r(x, c, t)$ be locally Lipschitz continuous in x (for $x > 0$) and piecewise continuous in t which ensures that the ODE in (3.4) has a unique solution [69].

Consider now an admissible transmission policy $r(\cdot)$ and define a *cost-to-go* function, $J_r(x, c, t)$, as the expected energy cost starting from state (x, c, t), $t < T$. Then,

$$J_r(x,c,t) = E\left[\int_t^T \frac{1}{c(s)} g(r(x(s), c(s), s))ds\right] \tag{3.5}$$

where the term within the brackets is the total energy expenditure obtained as the integral of the power cost over time. The expectation above is taken over $\{c(s), s \in (t, T]\}$ and conditional on the starting state $x(t) = x$, $c(t) = c$. Define a *minimum cost function*, $J(x, c, t)$, as the infimum of $J_r(x, c, t)$ over the set of all admissible transmission policies.

$$J(x,c,t) \triangleq \inf_{r(x,c,t)} J_r(x,c,t), \quad r(x,c,t) \text{ admissible} \tag{3.6}$$

Thus, now, stated concisely the optimization problem is to compute the minimum cost function $J(x, c, t)$ and obtain the optimal policy $r^*(x, c, t)$ that achieves this minimum cost.

3.3.2 Optimality Conditions

A standard approach towards studying continuous-time problems is to investigate their behavior over a small time interval. In the context of the BT-problem, this methodology is summarized as follows. Suppose that the system is in state (x, c, t). We first apply a transmission policy, $r(x, c, t)$, in the small interval $[t, t+h]$ and thereafter, starting from the state $(x(t+h), c(t+h), t+h)$ we assume that the optimal policy is followed. By assumption, the energy cost is optimal over $[t+h, T]$, hence, investigating the system over $[t, t+h]$ would give conditions for the optimality of the chosen rate at time t. Since t is arbitrary, we obtain formal conditions for an optimal policy.

Following the above approach, we now present the details of the analysis. Consider $t \in [0, T)$ and a small interval $[t, t+h]$, where $t+h < T$. Clearly, from Bellman's principle [63] the value function $J(x, c, t)$ satisfies,

$$J(x,c,t) = \min_{r(\cdot)} \left\{E \int_t^{t+h} \frac{1}{c(s)} g(r(x(s), c(s), s))ds + EJ(x_{t+h}, c_{t+h}, t+h)\right\} \tag{3.7}$$

where x_{t+h}, c_{t+h} is a short-hand notation for $x(t+h)$ and $c(t+h)$ respectively. The expression within the minimization bracket in (3.7) denotes the total cost with policy $r(\cdot)$ being followed over $[t, t+h]$ and the optimal policy thereafter. This cost must be clearly no less than the cost of applying the optimal policy directly from the starting state (x, c, t). Thus for an admissible policy $r(\cdot)$ we obtain,

$$J(x,c,t) \leq E\int_t^{t+h} \frac{1}{c(s)} g(r(x(s),c(s),s))ds + \ E\left[J(x_{t+h},c_{t+h},t+h)\right] \tag{3.8}$$

$$E[J(x_{t+h},c_{t+h},t+h)] - J(x,c,t) + E\int_t^{t+h} \frac{1}{c(s)} g(r(x(s),c(s),s))ds \geq 0 \tag{3.9}$$

Dividing (3.9) by h and taking the limit $h \downarrow 0$, we obtain,

$$\frac{E\int_t^{t+h} \frac{g(r(x(s),c(s),s))}{c(s)}ds}{h} \rightarrow \frac{g(r)}{c} \tag{3.10}$$

where, r, denotes the transmission rate for the policy at time t, i.e. $r = r(x,c,t)$.

Define, $\lim_{h\downarrow 0} \frac{EJ(x_{t+h},c_{t+h},t+h)-J(x,c,t)}{h} \triangleq A^r J(x,c,t)$, then taking this limit in (3.9) and using (3.10) we get,

$$A^r J(x,c,t) + \frac{1}{c} g(r) \geq 0 \tag{3.11}$$

The quantity $A^r J(x,c,t)$ is called the differential generator of the Markov process $(x(t), c(t))$ for transmission policy $r(\cdot)$. Intuitively, it can be viewed as a natural generalization of the ordinary time derivative for a function that depends on a stochastic process. An elaborate discussion on this topic can be found in [63–65]. For the process $(x(t), c(t))$, using the time evolution in (3.4), the quantity $A^r J(x,c,t)$ can be evaluated as,

$$A^r J(x,c,t) = \frac{\partial J(x,c,t)}{\partial t} - r(x,c,t)\frac{\partial J(x,c,t)}{\partial x} + \lambda(E_z[J(x,Z(c)c,t)] - J(x,c,t)) \tag{3.12}$$

where E_z is the expectation with respect to the Z variable as defined in (3.2). A heuristic computation for $A^r J(x,c,t)$ is given in Appendix B.12; for a detailed explanation see [63].

Now, in the above steps from (3.8)-(3.11), if policy $r(\cdot)$ is replaced with the optimal policy $r^*(\cdot)$, equation (3.11) holds with equality and we get,

$$A^{r^*} J(x,c,t) + \frac{1}{c} g(r^*) = 0 \tag{3.13}$$

Thus, we see that for a given system state (x, c, t), the optimal transmission rate r^* is that value of r that minimizes (3.11) and the minimum value of this expression equals zero. This gives,

$$\min_{r\in[0,\infty)}\left[\frac{g(r)}{c}+A^{r}J(x,c,t)\right]=0 \tag{3.14}$$

Substituting $A^r J$ from (3.12), we get a partial differential equation (PDE) in $J(x,c,t)$ which is also referred as the Hamilton-Jacobi-Bellman (HJB) equation. This is the *Optimality Equation* for the BT-problem.

$$\min_{r\in[0,\infty)}\left\{\frac{g(r)}{c}+\frac{\partial J(x,c,t)}{\partial t}-r\frac{\partial J(x,c,t)}{\partial x}+\lambda(E_z[J(x,Z(c)c,t)]-J(x,c,t))\right\}=0 \tag{3.15}$$

The boundary conditions for the above PDE are, $J(0,c,t)=0$, and $J(x,c,T)=\infty$, if $x>0$. The last condition follows due to the deadline constraint of T on the data.

While the above analysis gives the optimality equation, an important caveat is that it assumes $J(x,c,t)$ to be sufficiently smooth. Therefore, additionally, we also need converse arguments to verify that having a solution of (3.15) indeed gives the optimal solution. These technical details and the verification theorems are presented in Appendix B.1.

3.3.3 Optimal Transmission Policy

We have, so far, presented general results on the optimality condition for the BT-problem. We, now, give specific analytical results for the optimal policy and discuss some of the insights that can be drawn from it. However, before proceeding further a few additional notations regarding the channel process are required. Let there be total m channel states in the Markov model and denote the various states $c\in\mathcal{C}$ as $c^1,c^2,\ldots,c^m$. Given a channel state c^i, the values taken by the random variable $Z(c^i)$ (defined in (3.2)) are denoted as $\{z_{ij}\}$, where $z_{ij}=c^j/c^i$. The probability that $Z(c^i)=z_{ij}$ is denoted as p_{ij}. Clearly, if there is no transition from state c^i to c^j, $p_{ij}=0$. Also, without loss of generality we take the multiplicative constant $k=1$ in the function $g(r)=kr^n$ since any other value of k simply scales the total cost in (3.5) but the optimal policy results remain the same.

Theorem ***VI*** *Consider the BT-problem with $g(r)=r^n$, $n>1, n\in\mathbb{R}$ and a Markov channel model. The optimal policy, $r^*(x,c,t)$, and the minimum cost function, $J(x,c,t)$,* *are given by (note, $(x, c, t) \in [0, B] \times \mathcal{C} \times [0, T)$),*

$$r^*(x, c^i, t) = \frac{x}{f_i(T-t)}, \; i = 1, \ldots, m \tag{3.16}$$

$$J(x, c^i, t) = \frac{x^n}{c^i (f_i(T-t))^{n-1}}, \; i = 1, \ldots, m \tag{3.17}$$

The functions $\{f_i(s)\}_{i=1}^m$ are the solution of the following ordinary differential equation (ODE) system with the boundary conditions $f_i(0) = 0, f_i'(0) = 1, \forall i$ [2],

$$f_1'(s) = 1 + \frac{\lambda f_1(s)}{n-1} - \frac{\lambda}{n-1} \sum_{k=1}^{m} \frac{p_{1k}}{z_{1k}} \frac{(f_1(s))^n}{(f_k(s))^{n-1}} \tag{3.18}$$

$$\vdots$$

$$f_m'(s) = 1 + \frac{\lambda f_m(s)}{n-1} - \frac{\lambda}{n-1} \sum_{k=1}^{m} \frac{p_{mk}}{z_{mk}} \frac{(f_m(s))^n}{(f_k(s))^{n-1}} \tag{3.19}$$

Proof: See Appendix B.2. ■

The results in the above theorem can be interpreted as follows. From (3.16), the optimal rate given x amounts of data left, channel state c^i and time t, is $\frac{x}{f_i(T-t)}$, where the function $f_i(s)$ is associated with the channel state c^i. The corresponding minimum expected cost starting from state (x, c^i, t) is $\frac{x^n}{c^i f_i(T-t)^{n-1}}$. The boundary condition $f_i(0) = 0$ is due to the deadline constraint, since at the deadline $(T - t) = 0$ and we have $J(x, c^i, T) = \infty$, if $x \neq 0$. In full generality, the ODE system in (3.18)-(3.19) can be easily solved numerically using standard techniques (e.g. ODE solvers in MATLAB) and as shown in Appendix B.2, the system has a unique positive solution. Furthermore, this computation needs to be done only once before the system starts operating and $\{f_i(s)\}$ can be pre-determined and stored in a table in the transmitter's memory. Once $\{f_i(s)\}$ are known, the closed form structure of the optimal policy in (3.16) warrants no further computation. At time t, the transmitter simply looks at the amount of data left in the queue, x, the channel state, c^i, and using the appropriate $f_i(\cdot)$ function it computes the transmission rate as $\frac{x}{f_i(T-t)}$.

The solution in (3.16) provides several interesting observations and insights as follows. At time t, the optimal rate depends on the channel state c^i through the function $f_i(T-t)$ and this rate is linear in x with slope $\frac{1}{f_i(T-t)}$. Thus, as intuitively expected, the rate is

[2]For numerical evaluation of the ODE solution, the two boundary conditions can be combined by taking a small $\epsilon > 0$, letting $f_i(s) = s$, $s \in [0, \epsilon]$, $\forall i$ and then using an initial-value ODE solver to obtain $\{f_i(s)\}$, $s \geq \epsilon$.

proportionately higher when there is more data left in the queue. Furthermore, we can view the quantity $\frac{1}{f_i(T-t)}$ as the "urgency" of transmission under the channel state c^i and with $(T-t)$ time left until the deadline. This view gives a nice and intuitive separation form for the optimal rate:

$$\textit{optimal rate} = \textit{amount of data left} * \textit{urgency of transmission}$$

Due to the boundary condition, as t approaches T, $f_i(T-t), \forall i$ goes to zero, thus, as expected, the urgency of transmission, $\frac{1}{f_i(T-t)}$, increases as t approaches the deadline. Interestingly, if we set $\lambda = 0$ (no channel variations) then, $f_i(T-t) = T-t, \forall i$ and $r^*(x,c,t) = \frac{x}{T-t}$. Thus, with no channel variations the optimal policy is to transmit at a rate that just empties the buffer by the deadline. This observation is consistent with the results in Chapter 2 and also conforms with previous results in the literature for non-fading/time-invariant channels [10,21]. We refer to this transmission scheme as the "Direct Drain" (DD) policy.

Simulation Example: Consider the two state channel model with states "bad" and "good" as described in Section 3.2.2. Let $g(r) = r^2$ (i.e. $n = 2$) and for simplicity take $\lambda_{bg} = \lambda_{gb} = \lambda$. Denoting $\gamma = c_b/c_g$, we have, $Z(c_g) = \gamma$, *w.p.* 1, and $Z(c_b) = 1/\gamma$, *w.p.* 1. Denoting $f_b(s), f_g(s)$ as the respective functions in the bad and the good states, we have,

$$f_b'(s) = 1 + \lambda f_b(s) - \frac{\gamma\lambda(f_b(s))^2}{f_g(s)} \tag{3.20}$$

$$f_g'(s) = 1 + \lambda f_g(s) - \frac{\lambda(f_g(s))^2}{\gamma f_b(s)} \tag{3.21}$$

Figure 3-4 plots these functions, evaluated using MATLAB, for $T = 10, \lambda = 5, \gamma = 0.3$. First, as expected $f_g(T-t) \leq f_b(T-t), \forall t$, which implies that given x units of data in the buffer and time t, the rate $\frac{x}{f_g(T-t)}$ is higher under the good state than the bad state. Second, $f_g(T-t) \leq T-t \leq f_b(T-t)$, where the function, $T-t$, gives the rate, $\frac{x}{T-t}$, corresponding to the direct drain (DD) policy. Thus, the optimal policy both spreads the data over time and adapts the rate in response to the time-varying channel condition and this adaptation is governed by the respective functions $\{f_i(\cdot)\}$.

We now present illustrative simulation results to compare the performance of the optimal policy with the direct drain (DD) policy in terms of the energy expenditure. As stated

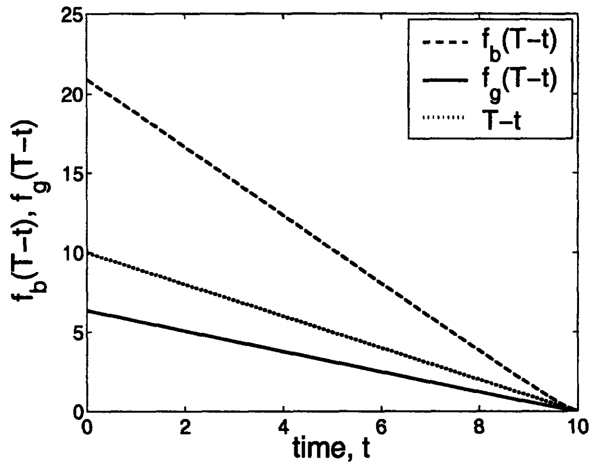


Figure 3-4: $f_b(T-t)$ and $f_g(T-t)$ plot for the bad and the good channel respectively. Other parameters include, $g(r) = r^2$, $T = 10, \lambda = 5, \gamma = 0.3$.

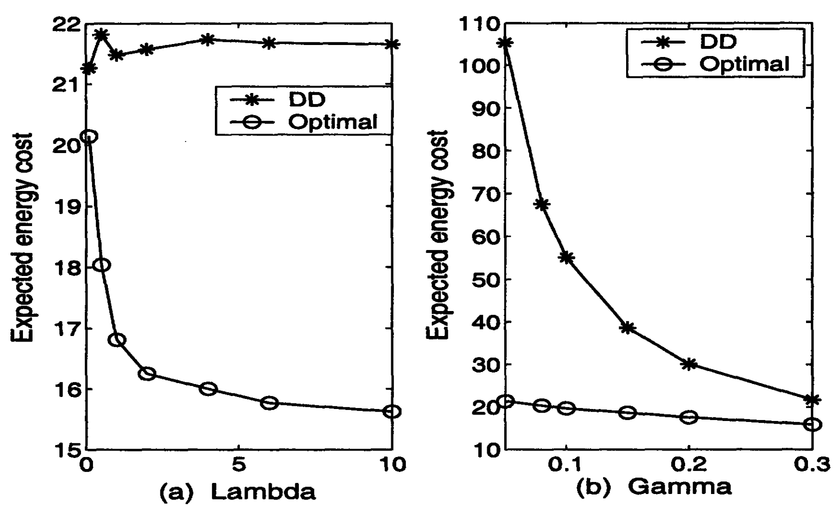


Figure 3-5: Expected energy cost for the optimal and the direct drain (DD) policy.

earlier, the DD policy transmits at a rate sufficient to just empty the buffer by the deadline. For the simulations, we consider the two-state channel model with $c_g = 1$, $c_b = \gamma$ and take $g(r) = r^2$. We let, $T = 10$ and partition the interval $[0, 10]$ into slots of length $dt = 10^{-3}$, thus, having $10,000$ time slots. The transmission rate chosen in each slot is obtained by evaluating the respective policies at the time corresponding to the start of that slot. A channel sample path is simulated using a Bernoulli process, where in a slot the channel transitions with probability λdt and with probability $1 - \lambda dt$ there is no transition. At each transition the new state is $\tilde{c} = Z(c)c$, which for the two-state model amounts to jumps between the two states. Expected energy cost is computed by taking an average over 10^4 sample paths.

Figure 3-5(a) plots the energy costs of the two policies as λ is varied with $\gamma = 0.3$ and $B = 10$. When λ is small the channel is essentially time-invariant over the deadline interval and the two policies are comparable. As λ increases, the optimal policy has a substantially lower energy cost than the DD policy since it adapts to the channel fluctuations; whereas, the DD policy which does not adapt incurs high energy cost during the bad channel states. In Figure 3-5(b), γ is varied with $\lambda = 5$ and $B = 10$. As γ decreases the good and the bad channel quality differ significantly and the optimal rate adaptation leads to a substantially lower energy cost with an order of magnitude difference as compared to the DD policy.

Constant Drift Channel: Theorem VI gives the optimal policy for a general Markov channel model. By considering a special structure on the channel model which we refer to as the "*Constant Drift*" channel, two specialized results can be obtained. First, we obtain the $f(\cdot)$ function in closed form for the Monomial class ($g(r) = r^n$), and second, we obtain the optimal policy for the Exponential class ($g(r) = \alpha^r - 1$).

In the constant drift channel model, we assume that the expected value of the random variable $1/Z(c)$ is independent of the channel state, i.e. $E[1/Z(c)] = \beta$, a constant. Thus, starting in state c, if $\tilde{c}$ denotes the next transition state we have $E\left[\frac{1}{\tilde{c}}\right] = E\left[\frac{1}{Z(c)}\right]\frac{1}{c} = \frac{\beta}{c}$. This means that if we look at the process $1/c(t)$, the expected value of the next state is a constant multiple of the present state. We refer to β as the "*drift*" parameter of the channel process. If $\beta > 1$, the process $1/c(t)$ has an upward drift; if $\beta = 1$, there is no drift and if $\beta < 1$, the drift is downwards. As a simple example of such a Markov model, suppose that the channel transitions at rate $\lambda > 0$ and at every transition the state either improves by a factor $u > 1$ with probability p_u, or worsens by a factor $1/u$ with probability p_d $(= 1 - p_u)$. Thus, given some state $c > 0$ the next channel state is either uc or c/u, and, $E[1/Z(c)] = p_u/u + up_d$. Here, the drift parameter $\beta = p_u/u + up_d$.

There are various situations where the above model is applicable over the time scale of the deadline interval. For example, when a mobile device is moving in the direction of the base station, the channel has an expected drift towards improving conditions and vice-versa. Similarly, in case of satellite channels, changing weather conditions such as cloud cover makes the channel drift towards worsening conditions and vice-versa. In cases when the time scale of these drift changes is longer than the packet deadlines, the constant drift channel serves as an appropriate model.

The next theorem, Theorem VII, gives the optimal policy result for the constant drift channel model for the monomial class of functions while Theorem VIII later gives the result for the exponential class.

Theorem ***VII*** *Consider the BT-problem with* $g(r) = r^n,\ n > 1, n \in \mathbb{R}$ *and a constant drift channel with drift* β. *The optimal policy,* $r^*(x,c,t)$, *and the minimum cost function,* $J(x,c,t)$, *are,*

$$r^*(x,c,t) = \frac{x}{f(T-t)} \tag{3.22}$$

$$J(x,c,t) = \frac{x^n}{c(f(T-t))^{n-1}} \tag{3.23}$$

where $f(T-t) = \frac{(n-1)}{\lambda(\beta-1)}(1-\exp(-\frac{\lambda(\beta-1)}{n-1}(T-t)))$.

Proof: See Appendix B.6. ■

The closed-form expression of $f(\cdot)$ above provides an interesting and intuitive observation related to the parameter β. Suppose that the present channel state is c, then for a fixed rate r, the expected power cost in the next channel state is $\left(E\left[\frac{g(r)}{Z(c)c}\right] = \frac{g(r)}{c}\beta\right)$ which is β times the present cost $\frac{g(r)}{c}$. This means that for higher values of parameter β, the channel on every transition drifts in an expected sense towards higher expected power cost or worsening conditions and vice-versa as β decreases. Hence, as expected, the urgency of transmission $1/f(t)$ is an increasing function with respect to β, since for larger β values it becomes more energy efficient to utilize the present channel conditions. Interestingly, when $\beta = 1$, the expected future power cost does not change and in this case the optimal policy reduces to the direct drain (DD) policy, i.e. $r^*(x,c,t) = \frac{x}{T-t}$ (where we have used L'Hopital's rule to evaluate $f(\cdot)$ for $\beta = 1$). Thus, we see that the direct drain policy is optimal both under no channel variations and under a constant drift channel with $\beta = 1$.

Theorem ***VIII*** *Consider the BT-problem with* $g(r) = \alpha^r - 1,\ \alpha > 1$ *and a constant drift channel with drift* β. *The optimal policy,* $r^*(x,c,t)$, *is the following,*

Case 1: $\beta \geq 1$,

$$r^*(x,c,t) = \begin{cases} \sqrt{\frac{2x\lambda(\beta-1)}{\ln\alpha}}, & 0 \leq x < \frac{\lambda(\beta-1)(T-t)^2}{2\ln\alpha} \\ \frac{x}{T-t} + \frac{\lambda(\beta-1)(T-t)}{2\ln\alpha}, & x \geq \frac{\lambda(\beta-1)(T-t)^2}{2\ln\alpha} \end{cases} \tag{3.24}$$

Case 2: $0 < \beta < 1$,

$$r^*(x,c,t) = \begin{cases} 0\,, & 0 \le x < \frac{\lambda(1-\beta)(T-t)^2}{2\ln\alpha} \\ \frac{x}{T-t} - \frac{\lambda(1-\beta)(T-t)}{2\ln\alpha}\,, & x \ge \frac{\lambda(1-\beta)(T-t)^2}{2\ln\alpha} \end{cases} \tag{3.25}$$

Proof: See Appendix B.7. ■

From above, we see that the optimal rate function has a different functional form than the monomial case. However some of the natural properties of the rate function hold true in this case as well - it is monotonically increasing in x, increasing as t approaches the deadline and also increasing in β.

3.4 BT-problem with Short-term Power Limits

In the previous section, Section 3.3, we considered the BT-problem without an explicit power limit constraint. The transmitter was allowed to choose any transmission rate, although, due to convexity of the power-rate function the required transmission power increased rapidly with the rate. In this section, we turn our attention to the case when there are explicit short-term power limits. We consider the setup where the average transmission power spent over a short time interval must be less than some amount P. With this constraint, clearly, there is a restriction on the average amount of data that can be transmitted over a certain time interval, thus having the possibility of data left in the queue at T. To minimize the leftover data we apply a penalty cost on it. The objective now is to obtain a rate control policy that minimizes the expected energy expenditure plus the penalty cost imposed on the leftover data.

To address the power-constrained problem, we consider, as before, a continuous-time stochastic control formulation and in addition utilize lagrange duality techniques [66,68] to obtain the optimal policy. These details are described in the following sections.

3.4.1 Problem Formulation

As in Section 3.3.1, let $x(t)$ be the amount of data left in the queue at time t and let (x,c,t) denote the system state, where, as before, this notation means that at time t we have, $x(t) = x$ and $c(t) = c$. Let $r(x,c,t)$ denote a particular transmission policy. Given a policy $r(x, c, t)$, the system evolves in time as a Piecewise-Deterministic-Process (PDP) as described earlier in Section 3.3.1. We are given $x(0) = B$ and $c(0) = c_0$. Until τ_1, where τ_1 is the first time instant after $t = 0$ at which the channel changes, the buffer is reduced at the rate $r(x(t), c_0, t)$. Hence, over the interval $[0, \tau_1)$, $x(t)$ satisfies the ordinary differential equation,

$$\frac{dx(t)}{dt} = -r(x(t), c_0, t) \tag{3.26}$$

Equivalently, $x(t) = x(0) - \int_0^t r(x(s), c_0, s)ds,\ t \in [0, \tau_1]$. Then, starting from the new state $(x(\tau_1), c(\tau_1), \tau_1)$, until the next channel transition, we have, $\frac{dx(t)}{dt} = -r(x(t), c(\tau_1), t)$, $t \in [\tau_1, \tau_2)$; and this procedure repeats until $t = T$ is reached.

At time T, the data that missed the deadline (amount $x(T)$) is assigned a penalty cost of $\frac{\tau g(x(T)/\tau)}{c(T)}$ for some $\tau > 0$. This peculiar cost can be viewed in the following two ways. First, it simply represents a specific penalty function where τ can be adjusted and in particular made small enough[3] so that the data that misses the deadline is small. This will ensure that with good source coding, the entire data can be recovered even if $x(T)$ misses the deadline. Second, note that $\frac{\tau g(x(T)/\tau)}{c(T)}$ is the amount of energy required to transmit $x(T)$ data in time τ with the channel state being $c(T)$. Thus, τ is the small time window in which the remaining data is completely transmitted out assuming that the channel state does not change over that period. In fact, viewing $T + \tau$ as the actual deadline, τ then models a small buffer window in which unlimited power can be used to meet the deadline, albeit, at an associated cost.

Let the interval $[0, T]$ be partitioned into L equal periods[4] and denote P as the short-term expected power constraint at the transmitter. Then, over each partition the power constraint requires that the expected energy cost, $E\left[\int \frac{1}{c(s)} g(r(x(s), c(s), s))ds\right]$, is less than $P(T/L)$, i.e. we require,

$$E\left[\int_{\frac{(k-1)T}{L}}^{\frac{kT}{L}} \frac{g(r(x(s), c(s), s))}{c(s)} ds\right] \leq \frac{PT}{L}, \quad k = 1, \ldots, L \tag{3.27}$$

Note that T/L is the duration of each partition interval and $[\frac{(k-1)T}{L}, \frac{kT}{L})$ is the k^{th} interval, $k = 1, \ldots, L$. Clearly, by varying L the duration of the partition interval can be varied and

[3]For a strictly convex $g(\cdot)$ function, making τ smaller increases the penalty cost.

[4]Extensions to arbitrary sized partitions is fairly straightforward and such a generality is omitted for mathematical simplicity.

the power constraint can be made either more or less restrictive.

Let Φ denote the set of all transmission policies, $r(x,c,t)$, that satisfy the following requirements,

(a) $0 \leq r(x,c,t) < \infty$, (non-negativity of rate)

(b) $r(x,c,t) = 0$, if $x = 0$ (no data left to transmit)[5].

We say that a policy $r(x,c,t)$ is *admissible* for the BT-problem with power constraints, if $r(x,c,t) \in \Phi$ and additionally it also satisfies the power constraint as given in (3.27). Denoting the optimization problem as $(\mathcal{P})$, it can now be summarized as follows,

$$
\begin{aligned}
(\mathcal{P}) \qquad & \inf_{r(\cdot)\in\Phi} \; E\left[\int_0^T \frac{1}{c(s)} g(r(x(s),c(s),s))ds + \frac{\tau g(\frac{x(T)}{\tau})}{c(T)}\right] \\
& \text{subject to } \; E\left[\int_0^{\frac{T}{L}} \frac{1}{c(s)} g(r(x(s),c(s),s))ds\right] \leq \frac{PT}{L} \\
& \qquad \vdots \\
& E\left[\int_{\frac{T(L-1)}{L}}^{T} \frac{1}{c(s)} g(r(x(s),c(s),s))ds\right] \leq \frac{PT}{L}
\end{aligned}
$$

The expectations above are conditional on (x_0, c_0)[6], the starting values at $t = 0$. For the analysis, we will keep the general notation x_0 but its value in our case is simply $x_0 = B$. Note that problem $(\mathcal{P})$ as stated above has at least one admissible solution since a policy that does not transmit any data and simply incurs the penalty cost is an admissible policy. Furthermore, as shown in Appendix B.11, this simple policy has a finite cost and hence the minimum value of the objective function above is finite.

3.4.2 Optimal Policy

In order to solve problem $(\mathcal{P})$, we consider a lagrange duality approach. The basic steps involved in such an approach are as follows: (a) form the lagrangian by incorporating the constraints into the objective function using lagrange multipliers, (b) obtain the dual function by minimizing over the primal space, and (c) maximize the dual function with respect to the lagrange multipliers. Finally, we need to show that there is no duality gap, that is, maximizing the dual function gives the optimal cost for the constrained problem.

[5] As before, to ensure that (3.26) has a unique solution, we also require that $r(x,c,t)$ be locally Lipschitz continuous in x ($x > 0$) and piecewise continuous in t.

[6] To avoid being cumbersome on notation, we will throughout represent conditional expectations without an explicit notation but rather mention the conditioning parameter whenever there is ambiguity.

There are, however, important subtleties in problem ($\mathcal{P}$) which make it non-standard. First, the domain of the rate functions $r(\cdot)$ is a functional space which makes ($\mathcal{P}$) an infinite dimensional optimization, and, second, ($\mathcal{P}$) is a stochastic optimization and by this we mean that there is a probability space involved over which the expectation is taken. We now present the technical details of the various steps mentioned above.

Dual Function

Consider the inequality constraints in ($\mathcal{P}$) and re-write them as follows,

$$E\left[\int_{\frac{(k-1)T}{L}}^{\frac{kT}{L}} \frac{g(r(x(s),c(s),s))}{c(s)}ds\right] - \frac{PT}{L} \leq 0, \quad k = 1,\ldots,L \tag{3.28}$$

Let $\bar{\nu} = (\nu_1,\ldots,\nu_L)$ be the lagrange multiplier vector for these constraints and since these are inequality constraints, the vector $\bar{\nu}$ must be non-negative, i.e. $\nu_1 \geq 0,\ldots,\nu_L \geq 0$. The *Lagrangian function* is then given as,

$$\mathcal{H}(r(\cdot),\bar{\nu}) = E\left[\int_0^T \frac{g(r(\cdot))}{c(s)}ds + \frac{\tau g(\frac{x(T)}{\tau})}{c(T)}\right] + \sum_{k=1}^{L}\nu_k\left(E\left[\int_{\frac{(k-1)T}{L}}^{\frac{kT}{L}} \frac{g(r(\cdot))}{c(s)}ds\right] - \frac{PT}{L}\right) \tag{3.29}$$

Re-arranging the above equation, it can be written in the form,

$$\mathcal{H}(r(\cdot),\bar{\nu}) = E\left[\int_0^T \frac{(1+\nu(s))g(r(\cdot))}{c(s)}ds + \frac{\tau g(\frac{x(T)}{\tau})}{c(T)}\right] - (\nu_1 + \ldots + \nu_L)\frac{PT}{L} \tag{3.30}$$

where $\nu(s)$ takes the value ν_k over the k^{th} partition interval, i.e. $\nu(s) = \nu_k, s \in [\frac{(k-1)T}{L}, \frac{kT}{L})$.

The *Dual function*, denoted as $\mathcal{L}(\bar{\nu})$, is defined as the infimum of the lagrangian function $\mathcal{H}(r(\cdot),\bar{\nu})$ over $r(x,c,t) \in \Phi$. Thus, we have,

$$\mathcal{L}(\bar{\nu}) = \inf_{r(\cdot)\in\Phi} \mathcal{H}(r(\cdot),\bar{\nu}) \tag{3.31}$$

A point to note here is that the policies $r(x,c,t)$ over which the above minimization is considered do not have to satisfy the power constraints, though, the other requirements still apply. This is because the short term power constraints (violation) have been added as a cost in the objective function of the dual problem. A well-known property of the dual function is that for a given lagrange vector $\bar{\nu} \geq 0$, the dual function $\mathcal{L}(\bar{\nu})$ gives a lower bound to the optimal cost in $(\mathcal{P})$. This standard property is referred to as *Weak Duality* and it applies in our case as well. Let $J(x_0, c_0)$ denote the optimal cost for problem $(\mathcal{P})$ (i.e. the minimum value of the objective function) with (x_0, c_0) being the starting state. We then have the following result.

Lemma 10 (Weak Duality) *Consider problem $(\mathcal{P})$ and let (x_0, c_0) be the starting state at $t = 0$. Then, for all $\bar{\nu} \geq 0$, we have, $\mathcal{L}(\bar{\nu}) \leq J(x_0, c_0)$.*

Proof: See Appendix B.8. ■

We next proceed to evaluate the dual function $\mathcal{L}(\bar{\nu})$ by solving the minimization problem in (3.31).

Evaluating the dual function: The approach we adopt to evaluate the dual function is to view the problem in L stages corresponding to the L partition intervals and solve for the optimal rate functions in each of the partitions with the necessary boundary conditions at the edges. An immediate observation from (3.30) is that the effect of the lagrange multipliers is to multiply the power-rate function $\frac{g(r(\cdot))}{c(s)}$ with a time-varying function $(1 + \nu(s))$. Thus, the difference over the various intervals is in a different multiplicative factor to the cost function, which for the k^{th} interval is, $1 + \nu(s) = 1 + \nu_k$. Intuitively, the lagrange multipliers re-adjust the cost function which causes the data transmission to be moved among the various time-periods. For example, if $\nu_k > \nu_l$, then it becomes more costly to transmit in the k^{th} period than the l^{th} period and this has the effect of (relatively) increasing the data transmission in the l^{th} period.

Since (3.31) involves a minimization over $r(\cdot)$ for fixed lagrange multipliers $\bar{\nu}$, the second term in (3.30), i.e. $\frac{(\nu_1 + \ldots + \nu_L)PT}{L}$, is irrelevant for the minimization and we will neglect it for now. Define,

$$H_\nu^r(x, c, t) \triangleq E\left[\int_t^T \frac{(1 + \nu(s))g(r(\cdot))}{c(s)} ds + \frac{\tau g(\frac{x(T)}{\tau})}{c(T)}\right] \tag{3.32}$$

$$H_\nu(x, c, t) \triangleq \inf_{r(\cdot) \in \Phi} H_\nu^r(x, c, t) \tag{3.33}$$

where the expectation in (3.32) is conditional on the state (x, c, t). In simple terms, $H_\nu^r(x, c, t)$ is the cost-to-go function starting from state (x, c, t) for policy $r(\cdot)$ and $H_\nu(x, c, t)$ is the corresponding optimal cost-to-go function. Relating back to (3.30), $H_\nu^r(x_0, c_0, 0)$ is the expectation term in (3.30) and $H_\nu(x_0, c_0, 0)$ is the minimization of this term over $r(\cdot) \in \Phi$.

Clearly from (3.30) and (3.31), having solved for $H_\nu(x, c, t)$, we then obtain the dual function as simply,

$$\mathcal{L}(\bar{\nu}) = H_\nu(x_0, c_0, 0) - \frac{(\nu_1 + \ldots + \nu_L)PT}{L} \tag{3.34}$$

Finally, in the process of obtaining $H_\nu(x, c, t)$, we also obtain the optimal rate function that achieves the minimum in (3.33). Comparing equations (3.32) and (3.33) with (3.5) and (3.6) respectively shows that the two problems are similar except that we now have a time-varying power-rate function $(1 + \nu(s))\frac{g(r)}{c}$ and a terminal cost imposed at time T. Thus, to solve this problem we can follow a similar line of argument as done earlier and the steps are detailed below starting with the optimality equation.

Consider the k^{th} partition interval so that $t \in [\frac{(k-1)T}{L}, \frac{kT}{L})$ and a small interval $[t, t+h)$, within this partition. Let some policy $r(\cdot)$ be followed over $[t, t+h)$ and the optimal policy thereafter, then using Bellman's principle [63] we have,

$$H_\nu(x, c, t) = \min_{r(\cdot)} \left\{ E \int_t^{t+h} \frac{(1 + \nu_k)g(r(x(s), c(s), s))}{c(s)} ds + EH_\nu(x_{t+h}, c_{t+h}, t + h) \right\} \tag{3.35}$$

where x_{t+h} is short-hand for $x(t + h)$ and the expectation is conditional on (x, c, t). The left side of the equation above denotes the cost if the optimal policy is followed right from the starting state (x, c, t), whereas, on the right side the expression within the minimization bracket is the total cost with policy $r(\cdot)$ being followed over $[t, t+h]$ and the optimal policy thereafter. Removing the minimization gives the following inequality,

$$H_\nu(x, c, t) \leq E \int_t^{t+h} \frac{(1 + \nu_k)g(r(x(s), c(s), s))}{c(s)} ds + E\left[H_\nu(x_{t+h}, c_{t+h}, t + h)\right] \tag{3.36}$$

Rearranging, dividing by h and taking the limit $h \downarrow 0$ gives,

$$A^r H_\nu(x, c, t) + \frac{(1 + \nu_k)g(r)}{c} \geq 0 \tag{3.37}$$

The above follows since $\frac{E\int_t^{t+h}\left(\frac{(1+\nu_k)g(r(\cdot))}{c_s}\right)ds}{h} \longrightarrow \frac{(1+\nu_k)g(r)}{c}$ where r is the value of the transmission rate at time t, i.e. $r = r(x, c, t)$. The quantity $A^r H_\nu(x, c, t)$ is defined as $A^r H_\nu(x, c, t) \triangleq \lim_{h\downarrow 0} \frac{EH_\nu(x_{t+h}, c_{t+h}, t+h) - H_\nu(x,c,t)}{h}$, and, as stated earlier in Section 3.3.2, $A^r H_\nu(x, c, t)$ is called the differential generator of the Markov process $(x(t), c(t))$ for policy $r(\cdot)$. In our case, using the time evolution as given in (3.26), it can be evaluated as [63–65],

$$A^r H_\nu(x,c,t) = \frac{\partial H_\nu(x,c,t)}{\partial t} - r\frac{\partial H_\nu(x,c,t)}{\partial x} + \lambda(E_z[H_\nu(x,Z(c)c,t)] - H_\nu(x,c,t)) \quad (3.38)$$

where E_z is the expectation with respect to the $Z(c)$ variable; $Z(c)$ is as defined in (3.2). Now, in the above steps from (3.36)-(3.37), if policy $r(\cdot)$ is replaced with the optimal policy $r^*(\cdot)$, equation (3.37) holds with equality and we get,

$$A^{r^*} H_\nu(x,c,t) + \frac{(1+\nu_k)g(r^*)}{c} = 0 \quad (3.39)$$

Hence, for a given system state (x,c,t), the optimal transmission rate, r^*, is the value that minimizes (3.37) and the minimum value of this expression equals zero. Thus, over the k^{th} partition interval with $t \in [\frac{(k-1)T}{L}, \frac{kT}{L})$, we get the following *Optimality Equation*,

$$\min_{r\in[0,\infty)} \left[\frac{(1+\nu_k)g(r)}{c} + A^r H_\nu(x,c,t)\right] = 0 \quad (3.40)$$

Substituting $A^r H_\nu(\cdot)$ from (3.38), we see that (3.40) is a partial differential equation in $H_\nu(x,c,t)$, also referred to as the Hamilton-Jacobi-Bellman (HJB) equation.

$$\min_{r\in[0,\infty)} \left\{\frac{(1+\nu_k)g(r)}{c} + \frac{\partial H_\nu}{\partial t} - r\frac{\partial H_\nu}{\partial x} + \lambda(E_z[H_\nu(x,Z(c)c,t)] - H_\nu(x,c,t))\right\} = 0 \quad (3.41)$$

The boundary conditions for $H_\nu(\cdot)$ are as follows. At $t = T$, $H_\nu(x,c,T) = \frac{Tg(\frac{x}{T})}{c}$, since starting in state (x,c) at time T, the optimal cost simply equals the penalty cost. At each of the partition interval $t = kT/L$, we require that $H_\nu(\cdot)$ be continuous at the edges, so that the functions evaluated for the various intervals are consistent.

We now solve the above optimality PDE equation to obtain the function $H_\nu(x,c,t)$, and the corresponding optimal rate function denoted as $r^*_\nu(x,c,t)$ (the subscript ν is used to indicate explicit dependence on the lagrange vector $\bar{\nu}$). Theorem IX summarizes the results while an intuitive explanation of the optimal rate function is presented later. As in Section 3.3.3, we consider m channel states in the Markov channel model and denote the various states $c \in \mathcal{C}$ as $c^1, c^2, \ldots, c^m$. Given a channel state c^i, the values taken by the random variable $Z(c^i)$ (defined in (3.2)) are denoted as $\{z_{ij}\}$, where $z_{ij} = c^j/c^i$. The probability that $Z(c^i) = z_{ij}$ is denoted as p_{ij}. Clearly, if there is no transition from state c^i

to c^j, $p_{ij} = 0$.

<u>***Theorem***</u> ***IX*** *(**BT-problem with Power Constraints**) Consider the minimization in (3.33) with $g(r) = r^n$, $(n > 1, n \in \mathbb{R})$. For $k = 1, \ldots, L$ and $t \in \left[\frac{(k-1)T}{L}, \frac{kT}{L}\right)$ (k^{th} partition interval), we have,*

$$
\begin{aligned}
r_\nu^*(x, c^i, t) &= \frac{x}{f_i^k(T-t)}, \qquad i = 1, \ldots, m && (3.42)\\
H_\nu(x, c^i, t) &= \frac{(1+\nu_k)x^n}{c^i(f_i^k(T-t))^{n-1}}, \quad i = 1, \ldots, m && (3.43)
\end{aligned}
$$

For a fixed k, the functions $\{f_i^k(s)\}_{i=1}^m$, $s \in \left[\frac{(L-k)T}{L}, \frac{(L-k+1)T}{L}\right]$ are the solution of the following ordinary differential equation (ODE) system,

$$
\begin{aligned}
(f_1^k(s))' &= 1 + \frac{\lambda f_1^k(s)}{n-1} - \frac{\lambda}{n-1}\sum_{j=1}^m \frac{p_{1j}}{z_{1j}} \frac{(f_1^k(s))^n}{(f_j^k(s))^{n-1}} && (3.44)\\
&\vdots \\
(f_m^k(s))' &= 1 + \frac{\lambda f_m^k(s)}{n-1} - \frac{\lambda}{n-1}\sum_{j=1}^m \frac{p_{mj}}{z_{mj}} \frac{(f_m^k(s))^n}{(f_j^k(s))^{n-1}} && (3.45)
\end{aligned}
$$

The following boundary conditions apply: if $k = L$, $f_i^L(0) = \tau(1+\nu_L)^{\frac{1}{n-1}}, \forall i$ and if $k = 1, .., L-1$, $f_i^k\left(\frac{(L-k)T}{L}\right) = \left(\frac{1+\nu_k}{1+\nu_{k+1}}\right)^{\frac{1}{n-1}} f_i^{k+1}\left(\frac{(L-k)T}{L}\right), \forall i$. The dual function in (3.34) is then given as (let $c_0 = c^j$, for some $j \in \{1, \ldots, m\}$),

$$
\mathcal{L}(\bar{\nu}) = \frac{(1+\nu_1)x_0^n}{c^j(f_j^1(T))^{n-1}} - \frac{(\nu_1 + \ldots + \nu_L)PT}{L} \qquad (3.46)
$$

Proof: See Appendix B.9. ∎

The above solution can be understood as follows. For each partition interval, k, there are m functions $\{f_i^k(s)\}_{i=1}^m$ corresponding to the respective channel states. The subscript in the notation for f refers to the channel state index while the superscript refers to the partition interval. Now, given that the present time t lies in the k^{th} interval, the optimal rate function has the closed form expression $\frac{x}{f_i^k(T-t)}$ as given in (3.42), while $H_\nu(\cdot)$ is as given in (3.43). The functions $\{f_i^k(s)\}_{i=1}^m$ for the k^{th} interval are the solution of the ODE system in (3.44)-(3.45) over $s \in \left[\frac{(L-k)T}{L}, \frac{(L-k+1)T}{L}\right]$ with the initial boundary condition given as, $f_i^k\left(\frac{(L-k)T}{L}\right) = \left(\frac{1+\nu_k}{1+\nu_{k+1}}\right)^{\frac{1}{n-1}} f_i^{k+1}\left(\frac{(L-k)T}{L}\right), \forall i$. This ensures that $H_\nu(x,c,t)$ is continuous at the partition edges, $t = \frac{kT}{L}$. For the L^{th} interval the boundary condition is, $f_i^L(0) = \tau(1+\nu_L)^{\frac{1}{n-1}}, \forall i$; this ensures that at $t = T$, $H_\nu(x, c^i, T) = \frac{(1+\nu_L)x^n}{c^i(f_i^L(0))^{n-1}} = \frac{x^n}{c^i\tau^{n-1}}$, same as the penalty cost function for $g(r) = r^n$ (as required).

The functions $\{f_i^k(s)\}$ can be evaluated starting with $k = L$ and initial boundary condition $f_i^L(0) = \tau(1+\nu_L)^{\frac{1}{n-1}}$, to obtain $\{f_i^L(s)\}_{i=1}^m$ over $s \in \left[0, \frac{T}{L}\right]$ (Note that $s = T - t$ as required in (3.42), and since over the L^{th} interval $t \in \left[\frac{(L-1)T}{L}, T\right)$ this gives $s = (T-t) \in \left(0, \frac{T}{L}\right]$). Then, consider $k = L-1$, and using the earlier mentioned boundary condition obtain $\{f_i^{L-1}(s)\}_{i=1}^m$, $s \in \left[\frac{T}{L}, \frac{2T}{L}\right]$. Proceeding backwards this way, we obtain all the functions $\{f_i^k(s)\}_{i,k}$.

A comparison of the optimal policy in (3.16), obtained earlier in Section 3.3.3, with the optimal rate function in (3.42) shows that the two forms are fairly identical. The ODE system here is exactly the same as before except for a different set of boundary conditions. In fact, if we take $P \to \infty$ and $\tau \to 0$, the lagrange vector $\bar{\nu} = 0$ and the solution here becomes the same as obtained earlier. This observation is intuitive since the problem in (3.33) is identical to that considered earlier in Section 3.3 except that now there is a penalty cost at $t = T$ and the power-rate function has the multiplicative factor $(1+\nu_k)$ coming from the lagrange multipliers. Finally, since the ODE system in (3.44)-(3.45) is identical to that in (3.18)-(3.19) (except for different boundary conditions), it can be easily solved numerically using standard techniques (e.g. ODE solvers in MATLAB). Also, as shown in Appendix B.9, the solution is unique and positive.

