THE COMPLEXITY THEORY OF SWITCHING NETWORKS

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THE COMPLEXITY THEORY
OF SWITCHING NETWORKS

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ABSTRACT

We consider switching networks of the type used for line switching in communication networks or for reconfiguration of modular computer systems. These networks are capable of establishing various combinations of simultaneous routes between inputs and outputs in such a way that routes from different inputs are disjoint. We examine the complexity (measured by the number of switch contacts) of such networks, as well as the complexity (measured by the number of arithmetic operations) of algorithms for controlling them.

We study three network applications: partial concentration, connection, and distribution. For partial concentration, certain specified outputs are to be joined to inputs (which are not specified) in such a way that each output is joined to a different input; the number of joinings to be established simultaneously (called the capacity) is less than the number of inputs, which, in turn, is less than the number of outputs. For connection, the particular input to which each output is to be joined is specified, and a different input is specified for each output. For distribution, the particular input to which each output is to be joined is specified, and the same input may be specified for more than one output.

For each of these applications, we study three modes of operation: rearrangeably nonblocking, incrementally nonblocking, and incrementally $\epsilon$-blocking. In rearrangeably nonblocking operation, the entire assignment of desired joinings is assumed to be specified and a network state realizing the assignment is sought. In incrementally nonblocking operation, we assume that requests to establish and disestablish joinings may arrive at any time and that routes satisfying these requests must be added to, and deleted from, the network state as they arrive, always in such a way that previously established routes are not disturbed. Incrementally $\epsilon$-blocking operation is similar to incrementally nonblocking operation, but a fixed probability (not exceeding $\epsilon$) is allowed that a randomly chosen request in a randomly chosen state cannot be satisfied.

For both rearrangeably and incrementally nonblocking partial concentration networks, we obtain upper and lower bounds having the same order of growth. The order of growth is different for the rearrangeable and incremental versions. Analogous results have already been obtained for connection networks, where the order of growth for the rearrangeable and incremental versions is the same. For distribution, we obtain upper and lower bounds with the same order of growth for rearrangeably nonblocking networks and bounds differing by a factor of the logarithm of the capacity for incrementally nonblocking networks. We consider $\epsilon$-blocking networks for all three applications and in each case we obtain upper bounds that have the same order of growth as for the rearrangeable versions. We also consider the $\epsilon$-dependence of the minimum number of contacts, and in each case establish a reciprocal relationship between this and the minimum number of contacts in incrementally nonblocking networks. Finally, we consider algorithms for controlling switching networks. For rearrangeably nonblocking networks, we obtain algorithms for realizing an assignment with a number of arithmetic operations proportional to the number of contacts in the network. For incrementally nonblocking networks, we obtain representations of the network state using a number of bits proportional to the number of contacts in the network, with the property that a request can be satisfied with a number of arithmetic operations bounded by a polynomial in the logarithm of the capacity.
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I. INTRODUCTION

This report studies the complexity theory of switching networks. In this introductory section we describe, in informal terms, the range of problems to which we address ourselves.

1.1 SWITCHING NETWORKS

A switching network is an interconnection of components used to establish various combinations of simultaneous routes that join inputs to outputs. Switching networks occupy a central position in switched-line communication systems and are of growing importance in certain modular computer systems.

We shall confine our attention to three of the many possible applications for switching networks: concentration, connection, and distribution. Imagine a set of equivalent devices (say, computational processing units) each capable of supplying some service, and another, larger, set of devices (say, communication terminals) each potentially capable of demanding this service. As long as the number of terminals actually demanding service does not exceed the number of processing units, we can meet all demands by joining each terminal actually demanding service to one of the processing units, and it is unimportant which terminal is joined to which processing unit, since the latter are equivalent. A switching network for accomplishing this (when the processing units are attached to its inputs and the terminals are attached to its outputs) will be called a concentration network. Next, imagine that the processing units are not equivalent and that each terminal demanding service specifies which of the processing units it is to be joined to. As long as no two terminals demand to be joined to the same processing unit, we can again meet all demands, and a switching network for accomplishing this will be called a connection network. Finally, imagine that the processing units are replaced by autonomous signal sources so that two or more terminals may demand to be joined to the same signal source. A switching network for meeting these demands will be called a distribution network.