Constant Drift Channel Model: As considered in Section 3.3.3, we now specialize the result in Theorem IX to the case of *Constant Drift* channel model. Under this model, the assumption on the channel process is that the expected value of $1/Z(c)$ is independent of the channel state. Thus, $E[1/Z(c)] = \beta$ (a constant), where β is the drift parameter. For a detailed explanation on the model, refer Section 3.3.3. Interestingly, as before, we can analytically solve the ODE system in (3.44)-(3.45) for this channel model and obtain a closed form result. This is summarized below.

Theorem X *Consider the minimization in (3.33) with $g(r) = r^n$ and the constant drift channel model with parameter β. For $k = 1, \ldots, L$, $t \in [\frac{(k-1)T}{L}, \frac{kT}{L})$, we have,*

$$H_\nu(x, c, t) = \frac{(1+\nu_k)x^n}{c(f^k(T-t))^{n-1}} \quad (3.47)$$

$$r^*_\nu(x, c, t) = \frac{x}{f^k(T-t)} \quad (3.48)$$

Let $\eta = \frac{\lambda(\beta-1)}{n-1}$, then [7]*,*

$$f^k(T-t) = \tau(1+\nu_k)^{\frac{1}{n-1}}e^{-\eta(T-t)} + \frac{1}{\eta}\Big\{\sum_{j=0}^{L-k-1}\left(\frac{1+\nu_k}{1+\nu_{L-j}}\right)^{\frac{1}{n-1}} \times \left(e^{-\eta\left((T-t)-(j+1)\frac{T}{L}\right)} - e^{-\eta\left((T-t)-j\frac{T}{L}\right)}\right)\Big\} + \frac{1}{\eta}\left(1 - e^{-\eta\left((T-t)-(L-k)\frac{T}{L}\right)}\right) \quad (3.49)$$

The dual function in (3.34) is given as,

$$\mathcal{L}(\bar{\nu}) = \frac{(1+\nu_1)x_0^n}{c_0(f^1(T))^{n-1}} - \frac{(\nu_1 + \ldots + \nu_L)PT}{L} \quad (3.50)$$

Proof: See Appendix B.10. ■

Strong Duality

In Theorems IX and X, we fixed a lagrange vector $\bar{\nu}$ and obtained the dual function $\mathcal{L}(\bar{\nu})$, and, the optimal rate function that achieves the minimum in (3.33). Now, from Lemma 10, given a lagrange vector $\bar{\nu} \geq 0$, the dual function is a lower bound to the optimal cost of the constrained problem $\mathcal{P}$. Thus, intuitively, it makes sense to maximize $\mathcal{L}(\bar{\nu})$ over $\bar{\nu} \geq 0$. Theorem XI below states that strong duality holds, i.e. maximizing $\mathcal{L}(\bar{\nu})$ over $\bar{\nu} \geq 0$ gives the optimal cost of $\mathcal{P}$, and, furthermore, the optimal rate function $r^*(x, c, t)$ for the constrained problem $\mathcal{P}$ is the same as $r^*_\nu(x, c, t)$ obtained in Theorem IX with $\bar{\nu} = \bar{\nu}^*$ ($\bar{\nu}^*$ is the maximizing lagrange vector).

As in Lemma 10, let $J(x_0, c_0)$ be the optimal cost of ($\mathcal{P}$) with initial state (x_0, c_0) at $t = 0$, where $x_0 \in [0, \infty), c_0 \in \mathcal{C}$. For the BT-problem, $x_0 = B$ and c_0 is the initial channel state; note that the starting state is known and fixed for the optimization in ($\mathcal{P}$).

[7]For $k = L$ the summation term in (3.49) is taken as zero.

Theorem ***XI (Strong Duality)*** *Consider the dual function defined in (3.31) for $\bar{\nu} \geq 0$, we then have,*

$$J(x_0, c_0) = \max_{\bar{\nu} \geq 0} \mathcal{L}(\bar{\nu}) \tag{3.51}$$

and the maximum on the right is achieved by some $\bar{\nu}^* \geq 0$. Let $r^*(x,c,t)$ denote the optimal transmission policy for problem $(\mathcal{P})$, then, $r^*(x,c,t)$ is as given in (3.42) for $\bar{\nu} = \bar{\nu}^*$.

Proof: See Appendix B.11. ■

For the maximization in (3.51), the dual functions are given in Theorems IX (general markov channel) and X (constant drift channel). It can also be shown using a standard argument that the dual function is concave [66] which makes the maximization much simpler since there is a unique global (and local) maxima. Using a standard gradient search algorithm the vector $\bar{\nu}^*$ can be obtained numerically as done for the simulations in Section 3.4.3.

Optimal Policy for $(\mathcal{P})$

Summarizing, the optimal policy for problem $(\mathcal{P})$ is obtained by combining Theorems IX and XI and is given as follows. For $k = 1, \ldots, L$ and $t \in [\frac{(k-1)T}{L}, \frac{kT}{L})$ (k^{th} partition interval),

$$r^*(x, c^i, t) = r^*_{\bar{\nu}^*}(x, c^i, t) = \frac{x}{f_i^k(T-t)}, \quad i = 1, \ldots, m \tag{3.52}$$

where the functions $\{f_i^k(s)\}$ are evaluated with $\bar{\nu} = \bar{\nu}^*$. As mentioned earlier, the computation for $\bar{\nu}^*$ and $\{f_i^k(s)\}$ needs to be done offline before the data transmission begins. In practice, if the transmitter has computational capabilities, these computations can be carried out at $t = 0$ for the given problem parameters, otherwise, the $\bar{\nu}^*$ and $\{f_i^k(s)\}$ can be pre-determined and stored in a table in the transmitter's memory. Having known $\{f_i^k(s)\}$, the closed-form structure of the optimal policy as given in (3.52) warrants no further computation and is simple to implement. At time t, the transmitter looks at the amount of data left in the buffer, x, the channel state, c^i, the partition interval k in which t lies and then computes the transmission rate as simply $\frac{x}{f_i^k(T-t)}$.

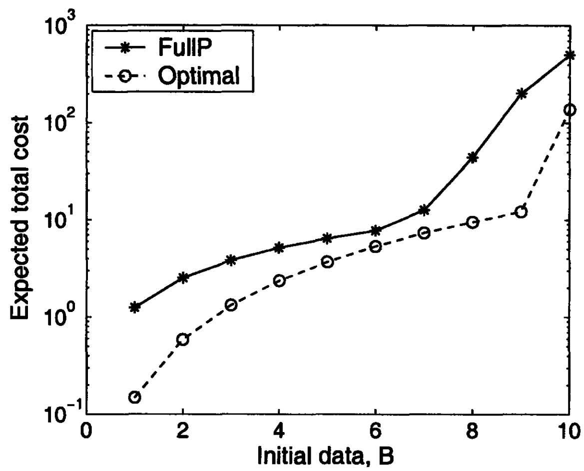


Figure 3-6: Total cost comparison of the optimal and the full power policy.

3.4.3 Simulation Results

In this section, we consider an illustrative example and present energy cost comparisons for the optimal and the Full Power (FullP) policy. In FullP policy, the transmitter always transmits at full power, P, and so given the system state (x, c, t) the rate is chosen as follows, $P = \frac{g(r(x,c,t))}{c} \Rightarrow r(x, c, t) = g^{-1}(cP) = (cP)^{1/n}$, for $g(r) = r^n$. The simulation setup is as follows. The channel model is the two-state model as described earlier in Section 3.2.2, with parameters $\lambda_{bg} = 1$, $\lambda_{gb} = 3/7$, $c_g = 1$ and $c_b = 0.2$; thus, $\lambda = \max(\lambda_{bg}, \lambda_{gb}) = 1$ and $\gamma = c_b/c_g = 0.2$. It can be easily checked that with the above parameters, in steady state the fraction of time spent in the good state is 0.7 and in the bad state it is 0.3. The deadline is taken as $T = 10$ and the number of partition intervals as $L = 20$. The power-rate function is $g(r) = r^2$ and the value of τ in the penalty cost function is taken as 0.01, which is 0.1% of the deadline; thus, a time window of 0.1% is provided at T. To simulate the process, the communication slot duration is taken as $dt = 10^{-3}$ implying that there are $T/10^{-3} = 10,000$ slots over the deadline interval. For each slot, the transmission rate is computed as given by the corresponding policy and the total cost is obtained as the sum of the energy costs in the slots plus the penalty cost. Expectation is then taken as an average over the sample paths.

Figure 3-6 is a plot of the expected total cost of the two policies with the initial data amount B varied from 1 to 10. The value of P is chosen such that at $B = 5$, even with bad channel condition over the whole time-period, the entire data can be served at full power. This implies, $P = \frac{1}{\gamma}(5/T)^2 = 1.25$ ($5/T$ is the rate required to serve 5 units in time T).

Thus, $B \leq 5$ gives the regime in which full power always meets the deadline and $B > 5$ is the regime in which data is left out which then incurs the penalty cost. It is evident from the plot that the optimal policy gives a significant gain in the total cost (note that the y-axis is on a log scale) and at around $B = 1$, FullP policy incurs almost 10 times the optimal cost. Thus, dynamic rate adaptation can yield significant energy savings.

3.5 Chapter Summary

In this chapter, we considered the problem of transmitting data with strict deadline constraint, over a wireless fading channel, with minimum energy expenditure. Specifically, the setup consisted of a wireless transmitter with B bits of data in the queue and a single deadline T by which this data must be transmitted. The channel gain varied stochastically over time and was modelled as a general Markov process. The objective was to obtain a transmission policy that minimized the expected energy expenditure while meeting the deadline constraint. We referred to this as the "BT-problem". To address this problem, we adopted a continuous-time approach and utilized ideas from stochastic control theory and lagrange duality to obtain the optimal solution. As we see later in the next chapter (Chapter 4), the solution to the BT-problem can be utilized towards many important generalizations involving variable deadlines and packet arrivals.

In Section 3.3, we considered the BT-problem without any explicit maximum power limit. For the monomial class of power-rate functions, the optimal transmission policy under a general Markov channel model was obtained in Theorem VI. It takes the simple form,

$$r^*(x, c^i, t) = \frac{x}{f_i(T-t)} \tag{3.53}$$

or more intuitively as,

*optimal rate = amount of data left * urgency of transmission*

The functions $\{f_i(s)\}_{i=1}^m$ are easily computed numerically as the solution of a certain system of ordinary differential equation and, as discussed in Section 3.3.3, this computation needs to be carried out offline. In fact, the pre-computed $\{f_i(s)\}$ can be stored in the transmitter's memory, in which case, during the system operation the transmission rate over time is obtained directly from the above formula. Under a special structure on the channel model, which we referred to as the constant drift channel, the optimal transmission policy was obtained in closed-form for the monomial as well as the exponential class of power-rate functions. These results were presented in Theorems VII and VIII.

In Section 3.4, we considered the BT-problem with short-term average power constraints. The optimal policy in this case is presented in Theorem IX and it takes the form,

$$r^*(x, c^i, t) = \frac{x}{f_i^k(T-t)} \tag{3.54}$$

Thus, in this case as well, the optimal policy has a similar form except with a different set of functions $\{f_i^k(s)\}$. As before, these functions can be obtained as the solution of a particular system of ordinary differential equation and this computation needs to be carried out offline.

In the next chapter, we build upon the work presented so far and consider various extensions to the BT-problem. The ideas presented in Chapters 2 and 3 provide important tools to address these generalizations; more specifically, the cumulative curves methodology presented earlier in Chapter 2 and the solution to the BT-problem obtained in this chapter will be utilized to address the problems in the next chapter.

Chapter 4

Stochastic Setup – Variable Deadlines and Arrivals

4.1 Introduction

In the previous chapter, we studied the canonical problem of transmitting B units of data by deadline T over a time-varying and stochastic channel, utilizing minimum energy. The optimal transmission policy for this problem was obtained using stochastic control techniques and various insights were also deduced from its functional form. In this chapter, we widen the scope of the energy minimization problem and consider various extensions of the BT-problem incorporating variable deadlines as well as packet arrivals to the queue.

We begin in Section 4.2 by formulating a general problem within the cumulative curves framework and then specialize it to two different setups as follows. The first setup is the variable-deadlines case where the transmitter has M packets in the queue, each packet has a distinct deadline associated with it and the goal is to transmit this data over a fading channel with minimum energy. In the second setup, we consider a stream of packet arrivals to the queue with known inter-arrival times and a single deadline T by which this data must be transmitted; the goal, as before, is to minimize the transmission energy expenditure. For both the setups, the graphical picture obtained from the cumulative curves lends itself into a very natural decomposition of the problem in terms of simpler BT-problems. Based on this decomposition, a transmission policy is obtained and is shown to be optimal for the class of constant drift channel model. Finally, an energy-efficient transmission policy is constructed for the case when packets could arrive arbitrarily and there is no information, statistical or otherwise, of the arrival process. This policy is referred to as the *BT-Adaptive* policy and simulation results are presented to illustrate its performance in terms of the energy expenditure.

Lastly, in Section 4.3, we study energy-efficient data transmission under a stochastic packet arrival model and consider the following problem. The transmitter queue has a stream of packets that arrive according to a homogeneous Poisson process and the goal is to transmit this data by a common deadline using minimum energy. As we see in that section, knowledge of the packet arrival statistics has an influence on the optimal transmission policy. And as intuitively expected, to take into account (expected) future packet arrivals, the transmitter chooses a rate higher than that required to just empty the present data in the buffer by the deadline.

4.2 Cumulative Curves Generalization

The cumulative curves framework was first introduced in Chapter 2, where we studied the energy minimization problem under a deterministic setup. In this section, we re-visit that methodology but also introduce stochastic channel fading into the picture. The underlying system model for the setup remains the same as considered in the previous chapters; however, for completeness we briefly re-state the model here and then proceed to the problem formulation.

4.2.1 System Model

Consider a wireless transmitter where the transmission rate can be controlled over time. Let $P(t)$ denote the transmission power, $|h(t)|^2$ the channel gain square and $r(t)$ the transmission rate at time t. We then have (see Section 3.2),

$$P(t) = \frac{g(r(t))}{c(t)} \tag{4.1}$$

where $c(t) \triangleq |h(t)|^2$ is referred to as the *channel state* and $g(\cdot)$ is a convex increasing function with $g(r) \geq 0,\ r \geq 0$. The channel state $c(t)$ is assumed to be a general continuous-time, discrete state-space, first-order Markov process, as considered previously in Chapter 3; the technical details and the notations for the channel model can be found in Section 3.2.2. For most analytical purposes, we assume that $g(r)$ belongs to the class of *Monomial functions* and takes the form $g(r) = kr^n,\ k > 0,\ n > 1\ (n, k \in \mathbb{R})$.

The data flow to the queue is described in terms of three cumulative curves – the arrival curve $A(t)$, the departure curve $D(t)$ and the minimum departure curve $D_{min}(t)$. The arrival curve, $A(t)$, denotes the cumulative amount of data arrived to the queue by time t; the minimum departure curve $D_{min}(t)$ denotes the minimum cumulative amount of data that must depart by time t to satisfy the deadline (or other QoS) constraints; and finally, the departure curve $D(t)$ denotes the cumulative amount of data departed by time t using a particular transmission policy. Clearly, to ensure that only the data that has already arrived to the queue is transmitted we require $D(t) \leq A(t)$ (causality constraint) and to satisfy the QoS constraints we need $D(t) \geq D_{min}(t)$. Thus, a departure curve must satisfy $D_{min}(t) \leq D(t) \leq A(t)$. For more details on the cumulative curves framework, the reader is referred to Section 2.2.1.

4.2.2 Problem Formulation

Consider a time interval $[0, T]$, and, let $A(t)$ be the arrival curve and $D_{min}(t)$ the minimum departure curve over this period. Given these two curves, the objective is to obtain the optimal transmission policy such that the departure curve for it satisfies $D_{min}(t) \leq D(t) \leq A(t)$ a.s. (almost surely) and the expected energy expenditure is minimized[1]. We now present the details of the stochastic optimal control formulation for the energy minimization problem.

Optimal Control Formulation: Let the system state be denoted as (D, c, t), where the notation means that at the present time t, the cumulative amount of data that has been transmitted is $D(t) = D$, and the channel state is $c(t) = c$. Let $r(D, c, t)$ denote a transmission policy, and as before, given such a policy, the system evolves in time as a Piecewise-Deterministic-Process (PDP) (see Section 3.3.1). It starts with the initial state $D(0) = 0$ and $c(0) = c_0$ $(c_0 \in C)$. Until τ_1, where τ_1 is the first time instant after $t = 0$ at which the channel changes, data is transmitted at the rate $r(D(t), c_0, t)$. Hence, over

[1]Note that while $A(t)$ and $D_{min}(t)$ are assumed known, the actual departure curve $D(t)$ would depend on the underlying channel sample path, if the transmission policy adapts the rate with the channel variations.

$t \in [0, \tau_1)$, $D(t)$ satisfies the differential equation,

$$\frac{dD(t)}{dt} = r(D(t), c_0, t) \tag{4.2}$$

Equivalently, $D(t) = D(0) + \int_0^t r(D(s), c_0, s)ds$, $t \in [0, \tau_1]$. Then, starting from the new state $(D(\tau_1), c(\tau_1), \tau_1)$ until the next channel transition, we have $\frac{dD(t)}{dt} = r(D(t), c(\tau_1), t)$, $t \in [\tau_1, \tau_2)$; and this procedure repeats in time.

A transmission policy, $r(D, c, t)$, is *admissible*, if it satisfies the following:

(a) $0 \leq r(D, c, t) < \infty$, (non-negativity of rate), and,

(b) $D_{min}(t) \leq D(t) \leq A(t), t \in [0, T]$, almost surely, (deadline and causality constraints)[2]

Consider now an admissible transmission policy $r(D, c, t)$ and define a *cost-to-go* function, $J_r(D, c, t)$, as the expected energy expenditure starting from an admissible state (D, c, t), i.e. $(D \in [D_{min}(t), A(t)],\ c \in \mathcal{C},\ t \in [0, T))$. Then,

$$J_r(D, c, t) = E\left[\int_t^T \frac{1}{c(s)} g(r(D(s), c(s), s))ds\right] \tag{4.3}$$

where the above expectation is taken over $\{c(s), s \in (t, T]\}$ and conditional on the starting state $D(t) = D$, $c(t) = c$. Define a *minimum cost function*, $J(D, c, t)$, as the infimum of $J_r(D, c, t)$ over the set of all admissible transmission policies.

$$J(D, c, t) = \inf_{r(D,c,t)} J_r(D, c, t), \quad r(D, c, t) \text{ admissible} \tag{4.4}$$

As in the case of the BT-problem, the optimization problem is to compute the minimum cost function $J(D, c, t)$ and obtain the optimal policy $r^*(D, c, t)$ that achieves it.

Following Section 3.3.2, the optimality Hamilton-Jacobi-Bellman (HJB) equation can be obtained directly by noting that the process evolution remains the same as the BT-problem except that for convenience we now use the cumulative data transmitted, $D(t)$, as the state variable instead of the amount of data left, $x(t)$, as done earlier. Thus, following

[2]As before, to ensure that (4.2) has a unique solution, we also require that $r(D, c, t)$ be locally Lipschitz continuous in D and piecewise continuous in t.

the arguments of Section 3.3.2, it is easy to see that the HJB equation is given as,

$$\min_{r\in[0,\infty)}\left\{\frac{g(r)}{c}+\frac{\partial J(D,c,t)}{\partial t}+r\frac{\partial J(D,c,t)}{\partial D}+\lambda(E_z[J(D,Z(c)c,t)]-J(D,c,t))\right\}=0 \quad (4.5)$$

The above formulation deals with a general setup involving arbitrary arrival and minimum departure curves. An analytical solution of the above in general is difficult due to the complexity of the boundary conditions imposed on the cost function. However, there are various specific setups that can solved for the optimal transmission policy under special scenarios and these are treated in the following sections. In fact, as we see later, an initial intuition for the transmission policy for these problems can be obtained from their analogues in the deterministic setup (without channel fading), which were considered earlier in Chapter 2.

Re-visiting the BT-problem: As a first example, let us re-visit the BT-problem considered in Section 3.3 and re-phrase it in terms of the cumulative curves. For the BT-problem, we have $A(t) = B,\ t \in [0,T]$ since the queue has B bits in the beginning at time 0 and no more data is added. The minimum departure curve is given as $D_{min}(t) = 0,\ t \in [0,T), D_{min}(T) = B$, since until the deadline $t < T$ there is no minimum data transmission requirement, while at T the entire B bits must have been transmitted. A schematic diagram of the curves is given in Figure 4-1(a). Using the results from Section 3.3.2 and noting that $D = B - x$, we get,

$$r^*(D,c^i,t) = \frac{B-D}{f_i(T-t)},\quad i=1,\ldots,m \quad (4.6)$$

$$J(D,c^i,t) = \frac{(B-D)^n}{c^i(f_i(T-t))^{n-1}},\quad i=1,\ldots,m \quad (4.7)$$

where the functions $\{f_i(s)\}$ are the solution of the ordinary differential equation system as given in (3.18)-(3.19). Similarly, using the results from Section 3.4, one can also re-phrase the optimal solution for the BT-problem with short-term power limits.

4.2.3 Variable Deadlines Setup

Consider the variable deadlines setup where the queue has M packets that are arranged and served in the earliest-deadline-first order. Let b_j be the number of bits in the j^{th} packet and T_j be the deadline for this packet; assume $0 < T_1 < T_2 < \ldots < T_M$. There

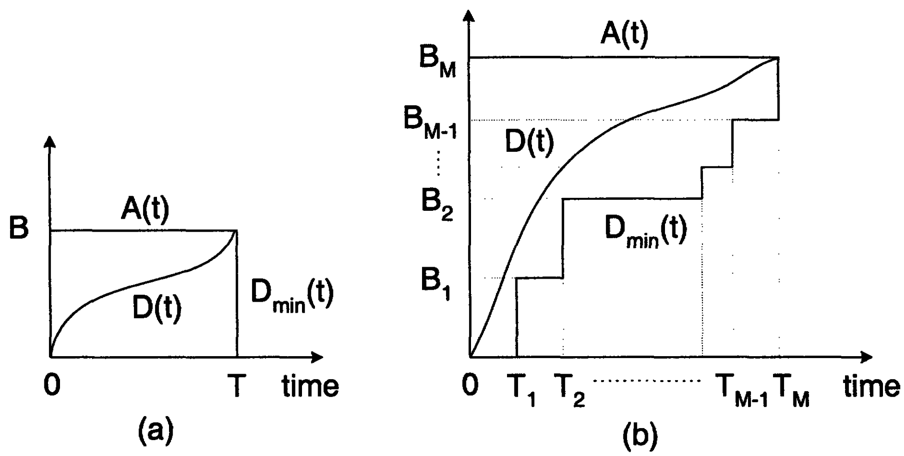


Figure 4-1: Cumulative curves for (a) BT-problem, (b) Variable deadlines case.

are no new arrivals and the objective is to obtain a transmission policy that serves this data over the time-varying channel with minimum expected energy cost while meeting the deadline constraints. In terms of the cumulative curves, the setup is visualized as depicted in Figure 4-1(b). Let B_j denote the cumulative amount of data in the first j packets; we then have $B_j = \sum_{l=1}^{j} b_l$. The cumulative curves for the problem are then given as follows: $A(t) = B_M, \forall t$, since a total B_M bits are in the queue at time 0 and no more data is added. The curve $D_{min}(t)$ is a piecewise-constant function with jumps at times T_j, i.e. $D_{min}(t) = 0,\ t \in [0, T_1)$; $D_{min}(t) = B_j,\ t \in [T_j, T_{j+1}),\ j = 1, \ldots, M-1$, and $D_{min}(T_M) = B_M$. The optimal control formulation for the problem is as given in the previous section (Section 4.2.2), while the optimality HJB equation is given in (4.5). The boundary conditions are, $J(B_M, c, t) = 0$ and $J(D, c, T_j) = \infty$, if $D < B_j$, $j = 1, \ldots, M$. The second condition follows from the deadline constraints, since, as t approaches T_j the cost function becomes unbounded if the required cumulative amount B_j has not been transmitted.

Before proceeding further, consider the analogue of the above problem in the deterministic case without channel fading. This was treated in Example 2 in Chapter 2, where the optimal departure curve was deterministic and as given in Figure 2-6. In functional form, this optimal solution can be re-phrased as follows. Consider the system state (D, t) and look at the straight line segments that connect the points (t, D) and (T_j, B_j) (jump points of $D_{min}(t)$) for all $\{j : B_j \geq D, T_j \geq t\}$. The optimal rate is then the maximum value among the slopes of these line segments, i.e.

$$r^*(D,t) = \max_{j:B_j \geq D, T_j \geq t} \frac{B_j - D}{T_j - t} \tag{4.8}$$

Note that in the above, each term $\frac{B_j-D}{T_j-t}$ gives the optimal rate to transmit $B_j - D$ bits of data by time $T_j - t$, under no channel fading.

Now utilizing the intuition from above and a natural decomposition of the variable deadlines problem in terms of multiple BT-problems, we can obtain a transmission policy under channel fading. This policy is later shown to be optimal under the constant drift channel model. A visual comparison of the two diagrams in Figure 4-1 suggests the following approach. First, instead of viewing the problem in terms of individual packets we can visualize it in terms of the cumulative amounts as $\{B_jT_j\}_{j=1}^{M}$ constraints; that is, a total of B_j bits must be transmitted by deadline T_j ($j = 1, \ldots, M$). Clearly, each B_jT_j constraint is like a BT-problem except that now there are multiple such constraints that all need to be satisfied. For every time t and channel state c, we know the optimal transmission rate to meet each of the B_jT_j constraint individually (assuming only this constraint existed), thus, to meet all the constraints a natural solution is to simply choose the maximum rate among them.

More precisely, the transmission policy is described as follows. Let the system be in state (D, c, t) where ($D \in [D_{min}(t), B_M]$, $c \in \mathcal{C}$, $t \in [0, T_M]$) and consider a particular B_jT_j constraint. Using the optimal rate function in (4.6), the rate function to satisfy an individual B_jT_j constraint, for channel state c^i, is given as $\frac{B_j-D}{f_i(T_j-t)}$; since $(B_j - D)$ is the amount of data left and $(T_j - t)$ is the time left until the deadline T_j. Let $\tilde{r}(D, c, t)$ denote the transmission rate for our proposed policy, then $\tilde{r}(\cdot)$ is the maximum value among the rates for all the B_jT_j constraints for which ($B_j \geq D$ and $T_j \geq t$). Thus, we get,

$$\tilde{r}(D, c^i, t) = \max_{j:(B_j \geq D, T_j \geq t)} \frac{B_j - D}{f_i(T_j - t)} \tag{4.9}$$

where, as before, the functions $\{f_i(s)\}_{i=1}^{m}$ are the solution of the following ODE system with the boundary conditions $f_i(0) = 0, f_i'(0) = 1, \forall i$,

$$f_i'(s) = 1 + \frac{\lambda f_i(s)}{n-1} - \frac{\lambda}{n-1} \sum_{k=1}^{m} \frac{p_{ik}}{z_{ik}} \frac{(f_i(s))^n}{(f_k(s))^{n-1}}, \; i = 1, \ldots, m \tag{4.10}$$

For the policy in (4.9), clearly, by construction, all the B_jT_j constraints are satisfied since at all times we choose the maximum rate among those needed to meet each of the remaining constraints. The rate function $\tilde{r}(\cdot)$ is also non-negative, locally Lipschitz continuous in D and piecewise-continuous in t. Hence, the policy in (4.9) is admissible and the departure curve obtained using (4.9) satisfies $D_{min}(t) \leq D(t) \leq A(t),\ a.s.,\ t \in [0, T_M]$. Furthermore, since the policy in (4.9) is based on the BT-solution, it inherits all the properties of that solution. The ODE system in (4.10) is identical to the BT-case, hence, as before the functions $\{f_i(s)\}_{i=1}^{m}$ can be obtained numerically using a standard ODE solver. This computation needs to be done only once before the system starts operating and the functions $\{f_i(s)\}$ can be pre-determined and stored in a table in the transmitter's memory. Once the $\{f_i(s)\}$ are known, the online computation is minimal. At time t, the transmitter looks at the cumulative amount of data transmitted, D, the channel state, c^i, and then using the corresponding $f_i(\cdot)$ function it simply computes the maximum among a set of values as given in (4.9).

The transmission policy in (4.9) applies for a general Markov channel model and under the specialization to a constant drift channel, it is in fact the optimal policy as shown in the following theorem. For a definition of this channel model, see Section 3.3.3.

***Theorem* XII (Variable Deadlines Case)** *Consider the variable deadlines problem with $g(r) = r^n$, $n > 1, n \in \mathbb{R}$ and the constant drift channel model with parameter β. The optimal rate, $r^*(D, c, t)$ for $D_{min}(t) \leq D \leq A(t),\ t \in [0, T_M)$ is given as,*

$$r^*(D, c, t) = \max_{j:(B_j \geq D, T_j \geq t)} \frac{B_j - D}{f(T_j - t)} \tag{4.11}$$

where, $f(s) = \frac{(n-1)}{\lambda(\beta-1)}(1 - \exp(-\frac{\lambda(\beta-1)}{n-1}s))$.

Proof: See Appendix C.1. ∎

Finally, as a corollary to Theorem XII it is easy to see that under no channel fading, which corresponds to setting $\lambda = 0$ in the above channel model, the function $f(s)$ reduces to, $f(s) = s$, and we recover back the result in (4.8).

4.2.4 Arrivals with a Single-Deadline Setup

In this section, we consider the case of packet arrivals with a single-deadline where the setup is as follows. There are M packet arrivals to the queue with the first packet arrival at time

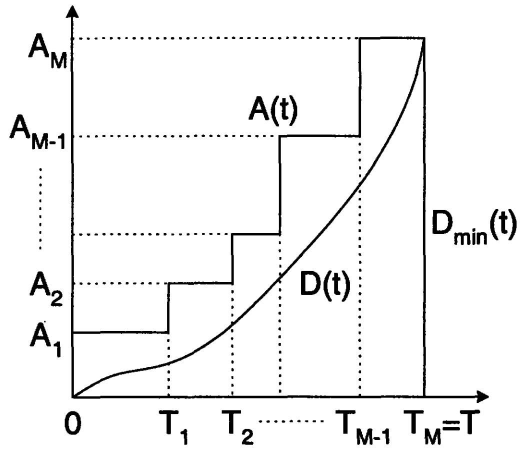


Figure 4-2: Cumulative curves for the arrivals with a single deadline case.

$T_0 = 0$ and the rest arriving at times $\{T_j\}_{j=1}^{M-1}$, where $0 < T_1 < T_2 < \ldots < T_{M-1}$. Let b_j be the number of bits in the j^{th} packet. The deadline constraint is that all of the data must be transmitted by time $T > T_{M-1}$. This problem has motivations in a sensor network scenario where the data collected at certain times must be transmitted to a central processing node within a particular time-interval. Clearly, minimizing the transmission energy cost here translates into a higher lifetime of the sensor node.

In terms of the cumulative curves, the picture is as follows. Let A_j denote the cumulative amount of data arrived in the first j packets; this is given as $A_j = \sum_{l=1}^{j} b_l$. The arrival curve $A(t)$ is then a piecewise-constant function with jumps at times T_j as depicted in Figure 4-2; i.e. $A(t) = A_{j+1},\ t \in [T_j, T_{j+1}),\ j = 0, \ldots, M-1$, and $A(T_M) = A_M$ (for notational convenience we define, $T_M \triangleq T$). The minimum departure curve is $D_{min}(t) = 0,\ t \in [0, T); D_{min}(T) = A_M$ since for $t < T$ there is no minimum data transmission requirement while at T the entire A_M bits must have been transmitted.

From Figure 4-2, we see that the cumulative curves picture can be viewed as a "dual" of the variable deadlines case. Earlier, we had constraints from $D_{min}(t)$, but now there are constraints from the arrival curve $A(t)$ and a final deadline constraint at time T. Once again, to gain some intuition, let us look at the analogue of the above problem in the deterministic case. This is presented in Example 1 in Chapter 2. Given a system state (D, t) the optimal solution under no channel fading is to compute the slopes, $\frac{A_j - D}{T_j - t}$, corresponding to the straight line segments that connect the points (t, D) and (T_j, A_j), for all $\{j : A_j \geq D, T_j \geq t\}$. The optimal rate is then the minimum value among these; i.e. $r^*(D, t) =$

$\min_{\{j:A_j \geq D, T_j \geq t\}} \frac{A_j - D}{T_j - t}$.

Now, utilizing the intuition from above and a similar reasoning as in the variable deadlines case, we can obtain a transmission policy as follows. First, note that a constraint $A_j T_j$ ($j = 1, \ldots, M-1$), requires that no more than A_j bits must be transmitted before time T_j, while $A_M T_M$ requires that the queue must be empty by time T_M. Starting from some system state (D, c^i, t) and without considering other constraints, emptying the buffer by time T_j (i.e. transmitting A_j bits by time T_j) is equivalent to a BT-problem with $B = A_j$ and $T = T_j$, and from (4.6) the rate for this is given as $\frac{A_j - D}{f_i(T_j - t)}$. Now, to ensure that none of the $A_j T_j$ constraints are violated, i.e. not more than A_j bits is transmitted by time T_j, a natural solution is to choose the minimum rate among them. More precisely, let $\tilde{r}(D, c, t)$ denote this policy we then have,

$$\tilde{r}(D, c^i, t) = \min_{j:(A_j \geq D, T_j \geq t)} \frac{A_j - D}{f_i(T_j - t)} \ , \ i = 1, \ldots, m \tag{4.12}$$

By construction all the arrival constraints are satisfied since at all times we choose the minimum rate among those needed to meet the $A_j T_j$ points. Furthermore, for $t > T_{M-1}$, $\tilde{r}(\cdot)$ reduces to choosing a rate that meets the $A_M T_M$ constraint, hence, the deadline constraint is also satisfied. The rate function $\tilde{r}(\cdot)$ is also non-negative, locally Lipschitz continuous in D and piecewise-continuous in t. Thus, the policy in (4.12) is admissible and furthermore, as in the variable deadlines case, we can also show that it is optimal for the constant drift channel model.

<u>Theorem</u> XIII (Arrivals with Single Deadline) *Consider the arrivals with a single deadline problem with $g(r) = r^n$, $n > 1, n \in \mathbb{R}$ and the constant drift channel model with parameter β. The optimal rate, $r^*(D, c, t)$, for $D_{min}(t) \leq D \leq A(t)$, $t \in [0, T_M)$ is,*

$$r^*(D, c, t) = \min_{j:(A_j \geq D, T_j \geq t)} \frac{A_j - D}{f(T_j - t)} \tag{4.13}$$

where $f(s) = \frac{(n-1)}{\lambda(\beta-1)}(1 - \exp(-\frac{\lambda(\beta-1)}{n-1} s))$.

Proof: See Appendix C.2. ■

As in the variable-deadlines case, setting $\lambda = 0$ which corresponds to no channel fading, recovers back the previous result obtained in the deterministic setup.

4.2.5 Arbitrary Packet Arrivals – BT-Adaptive Policy

Consider now an arbitrary stream of packet arrivals to the queue with each packet having a distinct deadline by which it must depart. Suppose that there is no information, statistical or otherwise, of the packet arrival process. Under this scenario, the transmitter chooses a transmission rate based solely on the data amount in the queue and the various deadlines. To address this problem, we utilize the solution for the variable-deadlines case and present an online transmission policy, referred to as the "***BT-Adaptive***" (BTA) policy. Later, we give numerical results comparing the energy expenditure of the BTA policy with a non-adaptive scheme.

BT-Adaptive (BTA) Policy

Consider packet arrivals to the queue with each packet having a distinct deadline associated with it. Assume that the arrivals occur at discrete times, then clearly, at the instant immediately following a packet arrival, the transmitter queue consists of (a) earlier remaining packets with their deadlines and (b) the new packet with its own deadline. Re-arranging the data in the earliest-deadline-first order we can view the queue as consisting of a total amount B_M of data with variable deadlines, identical to the case considered in Section 4.2.3. Not assuming any knowledge of the future arrivals and using (4.9), we have an energy-efficient policy to empty the transmitter buffer. As this policy is followed, at the next packet arrival instance the above procedure is then simply repeated. Summarizing, the BTA policy is as follows,

BT-Adaptive policy: Transmit the data in the queue with the rate as given in (4.9); at every packet arrival instant re-arrange the data in the earliest-deadline-first order to obtain a new set of B_jT_j values by including the new packet and its deadline; re-initialize D to zero and follow (4.9) thereafter.

Note that the BTA policy is not based on any specific arrival process, hence, an interesting feature of it is that it is robust to changes in the arrival statistics and can even accommodate multiple deadline classes of packet arrivals to the queue.

Simulation Results

In this section, we present simulation results to illustrate the performance of the BT-Adaptive policy. For comparison purposes we consider a policy that can be easily implemented in practice and refer to it as the "Head-of-Line Drain" (HLD) policy. In HLD policy, the data in the queue is arranged in the earliest-deadline-first order and the packets are served in that order. At time t, let H_t be the amount of data left in the head-of-the-line packet and T_H be the amount of time until its deadline, then the rate chosen is $r_t = \frac{H_t}{T_H}$. Thus, the transmitter serves the first packet in queue at a rate to transmit it out by its deadline, then moves to the next packet in line and so on. At every packet arrival instant, the data in the queue is re-arranged in the earliest-deadline-first order taking into account the new packet and its deadline and the above policy is then repeated.

The setup is as follows. The queue has packet arrivals and each packet has a deadline associated with it. On each simulation run, the total time over which the packets arrive and the system is operated is taken as $L = 10$ seconds. This interval $[0, 10]$ is partitioned into $10,000$ slots, thus each slot is of duration $dt = 1$ msec. The channel model is the two state model, described in Section 3.2.2, with the parameters, $c_g = 1$, $c_b = 0.2$, $\lambda_{bg} = \lambda_{gb} = \lambda = 50$. Thus, the average time spent in a state before the channel transitions is $1/50$ seconds, or 20 msec. A channel sample path is simulated using a Bernoulli process where in a slot the channel transitions with probability λdt; otherwise there is no transition. For simplicity, the packet arrival and the channel state transitions occur only at the slot boundaries. For both the BTA and the HLD policies, the rate chosen in a slot is obtained by evaluating the respective policies at the time corresponding to the start of that slot. We take the function $g(r) = r^2$. Energy cost per slot is $\frac{r^2 dt}{c}$ and the total expected energy cost is obtained as an average of the total cost over the set of simulated sample paths.

We first consider a Poisson packet arrival process with each packet having 1 unit of data and a deadline of 200 msec. Figure 4-3(a) is a plot of the expected energy cost, plotted on a log scale, versus the packet arrival rate. Note that a packet arrival rate of 10 implies that the average inter-arrival time of a packet is $1/10$ sec. or 100 msec. As is evident from the plot, the BTA policy has a much lower energy cost compared to the HLD policy and as the arrival rate increases the two costs are roughly an order of magnitude apart. This can be intuitively explained as follows. When the arrival rate is low, most of the time the queue

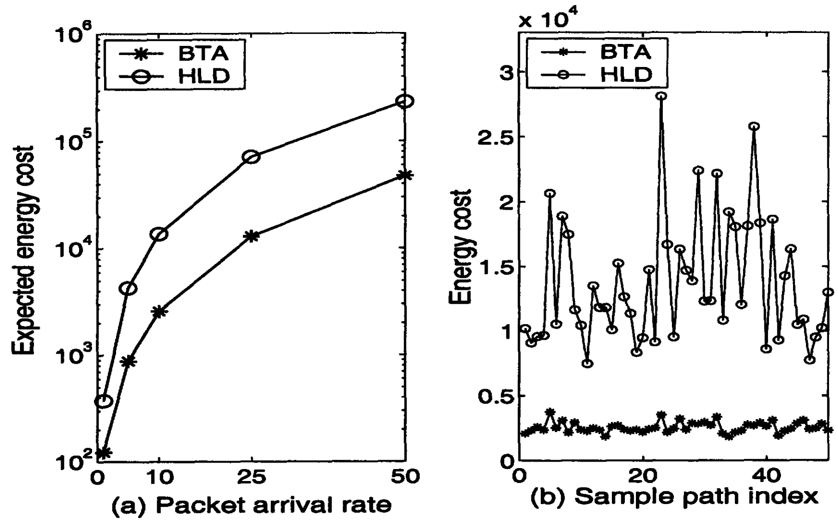


Figure 4-3: Energy cost comparison for Poisson arrival process for (a) different arrival rate, (b) different sample paths.

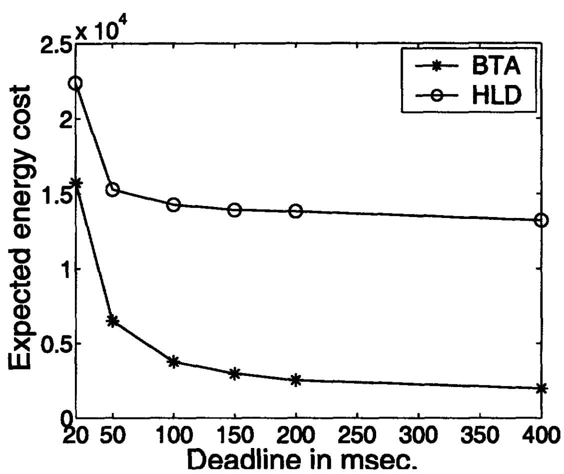


Figure 4-4: Energy cost versus packet deadline for Poisson arrival process.

has at most a single packet. Hence, both policies choose a rate based on the head-of-line packet with the BTA policy also adapting the rate with the channel state. As the arrival rate increases and due to the bursty nature of the Poisson process, the queue tends to have more packets. The BTA policy then adapts based on the channel and the deadlines of all the packets in the queue, whereas, the HLD policy chooses a rate based solely on the head-of-line packet. The energy efficiency of the BTA policy is not just in an average sense but even on individual sample paths. This is shown in Figure 4-3(b) for 50 sample paths for arrival rate of 10 packets/second. As seen in the figure, the BTA policy has a lower energy cost over individual sample paths as well.

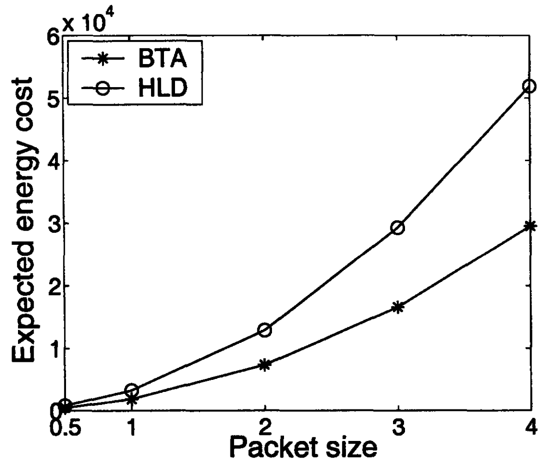


Figure 4-5: Energy cost versus packet size for Uniform arrival process.

In Figure 4-4, the packet arrival process is Poisson with rate 10 but now the packet deadline is varied. Clearly, as seen in the figure, the energy cost decreases as the packet deadline increases since lower transmission rates are required to meet the deadlines. Also, as the deadline increases the energy cost difference between the BTA and the HLD policy increases. This is because with a larger delay constraint there is more room for the adaptive techniques employed in the BTA policy to have a greater effect.

In Figure 4-5, we consider a Uniform packet arrival process where now the inter-arrival time between packets is uniformly distributed between 50 and 150 msec. The deadline for each packet is taken as 200 msec while the packet size is varied. First, as expected, the energy cost for both the policies increases with the packet size and second, the BTA policy has a much less energy cost compared with HLD policy even when the arrival process is less bursty as compared to the Poisson process.

4.3 Stochastic Arrivals

In the last section, we considered two different setups for the case of packet arrivals to the queue. First in which the packet arrival information was known in advance, and second in which no such information was assumed. In the latter case, the transmission rate is chosen based solely on the data in the buffer and the channel state, while from the results in the former case (see Section 4.2.4) it is clear that knowing the packet arrival information affects the optimal minimum-energy transmission policy. Thus, intuitively, it is quite expected that in the case of stochastic packet arrivals, statistical knowledge about the future packet arrivals should affect the transmission rate being chosen. To understand this behavior analytically, we now consider a deadline-constrained data transmission problem with a Poisson arrival process. For the sake of simplicity, we assume that there is no channel fading, thus, the stochastic variations in the system arise only from the packet arrivals.

We consider the following setup: The transmitter has a stream of packet arrivals according to a homogeneous Poisson process, there is no channel fading and there is a single deadline by which all the data must be transmitted. The objective is to minimize the (expected) transmission energy expenditure while ensuring that the deadline constraint is met. We now present in detail the optimal control formulation for the problem and give the transmission policy results following it.

4.3.1 Optimal Control Formulation

Consider a time interval $[0, T)$ and a stream of packet arrivals according to a homogeneous Poisson process with rate ξ and packet size B. The arrivals occur in time $[0, T)$ and the deadline constraint is that all this data must depart by time $T + \tau_0$; where $\tau_0 > 0$.

Let the system state be defined as (x, t), where, $x(t) = x$, denotes the buffer size at time t. Let $r(x, t)$ denote a transmission policy and given any such policy, the buffer $x(t)$ evolves in the following way,

$$dx = -r(x,t)dt + Bdq \tag{4.14}$$

Equation (4.14) above[3] can be understood by viewing dx as the change in the buffer size over a small interval dt. The term dq is the poisson differential and can be viewed as equal to 1 with probability ξdt, in which case B gets added to the buffer, and 0 with probability $1 - \xi dt$. We say that a policy $r(x, t)$ is *admissible* if it satisfies the following, (a) $r(x, t) \geq 0$ (non-negativity of rate), and, (b) $x(t) \geq 0$ (non-negativity of buffer size).

Consider now an admissible transmission policy $r(x, t)$ and let $J_r(x, t)$ be the expected energy cost starting in some state (x, t), $x \geq 0, t < T$. Taking the power-rate function from (4.1) as $P(t) = g(r(t))$, where we have taken $c(t) = 1$, $\forall t$, gives,

$$J_r(x,t) = E\left[\int_t^T g(r(x(s),s))ds + \tau_0 g\left(\frac{x(T)}{\tau_0}\right)\right] \tag{4.15}$$

[3] It is a formal representation of $x(t) = x(0) + \int_0^t -r(x,\tau)d\tau + \int_0^t Bdq$ where the integral is defined sample path-wise. See [63, 64], [88] chap 1.