Our convention regarding inputs and outputs may bear some further explanation. In the case of distribution networks, it is most natural to think of the signals as flowing from the inputs to the outputs, but to think of the requests to establish joinings as arising at the outputs. We have extended this convention to connection and concentration networks, where it is somewhat less natural, in order to increase the coherence of our definitions and to simplify comparisons.

In each of these three cases (concentration, connection, and distribution), we shall consider two different methods of operating the network: rearrangeable and incremental. In rearrangeable operation, the entire assignment of desired joinings is assumed to be specified and a network state realizing the assignment is sought. In incremental operation, we assume that requests to establish and disestablish joinings may arrive at any
time and that routes satisfying these requests must be added to, or deleted from, the
network state as they arrive, always in such a way that previously established routes
are not disturbed.

The application and method of operation determine a set of demands (assignments
to be realized or requests to be satisfied) that may be made upon the network. If the
network is such that any of these demands can be met, we say that it is nonblocking. If
certain assignments cannot be realized or, in certain states, certain requests cannot be
satisfied, the network may nevertheless be of practical value if the probability of such
a demand arising (that is, the probability of blocking) is sufficiently small. If this
probability does not exceed $\epsilon$, we say that the network is $\epsilon$-blocking. More gener-
ally, we may define a quantity called the capacity that will represent the number of join-
ings that may be simultaneously established, either without blocking or with small
probability of blocking.

Benes\footnote{1} is a good general reference for switching networks. With regard to com-
plexity theory, concentration networks were first considered by Pinsker,\footnote{2} connection
networks by Slepian\footnote{3} and by Clos,\footnote{4} and distribution networks by Masson and Jordan.\footnote{5}

1.2 COMPLEXITY THEORY

Complexity theory deals with the number of components necessary to con-
struct certain systems or the number of operations necessary to effect certain
processes. We may ask, for example, how many AND-gates and OR-gates are needed
in a network that multiplies two binary numbers, or how many scalar multiplications
and additions must be performed by an algorithm that multiplies two matrices. In a
typical problem of complexity theory, we are given a class of components (or opera-
tions) that are described behavioristically, some structural assumptions describing
the ways in which the components may be interconnected (or in which the oper-
ations may be combined), and a specification of the behavior to be exhibited by the
overall system (or process). In most cases this specification will have one or more free
parameters (in the examples stated above, the number of binary digits in the numbers or
the number of rows and columns in the matrices). We then study the number of compo-
nents (or operations) in systems (or processes) meeting the specification. In most
cases, the combinatorial complications are so formidable that exact expressions for
these numbers are unobtainable and we must content ourselves with bounds on them, or
on their asymptotic behavior as functions of the free parameters.

1.3 PROBLEMS

Two broad classes of problems in the complexity theory of switching networks can
be distinguished. First, there is the study of the number of components necessary for
the construction of the various types of switching networks. We shall take the components
to be switch contacts, interconnected according to the usual rules of electrical network theory. In some cases interesting lower bounds may be obtained by taking the components to be certain aggregates of switch contacts (such as crossbar switches) or by requiring that the interconnections possess certain symmetries or uniformities. Second, there is the study of the number of operations necessary to control these networks. Originally, the equipment responsible for controlling a switching network was distributed throughout the network. Recently, however, this equipment has been consolidated into progressively larger units, its structure evolving toward that of a general-purpose digital computer with random-access storage. In this case, the operations are the arithmetic and logical instructions executed by the machine, combined by means of the usual sequential, conditional, and iterative programming techniques.

1.3.1 Network Complexity Problems

For each type of switching network (concentration, connection, or distribution; rearrangeable or incremental; nonblocking or \( \varepsilon \)-blocking) we can study the asymptotic behavior of the minimum number of contacts as a function of the number of inputs, the number of outputs, the capacity and, when appropriate, the blocking probability. Our interest here will be in the asymptotic behavior of this function, rather than in its behavior in any particular bounded range of parameter values. Furthermore, although certain parameters (the number of inputs and outputs, for example) are constrained to have integral values, we shall generally proceed as though they were real variables. This will greatly simplify the analysis and it will be easy to verify in each case that the asymptotic behavior is not affected. We shall confine our attention to cases in which the number of inputs and the number of outputs are each asymptotically proportional to the capacity. In the case of concentration networks we must take the ratio of the number of inputs to the number of outputs to be less than unity to avoid triviality, and we shall distinguish between partial concentration networks (with capacity less than the number of inputs) and total concentration networks (with capacity equal to the number of inputs). In the cases of connection and distribution networks, we shall assume that the number of inputs and outputs are each equal to the capacity; other factors of proportionality lead to different coefficients in the asymptotic expansions, but not to different orders of growth. At the present time, in many problems there is a significant discrepancy between the best upper bounds demonstrated by explicit constructions and those that may be obtained by nonconstructive means. We shall mention these discrepancies and distinguish the two kinds of upper bounds.

1.3.2 Algorithmic Complexity Problems

Associated with each of the nonblocking network complexity problems there is an algorithmic complexity problem. Given a rearrangeably nonblocking network and an assignment, we ask to find a state that realizes the assignment (or, more generally, given a rearrangeably nonblocking network, a current state, and an assignment, we ask
to find a new state that differs in as few routes as possible from the current state and which realizes the assignment). The solution depends, of course, upon the representations chosen for assignments and states; it is generally assumed that an assignment is represented by a list of the input-output pairs to be joined and that a state is represented by a list of the contacts used. In the case of incrementally nonblocking networks we assume that we are given the current state and a request, and ask to find an admissible route that satisfies the request and the new state that results from adding this route to the current state. We assume that a request is represented by the input-output pair to be joined and that a route is to be represented by a list of the contacts used. We are free, however, to choose any representation for the state we wish (since this is neither an input nor an output of the algorithm, but an internal variable). In either case, we can study the asymptotic behavior of the minimum number of operations that must be performed by any algorithm for controlling the given network.

1.4 INFORMATION-THEORETIC LOWER BOUNDS

For any network complexity problem, a lower bound can be obtained by means of a technique introduced by Shannon.6 This technique depends upon two simple observations. First, in a rearrangeably nonblocking switching network, the number of realizable assignments cannot exceed the number of states, since distinct assignments are realized by distinct states. Second, if a network is formed by interconnecting a number of components, the number of states of the overall network cannot exceed the product of the numbers of states of the components, since the state of the overall network is determined by those of the components. If we assume that all of the interconnected components are identical, the result is that the number of components is at least the logarithm of the number of realizable assignments (the base of the logarithm being the number of states of each component). If we let the capacity N tend to infinity while the number of inputs and the number of outputs remain proportional to it, we obtain O(N), O(N log N) and O(N log N) as lower bounds on the order of growth of the number of identical components in rearrangeably nonblocking concentration, connection, and distribution networks, respectively. These bounds are generally referred to as the information-theoretic lower bounds. Since an incrementally nonblocking network is also rearrangeably nonblocking, these bounds also apply to incrementally nonblocking networks.

1.5 RESULTS

Our results may be divided into three main classes: combinatorial results on non-blocking networks, probabilistic results on c-blocking networks, and results on algorithms for controlling networks. These results, which appear in the following sections, are summarized here, and the most important previous results are indicated.
1.5.1 Combinatorial Network Complexity Results