The expectation above is taken over the poisson arrival process $\{q(s), s \in (t,T)\}$. The first term within the bracket is the total transmission energy cost over $[t,T]$ for the policy $r(x,t)$ and the second term is the terminal energy cost at time T. The terminal cost is the amount of energy needed to empty the buffer at time T by the deadline $T+\tau_0$, using transmission rate $x(T)/\tau_0$. Define a *minimum cost function* $J(x,t)$ as the infimum of $J_r(x,t)$ over the set of all admissible policies.

$$J(x,t) \triangleq \inf_{r(x,t)} J_r(x,t), \; r(x,t) \text{ admissible} \tag{4.16}$$

The optimization problem is to compute the minimum cost function $J(x,t)$ and obtain the optimal policy $r^*(x,t)$ that achieves it. Specifically, the minimum cost starting at time 0, is then given as $J(x_0,0)$, where x_0 is the initial amount of data in the buffer at time 0.

The Hamilton-Jacobi-Bellman (HJB) equation for the above problem can be obtained using an identical set of arguments as in Section 3.3.2. However, the difference now is that the differential generator, $A^r J(x,t) \triangleq \lim_{h \downarrow 0} \frac{EJ(x_{t+h},t+h)-J(x,t)}{h}$, for the process (x,t) takes the form [63, 64, 87],

$$A^r J(x,t) = \frac{\partial J(x,t)}{\partial t} - r(x,t)\frac{\partial J(x,t)}{\partial x} + \xi(J(x+B,t) - J(x,t)) \tag{4.17}$$

Using the above, the HJB equation then takes the following form ($\forall\ x > 0, t \in (0,T)$),

$$\min_{r \in [0,\infty)} \left\{ g(r) + \frac{\partial J(x,t)}{\partial t} - r\frac{\partial J(x,t)}{\partial x} + \xi(J(x+B,t) - J(x,t)) \right\} = 0 \tag{4.18}$$

The optimal transmission rate r^* for a given system state (x,t) is the value of r that achieves the minimum in (4.18). The boundary conditions on $J(x,t)$ for the partial differential equation in (4.18) are as follows. For the boundary $(x = 0, 0 \le t < T)$ we get the following condition from the analysis in Appendix C.3,

$$g(0) + \frac{\partial J(0,t)}{\partial t} + \xi(J(B,t) - J(0,t)) = 0 \tag{4.19}$$

The boundary condition for $(x \ge 0, t = T)$ is simply the terminal energy cost equal to emptying the buffer by the deadline $T+\tau_0$ and is given as,

$$J(x,T) = \tau_0 g(x/\tau_0) \tag{4.20}$$

4.3.2 Constraint Relaxation

An analytical solution to the partial differential equation in (4.18) with the boundary conditions in (4.19) and (4.20) is difficult to obtain. Therefore, we consider a relaxation of the problem where we ignore the boundary condition in (4.19) and solve the PDE without it. In terms of the original problem, this relaxation corresponds to ignoring the non-negativity constraints on $x(t)$. Thus, the relaxed solution is infeasible for the original problem but it can be made feasible by setting $r(x,t) = 0$, if $x = 0$, which would ensure that once empty the buffer does not become negative. Later in this section, we present a comparison of the relaxed solution with the optimal solution, obtained numerically by solving the PDE using a finite-difference method (see Figure 4-7). Indeed, we see from that comparison that the relaxed policy is very close to the optimal solution.

We now proceed to solve for a particular solution of (4.18) with only the boundary condition in (4.20). Consider first the class of exponential power-rate functions, namely $g(r) = \alpha^r - 1,\ \alpha > 1$. Let us take the solution $r(x,t)$ to be of the form,

$$r(x,t) = \frac{x}{T + \tau_0 - t} + f(t) \tag{4.21}$$

where $f(t)$ is a function that needs to be determined. Using the first-order derivative condition for the minimization in (4.18) gives $\frac{\partial J(x,t)}{\partial x} = g'(r(x,t))$. Integrating gives,

$$J(x,t) = \alpha^{f(t)}(T + \tau_0 - t)\alpha^{\frac{x}{T+\tau_0-t}} + c(t) \tag{4.22}$$

where $c(t)$ is the constant of integration that depends on t. Incorporating the boundary condition in (4.20) we get,

$$f(T) = 0 \quad \text{and} \quad c(T) = -\tau_0 \tag{4.23}$$

Next, substituting $J(x,t)$ and $r(x,t)$ from (4.22) and (4.21) respectively, into (4.18), we require that the PDE be satisfied. This entails,

$$c'(t) - 1 + \alpha^{f(t)}(T + \tau_0 - t)\ln(\alpha)\alpha^{\frac{x}{T+\tau_0-t}} \left\{ f'(t) - \frac{f(t)}{T + \tau_0 - t} + \frac{\xi}{\ln(\alpha)}\left(\alpha^{\frac{B}{T+\tau_0-t}} - 1\right) \right\} = 0 \tag{4.24}$$

Since the above equation holds for all values of x the coefficients must equate to zero.

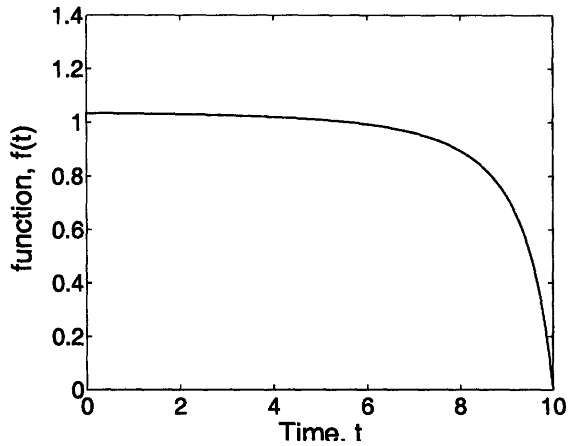


Figure 4-6: Plot of $f(t)$ for $T = 10,\ \tau_0 = 1,\ \xi = 1,\ B = 1$ and $g(r) = e^r - 1$.

Thus, we get the following set of ordinary differential equations (ODE).

$$c'(t) = 1 \tag{4.25}$$

$$f'(t) - \tfrac{f(t)}{T+\tau_0-t} + \tfrac{\xi}{\ln(\alpha)}\left(\alpha^{\frac{B}{T+\tau_0-t}} - 1\right) = 0 \tag{4.26}$$

Combining (4.25) and (4.23) we get $c(t) = t - T - \tau_0$ while $f(t)$ can be obtained from the following lemma.

***<u>Lemma</u>* 11** *Let $f(t)$ satisfy the ODE in (4.26) and the boundary condition $f(T) = 0$, then, denoting $\beta_t = T + \tau_0 - t$, the function $f(t)$ is given as,*

$$\begin{aligned} f(t) = {} & \frac{\xi/\ln(\alpha)}{\beta_t}\left(B\ln(\alpha)(T-t) + \frac{(B\ln(\alpha))^2}{2}\ln\left(\frac{\beta_t}{\tau_0}\right)\right) \\ & + \frac{\xi/\ln(\alpha)}{\beta_t}\left(\sum_{n=3}^{\infty}\frac{(B\ln(\alpha))^n}{n!(n-2)}\left(\frac{1}{\tau_0^{n-2}} - \frac{1}{\beta_t^{n-2}}\right)\right) \end{aligned} \tag{4.27}$$

Proof: Appendix C.4 ■

An illustrative plot of the function $f(t)$ for $g(r) = e^r - 1$, $T = 10$, $\tau_0 = 1$, $\xi = 1$ and $B = 1$ is shown in Figure 4-6.

The solution thus obtained by combining (4.27) and (4.21) satisfies $r(x,t) > 0$, if $x > 0$ but does not satisfy $r(x,t) = 0$, if $x = 0$. However, a feasible solution can be easily constructed as follows.

$$r(x,t) = \begin{cases} \frac{x}{T+\tau_0-t} + f(t), & \text{if } x > 0 \\ 0, & \text{if } x = 0 \end{cases} \tag{4.28}$$

We refer to the above policy as the *Relaxed Policy* (RP). The closed-form structure of the RP policy provides some interesting and intuitive insights. First, the transmission rate at time t for buffer size x equals $x/(T+\tau_0-t)$, which is the minimum constant rate required to serve x amounts of data by time $T+\tau_0$, plus an additional rate $f(t)$. This is natural as there is anticipation of future arrivals and the convexity of the cost function dictates that these (expected) future arrivals should be taken into account. This is because transmitting at a uniform average rate has a less total energy cost as compared to transmitting at a lower rate and then increasing the rate later on. Second, $f(t)$ depends on the underlying function $g(.)$ (as observed by the dependence on α). The intuition behind this is that if $g(.)$ has a very fast increasing slope then it is beneficial to reduce the buffer at a higher rate, as future data arrivals close to the deadline will incur a lot of energy expenditure. Third, $r(x,t)$ depends on time t through $T-t$. This observation is intuitive and follows from the fact that the Poisson process is memoryless and the future arrival statistics depend only on the remaining time.

Thus far, we have assumed $g(r) = \alpha^r - 1,\ \alpha > 1$. Proceeding as above, we can also obtain solutions for other convex functions as well. One such example is $g(r) = r^2$. For this function, the above methodology leads to a very intuitive solution and for which $f(t)$ is given as,

$$f(t) = \frac{\xi B(T-t)}{T+\tau_0-t} \tag{4.29}$$

Note that $\xi B(T-t)$ is the expected future amount of data to arrive and $T+\tau_0-t$ is the time left. Thus, the excess rate can be interpreted as the rate required to drain the expected future amount of data in the remaining time.

To understand how the RP policy compares with the optimal solution, we present an illustrative numerical comparison, where the optimal solution is obtained by numerically solving (4.18) with the boundary conditions in (4.19) and (4.20), using a finite difference method. The partial differentials are approximated with a finite difference and the functions are evaluated starting from the boundaries. For a rigorous treatment on such techniques,

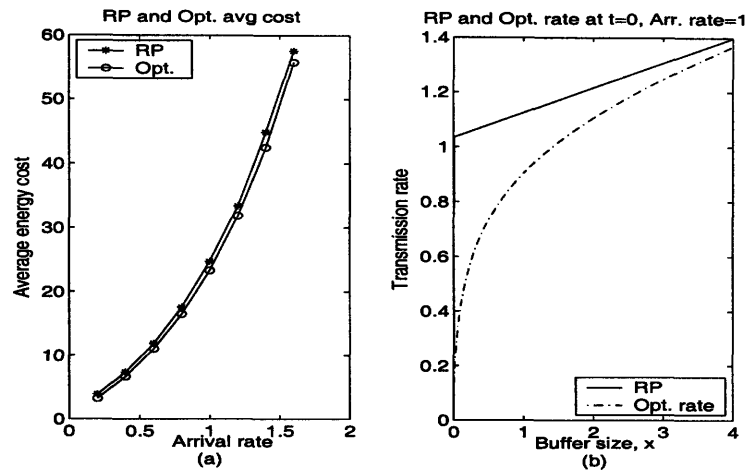


Figure 4-7: (a) Comparison of expected energy cost for RP and Optimal policy. (b) Comparison of rate at $t = 0$ as a function of the buffer size x.

the reader is referred to [88]. We now give the details of the numerical computation.

The set of equations for the numerical evaluation are as follows. First, from the boundary conditions, we have, $J(x,T) = \tau_0 g(x/\tau_0), x \geq 0$ and $\left\{g(0)+\frac{\partial J(0,t)}{\partial t}+\xi(J(B,t)-J(0,t))\right\} = 0$. From (4.18), the first-order derivative condition for the minimization gives $g'(r^*(x,t)) = \frac{\partial J(x,t)}{\partial x}$ while the PDE takes the form $\left\{g(r^*(x,t)) + \frac{\partial J(x,t)}{\partial t} - r^*(x,t)\frac{\partial J(x,t)}{\partial x} + \xi(J(x+B,t) - J(x,t))\right\} = 0$. Approximating the partial differentials with finite differences, these equations become,

$$g'(r^*(x,t)) = \frac{J(x,t+\delta t) - J(x-\delta x, t+\delta t)}{\delta x} \tag{4.30}$$

$$J(0,t) = g(0)\delta t + \xi\delta t J(B,t+\delta t) + (1-\xi\delta t)J(0,t+\delta t) \tag{4.31}$$

$$\begin{aligned}(r^*/\delta x + 1/\delta t)J(x,t) &= g(r^*) + \xi\{J(x+B,t+\delta t) - J(x,t+\delta t)\} \\ &\quad + r^*\frac{J(x-\delta x,t)}{\delta x} + \frac{J(x,t+\delta t)}{\delta t}\end{aligned} \tag{4.32}$$

where δx and δt are the step sizes for x and t respectively. Starting at $t = T$, we have $J(x,T) = \tau_0 g(x/\tau_0)$. Now iterating backwards, each time decrementing t by δt, we can evaluate $J(0,t)$, $r^*(x,t)$ and $J(x,t)$, for $x = (\delta x, .., B)$. For the purposes of this simulation, we used the following parameters in the numerical evaluation; $T = 10$, $\tau_0 = 1$, $B = 1$, $g(r) = \exp(r) - 1$, $\delta x = 0.01$, $\delta t = 0.02$. Figure 4-7(a) compares the optimal energy cost evaluated numerically with the expected energy cost for RP at $t = 0$ and with $x_0 = 1$. The expected energy cost for RP is obtained using simulations as explained later in Section 4.3.3. As we see from the plot, RP performs very close to the optimal and the two energy costs have a very small difference.

Figure 4-7(b) compares the optimal rate and the RP rate (Equation 4.28) as a function of the buffer size for $t = 0$ and $\xi = 1$. As seen from the figure, at moderate buffer sizes the two rates are fairly close to each other and in fact converge as x increases, but at very low buffer values the optimal rate is reduced as the boundary $x = 0$ is closer and the boundary effect becomes prominent. The asymptotic (large x) convergence of the two rates is quite intuitive since the buffer non-negativity constraint becomes less important for large x. Thus, we see that the RP and the optimal rate tend to differ only for small values of buffer size and the effect of this on the energy consumption is minimal.

4.3.3 Simulation Results

From (4.28), we see that the relaxed policy takes into account the future arrivals in computing the transmission rate. To understand how this policy compares with a non-anticipative policy which does not take into account the arrival statistics, we present an illustrative simulation example in this section. We consider the *Direct Drain* (DD) policy in which the transmission rate at any time t is chosen as $x(t)/(T - t + \tau_0)$. Thus, in this policy the transmitter simply looks at the buffer size, $x(t)$, and chooses a rate that would empty the buffer by the deadline.

In the simulation setup, we consider the following parameters. $T = 10$, $\tau_0 = 1$, $B = 1$, $g(r) = \exp(r) - 1$ and the initial buffer size $x_0 = 1$. The time interval $[0, T = 10]$ is divided into discrete intervals of length $dt = 10^{-3}$; thus, having $10,000$ time slots. The arrival rate ξ is varied between $\xi = 0.2 - 1.6$ in steps of 0.2. The Poisson arrival process is simulated using a Bernoulli model. In each time slot an arrival occurs with probability ξdt and there are no arrivals with probability $1 - \xi dt$. At the beginning of each time slot the buffer size x and the time t is known. The rate of transmission in that slot for RP is chosen from (4.28)[4] while the rate of transmission for DD policy, as mentioned earlier, is chosen as $\frac{x}{T-t+\tau_0}$. The same set of sample paths are applied to both the policies and the energy cost is computed as $\sum_i 10^{-3}(\exp(r_i) - 1) + (\exp(x(T)) - 1)$ (note $\tau_0 = 1$). The average is then taken over a set of 10^4 sample paths.

Figure 4-8 compares the expected energy expenditure of the two policies for the set of ξ

[4] If x is very small then the rate chosen might make the buffer go negative. In this case the rate is simply taken as $x/10^{-3}$.

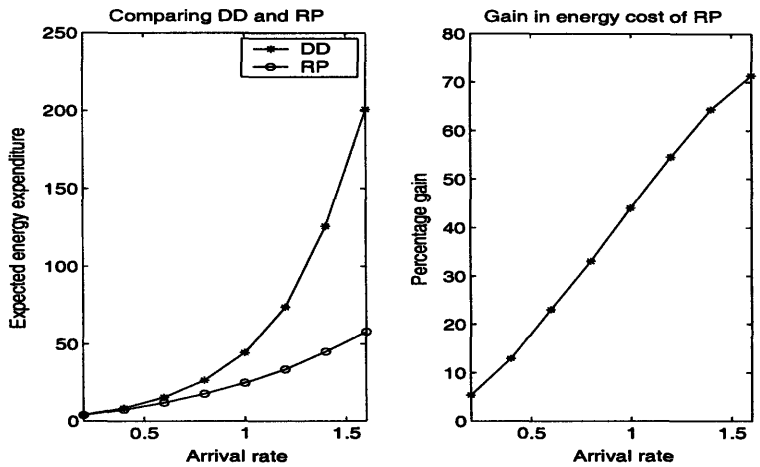


Figure 4-8: Plot comparing the expected energy expenditure of RP and DD policies and the percentage gain ((DDcost-RPcost)*100/DDcost).

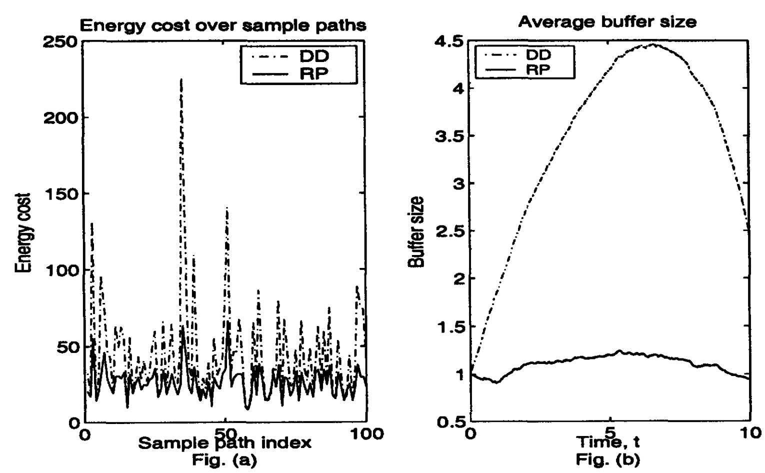


Figure 4-9: (a) Comparison of energy expenditure for 100 sample paths at $\xi = 1$. (b) Comparison of average buffer size over time for $\xi = 1$.

considered. As shown in the gain plot, RP significantly outperforms DD policy and in fact the curve is upward sloping. Figure 4-9(a) plots the energy expenditure for the first 100 sample paths for $\xi = 1$. It is clear from the figure that RP has lower energy cost than DD policy for almost all sample paths. Thus, even on a sample path comparison RP performs better. Finally, Figure 4-9(b) compares the average buffer size at time t of the two policies for $\xi = 1$. As seen from the figure, RP tends to have a more uniform and much smaller average buffer size as compared to DD policy. Observe that for $\xi = 1$, on average one packet arrives in unit time and starting from $x_0 = 1$, RP tends to transmit at that rate in an average sense. As DD policy does not adjust the rate in anticipation of the arrivals, it transmits at low rates initially and hence the buffer tends to increase. Then, as the deadline gets closer the rate increases and the average buffer size drops. And, this results in a much higher total energy cost.

4.4 Chapter Summary

In this chapter, we considered various extensions of the deadline-constrained energy-efficient data transmission problem, from the canonical BT-problem treated earlier in Chapter 3. The cumulative curves methodology and the solution to the BT-problem provided important tools to address the various generalizations considered here. In fact, the graphical visualization of the problems, combined with the intuition gained through the deterministic setup in Chapter 2, yielded the transmission policies in a very natural way, and these were also shown to be optimal for the constant drift channel model.

First, in Section 4.2.3, we considered the variable deadlines setup. Here, the transmitter queue has data with individual packet deadline constraints and the goal was to transmit the data over a fading channel with minimum energy while meeting the deadline constraints. Based on a graphical decomposition of the problem into multiple BT-problems, we obtained a transmission policy which takes a very simple form and is given as,

$$r(D, c^i, t) = \max_{j:(B_j \geq D, T_j \geq t)} \frac{B_j - D}{f_i(T_j - t)}, \quad i = 1, \ldots, m \tag{4.33}$$

The above policy was shown to be optimal for the constant drift channel model (Theorem XII). It draws its intuition from the analogous problem in the deterministic setup considered in Chapter 2, where the optimal policy was to choose the maximum slope straight-line segment among those that meet the B_jT_j constraints.

In Section 4.2.4, we considered the following setup. There is a stream of packet arrivals at known inter-arrival times and a single deadline constraint by which the data must be transmitted. The goal as before is to transmit the data over a fading channel with minimum energy while meeting the deadline constraint. Using the graphical visualization of the problem and the intuition from the analogous setup in the deterministic setup, we obtained a transmission policy which was shown to be optimal for the constant drift channel model (see Theorem XIII) and is given as,

$$r(D, c^i, t) = \min_{j:(A_j \geq D, T_j \geq t)} \frac{A_j - D}{f_i(T_j - t)}, \quad i = 1, \ldots, m \tag{4.34}$$

We then considered the case of arbitrary packet arrivals to the queue without any arrival information. Each arriving packet had a deadline by which it must depart and the goal was to transmit the data over a fading channel while meeting the deadline constraints. In this case, we obtained the BT-Adaptive policy (BTA) and showed through simulations, that BTA has a much lower energy cost as compared to the Head-of-Line drain policy.

Finally, in Section 4.3, we studied a problem involving stochastic packet arrivals to the queue. We considered a stream of Poisson packet arrivals to the queue and a single deadline by which all this data must be transmitted. The objective is to transmit the data within the deadline constraint and minimize the energy expenditure. Using an optimal control formulation and a relaxation of the boundary condition, we obtained in closed-form the transmission policy that satisfies the HJB equation. We referred to it as the Relaxed Policy and is given as follows,

$$r(x,t) = \begin{cases} \frac{x}{T+\tau_0-t} + f(t), & \text{if } x > 0 \\ 0, & \text{if } x = 0 \end{cases} \tag{4.35}$$

Thus, we see from the above equation that the transmitter chooses a rate to empty the buffer by the deadline plus an additional rate $f(t)$ which accounts for the (expected) future packet arrivals. Finally, we presented simulation results comparing the relaxed policy with the direct drain policy and these results showed that substantial gains can be achieved in the energy expenditure.

Chapter 5

Multi-user Scheduling with Throughput-rate Guarantees

5.1 Introduction

In Chapters 2, 3 and 4, we studied energy-efficient transmission rate-control with hard deadline (or other strict Quality-of-Service) constraints on data. Such a setup applies to scenarios involving real-time data communication, where there are stringent restrictions on maximum packet delays. However, in addition to such applications, there are other services where instead of hard latency requirements, a more appropriate quality-of-service metric is the long-term throughput rate; these include, for example, file-transfers (FTP), web-browsing etc. In this chapter, we study a multiple-user setup involving long-term throughput rate as the quality-of-service metric. The goal here is to develop a multi-user scheduling algorithm that provides the required throughput rates using minimal time-slot utilization.

The setup for this chapter is as follows. There is a single server that represents the wireless base station transmitting to multiple users that represent the mobile handsets. The system operates in a time-slotted manner and in each time-slot the base station can serve only one user. This setup is referred to in the literature as the *Wireless Downlink Scenario*, where "downlink" refers to the communication link from the base-station to the mobile user. We further assume that the set of users are divided into two classes: (i) throughput-rate guaranteed, QoS users and (ii) "best effort" (BE) users. The QoS users in the system represent session applications such as FTP, high data-rate web-browsing, throughput-constrained data transfers etc., which require the base station to provide a certain long-term data rate on the downlink. In contrast, the BE users represent on-the-fly applications such as email transfers, low priority and latency tolerant data transfers etc. which do not have rate requirements and are short-lived. The goal of this work is to design a scheduling policy for the users, that provides the required throughput rates to the QoS users with the least time-slot utilization and maximizes the remaining fraction of time-slots assigned for the BE class.

Since we are concerned with a wireless channel, the communication rate at which the base station can reliably transmit to the various users fluctuates over time. Furthermore, at any given time-slot, the different users also have different rates among them and the transmitter can take advantage of this diversity to decide which user to transmit to based on certain required objectives. In the literature, such an approach is referred to as *Multi-user Diversity* [4, 80] or *Opportunistic Scheduling* [75–79]. Clearly, if the goal is to simply maximize the sum throughput, the transmitter must always select the user with the highest communication rate in that time-slot. While this simple policy maximizes the total throughput, to achieve throughput-rate guarantees for multiple users one needs to look into more sophisticated scheduling algorithms.

The contents of the chapter are organized as follows. In Section 5.2, we present the system model and the problem description. In Section 5.3, we present a geometric approach to the problem and obtain the optimal policy through it. The throughput results for Rayleigh fading are presented in Section 5.4 and simulation results comparing the optimal and the random scheduling policy for Rayleigh and Nakagami fading are presented in Section 5.5. Finally, Section 5.6 summarizes the work in this chapter.

5.2 System and Problem Description

5.2.1 System Model

As mentioned earlier, we consider the wireless downlink scenario, namely, communication from the base station (the transmitter) to the mobile handsets (the receivers, also referred to as users) in a time-slotted system. There are multiple users in the system, each user experiencing time-varying channel condition. The channel state of a user remains constant for a single time slot but changes over multiple time slots. We assume that the channel stochastic process is stationary and ergodic. This assumption does not preclude channel correlations over time and among the users, thus allowing the possibility of channel states over multiple time-slots to be dependent. At the beginning of a time-slot, the transmitter knows the channel state of each user for that particular slot[1]. In a time-slot, it serves at most one user with full power P. Since the users have different channel conditions the rate of communication per time slot to the users is variable. Clearly, the transmitter can exploit this variability and select the appropriate user for transmission in a time-slot based on some performance measure. The above system models a TDMA system and the recently proposed 1xEV-DO data system [3] and is a commonly used model in the literature for the wireless downlink [75–79].

Let $\bar{\mathbf{r}} = \{r_i\}$ denote the vector of communication rates to the users in a generic time-slot, say for example the k^{th} time-slot. This means that if user i is chosen to be served in time-slot k, the throughput for that user in that slot is simply r_i. We will refer to r_i as the "*channel rate*" for user i and $\bar{\mathbf{r}}$ as the "*channel rate vector*". The transmitter has knowledge of $\bar{\mathbf{r}}$ at the beginning of slot k but does not know this vector for future slots. In the k^{th} time-slot, $\bar{\mathbf{r}}$ is a particular realization from the set comprising all possible channel rate vectors, whose probability distribution depends on the stochastic model of the underlying channels' states; and it is assumed to be a stationary process. A scheduling policy, denoted as $\Gamma^k(\bar{\mathbf{r}})$, is a rule that specifies which user the transmitter serves in time-slot k given that the channel rate vector in that slot is $\bar{\mathbf{r}}$. A *stationary scheduling policy*, denoted $\Gamma(\bar{\mathbf{r}})$, is one that depends solely on $\bar{\mathbf{r}}$ but does not depend on the time index. Clearly, such a policy can be represented as a map from the set of channel rate vectors to the user index; namely, each $\bar{\mathbf{r}}$ is mapped to a unique user index. As the underlying processes are stationary, we restrict attention in this work to stationary scheduling policies and such a restriction suffices.

5.2.2 Problem Description

The set of users in the system are divided into two service classes: (i) throughput rate guaranteed (QoS) users and (ii) "best effort" (BE) users. As mentioned earlier, QoS users represent session applications that require the base station to provide a certain data rate on the downlink, whereas, the BE users represent low priority data transfer applications which

[1]This is a simplifying assumption that models one step channel prediction

do not have a rate requirement and are short-lived. The number of BE users is assumed large and being short-lived it changes rapidly over time. In such a setup, the objective at the base station is to provide the throughput rates to the QoS users with the least time-slot utilization so that the remaining fraction of time-slots allocated for serving the BE class is maximized[2]. The scheduling problem now is to obtain a rule that assigns time-slots dynamically over time to meet the above objective.

Let there be N QoS users in the system and denote the channel rate vector for these users as $\bar{\mathbf{r}} = (r_1, \ldots, r_N)$. Let $X_i(\bar{\mathbf{r}})$ denote the throughput per time-slot of user i. We have[3],

$$X_i(\bar{\mathbf{r}}) = \begin{cases} r_i, & \text{if } \Gamma(\bar{\mathbf{r}}) = i \text{ (i.e. user } i \text{ selected)} \\ 0, & \text{otherwise} \end{cases} \tag{5.1}$$

The expected throughput per time slot is $E[X_i(\bar{\mathbf{r}})]$. Under stationarity of the scheduling rule, it is easy to see that $X_i(\bar{\mathbf{r}})$ is stationary and ergodic and that $E[X_i(\bar{\mathbf{r}})]$ equals the long term time-average throughput per slot (called *throughput rate*) of user i. Let $\bar{\mathbf{R}} = (R_1, .., R_N)$ be the guaranteed throughput rates to the QoS users. We will assume that $\bar{\mathbf{R}}$ is *feasible* and by feasibility we mean that there exists at least one scheduling policy that achieves the throughput rates, i.e. $E[X_i(\bar{\mathbf{r}})] \geq R_i, \forall i = 1, .., N$ for some policy.

Let $I_i(\bar{\mathbf{r}})$ be the indicator function for selection of user i,

$$I_i(\bar{\mathbf{r}}) = \begin{cases} 1, & \text{if } \Gamma(\bar{\mathbf{r}}) = i \\ 0, & \text{otherwise} \end{cases} \tag{5.2}$$

With this notation we can re-write $X_i(\bar{\mathbf{r}})$ as $X_i(\bar{\mathbf{r}}) = r_i I_i(\bar{\mathbf{r}})$. The optimization problem can now be formally stated as follows,

$$\begin{aligned} \min \quad & \sum_{i=1}^{N} E[I_i(\bar{\mathbf{r}})] \\ \text{subject to} \quad & E[r_i I_i(\bar{\mathbf{r}})] \geq R_i, \ i = 1, .., N \end{aligned} \tag{5.3}$$

[2]We assume that among the BE users a greedy algorithm is used to share the slots that are allocated for the BE class. With a large population of BE users there is a high probability of at least one user experiencing good channel condition. Thus, maximizing the time-slot allocation is then equivalent to maximizing the sum total throughput of BE users.

[3]For notational simplicity, explicit dependence of $X_i(\cdot)$ on Γ is not indicated. Also, since the service of BE users is simply the fraction of allocated time-slots to that class, their channel rate vector is not required for the optimization.

The expectation above is taken over the joint distribution of the channel rate vector, $\bar{\mathbf{r}}$, for the N QoS users. Note that minimizing $\sum_{i=1}^{N} E[I_i(\bar{\mathbf{r}})]$ is equivalent to maximizing $1 - \sum_{i=1}^{N} E[I_i(\bar{\mathbf{r}})]$ which equals the fraction of time-slots available for the BE users. We assume that $\bar{\mathbf{R}} > 0$, i.e. $(R_1 > 0, .., R_N > 0)$. If some $R_k = 0$, we can neglect that user and the problem reduces to $N-1$ dimensions. We also assume that $\bar{\mathbf{R}}$ is away from the boundary of the set, which is characterized later, comprising all feasible throughput rate vectors. This assumption is solely to simplify the mathematical exposition by avoiding the limiting conditions at the boundary and does not affect the results presented throughout the chapter.

5.3 Optimal Policy

The QoS users experience different time-varying channel conditions, hence, intuitively the optimal policy must exploit this variability giving preference to users with better channel conditions. This would ensure a high throughput per slot and would lead to a fewer fraction of time-slots being utilized to provide the throughput guarantee. However, simply choosing the best user is not sufficient since the throughput requirements of the QoS users and their channel statistics might be very different which necessitates that these parameters must also be taken into account.

Let Ω be the set comprising all possible channel rate vectors, $\bar{\mathbf{r}}$; we have $\Omega \subseteq \mathbb{R}^{+N}$. Let the joint probability density function be $f(\bar{\mathbf{r}})$ [4] so that the probability of a subset $Z \subset \Omega$ is given as $\int_Z f(\bar{\mathbf{r}})d\bar{\mathbf{r}}$. We assume that $f(\bar{\mathbf{r}})$ is such that subsets with zero volume in Ω (or individual points) have zero probability, thus, excluding point mass distributions. Since a scheduling policy maps $\bar{\mathbf{r}} \in \Omega$ to a unique user index, we will represent a scheduling policy as a partition of the set Ω into $N+1$ regions denoted as $Z_1, .., Z_N, Z_f$. In a particular time-slot, if the channel rate vector $\bar{\mathbf{r}}$ lies within region Z_i, user i is selected for service whereas if $\bar{\mathbf{r}} \in Z_f$, no QoS user is selected and the slot is used to serve the BE users. The problem thus reduces to choosing these regions optimally to minimize the objective function and to satisfy the throughput rate constraint, $\int_{Z_i} r_i f(\bar{\mathbf{r}})d\bar{\mathbf{r}} \geq R_i,\ i = 1, \ldots, N$.

To eliminate uninteresting partitions the following technical assumptions are made. The set Ω can be partitioned into a finite set of components, where, each component is a con-

[4] To avoid excessive notations, $\bar{\mathbf{r}}$, depending on the context denotes both a random vector and a particular realization for a generic time-slot.

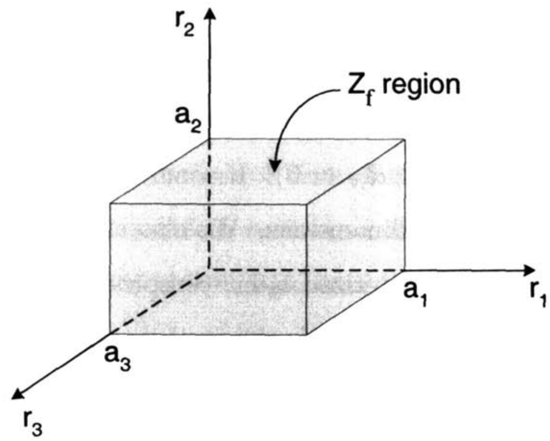


Figure 5-1: The Z_f region for $N = 3$, threshold vector $\bar{\mathbf{a}} = (a_1, a_2, a_3)$ and $\Omega = \mathbb{R}^{+N}$. Note that $Z_f = \{\bar{\mathbf{r}} : 0 \leq r_i \leq a_i,\ \forall i = 1, \ldots, N\}$.

nected set with non-zero volume and every point of this set is arbitrarily close to an interior point. Such an assumption removes the trivial point/zero volume sets. A scheduling policy is a partition as above and each region Z_i is a finite union of the component sets of the partition. Further, we assume that for set Ω, non-zero volume sub-sets that have zero probability have already been removed as their mapping plays no role in the optimization.

In the rest of the chapter, the notation $\bar{\mathbf{r}} \rightarrow Z$ ($\bar{\mathbf{r}} \not\rightarrow Z$) means that there is a neighborhood around $\bar{\mathbf{r}}$ that is mapped (is not mapped) to region Z and the probability of this neighborhood is non-zero. Formally, $\bar{\mathbf{r}} \rightarrow Z$ implies that there exists $\epsilon > 0$ such that all $\hat{\mathbf{r}} \in \Omega$, $||\hat{\mathbf{r}} - \bar{\mathbf{r}}|| < \epsilon \Rightarrow \hat{\mathbf{r}} \in Z$ and $\int_{||\hat{\mathbf{r}}-\bar{\mathbf{r}}||<\epsilon} f(\hat{\mathbf{r}}) d\hat{\mathbf{r}} > 0$; where the norm $|| \cdot ||$ is the Euclidean distance norm in $\mathbb{R}^N$. The following two lemmas give the properties of the optimal $Z_1, \ldots, Z_N, Z_f$ regions. The first lemma deals with the region Z_f and it states that if $\bar{\mathbf{r}}$ is mapped to Z_i, all rate vectors with the i^{th} component larger than r_i cannot be mapped to Z_f.

***Lemma* 12** *Under the optimal policy, suppose* $\bar{\mathbf{r}} = (r_1, .., r_N) \rightarrow Z_i$ *then* $\hat{\mathbf{r}} = (\hat{r}_1, .., (\hat{r}_i > r_i), .., \hat{r}_N) \not\rightarrow Z_f$.

Proof: Appendix D.1 ■

A careful observation of Lemma 12 yields a special structure on Z_f as follows. Let a_1 be the infimum value of the first component among all vectors $\bar{\mathbf{r}} \rightarrow Z_1$; i.e. $a_1 = \inf_{(\bar{\mathbf{r}} \rightarrow Z_1)} r_1$. Now, any $\hat{\mathbf{r}} \rightarrow Z_f$ must be such that $\hat{r}_1 \leq a_1$; otherwise Lemma 12 will be violated. As this holds for all Z_i, an optimal policy has constants $\{a_i\}$ where $a_i = \inf_{(\bar{\mathbf{r}} \rightarrow Z_i)} r_i$ such that if

$r_i \le a_i, \forall i$, then $\bar{\mathbf{r}} \in Z_f$. The region Z_f is shown in Figure 5-1. This implication is quite intuitive as it suggests that when the channel rate vector of the QoS users is below some threshold vector (bad channel conditions), the QoS users must not be scheduled and the slot must be used to serve the BE users.

The vector $\bar{\mathbf{a}}$ depends on the required throughput vector $\bar{\mathbf{R}}$ for the QoS users and the density function $f(\bar{\mathbf{r}})$. Given that $\bar{\mathbf{R}}$ does not lie on the boundary of feasible throughput rates, it follows that $\bar{\mathbf{a}}$ is at least a positive vector ($a_1 > 0, \ldots, a_N > 0$) and the region $Z_f = \{\bar{\mathbf{r}} \mid \bar{\mathbf{r}} \in \Omega, r_i \le a_i \forall i\}$ is not null (non-zero probability). We now proceed to obtain the structure of the regions $Z_i,\ i = 1, \ldots, N$.

Lemma ***13*** *Consider regions Z_i, Z_j, $j \neq i$ and the corresponding thresholds a_i, a_j. Suppose $\bar{\mathbf{r}} \notin Z_f$ and satisfies,*

$$\frac{r_i}{a_i} > \frac{r_j}{a_j} \tag{5.4}$$

then under the optimal policy $\bar{\mathbf{r}} \not\to Z_j$

Proof: Appendix D.2 ■

The above lemma states that if the weighted comparison of i^{th} and j^{th} component of $\bar{\mathbf{r}}$ is in favour of the i^{th} component (user i), it is not optimal to serve user j. The weights are the inverse values of the corresponding components of the threshold vector $\bar{\mathbf{a}}$. The above implication is intuitive as condition (5.4) means that in some sense user i has a better channel condition than user j and hence serving user j is not optimal. Combining the above two lemmas, we obtain the following geometric structure for the optimal policy.

Theorem ***XIV (Optimal Structure)*** *Consider a channel rate vector $\bar{\mathbf{r}} = (r_1, \ldots, r_N)$, then, under the optimal policy there exists a threshold vector $\bar{\mathbf{a}}$ with the following structure.*

1. *$\bar{\mathbf{r}} \to Z_f$ if it satisfies,*

$$r_i < a_i, \ \forall i = 1, \ldots, N \tag{5.5}$$

2. *$\bar{\mathbf{r}} \to Z_i$, $(i = 1, \ldots, N)$ if it satisfies,*

$$\frac{r_i}{a_i} > \frac{r_j}{a_j}, \quad \forall j = 1, \ldots, N, j \neq i \tag{5.6}$$

$$r_i > a_i \tag{5.7}$$

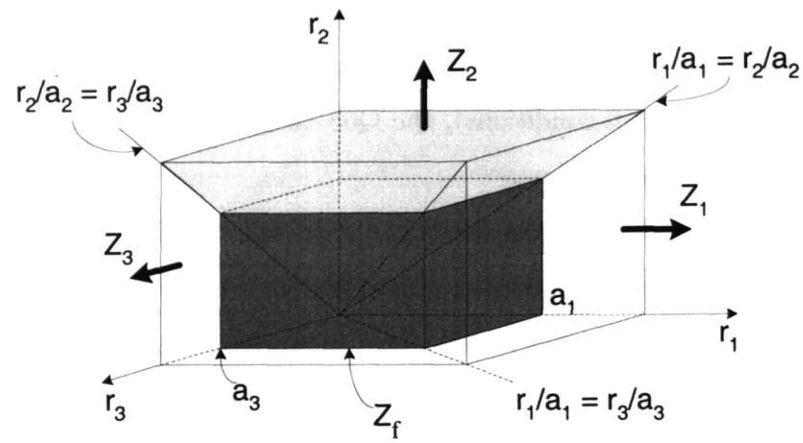


Figure 5-2: Optimal policy structure for $N = 3$, threshold vector $\bar{\mathbf{a}} = (a_1, a_2, a_3)$ and $\Omega = \mathbb{R}^{+N}$. The Z_i regions are top truncated pyramids.

3.

$$\int_{Z_i} r_i f(\bar{\mathbf{r}}) d\bar{\mathbf{r}} = R_i, \ \forall i = 1, \ldots, N \tag{5.8}$$

Proof: Conditions 1 and 2 follow from Lemmas 12 and 13. Condition 3 states the obvious requirement that for optimality the throughput constraint must be met with equality; since, otherwise the excess fraction of slots that lead to a throughput above R_i can be assigned to the BE users. ■

The set of $\bar{\mathbf{r}}$ that lie on the boundaries for which there is equality in (5.5) and (5.6) can be mapped to any Z_i without affecting optimality. It can also be observed that the set of conditions in Theorem XIV are exhaustive and map every $\bar{\mathbf{r}} \in \Omega$ to a unique user index. Thus, given $\bar{\mathbf{a}}$, we have a unique partition of Ω into regions $Z_1, \ldots, Z_N, Z_f$. In Figure 5-2, we present a geometric picture of these regions for $N = 3$. As seen from the figure the Z_i regions are top truncated pyramid-like (see, for example the light shaded Z_2 region) and it can be verified that in this region, (5.6) is satisfied.

Next, we present the sufficiency argument by proving that a scheduling policy of the form as in Theorem XIV minimizes the objective function in (5.3) and hence is optimal. First, observe that a scheduling policy outlined in Theorem XIV can be re-written in a simplified way as a maximum weighted rule (with ties broken arbitrarily) as follows,

$$\Gamma(\bar{\mathbf{r}}) = \begin{cases} Z_f \text{ (serve BE class)}, & \text{if } r_i \leq a_i, \forall i = 1, .., N \\ \text{argmax}_i \, \frac{r_i}{a_i}, & \text{otherwise} \end{cases} \tag{5.9}$$

where $\{a_i\}$ are such that $E[r_i I_i] = R_i, \forall i = 1, \ldots, N$.

<u>***Theorem***</u> ***XV (Sufficiency)*** *Consider the optimization problem in (5.3) and let* $\bar{\mathbf{R}}$ *be feasible, then, policy* Γ *defined in (5.9) is optimal.*

Proof: Appendix D.3. ■

Thus, Theorem XIV states that the optimal policy must satisfy certain conditions which impose a weighted comparison structure on it and conversely, Theorem XV completes the argument by stating that a policy with that structure is optimal.

The results presented so far for the optimal policy assumed that $\bar{\mathbf{R}}$ was feasible, that is, it assumed that the optimization problem in (5.3) had a solution and the throughput rate $\bar{\mathbf{R}}$ could be guaranteed by some scheduling policy. We now go back and characterize the set of all such feasible throughput rate vectors. Let Π denote this set; we claim that the interior of Π is generated by considering each threshold vector $\bar{\mathbf{a}} > 0$ and obtaining the corresponding $\bar{\mathbf{R}}$ that can be achieved for the policy in (5.9) for that particular $\bar{\mathbf{a}}$. To see why this is true consider the following. Given any $\bar{\mathbf{a}} > 0$, we first construct a policy as given in (5.9). Since this is a valid scheduling policy the corresponding $\bar{\mathbf{R}}$ with $R_i = E[r_i I_i]$ is feasible, hence, Π must at least include all such $\bar{\mathbf{R}}$. Now, conversely, pick a feasible $\bar{\mathbf{R}}$ in the interior of Π, then, from Theorem XIV we see that a scheduling policy can be re-mapped to have the optimal geometric structure or equivalently there exists $\bar{\mathbf{a}} > 0$ for which the policy in (5.9) is optimal.

For a given $\bar{\mathbf{R}}$, we know from (5.8) that the threshold vector $\bar{\mathbf{a}}$ for the optimal policy is chosen such that $\int_{Z_i} r_i f(\bar{\mathbf{r}}) d\bar{\mathbf{r}} = R_i, \; i = 1, .., N$. This can be solved using numerous techniques of finding the positive root of a non-linear vector equation. In practice, however, the density function $f(\bar{\mathbf{r}})$ may not be known apriori in which case the vector $\bar{\mathbf{a}}$ can be adjusted in real time using stochastic approximation algorithms, similar to those outlined in [75, 76]. For a comprehensive treatment of stochastic approximation algorithms see [83]. We now consider the special case of Rayleigh fading in the next section and obtain explicit expressions for various system metrics.

5.4 Dimensioning

In this section, we apply the general results obtained in the last section to a Rayleigh fading scenario. From a practical perspective while such a fading model might be restrictive, nevertheless, from a systems viewpoint the closed form formulas obtained provide important tradeoff limits between the allocation of resources to the QoS and the BE users and can be used as a first cut calculation in system design. For other fading distributions a similar analysis can be carried out, albeit, closed form expressions may not always be possible and certain quantities would need to be evaluated numerically, as done in Section 5.5 for an illustrative Nakagami fading scenario.

To proceed, we consider the following specializations to the earlier model. The users experience independent identically distributed (i.i.d) flat Rayleigh fading, hence, $|h|^2$ is Exponentially distributed, where $|h|$ is the magnitude of the channel gain/fade state. The rate per time slot of a user is assumed proportional to the fade state (square magnitude); i.e. $r = k(|h|^2 P)$, where k is a constant and P is the transmission power. A linear power-rate relationship is a good model in various scenarios such as the low SNR regime in which most CDMA systems operate, ultra-wideband transmission and fixed modulation schemes and has been studied earlier in the literature [81]. As r is proportional to $|h|^2$, the distribution of r is also Exponential and is given as $f(r) = e^{-r/\mu}/\mu,\ r \geq 0$ where $\mu = E[r]$ is the average throughput rate of a user if it is served in all the time-slots. Lastly, we take the guaranteed throughput rate the same for all N QoS users, namely, $\bar{\mathbf{R}} = (R, \ldots, R)$.