Connection networks were the first to be considered from the point of view of complexity theory. For these, Benes showed that rearrangeably nonblocking networks could be built with $O(N \log N)$ contacts. Incrementally nonblocking networks have presented a more challenging problem: Bassalygo and Pinsker have shown by a nonconstructive argument that they can be built with $O(N \log N)$ contacts, but the best known construction, that of Cantor, requires $O(N \log N^2)$ contacts. Our combinatorial results concern partial concentration and distribution networks. In the case of partial concentration, we show by a nonconstructive argument that rearrangeably nonblocking networks can be built with $O(N)$ contacts. By the information-theoretic lower bound, this is the best possible order of growth. We then show that incrementally nonblocking networks cannot be built with fewer than $O(N \log N)$ contacts, so that in this case the information-theoretic lower bound cannot be achieved. Since a connection network is also a partial concentration network, the work on connection networks cited above provides nonconstructive and constructive upper bounds of $O(N \log N)$ and $O(N \log N^2)$ contacts, respectively. In the case of distribution, the best previously published results were those of Masson and Jordan, who gave constructive upper bounds of $O(N^{5/3})$ contacts for both rearrangeably and incrementally nonblocking networks. We give a construction for rearrangeably nonblocking networks with $O(N \log N)$ contacts. Again, by the information-theoretic lower bound, this is the best possible order of growth. We also give a construction for incrementally nonblocking distribution networks in terms of incrementally nonblocking connection networks. In this case, the work on connection networks can be used to provide nonconstructive and constructive upper bounds of $O(N \log N^2)$ and $O(N \log N^3)$ contacts, respectively.

1.5.2 Probabilistic Network Complexity Results

Incrementally $\epsilon$-blocking networks are of interest for a number of reasons. First, a small probability of blocking is acceptable in many practical situations. Second, the best known $\epsilon$-blocking networks require fewer contacts than the best known nonblocking networks in many cases. Finally, our upper bounds for $\epsilon$-blocking networks are obtained by constructive means. Ikeno has shown that for any fixed blocking probability greater than zero and a capacity proportional to the number of inputs and outputs (with any coefficient of proportionality less than one), an upper bound asymptotic to $4eN \log_e N$, where $e = 2.718 \ldots$ is the base of the natural logarithm, can be obtained, by using an approximate probabilistic model introduced by Lee. We improve this by giving a construction that provides an upper bound asymptotic to $6N \log_2 N$, without using any approximations. Marcus considers variable $\epsilon > 0$ and a capacity equal to the number of inputs and outputs and obtains an upper bound of $O(N \log N) + O(N \log \log N)$ contacts. We generalize and improve this by obtaining upper bounds of $O(N \log N + O(N \log N))$ and $O(N \log N + O(N \log N))$ contacts for incrementally $\epsilon$-blocking partial networks.
concentration, connection, and distribution networks. We show that for \( v \geq 1 \),
\( O((\log \log 1/\varepsilon)^v) \) is an upper bound for \( E(\varepsilon) \) if and only if \( O(N(\log N)^v) \) is an upper bound for the complexity of incrementally nonblocking partial concentration, connection, and distribution networks.

1.5.3 Algorithmic Complexity Results

We discuss algorithms for controlling switching networks, confining our attention to networks obtained by constructive means. Algorithms for controlling rearrangeably nonblocking connection networks have been discussed by Tsao-Wu and Opferman,\(^{12}\) who give an algorithm requiring \( O(N \log N) \) arithmetic operations to find a state realizing a given assignment. We complement this by giving an algorithm for controlling incrementally nonblocking connection networks obtained by Cantor's construction\(^{8}\) which requires \( O((\log N)^2) \) arithmetic operations to find an admissible route satisfying a given request. The same technique can be used to provide analogous algorithms for all of the incrementally nonblocking connection networks for which explicit constructions have been given. In the case of distribution networks we give an algorithm requiring \( O(N \log N) \) arithmetic operations to find a state realizing a given assignment in our rearrangeably nonblocking network and an algorithm requiring \( O((\log N)^3) \) arithmetic operations to find an admissible route satisfying a given request in our constructive incrementally nonblocking network.
II. DEFINITIONS

We shall now present the graph-theoretic model which will be used in our discussion of switching networks. We also define some basic switching networks and some operations by means of which new switching networks can be obtained from old ones.

2.1 NETWORK MODEL

Our network model is similar to one introduced by Benes, but it has been extended to deal with a broader class of problems. (Interspersed with the definitions are some parenthetical remarks that may help the reader relate the graph-theoretic constructs to their counterparts in an actual switching network.)