5.4.1 Throughput Characterization

Let γ denote the fraction of time-slots allocated to the BE users. We first obtain the threshold vector in terms of γ as follows. Due to symmetry in $f(\bar{\mathbf{r}})$ and $\bar{\mathbf{R}}$, clearly, the regions $Z_i,\ i = 1, .., N$ are identical, hence, the $\{a_i\}$'s for the optimal policy are equal and the threshold vector is given as $\bar{\mathbf{a}} = (a, .., a)$. Now, the threshold value a in terms of γ is as follows.

***Lemma* 14** *Let γ be the fraction of time-slots allocated to the BE users, then the threshold value a for the optimal policy is given by,*

$$a = \mu \ln \left(\frac{1}{1 - \gamma^{1/N}} \right) \tag{5.10}$$

Proof: From Theorem XIV, the region Z_f is given as $Z_f = \{\bar{\mathbf{r}} : 0 \leq r_i \leq a, \ \forall i = 1, \ldots, N\}$. By ergodicity, the probability of this region equals γ and by the i.i.d channel assumption, $f(\bar{\mathbf{r}}) = \prod_i f_i(r_i) = \prod_i f(r_i)$. Thus we get,

$$\int_0^a \cdots \int_0^a \prod_i f(r_i) dr_i = \gamma \tag{5.11}$$

Evaluating the integrals for the exponential distribution gives,

$$\gamma = \left(1 - e^{-a/\mu}\right)^N \tag{5.12}$$

Re-writing the above expression gives the result in (5.10). ∎

Observe from (5.10) that $\gamma = 0 \Rightarrow a = 0$ which agrees with the fact that $\gamma = 0$ (no slot for the BE users) implies Z_f is null and similarly, $\gamma = 1 \Rightarrow a \to \infty$ which agrees with the fact that $\gamma = 1$ (all slots for the BE users) implies $Z_f = \mathbb{R}^{+N}$.

Now, using the optimal structure of region Z_i we can obtain an expression for the required throughput rate R in terms of the threshold value a.

***Lemma* 15** *Under the optimal policy, the throughput-rate guarantee, R, for a given threshold value a is given by,*

$$R = \sum_{k=0}^{N-1} \binom{N-1}{k} (-1)^k \left(a + \frac{\mu}{k+1}\right) \frac{e^{-(k+1)a/\mu}}{k+1} \tag{5.13}$$

Proof: Given a threshold vector $\bar{\mathbf{a}} = (a, \ldots, a)$, the region Z_i is given as, $Z_i = \{\bar{\mathbf{r}} : a \leq r_i < \infty, \ 0 \leq r_j \leq r_i, \ j \neq i\}$. As $R = E[r_i I_i]$ we get,

$$R = \int_a^\infty \int_0^{r_i} \cdots \int_0^{r_i} r_i f(r_i) dr_i \prod_{j \neq i} f(r_j) dr_j \tag{5.14}$$

where $f(\bar{\mathbf{r}}) = \prod_i f_i(r_i) = \prod_i f(r_i)$ by the i.i.d assumption. For the exponential distribution, (5.14) simplifies to,

$$R = \int_a^\infty \frac{r_i e^{-r_i/\mu}}{\mu} \left(1 - e^{-r_i/\mu}\right)^{N-1} dr_i \tag{5.15}$$

Using the binomial expansion, $(1 - e^{-r_i/\mu})^{N-1} = \sum_{k=0}^{N-1} \binom{N-1}{k} (-1)^k e^{-kr_i/\mu}$, (5.15) can be solved to get (5.13). ■

Note from (5.13) that R is monotonically decreasing in a, hence there is a one to one relationship between R and a. Stated equivalently, given a certain R value, there is a unique threshold $a \geq 0$ that achieves it. Eliminating a from (5.10) and (5.13) we obtain a unified relationship among the system quantities: (i) Throughput rate R, (ii) Fraction of time-slots, γ, allocated to the BE users (iii) Number of QoS users, N, and (iv) The average channel condition, μ, of the users.

Theorem XVI *Under the model assumptions stated earlier with N QoS users in the system and $\gamma \in [0,1]$ fraction of time-slots allocated to the BE users, the maximum throughput rate R for each QoS user is given by,*

$$\frac{R}{\mu} = \sum_{k=0}^{N-1} \binom{N-1}{k} (-1)^k \left(\frac{-\ln(1-\gamma^{1/N})}{k+1} + \frac{1}{(k+1)^2} \right) (1 - \gamma^{\frac{1}{N}})^{(k+1)} \tag{5.16}$$

Proof: The result follows from Lemmas 14 and 15. ■

From (5.16), we see that R depends linearly on μ, thus as expected, for a given N, γ, the throughput guarantee is higher if μ is increased. Now, re-phrasing (5.16), theoretical limits for various performance measures can be deduced as follows.

Maximum Throughput Rate: By setting $\gamma = 0$, we can obtain the maximum throughput rate $R_{max}(N)$ for each QoS user when no slots are allocated for the BE users. This is given as,

$$R_{max}(N) = \mu \left(\sum_{k=0}^{N-1} \binom{N-1}{k} (-1)^k \frac{1}{(k+1)^2} \right) \tag{5.17}$$

Figure 5-3 is a plot of R/μ versus N for different γ values. The function $R_{max}(N)/\mu$ is the topmost curve corresponding to $\gamma = 0$. As $R_{max}(N)$ is monotonically decreasing in N, its maximum value is at $N = 1$ and equals $R_{max}/\mu = 1$. This is expected as the maximum

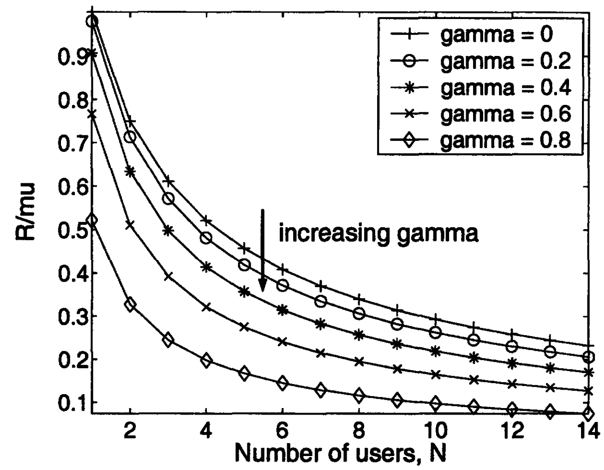


Figure 5-3: Plot of R/μ versus N for the optimal policy for various γ values.

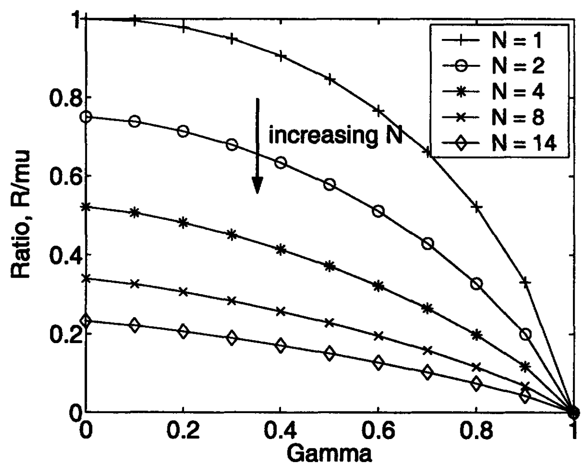


Figure 5-4: Plot of R/μ versus γ for values of $N = 1, 2, 4, 8, 14$.

rate achievable when all the slots are assigned to just one QoS user equals $E[r]$ $(= \mu)$.

Maximum Number of QoS Users: Fix a value of R_0 and γ, the maximum number of QoS users such that throughput of each is at least R_0 is given by,

$$N_{max}(R_0, \gamma) = \max_{N} \; (R \geq R_0) \tag{5.18}$$

Obviously if the values of R_0, γ are such that there is no integer $N \geq 1$ that achieves it, the system values in this case are infeasible. Figure 5-4 is a plot of R/μ versus γ for various values of N. Infeasibility arises when $(\gamma, R_0/\mu)$ point lies above the $N = 1$ curve (in Fig. 5-4).

Maximum Value of γ: Given R and N, the value of γ that solves the equation in

(5.16) gives the maximum fraction of slots that can be allocated to the BE users. Figure 5-4 with its axes inverted gives a plot of γ versus R/μ for different N.

5.4.2 Comparison with Random-scheduling Policy

To understand how much gain can be achieved, we present an analytical comparison of the optimal policy with the random scheduling policy. The random policy assigns a time-slot to the BE users with probability γ and to the QoS users with probability $1-\gamma$. Among the QoS users the slot is then randomly assigned to one of the users with equal probability $1/N$. Clearly, this policy does not exploit the varying channel conditions for scheduling the users. Due to the random nature of the assignment each QoS user gets $(1-\gamma)/N$ fraction of time-slots and since the users have statistically identical channel conditions, the throughput rate of each QoS user, denoted R_r, is given as,

$$R_r = \mu\frac{(1-\gamma)}{N} \tag{5.19}$$

Let us now fix a value of γ for both the optimal and the random policies, i.e. under both policies, γ fraction of slots are assigned to the BE class. Let R^{opt}, R_r denote the corresponding throughput rate provided to each QoS user. Then, as shown below, the gain defined as R^{opt}/R_r is on the order of $\ln(N)$. To show this result, we need the following lemma.

***Lemma* 16** *For any $\gamma \in (0,1)$, we have the following relationship*[5]

$$\ln\left(\frac{1}{1-\gamma^{\frac{1}{N}}}\right) = \Theta(\ln(N)) \tag{5.20}$$

Proof: Appendix D.4 ■

[5]The following notation is followed: (i) $f(N) = O(g(N))$ means that there exists a constant c and integer N_0 such that $f(N) \leq cg(N)$ for all $N > N_0$, (ii) $f(N) = \Theta(g(N))$ means that $f(N) = O(g(N))$ and $g(N) = O(f(N))$

Theorem *XVII The throughput gain of the optimal policy as compared to the random policy, defined as R^{opt}/R_r, for $\gamma \in (0,1)$ satisfies the relationship,*

$$\frac{R^{opt}}{R_r} = \Theta(\ln(N)) \tag{5.21}$$

Proof: Appendix D.5 ■

Observe that as $N \to \infty$ the throughput rate per QoS user for both the optimal and the random policy tends to zero. Equation (5.19) states that R_r decreases as $1/N$ whereas (D.24) in Appendix D.5 states that by using the optimal policy R^{opt} decreases more slowly as $\ln(N)/N$. Hence, we get a gain on the order of $\ln(N)$. The above logarithmic behavior can be attributed to the exponential distribution of the rate under Rayleigh fading and while such channel statistics are simplified models, in practice one could expect gains along these orders for moderate QoS user population.

5.5 Simulation Results

To validate the theoretical results derived in the earlier sections, we present simulation results obtained for two fading distributions, Rayleigh and Nakagami. The setup for the simulations is as follows: we consider a time duration of 10 seconds and divide it into 10,000 slots, thus, each time-slot is of length 1 millisecond. For the sake of simplicity, the QoS users all experience i.i.d channel fading. We assume a linear relationship between the channel rate and the fade state (squared magnitude); i.e. $r \propto |h|^2$. Thus, for Rayleigh fading the rate, r, at which data can be transmitted in a slot is Exponentially distributed with density $f(r) = \frac{e^{-r/\mu}}{\mu}$, $r \geq 0$; while for Nakagami fading, r has a Gamma distribution given as $f(r) = \left(\frac{m}{\mu}\right)^m \frac{r^{m-1}}{\Gamma(m)} e^{-mr/\mu}$, $r \geq 0$, where m is the fading parameter [82]. The mean channel rate, μ, for each user is taken as, $\mu = 800$ Kbits/sec for both the distributions. At each time-slot, a random vector of channel rates for the QoS users is drawn from the respective distribution. Given this channel rate vector, the particular scheduling policy decides which QoS user to serve or to allocate the slot to the BE class. In the former case, the chosen QoS user, say user i, receives a throughput rate of r_i while for the others the throughput rate is 0 in that slot. In the latter case, all QoS users get a 0 throughput in that slot.

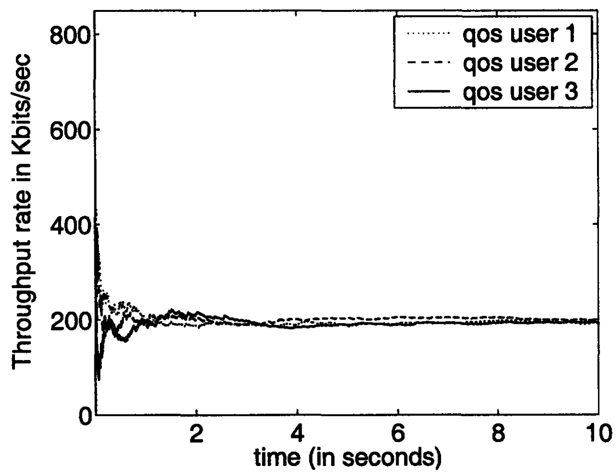


Figure 5-5: Running time-average of throughput rate for Rayleigh fading with 3 QoS users, $R = 200$ Kbits/sec.

We simulate the optimal, the random, the greedy Time Division Multi-Access (TDMA) and an opportunistic scheduling policy studied in [80] which we refer to as "*Opportunistic Proportional Fair*" (OPF) policy. In case of the optimal policy, the scheduling decision is taken as given in (5.9) where the threshold vector $\bar{\mathbf{a}}$ is computed using the formulas in Section 5.4. The random policy makes a scheduling decision as described in Section 5.4.2. For the greedy TDMA and the OPF policy the scheduling decision is taken as follows. Let T_k denote the running time-average of the throughput rate for the k^{th} QoS user. At the beginning of each time-slot, consider all QoS users for which $T_k < R$ where R is the required throughput guarantee. In the greedy TDMA policy the user with the maximum channel rate is selected whereas for the OPF policy the user that maximizes the metric r_k/T_k is selected. If for all QoS users $T_k \geq R$, the slot is allocated to the BE class.

We first numerically validate the theoretical results obtained in Section 5.4. We consider Rayleigh fading with 3 QoS users each having a throughput rate guarantee of $R =$ 200 Kbits/sec. Figure 5-5 gives a plot of the running time-average of throughput rate under the optimal policy. As can be seen from the plot, the long-term required rate is achieved very quickly in time within almost a second and is maintained thereafter within a close range. Thus, within a very short time interval the required throughput rate can be provided to the QoS users. A similar trend is observed when the parameter values are varied. In Figure 5-6, we fix $\gamma = 0.3$, i.e. the BE class is assigned 30% of the slots. The figure gives a plot of the simulated throughput gain R^{opt}/R_r as a function of N; where R^{opt}, R_r is the throughput rate of each QoS user under the optimal and the random policy

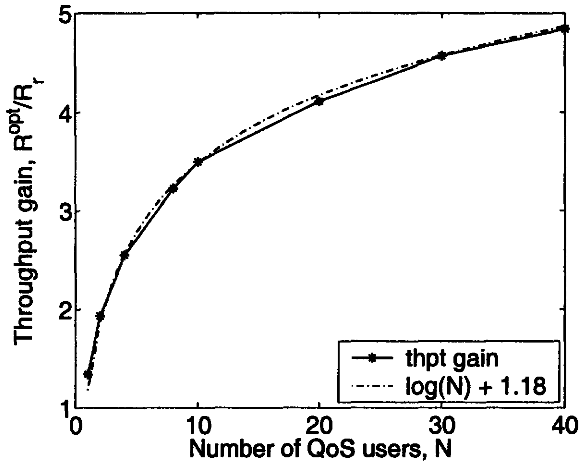


Figure 5-6: Throughput gain, R^{opt}/R_r, for Rayleigh fading with $\gamma = 0.3$.

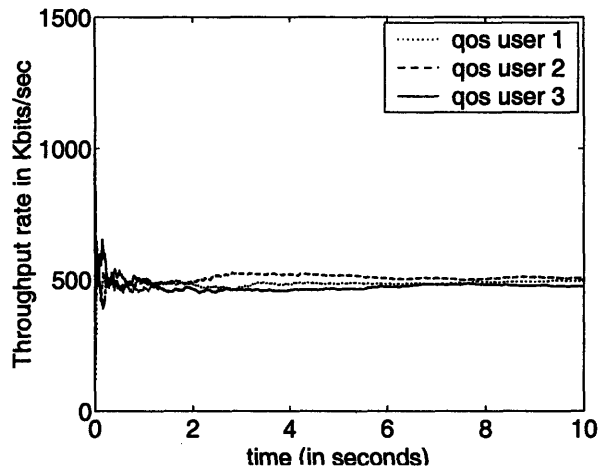


Figure 5-7: Running time-average of throughput rate for Nakagami fading with fade parameter $m = 0.6$, $\gamma = 0.3$ and 3 QoS users.

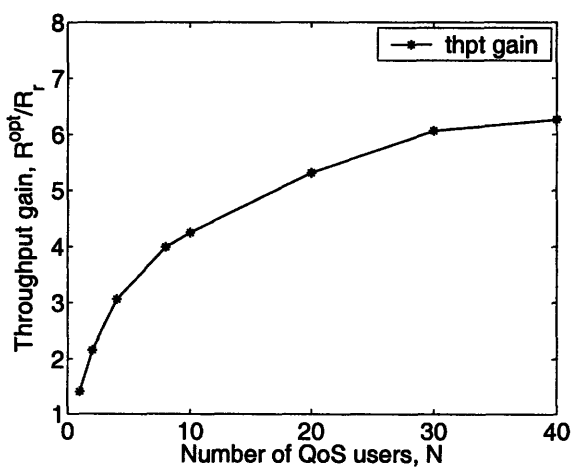


Figure 5-8: Throughput gain, R^{opt}/R_r, for Nakagami fading with fade parameter $m = 0.6$ and $\gamma = 0.3$.

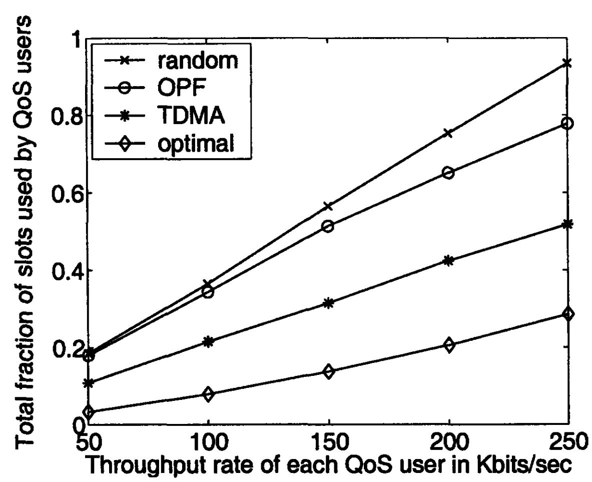


Figure 5-9: Comparison of the fraction of slots utilized by the random, OPF, TDMA and optimal policies.

respectively. In conformation with the result in (5.21), we see from the plot that $\frac{R^{opt}}{R_r}$ grows logarithmic in N. We next consider Nakagami fading with the fading parameter $m = 0.6$. In Figure 5-7, we fix $\gamma = 0.3$ and plot the running time-average of the throughput rate for the optimal policy with 3 QoS users. For the case of Nakagami fading, (5.11) becomes, $\int_0^{\frac{ma}{\mu}} t^{m-1}e^{-t}dt = \gamma^{\frac{1}{N}}\Gamma(m)$ from which the optimal threshold a is evaluated numerically by finding the root of the above non-linear equation. The long-term rate provided to each QoS user in this case is $R = 494$ Kbits/sec. Again as before, the throughput rate is achieved very quickly in time and is maintained thereafter within a close range. In Figure 5-8 we compare the throughput gain of the optimal policy versus the random policy. As seen from the plot the optimal policy achieves a substantial gain in throughput even with Nakagami distribution. In fact, the gain is higher now because the Gamma distribution with $m = 0.6$ has a larger variance than the Exponential with the same mean. As a result, the optimal policy which opportunistically exploits rate variations gives a higher gain in comparison to random assignment.

We now present simulation results that compare the performance of the optimal, random, TDMA and OPF policies. We consider 3 QoS users with Rayleigh fading and the mean channel rate of each QoS user, $\mu = 800$ Kbits/sec. Figure 5-9 plots the total fraction of slots utilized by the QoS users under each policy versus the throughput rate requirement of each QoS user. The quantity, (1− total fraction of slots used by QoS users), is the time-slot allocation to the BE class. First, as expected the random policy has the worst performance and utilizes the maximum time-slots to provide the throughput rate guarantees. Since the

OPF, TDMA and optimal policy exploit the channel variations and opportunistically schedule the users, the time-slot utilization is lower as compared to the random policy. The OPF policy performs worse than the TDMA policy which is expected since the TDMA policy by being greedy has a high throughput per slot and hence utilizes fewer time-slots. Finally, as expected the optimal policy uses a substantially lower fraction of time-slots than all the policies.

5.6 Chapter Summary

In this chapter, we addressed the issue of downlink scheduling over a wireless channel incorporating the QoS and best-effort services. We considered a set of N rate-guaranteed users and the objective was to serve these users with the least time-slot utilization, thereby, maximizing the time-slot allocation to the BE users. Using a geometric approach, we obtained the optimal policy which is given in a compact form as,

$$\Gamma(\bar{\mathbf{r}}) = \begin{cases} Z_f \text{ (serve BE class)}\ , & \text{if } r_i \leq a_i, \forall i = 1, .., N \\ \text{argmax}_i\ \frac{r_i}{a_i}\ , & \text{otherwise} \end{cases} \tag{5.22}$$

Equivalently, the optimal policy also solves the problem of maximizing the rate guarantee for the QoS users given that a certain fraction of time-slots must be allocated to the BE users. Under a symmetric Rayleigh-fading setup, we specialized the optimal policy results to obtain analytical expressions for the various performance metrics. We also presented an analytical comparison with the random-scheduling policy and showed that throughput gains on the order of $\ln(N)$ can be achieved by exploiting multi-user diversity. Finally simulation results showed substantial gains achieved by the optimal policy as compared to other well-known policies in the literature.

Chapter 6

Conclusion

In this thesis, we developed dynamic rate-control and scheduling algorithms to meet quality-of-service requirements on data using minimum resource utilization. Two different setups were addressed, first involving energy-efficient transmission of data with strict packet-deadline constraints, and second, involving multi-user scheduling to provide throughput-rate guarantees with minimum time-slot utilization.

In the first setup, we considered a wireless link model with data packets having strict deadline constraints, and presented dynamic rate-control algorithms to minimize the total transmission energy expenditure. A novel framework based on cumulative curves was presented which provided an intuitive and appealing visualization of the problem and the optimal solution. For the deterministic case where the data arrival information is known in advance, the optimal solution had a neat representation as a "stretched string". Stochastic channel fading was also addressed by first introducing the canonical problem of transmitting B bits of data by deadline T over a Markov fading channel. This problem was referred to as the BT-problem and its optimal solution was obtained in closed-form using a stochastic optimal-control formulation. Various insights from the optimal policy were discussed and an extension involving average power limit was also treated. Further, we utilized the solution of the BT-problem and the cumulative curves framework to obtain optimal solutions under different scenarios involving variable packet deadlines and arrivals. These were obtained through a natural and intuitive decomposition approach.

In the second setup, we considered a wireless down-link model and addressed the issue of multi-user scheduling. Here, the quality-of-service requirement was to provide an average throughput to a certain class of users, referred to as the QoS users, and the objective was to obtain a multi-user scheduling policy that achieves this with the least time-slot utilization. Using a geometric approach, we obtained the optimal policy and further specialized it to the case of symmetric Rayleigh fading to obtain closed-form relationships for the various performance metrics. These relationships provided interesting tradeoffs between the various system parameters as well as facilitated analytical comparison of the optimal policy with the random-scheduling policy.

The formulations considered in this thesis have a wide range of applicability in wireless data, ad-hoc, satellite and sensor networks, especially for applications involving time-constrained data communication. The theoretical approach that we adopted provided simple optimal solutions for the various setups, it facilitated a fundamental understanding of the issues involved and highlighted the interplay between packet deadline constraints and minimizing the transmission energy expenditure. Various analysis methodologies, such as the cumulative curves framework and the stochastic optimal control techniques, also provide unique approaches in addressing other problems in wireless networks. The work in this thesis and the methodologies presented therein, thus, open up new research possibilities into addressing issues related to packet deadlines and quality-of-service for broader communication networks.

Appendix A

Proofs for Chapter 2

A.1 Proof of Theorem II – *Uniqueness*

Let us assume that the admissible departure curve satisfying the optimality criterion, $\tilde{D}(t)$, is not unique. Let $D_1(t)$ and $D_2(t)$ be two such distinct curves. From the boundary conditions we have $D_1(0) = D_2(0) = 0$ and $D_1(T) = D_2(T) = D_{min}(T)$. Since $D_1(t) \not\equiv D_2(t)$ the two curves must differ over some time interval in $[0, T]$. Let $t = a$ be the first instant at which the two curves differ and $t = b$ be the first time instant after $t = a$ at which they are equal again. Note that $b \leq T$ as at time T, $D_1(T) = D_2(T)$. Without loss of generality let $D_1(t) > D_2(t), t \in (a, b)$. From the admissibility of the two curves we have,

$$D_{min}(t) \leq D_2(t) < D_1(t) \leq A(t), \; t \in (a, b) \tag{A.1}$$

By assumption, since both curves $D_1(t)$ and $D_2(t)$ satisfy the optimality criterion, Lemmas 2-4 apply for points of slope changes. As $D_1(t)$ is strictly greater than $D_{min}(t)$ in $t \in (a, b)$ it follows from Lemmas 2-4 that its slope cannot decrease in (a, b). This implies that $D_1(t)$ is convex in (a, b) (it could be linear as well). Similarly as $D_2(t)$ is strictly less than $A(t)$ in $t \in (a, b)$, its slope cannot increase and hence it must be concave in (a, b). It is clear that starting with $D_1(a) = D_2(a)$ and having $D_1(t)$ convex and $D_2(t)$ concave in $t \in (a, b)$, the two curves cannot be equal again at $t = b$ which leads to a contradiction. Finally, if both curves are linear in (a, b) with equality at $t = a$ and $t = b$, then this violates the assumption that $D_1(t) \neq D_2(t), t \in (a, b)$.

To show that $\tilde{D}(t)$ minimizes the energy cost in (2.3), we proceed as follows. Let

$\mathcal{B}$ denote the space of continuous functions defined on $[0,T]$ with the supremum norm, $||f|| = \sup_{t\in[0,T]} f(t)$; this space is then a Banach space [70]. Let Ω denote the set of all admissible departure curves, we then have $\Omega \subset \mathcal{B}$. First, we claim that Ω is a convex set. To see this, consider $D_1(t), D_2(t) \in \Omega$ and let $D_3(t) = xD_1(t) + (1-x)D_2(t),\ x \in [0,1]$. Since $D_1(t), D_2(t)$ are continuous, non-decreasing and have bounded right-derivative, it is easy to see that $D_3(t)$ also has these properties. Further, we also have $xD_{min}(t) \le xD_1(t) \le xA(t)$ and $(1-x)D_{min}(t) \le (1-x)D_2(t) \le (1-x)A(t)$, which gives, $D_{min}(t) \le D_3(t) \le A(t)$; thus, the causality and the QoS constraints are also satisfied. Next, we show that Ω is compact. To see this, consider a sequence of admissible departure curves $\{D_n(t)\}_{n=1}^{\infty}$. Since $D'(t) \le \mathcal{M},\ \forall D(t) \in \Omega$, we have, $|D_n(t_2) - D_n(t_1)| \le \mathcal{M}|t_2 - t_1|$, which makes the sequence of functions $\{D_n(t)\}$ form an equi-continuous family of functions. From [70] (Thm. 7.25, pg. 158), it then follows that there is a subsequence that converges in the supremum norm. Thus, this limit function is continuous and since $D_n(t)$ satisfies the causality and the QoS constraints for all n, it is satisfied by the limit function as well. Hence, the limit function lies in Ω and we see that Ω is compact. Now, consider the energy cost function $\mathcal{E}(D(t))$ as given in (2.3) with $g(\cdot)$ being strictly convex. We next show that $\mathcal{E}(D(t))$ is also strictly convex. Consider $D_1(t), D_2(t) \in \Omega$ and let $D_3(t) = xD_1(t) + (1-x)D_2(t),\ x \in [0,1]$. Then, $\mathcal{E}(D_3(t)) = \int_0^T g(xD_1'(t) + (1-x)D_2'(t))dt < \int_0^T (xg(D_1'(t)) + (1-x)g(D_2'(t)))\,dt$. Thus, we see that, $\mathcal{E}(D_3(t)) < x\mathcal{E}(D_1(t)) + (1-x)\mathcal{E}(D_2(t))$. From above, we see that (2.3) involves an optimization of a strictly convex functional over a compact convex set. Thus, it has a unique minimizer in Ω [66]. From Theorem I, the necessary condition for any admissible departure curve to be the minimizer is that it must satisfy the optimality criterion and since such a curve is unique, it must be the optimal solution.

A.2 Proof of Theorem III – *Minimal Maximum Power*

Consider an admissible departure curve $D(t) \in \Gamma$ that is not optimal. Let $[a,b]$ be the interval over which the optimality criterion is violated. Then, based on the construction in Theorem I we obtain a new curve $\tilde{D}(t)$ that is also admissible. The line segment $L(t)$ between $[a,b]$ in $\tilde{D}(t)$ always has a slope that is less than the maximum slope of $D(t)$ between $[a,b)$. As $\tilde{D}(t) = D(t), t \notin (a,b)$, the overall maximum slope of $\tilde{D}(t)$ cannot exceed that of $D(t)$.

$$\max_{t \in [0,T)} \tilde{D}'(t) \leq \max_{t \in [0,T)} D'(t) \tag{A.2}$$

If $\tilde{D}(t) = D^{opt}(t)$ then we are done. If not then repeat the process for $\tilde{D}(t)$ now. For $g(.)$ strictly convex, the energy expenditure strictly decreases at each iteration. Thus, we obtain a sequence of curves with decreasing energy metric that is lower bounded by the optimal cost. As the optimal curve that achieves the lower bound is unique, it follows that the above sequence eventually converges to $D^{opt}(t)$. The result then follows from a repeated application of (A.2).

A.3 Proof of Lemma 5

Take two lines $L_1(t)$, $L_2(t) \in \mathcal{F}$ with slopes s_1, s_2 respectively. Without loss of generality, let $s_1 > s_2$. Let ϵ_1 and ϵ_2 be the respective durations over which they are admissible. Take $\epsilon = \min[\epsilon_1, \epsilon_2]$, then, over $[t_0, t_0 + \epsilon)$ we can view $L_1(t)$, $L_2(t)$ equivalently as new $A(t)$ and $D_{min}(t)$ respectively. Any line with slope s such that $s_2 \leq s \leq s_1$ is then admissible for duration ϵ and hence belongs to $\mathcal{F}$.

A.4 Proof of Lemma 7

(a) Let $\tilde{t}$ be the point at which $L_D(t)$ intersects $D_{min}(t)$ first. By definition, $L_D(t) < A(t), \forall t \in (t_0, \tilde{t})$. The proof now follows in two parts. First, we show that any line in $\mathcal{F}$ with slope less than L'_D must intersect $D_{min}(t)$ at or before $\tilde{t}$ and second that this line does not intersect $A(t)$ in $(t_0, \tilde{t})$. Consider $L(t) \in \mathcal{F}$ with slope less than L'_D, then, $L(t) < L_D(t), \forall t > t_0$. Hence, at time $\tilde{t}$ we have $L(\tilde{t}) < L_D(\tilde{t}) = D_{min}(\tilde{t})$. If instead, $\tilde{t}$ is the discontinuity point for $D_{min}(t)$, then, $L_D(t) - D_{min}(t)$ changes sign at $\tilde{t}$ and so $L(t) - D_{min}(t)$ must have changed sign earlier at $t \leq \tilde{t}$. Thus, we see that $L(t)$ must intersect $D_{min}(t)$ at or before $\tilde{t}$. Next, since $L(t) < L_D(t) < A(t)$ in $t \in (t_0, \tilde{t})$, the line $L(t)$ cannot intersect $A(t)$ first. This completes the proof of part (a) in the lemma. Along similar lines as above part (b) follows.

A.5 Proof of optimality of the algorithm

From Theorem II we know that $D^{opt}(t)$ is unique. Hence it suffices to prove that (a) at every iteration one of the steps of the algorithm is satisfied and (b) the constructed curve satisfies the optimality criterion.

Proof of claim (a): At every admissible point (t_0, α), β_0 is defined as given in (2.16). Line L_0 is either tangent to $A(t)$ $(or D_{min}(t))$ or not. If it is not tangent then by Lemma 6 it must intersect either $A(t)$ or $D_{min}(t)$ first, and step (2) of the algorithm is then satisfied. If L_0 is the tangent, step (3) is followed. Finally, the new point (t_1, γ) obtained from the algorithm is also admissible.

Proof of claim (b): Let $D_c(t)$ denote the constructed curve. It is obvious from the construction that at all points where the slope changes Lemma 2 is satisfied. We next show that Lemmas 3 and 4 are also satisfied. Let t_0 be the starting instant at some iteration. Let step 2 be satisfied at t_0, then, the sets $\mathcal{F}_{D_m}$ and $\mathcal{F}_A$ are non-empty. Suppose L_o intersects $D_{min}(t)$ first, i.e. at t_1 (as in the algorithm) we have $L_o(t_1) = D_{min}(t_1)$. Also, suppose that $L_o(t_1) \neq A(t_1^-)$. From the chosen t_1 in step 2, it is clear that $L_o(t) < A(t)$ in $(t_0, t_1]$. Thus, if we pick a line $L_1 \in F_A$ with slope close to L_o' $(= \beta_o)$, then L_1 would intersect $A(t)$ beyond t_1. More precisely, there exists an $\epsilon > 0$ such that $L_1 \in F_A$ with slope $\beta_o < L_1' < \beta_o + \epsilon$ intersects $A(t)$ first at $\tilde{t} > t_1$. Now, it follows that at the next iteration, starting from time t_1, the new set $\mathcal{F}_A$ must at least contain all lines with slopes in $(\beta_o, \beta_o + \epsilon)$, hence, the optimal line starting at time t_1 cannot have slope greater than β_o (β_o here refers to the optimal slope for the iteration at t_0). Thus, we see that Lemma 3 is satisfied at t_1.

Similarly, if at t_0 step 2 is satisfied but t_1 is such that we have $L_o(t_1) = A(t_1)(or\ A(t_1^-))$, then, using a similar argument as above it can be seen that starting from time t_1, the new set $\mathcal{F}_{D_m}$ must at least contain all lines with slopes in $(\beta_o, \beta_o - \epsilon)$. Hence, the optimal line starting at time t_1 cannot have slope less than β_o and now Lemma 4 is satisfied at t_1. Note, that if at t_1 we have $L_o(t_1) = D_{min}(t_1) = A(t_1^-)$, it does not matter how the slope changes beyond t_1.

Now, suppose instead that step 3 is satisfied at t_0 then L_o is tangent to $D_{min}(t)(or A(t))$. If $L_o(t)$ is tangent to $D_{min}(t)$, over $t \in [t_0, t_1]$ we have $L_o(t) = D_{min}(t)$. We claim that the curve $D_{min}(t)$ over $t \in [t_0, t_1]$ must be concave. To see this, suppose instead that there exists $[\tilde{t}_1, \tilde{t}_2]$, $t_0 \leq \tilde{t}_1, \tilde{t}_2 \leq t_1$, such that the curve $D_{min}(t)$ is strictly convex. Then, at $\tilde{t}_1$ the tangent cannot be the optimal line; this is because, a line with slope greater than the tangent will be in the set $\mathcal{F}_{D_m}$ for point $\tilde{t}_1$ and the tangent slope would not be the maximum over $\mathcal{F}_{D_m}$. Thus, $L_o(t)$ must be concave over $t \in [t_0, t_1]$ which shows that Lemma 3 is satisfied over the entire interval $[t_0, t_1]$. Furthermore, using the argument from the preceding paragraphs it also follows that on the next iteration, starting at t_1 Lemma 3 will be satisfied. Using an analogous argument as above, it is easy to see that if $L_o(t)$ is tangent to $A(t)$, it is convex over $t \in [t_0, t_1]$ and Lemma 4 is satisfied.

Thus, from the above, we see that starting at $(0,0)$, at every iteration of the algorithm (every constructed segment of $D_c(t)$) Lemmas 2-4 are satisfied. This implies that around every point where the slope of $D_c(t)$ changes we cannot construct an admissible line segment, hence, $D_c(t)$ satisfies the optimality criterion.

A.6 Algorithm for constructing $D^{opt}(t)$ when $A(t)$ and $D_{min}(t)$ are piecewise constant functions

Consider $A(t)$ and $D_{min}(t)$ as piecewise-constant functions. Let the arrival curve $A(t)$ be denoted as,

$$A(t) \equiv \{t^i, A^i\}_{i=1}^{N}\ ,\quad 0 < t^1 < t^2 < \ldots < t^N = T \tag{A.3}$$

where $\{t^i\}_{i=1}^{N-1}$ are the jump points of the curve $A(t)$, A^i is the cumulative data just before time t^i (the value of $A(t)$ just before the jump), t^N is taken as the final time T and A^N is the value at T. Similarly, denote the $D_{min}(t)$ curve using subscripts as,

$$D_{min}(t) \equiv \{t_j, B_j\}_{j=1}^{M}\ ,\quad 0 < t_1 < t_2 < \ldots < t_M = T \tag{A.4}$$

where $\{t_j\}_{j=1}^{M}$ are the jump points of the curve $D_{min}(t)$, B_j is the minimum data that must depart by time t_j (the value of $D_{min}(t)$ at the jump) and t_M is taken as the final time T. By the assumption $D_{min}(T) = A(T)$, we have, $A^N = B_M$.

Arrange the jump times $\{t^i\}_{i=1}^{N}$ and $\{t_j\}_{j=1}^{M}$ of $A(t)$ and $D_{min}(t)$ respectively in increasing order with ties broken arbitrarily. Denote this arranged sequence as $\{\pi_p\}_{p=1}^{N+M}$. For any p, if π_p is a jump point of $A(t)$, denote $i(p)$ as the corresponding index of this jump point within $A(t)$ curve. Similarly, if π_p is a jump point of $D_{min}(t)$, denote $j(p)$ as the corresponding index of this jump point within $D_{min}(t)$ curve. Denote the output vector of slopes and times as $\bar{\mathbf{s}}$ and $\bar{\mathbf{t}}$ respectively. Vector $\bar{\mathbf{s}}$ contains the slopes of the piecewise-linear segments of $D^{opt}(t)$ curve and $\bar{\mathbf{t}}$ contains the times when the last segment stops and the new segment starts, i.e. the l^{th} line segment has slope $s(l)$ (l^{th} entry) and is over time interval $[t(l-1), t(l)]$, where we take $t(0) = 0$.

The following algorithm obtains the vectors $\bar{\mathbf{s}}$ and $\bar{\mathbf{t}}$ by looking at the jump points $\{\pi_p\}_{p=1}^{N+M}$ sequentially. Explanations about the various steps are given later. To begin, $\bar{\mathbf{s}}$ and $\bar{\mathbf{t}}$ are taken as empty vectors.

1. Initialize $p = 1$, $t_s = 0$, $V_s = 0$, $r^A = \infty$ (or some large value), $r_D = 0$, $I^A = I_D = I^\pi = I_\pi = 0$.

2. If π_p is a jump point of $A(t)$ curve and $\frac{A^{i(p)}-V_s}{t^{i(p)}-t_s} \leq r^A$, Update: $I^A = i(p)$, $I^\pi = p$, $r^A = \frac{A^{i(p)}-V_s}{t^{i(p)}-t_s}$.

   If π_p is a jump point of $D_{min}(t)$ and $\frac{B_{j(p)}-V_s}{t_{j(p)}-t_s} \geq r_D$, Update: $I_D = j(p)$, $I_\pi = p$, $r_D = \frac{B_{j(p)}-V_s}{t_{j(p)}-t_s}$.

3. If $r_D \geq r_A$ and π_p is a jump point of $A(t)$ curve, append r_D to $\bar{\mathbf{s}}$ and t_{I_D} to $\bar{\mathbf{t}}$, Update: $t_s = t_{I_D}$, $V_s = B_{I_D}$, $p = I_\pi$, $r^A = \infty$, $r_D = 0$.

   If $r_D \geq r_A$ and π_p is a jump point of $D_{min}(t)$ curve, append r^A to $\bar{\mathbf{s}}$ and t_{I^A} to $\bar{\mathbf{t}}$, Update: $t_s = t^{I^A}$, $V_s = A^{I^A}$, $p = I^\pi$, $r^A = \infty$, $r_D = 0$.

4. Increment $p \equiv p + 1$ and repeat steps 2-4. Stop if $p = N + M + 1$.

The variables (t_s, V_s) represent the start time and value respectively at each recursive step of the algorithm (the origin point) and initialized to $(0, 0)$. The variable r^A keeps track of the minimum slope value, among straight line segments between (t_s, V_s) and (t^i, A^i) (the jump points of $A(t)$ curve); it is initialized to a high value for computational purposes. The variables I^A, I^π keep track of the indices involved for the minimizing point. The corresponding variables for $D_{min}(t)$ are r_D which keeps track of the maximizing slope over the jump points of $D_{min}(t)$ curve and I_D, I_π are the corresponding indices.

In Step 2, depending on whether the jump point π_p belongs to $A(t)$ or $D_{min}(t)$ curve the corresponding variables are updated. In Step 3, the condition $r_D \geq r^A$ is checked. If true and if the present jump point π_p belongs to $A(t)$ curve, it implies that the line segments with slopes lying in $[r^A, r_D]$ intersect the $D_{min}(t)$ curve first; whereas, line segments with slope greater than r_D intersect the $A(t)$ curve first. Thus, from (2.16) the optimal slope β_o is r_D

and hence the line segment with this slope is appended to the optimal curve constructed so far. Similarly, if $r_D \geq r^A$ is true and if the present jump point π_p belongs to $D_{min}(t)$ curve, it implies that the line segments with slope lying in $[r^A, r_D]$ intersect the $A(t)$ curve first; whereas, line segments with slope less than r^A intersect the $D_{min}(t)$ curve first. Thus, here the optimal slope β_o is r^A and the line segment with this slope is appended to the optimal curve. Finally, in Step 3, the variables p and (t_s, V_s) are also updated to represent the endpoint of the chosen line segment. The algorithm then repeats in a similar manner starting from the new point (t_s, V_s).

A.7 Proof of Lemma 8

As presented in Section 2.3.2, the two curves $A(t)$ and $D_{min}(t)$ for the BT-problem are, $A(t) = B$, $\forall t \in [0,T]$, and $D_{min}(t) = 0, t \in [0,T)$, $D_{min}(T) = B$. The admissibility criterion is $0 \leq D(t) \leq B$ and $D(T) = B$. Re-phrasing the BT-problem as a calculus of variations problem we get [71],

$$\begin{aligned} \min_{r(t)} \quad & \mathcal{E}(D(t)) = \int_0^T g(r(t),t)dt \\ \text{subject to} \quad & D'(t) = r(t), \;\; D(T) = B \\ & r(t) \geq 0, \;\; t \in [0,T] \end{aligned} \tag{A.5}$$

Uisng [71], the Hamiltonian for the above is, $H(D,r,t) = g(r,t) + \lambda(t)r$, and from Pontryagin's maximum principle (which is also a sufficient condition in our case due to convexity) the optimal value $r^{opt}(t)$ satisfies, $r^{opt}(t) = \arg\max_{r\geq 0} H(D^{opt}, r, t) = \arg\max_{r\geq 0}(g(r,t) + \lambda(t)r)$. We also have, $\dot{\lambda}(t) = -\frac{\partial H}{\partial D} = 0$, which implies $\lambda(t) =$ constant. Taking $k = -\lambda(t)$ as the constant and substituting back in the $r^{opt}(t)$ equation, we get, $r^{opt}(t) = \arg\max_{r\geq 0}(g(r,t) - kr)$. The solution to this maximization is as given in (2.18). Since, $g(r,t)$ is strictly convex and increasing in r, we have that r^* is unique. Finally, to satisfy the deadline constraint we require that the value of k must be such that $\int_0^T r^{opt}(t)dt = B$.

Appendix B

Proofs for Chapter 3

B.1 Verification Theorem for the BT-problem in Section 3.3

In Section 3.3, we obtained heuristically the optimality equation as given by (3.15). To present a rigorous argument we need to verify that a solution of (3.15), i.e. functional forms $J(x, c, t)$ and $r^*(x, c, t)$ that satisfy (3.15) with the required boundary conditions, indeed give the optimal solution for the BT-problem. However, the standard verification theorems in [63] that provide conditions to check for the optimality of the solution to the HJB equation do not directly apply for the BT-problem. This is because the non-standard boundary condition $x(T) = 0$ leads to a singularity in $J(x, c, t)$ at $t = T$ (since, $J(x, c, t) \xrightarrow{(t \to T)} \infty$, if $x > 0$). To overcome this technical difficulty and obtain a verification theorem for the BT-problem, we consider a particular relaxation and take appropriate limits as discussed next.

Consider the following modification to the problem. Instead of emptying the buffer by time T, extend the deadline to $T + \tau_k$ for some $\tau_k > 0$. In the interval $[T, T + \tau_k]$ the channel does not change and whatever data, $x(T)$, left at time T is transmitted out at the constant rate $x(T)/\tau_k$. Thus, now the system runs over time $[0, T]$ and the data left at T has a terminal energy cost of emptying it in the next τ_k interval. This terminal cost is given as,

$$h_k(x(T), c(T)) = \frac{\tau_k \, g\left(\frac{x(T)}{\tau_k}\right)}{c(T)} \tag{B.1}$$

We now consider a sequence $\{\tau_k\}_1^\infty$ such that $\tau_k \downarrow 0$. This gives a sequence of modified problems which we denote as $\{\mathcal{P}_k\}$ and the corresponding minimum-cost functions are denoted as $\{J^k(x,c,t)\}$.