By a graph we mean a finite set of vertices, together with a set of unordered pairs of vertices called edges. Two vertices, \( v_1 \) and \( v_2 \), are adjacent if \( \{v_1, v_2\} \) is an edge. A vertex \( v_0 \) and an edge \( \{v_1, v_2\} \) are incident if \( v_0 \in \{v_1, v_2\} \). (As in electrical network theory, each vertex represents one or more terminals connected by conducting wires, while each edge represents a switch contact that may be either open or closed.) By a switching graph we mean a graph in which two disjoint sets of vertices, called inputs and outputs, are distinguished. Vertices that are neither inputs nor outputs are called links. (Inputs and outputs represent the terminals through which a switching network is connected to its environment, while links represent internal terminals not accessible from outside the network. Note that a link is a vertex, not an edge as in some network models.)

By a path in a switching graph we mean a sequence of distinct vertices \( v_0, v_1, \ldots, v_n \) such that \( v_0 \) is an input, \( v_1 \) through \( v_{n-1} \) are links, and \( v_n \) is an output, and such that for all \( 0 \leq i < n \), \( v_i \) and \( v_{i+1} \) are adjacent. In a path \( v_0, v_1, \ldots, v_n \), the vertex \( v_0 \) is called the origin, the vertex \( v_n \) is called the termination, and the number \( n \) is called the length. (Paths represent sets of contacts (the edges between successive vertices) that may be closed simultaneously to join an input to an output.) By a tree in a switching graph we mean a set of paths with a common origin and distinct terminations, and such that any two paths \( v_0, v_1, \ldots, v_n \) and \( w_0, w_1, \ldots, w_m \) having a link in common \( (v_i = w_j) \) have the predecessors of this link in common \( (v_{i-1} = w_{j-1}) \). In a tree, the common origin of the paths is called the root and their terminations are called leaves. (Trees represent sets of contacts (the paths of the tree) that may be closed simultaneously to connect an input to one or more outputs.)

2.1.1 Networks, Routes, and States

A switching network is a switching graph, together with a finite set of trees in the switching graph called fans. (In a switching network the switching graph represents the physical network, while the fans represent certain constraints on the manner in which the network is operated. Fans will be used to define which paths may be used as routes and which routes may be combined to form states.) An elementary switching network is one in which each fan is reduced to a single path. (Elementary switching networks
will be used for concentration and connection problems, while general switching networks will be used for distribution problems.)

A route in a switching network is a path that appears in some fan of that network. (In most switching networks it would be possible to find a path including most of the vertices of the network, thereby blocking all other traffic in the network. A realistic model of a switching network must include some means of ensuring that such needlessly circuitous paths are not used. In our model this is done by the fans, and the routes are the paths that actually can be used during the operation of the network.) Two routes in a switching network are compatible if they both appear in a common fan of that network or if they have no vertex in common. Thus, two routes in an elementary switching network are compatible if they are identical or have no vertex in common.

A state is a set of mutually compatible routes. In a given state, a route is admissible if it is compatible with every route of the state. (States are those combinations of routes that can actually occur during the operation of the network. The compatibility condition serves the twofold purpose of ensuring that routes from distinct inputs do not include a common vertex and of ensuring that the routes from a common input to distinct outputs include enough common vertices.) In a given state, a vertex is busy if it occurs in some route of the state, otherwise it is idle. Finally, in a given state, two vertices are joined if they both occur in a common route of the state.

2.1.2 Assignments and Requests

A concentration assignment is a subset Y of the outputs; a state S realizes Y if Y is the set of busy outputs in S. In a given state, a concentration request is an idle output y; a route satisfies y if it terminates at y.

A connection assignment (respectively, a distribution assignment) is an injective map (respectively, an arbitrary map) F from a subset Y of the outputs into the inputs; a state S realizes F if Y is the set of busy outputs in S and if, for every y in Y, y and F(y) are joined in S. In a given state, a connection request (respectively, a distribution request) is an ordered pair (x, y) comprising an idle input (respectively, an arbitrary input) x and an idle output y; a route satisfies (x, y) if it originates at x and terminates at y.