Note that the relaxation does not change the system dynamics over time but only affects the terminal cost applied to the leftover data at time T. In the BT-problem, we had an infinite cost on any data left at T but now each problem $\{\mathcal{P}_k\}$ has a smooth function $h_k(x(T),c(T))$ associated with it. Clearly, then, the optimality equation for each $\mathcal{P}_k$ is the same as (3.15) except that the boundary conditions for the PDE now become $J^k(0,c,t) = 0$ and $J^k(x,c,T) = h_k(x,c)$. The admissibility of a policy for problem $\mathcal{P}_k$ includes the constraints required for the BT-problem with the exception of $x(T) = 0$ which is no longer a necessary requirement. Furthermore, from the increasing and convexity properties of $g(r)$, it is easy to see that for a fixed (x,c), $h_k(x,c) = 0$, if $x = 0, \forall k$ and $h_k(x,c) \xrightarrow{(k\to\infty)} \infty$, if $x > 0$. Thus, as we look at the modified problems $\mathcal{P}_k$ with large values of k (smaller values of τ_k), there is an increasingly higher penalty cost applied to the data left at time T. And as $k \to \infty$, this penalty cost goes to infinity; thus, in the limit we have a situation equivalent to the BT-problem. The rest of the proof delves into the technical details involved in taking the limits. Specifically, we show that having obtained the optimal cost function for the modified problem $\mathcal{P}_k$ and then taking the limit $k \to \infty$ gives the optimal solution for the BT-problem.

We will use the notation $\tilde{\Gamma}$ to denote the set of all admissible policies for problem $\mathcal{P}_k$ (note that for all $\mathcal{P}_k$, the set $\tilde{\Gamma}$ is the same since the problems only differ in the terminal cost function $h_k(\cdot,\cdot)$). The cost-to-go function for a policy $r(\cdot)$ for problem $\mathcal{P}_k$ will be denoted as $J_r^k(x,c,t)$; i.e. $J_r^k(x,c,t) = E\left[\int_t^T \frac{g(r(x(s),c(s),s))}{c(s)}ds + h_k(x(T),c(T))\right]$. We start with Lemma 17 which gives the verification result for problem $\mathcal{P}_k$. It states that a solution of the PDE equation (3.15) satisfying the relevant boundary conditions indeed gives the minimum cost function and that the transmission policy obtained from the minimizing r in (3.15) is the optimal policy.

***Lemma* 17 (Verification Result for $\mathcal{P}_k$)** *Let $J^k(x,c,t)$ defined on $[0,B] \times \mathcal{C} \times [0,T]$, solve the equation in (3.15) with the boundary conditions $J^k(0,c,t) = 0, \forall c \in \mathcal{C}, t \in [0,T)$ and $J^k(x,c,T) = h_k(x,c)$. Let $r_k^*(x,c,t)$ be an admissible policy for $\mathcal{P}_k$ such that r_k^* is the minimizing value of r in (3.15) for $J^k(x,c,t)$. Then,*

1. $J^k(x,c,t) \leq J_r^k(x,c,t), \ \forall\, r(\cdot) \in \tilde{\Gamma}$

2. $r_k^*(x,c,t)$ *is an optimal policy,* $J^k(x,c,t)$ *is the minimum cost function and,*

$$J^k(x,c,t) = E\left[\int_t^T \frac{g(r_k^*(x(s),c(s),s))}{c(s)}ds + h_k(x(T),c(T))\right] \tag{B.2}$$

Proof: See [63], Chap III, Theorem 8.1. ■

Now, define the function $J(x,c,t) \triangleq \lim_{k\to\infty} J^k(x,c,t)$. The next theorem shows that this limit exists and if it satisfies (3.15), it is the optimal solution for the BT-problem. We will use the notation Γ in the theorem to denote the set of all admissible policies for the BT-problem.

<u>Theorem</u> XVIII (Verification Thm. for the BT-problem) *Consider* $(x,c,t) \in [0,B] \times \mathcal{C} \times [0,T)$ *and define* $J(x,c,t) \triangleq \lim_{k\to\infty} J^k(x,c,t)$. *Let* $J(x,c,t)$ *satisfy the HJB equation in (3.15) and* $r^*(x,c,t)$ *be an admissible policy for the BT-problem such that* r^* *is the minimizing value of* r *in (3.15) for* $J(x,c,t)$. *Then,*

1. $J(x,c,t) \le J_r(x,c,t), \ \forall\, r(\cdot) \in \Gamma$

2. $r^*(x,c,t)$ *is the optimal policy,* $J(x,c,t)$ *is the minimum cost function and,*

$$J(x,c,t) = E\left[\int_t^T \frac{1}{c(s)} g(r^*(x(s),c(s),s))ds\right] \tag{B.3}$$

Proof: We divide the proof into various steps each giving arguments for the various claims in the theorem statement.

Step 1: <u>*The limit,* $J(x,c,t) = \lim_{k\to\infty} J^k(x,c,t)$ *exists and is finite*</u>

Consider the relaxed problem $\mathcal{P}_k$ and the corresponding minimum cost function $J^k(x,c,t)$. We now make two claims, first that $J^k(x,c,t)$ is non-decreasing in k for each (x,c,t) and second that $J^k(x,c,t)$ is bounded for all k. These two claims are proved as follows. First, note that the sequence τ_k is decreasing and hence $h_k(x,c)$ is monotonically point-wise increasing in x with increasing k. Fix an admissible policy $r(\cdot) \in \tilde{\Gamma}$, then for every channel sample path the total energy cost is higher as k increases because the terminal cost is higher. Hence, for all $r(\cdot) \in \tilde{\Gamma}$ the expected energy cost increases with k; taking the infimum over $r(\cdot)$ proves the first claim. To prove the second claim consider a simple policy, $\pi(\cdot)$, that empties the data at a constant rate by time $\tilde{t}$, where $t < \tilde{t} < T$. For such a policy,

$$\begin{aligned} J_\pi(x,c,t) &= E\left[\int_t^{\tilde{t}} \frac{g(x/(\tilde{t}-t))}{c(s)} ds\right] = g\left(\frac{x}{\tilde{t}-t}\right) E\left[\int_t^{\tilde{t}} \frac{1}{c(s)} ds\right] \\ &\leq g\left(\frac{x}{\tilde{t}-t}\right) \sum_{j=0}^{\infty} \left(\frac{(\tilde{t}-t)}{c(z_l)^j}\right) \frac{\left(\lambda(\tilde{t}-t)\right)^j e^{-\lambda(\tilde{t}-t)}}{j!} \\ &= \frac{(\tilde{t}-t)}{c} g\left(\frac{x}{\tilde{t}-t}\right) e^{\frac{\lambda(\tilde{t}-t)}{z_l}} e^{-\lambda(\tilde{t}-t)} < \infty \end{aligned}$$

The inequality above follows by first conditioning that the channel makes j transitions over $[t, \tilde{t}]$, taking $c(s) = (z_l)^j c$, where $(z_l)^j c$ is the worst possible channel quality starting with state c and making j transitions, and finally taking expectation with respect to j (number of transitions, j, is Poisson distributed with rate $\lambda(\tilde{t}-t)$ and $z_l > 0$ is the least value that any $Z(c)$ can take). Since, $J^k(x,c,t) \leq J_\pi(x,c,t), \forall k$, the bounded-ness claim follows. Combining the above two claims (non-decreasing and bounded), we see that the point-wise limit $J(x,c,t) = \lim_{k\to\infty} J^k(x,c,t)$ exists.

Step 2: *Result 1 stated in the theorem, i.e. $J(x,c,t) \leq J_r(x,c,t)$, $\forall$ BT-admissible policies*

From the notation considered, Γ denotes the set of admissible policies for the BT-problem and $\tilde{\Gamma}$ the set of admissible policies for problems $\{\mathcal{P}_k\}$. We have $\Gamma \subset \tilde{\Gamma}$ because a policy that empties the data by the deadline is clearly an admissible policy for the modified problems $\{\mathcal{P}_k\}$ in which case such a policy simply incurs zero terminal energy cost. Thus, for all $r(\cdot) \in \Gamma$, $x(T) = 0$ and the terminal energy cost is zero. This gives for all k,

$$J_r^k(x,c,t) = J_r(x,c,t),\ \forall\, r(\cdot) \in \Gamma \tag{B.4}$$

where $J_r(\cdot)$ above is defined in (3.5). From Lemma 17 we know that,

$$J^k(x,c,t) \leq J_r^k(x,c,t),\ \forall\, r(\cdot) \in \tilde{\Gamma} \supset \Gamma \tag{B.5}$$

Thus from (B.4) and (B.5) we have,

$$J^k(x,c,t) \leq J_r(x,c,t),\ \forall\, r(\cdot) \in \Gamma \tag{B.6}$$

Since the above inequality holds for all k, taking limits we get,

$$J(x,c,t) \triangleq \lim_{k\to\infty} J^k(x,c,t) \le J_r(x,c,t),\ \forall\, r(\cdot) \in \Gamma \tag{B.7}$$

Step 3: Result 2 stated in the theorem

From the theorem statement, we know that $J(x,c,t)$ satisfies (3.15) and $r^*(x,c,t)$ is an admissible policy for the BT-problem. Now, using Dynkin's formula, [63], on $J(x,c,t)$ for policy $r^*(\cdot)$ we get $\forall\, \tau,\ t < \tau < T$,

$$\begin{aligned}
J(x,c,t) &= EJ(x_\tau, c_\tau, \tau) - E\int_t^\tau A^{r^*} J(x_s, c_s, s)ds && \text{(B.8)}\\
&= EJ(x_\tau, c_\tau, \tau) + E\int_t^\tau \frac{g(r^*(x_s, c_s, s))}{c_s} ds && \text{(B.9)}\\
&\ge E\int_t^\tau \frac{g(r^*(x_s, c_s, s))}{c_s} ds && \text{(B.10)}
\end{aligned}$$

where we have used x_s, c_s as short-hand notations for $x(s)$ and $c(s)$ respectively. The equality in (B.9) follows since r^* is the minimizing value in (3.14) which gives $A^{r^*}J(x,c,t) + \frac{1}{c}g(r^*) = 0$ or equivalently $\frac{1}{c_s}g(r^*(x_s,c_s,s)) = -A^{r^*}J(x_s,c_s,s)$. The inequality in (B.10) follows since $J(\cdot)$ is non-negative. Since the above holds for all $\tau < T$, taking limits and using the Monotone Convergence theorem we get,

$$J(x,c,t) \ge E\int_t^T \frac{g(r^*(x_s, c_s, s))}{c_s} ds \tag{B.11}$$

Combining the above inequality with that in (B.7) shows that we have equality for policy $r^*(x,c,t)$, i.e. $J(x,c,t) = E\int_t^T \frac{g(r^*(x_s,c_s,s))}{c_s} ds$. This completes the proof that $J(x,c,t)$ is the minimum cost function and $r^*(\cdot)$ is the optimal policy. ■

B.2 Proof of Theorem VI – *BT-problem*

To prove optimality, we check all the conditions required in the verification results of Appendix B.1 and proceed as follows. We first consider the relaxed problem $\mathcal{P}_k$ and obtain the optimal solution by verifying the conditions in Lemma 17. Then, we take the limit $k \to \infty$ and check the conditions required in Theorem XVIII. These limits give us the optimal solution for the BT-problem.

Step 1: *Optimal solution for the modified problem $\mathcal{P}_k$*

Let us suppose that the functional form for the optimal rate $r_k^*(x,c,t)$ is given as,

$$r_k^*(x,c^i,t) = \frac{x}{f_i^k(T-t)}, \quad i = 1,\dots,m \tag{B.12}$$

Assuming this functional form we now obtain the minimum cost function $J^k(x,c,t)$. To proceed, note that $r_k^*(\cdot)$ must be the minimizing value of r in (3.15). Thus, using the first-order condition for the minimization (i.e. first derivative with respect to r equal to zero) we get, $\forall(x,c,t) \in (0,B] \times \mathcal{C} \times [0,T)$,

$$\frac{\partial}{\partial r}\left(\frac{g(r)}{c^i} + \frac{\partial J^k(x,c^i,t)}{\partial t} - r\frac{\partial J^k(x,c^i,t)}{\partial x} + \lambda(E_z[J^k(x,Z(c^i)c^i,t)] - J^k(x,c^i,t))\right)\Big|_{r_k^*} = 0 \tag{B.13}$$

This gives, $\frac{\partial J^k(x,c^i,t)}{\partial x} = \frac{g'(r_k^*(x,c^i,t))}{c^i}$ and upon integration with the boundary condition $J^k(0,c^i,t) = 0$, we get,

$$J^k(x,c^i,t) = \frac{x^n}{c^i(f_i^k(T-t))^{n-1}}, \quad i = 1,\dots,m \tag{B.14}$$

In order for the functional forms in (B.12) and (B.14) to be the optimal solution we need to satisfy the conditions in Lemma 17.

– First, the boundary condition $J^k(x,c^i,T) = h_k(x,c^i) = \frac{(x/\tau_k)^n \tau_k}{c^i}$, requires,

$$f_i^k(0) = \tau_k, \quad \forall i = 1,\dots,m \tag{B.15}$$

The other boundary condition $J^k(0,c^i,t) = 0$, $\forall i,t$, is also satisfied as can be easily checked.

– Second, $J^k(\cdot)$ and $r_k^*(\cdot)$ must solve the PDE equation in (3.15) for all values of the system state $(x,c,t) \in ([0,\infty) \times (c_1,\dots,c_m) \times [0,T))$. That is, we require,

$$\frac{g(r_k^*(x,c^i,t))}{c^i} + \frac{\partial J^k(x,c^i,t)}{\partial t} - r_k^*(x,c^i,t)\frac{\partial J^k(x,c^i,t)}{\partial x} + \lambda(E_z[J^k(x,Z(c^i)c^i,t)] - J^k(x,c^i,t)) = 0 \tag{B.16}$$

Substituting (B.12) and (B.14) in the equation above, we get,

$$\begin{aligned}\frac{x^n}{c^i(f_i^k(T-t))^n}+\frac{-x^n(1-n)(f_i^k)'(T-t)}{c^i(f_i^k(T-t))^n}-\frac{x}{f_i^k(T-t)}\frac{nx^{n-1}}{c^i(f_i^k(T-t))^{n-1}}&\\ +\lambda\sum_{j=1}^{m}\frac{p_{ij}}{z_{ij}c^i}\frac{x^n}{(f_j^k(T-t))^{n-1}}-\lambda\frac{x^n}{c^i(f_i^k(T-t))^{n-1}}&=0\end{aligned}$$

Cancelling out $\frac{x^n}{c^i}$, simplifying the above and setting $s=T-t$ gives the ODE system,

$$(f_i^k)'(s)=1+\frac{\lambda f_i^k(s)}{n-1}-\frac{\lambda}{n-1}\sum_{j=1}^{m}\frac{p_{ij}}{z_{ij}}\frac{(f_i^k(s))^n}{(f_j^k(s))^{n-1}},\quad \forall i=1,\ldots,m \tag{B.17}$$

Thus, from above we see that for $r_k^*(\cdot)$ and $J^k(\cdot)$, as given in (B.12) and (B.14), to satisfy the optimality PDE equation in (3.15) we require that the functions $\{f_i^k(s)\}_{i=1}^m$ satisfy the above ODE system with the boundary conditions in (B.15). The question that remains is whether a set of positive functions exist that solve the ODE system in (B.17). The following lemma shows that indeed such a set exists and also that these functions are unique.

<u>Lemma</u> 18 (Existence and Uniqueness of the ODE solution in (B.17)) *The ODE system in (B.17) with the boundary conditions $f_i^k(0)=\tau_k$, $\forall i$, $\tau_k>0$, has a unique positive solution for $s\in[0,T]$.*

Proof: See Appendix B.3. ■

Thus, we see that $J^k(x,c^i,t)$ as given in (B.14) solves (3.15) with the minimizing rate function $r_k^*(x,c^i,t)$ as given in (B.12). This rate function is a valid transmission policy as it satisfies all the admissibility requirements for problem $\mathcal{P}_k$: that is, $r^*\geq 0$, ($r^*=0$ for $x=0$), $r_k^*(x,c,t)$ is locally-Lipschitz-continuous in x and continuous in t. From Lemma 17, it then follows that $J^k(x,c^i,t)$ and $r_k^*(x,c^i,t)$ are the optimal solution for problem $\mathcal{P}_k$.

Step 2: <u>*Optimal solution for the BT-problem (taking* $\lim_{k\to\infty}$ *in the Step 1 results)*</u>

Consider $(x,c,t)\in[0,B]\times\mathcal{C}\times[0,T)$ and the limit $J(x,c,t)=\lim_{k\to\infty}J^k(x,c,t)$. From Theorem XVIII we know that this limit exists and using (B.14) we obtain,

$$J(x,c^i,t)=\lim_{k\to\infty}J^k(x,c^i,t)=\frac{x^n}{c^i(f_i(T-t))^{n-1}},\ i=1,\ldots,m \tag{B.18}$$

where we define,

$$f_i(s) \triangleq \lim_{k\to\infty} f_i^k(s), \quad s \in [0,T], \ \forall i \tag{B.19}$$

Note that since the limit function $J(x, c^i, t)$ is positive and bounded for all $t \in [0, T)$, the function $f_i(s)$ is positive for all $s > 0$.

For optimality we now check the conditions required in Theorem XVIII. First, we need to show that $J(\cdot)$ as obtained in (B.18) satisfies the HJB equation in (3.15). Substituting the above form of $J(x, c^i, t)$ in (3.15) and using the first-order condition for the minimization we get, $\frac{\partial J(x,c^i,t)}{\partial x} = \frac{g'(r^*(x,c^i,t))}{c^i}$ which gives,

$$r^*(x, c^i, t) = \frac{x}{f_i(T-t)}, \ i = 1, \ldots, m \tag{B.20}$$

Furthermore, to satisfy the PDE equation we require (see the steps presented in Step 1),

$$f_i'(s) = 1 + \frac{\lambda f_i(s)}{n-1} - \frac{\lambda}{n-1}\sum_{j=1}^{m} \frac{p_{ij}}{z_{ij}} \frac{(f_i(s))^n}{(f_j(s))^{n-1}}, \quad \forall i = 1, \ldots, m \tag{B.21}$$

Thus, equivalently, in order to prove that $J(x, c, t)$ satisfies the HJB equation, we need to show that the functions $\{f_i(s)\}$ as defined in (B.19) satisfy the above ODE system with the boundary conditions $f_i(0) = 0$ and $f_i'(0) = 1$, $\forall i$. These boundary conditions follow by taking the limit $k \to \infty$ in $f_i^k(s)$; specifically, $f_i^k(0) = \tau_k \to 0$ and $(f_i^k)'(0) = \left(1 + \frac{\lambda\tau_k}{n-1} - \frac{\lambda\tau_k}{n-1}\sum_{j=1}^{m}\frac{p_{ij}}{z_{ij}}\right) \to 1$ (Note that as $k \to \infty$, $\tau_k \downarrow 0$). The following lemma shows that this is indeed true and $\{f_i(s)\}$ as defined in (B.19) satisfy the ODE system in (B.21) with the above mentioned boundary conditions; furthermore $\{f_i(s)\}$ are also the unique solution of that ODE system.

***Lemma* 19** *The functions $\{f_i(s)\}$, $s \in [0,T]$ as defined in (B.19) are the unique solution of the ODE system in (B.21) with the boundary conditions $f_i(0) = 0$ and $f_i'(0) = 1$, $\forall i$.*

Proof: See Appendix B.4. ■

Finally, we check the admissibility of policy $r^*(x, c, t)$ as given in (B.20). To see this, note that the rate r^* is non-negative and is zero when $x = 0$, $r^*(x, c, t)$ is locally-Lipschitz-continuous in x and continuous in t. The policy $r^*(\cdot)$ also satisfies the deadline constraint $x(T) = 0$ since the boundary condition, $f_i'(0) = 1$, $\forall i$, implies that very close to the deadline T, the policy behaves as $r^*(x, c^i, t) = \frac{x}{T-t}$; thereby emptying the buffer by the deadline.

B.3 Proof of Lemma 18 – *Existence and Uniqueness of the solution to the ODE in (B.17)*

To ease the notations, let us abstract the ODE system in (B.17) as follows. Let $x_i(s)\triangleq f_i^k(s)$, $a\triangleq\frac{\lambda}{n-1}$, $b_{ij}\triangleq\frac{\lambda p_{ij}}{(n-1)z_{ij}}$, then (B.17) can be re-written as,

$$x_i'(s) = 1 + ax_i(s) - \sum_{j=1}^{m} b_{ij}\frac{x_i(s)^n}{x_j(s)^{n-1}}, \quad \forall i = 1, \ldots, m \tag{B.22}$$

where for $p_{ij} \geq 0, (\lambda, z_{ij}) > 0$ and $n > 1$, we have, $a > 0$ and $b_{ij} \in [0, \infty)$. Thus, we now have to find a vector of functions $\bar{\mathbf{x}}(s)\triangleq(x_1(s), \ldots, x_m(s))$ such that each $x_i(s)$ satisfies the equation in (B.22) with the initial condition $\bar{\mathbf{x}}(0) = (\tau_k, \ldots, \tau_k)$ (since $f_i^k(0) = \tau_k,\ \forall i$).

Let us define $G_i(\bar{\mathbf{x}}(s)) \triangleq 1 + ax_i(s) - \sum_{j=1}^{m} b_{ij}\frac{x_i(s)^n}{x_j(s)^{n-1}}$, then, in a very compact form we get,

$$\bar{\mathbf{x}}'(s) = \bar{\mathbf{G}}(\bar{\mathbf{x}}(s)) \tag{B.23}$$

where $\bar{\mathbf{x}}'(s)$ denotes the column vector $(x_1'(s), \ldots, x_m'(s))$ and $\bar{\mathbf{G}}(\cdot)$ denotes the column vector $(G_1(\cdot), \ldots, G_m(\cdot))$. Now consider the open positive orthant and denote it as $\mathcal{U}$, thus, $\mathcal{U} = (x_1 > 0, \ldots, x_m > 0)$. For $\bar{\mathbf{x}} \in \mathcal{U}$, each $G_i(\bar{\mathbf{x}})$ is a continuously differentiable function. Hence, $\bar{\mathbf{G}}(\bar{\mathbf{x}})$ is continuously differentiable which means that it is locally Lipschitz continuous in $\bar{\mathbf{x}}$ over the set $\mathcal{U}$. Therefore, starting with $\bar{\mathbf{x}}(0) = (\tau_k, \ldots, \tau_k) \in \mathcal{U}$, the ODE in (B.23) has a unique local solution $\bar{\mathbf{x}}(s)$ that lies in $\mathcal{U}$ [69]. The only question now remains is whether the local solution leaves the open positive orthant, i.e. whether $\bar{\mathbf{x}}(s) \notin \mathcal{U}$ for some finite $s > 0$. And the answer is no; the local solution remains inside $\mathcal{U}$, which then proves the claim that the ODE in (B.23) has a unique positive solution for all $s > 0$. To prove the last requirement that $\bar{\mathbf{x}}(s) \in \mathcal{U},\ \forall s > 0$, we proceed as follows.

First, since $\bar{\mathbf{G}}(\bar{\mathbf{x}})$ is locally Lipschitz continuous in $\bar{\mathbf{x}}$, a unique local solution that lies in $\mathcal{U}$ exists for the ODE in (B.23). Suppose now that $0 < s_0 < \infty$ is the first instant at which for some i, we have $x_i(s_0) = 0$ or $x_i(s_0) = \infty$, i.e. s_0 is the first instant at which $\bar{\mathbf{x}}(s)$ leaves the positive orthant $\mathcal{U}$. Over the interval $s \in [0, s_0)$ we have,

$$\begin{aligned} x_i'(s) &= 1 + ax_i(s) - \sum_{j=1}^{m} b_{ij}\frac{x_i(s)^n}{x_j(s)^{n-1}} && \text{(B.24)} \\ &\leq 1 + ax_i(s) && \text{(B.25)} \end{aligned}$$

From (B.25) above we get,

$$x_i(s) \;\le\; \frac{(1 + a\tau_k)e^{as} - 1}{a} \tag{B.26}$$

Thus, each $x_i(s)$ is bounded above by an exponential function that goes to infinity only when $s \to \infty$, and this exponential function is given as $\frac{(1+a\tau_k)e^{as}-1}{a}$.

We now compute the lower bound on the functions $\{x_i(s)\}$. Let $x_l(s)$ take the smallest value among $\{x_i(s)\}$ over an interval $[0, s_1]$, $s_1 \le s_0$; this is true since $x_i(s)$ are continuous functions and if a certain function takes the smallest value, it will remain the minimum over some interval. We then have, $x_l(s)/x_j(s) \le 1$, $\forall j = 1, \ldots, m$ over $s \in [0, s_1]$. This gives,

$$\begin{aligned}
x_l'(s) &= 1 + a x_l(s) - \sum_{j=1}^{m} b_{lj} \frac{x_l(s)^n}{x_j(s)^{n-1}} && \text{(B.27)}\\
&\ge -\sum_{j=1}^{m} b_{lj} \frac{x_l(s)^n}{x_j(s)^{n-1}} && \text{(B.28)}\\
&= -x_l(s) \sum_{j=1}^{m} b_{lj} \left(\frac{x_l(s)}{x_j(s)}\right)^{n-1} && \text{(B.29)}\\
&\ge -x_l(s) \sum_{j=1}^{m} b_{lj}, \quad (\text{since } x_l(s)/x_j(s) \le 1,\ \forall j) && \text{(B.30)}\\
&= -c_l x_l(s), \ (\text{taking } c_l = \textstyle\sum_{j=1}^{m} b_{lj}) && \text{(B.31)}
\end{aligned}$$

From (B.31) above we get,

$$x_l(s) \;\ge\; \tau_k e^{-s c_l} \tag{B.32}$$

Thus, $x_l(s)$ is bounded below by an exponential function that goes to zero only when $s \to \infty$. Using a recursive argument starting with $s = s_1$ and following the new minimum function, it follows that over the interval $[0, s_0)$ all functions $\{x_i(s)\}$ are lower bounded by $\tau_k e^{-s c_{max}}$, where $c_{max} = \max_{l=1,\ldots,m} c_l$.

From the arguments above we therefore deduce that the unique local solution, $\bar{\mathbf{x}}(s)$, is upper and lower bounded by two respective positive exponential functions. Hence, the local solution never leaves the set $\mathcal{U}$. Thus, by contradiction s_0 cannot be finite and it then follows that the ODE in (B.23) has a unique positive global solution, $\bar{\mathbf{x}}(s)$, for all $s > 0$, i.e. we have a unique $\bar{\mathbf{x}}(s) \in \mathcal{U}$, $\forall s > 0$ that satisfies (B.23) with $\bar{\mathbf{x}}(0) = (\tau_k, \ldots, \tau_k)$.

B.4 Proof of Lemma 19 – *Functions $\{f_i(s)\}$ are the unique solution of the ODE system in (B.21)*

We know from Lemma 18 that $f_i^k(s)$ is a continuously differentiable function, hence, $(f_i^k)'(s)$ exists for all $s \in [0,T]$ and from (B.17) it is given as,

$$(f_i^k)'(s) = 1 + \frac{\lambda f_i^k(s)}{n-1} - \frac{\lambda}{n-1}\sum_{j=1}^{m}\frac{p_{ij}}{z_{ij}}\frac{(f_i^k(s))^n}{(f_j^k(s))^{n-1}}, \quad \forall i = 1,\ldots,m \tag{B.33}$$

Take the limit $k \to \infty$ in the above equation and denote this point-wise limit as $h_i(s)$, i.e. $h_i(s) \triangleq \lim_{k\to\infty}(f_i^k)'(s)$. The limit exists since $f_i^k(s)$ is pointwise convergent for all i (see Step 2 of Section B.2). Thus, we get,

$$\begin{aligned} h_i(s) &= 1 + \frac{\lambda f_i(s)}{n-1} - \frac{\lambda}{n-1}\sum_{j=1}^{m}\frac{p_{ij}}{z_{ij}}\frac{(f_i(s))^n}{(f_j(s))^{n-1}}, \quad s \in (0,T] && \text{(B.34)} \\ &= 1, \quad s = 0 && \text{(B.35)} \end{aligned}$$

To prove the lemma, we need to show that $f_i(s)$ as defined in (B.19) satisfies $f_i'(s) = h_i(s)$. To do this, we use the following result [70] (Thm. 7.17, pg. 152).

***Lemma* 20** *[70] Suppose $\{f_n\}$ is a sequence of functions, differentiable on $[a,b]$ and such that $\{f_n(x_0)\}$ converges for some point x_0 on $[a,b]$. If $\{f_n'\}$ converges uniformly on $[a,b]$, then $\{f_n\}$ converges uniformly on $[a,b]$, to a function f, and $f'(x) = \lim_{n\to\infty} f_n'(x)$, $(a \leq x \leq b)$.*

For our case, for all i, $\{f_i^k(s)\}_{k=1}^{\infty}$ forms a sequence of differentiable functions on $[0,T]$ and $f_i^k(s)$ converges point-wise to $f_i(s)$. We show in Lemma 21 below that $(f_i^k)'(s)$ has a uniformly convergent subsequence over $s \in [0,T]$. Considering this subsequence, combined with Lemma 20 above (for our case, the sequence $\{f_n'\}$ in Lemma 20 is the uniformly convergent subsequence $\{(f_i^k)'(s)\}$ and the limit function f is $f_i(s)$), we obtain, $f_i'(s) = h_i(s) = 1 + \frac{\lambda f_i(s)}{n-1} - \frac{\lambda}{n-1}\sum_{j=1}^{m}\frac{p_{ij}}{z_{ij}}\frac{(f_i(s))^n}{(f_j(s))^{n-1}}$ (from (B.34)). Thus, this proves that $f_i(s)$ is differentiable over $[0,T]$ and is a solution of the ODE in (B.21) with $f_i(0) = 0$ and $f_i'(0) = 1$.

***Lemma* 21** ***(Uniform convergence of** $(f_i^k)'(s)$**)** *The functions $\{(f_i^k)'(s)\}_{k=1}^{\infty}$ have a uniformly convergent subsequence on $s \in [0,T]$ for all i.*

Proof: See Section B.5. ∎

We now prove uniqueness using a contradiction argument. Suppose that the solution is not unique and let $\bar{\mathbf{f}}(s) = (f_1(s), \ldots, f_m(s))$ and $\bar{\mathbf{y}}(s) = (y_1(s), \ldots, y_m(s))$ be two solutions with $f_i(0) = 0$, $f_i'(0) = 1$, $y_i(0) = 0$, $y_i'(0) = 1$, $\forall i = 1, \ldots, m$. We first show that if we look at s close to 0, the two solutions $\bar{\mathbf{y}}(s)$ and $\bar{\mathbf{f}}(s)$ are in the positive orthant and close to each other. Start at $s = 0$ and consider $\epsilon > 0$, then by the mean value theorem [70] we have, $y_i(\epsilon) = \epsilon y_i'(\eta)$ with $\eta \in (0, \epsilon)$. By the continuity of the derivative, we further have $y_i(\epsilon) = \epsilon(y_i'(0) + \gamma_i(\epsilon)) = \epsilon(1 + \gamma_i(\epsilon))$, where $\gamma_i(\epsilon) \xrightarrow{\epsilon \to 0} 0$ and this holds for all i. Thus for ϵ small enough we must have $\bar{\mathbf{y}}(\epsilon) > 0$; in other words there exists a $\tilde{\epsilon}$ such that for all $0 < \epsilon < \tilde{\epsilon}$ the solution $\bar{\mathbf{y}}(\epsilon)$ is in the positive orthant. Similarly, since $\bar{\mathbf{f}}(s) = (f_1(s), \ldots, f_m(s))$ is also a solution, the above set of arguments hold for it as well and we have, $\bar{\mathbf{f}}(\epsilon) > 0$. From above we also see that $||\bar{\mathbf{y}}(\epsilon)|| < \gamma_y(\epsilon)$, where $\gamma_y(\epsilon) \xrightarrow{\epsilon \to 0} 0$ and a similar inequality holds for $\bar{\mathbf{f}}(\epsilon)$ as well. Thus, $||\bar{\mathbf{y}}(\epsilon) - \bar{\mathbf{f}}(\epsilon)|| < ||\bar{\mathbf{y}}(\epsilon)|| + ||\bar{\mathbf{f}}(\epsilon)|| < (\gamma_y(\epsilon) + \gamma_f(\epsilon))$.

Now, pick $\epsilon \in (0, \tilde{\epsilon})$ and consider the two solutions of the ODE over time $s \in [\epsilon, T]$ starting from the initial state $\bar{\mathbf{y}}(\epsilon)$ and $\bar{\mathbf{f}}(\epsilon)$ respectively. Following the proof of Lemma 18 (Appendix B.3), we see that starting from an initial state in the positive orthant, the ODE has a unique solution that lies in the positive orthant. Furthermore, from [69], the solution is continuous with respect to the initial conditions. Thus, in simple terms, this implies that starting with close enough initial conditions the two solutions $\bar{\mathbf{y}}(s)$ and $\bar{\mathbf{f}}(s)$ must be close enough for all $s \in [\epsilon, T]$. Mathematically, for any $\zeta > 0$, there exists an $\epsilon \in (0, \tilde{\epsilon})$ such that $\max_{s \in [\epsilon, T]} ||\bar{\mathbf{y}}(s) - \bar{\mathbf{f}}(s)|| < \zeta$. By taking ζ going to zero, we see that $\bar{\mathbf{y}}(s)$ and $\bar{\mathbf{f}}(s)$ cannot be distinct over $s \in [0, T]$ and this completes the proof.

B.5 Proof of Lemma 21 – *Uniform convergence of* $(f_i^k)'(s)$

We prove the result as follows: First, we show that $|(f_i^k)'(s)|$ is bounded for all $s \in [0, T]$ and all i, k and then use this result to show a uniformly convergent subsequence for $f_i^k(s)$ and $\frac{(f_i^k(s))^n}{(f_j^k(s))^{n-1}}$, $\forall j$ on $s \in [0, T]$. Using the relationship in (B.33) the result of the lemma then directly follows.

To proceed, consider first the term $\frac{(f_i^k(s))^n}{(f_j^k(s))^{n-1}}$ and for ease of notation set $H_{ij}^k(s) \triangleq \frac{(f_i^k(s))^n}{(f_j^k(s))^{n-1}}$. We now prove that $H_{ij}^k(s)$ is positive and bounded for $s \in [0, T]$ by showing that the vector $(f_1^k(s), \ldots, f_m^k(s))$, $s \in [0, T]$ lies within a conic set in the positive orthant.

Let $x_i(s) \triangleq f_i^k(s)$, $a \triangleq \frac{\lambda}{n-1}$, $b_{ij} \triangleq \frac{\lambda p_{ij}}{(n-1) z_{ij}}$, then in a more generic form (B.33) can be written as,

$$x_i'(s) = 1 + ax_i(s) - \sum_{j=1}^{m} b_{ij} \frac{x_i(s)^n}{x_j(s)^{n-1}}, \quad \forall i = 1, \ldots, m \tag{B.36}$$

where for $p_{ij} \geq 0, (\lambda, z_{ij}) > 0$ and $n > 1$, we have, $a > 0$ and $b_{ij} \in [0, \infty)$. Consider the following set,

$$\delta \leq \frac{x_i}{x_l} \leq \frac{1}{\delta}, \quad \forall i, l \tag{B.37}$$

where $0 < \delta < 1$ is appropriately chosen as presented later. Let $\mathcal{H}$ denote the set of all those values of $(x_1, \ldots, x_m)$ in the positive orthant that satisfy the relationship in (B.37) (we also include the origin in $\mathcal{H}$). Graphically, $\mathcal{H}$ looks like a conic region and it consists of all straight lines with bounded slopes lying in $[\delta, 1/\delta]$.

We now show that for an appropriately chosen δ, the gradient vector at the boundaries of set $\mathcal{H}$ points inwards into the set. Without loss of generality consider the hyperplane boundary defined by the constraint $\frac{x_i}{x_l} = \delta$ or $x_i - \delta x_l = 0$. The vector $\bar{\mathbf{e}}$ with components $e_i = 1, e_l = -\delta, e_q = 0, \forall q \neq i, l$ is normal to this plane and points inwards into the set. Consider the dot product of the gradient vector $(x_1'(s), \ldots, x_m'(s))$ with $\bar{\mathbf{e}}$ and denote it as w. Using (B.36), we then have,

$$\begin{aligned} w &= \left(1 + ax_i(s) - \sum_{j=1}^{m} b_{ij} \frac{x_i(s)^n}{x_j(s)^{n-1}}\right) - \delta\left(1 + ax_l(s) - \sum_{j=1}^{m} b_{lj} \frac{x_l(s)^n}{x_j(s)^{n-1}}\right) \\ &= (1-\delta) + \sum_{j=1}^{m}\left(-b_{ij}\delta^n \frac{x_l(s)^n}{x_j(s)^{n-1}} + \delta b_{lj} \frac{x_l(s)^n}{x_j(s)^{n-1}}\right), \text{ (since } x_i(s) = \delta x_l(s)) \end{aligned}$$

But since $\bar{\mathbf{x}}(s)$ is within $\mathcal{H}$ (at the boundary), it also satisfies $\delta \leq \frac{x_l}{x_j} \leq \frac{1}{\delta}$, $\forall j$. Using these inequalities in the equation above we get,

$$w \geq (1-\delta) + \sum_{j=1}^{m} (-b_{ij}\delta x_l(s) + \delta^n b_{lj} x_l(s)) \tag{B.38}$$

$$= (1-\delta) + \left(\sum_{j=1}^{m} \delta^{n-1} b_{lj}\right) \delta x_l(s) - \left(\sum_{j=1}^{m} b_{ij}\right) \delta x_l(s) \tag{B.39}$$

Now, $x_l(s)$ is a continuously differentiable function over the compact interval $[0, T]$. Hence, $x_l(s)$ is bounded over $[0, T]$ (and this applies for all k) [70] (Thm. 4.15, pg. 89). But, we know that $f_l^k(s)$ is positive and monotonically non-increasing in k. This follows from Step 1 of the proof of Theorem XVIII, where for a fixed (x,c,t), $J^k(x,c,t)$ is non-decreasing in k and from (B.14) this translates into the non-increasing property for $f_l^k(s)$, $\forall l$. Hence, by taking $k=1$ and by considering a maximum bound over $\{f_l^1\}_{l=1}^m$, we have $x_l(s) < M$, where the bound $M > 0$ is independent of k and l. Using this in the above equation we get,

$$\begin{aligned} w &\geq 1-\delta+\left(\sum_{j=1}^m \delta^{n-1} b_{lj}\right)\delta x_l(s) - \left(\sum_{j=1}^m b_{ij}\right)\delta M && \text{(B.40)} \\ &= 1+\left(\sum_{j=1}^m \delta^{n-1} b_{lj}\right)\delta x_l(s) - \left(1+\sum_{j=1}^m b_{ij} M\right)\delta && \text{(B.41)} \\ &\geq 1-\delta\left(1+M\sum_{j=1}^m b_{ij}\right), \text{ (since } x_l(s)\geq 0) && \text{(B.42)} \end{aligned}$$

Now taking $0 < \delta < \frac{1}{1+M\sum_{j=1}^m b_{ij}}$, we get $w > 0$ (note that $\frac{1}{1+M\sum_{j=1}^m b_{ij}}$ is a positive number). Thus, since the dot product is positive, the gradient vector points inwards into the set $\mathcal{H}$ at that particular boundary. Repeating this argument over all i,l pairs and taking δ as, $0 < \delta < \min_i \left(\frac{1}{1+M\sum_{j=1}^m b_{ij}}\right)$, shows that the gradient vector at all the boundaries points inwards into the set $\mathcal{H}$. Thus, since the initial vector, $(x_1(0),\ldots,x_m(0)) = (\tau_k,\ldots,\tau_k)$ lies in $\mathcal{H}$ and the gradient vector points inwards into $\mathcal{H}$ at all the boundaries, it proves that the solution vector $(x_1(s),\ldots,x_m(s)) \in \mathcal{H}$ for all $s \in [0,T]$. Rephrasing the above statement, the set $\mathcal{H}$ as defined in (B.37) is such that the functions $(f_1^k(s),\ldots,f_m^k(s))$ lie inside $\mathcal{H}$ for all k and $s \in [0,T]$. Once we have this set $\mathcal{H}$, we can now bound the derivative $(f_i^k)'(s)$, $\forall k$.

$$\begin{aligned} |(f_i^k)'(s)| &= \left|1+\frac{\lambda f_i^k(s)}{n-1} - \frac{\lambda}{n-1}\sum_{j=1}^m \frac{p_{ij}}{z_{ij}}\frac{(f_i^k(s))^n}{(f_j^k(s))^{n-1}}\right| \\ &\leq 1+\frac{\lambda}{n-1}\left|f_i^k(s)\right| + \frac{\lambda}{n-1}\sum_{j=1}^m \frac{p_{ij}}{z_{ij}}\left|\frac{(f_i^k(s))^n}{(f_j^k(s))^{n-1}}\right| \\ &\leq 1+\frac{\lambda}{n-1}M+\frac{\lambda}{n-1}\sum_{j=1}^m \frac{p_{ij}}{z_{ij}}\frac{1}{\delta^{n-1}}M, \quad \left(\text{since, } \tfrac{(f_i^k(s))^{n-1}}{(f_j^k(s))^{n-1}} \leq \tfrac{1}{\delta^{n-1}};\ f_i^k(s)\leq M\right) \\ &\leq \tilde{M}, \quad \left(\text{where } \tilde{M} = 1+\tfrac{\lambda}{n-1}M+\tfrac{\lambda}{n-1}\tfrac{1}{\delta^{n-1}}M\max_i\left(\textstyle\sum_{j=1}^m \tfrac{p_{ij}}{z_{ij}}\right)\right) \end{aligned}$$

where the bound $\tilde{M}$ is independent of i,k,s. Thus, we have that $\{f_i^k(s)\}$, $\forall s \in [0,T], i, k$ have a bounded derivative.

Uniform convergence of $f_i^k(s)$: First, note that since $(f_i^k)'(s)$ is bounded, $h_i(s)$ as defined in (B.34) is bounded over $s \in [0,T]$ for all i. We now show that $f_i(s)$, $\forall i$ (as defined in (B.19)) is continuous over $s \in [0,T]$. Pick $s \in (0,T)$, then,

$$\lim_{\epsilon \to 0} (f_i(s+\epsilon) - f_i(s)) = \lim_{\epsilon \to 0} \left(\lim_{k \to \infty} f_i^k(s+\epsilon) - \lim_{k \to \infty} f_i^k(s) \right) \quad \text{(B.43)}$$

$$= \lim_{\epsilon \to 0} \left(\lim_{k \to \infty} (f_i^k(s+\epsilon) - f_i^k(s)) \right) \quad \text{(B.44)}$$

Since $f_i^k(s)$ is differentiable, from the Mean Value Theorem [70], $f_i^k(s+\epsilon) - f_i^k(s) = \epsilon\ (f_i^k)'(\zeta(\epsilon))$, where $\zeta(\epsilon) \in [s, s+\epsilon]$. Substituting this in the equation above gives,

$$\lim_{\epsilon \to 0} (f_i(s+\epsilon) - f_i(s)) = \lim_{\epsilon \to 0} \left(\lim_{k \to \infty} (\epsilon (f_i^k)'(\zeta(\epsilon))) \right) \quad \text{(B.45)}$$

$$= \lim_{\epsilon \to 0} \epsilon h_i(\zeta(\epsilon)) = 0 \quad \text{(B.46)}$$

The last equality above follows since $h_i(\zeta)$ is bounded. By using a similar argument as above for $s = 0, T$ and looking at the limits from the right and the left respectively, we see that $f_i(s)$ is continuous at the boundaries as well. Thus, this proves that $f_i(s)$ is continuous over $s \in [0,T]$. Now from the proof of Theorem XVIII we know that $J^k(x,c,t)$ is monotonically non-decreasing in k and using (B.14) this implies that $f_i^k(s)$ is monotonically non-increasing in k for a fixed s. Thus, $f_i^k(s)$ is a monotonically non-increasing sequence in k which converges point-wise to a continuous function $f_i(s)$ over the compact interval $[0,T]$. From [70] (Thm. 7.13) it follows that $f_i^k(s)$ converges uniformly to $f_i(s)$ on $s \in [0,T]$.

Uniform convergence of $\frac{(f_i^k(s))^n}{(f_j^k(s))^{n-1}}$: We first show that $H_{ij}^k(s)$ over $s \in [0,T]$ is continuously differentiable with bounded derivative as given below,

$$\frac{\partial H_{ij}^k(s)}{\partial s} = \frac{n(f_i^k(s))^{n-1}}{(f_j^k(s))^{n-1}} (f_i^k)'(s) + \frac{(1-n)(f_i^k(s))^n}{(f_j^k(s))^n} (f_j^k)'(s) \quad \text{(B.47)}$$

$$\left| \frac{\partial H_{ij}^k(s)}{\partial s} \right| \leq \frac{n\tilde{M}}{\delta^{n-1}} + (1-n)\frac{\tilde{M}}{\delta^n} \quad \text{(B.48)}$$

$$\leq \hat{M}, \quad \left(\text{where } \hat{M} = \tfrac{n\tilde{M}}{\delta^{n-1}} + (1-n)\tfrac{\tilde{M}}{\delta^n} \right) \quad \text{(B.49)}$$

It now follows that $|H_{ij}^k(s) - H_{ij}^k(s')| \leq \hat{M}|s - s'|$, where $s, s' \in [0,T]$. Since the bound $\hat{M}$ is independent of s, k, i, j, the functions $\{H_{ij}^k(s)\}_{k=1}^{\infty}$ form an equicontinuous family of functions for every i, j pair. It then follows [70] (Thm. 7.25, pg. 158) that $H_{ij}^k(s) = \frac{(f_i^k(s))^n}{(f_j^k(s))^{n-1}}$

has a uniformly convergent subsequence that converges to the point-wise limit $\frac{(f_i(s))^n}{(f_j(s))^{n-1}}$.

From (B.33), $(f_i^k)'(s)$ is a linear combination of the terms $f_i^k(s)$ and $\frac{(f_i^k(s))^n}{(f_j^k(s))^{n-1}}$. We have shown that each of them has a uniformly convergent subsequence, thus, it follows that $\{(f_i^k)'(s)\}_{k=1}^{\infty}$ contains a uniformly convergent subsequence over $s \in [0,T]$, $\forall i$.