2.1.3 Capacity

By the order of a concentration assignment we mean its cardinality; by the order of a connection or distribution assignment we mean the cardinality of its domain. We say that a switching network with at least N outputs is rearrangeably nonblocking with capacity N if, for every assignment Y (or F) of order at most N, there is a state realizing this assignment. By the order of a state we mean its cardinality. We say that a switching network with at least N outputs is incrementally nonblocking with capacity N if, for any state S of order less than N and any request y (or (x, y)) in S, there is an admissible route satisfying this request. An easy induction on the capacity shows that
an incrementally nonblocking switching network with capacity $N$ is rearrangeably non-blocking with capacity $N$; it is not hard to show that the converse of this statement is false.

A switching network is rearrangeably $\varepsilon$-blocking if, for a randomly chosen assignment, the probability that no state realizes this assignment does not exceed $\varepsilon$. A switching network is incrementally $\varepsilon$-blocking if, for a randomly chosen state and a randomly chosen request in this state, the probability that no admissible route satisfies this request does not exceed $\varepsilon$. In either case, $\varepsilon$ is called the blocking probability and we define the capacity to be the expected number of busy outputs.

To make our use of the phrase "randomly chosen" precise, we must specify probabilities for the assignments, for the states, and for the requests in a given state. Such a specification is usually made by assuming that the environment of the network is symmetric with respect to the various inputs and outputs (which leads to the assumption that assignments, states, and requests that are equivalent under permutation of inputs and of outputs are equiprobable), and by considering the worst case with respect to the number of joinings in an assignment or the number of routes in a state (which usually leads to the assumption that these numbers are always equal to the capacity). Our assumptions will be described in more detail in the sections in which they are used.

2.1.4 Graded and Uniform Networks

For certain purposes, it is useful to consider networks with a more restricted structure than those allowed by our original definition. We say that a network is graded if its vertices are partitioned into a sequence of sets (called ranks) in such a way that the first rank comprises the inputs, the last rank comprises the outputs, and adjacent vertices appear in consecutive ranks. In a graded network, the edges are partitioned into a sequence of sets (called stages) according to the pair of consecutive ranks in which their vertices appear. The number of stages is one less than the number of ranks. We say that a graded network is uniform if the length of each route is equal to the number of stages (so that each route has one vertex from each rank and one edge from each stage) and if each vertex is adjacent to at most two vertices in the preceding rank and at most two vertices in the succeeding rank. The choice of the number two in the last condition is not especially significant: any finite number would serve, but two fits well with our use of the exponential and the logarithm to that base.

2.2 BASIC CONSTRUCTIONS

We shall now present some basic switching networks and operations on switching networks which will be useful in our work.

2.2.1 Crossbar Switches

Conceptually, the simplest switching network is the crossbar switch. (This is an example of one physical switching network having two graph-theoretic models
corresponding to two modes of operation.)

The network $C_{a,b}$ has $a+b$ vertices, of which $a$ are inputs, $b$ are outputs and none are links, and it has $ab$ edges, one between each input and output. Each of the $ab$ sequences comprising one input followed by one output is a path of $C_{a,b}$. Each of the $ab$ sets of just one path is a fan of $C_{a,b}$. (The seven states of $C_{2,2}$ are shown in Fig. 1, where the heavy lines represent closed contacts and the light lines open contacts.) $C_{a,b}$ is evidently an elementary switching network that is both an incrementally nonblocking concentration network and an incrementally nonblocking connection network (with capacity $\min(a, b)$ in each case), giving an upper bound of $O(N^2)$ contacts for concentration and connection networks with capacity $N$.

![Fig. 1. States of $D_{2,2}$ and $C_{2,2}$. Light lines represent open contacts, heavy lines represent closed contacts. (a) The seven states of $C_{2,2}$. (b) The two states of $D_{2,2}$ that are not states of $C_{2,2}$.]

The network $D_{a,b}$ has the same switching graph (and thus the same paths) as $C_{a,b}$ but has different fans. Each of the $a$ sets comprising all of the $b$ paths having a common input is a fan of $D_{a,b}$. (The nine states of $D_{2,2}$ are shown in Fig. 1.) $D_{a,b}$ is evidently an incrementally nonblocking distribution network (with capacity $b$), giving an upper bound of $O(N^2)$ contacts for distribution networks with capacity $N$.

2.2.2 Transposition

If $G$ is an elementary switching network, its transpose, $G'$, is obtained by exchanging its inputs with its outputs and by reversing the paths constituting its fans. The transpose of a switching network that is not elementary is not defined.

2.2.3 Concatenation

If $G_1$ is an elementary switching network, $G_2$ is a switching network and $\phi$ is a
bijection from the outputs of $G_1$ to the inputs of $G_2$, their concatenation, $(G_1, G_2)_\phi$, is obtained by identifying the output $y$ of $G_1$ with the input $\phi(y)$ of $G_2$ for each output $y$ of $G_1$ (see Fig. 2a). (In Fig. 2 boxes represent (copies of) networks, dots on the left sides of boxes represent inputs and dots on the right sides represent outputs. Lines between boxes represent input-output pairs that are identified to form links of the composite network, inputs not so identified are inputs of the composite network and outputs not so identified are outputs of the composite network.) A fan of $(G_1, G_2)_\phi$ is obtained by taking a fan $F_1$ of $G_1$ with its unique leaf at $y$ and a fan $F_2$ of $G_2$ with its root at $\phi(y)$ and by concatenating the unique path of $F_1$ with each path of $F_2$, identifying the common vertex.

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**Fig. 2.** Representation of networks. Lines between boxes represent input-output pairs that are identified to form links.

(a) Concatenation $(G_1, G_2)$.

(b) Product $G_1 \times G_2$.

(c) Triple product $(G_1, G_2, G_3)$.

(d) Iteration $(G_1 \ldots G_2)$.
2.2.4 Products

If $G_1$ and $G_2$ are switching networks, their product, $G_1 \times G_2$, is obtained by taking a copy $G_{1,x}$ of $G_1$ for each input $x$ of $G_2$ and a copy $G_{2,y}$ of $G_2$ for each output $y$ of $G_1$ and by identifying the output $y$ of $G_{1,x}$ with the input $x$ of $G_{2,y}$ for each output $y$ of $G_1$ and each input $x$ of $G_2$ (see Fig. 2b). A fan of $G_1 \times G_2$ is obtained by taking a fan $F_1$ of some copy of $G_1$ and for each leaf $y$ of $F_1$ a fan $F_{2,y}$ of the copy $G_{2,y}$ of $G_2$ and by concatenating the path in $F_1$ terminating at $y$ with each path in $F_{2,y}$, identifying the common vertex. Note that $(G_1 \times (G_2 \times G_3)) \ni (G_1 \times G_2) \times G_3$, where $\ni$ denotes switching network isomorphism, defined in the obvious way. The $r$-fold product of identical factors $G \times G \times \ldots \times G$ will be denoted $G^r$.

2.2.5 Triple Products

If $G_1$, $G_2$, and $G_3$ are switching networks and $\phi$ is a bijection from the inputs of $G_3$ to the outputs of $G_1$, their triple product, $(G_1, G_2, G_3)_\phi$, is obtained by taking a copy $G_{1,x}$ of $G_1$ for every input $x$ of $G_2$, a copy $G_{2,y}$ of $G_2$ for every output $y$ of $G_1$, and a copy $G_{3,y}$ of $G_3$ for every output $y$ of $G_2$, by identifying the output $y$ of $G_{1,x}$ with the input $x$ of $G_{2,y}$ for each output $y$ of $G_1$ and each input $x$ of $G_2$ and by identifying the output $y$ of $G_{2,y}$ with the input $x$ of $G_{3,y}$ for each output $y$ of $G_2$ and each input $x$ of $G_3$ (see Fig. 2c). Observe that $(G_1, G_2, G_3)_\phi$ contains an isomorphic image of $G_1 \times G_2$. A fan of $(G_1, G_2, G_3)_\phi$ is obtained by taking a fan $F_{1 \times 2}$ of $G_1 \times G_2$ which has at most one leaf on each copy of $G_3$ and for each leaf $y$ of $F_{1 \times 2}$ a fan $F_{3,y}$ of the copy $G_{3,y}$ of $G_3$ and by concatenating the path in $F_{1 \times 2}$ terminating at $y$ with each path in $F_{3,y}$, identifying the common vertex. (Note that $(G_1, G_2, G_3)$ is not isomorphic to $G_1 \times G_2 \times G_3$.