B.6 Proof of Theorem VII – *Constant Drift Channel, Monomial Case*

The proof for this result is identical to that of Theorem VI but now we can evaluate the functions $\{f_i(s)\}$ in closed form. To see this, start with problem $\mathcal{P}_k$ and suppose that for all channel states the function $f_i^k(s)$ is the same, i.e. $f_i^k(s) = f^k(s)$. The ordinary differential for $f^k(s)$ then becomes,

$$\begin{aligned} (f^k)'(s) &= 1 + \frac{\lambda f^k(s)}{n-1} - \frac{\lambda}{n-1} f^k(s) \left(\sum_j \frac{p_{ij}}{z_{ij}} \right) && \text{(B.50)} \\ &= 1 - \frac{\lambda f^k(s)}{n-1}(\beta - 1) && \text{(B.51)} \end{aligned}$$

where $\sum_j \frac{p_{ij}}{z_{ij}} = E[1/Z(c^i)] = \beta$, $\forall i$, by the constant drift channel assumption. The solution to the above ODE with the boundary condition $f^k(0) = \tau_k$ is given as,

$$f^k(s) = \frac{(n-1)}{\lambda(\beta-1)} \left(1 - \exp(-\frac{\lambda(\beta-1)s}{n-1}) \right) + \tau_k \exp(-\frac{\lambda(\beta-1)s}{n-1}), \; s \geq 0 \qquad \text{(B.52)}$$

From Lemma 18 the above function is the unique solution of the ODE in (B.51) and it can be easily checked that the functional forms $r_k^*(x,c,t) = \frac{x}{f^k(T-t)}$ and $J^k(x,c,t) = \frac{x^n}{c(f^k(T-t))^{n-1}}$ satisfy the conditions in Lemma 17. To obtain the solution for the *BT*-problem, we take the limit $(k \to \infty)$ which gives the optimal solution in (3.22) and (3.23) with $f(s) \triangleq \lim_{k\to\infty} f^k(s) = \frac{(n-1)}{\lambda(\beta-1)} \left(1 - \exp(-\frac{\lambda(\beta-1)s}{n-1})\right)$.

B.7 Proof of Theorem VIII – *Constant Drift Channel, Exponential Case*

A direct non-constructive proof for showing optimality is to plug the functional forms given in the theorem statement into the PDE equation in (3.15) and check if it satisfies the equation. However, such a proof would not reveal how the particular functional form can be obtained. To present a constructive proof, we utilize discrete dynamic programming and proceed as follows. From the steps in Appendix B.1, we first solve for the optimal functions, $\{J^k(x,c,t), r_k^*(x,c,t)\}$, of the relaxed problem $\mathcal{P}_k$, take the limit $\tau_k \downarrow 0$ and verify the conditions of Theorem XVIII. Now, to solve problem $\mathcal{P}_k$, we consider a discrete approximation of the time interval $[0,T]$ with step size dt. Using dynamic programming (DP), we obtain the optimal policy and the minimum cost function and take the limit $dt \to 0$. Thus, there are two limiting operations involved, first $dt \to 0$ to solve for the optimal functions for problem $\mathcal{P}_k$ and then $\tau_k \downarrow 0$ to solve for the optimal functions for the BT-problem. We treat the two cases, $\beta \geq 1$ and $\beta < 1$ separately.

Case 1: $\beta \geq 1$. Consider a discrete approximation of time with step size $dt > 0$. Starting at time T and recursing backwards, let $[T - jdt, T-(j-1)dt]$, $j \geq 1$ denote the j^{th} stage and $V_j(x,c)$ the corresponding cost-to-go function starting with x amounts of data and channel state c. Denote the j^{th} stage optimal transmission rate as $r_j(x,c)$. Let V_0 denote the terminal energy cost over $[T, T+\tau_k]$, then, $V_0(x,c) = h_k(x,c) = \frac{(\alpha^{x/\tau_k}-1)}{c}\tau_k$. The first step DP recursion is,

$$V_1(x,c) = \min_{0 \leq r \leq x/dt} \left\{ \frac{(\alpha^r - 1)dt}{c} + (1-\lambda dt)V_0(x - rdt, c) + \lambda dt E_z(V_0(x-rdt, Zc)) \right\} \quad \text{(B.53)}$$

The constraint $0 \leq r \leq x/dt$ follows from the non-negativity of the rate and the buffer respectively. Substituting $V_0(\cdot)$ and using standard lagrangian techniques, it is easy to show that the above minimization has the following solution. Let $\rho = 1 + \lambda dt(\beta - 1)$, ($\rho \geq 1$, since $\beta \geq 1$),

$$r_1(x,c) = \begin{cases} \frac{x}{\tau_k + dt} + \frac{\tau_k \ln \rho}{(\tau_k+dt)\ln \alpha}, & x \geq \frac{dt \ln \rho}{\ln \alpha} \\ \frac{x}{dt}, & 0 \leq x < \frac{dt \ln \rho}{\ln \alpha} \end{cases} \quad \text{(B.54)}$$

$$V_1(x,c) = \begin{cases} \frac{\tau_k + dt}{c}\alpha^{r_1(x,c)} - \frac{(dt + \rho\tau_k)}{c}, & x \geq \frac{dt \ln \rho}{\ln \alpha} \\ \frac{dt}{c}(\alpha^{r_1(x,c)} - 1), & 0 \leq x < \frac{dt \ln \rho}{\ln \alpha} \end{cases} \quad \text{(B.55)}$$

Following the DP recursion for the next stage, we get,

$$V_2(x,c) = \min_{0 \leq r \leq x/dt} \left\{ \frac{(\alpha^r - 1)dt}{c} + (1-\lambda dt)V_1(x - rdt, c) + \lambda dt E_z(V_1(x-rdt, Zc)) \right\} \quad \text{(B.56)}$$

Now, to solve the above minimization, first assume $x - rdt \geq \frac{dt \ln \rho}{\ln \alpha}$. Substituting the corresponding form of $V_1(\cdot)$ into (B.56) and solving the minimization by standard differentiation, the optimal rate can be obtained as $r = \frac{x}{\tau_k + 2dt} + \frac{(2\tau_k + dt) \ln \rho}{(\tau_k + 2dt) \ln \alpha}$. With this optimal r, substituting in $x - rdt \geq \frac{dt \ln \rho}{\ln \alpha}$, we get the threshold, $x \geq \frac{3dt \ln \rho}{\ln \alpha}$. Note that for the above threshold $rdt \leq x$, thus, buffer non-negativity constraint is also satisfied. Next, assume $x - rdt < \frac{dt \ln \rho}{\ln \alpha}$, and substitute the relevant form of $V_1(\cdot)$ into (B.56). Proceeding as before, we get, $r = \frac{x}{2dt} + \frac{\ln \rho}{2 \ln \alpha}$. Using this r in $x - rdt < \frac{dt \ln \rho}{\ln \alpha}$ and the buffer non-negativity constraint $x - rdt \geq 0$ we get the threshold on x as, $\frac{dt \ln \rho}{\ln \alpha} \leq x \leq \frac{3dt \ln \rho}{\ln \alpha}$. Finally, for $x < \frac{dt \ln \rho}{\ln \alpha}$, all the data is drained in the second stage and the rate is $r = \frac{x}{dt}$. Thus, the solution of the minimization in (B.56) is,

$$r_2(x,c) = \begin{cases} \frac{x}{\tau_k + 2dt} + \frac{(2\tau_k + dt) \ln \rho}{(\tau_k + 2dt) \ln \alpha}, & x \geq \frac{3dt \ln \rho}{\ln \alpha} \\ \frac{x}{2dt} + \frac{\ln \rho}{2 \ln \alpha}, & \frac{dt \ln \rho}{\ln \alpha} \leq x < \frac{3dt \ln \rho}{\ln \alpha} \\ \frac{x}{dt}, & x < \frac{dt \ln \rho}{\ln \alpha} \end{cases} \tag{B.57}$$

$$V_2(x,c) = \begin{cases} \frac{\tau_k + 2dt}{c} \alpha^{r_2(x,c)} - \frac{(dt + \rho dt + \rho^2 \tau_k)}{c}, & x \geq \frac{3dt \ln \rho}{\ln \alpha} \\ \frac{2dt}{c} \alpha^{r_2(x,c)} - \frac{(dt + \rho dt)}{c}, & \frac{dt \ln \rho}{\ln \alpha} \leq x < \frac{3dt \ln \rho}{\ln \alpha} \\ \frac{dt}{c}(\alpha^{r_2(x,c)} - 1), & x < \frac{dt \ln \rho}{\ln \alpha} \end{cases} \tag{B.58}$$

Continuing the DP recursion, the solution for the j^{th} stage is,

$$r_j(x,c) = \begin{cases} \frac{x}{\tau_k + jdt} + \frac{(j\tau_k + \frac{j(j-1)}{2} dt) \ln \rho}{(\tau_k + jdt) \ln \alpha}, & x \geq \frac{j(j+1) dt \ln \rho}{2 \ln \alpha} \\ \frac{x}{(j-i+1)dt} + \frac{(j-i) \ln \rho}{2 \ln \alpha}, & \frac{(j-i)(j-i+1) dt \ln \rho}{2 \ln \alpha} \leq x < \frac{(j-i+1)(j-i+2) dt \ln \rho}{2 \ln \alpha} \end{cases} \tag{B.59}$$

$$V_j(x,c) = \begin{cases} \frac{\tau_k + jdt}{c} \alpha^{r_2(x,c)} - \frac{(1 + \rho + \ldots + \rho^{j-1}) dt + \rho^j \tau_k}{c}, & x \geq \frac{j(j+1) dt \ln \rho}{2 \ln \alpha} \\ \frac{(j-i+1)dt}{c} \alpha^{r_2(x,c)} - \frac{(1 + \rho + \ldots + \rho^{j-i}) dt}{c}, & \frac{(j-i)(j-i+1) dt \ln \rho}{2 \ln \alpha} \leq x < \frac{(j-i+1)(j-i+2) dt \ln \rho}{2 \ln \alpha} \end{cases} \tag{B.60}$$

where $i = 1, \ldots, j$. Now, take the limits, $dt \to 0$, $jdt \to (T - t)$. Under this limiting operation, we have $j \ln \rho \to \lambda(\beta - 1)(T - t)$. Applying these limits we get (let, $\zeta = \lambda(\beta - 1)$),

$$r_k^*(x,c,t) = \begin{cases} \sqrt{\frac{2x\zeta}{\ln\alpha}}\,, & 0 \le x < \frac{\zeta(T-t)^2}{2\ln\alpha} \\ \frac{x}{\tau_k+T-t} + \frac{\zeta(T-t)(\tau_k+\frac{T-t}{2})}{(\tau_k+T-t)\ln\alpha}\,, & x \ge \frac{\zeta(T-t)^2}{2\ln\alpha} \end{cases} \tag{B.61}$$

$$J^k(x,c,t) = \begin{cases} \frac{1}{c}\left(\sqrt{\frac{2x\ln\alpha}{\zeta}}e^{\sqrt{2x\zeta\ln\alpha}} + \frac{1-e^{\sqrt{2x\zeta\ln\alpha}}}{\zeta}\right), & 0 \le x < \frac{\zeta(T-t)^2}{2\ln\alpha} \\ \frac{1}{c}\left((T-t+\tau_k)\alpha^{r_k^*(x,c,t)} - \frac{(1+\zeta\tau_k)e^{\zeta(T-t)}-1}{\zeta}\right), & x \ge \frac{\zeta(T-t)^2}{2\ln\alpha} \end{cases} \tag{B.62}$$

The function $J^k(x,c,t)$ given in (B.62) is continuously differentiable, satisfies the HJB equation in (3.15) and the boundary conditions for problem $\mathcal{P}_k$. The policy $r_k^*(x,c,t)$ is admissible and is the minimizing r for the HJB equation. Thus, by Lemma 17, (B.61) and (B.62) form the optimal solution for $\mathcal{P}_k$. To obtain $J(x,c,t)$ take the limit $\tau_k \downarrow 0$ in (B.62). This gives,

$$J(x,c,t) = \begin{cases} \frac{1}{c}\left(\sqrt{\frac{2x\ln\alpha}{\zeta}}e^{\sqrt{2x\zeta\ln\alpha}} + \frac{1-e^{\sqrt{2x\zeta\ln\alpha}}}{\zeta}\right), & 0 \le x < \frac{\zeta(T-t)^2}{2\ln\alpha} \\ \frac{1}{c}\left((T-t)\alpha^{\frac{x}{T-t}+\frac{\zeta(T-t)}{2\ln\alpha}} - \frac{e^{\zeta(T-t)}-1}{\zeta}\right), & x \ge \frac{\zeta(T-t)^2}{2\ln\alpha} \end{cases} \tag{B.63}$$

Taking limits in (B.61) gives $r^*(\cdot)$ as in (3.24), i.e.

$$r^*(x,c,t) = \begin{cases} \sqrt{\frac{2x\lambda(\beta-1)}{\ln\alpha}}\,, & 0 \le x < \frac{\lambda(\beta-1)(T-t)^2}{2\ln\alpha} \\ \frac{x}{T-t} + \frac{\lambda(\beta-1)(T-t)}{2\ln\alpha}\,, & x \ge \frac{\lambda(\beta-1)(T-t)^2}{2\ln\alpha} \end{cases} \tag{B.64}$$

To check optimality, we need to verify the conditions of Theorem XVIII. It is easy to check that $J(x,c,t)$ in (B.63) satisfies the HJB equation with $r^*(\cdot)$ the minimizing value. Policy $r^*(\cdot)$, satisfies the admissibility criteria including the deadline constraint, since, the rate $r^*(x,c,t) > \frac{x}{T-t}, \forall x > 0, t < T$.

Case 2: $0 < \beta < 1$. The result follows using the same methodology as in the previous case and is omitted here for brevity. The function $J(x,c,t)$ in this case is (let, $\eta = \lambda(1-\beta)$),

$$J(x,c,t) = \begin{cases} \frac{e^{-\eta(T-t)}}{c}\left(\sqrt{\frac{2x\ln\alpha}{\eta}}e^{\sqrt{2x\eta\ln\alpha}} + \frac{1-e^{\sqrt{2x\eta\ln\alpha}}}{\eta}\right), & 0 \le x < \frac{\eta(T-t)^2}{2\ln\alpha} \\ \frac{1}{c}\left((T-t)\alpha^{\frac{x}{T-t}-\frac{\eta(T-t)}{2\ln\alpha}} + \frac{e^{-\eta(T-t)}-1}{\eta}\right), & x \ge \frac{\eta(T-t)^2}{2\ln\alpha} \end{cases}$$

while $r*(x,c,t)$ is as given in (3.25), i.e.

$$r^*(x,c,t) = \begin{cases} 0\,, & 0 \le x < \frac{\lambda(1-\beta)(T-t)^2}{2\ln\alpha} \\ \frac{x}{T-t} - \frac{\lambda(1-\beta)(T-t)}{2\ln\alpha}\,, & x \ge \frac{\lambda(1-\beta)(T-t)^2}{2\ln\alpha} \end{cases} \tag{B.65}$$

B.8 Proof of Lemma 10 – *Weak Duality*

Consider a policy $\tilde{r}(\cdot)$ that is admissible for problem $(\mathcal{P})$, i.e. $\tilde{r}(\cdot) \in \Phi$ and $\tilde{r}(\cdot)$ satisfies the power constraints in (3.27). Note that problem $(\mathcal{P})$ has at least one admissible solution since a policy that does not transmit any data and simply incurs the penalty cost is an admissible policy. Now, fix a lagrange vector $\bar{\nu} \ge 0$. Since $\tilde{r}(\cdot)$ is admissible, using (3.27) we get,

$$\nu_k \left(E\left[\int_{\frac{(k-1)T}{L}}^{\frac{kT}{L}} \frac{g(\tilde{r}(x(s),c(s),s))}{c(s)} ds \right] - \frac{PT}{L} \right) \le 0, \quad k = 1,\ldots,L \tag{B.66}$$

From (3.29) it directly follows that,

$$\mathcal{H}(\tilde{r}(\cdot),\bar{\nu}) \le J(\tilde{r}(\cdot),x_0,c_0) \tag{B.67}$$

where, $J(\tilde{r}(\cdot),x_0,c_0) = E\left[\int_0^T \frac{1}{c(s)} g(\tilde{r}(x(s),c(s),s))ds + \frac{\tau g(\frac{x(T)}{\tau})}{c(T)}\right]$. Now, since $\tilde{r}(\cdot) \in \Phi$, using (3.31) we clearly have, $\mathcal{L}(\bar{\nu}) \le \mathcal{H}(\tilde{r}(\cdot),\bar{\nu})$ and plugging in (B.67) above we get,

$$\mathcal{L}(\bar{\nu}) \le J(\tilde{r}(\cdot),x_0,c_0) \tag{B.68}$$

Since the above holds for all admissible $\tilde{r}(\cdot)$, taking the infimum of the right-side of the inequality above, over the set of admissible policies for problem $(\mathcal{P})$ gives,

$$\mathcal{L}(\bar{\nu}) \le \inf_{\tilde{r}()\ admissible} J(\tilde{r}(\cdot),x_0,c_0) = J(x_0,c_0) \tag{B.69}$$

B.9 Proof of Theorem IX – *BT-problem with Power Constraints*

Consider first the L^{th} partition interval, i.e. $k = L$ and suppose that we start with the system state (x,c,t) lying in this interval. Note that the system state space for this interval is $(x,c,t) \in [0,B] \times \mathcal{C} \times \left[\frac{(L-1)T}{L}, T\right)$ and over this period, equations (3.42) and (3.43) take the form,

$$r_\nu^*(x, c^i, t) = \frac{x}{f_i^L(T-t)}, \qquad i = 1, \ldots, m \tag{B.70}$$

$$H_\nu(x, c^i, t) = \frac{(1+\nu_L)x^n}{c^i(f_i^L(T-t))^{n-1}}, \quad i = 1, \ldots, m \tag{B.71}$$

The cost function over this period is, $P(r, c) = (1+\nu_L)\frac{g(r)}{c}$, and the optimality equation is given as in (3.41) with $\nu_k = \nu_L$. Over this period, a direct comparison shows that the minimization problem in (3.33) is identical to the relaxed BT-problem considered in Appendix B.1, hence, Lemma 17 applies in the following form.

***Lemma** 22* *Consider the L^{th} partition interval and let $H_\nu(x, c, t)$ defined on $(x, c, t) \in [0, B] \times \mathcal{C} \times \left[\frac{(L-1)T}{L}, T\right]$, solve the equation in (3.41) with the boundary conditions $H_\nu(0, c, t) = 0$ and $H_\nu(x, c, T) = \frac{\tau g(\frac{x}{\tau})}{c}$. Then,*

1. *$H_\nu(x, c, t) \leq H_\nu^r(x, c, t), \ \forall\ r(\cdot) \in \Phi$*

2. *Let $r_\nu^*(x, c, t) \in \Phi$ be such that r_ν^* is the minimizing value of r in (3.41), then, $r_\nu^*(x, c, t)$ is an optimal policy, $H_\nu(x, c, t)$ is the minimum cost-to-go function and,*

$$H_\nu(x, c, t) = E\left[\int_t^T \frac{g(r_\nu^*(x(s), c(s), s))}{c(s)} ds + \frac{\tau g(\frac{x(T)}{\tau})}{c(T)}\right] \tag{B.72}$$

By verifying the requirements in the above lemma, we now show that (B.70) and (B.71) are the optimal solution for the L^{th} interval. First note that $g(r) = r^n$ and from the boundary conditions on $f_i^L(s)$ in Theorem IX, we have $f_i^L(0) = \tau(1+\nu_L)^{\frac{1}{n-1}}, \forall i$. Using this it is easy to check that the boundary conditions $H_\nu(0, c, t) = 0$ and $H_\nu(x, c, T) = \frac{\tau g(\frac{x}{\tau})}{c}$ are satisfied.

Now, substituting (B.70) and (B.71) into the PDE equation in (3.41) gives,

$$\begin{aligned}\frac{(1+\nu_L)x^n}{c^i(f_i^L(T-t))^n} + \frac{-(1+\nu_L)x^n(1-n)(f_i^L)'(T-t)}{c^i(f_i^L(T-t))^n} - \frac{x}{f_i^L(T-t)}\frac{n(1+\nu_L)x^{n-1}}{c^i(f_i^L(T-t))^{n-1}} & \\ +\lambda\sum_{j=1}^m \frac{p_{ij}}{z_{ij}c^i}\frac{(1+\nu_L)x^n}{(f_j^L(T-t))^{n-1}} - \lambda\frac{(1+\nu_L)x^n}{c^i(f_i^L(T-t))^{n-1}} &= 0\end{aligned}$$

Cancelling out $\frac{(1+\nu_L)x^n}{c^i}$, simplifying the above and setting $s = T - t$ gives the following

ODE system (note, $t \in \left[\frac{(L-1)T}{L}, T\right]$ implies that $s = (T-t) \in [0, T/L]$),

$$(f_i^L(s))' = 1 + \frac{\lambda f_i^L(s)}{n-1} - \frac{\lambda}{n-1}\sum_{j=1}^{m}\frac{p_{ij}}{z_{ij}}\frac{(f_i^L(s))^n}{(f_j^L(s))^{n-1}}, \ i = 1, \ldots, m \tag{B.73}$$

Thus, from above we see that for $r_\nu^*(\cdot)$ and $H_\nu(\cdot)$ as given in (B.70) and (B.71) respectively to satisfy the optimality PDE equation we require that the functions $\{f_i^L(s)\}_{i=1}^m$, $s \in [0, T/L]$, satisfy the above ODE system with the boundary conditions $f_i^L(0) = \tau(1+\nu_L)^{\frac{1}{n-1}}, \forall i$. The following lemma shows that indeed such a set of positive functions exists and also that they are unique.

***<u>Lemma</u>* 23 (Existence and Uniqueness of the ODE solution in (B.73))** *The ODE system in (B.73) with the boundary conditions $f_i^L(0) = \tau(1 + \nu_L)^{\frac{1}{n-1}}, \forall i$, has a unique positive solution for $s \in [0, T/L]$.*

Proof: The proof is identical to that in Appendix B.3 ■

This completes the verification that $H_\nu(x, c, t)$ and $r^*(x, c, t)$ satisfy the optimality PDE equation. Furthermore, it is easy to check that the rate r^* as given in (B.70) is the minimizing value of r in (3.41) (take the first derivative with respect to r and set it to zero). The admissibility of $r_\nu^*(x, c, t)$ follows by noting that the functional form in (B.70) is continuous and locally Lipschitz in x, continuous in t and satisfies $r_\nu^*(0, c, t) = 0$. Thus, we have verified all the requirements in Lemma 22 and this proves that (B.70) and (B.71) give the optimal solution over the L^{th} partition interval.

Now, consider the $(L-1)^{th}$ partition interval, i.e. $k = L-1$. The system state space for this interval is $(x, c, t) \in [0, B] \times \mathcal{C} \times \left[\frac{(L-2)T}{L}, \frac{(L-1)T}{L}\right)$ and over this period, equations (3.42) and (3.43) take the form,

$$r_\nu^*(x, c^i, t) = \frac{x}{f_i^{L-1}(T-t)}, \qquad i = 1, \ldots, m \tag{B.74}$$

$$H_\nu(x, c^i, t) = \frac{(1+\nu_{L-1})x^n}{c^i(f_i^{L-1}(T-t))^{n-1}}, \ i = 1, \ldots, m \tag{B.75}$$

Suppose that we start with a system state in this $(L-1)^{th}$ partition interval. Once we reach the L^{th} interval, i.e. $t = \frac{(L-1)T}{L}$, we know from the preceding arguments that (B.71) gives the minimum cost and (B.70) gives the optimal rate to be followed thereafter. Thus, for the optimization over the $(L-1)^{th}$ interval, we can abstract the L^{th} interval as a terminal cost of $H_\nu\left(x, c^i, (L-1)T/L\right) = \frac{(1+\nu_L)x^n}{c^i(f_i^L(T/L))^{n-1}}$, applied at $t = \frac{(L-1)T}{L}$. The minimization problem in (3.33) over the $(L-1)^{th}$ interval is therefore identical to that over the L^{th} interval (discussed earlier) except that we now require $H_\nu(x, c^i, (L-1)T/L) = \frac{(1+\nu_L)x^n}{c^i(f_i^L(T/L))^{n-1}}$. Using (B.75), this boundary condition translates into $f_i^{L-1}\left(\frac{T}{L}\right) = \left(\frac{1+\nu_{L-1}}{1+\nu_L}\right)^{\frac{1}{n-1}} f_i^L\left(\frac{T}{L}\right), \forall i$ (as outlined in the Theorem IX statement). Now, following an identical set of arguments as done for the L^{th} partition interval, it is easy to check that (B.74) and (B.75) give the optimal solution over the $(L-1)^{th}$ interval.

Finally, recursively going backwards and considering the partition intervals $k = L-2, L-3, \ldots, 1$, it follows that (3.42) and (3.43) with the boundary conditions as presented in the theorem statement give the optimal solution.

B.10 Proof of Theorem X – *BT-problem with Power Constraints and Constant Drift Channel*

The proof for this theorem is identical to that of the general case in Theorem IX except that now the functions $\{f_i^k(s)\}$ can be evaluated in closed form. Therefore, to avoid repetition we only present the details regarding the functions $\{f_i^k(s)\}$. As before, start with the L^{th} partition interval and suppose that for all the channel states the $f_i^L(s)$ function is the same, i.e. $f_i^L(s) = f^L(s)$. The ordinary differential for $f^L(s)$ then becomes,

$$
\begin{aligned}
(f^L)'(s) &= 1 + \frac{\lambda f^L(s)}{n-1} - \frac{\lambda}{n-1} f^L(s)\left(\sum_j \frac{p_{ij}}{z_{ij}}\right) && \text{(B.76)}\\
&= 1 - \frac{\lambda f^L(s)}{n-1}(\beta - 1) && \text{(B.77)}
\end{aligned}
$$

where $\sum_j \frac{p_{ij}}{z_{ij}} = E[1/Z(c^i)] = \beta$, $\forall i$, by the constant drift channel assumption. The solution to the above ODE evaluated over $s \in [0, T/L]$ with the boundary condition $f^L(0) = \tau(1+\nu_L)^{\frac{1}{n-1}}$ is given as (let $\eta = \frac{\lambda(\beta-1)}{n-1}$),

$$
f^L(s) = \tau(1+\nu_L)^{\frac{1}{n-1}} e^{-\eta s} + \frac{1}{\eta}\left(1 - e^{-\eta s}\right) \qquad \text{(B.78)}
$$

Clearly, for $k = L$, equation (3.49) is the same as (B.78) above (set $s = T - t$).

Now, consider the $(L-1)^{th}$ partition interval and following the same argument as for the L^{th} partition interval, it is easy to see that $f^{L-1}(s)$ satisfies the same ODE as given in (B.77). This ODE must now be evaluated over $s \in \left[\frac{T}{L}, \frac{2T}{L}\right]$ with the following boundary condition,

$$
\begin{aligned}
f^{L-1}\left(\frac{T}{L}\right) &= \left(\frac{1+\nu_{L-1}}{1+\nu_L}\right)^{\frac{1}{n-1}} f^L\left(\frac{T}{L}\right) \\
&= \tau(1+\nu_{L-1})^{\frac{1}{n-1}} e^{-\frac{\eta T}{L}} + \frac{1}{\eta}\left(\frac{1+\nu_{L-1}}{1+\nu_L}\right)^{\frac{1}{n-1}}\left(1-e^{-\eta\frac{T}{L}}\right)
\end{aligned}
$$

Evaluating the ODE with the above boundary condition gives $f^{L-1}(s)$ as follows,

$$
f^{L-1}(s) = \tau(1+\nu_{L-1})^{\frac{1}{n-1}} e^{-\eta s} + \frac{1}{\eta}\left(\frac{1+\nu_{L-1}}{1+\nu_L}\right)^{\frac{1}{n-1}}\left(e^{-\eta(s-\frac{T}{L})} - e^{-\eta s}\right) + \frac{1}{\eta}\left(1-e^{-\eta(s-\frac{T}{L})}\right) \tag{B.79}
$$

Again for $k = L-1$, equation (3.49) is the same as (B.79) above with $s = T-t$. Recursing backwards and following the same steps as earlier, it can be shown that $f^k(s)$ can be written in the general form as given in (3.49).

B.11 Proof of Theorem XI – *Strong Duality*

The optimization problem $(\mathcal{P})$ as stated earlier is given as,

$$
\begin{aligned}
(\mathcal{P}) \qquad & \min_{r(\cdot)\in\Phi} \; E\left[\int_0^T \frac{1}{c(s)} g(r(x(s),c(s),s))ds + \frac{\tau g(\frac{x(T)}{\tau})}{c(T)}\right] \\
& \text{subject to } \; E\left[\int_{\frac{(k-1)T}{L}}^{\frac{kT}{L}} \frac{g(r(x(s),c(s),s))}{c(s)} ds\right] \leq \frac{PT}{L}, \quad k = 1,\dots,L
\end{aligned} \tag{B.80}
$$

Before proceeding to show strong duality holds, we first interchange the expectations and the integrals and re-write the above problem in a standard form as in [66]. But to do that, we need the following. Let $I_{[a,b]}(s)$ be the indicator function for the interval $s \in [a,b]$; it is defined as,

$$
I_{[a,b]}(s) \triangleq \begin{cases} 1, & \text{if } s \in [a,b] \\ 0, & \text{otherwise} \end{cases} \tag{B.81}
$$

Also define,

$$K^{\tau}(s) \triangleq \frac{g(r(x(s), c(s), s))}{c(s)} I_{[0,T]}(s) + g\left(\frac{x_0 - \int_0^T r(x(t), c(t), t)dt}{\tau}\right) \frac{1}{c(T)} I_{[T,T+\tau]}(s) \quad \text{(B.82)}$$

Let $F(r(\cdot))$ denote the total cost for policy $r(\cdot)$ (i.e. the objective function in $(\mathcal{P})$). From (B.80), it is given as,

$$F(r(\cdot)) = E\left[\int_0^T \frac{1}{c(s)} g(r(x(s), c(s), s))ds + \frac{\tau g(\frac{x(T)}{\tau})}{c(T)}\right] \quad \text{(B.83)}$$

Using (B.82), we then have,

$$F(r(\cdot)) = E\left[\int_0^{T+\tau} K^{\tau}(s)ds\right] \quad \text{(B.84)}$$

For any policy $r(\cdot) \in \Phi$, it is clear that $K^{\tau}(s),\ s \in [0, T+\tau]$ is a collection of non-negative random variables which depend on the underlying channel stochastic process. Hence, using Fubini's theorem [85], we can interchange the expectation and the integral which gives,

$$F(r(\cdot)) = \int_0^{T+\tau} E[K^{\tau}(s)]ds \quad \text{(B.85)}$$

Similarly, we can interchange the expectation and the integral for the power constraint inequalities in (B.80). Thus, we can now re-write the optimization problem $(\mathcal{P})$ as,

$$\begin{aligned} \min_{r(\cdot)\in\Phi} \quad & F(r(\cdot)) \qquad \text{(B.86)} \\ \text{subject to} \quad & \int_{\frac{(k-1)T}{L}}^{\frac{kT}{L}} E\left[\frac{g(r(x(s), c(s), s))}{c(s)}\right] ds - \frac{PT}{L} \leq 0, \quad k = 1, \ldots, L \end{aligned}$$

where $F(r(\cdot))$ is as given in (B.85). Now, having written the optimization problem $(\mathcal{P})$ in the above form, the strong duality result in [66] (Theorem 1, sec. 8.6, pp. 224) gives the results as stated in Theorem XI which then completes the proof. However, as a final step we need to verify the technical conditions required in [66]. These are presented below with a description of the technical requirement and the proof of its validity in our case.

(1) *$F(r(\cdot))$ is a convex functional over $r(\cdot) \in \Phi$*

Consider two policies $r_1(x, c, t), r_2(x, c, t) \in \Phi$ and let $0 \leq \alpha \leq 1$. Let $\tilde{r}(x, c, t) =$

$\alpha r_1(x,c,t) + (1-\alpha) r_2(x,c,t)$; since $r_1(\cdot), r_2(\cdot) \in \Phi$ it is easy to check that $\tilde{r}(\cdot)$ also lies in Φ. Now,

$$\begin{aligned}
K^{\tilde{r}}(s) &= \frac{g(\tilde{r}(x(s),c(s),s))}{c(s)} I_{[0,T]}(s) + g\left(\frac{x_0 - \int_0^T \tilde{r}(x(t),c(t),t)dt}{\tau}\right) \frac{1}{c(T)} I_{[T,T+\tau]}(s) \\
&\leq \alpha \left(\frac{g(r_1(x(s),c(s),s))}{c(s)} I_{[0,T]}(s) + g\left(\frac{x_0 - \int_0^T r_1(x(t),c(t),t)dt}{\tau}\right) \frac{1}{c(T)} I_{[T,T+\tau]}(s) \right) \\
&+ (1-\alpha) \left(\frac{g(r_2(x(s),c(s),s))}{c(s)} I_{[0,T]}(s) + g\left(\frac{x_0 - \int_0^T r_2(x(t),c(t),t)dt}{\tau}\right) \frac{1}{c(T)} I_{[T,T+\tau]}(s) \right) \\
&= \alpha K^{r_1}(s) + (1-\alpha) K^{r_2}(s)
\end{aligned}$$

where the inequality above follows since $g(r)$ is a convex function of r. Thus, $K^r(s)$ is a convex functional over $r(\cdot) \in \Phi$ and this implies that $E[K^r(s)]$ is a convex functional. It then directly follows that $F(r(\cdot))$ is a convex functional over $r(\cdot) \in \Phi$.

(2) *Let* $G_k(r(\cdot)) = \left(\int_{\frac{(k-1)T}{L}}^{\frac{kT}{L}} E\left[\frac{g(r(x(s),c(s),s))}{c(s)}\right] ds - \frac{PT}{L} \right)$, $k = 1, \ldots, L$, *then,* $G_k(r(\cdot))$ *is a convex functional over* $r(\cdot) \in \Phi$. The proof for this is identical to the previous case.

(3) *Minimum cost for problem* $(\mathcal{P})$ *is finite*

To see this consider the simple policy which does not transmit any data and only incurs the terminal cost. The expected total cost for this policy is given as,

$$\begin{aligned}
\text{total cost} &= E\left[\frac{g(x_0/\tau)}{c(T)} \tau\right] = \tau g\left(\frac{x_0}{\tau}\right) E\left[\frac{1}{c(T)}\right] \\
&\leq \tau g\left(\frac{x_0}{\tau}\right) \sum_{j=0}^{\infty} \left(\frac{1}{c_0 (z_l)^j}\right) \frac{(\lambda T)^j e^{-\lambda T}}{j!} \\
&= \frac{\tau}{c_0} g\left(\frac{x_0}{\tau}\right) e^{\frac{\lambda T}{z_l}} e^{-\lambda T} < \infty
\end{aligned}$$

The inequality above follows by first conditioning that the channel makes j transitions over $[0,T]$, taking $c(T) = (z_l)^j c_0$, where $(z_l)^j c_0$ is the worst possible channel quality starting with state c_0 and making j transitions, and finally taking expectation with respect to j (number of transitions, j, is Poisson distributed with rate λT and $z_l > 0$ is the least value that any $Z(c)$ can take). Since there exists an admissible policy with a finite cost, it follows that the minimum cost over all admissible $r(\cdot)$ is finite.

(4) *Let* $G_k(r(\cdot)) = \left(\int_{\frac{(k-1)T}{L}}^{\frac{kT}{L}} E\left[\frac{g(r(x(s),c(s),s))}{c(s)}\right] ds - \frac{PT}{L} \right)$, $k = 1, \ldots, L$, *then, a policy*

$r(\cdot) \in \Phi$ *exists such that* $G_k(r(\cdot)) < 0,\ \forall k$ (the interior-point policy). Take $r(\cdot)$ as the policy that does not transmit at all and only incurs the terminal cost.

B.12 Computation of $A^r J(x, c, t)$ given in (3.12)

From the definition of the differential generator, we have,

$$A^r J(x,c,t) = \lim_{h \downarrow 0} \frac{EJ(x_{t+h}, c_{t+h}, t+h) - J(x,c,t)}{h} \tag{B.87}$$

To compute $A^r J(x,c,t)$ we first evaluate the term $EJ(x_{t+h}, c_{t+h}, t+h)$. Consider the Markov model for the channel process obtained after the uniformization, as discussed in Section 3.2.2. Pick a small $h > 0$, then, over the period $[t, t+h]$, the channel state does not change with probability $1 - \lambda h$, there is a single channel transition with probability λh in which case the new state is given as $\tilde{c} = Z(c)c$, and with probability $o(h)$ there are more than one transitions. Thus, we get[1],

$$EJ(x_{t+h}, c_{t+h}, t+h) = (1-\lambda h)J(x_{t+h}, c, t+h) + \lambda h E_z[J(x_{t+h}, Z(c)c, t+h)] + o(h) \tag{B.88}$$

Using the Taylor series expansion we get,

$$J(x_{t+h}, c, t+h) = J(x,c,t) + h\frac{\partial J(x,c,t)}{\partial t} + dx\frac{\partial J(x,c,t)}{\partial x} + o(h) \tag{B.89}$$

But, from the process evolution as given in 3.4, we get $dx = -r(x,c,t)h$, and the above equation becomes,

$$J(x_{t+h}, c, t+h) = J(x,c,t) + h\frac{\partial J(x,c,t)}{\partial t} - hr(x,c,t)\frac{\partial J(x,c,t)}{\partial x} + o(h) \tag{B.90}$$

Similarly, as above, we can evaluate $J(x_{t+h}, Z(c)c, t+h)$ which is given as,

$$J(x_{t+h}, Z(c)c, t+h) = J(x, Z(c)c, t) + h\frac{\partial J(x, Z(c)c, t)}{\partial t} - hr(x, Z(c)c, t)\frac{\partial J(x, Z(c)c, t)}{\partial x} + o(h) \tag{B.91}$$

[1]For this heuristic computation, we will assume that a channel transition, if it occurs, takes place at time t itself and the channel state is then retained over the entire period $[t, t+h]$.

Substituting (B.90) and (B.91) in (B.88) and keeping only upto first-order terms in h, we get,

$$\begin{aligned} EJ(x_{t+h}, c_{t+h}, t+h) = J(x,c,t) &+ h\frac{\partial J(x,c,t)}{\partial t} - hr(x,c,t)\frac{\partial J(x,c,t)}{\partial x} \\ &+ \lambda h\left(E_z[J(x, Z(c)c, t)] - J(x,c,t)\right) + o(h) \end{aligned} \quad \text{(B.92)}$$

Substituting (B.92) in (B.87) and taking the limit gives the result in (3.12).

Appendix C

Proofs for Chapter 4

C.1 Proof of Theorem XII – *Variable Deadlines Case*

To elucidate the steps involved, we first consider the two packet case ($M = 2$), and then extend it to the general scenario for any value of M.

Two Packet Case: The proof outline is as follows. We first start with the functional form for $r^*(D, c, t)$ as given in (4.11), obtain the minimum cost function $J(D, c, t)$ and check that these satisfy the HJB equation in (4.5). While this simply constitutes a check that the HJB equation is satisfied, to finally complete the optimality proof, we consider a sequence of relaxed problems $\{\mathcal{P}_k\}$ along similar lines as done in Appendix B.1 and then take the appropriate limits. We begin first with the verification that the given rate functional satisfies the HJB equation.

Step 1 – Verification of the HJB Equation: Start with the rate function in (4.11) and consider first the state space $(D, c, t) \in [B_1, B_2] \times \mathcal{C} \times [T_1, T_2)$. Thus, we are looking at time $t \geq T_1$ and all admissible D values over this time. Starting from any (D, c, t) in this state space, clearly, the problem is identical to the BT-problem, where $(B_2 - D)$ bits remain in the buffer and these need to be transmitted in time $(T_2 - t)$. From Theorem VII, the optimal rate function is given as, $r^*(D, c, t) = \frac{B_2 - D}{f(T_2 - t)}$. In conformation, over this state space the rate function in (4.11) also reduces to the same form. Thus, over this state space the policy given in (4.11) is trivially the optimal policy. The corresponding minimum cost function is given as,

$$J(D, c, t) = \frac{(B_2 - D)^n}{c(f(T_2 - t))^{n-1}} \tag{C.1}$$

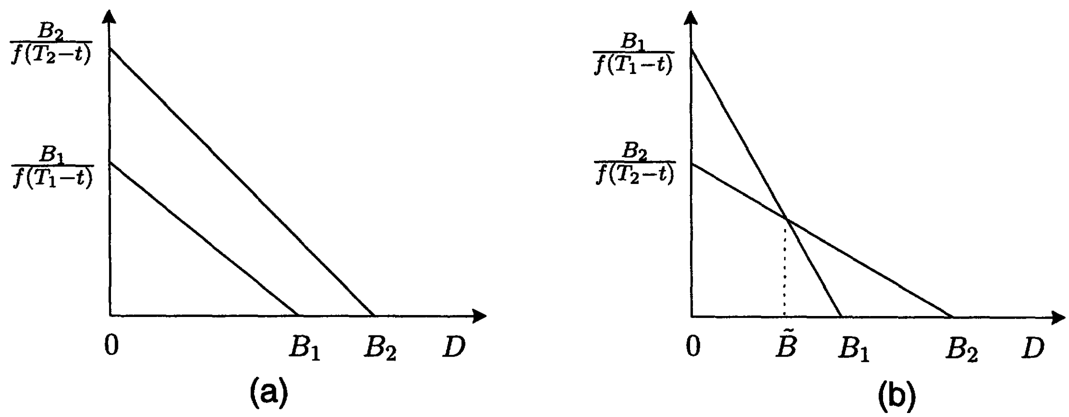


Figure C-1: Proof of Theorem XII for the two packet case, (a) case $\frac{B_2}{f(T_2-t)} > \frac{B_1}{f(T_1-t)}$ and (b) case $\frac{B_2}{f(T_2-t)} \leq \frac{B_1}{f(T_1-t)}$.

Next, consider the state space $(D, c, t) \in [0, B_2] \times \mathcal{C} \times [0, T_1)$; thus now we are considering the region $0 \leq t < T_1$ and all admissible D values over this time which are $[0, B_2]$. Fix a value of t and channel state c, then, as a function of D the rate $r^*(\cdot)$ in (4.11) has the following two possibilities (Note that at $D = 0$, we have $r^*(0, c, t) = \max\left(\frac{B_1}{f(T_1-t)}, \frac{B_2}{f(T_2-t)}\right)$).

1. Suppose $\frac{B_2}{f(T_2-t)} > \frac{B_1}{f(T_1-t)}$. For a fixed t, we see that both $\frac{B_1-D}{f(T_1-t)}$ and $\frac{B_2-D}{f(T_2-t)}$ are linear in D. Figure C-1(a) gives a schematic picture of the two curves and from the figure it is clear that since $B_2 > B_1$, the two curves do not intersect over $D \in [0, B_2]$. Thus, in this case the maximizing function for all $D \in [0, B_2]$ is $\frac{B_2-D}{f(T_2-t)}$ and so, $r^*(D, c, t) = \frac{B_2-D}{f(T_2-t)}$.

2. Suppose $\frac{B_2}{f(T_2-t)} \leq \frac{B_1}{f(T_1-t)}$. In this case, the two functions $\frac{B_1-D}{f(T_1-t)}$ and $\frac{B_2-D}{f(T_2-t)}$ are plotted in Figure C-1(b). From the figure it is clear that since $B_1 < B_2$ the two curves must intersect at some $\tilde{B} \in [0, B_1]$ which satisfies $\frac{B_1-\tilde{B}}{f(T_1-t)} = \frac{B_2-\tilde{B}}{f(T_2-t)}$. This gives, $\tilde{B} = \frac{\left(\frac{B_1}{f(T_1-t)} - \frac{B_2}{f(T_2-t)}\right)}{\left(\frac{1}{f(T_1-t)} - \frac{1}{f(T_2-t)}\right)}$ and we get $r^*(D, c, t) = \frac{B_1-D}{f(T_1-t)}$ for $D \in [0, \tilde{B}]$, and $r^*(D, c, t) = \frac{B_2-D}{f(T_2-t)}$ for $D \in [\tilde{B}, B_2]$.

Define,

$$\tilde{B}(t) = \begin{cases} 0, & \text{if } \frac{B_1}{f(T_1-t)} < \frac{B_2}{f(T_2-t)} \\ \frac{\frac{B_1}{f(T_1-t)} - \frac{B_2}{f(T_2-t)}}{\frac{1}{f(T_1-t)} - \frac{1}{f(T_2-t)}}, & \text{otherwise} \end{cases} \tag{C.2}$$

Using the above definition, the rate function can be written in the following form,

$$r^*(D,c,t) = \begin{cases} \frac{B_2-D}{f(T_2-t)}, & \tilde{B}(t) \le D \le B_2 \\ \frac{B_1-D}{f(T_1-t)}, & 0 \le D < \tilde{B}(t) \end{cases} \tag{C.3}$$

The above compact form covers both cases 1 and 2 above – for the first case $\tilde{B}(t) = 0$ and for the second case we get $\tilde{B}(t)$ as required. Note that for the constant drift channel, since the function $f(\cdot)$ is the same for all the channel states, the intersection point $\tilde{B}(\cdot)$ as defined in (C.2) depends only on time t and not on the channel state[1].

In order for the HJB equation to be satisfied, the rate function $r^*(D,c,t)$ above must be the minimizing value in (4.5). Using the first-order condition for the minimization then gives, $\frac{\partial J(D,c,t)}{\partial D} = -\frac{g'(r^*(D,c,t))}{c}$. Integrating this with respect to D and using the boundary condition $J(B_2,c,t) = 0$, we obtain,

$$J(D,c,t) = \begin{cases} \frac{(B_2-D)^n}{c(f(T_2-t))^{n-1}}, & \tilde{B}(t) \le D \le B_2 \\ \frac{(B_1-D)^n}{c(f(T_1-t))^{n-1}} + \frac{(B_2-\tilde{B}(t))^n}{c(f(T_2-t))^{n-1}} - \frac{(B_1-\tilde{B}(t))^n}{c(f(T_1-t))^{n-1}}, & 0 \le D < \tilde{B}(t) \end{cases} \tag{C.4}$$

It is easy to see that $J(D,c,t)$ in (C.4) is continuous at the boundary $D = \tilde{B}(t)$. It is also continuously differentiable with respect to D including at the boundary $D = \tilde{B}(t)$ and this can be checked directly. Furthermore, for values of $\tilde{B}(t) > 0$, the function $\tilde{B}(t)$ is continuously differentiable with respect to t and this implies that $J(D,c,t)$ is also continuously differentiable in t. Finally, at the boundary $t = T_1$, we have $\tilde{B}(t) = B_1$ and this makes (C.4) consistent with (C.1) for $D \ge B_1$. To verify that the HJB equation is satisfied, we now only need to check that $r^*(D,c,t)$ and $J(D,c,t)$ as given in (C.3) and (C.4) respectively, satisfy the following PDE,

$$\left\{\frac{g(r^*(D,c,t))}{c} + \frac{\partial J(D,c,t)}{\partial t} + r^*(D,c,t)\frac{\partial J(D,c,t)}{\partial D} + \lambda(E_z[J(D,Z(c)c,t)] - J(D,c,t))\right\} = 0 \tag{C.5}$$

where in the above equation, $g(r) = r^n$.

Consider first $D \in [\tilde{B}(t), B_2]$, then, from (C.4) we have $J(D,c,t) = \frac{(B_2-D)^n}{c(f(T_2-t))^{n-1}}$ and from (C.3) we have $r^*(D,c,t) = \frac{B_2-D}{f(T_2-t)}$. The differentials are $\frac{\partial J}{\partial D} = \frac{-n(B_2-D)^{n-1}}{c(f(T_2-t))^{n-1}}$, $\frac{\partial J}{\partial t} =$

[1]For the general Markov channel model this is not true and hence some of the steps in this proof, following this stage, do not apply in that setting.

$\frac{(n-1)(B_2-D)^n f'(T_2-t)}{c(f(T_2-t))^n}$. Substituting these in the left hand side (LHS) of (C.5) and simplifying it then gives,

$$\begin{aligned} LHS &= \frac{(B_2-D)^n}{c(f(T_2-t))^n} + \frac{(n-1)(B_2-D)^n f'(T_2-t)}{c(f(T_2-t))^n} - \frac{(B_2-D)}{f(T_2-t)} \frac{n(B_2-D)^{n-1}}{c(f(T_2-t))^{n-1}} \\ &\quad + \lambda(\beta-1)\frac{(B_2-D)^n}{c(f(T_2-t))^{n-1}} \qquad \text{(C.6)} \\ &= \frac{(n-1)(B_2-D)^n}{c(f(T_2-t))^n}\left(f'(T_2-t) - \left(1 - \frac{\lambda(\beta-1)}{n-1} f(T_2-t)\right)\right) \\ &= 0 \qquad \text{(C.7)} \end{aligned}$$

The last equation above follows since $f'(s) = 1 - \frac{\lambda(\beta-1)}{n-1} f(s)$. Thus, we see that (C.5) is satisfied over $D \in [\tilde{B}(t), B_2]$. If $\tilde{B}(t) = 0$ then we are done. So, now let us suppose that $\tilde{B}(t) > 0$.

Consider $D \in [0, \tilde{B}(t))$, then, from (C.3) we have $r^*(D,c,t) = \frac{B_1-D}{f(T_1-t)}$ and from (C.4) we have $J(D,c,t) = Q(c,t) + H(D,c,t)$, where for simplicity of exposition, we define $Q(c,t) = \left(\frac{(B_2-\tilde{B}(t))^n}{c(f(T_2-t))^{n-1}} - \frac{(B_1-\tilde{B}(t))^n}{c(f(T_1-t))^{n-1}}\right)$ and $H(D,c,t) = \frac{(B_1-D)^n}{c(f(T_1-t))^{n-1}}$. Substituting this in the left hand side of (C.5) we get,

$$\begin{aligned} LHS &= \left(\frac{\partial Q(c,t)}{\partial t} + \lambda(E_z[Q(Z(c)c,t)] - Q(c,t))\right) + \\ &\left(\frac{g(r^*(D,c,t))}{c} + \frac{\partial H(D,c,t)}{\partial t} + r^*(D,c,t)\frac{\partial H(D,c,t)}{\partial D} + \lambda(E_z[H(D,Z(c)c,t)] - H(D,c,t))\right) \end{aligned}$$

Using identical steps as in (C.6)-(C.7) the terms within the second bracket above equal zero. Now consider the term within the first bracket. Let $Q(c,t) = Q_2(c,t) - Q_1(c,t)$, where $Q_2(c,t) = \frac{(B_2-\tilde{B}(t))^n}{c(f(T_2-t))^{n-1}}$ and $Q_1(c,t) = \frac{(B_1-\tilde{B}(t))^n}{c(f(T_1-t))^{n-1}}$. We have,

$$\begin{aligned} \frac{\partial Q_2(c,t)}{\partial t} + \lambda(E_z[Q_2(Z(c)c,t)] - Q_2(c,t)) &= \frac{(n-1)(B_2-\tilde{B}(t))^n}{c(f(T_2-t))^n} \times \\ &\left(-\frac{\tilde{B}'(t)f(T_2-t)n}{(B_2-\tilde{B}(t))(n-1)} + f'(T_2-t) + \frac{\lambda(\beta-1)f(T_2-t)}{n-1}\right) \\ &= \frac{(n-1)(B_2-\tilde{B}(t))^n}{c(f(T_2-t))^n}\left(-\frac{\tilde{B}'(t)f(T_2-t)n}{(B_2-\tilde{B}(t))(n-1)} + 1\right) \end{aligned}$$

The last equality above follows since $f'(s) = 1 - \frac{\lambda(\beta-1)}{n-1} f(s)$. A similar expression as above is obtained for the term $Q_1(c,t)$. Combining the two then gives,

$$
\begin{aligned}
\frac{\partial Q(c,t)}{\partial t} + \lambda(E_z[Q(Z(c)c,t)] - Q(c,t)) &= \frac{(n-1)(B_2 - \tilde{B}(t))^n}{c(f(T_2-t))^n}\left(-\frac{\tilde{B}'(t)f(T_2-t)n}{(B_2-\tilde{B}(t))(n-1)}+1\right) \\
&\quad -\frac{(n-1)(B_1 - \tilde{B}(t))^n}{c(f(T_1-t))^n}\left(-\frac{\tilde{B}'(t)f(T_1-t)n}{(B_1-\tilde{B}(t))(n-1)}+1\right) \\
&= 0, \quad \left(\text{since, } \frac{B_1-\tilde{B}(t)}{f(T_1-t)} = \frac{B_2-\tilde{B}(t)}{f(T_2-t)}\right)
\end{aligned}
$$

This completes the verification that the functions in (C.3) and (C.4) satisfy the PDE equation in (C.5). We now complete the optimality proof by considering a sequence of relaxed problems and taking the appropriate limit as outlined next.

Step 2 - Verification of Optimality: To verify optimality, we view the problem in two stages - first, over the state space $(D,c,t) \in [0,B_2] \times \mathcal{C} \times [0,T_1)$ (transmission over time-period $[0,T_1]$) and second over the state space $(D,c,t) \in [B_1,B_2] \times \mathcal{C} \times [T_1,T_2)$ (transmission over time-period $[T_1,T_2]$). As mentioned in Step 1 of the proof, over the state space $(D,c,t) \in [B_1,B_2] \times \mathcal{C} \times [T_1,T_2)$, the problem is identical to the BT-problem, where $(B_2 - D)$ bits remain in the buffer and these need to be transmitted in time $(T_2 - t)$. The rate function in (4.11) reduces to $r^*(D,c,t) = \frac{B_2-D}{f(T_2-t)}$ and this has been shown to be the optimal policy; see Appendices B.1 and B.6. Thus, the optimality of $r^*(D,c,t)$ and $J(D,c,t)$ over the second stage follows directly from that of the BT-problem.

Now consider the first stage, i.e. the state space $(D,c,t) \in [0,B_2] \times \mathcal{C} \times [0,T_1)$. This stage corresponds to transmission over time-period $[0,T_1]$. Once we reach time $t = T_1$, we know from the preceding paragraph the optimal policy to be followed thereafter in the second stage. Thus, for the optimization over the first stage, we can abstract the second stage energy cost as a terminal cost incurred at time T_1 given the particular terminal state. Specifically, the terminal cost function is given as, $h(D,c) = \frac{(B_2-D)^n}{c(f(T_2-T_1))^{n-1}}$, $D \in [B_1,B_2]$ (since this is the minimum (expected) energy cost required to transmit the remaining $(B_2 - D)$ bits by time (T_2-T_1)), and $h(D,c) = \infty$, $D \in [0,B_1)$ (since there is a deadline constraint of T_1 for the first B_1 bits, and an infinite penalty cost is incurred if $D < B_1$). Since this is a non-continuous terminal cost function, we cannot directly apply standard verification results to show optimality. To circumvent this problem, we consider a sequence of relaxed problems $\{\mathcal{P}_k\}$, where the hard deadline constraint on the B_1 bits is relaxed and instead a sequence of smooth terminal cost functions is assigned, which monotonically converge to the required function above. This is analogous to the steps followed earlier for the BT-problem and they are outlined below.

Consider a sequence of numbers $\{\tau_k\}_{k=1}^{\infty}$, where $\tau_k > 0$ and $\tau_k \downarrow 0$. Define a sequence of functions $\{f^k(s)\}_{k=1}^{\infty}$, where each $f^k(s)$ satisfies the ODE, $(f^k)'(s) = 1 - \frac{\lambda(\beta-1)}{n-1} f^k(s)$ with the initial condition $f^k(0) = \tau_k$. Thus,

$$f^k(s) = \frac{(n-1)}{\lambda(\beta-1)}\left(1 - \exp(-\frac{\lambda(\beta-1)s}{n-1})\right) + \tau_k \exp(-\frac{\lambda(\beta-1)s}{n-1}),\ s \geq 0 \tag{C.8}$$

Consider now a sequence of relaxed problems, $\{\mathcal{P}_k\}$, over the state space $(D, c, t) \in [0, B_2] \times \mathcal{C} \times [0, T_1)$. Each problem $\mathcal{P}_k$ is identical to the BT-problem in terms of the system dynamics except that at time T_1, instead of the hard deadline, a terminal cost is assigned. This terminal cost function is denoted as $h^k(D, c)$ and is taken as follows,

$$h^k(D,c) = \begin{cases} \frac{(B_2-D)^n}{c(f^k(T_2-T_1))^{n-1}}, & \tilde{B}^k(T_1) \leq D < B_2 \\ \frac{(B_1-D)^n}{c(f^k(0))^{n-1}} + \frac{(B_2-\tilde{B}^k(T_1))^n}{c(f^k(T_2-T_1))^{n-1}} - \frac{(B_1-\tilde{B}^k(T_1))^n}{c(f^k(0))^{n-1}}, & 0 \leq D < \tilde{B}^k(T_1) \end{cases} \tag{C.9}$$

where in the above equation, the function $\tilde{B}^k(t),\ t \in [0, T_1]$ for the relaxed problem $\mathcal{P}_k$ is correspondingly defined as,

$$\tilde{B}^k(t) = \begin{cases} 0, & \text{if } \frac{B_1}{f^k(T_1-t)} < \frac{B_2}{f^k(T_2-t)} \\ \dfrac{\frac{B_1}{f^k(T_1-t)} - \frac{B_2}{f^k(T_2-t)}}{\frac{1}{f^k(T_1-t)} - \frac{1}{f^k(T_2-t)}}, & \text{otherwise} \end{cases} \tag{C.10}$$

Note that since $f^k(0) = \tau_k$, as we consider larger values of k then τ_k goes to zero and $\tilde{B}^k(T_1)$ converges to B_1 while $f^k(s)$ converges to $f(s)$. Thus, we see that the terminal cost function $h^k(D, c)$ converges to the desired function as mentioned earlier.

For the relaxed problem $\mathcal{P}_k$ the system operates as follows. Given a transmission policy, denoted as $r_k(D, c, t)$, the system starts with $D(0) = 0$. As this policy is followed, at time T_1, the terminal cost $h^k(D(T_1), c(T_1))$ is incurred and the system stops. Also, during the period $t \in [0, T_1]$, if $D(t) = B_2$ then all the data has been transmitted and there is no terminal cost incurred. Thus, we see that the relaxed problem $\mathcal{P}_k$ is a well-posed, continuous-time control problem with smooth terminal cost functions. Consider now the following rate function,

$$r_k^*(D,c,t) = \max_{j:(B_j \geq D, T_j \geq t)} \frac{B_j - D}{f^k(T_j - t)} \tag{C.11}$$

and the following minimum cost function which is denoted as $J^k(D,c,t)$,

$$J^k(D,c,t) = \begin{cases} \frac{(B_2-D)^n}{c(f^k(T_2-t))^{n-1}}, & \tilde{B}^k(t) \leq D \leq B_2 \\ \frac{(B_1-D)^n}{c(f^k(T_1-t))^{n-1}} + \frac{(B_2-\tilde{B}^k(t))^n}{c(f^k(T_2-t))^{n-1}} - \frac{(B_1-\tilde{B}^k(t))^n}{c(f^k(T_1-t))^{n-1}}, & 0 \leq D < \tilde{B}^k(t) \end{cases} \tag{C.12}$$

Following an identical set of arguments as done in the first step of this proof, it can be seen that the above functions satisfy the HJB equation (note that the functional forms are analogous to those earlier except with $f^k(s)$ replacing $f(s)$ and $\tilde{B}^k(t)$ replacing $\tilde{B}(t)$). It is also easy to see that the minimum cost function also satisfies the boundary conditions, i.e. it equals the terminal cost function $h^k(D,c)$ and also equals zero for $D = B_2$. Using the standard verification result, outlined earlier in Lemma 17, it can be seen that the rate function in (C.11) gives the optimal transmission policy for the relaxed problem $\mathcal{P}_k$.

Now consider the limit $k \to \infty$, then, $J^k(D,c,t)$ converges to $J(D,c,t)$ (given in (C.4)) and $r_k^*(D,c,t)$ converges to $r^*(D,c,t)$. Utilizing the result of Theorem XVIII (an analogous version as stated below), the optimality of $J(D,c,t)$ and $r^*(D,c,t)$ for the first stage of the two-packet problem follows.

***Theorem* XIX (Two Packet Case):** *Consider $(D,c,t) \in [0,B_2] \times \mathcal{C} \times [0,T_1)$ and define $J(D,c,t) \triangleq \lim_{k\to\infty} J^k(D,c,t)$. Let $J(D,c,t)$ satisfy the HJB equation in (4.5) and let $r^*(D,c,t)$ be an admissible policy for the first stage of the two-packet problem, such that r^* is the minimizing value of r in (4.5). Then,*

1. *$J(D,c,t) \leq J_r(D,c,t), \ \forall \ r(\cdot)$ admissible (where $J_r(D,c,t)$ denotes the cost-to-go function for that policy)*
2. *$r^*(D,c,t)$ is the optimal policy and $J(D,c,t)$ is the minimum cost function*

Proof: The proof is identical to that of Theorem XVIII. ■

The requirements of the above verification theorem are satisfied. First, from Step 1 we know that $J(D,c,t)$ and $r^*(D,c,t)$ satisfy the HJB equation. The function $J(D,c,t)$ also satisfies the boundary condition, i.e. $J(D,c,T_1) = h(D,c), \ D \in [B_1,B_2]$ (where $h(D,c)$

gives the optimal cost for the second stage). The rate function $r^*(D,c,t)$ is admissible; to see this note that it is non-negative and locally Lipschitz-continuous in D. Furthermore, the deadline constraint of T_1 for the B_1 bits is also satisfied. This is because from the BT-problem we know that $\frac{B_1-D}{f(T_1-t)}$ is an admissible rate function that meets the required deadline constraint. Here, since $r^*(D,c,t)$ is chosen as the maximum among $\frac{B_1-D}{f(T_1-t)}$ and $\frac{B_2-D}{f(T_2-t)}$, the transmission rate selected ensures that at least B_1 bits have been transmitted by time T_1 (almost surely).

General M Packet Case: The proof for the general M packet case is a direct extension of the ideas presented in the two-packet case. To proceed, consider the state space $(D,c,t) \in [B_{L-1}, B_M] \times \mathcal{C} \times [T_{L-1}, T_L)$ where $L = 1,\ldots,M$ (let $B_0 = 0$ and $T_0 = 0$). Thus, we are viewing the problem in M stages where the L^{th} stage corresponds to looking at time $T_{L-1} \leq t < T_L$ and all admissible D values over this time period which are $D \in [B_{L-1}, B_M]$. Over this state space the rate function $r^*(\cdot)$, from (4.11), is given as, $r^*(D,c,t) = \max_{j=L,\ldots,M} \frac{B_j-D}{f(T_j-t)}$. Thus, we need to only look at rate values for $j = L,\ldots,M$.

Now in the above state space, fix a value of t and c. Then, as in the two-packet case, the rate $r^*(\cdot)$ as a function of D is a piecewise linear curve with at most $M-L+1$ segments. To see this, first note that $\frac{B_j-D}{f(T_j-t)}, \forall j$, is a linear function of D (with t fixed) and we also have $B_L < \ldots < B_M$. From Figure C-1, we see that in the two packet case if $\frac{B_2-D}{f(T_2-t)}$ becomes the maximizing function then it remains the maximizing function for all D values thereafter. In the general M packet case, this observation translates in the following way: if $\frac{B_j-D}{f(T_j-t)}$ is the maximizing function then for all D values thereafter the functions $\frac{B_p-D}{f(T_p-t)}$, $p < j$ cannot be the maximizing function, and only functions $\frac{B_p-D}{f(T_p-t)}$ with $p > j$ can replace it as the maximizing function. Thus, we see that each function plays the maximizing role at most once and further that the indices j of these maximizing functions must be in increasing order. This implies that $r^*(\cdot)$ is piecewise linear with at most $M-L+1$ segments and takes the following general form: function $\frac{B_L-D}{f(T_L-t)}$ is maximum over $D \in [B_{L-1}, \tilde{B}_L(t))$, then, function $\frac{B_{L+1}-D}{f(T_{L+1}-t)}$ is maximum over $D \in [\tilde{B}_L(t), \tilde{B}_{L+1}(t))$ and so on, where $\{\tilde{B}_l(t)\}_{l=L}^{M-1}$ are the rate change points. Note that $\tilde{B}_L(t)$ could be equal to B_{L-1} which covers the case in which $\frac{B_L-D}{f(T_L-t)}$ is not the maximizing function for all $D \in [B_{L-1}, B_M]$. Similarly, $\tilde{B}_{L+1}(t)$ could equal $\tilde{B}_L(t)$ and so on.

Mathematically, the rate change points can be defined as follows. Let $\tilde{b}_{pq}^L(t)$ denote the pairwise intersection points for the L^{th} stage, then (see the example of two-packet case),

$$\tilde{b}^L_{pq}(t) = \begin{cases} B_{L-1}, & \frac{B_p}{f(T_p-t)} < \frac{B_q}{f(T_q-t)} \\ \frac{\left(\frac{B_p-B_{L-1}}{f(T_p-t)} - \frac{B_q-B_{L-1}}{f(T_q-t)}\right)}{\left(\frac{1}{f(T_p-t)} - \frac{1}{f(T_q-t)}\right)}, & \text{otherwise} \end{cases} \tag{C.13}$$

where in the above $q > p, \{p, q = L, \ldots, M\}$. Using this, we have for, $l = L, \ldots, M-1$, (take $\tilde{B}_{L-1}(t) = B_{L-1}$)

$$\tilde{B}_l(t) = \max\left(\tilde{B}_{l-1}(t), \min_{q=l+1,\ldots,M} \tilde{b}^L_{lq}(t)\right) \tag{C.14}$$

We can now write the rate function $r^*(D,c,t)$ for $(D,c,t) \in [B_{L-1}, B_M] \times \mathcal{C} \times [T_{L-1}, T_L)$ in the following form,

$$r^*(D,c,t) = \begin{cases} \frac{B_M-D}{f(T_M-t)}, & \tilde{B}_{M-1}(t) \leq D \leq B_M \\ \frac{B_{M-1}-D}{f(T_{M-1}-t)}, & \tilde{B}_{M-2}(t) \leq D < \tilde{B}_{M-1}(t) \\ \vdots & \\ \frac{B_L-D}{f(T_L-t)}, & B_{L-1} \leq D < \tilde{B}_L(t) \end{cases} \tag{C.15}$$

Using $\frac{\partial J(D,c,t)}{\partial D} = -\frac{g'(r^*(D,c,t))}{c}$ and integrating with respect to D with the boundary condition $J(B_M, c, t) = 0$, gives,

$$J(D,c,t) = \begin{cases} \frac{(B_M-D)^n}{c(f(T_M-t))^{n-1}}, & \tilde{B}_{M-1}(t) \leq D \leq B_M \\ \vdots & \\ \frac{(B_K-D)^n}{c(f(T_K-t))^{n-1}} + \sum_{j=K}^{M-1}\left(\frac{(B_{j+1}-\tilde{B}_j(t))^n}{c(f(T_{j+1}-t))^{n-1}} - \frac{(B_j-\tilde{B}_j(t))^n}{c(f(T_j-t))^{n-1}}\right), & \tilde{B}_{K-1}(t) \leq D < \tilde{B}_K(t) \\ \vdots & \\ \frac{(B_L-D)^n}{c(f(T_L-t))^{n-1}} + \sum_{j=L}^{M-1}\left(\frac{(B_{j+1}-\tilde{B}_j(t))^n}{c(f(T_{j+1}-t))^{n-1}} - \frac{(B_j-\tilde{B}_j(t))^n}{c(f(T_j-t))^{n-1}}\right), & B_{L-1} \leq D < \tilde{B}_L(t) \end{cases} \tag{C.16}$$

The above functional form is for the L^{th} stage, i.e. over state space $(D,c,t) \in [B_{L-1}, B_M] \times \mathcal{C} \times [T_{L-1}, T_L)$. It can be checked directly that the function $J(D,c,t)$ in (C.16) is continuous in D and t; it is also continuously differentiable and is consistent at the boundaries of the various stages $L = 1, \ldots, M$. Due to the similarity of its functional form with that of (C.4), it can also be seen that following an identical set of steps as outlined in the two-packet case, the functions $J(D,c,t)$ and $r^*(D,c,t)$ satisfy the HJB equation in (C.5). Finally, the verification of optimality follows in an identical manner as the two-packet case, where each stage is considered separately in a recursive fashion starting from the last stage. Since these results are the same as outlined in the two-packet case, the steps have been omitted here to avoid repetition.

C.2 Proof of Theorem XIII – *Arrivals with Single-Deadline*

The proof for this case follows a similar methodology as presented in Appendix C.2 for the variable-deadlines scenario. We start with the functional form for $r^*(D,c,t)$ as given in (4.13), obtain the minimum cost function $J(D,c,t)$ and verify that these satisfy the HJB equation in (4.5). Finally, we verify optimality by viewing the problem in multiple stages. To elucidate the steps we begin by first considering the two-packet case and then extend it to the general M packet scenario.

Before proceeding further, we first verify that the transmission policy in (4.13) is admissible. For any $D \in [D_{min}(t), A(t)]$, $c \in \mathcal{C}$, $t \in [0, T_M]$, the rate function $r^*(D,c,t)$ is clearly non-negative. Furthermore, since each of the function $\frac{A_j - D}{f(T_j - t)}$ is locally-Lipschitz-continuous in D and $r^*(\cdot)$ is a minimum among a set of finite number of such functions, $r^*(\cdot)$ is also locally Lipschitz continuous in D. Similarly, it is also piecewise continuous in t. Finally, by construction all the causality constraints are satisfied since at all times we choose the minimum rate among those needed to meet the A_jT_j points. Thus, the departure curve never exceeds the arrival curve (almost surely). Furthermore, for $t > T_{M-1}$, $r^*(D,c,t)$ reduces to choosing a rate that meets the A_MT_M constraint, hence, the single deadline constraint is also satisfied.

Two Packet Case: We begin with the two-packet case and first verify that the proposed policy satisfies the HJB equation.

Step 1 – Verification of the HJB Equation: Consider first the state space $(D,c,t) \in [0, A_2] \times \mathcal{C} \times [T_1, T_2)$; thus, we are looking at time $t \geq T_1$ and all admissible D values over this time-period. Starting from (D,c,t), clearly, the problem is identical to the BT-problem with $B = (A_2 - D)$ and $T = (T_2 - t)$. The optimal rate function from (4.6) is therefore given as, $r^*(D,c,t) = \frac{A_2 - D}{f(T_2 - t)}$. In conformation, the rate function in (4.13) over this state space also reduces to the same form. Thus, over this state space the policy in (4.13) is trivially

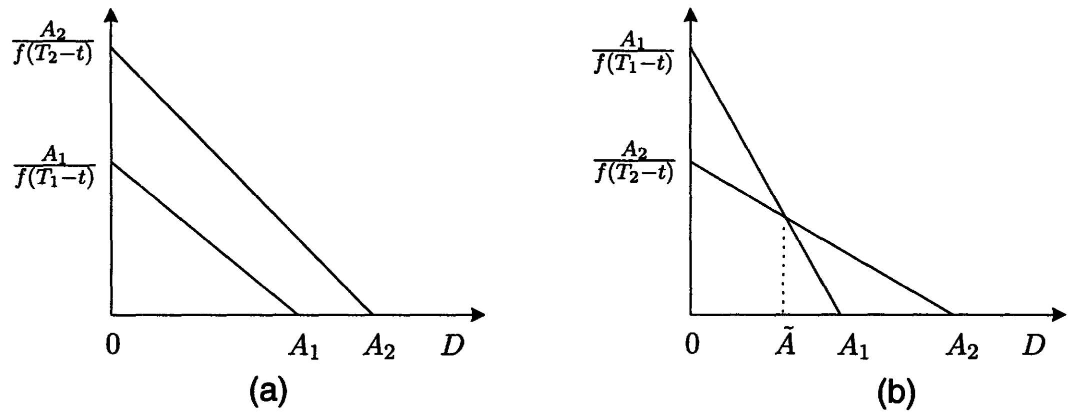


Figure C-2: Proof of Theorem XIII for the two packet case, (a) case $\frac{A_2}{f(T_2-t)} > \frac{A_1}{f(T_1-t)}$ and (b) case $\frac{A_2}{f(T_2-t)} \leq \frac{A_1}{f(T_1-t)}$.

the optimal policy and the corresponding minimum cost function over this state space is,

$$J(D,c,t) = \frac{(A_2 - D)^n}{c(f(T_2-t))^{n-1}} \tag{C.17}$$

Next, consider the state space $(D,c,t) \in [0,A_1] \times \mathcal{C} \times [0,T_1)$; thus now we are considering the region $0 \leq t < T_1$ and all admissible D values over this time which are $[0,A_1]$. Fix a value of t and c, then, as a function of D the rate $r^*(D,c,t)$ in (4.13) has the following two possibilities. Note that at $D=0$, we have $r^*(0,c,t) = \min\left(\frac{A_1}{f(T_1-t)}, \frac{A_2}{f(T_2-t)}\right)$.

1. Suppose $\frac{A_2}{f(T_2-t)} > \frac{A_1}{f(T_1-t)}$. For a fixed t, we see that both $\frac{A_1-D}{f(T_1-t)}$ and $\frac{A_2-D}{f(T_2-t)}$ are linear in D. Figure C-2(a) gives a schematic picture of the two curves and from the figure it is clear that since $A_2 > A_1$, the two curves do not intersect over $D \in [0,A_1]$. Thus, in this case the minimizing function for all $D \in [0,A_1]$ is $\frac{A_1-D}{f(T_1-t)}$ and so, $r^*(D,c,t) = \frac{A_1-D}{f(T_1-t)}$.

2. Suppose $\frac{A_2}{f(T_2-t)} \leq \frac{A_1}{f(T_1-t)}$. In this case, the two functions $\frac{A_1-D}{f(T_1-t)}$ and $\frac{A_2-D}{f(T_2-t)}$ are plotted in Figure C-2(b). From the figure it is clear that since $A_1 < A_2$ the two curves must intersect at some $\tilde{A} \in [0,A_1]$ which satisfies $\frac{A_1-\tilde{A}}{f(T_1-t)} = \frac{A_2-\tilde{A}}{f(T_2-t)}$. This gives, $\tilde{A} = \frac{\left(\frac{A_1}{f(T_1-t)} - \frac{A_2}{f(T_2-t)}\right)}{\left(\frac{1}{f(T_1-t)} - \frac{1}{f(T_2-t)}\right)}$ and we get $r^*(D,c,t) = \frac{A_2-D}{f(T_2-t)}$ for $D \in [0,\tilde{A}]$ and $r^*(D,c,t) = \frac{A_1-D}{f(T_1-t)}$ for $D \in [\tilde{A},A_1]$.

Define,

$$\tilde{A}(t) = \begin{cases} 0, & \text{if } \frac{A_1}{f(T_1-t)} < \frac{A_2}{f(T_2-t)} \\ \frac{\frac{A_1}{f(T_1-t)} - \frac{A_2}{f(T_2-t)}}{\frac{1}{f(T_1-t)} - \frac{1}{f(T_2-t)}}, & \text{otherwise} \end{cases} \tag{C.18}$$

Using the above definition, the rate function can be written in the form,

$$r^*(D,c,t) = \begin{cases} \frac{A_2-D}{f(T_2-t)}, & 0 \le D < \tilde{A}(t) \\ \frac{A_1-D}{f(T_1-t)}, & \tilde{A}(t) \le D \le A_1 \end{cases} \tag{C.19}$$

Note that the above compact form covers both cases 1 and 2 above – for the first case $\tilde{A}(t) = 0$ and for the second case we get $\tilde{A}(t)$ as required. Furthermore, for the constant drift channel, since the function $f(\cdot)$ is the same for all the channel states, the intersection point $\tilde{A}(\cdot)$ as defined in (C.18) depends only on time t and not on the channel state.

For $r^*(D,c,t)$ to satisfy the HJB equation, it must be the minimizing value of r in (4.5). Using the first-order condition for the minimization then gives, $\frac{\partial J(D,c,t)}{\partial D} = -\frac{g'(r^*(D,c,t))}{c}$. Given $r^*(D,c,t)$ as in (C.19), we can obtain $J(D,c,t)$ by integrating with respect to D. But to do that we first obtain the boundary condition at $D = A_1$ as follows. Note that since we are looking at $t < T_1$, if $D = A_1$, it means that the transmitter queue is empty and hence the transmission rate must be zero until the next packet arrival instant, which happens at time $t = T_1$. Since the transmission rate is zero until T_1, no energy cost is incurred over the time period $[t, T_1)$. Now starting at time $t = T_1$, with $D = A_1$ and channel state $c = c^i$, the optimal cost-to-go is given as $\frac{(A_2-A_1)^n}{c^i(f(T_2-T_1))^{n-1}}$ (this follows from (C.17)). Thus, the optimal cost-to-go starting from time t onwards is then given as,

$$J(A_1, c^i, t) = E\left[\frac{H}{c(T_1)}\right]\Big|_{c(t)=c^i} \tag{C.20}$$

where in the above equation $H \triangleq \frac{(A_2-A_1)^n}{(f(T_2-T_1))^{n-1}}$ and the channel states $c \in C$ are denoted as $\{c^i\}$. Let,

$$p_j(i,t) = \text{ probability that } \{c(T_1) = c^j\} \text{ starting in state } c^i \text{ at time } t \ (t < T_1)$$

Using the above notation and denoting $z_{ij} = c^j/c^i$, we can re-write (C.20) as,

$$J(A_1, c^i, t) = \frac{1}{c^i}\sum_j \frac{p_j(i,t)}{z_{ij}} H \tag{C.21}$$

Now, using the boundary condition in (C.21), the differential equation, $\frac{\partial J(D,c,t)}{\partial D} = -\frac{g'(r^*(D,c,t))}{c}$, evaluates as follows,

$$J(D, c^i, t) = \begin{cases} \frac{(A_1-D)^n}{c^i(f(T_1-t))^{n-1}} + \frac{1}{c^i}\sum_j \frac{p_j(i,t)}{z_{ij}}H, & \tilde{A}(t) \le D \le A_1 \\ \frac{(A_2-D)^n}{c^i(f(T_2-t))^{n-1}} + \frac{(A_1-\tilde{A}(t))^n}{c^i(f(T_1-t))^{n-1}} - \frac{(A_2-\tilde{A}(t))^n}{c^i(f(T_2-t))^{n-1}} + \frac{1}{c^i}\sum_j \frac{p_j(i,t)}{z_{ij}}H, & 0 \le D < \tilde{A}(t) \end{cases} \tag{C.22}$$

It is easy to see that $J(D, c^i, t)$ in (C.22) is continuous at the boundary $D = \tilde{A}(t)$. It is also continuously differentiable with respect to D including at the boundary $D = \tilde{A}(t)$ and this can be checked directly. Furthermore, for values of $\tilde{A}(t) > 0$, the function $\tilde{A}(t)$ is continuously differentiable with respect to t and this implies that $J(D, c^i, t)$ is also continuously differentiable in t. Finally, as $t \to T_1$, we have $\tilde{A}(t) \to A_1$ and this makes (C.22) consistent with (C.17) for $D \le A_1$ (since, $p_j(i, T_1) = 1$ for $j = i$, and, $p_j(i, T_1) = 0$ otherwise). Thus, we see that $J(D, c^i, t)$ in (C.22) satisfies the technical requirements of continuity and differentiability and the boundary conditions. We now need to check that $r^*(D, c, t)$ and $J(D, c, t)$ satisfy the following HJB equation as given in (4.5), i.e.,

$$\left\{\frac{g(r^*(D,c,t))}{c} + \frac{\partial J(D,c,t)}{\partial t} + r^*(D,c,t)\frac{\partial J(D,c,t)}{\partial D} + \lambda(E_z[J(D, Z(c)c, t)] - J(D,c,t))\right\} = 0 \tag{C.23}$$

where in the above equation, $g(r) = r^n$.

The functional form of $J(D, c, t)$ in (C.22) is closely related to that in (C.4), except for the additional term $\frac{1}{c^i}\sum_j \frac{p_j(i,t)}{z_{ij}}H$. Thus, we can utilize the proof in Appendix C.1 for this case as well. Consider first $D \in [\tilde{A}(t), A_1]$, then, from (C.22) we have $J(D, c^i, t) = K(D, c^i, t) + L(c^i, t)$, where, $K(D, c^i, t) \triangleq \frac{(A_1-D)^n}{c^i(f(T_1-t))^{n-1}}$ and $L(c^i, t) \triangleq \frac{1}{c^i}\sum_j \frac{p_j(i,t)}{z_{ij}}H$. Substituting this into the left hand side of (C.23) gives,

$$\begin{aligned} LHS = &\left(\frac{\partial L(c^i,t)}{\partial t} + \lambda(E_z[L(Z(c^i)c^i, t)] - L(c^i, t))\right) + \left\{\frac{g(r^*(D, c^i, t))}{c^i}\right. \\ &\left. + \frac{\partial K(D,c^i,t)}{\partial t} + r^*(D, c^i, t)\frac{\partial K(D,c^i,t)}{\partial D} + \lambda(E_z[K(D, Z(c^i)c^i, t)] - K(D, c^i, t))\right\} \end{aligned}$$

Using an identical set of arguments as in Appendix C.1, it can be seen that the terms within the curly bracket above equal zero. Thus, to verify that (C.23) is satisfied we only need to show that the terms within the first bracket equal zero. Similarly, for $D \in [0, \tilde{A}(t)]$, we can utilize the results in Appendix C.1 and it can be seen analogously that to verify (C.23) is satisfied, we need to show a similar result.

We now proceed to verify that $\left(\frac{\partial L(c^i,t)}{\partial t} + \lambda(E_z[L(Z(c^i)c^i,t)] - L(c^i,t))\right)$ equals zero. First, using the Chapman-Kolmogorov equations we get,

$$p_j(i,t) = (1-\lambda dt)p_j(i,t+dt) + \lambda dt \sum_k p_{ik} p_j(k,t+dt) + o(dt)$$

where $\{p_{ij}\}$ denote the transition probabilities among the various channel states obtained after the uniformization (as considered in Section 3.2.2). Taking the limit $dt \to 0$ in the above equation, we get,

$$-\frac{dp_j(i,t)}{dt} = -\lambda p_j(i,t) + \lambda \sum_k p_{ik} p_j(k,t) \tag{C.24}$$

Consider now $\frac{\partial L(c^i,t)}{\partial t}$ which can evaluated as follows,

$$\frac{\partial L(c^i,t)}{\partial t} = \frac{1}{c^i} \sum_j \left(\lambda \frac{p_j(i,t)}{z_{ij}} - \lambda \sum_k p_{ik} \frac{p_j(k,t)}{z_{ij}} \right) H \tag{C.25}$$

$$= \lambda \frac{1}{c^i} \sum_j \frac{p_j(i,t)}{z_{ij}} H - \lambda \frac{1}{c^i} \sum_j \sum_k p_{ik} \frac{p_j(k,t)}{z_{ij}} H \tag{C.26}$$

The first term on the right-hand side of the equation above is easily seen to be $\lambda L(c^i,t)$, whereas the second term is $\lambda E_z[L(Z(c^i)c^i,t)]$. To see this, consider,

$$\lambda E_z[L(Z(c^i)c^i,t)] = \lambda \sum_k p_{ik} \left(\sum_j \frac{1}{c^k} \frac{p_j(k,t)}{z_{kj}} H \right) \tag{C.27}$$

$$= \lambda \sum_k \sum_j \frac{1}{c^i} \frac{p_{ik} p_j(k,t)}{z_{ik} z_{kj}} H, \quad (\text{since, } c^k = c^i z_{ik}) \tag{C.28}$$

$$= \lambda \frac{1}{c^i} \sum_j \sum_k \frac{p_{ik} p_j(k,t)}{z_{ij}} H \tag{C.29}$$

where the last equality above follows from $z_{ij} = z_{ik} z_{kj}$ and interchanging the two summations. The interchange is valid assuming that from every state the channel can jump to a finite number of new states, in which case, the summations are over finite number of non-zero terms. Thus, from (C.26) and (C.29), the required verification result follows. This completes the proof that the functions in (C.19) and (C.22) satisfy the HJB equation in (C.23).

Step 2 – Verification of Optimality: To verify optimality, we view the problem in two stages - first, over the state space $(D, c, t) \in [0, A_1] \times \mathcal{C} \times [0, T_1)$ (transmission over time-period $[0, T_1]$) and second over the state space $(D, c, t) \in [0, A_2] \times \mathcal{C} \times [T_1, T_2)$ (transmission over time-period $[T_1, T_2]$). As mentioned in Step 1 of the proof, over the state space $(D, c, t) \in [0, A_2] \times \mathcal{C} \times [T_1, T_2)$, the problem is identical to the BT-problem, where $(A_2 - D)$ bits remain in the buffer and these need to be transmitted in time $(T_2 - t)$. The rate function in (4.13) reduces to $r^*(D, c, t) = \frac{A_2 - D}{f(T_2 - t)}$ and this has been shown to be the optimal policy; see Appendices B.1 and B.6. Thus, the optimality of $r^*(D, c, t)$ and $J(D, c, t)$ over the second stage follows directly from that of the BT-problem.

Now consider the first stage, i.e. the state space $(D, c, t) \in [0, A_1] \times \mathcal{C} \times [0, T_1)$. This stage corresponds to transmission over time-period $[0, T_1]$. Once we reach time $t = T_1$, we know from the preceding paragraph the optimal policy to be followed thereafter in the second stage. Thus, for the optimization over the first stage, we can abstract the second stage energy cost as a terminal cost incurred at time T_1 given the particular terminal state. Specifically, the terminal cost function is given as, $h(D, c) = \frac{(A_2 - D)^n}{c(f(T_2 - T_1))^{n-1}}$, $D \in [0, A_1]$ (since this is the minimum (expected) energy cost required to transmit the remaining $(A_2 - D)$ bits by time $(T_2 - T_1)$). Thus, for the first stage the system starts with $D(0) = 0$ and the chosen transmission policy is followed until time T_1 at which point the terminal cost $h(D(T_1), c(T_1))$ is incurred. Also, during the period $t \in [0, T_1]$, if $D(t) = A_1$ then all the data has been transmitted and for an admissible policy the transmission rate must be zero until T_1. Having formulated the problem in the above form, the verification of optimality of $r^*(D, c, t)$ and $J(D, c, t)$ as given in (C.19) and (C.22) respectively, follows from the following standard result. This is an analogous version of Lemma 17.

Lemma 24 ***(Two-Packet Case)*** *Consider the first stage of the two-packet problem. Let $J(D, c, t)$ and $r^*(D, c, t)$ defined on $[0, A_1] \times \mathcal{C} \times [0, T_1]$, solve the equation in (4.5) with the boundary condition $J(D, c, T_1) = h(D, c)$. Let $r^*(D, c, t)$ be an admissible policy such that r^* is the minimizing value of r in (4.5). Then,*

1. *$J(D, c, t) \leq J_r(D, c, t), \ \forall\, r(\cdot)$ admissible.*

2. *$r^*(D, c, t)$ is an optimal policy and $J(D, c, t)$ is the minimum cost function.*

Proof: See [63], Chap III, Theorem 8.1. ■

In Step 1, we have shown that $r^*(D,c,t)$ and $J(D,c,t)$ as given in (C.19) and (C.22) respectively, satisfy the HJB equation and that r^* is the minimizing value of r. Policy $r^*(D,c,t)$ is an admissible policy as argued in beginning of the proof. Hence, the above lemma applies and this completes the proof of optimality.

General M Packet Case: The proof for the general M packet case is a direct extension of the ideas presented in the two-packet case. To proceed, consider the state space $(D,c,t) \in [0, A_L] \times \mathcal{C} \times [T_{L-1}, T_L)$ where $L = 1, \ldots, M$ (note $T_0 = 0$). Thus, we are viewing the problem in M stages where the L^{th} stage corresponds to looking at time $T_{L-1} \le t < T_L$ and all admissible D values over this time period which are $D \in [0, A_L]$. Over this state space the rate $r^*(\cdot)$, from (4.13), is given as, $r^*(D,c,t) = \min_{j=L,\ldots,M} \frac{A_j - D}{f(T_j - t)}$. Thus, we need to only look at $A_j T_j$ constraint points for $j = L, \ldots, M$.

Now in the above state space, fix a value of t and $c = c^i$. Then, as in the two-packet case, the rate function $r^*(\cdot)$ as a function of D is a piecewise linear curve with at most $M - L + 1$ segments. To see this, first note that $\frac{A_j - D}{f(T_j - t)}, \forall j$, is a linear function of D (with t fixed) and we also have $A_L < \ldots < A_M$. From Figure C-2, we see that in the two packet case, if $\frac{A_1 - D}{f(T_1 - t)}$ becomes the minimizing function, it remains as the minimum function for all $D \in [0, A_1]$. In the general M packet case, this observation translates in the following way: if $\frac{A_j - D}{f(T_j - t)}$ is the minimizing function, then for all D values thereafter, the functions $\frac{A_p - D}{f(T_p - t)}$, $p > j$ cannot be the minimizing function, and only functions $\frac{A_p - D}{f(T_p - t)}$ with $p < j$ can replace it as the minimizing function. Thus, we see that each function plays the minimizing role at most once and further that the indices j of these minimizing functions must be in decreasing order. This implies that $r^*(\cdot)$ is piecewise linear with at most $M - L + 1$ segments and takes the following general form: function $\frac{A_M - D}{f(T_M - t)}$ is minimum over $D \in [0, \tilde{A}_1(t))$, then, $\frac{A_{M-1} - D}{f(T_{M-1} - t)}$ is minimum over $D \in [\tilde{A}_1(t), \tilde{A}_2(t))$ and so on, where $\{\tilde{A}_l(t)\}_{l=1}^{M-L}$ are the rate change points. Note that $\tilde{A}_1(t)$ could be equal to 0 which covers the case where $\frac{A_M - D}{f(T_M - t)}$ is not the minimizing function for $D \in [0, A_L]$. Similarly, $\tilde{A}_2(t)$ could equal $\tilde{A}_1(t)$ and so on.

Mathematically, the rate change points can be defined as follows. Let $\tilde{a}_{pq}(t)$ denote the pairwise intersection points, then (see the example of two-packet case),

$$\tilde{a}_{pq}(t) = \begin{cases} 0, & \frac{A_q}{f(T_q - t)} < \frac{A_p}{f(T_p - t)} \\ \frac{\left(\frac{A_q}{f(T_q - t)} - \frac{A_p}{f(T_p - t)}\right)}{\left(\frac{1}{f(T_q - t)} - \frac{1}{f(T_p - t)}\right)}, & \text{otherwise} \end{cases} \tag{C.30}$$

where in the above $p > q, \{p, q = L, \ldots, M\}$. Using this, we have for, $l = 1, \ldots, M - L$, (take $\tilde{A}_0(t) = 0$)

$$\tilde{A}_l(t) = \max\left(\tilde{A}_{l-1}(t), \min_{q=L,\ldots,(M-l)} \tilde{a}_{(M-l+1)q}(t)\right) \tag{C.31}$$

We can now write the rate function $r^*(D, c, t)$ for $(D, c, t) \in [0, A_L] \times \mathcal{C} \times [T_{L-1}, T_L)$ in the following form,

$$r^*(D, c^i, t) = \begin{cases} \frac{A_M - D}{f(T_M - t)}, & 0 \le D \le \tilde{A}_1(t) \\ \frac{A_{M-1} - D}{f(T_{M-1} - t)}, & \tilde{A}_1(t) \le D < \tilde{A}_2(t) \\ \vdots & \\ \frac{A_L - D}{f(T_L - t)}, & \tilde{A}_{M-L}(t) \le D \le A_L \end{cases} \tag{C.32}$$

Using $\frac{\partial J(D,c^i,t)}{\partial D} = -\frac{g'(r^*(D,c^i,t))}{c^i}$ and integrating with respect to D we obtain $J(D, c^i, t)$. The corresponding boundary condition is $J(A_L, c^i, t) = \frac{1}{c^i} \sum_j \frac{p_j^L(i,t)}{z_{ij}} H_L$, where as in the two-packet case, $p_j^L(i, t)$ denotes the probability that $c(T_L) = c^j$ starting in channel state c^i at time $t < T_L$, and, $\left(\frac{H_L}{c^j}\right)$ gives the cost-to-go starting in state $D = A_L$, $c = c^j$ and $t = T_L$. The term H_L abstracts the numerator term and its exact form is not necessary for the analysis; however the point to note is that H_L does not depend on the channel state which only appears in the denominator (see the two-packet case as an example). Thus, we get,

$$J(D, c^i, t) = \begin{cases} \frac{(A_L - D)^n}{c^i (f(T_L - t))^{n-1}} + \frac{1}{c^i} \sum_j \frac{p_j^L(i,t)}{z_{ij}} H_L, & \tilde{A}_{M-L}(t) \le D \le A_L \\ \vdots & \\ \frac{(A_K - D)^n}{c^i (f(T_K - t))^{n-1}} + \sum_{q=L}^{K-1} \left(\frac{(A_q - \tilde{A}_{M-q}(t))^n}{c^i (f(T_q - t))^{n-1}} - \frac{(A_{q+1} - \tilde{A}_{M-q}(t))^n}{c^i (f(T_{q+1} - t))^{n-1}} \right) & \\ \qquad + \frac{1}{c^i} \sum_j \frac{p_j^L(i,t)}{z_{ij}} H_L, & \tilde{A}_{M-K}(t) \le D < \tilde{A}_{M-K+1}(t) \\ \vdots & \\ \frac{(A_M - D)^n}{c^i (f(T_M - t))^{n-1}} + \sum_{q=L}^{M-1} \left(\frac{(A_q - \tilde{A}_{M-q}(t))^n}{c^i (f(T_q - t))^{n-1}} - \frac{(A_{q+1} - \tilde{A}_{M-q}(t))^n}{c^i (f(T_{q+1} - t))^{n-1}} \right) & \\ \qquad + \frac{1}{c^i} \sum_j \frac{p_j^L(i,t)}{z_{ij}} H_L, & 0 \le D < \tilde{A}_1(t) \end{cases} \tag{C.33}$$

The above functional form is for the L^{th} stage, i.e. over state space $(D, c, t) \in [0, A_L] \times \mathcal{C} \times [T_{L-1}, T_L)$. It can be checked directly that the function $J(D, c, t)$ in (C.33) is continuous in D and t; it is also continuously differentiable and is consistent at the boundaries of the various stages $L = 1, \ldots, M$. Due to the similarity of its functional form with that of (C.22), it can also be seen that following an identical set of steps as outlined in the two-packet case, the functions $J(D, c, t)$ and $r^*(D, c, t)$ satisfy the HJB equation in (C.23). Finally, the verification of optimality follows in an identical manner as outlined for the two-packet case, where each stage is considered separately in a recursive fashion starting from the last stage. Since these results are the same as presented in the two-packet case, the steps have been omitted here to avoid repetition.

C.3 Boundary Condition for the Poisson Arrivals Problem

Consider time $t < T$ and suppose that $x = 0$. To satisfy the buffer non-negativity constraints, we must have, $r(x, t) = 0$ when $x = 0$. Thus, until the next packet arrival instant, the transmitter chooses a transmission rate of zero. Now, starting at the boundary point $(x = 0, t < T)$, let $\gamma > t$ be the first packet arrival instant after t. Let $\tau = \gamma \wedge T$, where $\wedge$ denotes the minimum operation. Consider $\delta > 0$ and let $\tilde{t} = t + \delta$, then, from Bellman's principle we can write $J(0, t)$ as,

$$J(0,t) = E\left[\int_t^{\tilde{t}\wedge\tau} g(0)ds + J(x_{\tilde{t}\wedge\tau}, \tilde{t}\wedge\tau)\right] \tag{C.34}$$

where $g(0)$ is the power cost for rate 0 (for most practical purposes, $g(0) = 0$) and $J(x_{\tilde{t}\wedge\tau}, \tilde{t}\wedge\tau)$ is the optimal cost starting from time $\tilde{t}\wedge\tau$ onwards. Using the indicator functions $I_{(\tau\leq\tilde{t})}$ and $I_{(\tau>\tilde{t})} = 1 - I_{(\tau\leq\tilde{t})}$ to condition on the respective events we can re-write (C.34) as,

$$J(0,t) = E\left[\left(\int_t^{\tilde{t}} g(0)ds + J(x_{\tilde{t}}, \tilde{t})\right) I_{(\tau>\tilde{t})}\right] + E\left[\left(\int_t^{\tau} g(0)ds + J(x_\tau, \tau)\right) I_{(\tau\leq\tilde{t})}\right] \tag{C.35}$$

$$g(0) + \frac{E\left[J(x_{\tilde{t}}, \tilde{t})\right] - J(0,t)}{\delta} + \frac{1}{\delta}E\left[h(\tilde{t},\tau)I_{(\tau\leq\tilde{t})}\right] = 0 \tag{C.36}$$

where $h(\tilde{t}, \tau) = J(x_\tau, \tau) - J(x_{\tilde{t}}, \tilde{t}) - \int_\tau^{\tilde{t}} g(0)ds$. Consider the event $\tau \leq \tilde{t}$, then, as $\delta \downarrow 0$, $Eh(\tilde{t}, \tau) \to 0$ and $\frac{P(t<\tau\leq\tilde{t})}{\delta} \to \xi$. Hence, the third term in (C.36) goes to zero. Also in the limit $\delta \downarrow 0$, the second term is the differential generator as in (4.17) with $x = 0$ and $r = 0$.

Thus we get the boundary condition,

$$g(0) + \frac{\partial J(0,t)}{\partial t} + \xi\{J(B,t) - J(0,t)\} = 0$$

C.4 Proof of Lemma 11

We can re-write (4.26) as,

$$\frac{d}{dt}((T - t + \tau_0)f(t)) = -\frac{\xi(T - t + \tau_0)}{\ln(\alpha)}\left(\alpha^{\frac{B}{T+\tau_0-t}} - 1\right) \tag{C.37}$$

Taking a Taylor series expansion of $\alpha^{\frac{B}{T+\tau_0-t}} - 1$ and using the Monotone Convergence Theorem we get,

$$(T - t + \tau_0)f(t) = \sum_{n=1}^{\infty} \int -\frac{(\xi/\ln(\alpha))(B\ln(\alpha))^n}{n!(T - t + \tau_0)^{n-1}} dt + c \tag{C.38}$$

where c is the constant of integration. Integrating each term and substituting $f(T) = 0$ we get the result. The series term in (4.27) which is denoted as S_m has non-negative terms and hence is non-decreasing. For $t \leq T$, $S_m \leq \sum_{n=3}^{m} \frac{(B\ln(\alpha))^n \tau_0^2}{n!\tau_0^n} \leq \sum_{n=0}^{m} \frac{(B\ln(\alpha))^n \tau_0^2}{n!\tau_0^n} \xrightarrow{m\to\infty}$ $\tau_0^2 \exp(B\ln(\alpha)/\tau_0)$. Thus the series is convergent.

Appendix D

Proofs for Chapter 5

D.1 Proof of Lemma 12

The proof is based on a contradiction argument where we begin by supposing that for the optimal policy there is a $\hat{\mathbf{r}} \rightarrow Z_f$ with $\hat{r}_i > r_i$. By re-mapping the regions we will show that the objective function in (5.3) decreases, thus, contradicting the optimality claim and proving $\hat{\mathbf{r}} \not\rightarrow Z_f$.

We are given that $\bar{\mathbf{r}} \rightarrow Z_i$, hence, there is a neighborhood of $\bar{\mathbf{r}}$, which we denote as S_1, that is mapped to Z_i, i.e. $S_1 \in Z_i$ and $S_1 = \{\bar{\mathbf{x}} \mid \bar{\mathbf{x}} \in \Omega, ||\bar{\mathbf{x}} - \bar{\mathbf{r}}|| < \delta_1\}$ for some $\delta_1 > 0$. Further, by assumption $\hat{\mathbf{r}} \rightarrow Z_f$, there is a neighborhood of $\hat{\mathbf{r}}$ given as, $S_2 = \{\bar{\mathbf{x}} \mid \bar{\mathbf{x}} \in \Omega, ||\bar{\mathbf{x}} - \hat{\mathbf{r}}|| < \delta_2\}$ for some $\delta_2 > 0$, such that $S_2 \in Z_f$.

Now re-map the regions as follows. Map $S_1 \Rightarrow Z_f$ and $S_2 \Rightarrow Z_i$. To ensure the new mapping is feasible we must satisfy the QoS rate constraint for user i which entails the following equality.

$$\int_{S_2} x_i f(\bar{\mathbf{x}}) d\bar{\mathbf{x}} = \int_{S_1} x_i f(\bar{\mathbf{x}}) d\bar{\mathbf{x}} \tag{D.1}$$

The left side above is the throughput achieved over region S_2 under the new map and the right side is the throughput lost by re-mapping S_1 to Z_f. A set of $\delta_1, \delta_2 > 0$ exist that satisfy (D.1); to see this note that the integral over any region $\{S_k\}_{k=1}^2$ is a positive, continuous function with respect to δ_k, non-increasing as δ_k decreases and tends to zero as $\delta_k \downarrow 0$. Hence, starting with the largest δ_1, δ_2 values (that satisfy the S_1, S_2 definition) and then decreasing these values one can obtain $\{\delta_1, \delta_2 > 0\}$ such that each integral above is positive and the two are equal. Now, viewing δ_2 as a function of δ_1, it's clear that if a solution exists for some δ_1^0 then for all $\delta_1 \leq \delta_1^0$ a solution exists by the continuity and decreasing property of the integrals. We now proceed by choosing $\delta_1 \leq \delta_1^0$.

Using the First Mean Value theorem, [84], we can take the x_i outside the integrals as follows, $\int_{S_1} x_i f(\bar{\mathbf{x}})d\bar{\mathbf{x}} = (r_i + \epsilon_1)\int_{S_1} f(\bar{\mathbf{x}})d\bar{\mathbf{x}}$ and $\int_{S_2} x_i f(\bar{\mathbf{x}})d\bar{\mathbf{x}} = (\hat{r}_i + \epsilon_2)\int_{S_2} f(\bar{\mathbf{x}})d\bar{\mathbf{x}}$, where the $\{\epsilon_k\}_{k=1}^2$ depend on $\{\delta_k\}_{k=1}^2$ or equivalently on δ_1 (as δ_2 depends on δ_1 through (D.1)). With this, we can re-write (D.1) as,

$$(\hat{r}_i + \epsilon_2)\int_{S_2} f(\bar{\mathbf{x}})d\bar{\mathbf{x}} = (r_i + \epsilon_1)\int_{S_1} f(\bar{\mathbf{x}})d\bar{\mathbf{x}} \tag{D.2}$$

Now, looking at the objective function in (5.3), the change in its value due to the re-map equals the probability of region S_2 (added from Z_f to Z_i) minus the probability of region S_1 (removed from Z_i). Thus,

$$\begin{aligned}\Delta J &= -\int_{S_1} f(\bar{\mathbf{x}})d\bar{\mathbf{x}} + \int_{S_2} f(\bar{\mathbf{x}})d\bar{\mathbf{x}} \\ &= -\left(\frac{\hat{r}_i + \epsilon_2}{r_i + \epsilon_1} - 1\right)\int_{S_2} f(\bar{\mathbf{x}})d\bar{\mathbf{x}} \end{aligned} \tag{D.3}$$

Let $c = \hat{r}_i - r_i$, then, $c > 0$ (since by assumption $\hat{r}_i > r_i$). Using the First Mean Value theorem, we also have $\epsilon_k \to 0$ as $\delta_k \to 0$. Thus, for any c we can scale δ_1 to be small enough such that $\left(\frac{\hat{r}_i+\epsilon_2}{r_i+\epsilon_1} - 1\right) > 0$. Further, since the integral in (D.3) is the probability of S_2 which is strictly positive (regions with zero probability are uninteresting and have been removed from Ω), we finally get, $\Delta J < 0$. This completes the contradiction argument.

D.2 Proof of Lemma 13

The proof is based on a contradiction argument. To begin, consider $\bar{\mathbf{r}} \notin Z_f$ and suppose that for the optimal policy, $\bar{\mathbf{r}} \to Z_j$ such that,

$$\frac{r_i}{a_i} > \frac{r_j}{a_j} \tag{D.4}$$

We now give a re-mapping of the regions such that the objective function in (5.3) decreases or equivalently the probability of Z_f region increases, thus, proving that the earlier mapping cannot be optimal.

As the lemma involves only the i^{th} and j^{th} component, we will focus only on these

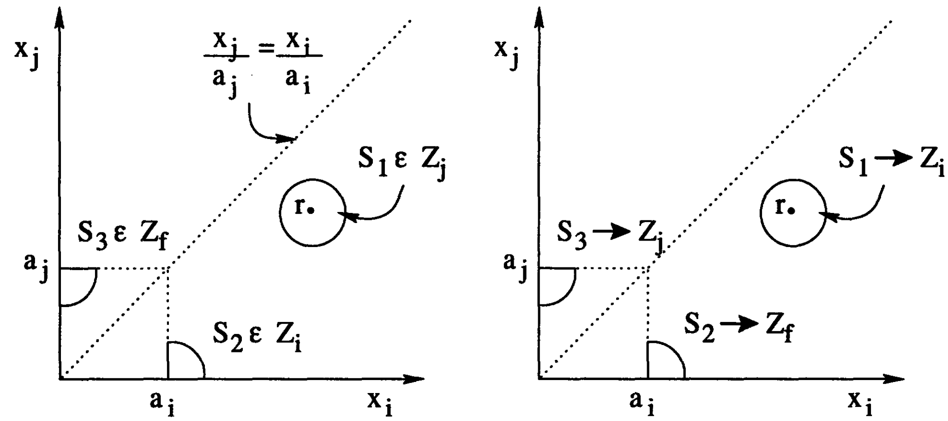


Fig. (a): Original mapping Fig. (b): New mapping

Figure D-1: Figure showing the mappings for the proof of Lemma 13.

components. Let $\bar{\mathbf{x}} \in \Omega$ denote a generic rate vector. Since by assumption $\bar{\mathbf{r}} \to Z_j$, there is a neighborhood around $\bar{\mathbf{r}}$ given as $S_1 = \{\bar{\mathbf{x}} \mid \bar{\mathbf{x}} \in \Omega, ||\bar{\mathbf{x}} - \bar{\mathbf{r}}|| < \delta_1\}$ for some $\delta_1 > 0$, such that $S_1 \in Z_j$. Next, since the optimal policy satisfies Lemma 12 (its violation would make the policy non-optimal to start with) we know that a_i is the infimum value of the i^{th} component among $\bar{\mathbf{x}} \to Z_i$. Thus, there exists a point $\bar{\mathbf{m}}$ with $m_i = a_i$ and a region around $\bar{\mathbf{m}}$, denoted S_2, that maps to Z_i; i.e. $S_2 \in Z_i$ and $S_2 = \{\bar{\mathbf{x}} \mid \bar{\mathbf{x}} \in \Omega, 0 < (x_i - m_i) < \delta_2\}$ for some $\delta_2 > 0$. Finally, since $\bar{\mathbf{R}}$ does not lie on the boundary of feasible throughput vectors the region Z_f is not null. Hence, there exists $\bar{\mathbf{n}}$ with $n_j = a_j > 0$ and a region around $\bar{\mathbf{n}}$, denoted S_3, that maps to Z_f; namely, $S_3 \in Z_f$ and $S_3 = \{\bar{\mathbf{x}} \mid \bar{\mathbf{x}} \in \Omega, 0 < (n_j - x_j) < \delta_3\}$ for some $\delta_3 > 0$. The regions S_1, S_2, S_3 are depicted in Figure D-1(a).

Now re-map these regions as follows. Map $S_1 \Rightarrow Z_i$, $S_2 \Rightarrow Z_f$ and $S_3 \Rightarrow Z_j$ as shown in Figure D-1(b). To ensure the new mapping is feasible we must satisfy the QoS rate constraints for user i and user j, which entails the following equalities.

$$\int_{S_2} x_i f(\bar{\mathbf{x}}) d\bar{\mathbf{x}} = \int_{S_1} x_i f(\bar{\mathbf{x}}) d\bar{\mathbf{x}} \quad \text{(D.5)}$$

$$\int_{S_3} x_j f(\bar{\mathbf{x}}) d\bar{\mathbf{x}} = \int_{S_1} x_j f(\bar{\mathbf{x}}) d\bar{\mathbf{x}} \quad \text{(D.6)}$$

Equation (D.5) matches the throughput lost for user i due to the re-map of $S_2 \Rightarrow Z_f$ and the throughput gained by $S_1 \Rightarrow Z_i$, while (D.6) gives a similar equality for user j. To see why a set of $\{\delta_k\}_{k=1}^3$ exist that solve the above equations, note that the integral over any region S_k is a continuous, positive function of δ_k, decreasing (or non-increasing) as δ_k decreases and tends to zero as $\delta_k \downarrow 0$. Hence, starting with the largest δ_1 (that satisfies the S_1 definition), decrease it until a δ_2 is obtained that solves (D.5). By the non-nullity of S_1, S_2 and the above property of the integrals such a solution $\delta_1, \delta_2 > 0$ exists. Similarly obtain a δ_1, δ_3 that solves (D.6). Finally, taking δ_1 as the minimum of the two solutions, re-obtain δ_2, δ_3 such that both (D.5) and (D.6) are satisfied. Now, viewing δ_2, δ_3 as functions of δ_1, it's clear that if a solution exists for some δ_1^0, then, for all $\delta_1 \leq \delta_1^0$ a solution exists by the continuity and decreasing property of the integrals. We now proceed by choosing $\delta_1 \leq \delta_1^0$.

Using the First Mean Value theorem, [84], we can re-write the above integrals as,

$$
\begin{aligned}
(a_i + \epsilon_2) \int_{S_2} f(\bar{\mathbf{x}}) d\bar{\mathbf{x}} &= (r_i + \epsilon_1) \int_{S_1} f(\bar{\mathbf{x}}) d\bar{\mathbf{x}} && \text{(D.7)} \\
(a_j + \epsilon_3) \int_{S_3} f(\bar{\mathbf{x}}) d\bar{\mathbf{x}} &= (r_j + \epsilon_4) \int_{S_1} f(\bar{\mathbf{x}}) d\bar{\mathbf{x}} && \text{(D.8)}
\end{aligned}
$$

where the $\{\epsilon_k\}$ above depend on the $\{\delta_k\}$ or equivalently on δ_1 (as δ_2, δ_3 depend on δ_1 through (D.5) and (D.6)). Next, looking at the objective function in (5.3), the change in its value due to the re-map equals the probability of region S_3 (added from Z_f to Z_j) minus the probability of region S_2 (removed from Z_i). Thus,

$$
\begin{aligned}
\Delta J &= -\int_{S_2} f(\bar{\mathbf{x}}) d\bar{\mathbf{x}} + \int_{S_3} f(\bar{\mathbf{x}}) d\bar{\mathbf{x}} \\
&= -\left(\frac{r_i + \epsilon_1}{a_i + \epsilon_2} - \frac{r_j + \epsilon_4}{a_j + \epsilon_3} \right) \int_{S_1} f(\bar{\mathbf{x}}) d\bar{\mathbf{x}} && \text{(D.9)}
\end{aligned}
$$

Let $c = \frac{r_i}{a_i} - \frac{r_j}{a_j}$, then, from (D.4) we have $c > 0$. From the First Mean Value theorem we also have $\epsilon_k \to 0$ as $\delta_k \to 0$. Thus, for any given c we can scale δ_1 to be small enough such that $\left(\frac{r_i + \epsilon_1}{a_i + \epsilon_2} - \frac{r_j + \epsilon_4}{a_j + \epsilon_3} \right) > 0$. Further, since the integral in (D.9) is the probability of S_1 which is strictly positive, we finally get $\Delta J < 0$. This completes the proof.

D.3 Proof of Theorem XV

We will prove optimality of policy Γ, defined in (5.9), by showing that for any other feasible policy $\tilde{\Gamma}$ we have $\sum_{i=1}^{N} E[I_i] \leq \sum_{i=1}^{N} E[\tilde{I}_i]$ where $I_i(\bar{\mathbf{r}})$ and $\tilde{I}_i(\bar{\mathbf{r}})$ are the indicator functions for the respective policies. We know that policy Γ satisfies the throughput rate constraints with equality, i.e. $E[r_i I_i] = R_i$. If $\tilde{\Gamma}$ does not, it is trivial to prove that $\tilde{\Gamma}$ cannot be optimal. Now, suppose $\tilde{\Gamma}$ also satisfies the rate constraints with equality, i.e. $E[r_i \tilde{I}_i] = R_i$, then, the objective function for policy $\tilde{\Gamma}$ can be re-written as,

$$\sum_{i=1}^{N} E[\tilde{I}_i] = \sum_{i=1}^{N} E[\tilde{I}_i] - \sum_{i=1}^{N} \frac{1}{a_i}(E[r_i\tilde{I}_i] - R_i) \tag{D.10}$$

where $\{a_i\}$ is the threshold vector for policy Γ. Note that the second term in (D.10) is zero. Re-arranging (D.10) we get,

$$\sum_{i=1}^{N} E[\tilde{I}_i] = E\left[\sum_{i=1}^{N}\left(1 - \frac{r_i}{a_i}\right)\tilde{I}_i\right] + \sum_{i=1}^{N}\frac{R_i}{a_i} \tag{D.11}$$

For any vector $\bar{\mathbf{r}}$ we have the following two cases.

Case 1: Suppose $r_i \leq a_i, \forall i$, then, policy Γ does not choose any QoS user (Equation (5.9)) and $I_i = 0, \forall i = 1, \ldots, N$. Now, since $r_i \leq a_i$, we have $(1 - \frac{r_i}{a_i}) \geq 0, \forall i$. This implies that whether $\tilde{\Gamma}$ chooses or does not choose a QoS user we have the following inequality,

$$\sum_{i=1}^{N}\left(1 - \frac{r_i}{a_i}\right)\tilde{I}_i \geq 0 = \sum_{i=1}^{N}\left(1 - \frac{r_i}{a_i}\right)I_i \tag{D.12}$$

Case 2: Suppose $r_i > a_i$ for some index i. Let j be the chosen user for policy Γ, then, from (5.9) we see that r_j/a_j has the maximum value. Thus, $(1 - \frac{r_j}{a_j}) \leq (1 - \frac{r_i}{a_i}), \forall i$ and also $(1 - \frac{r_j}{a_j}) < 0$. Again irrespective of what $\tilde{\Gamma}$ chooses,

$$\sum_{i=1}^{N}\left(1 - \frac{r_i}{a_i}\right)\tilde{I}_i \geq \left(1 - \frac{r_j}{a_j}\right) = \sum_{i=1}^{N}\left(1 - \frac{r_i}{a_i}\right)I_i \tag{D.13}$$

From (D.11), (D.12) and (D.13) we get,

$$\sum_{i=1}^{N} E[\tilde{I}_i] \geq E\left[\sum_{i=1}^{N}\left(1 - \frac{r_i}{a_i}\right)I_i\right] + \sum_{i=1}^{N}\frac{R_i}{a_i} = \sum_{i=1}^{N} E[I_i]$$

where the last equality follows from (D.10) replacing $\tilde{I}_i$ with I_i. This completes the proof.

D.4 Proof of Lemma 16

To prove the lemma we need to show the following two relationships, $\ln(1/(1-\gamma^{\frac{1}{N}})) = O(\ln(N))$ and $\ln(N) = O(\ln(1/(1-\gamma^{\frac{1}{N}})))$. We begin by proving the first relationship. Since $\gamma \in (0,1)$ and $N \geq 1$ is a positive integer, we have $0 < \gamma^{\frac{1}{N}} < 1$. Taking a power series expansion of $\left(\frac{1}{1-\gamma^{\frac{1}{N}}}\right)$ we get,

$$\begin{aligned}
\ln\left(\frac{1}{1-\gamma^{\frac{1}{N}}}\right) &= \ln\left(1+\gamma^{1/N}+\ldots+\gamma^{(N-1)/N}+\gamma(1+\gamma^{1/N}+\ldots)+\gamma^2(\ldots)\right) && \text{(D.14)}\\
&= \ln\left(\frac{1+\gamma^{1/N}+\ldots+\gamma^{(N-1)/N}}{1-\gamma}\right) && \text{(D.15)}\\
&\leq \ln\left(\frac{N}{1-\gamma}\right) = \ln(N)-\ln(1-\gamma) && \text{(D.16)}
\end{aligned}$$

The inequality above follows, since $\gamma < 1 \Rightarrow \left(1+\gamma^{1/N}+\ldots+\gamma^{(N-1)/N}\right) \leq N$; thus we get $\ln(1/(1-\gamma^{\frac{1}{N}})) = O(\ln(N))$. To prove the reverse relationship, i.e. $\ln(N) = O(\ln(1/(1-\gamma^{\frac{1}{N}})))$, proceed as follows. Using the standard inequality, $\ln(N) \leq 1+\frac{1}{2}+\ldots+\frac{1}{N-1}$, we get,

$$\begin{aligned}
\gamma\ln(N) &\leq \gamma+\frac{\gamma}{2}+\ldots+\frac{\gamma}{N-1}\\
&\leq \gamma^{1/N}+\frac{\gamma^{2/N}}{2}+\ldots+\frac{\gamma^{N/N}}{N-1} \quad \text{(since } 0<\gamma<1\text{)}\\
&\leq \ln\left(\frac{1}{1-\gamma^{\frac{1}{N}}}\right) && \text{(D.17)}
\end{aligned}$$

where the last inequality above follows by truncating the power series expansion of $-\ln(1-\gamma^{\frac{1}{N}})$. Thus, $\ln(N) \leq \frac{1}{\gamma}\ln(1/(1-\gamma^{\frac{1}{N}}))$ which gives $\ln(N) = O(\ln(1/(1-\gamma^{\frac{1}{N}})))$.

D.5 Proof of Theorem XVII

Starting with (5.16) we can write it as,

$$\frac{R}{\mu} = \ln\left(\frac{1}{1-\gamma^{\frac{1}{N}}}\right)\sum_{k=0}^{N-1}\binom{N-1}{k}\frac{(-1)^k(1-\gamma^{\frac{1}{N}})^{(k+1)}}{k+1}+\sum_{k=0}^{N-1}\binom{N-1}{k}\frac{(1-\gamma^{\frac{1}{N}})^{(k+1)}}{(k+1)^2} \quad \text{(D.18)}$$

Consider the first term in (D.18) above; it can be evaluated as follows. Let $\alpha = (1-\gamma^{\frac{1}{N}})$, then, since $\gamma \in (0,1)$ we have $\alpha \in (0,1)$.

$$\begin{aligned}
\sum_{k=0}^{N-1}\binom{N-1}{k}(-1)^k\frac{\alpha^{(k+1)}}{k+1} &= \sum_{k=0}^{N-1}\binom{N-1}{k}\int_0^\alpha(-x)^k dx \\
&\overset{(a)}{=} \int_0^\alpha (1-x)^{N-1}dx \qquad \text{(D.19)}\\
&= \frac{1-(1-\alpha)^N}{N} = \frac{1-\gamma}{N} \qquad \text{(D.20)}
\end{aligned}$$

Equality (a) above follows by interchanging the summation and the integral and using the Binomial expansion. Thus, we get, $\ln\left(\frac{1}{\alpha}\right)\sum_{k=0}^{N-1}\binom{N-1}{k}(-1)^k\frac{\alpha^{(k+1)}}{k+1} = \ln\left(\frac{1}{\alpha}\right)\frac{1-\gamma}{N}$. Now, consider the second term in (D.18) and proceed as follows. First, since (D.20) holds for all α, we get the identity, $\sum_{k=0}^{N-1}\binom{N-1}{k}(-1)^k\frac{x^{(k+1)}}{k+1} = \frac{1-(1-x)^N}{N}$. Dividing both sides by x and integrating from 0 to α, gives,

$$\begin{aligned}
\int_0^\alpha\sum_{k=0}^{N-1}\binom{N-1}{k}(-1)^k\frac{x^k}{k+1} &= \int_0^\alpha\left(\frac{1-(1-x)^N}{Nx}\right)dx \\
\Rightarrow \sum_{k=0}^{N-1}\binom{N-1}{k}\frac{-1^k\alpha^{k+1}}{(k+1)^2} &= \int_0^\alpha\left(\frac{1-(1-x)^N}{Nx}\right)dx \\
&\le \int_0^\alpha dx = \alpha = (1-\gamma^{\frac{1}{N}}) \qquad \text{(D.21)}
\end{aligned}$$

The inequality above follows by noting that $\frac{1-(1-x)^N}{Nx}$ is positive, monotonically non-increasing over $x \in [0,1]$, for fixed $N \ge 1$, and has a maximum value equal to 1 at $x = 0$. To show the monotonic behavior of $\frac{1-(1-x)^N}{Nx}$, we claim that its derivative with respect to x is always non-positive for all $x \in [0,1]$. To prove this, first note that the derivative is given as $\frac{xN(1-x)^{N-1}-(1-(1-x)^N)}{Nx^2}$. Now set $1-x=y$ then, we need to show $N(1-y)y^{N-1} \le 1-y^N$ which is equivalent to showing $(1-y^N)/(1-y) \ge Ny^{N-1}$. But note that $(1-y^N)/(1-y) = 1+y+y^2+..+y^{N-1} \ge y^{N-1}+..+y^{N-1}$, since we replaced all the terms with y^{N-1}, which is the smallest term as $y \in [0,1]$. This verifies the claim.

Equation (D.21) further gives, $\frac{N}{1-\gamma}\left(\sum_{k=0}^{N-1}\binom{N-1}{k}\frac{-1^k\alpha^{k+1}}{(k+1)^2}\right) \le \frac{N}{1-\gamma}(1-\gamma^{\frac{1}{N}}) \xrightarrow{N\to\infty} \frac{-\ln(\gamma)}{1-\gamma}$ (which is finite for $0<\gamma<1$) and since $\frac{N}{1-\gamma}(1-\gamma^{\frac{1}{N}})$ is monotonically non-decreasing in N with a finite limiting value, it is bounded for all N. To see the monotonic behavior differentiate w.r.t N, which gives $\frac{1-\gamma^{1/N}-\frac{\gamma^{1/N}\ln(1/\gamma)}{N}}{1-\gamma}$. To show that this is positive we need to show that $\gamma^{1/N}(1+\ln((1/\gamma)^{1/N})) \leq 1$. Using $\ln(x) \leq x-1$ we get, $\gamma^{1/N}(1+\ln((1/\gamma)^{1/N})) \leq \gamma^{1/N}(1/\gamma)^{1/N} = 1$, hence the above is true. Thus, we get,

$$\frac{N}{1-\gamma}\left(\sum_{k=0}^{N-1}\binom{N-1}{k}\frac{-1^k\alpha^{k+1}}{(k+1)^2}\right) \leq \frac{\ln(1/\gamma)}{1-\gamma} \tag{D.22}$$

Now, using the above simplifications we can re-write (D.18) as,

$$\frac{R}{\mu} = \frac{1-\gamma}{N}\left(\ln\left(\frac{1}{\alpha}\right) + \frac{N}{1-\gamma}\sum_{k=0}^{N-1}\binom{N-1}{k}\frac{-1^k\alpha^{k+1}}{(k+1)^2}\right) \tag{D.23}$$

For $\gamma \in (0,1)$, the first term within brackets above, grows as $\ln(\frac{1}{\alpha}) = \Theta(\ln(N))$ (using Lemma 16) whereas the second term is bounded (from (D.22)). Hence, for large N, R^{opt} can be expressed as,

$$\frac{R^{opt}}{\mu} = \frac{1-\gamma}{N}\Theta(\ln(N)) \tag{D.24}$$

From (5.19) and (D.24) we get the result in (5.21),

Bibliography

[1] A. K. Katsaggelos, Y. Eisenberg, F. Zhai, R. Berry and T. Pappas, "Advances in efficient resource allocation for packet-based real-time video transmission", *Proceedings of the IEEE*, vol. 93, no. 1, Jan. 2005.

[2] A. Ephremides, "Energy concerns in wireless networks", *IEEE Wireless Communications*, vol. 9, issue 4, pp. 48-59, August 2002.

[3] A. Jalali, R. Padovani and R. Pankaj, "Data throughput of CDMA-HDR a high efficiency high data rate personal communication wireless system", *IEEE Vehicular Technology Conf.*, vol. 3, 2000.

[4] D. Tse and P. Viswanath, *Fundamentals of Wireless Communication*, Cambridge University Press, 2005.

[5] W. C. Jakes, *Microwave Mobile Communications*, Wiley-IEEE press, May 1994.

[6] S. Nanda, K. Balachandran and S. Kumar, "Adaptation techniques in wireless packet data services", *IEEE Communications Magazine*, Jan. 2000.

[7] L. Tsaur and D. C. Lee, "Closed-loop architecture and protocols for rapid dynamic spreading gain adaptation in CDMA networks", *IEEE Proc. of INFOCOM*, 2004.

[8] R. Berry, "Power and delay optimal transmission scheduling: small delay asymptotics", *Proceedings of the International Symposium on Information Theory*, July 2003.

[9] R. Berry, R. Gallager, "Communication over fading channels with delay constraints", *IEEE Tran. on Information Theory*, vol. 48, no. 5, May 2002.

[10] B. Prabhakar, E. Biyikoglu, A. El Gamal, "Energy-efficient transmission over a wireless link via lazy packet scheduling", *IEEE INFOCOM 2001*, vol. 1, pp 386 - 394, April 2001.

[11] A. El Gamal, C. Nair, B. Prabhakar, E. Uysal-Biyikoglu and S. Zahedi, "Energy-efficient scheduling of packet transmissions over wireless networks", *Proceedings of the IEEE Infocom 2002*, pp. 1773-1782, 2002.

[12] E. Uysal-Biyikoglu and A. El Gamal, "On adaptive transmission for energy-efficiency in wireless data networks," in *IEEE Tran. on Information Theory*, Dec. 2004.

[13] A. Fu, E. Modiano, J. Tsitsiklis, "Optimal energy allocation for delay constrained data transmission over a time-varying channel", *IEEE INFOCOM 2003*, vol. 2, pp 1095 - 1105, April 2003.

[14] M. Zafer, E. Modiano, "A Calculus approach to minimum energy transmission policies with quality of service guarantees", *Proceedings of the IEEE INFOCOM 2005*, vol. 1, pp. 548-559, March 2005.

[15] M. Zafer and E. Modiano, "A calculus approach to energy-efficient data transmission with quality-of-service constraints", *in preparation.*

[16] M. Zafer and E. Modiano, "Optimal rate control for delay-constrained data transmission over a wireless channel", *submitted to IEEE Transactions on Information Theory.*

[17] M. Zafer and E. Modiano, "Delay-constrained energy efficient data transmission over a wireless fading channel", *Information Theory and Applications Workshop*, San Deigo, January 2007.

[18] M. Zafer and E. Modiano, "Optimal adaptive data transmission over a fading channel with deadline and power constraints", *Conference on Information Sciences and Systems*, Princeton, New Jersey, March 2006.

[19] M. Zafer and E. Modiano, "Continuous-time optimal rate control for delay constrained data transmission", *43rd Annual Allerton Conference on Communication, Control and Computing, Monticello*, Sept. 2005.

[20] A. Tarello, J. Sun, M. Zafer and E. Modiano, "Minimum energy transmission scheduling subject to deadline constraints", *WiOpt 2005*, April 2005.

[21] M. Khojastepour, A. Sabharwal, "Delay-constrained scheduling: power efficiency, filter design and bounds", *IEEE INFOCOM 2004*, March 2004.

[22] P. Nuggehalli, V. Srinivasan, R. Rao, "Delay constrained energy efficient transmission strategies for wireless devices", *IEEE INFOCOM 2002*, vol. 3 , pp. 1765 - 1772,June 2002.

[23] B. Hajek and P. Seri, "Lex-optimal on-line multiclass scheduling with hard deadlines", *Mathematics of Operations Research*, vol. 39, pp. 562-596, August 2005.

[24] J. R. Jackson, "Scheduling a production line to minimize maximum tardiness", Research Report 43, Management Science Research Project, University of California, Los Angeles, CA, 1955.

[25] T. L. Ling, N. Shroff, "Scheduling real-time traffic in ATM networks", In proceedings of IEEE INFOCOM 1996, San Francisco, CA, 1996.

[26] L. Miao and C. Cassandras, "Optimal transmission scheduling for energy-efficient wireless networks", *IEEE INFOCOM 2006*, April 2006.

[27] B. Collins, R. Cruz, "Transmission policies for time varying channels with average delay constraints", *Proceedings of Allerton conference on communication, control and computing*, Monticello, IL, 1999.

[28] D. Rajan, A. Sabharwal and B. Aazhang, "Delay bounded packet scheduling of bursty traffic over wireless channels", *IEEE Transactions on Information Theory*, , vol. 50, 1, pp. 125-144, Jan. 2004.

[29] M. Goyal, A. Kumar and V. Sharma, "Power constrained and delay optimal policies for scheduling transmission over a fading channel", *Proc. of the IEEE INFOCOM 2003.*

[30] M. Zafer and E. Modiano, "Blocking probability and channel assignment in wireless networks", *IEEE Transactions on Wireless Communications*, vol. 5, no. 4, April 2006.

[31] M. Zafer, "Channel assignment algorithms and blocking probability analysis for connection-oriented traffic in wireless networks", *Masters Thesis, Massachusetts Institute of Technology*, Laboratory for Information and Decision Systems, 2003.

[32] B. Ata, "Dynamic power control in a wireless static channel subject to a quality of service constraint", *Operations Research 53* (2005), no 5, 842-851.

[33] G. J. Foschini, "A simple distributed autonomous power control algorithm and its convergence", *IEEE Transactions on Vehicular Technology*, 42(4), 1993.

[34] S. Grandhi, J. Zander and R. Yates, "Constrained power control", *International Journal of Wireless Personal Communications*, 1(4), 1995.

[35] P. Viswanath, V. Anantharam and D. Tse, "Optimal sequences, power control and capacity of synchronous CDMA systems with linear MMSE multiuser receivers", *IEEE Transactions on Information Theory*, vol. 45(6), Sept., 1999.

[36] L. Tassiulas and A. Ephremides, "Stability properties of constrained queueing systems and scheduling policies for maximum throughput in multihop radio networks", *IEEE Transactions on Automatic Control*, vol. 37, no. 12, Dec. 1992.

[37] L. Tassiulas and A. Ephremides, "Dynamic server allocation to parallel queues with randomly varying connectivity", *IEEE Transactions on Information Theory*, vol. 39, pp. 466-478, March 1993.

[38] R. Buche and H. Kushner, "Stability and control of mobile communications systems with time varying channels", *IEEE Transactions on Automatic Control*, vol. 49, 11, pp. 1954-1962, Nov. 2004.

[39] M. Neely, E. Modiano and C. Rohrs, "Power and server allocation in a multi-beam satellite with time varying channels", *IEEE INFOCOM 2002*, vol. 3, pp 1451-1460, June 2002.

[40] M. J. Neely, "Dynamic power allocation and routing for satellite and wireless networks with time varying channels", *Ph.D. Dissertation, Massachusetts Institute of Technology, LIDS*, November 2003.

[41] M. J. Neely, "Energy optimal control for time varying wireless networks", *IEEE INFOCOM*, March 2005.

[42] A. L. Stolyar, "Maximizing queueing network utility subject to stability: greedy primal-dual algorithm", *Queueing Systems* vol. 50, no.4, pp.401-457, 2005.

[43] A. Eryilmaz, R. Srikant and J. Perkins, "Stable scheduling policies for fading wireless channels," *IEEE/ACM Transactions on Networking*, vol. 13, no. 2, pp. 411-424, April 2005.

[44] E. Modiano, D. Shah and G. Zussman, "Maximizing throughput in wireless networks via gossiping", *in Proceedings of ACM Sigmetric*, June 2006.

[45] E. M. Yeh and A. S. Cohen, "Throughput and delay optimal resource allocation in multi-access fading channels" *Proceedings of the international symposium on information theory*, May 2003.

[46] R. L. Cruz and A. V. Santhanam, "Optimal routing, link scheduling and power control in multi-hop wireless networks", *IEEE INFOCOM 2003*, April 2003.

[47] C. Hsu, A. Ortega and M. Khansari, "Rate control for robust video transmission over burst-error wireless channels", *IEEE Journal on Selected Areas in Communications*, vol. 17, no. 5, May 1999.

[48] Y. Eisenberg, C. Luna, T. Pappas, R. Berry and A. Katsaggelos, "Joint source coding and transmission power management for energy efficient wireless video communications", *IEEE Transactions on Circuits and Systems for Video Technology*, vol. 12, no. 6, June 2002.

[49] A. Goldsmith, "The capacity of downlink fading channels with variable rate and power", *IEEE Tran. Vehicular Technology*, vol. 45, pp. 1218-1230, Oct. 1997.

[50] R. L. Cruz, "A calculus for network delay, Part I: Network elements in isolation",*IEEE Transactions on Information Theory"*, vol. 37, no. 1, pp. 114-131, Jan., 1991.

[51] R. L. Cruz, "A calculus for network delay, Part II: Network analysis",*IEEE Transactions on Information Theory"*, vol. 37, no. 1, pp. 132-141, Jan., 1991.

[52] R. L. Cruz, "Quality of service guarantees in virtual circuit switched networks", *IEEE JSAC*, vol. 13, no.6, pp. 1048-1056, Aug. 1995.

[53] J. Le Boudec and P. Thiran, "Network Calculus", *Springer Verlag*, LNCS 2050, 2001.

[54] F. Babich and G. Lombardi, "A Markov model for the mobile propagation channel", *IEEE Transactions on Vehicular Technology*, vol. 49, no. 1, pp. 63-73, Jan. 2000.

[55] H. S. Wang, "On verifying the first-order Markovian assumption for a Rayleigh fading channel model," *IEEE Transactions on Vehicular Technology*, vol. VT-45, pp. 353357, May 1996.

[56] F. Babich and G. Lombardi, "A measurement based Markov model for the indoor propagation channel," *in Proceedings of IEEE VTC'97*, Phoenix, AZ, May 57, 1997, pp. 7781.

[57] Q. Zhang, S. Kassam, "Finite-State Markov Model for Rayleigh Fading Channels", *IEEE Tran. on Communications*, vol. 47, no. 11, Nov. 1999.

[58] E. N. Gilbert, "Capacity of burst-noise channel", *Bell Syst. Tech. J.*,vol. 39, no. 9, pp. 1253-1265, Sept. 1960.

[59] E. O. Elliott, "Estimates of error rates for codes on burst-noise channels", *Bell Syst. Tech. J.*, vol. 42, no. 9, pp. 1977-1997, Sept. 1963.

[60] A. Goldsmith and P. Varaiya, "Capacity, mutual Information and coding for finite state markov channels", *IEEE Transactions on Information Theory*, 1996.

[61] D. P. Bertsekas, *Dynamic Programming and Optimal Control, Vol. 1 and 2*, Athena Scientific, 2005.

[62] Hurewicz W., *Lectures on Ordinary Differential Equations*, Dover Publications.

[63] W. Fleming and H. Soner, *Controlled Markov Processes and Viscosity Solutions*, Springer-Verlag, 1993.

[64] M. Davis, *Markov Models and Optimization*, Chapman and Hall, 1993.

[65] B. Oksendal, *Stochastic Differential Equations*, Springer, 5^{th} edn., 2000.

[66] D. Luenberger, *Optimization by vector space methods*, John Wiley & sons, 1969.

[67] T. Cover and J. Thomas, *Elements of Information Theory*, John Wiley & Sons, 1991.

[68] D. P. Bertsekas, A. Nedic and A. Ozdaglar, *Convex analysis and optimization*, Athena Scientific, 2003.

[69] J. Cronin, *Differential Equations: Introduction and Qualitative Theory*, New York: Marcel Dekker, Inc., 1980.

[70] W. Rudin, *Principles of Mathematical Analysis*, McGraw-Hill, 3^{rd} edition, Singapore, 1976.

[71] E.R. Pinch, *Optimal Control and the Calculus of Variations*, Oxford University Press, 1993.

[72] M. Andrews, K. Kumaran, K. Ramanan, A. Stolyar, P. Whiting, and R. Vijaykumar, "Providing quality of service over a shared wireless link", *IEEE Communications Magazine*, pp. 150-154, Feb. 2001.

[73] M. Zafer and E. Modiano, "Joint scheduling of rate-guaranteed and best-effort services over a wireless channel", *IEEE CDC-ECC 2005*, Seville, Spain, Dec. 2005.

[74] M. Zafer and E. Modiano, "Joint scheduling of rate-guaranteed and best-effort users over a wireless fading channel", *IEEE Transactions on Wireless Communications.*

[75] X. Liu, E. Chong and N. Shroff, "A framework for opportunistic scheduling in wireless networks " *Computer Networks*, 41, pp. 451-474, 2003.

[76] S. Borst and P. Whiting, "Dynamic rate control algorithms for HDR throughput optimization", *IEEE INFOCOM*, Alaska, April 2001.

[77] S. Borst, "User-level performance of channel-aware scheduling algorithms in wireless data networks", *IEEE/ACM Transactions on Networking*, vol. 13, no. 3, pp. 636-647, 2005.

[78] Y. Liu and E. Knightly, "Opportunistic fair scheduling over multiple wireless channels", *IEEE INFOCOM*, San Francisco, 2003.

[79] S. Shakkottai and A. Stolyar, "Scheduling Algorithms for a Mixture of Real-Time and Non-Real-Time Data in HDR", *Proc. International Teletraffic Congress (ITC-17)*, Brazil, Sept. 2001.

[80] P. Viswanath, D. Tse and R. Laroia, "Opportunistic Beamforming using Dumb Antennas", *IEEE Trans. on Information Theory*, 48(6), June, 2002.

[81] P. Liu, R. Berry and M. Honig, "Delay-Sensitive Packet Scheduling in Wireless Networks", *IEEE WCNC*, New Orleans, 2003.

[82] M. Nakagami, "The m-distribution - A general formula of intensity distribution of fading," *Statistical Methods in Radio Wave Propagation*, W. C. Hoffman, Ed. London, England: Pergamon, 1960

[83] H. Kushner and G. Yin, "Stochastic approximation algorithms and applications", Springer, New York, 1997.

[84] Gradshteyn I.S. and Ryzhik I.M., "*Table of Integrals, Series and Products*", Academic Press,

[85] G. R. Grimmett, and D. R. Stirzaker, *Probability and Random Processes*, 3rd ed. New York, Oxford University Press, 2001.

[86] R. G. Gallager, *Discrete Stochastic Processes*, Kluwer Publishers, 1995.

[87] Kushner H., *Stochastic Stability and Optimal Control*, Academic Press, NY, 1967.

[88] Kushner H., Dupuis P., *Numerical Methods for Stochastic Control Problems in Continuous Time*, Springer-Verlag, 1992.