In fact, $(C_n \times C_n \times C_n, n, n, n, n, n, n, n)$ has $n^2$ inputs and outputs, while $C_n \times C_n \times C_n \times C_n$ has $n^3$.

2.2.6 Iteration

If $G_1$ and $G_2$ are elementary switching networks, $\phi$ is a bijection from the outputs of $G_1$ to the inputs of $G_2$ and $\psi$ is a bijection from the outputs of $G_2$ to the inputs of $G_1$, their iteration, $(G_1 \ldots G_2)_{\phi, \psi}$, is obtained from $G_1 \times C_{1,1}$ and $G_2$ by identifying the output $y$ of $G_1$ with the input $\phi(y)$ of $G_2$ for each output $y$ of $G_1$, and by identifying the output $y$ of $G_2$ with the input $\psi(y)$ of $G_2$ for each output $y$ of $G_2$ (see Fig. 2d). We assume that $G_2$ contains no paths of length one, so that $(G_1 \ldots G_2)_{\phi, \psi}$ will have no self-loops or multiple edges. (The copies of $C_{1,1}$ have been included because only the last vertex in a path may be an output; they do not have any practical significance.) Observe that $(G_1 \ldots G_2)_{\phi, \psi}$ contains an isomorphic image of $G_1 \times C_{1,1}$ and a homomorphic image of $G_2 \times C_{1,1}$ (homomorphic rather than isomorphic because the inputs and outputs of $G_2$ have been identified). A fan of $(G_1 \ldots G_2)_{\phi, \psi}$ is either a fan of $G_1 \times C_{1,1}$ or is obtained by taking a fan $F$ of $(G_1 \ldots G_2)_{\phi, \psi}$ and adding to it any path obtained by taking
the longest path in $F$ and replacing its last two vertices (which are a path in $C_{1,1}$) by a path in $G_2 \times C_{1,1}$. Thus, a fan in $(G_1, G_2) \phi, \psi$ comprises paths that go through $G_1$, around $G_2$ zero or more times, and then through some copy of $C_{1,1}$.
III. PARTIAL CONCENTRATION NETWORKS

In this section we consider concentration networks, confining our attention to partial concentration networks. In incrementally nonblocking networks, our lower bound for partial concentration clearly applies for total concentration also. Pinsker\textsuperscript{2} has shown that for rearrangeably nonblocking networks the same is true of upper bounds.

While rearrangeably nonblocking partial concentration networks can be built with $O(N)$ contacts, their incrementally nonblocking counterparts cannot be built with less than $O(N\log N)$ contacts. The first of these results was obtained independently by Pinsker,\textsuperscript{2} using an argument similar to ours. Each of these results is the best possible, in terms of order of growth: the information-theoretic lower bound for this problem is $O(N)$ contacts, while the proof\textsuperscript{7} that incrementally nonblocking connection networks can be built with $O(N\log N)$ contacts applies to incrementally nonblocking concentration networks also. We shall conclude with a discussion of incrementally $\epsilon$-blocking partial concentration networks, giving nonconstructive and constructive upper bounds of $O(N\log \log 1/\epsilon)$ and $O(N(\log \log 1/\epsilon)^2)$ contacts, respectively, and giving, in effect, a lower bound of $O(N\log \log 1/\epsilon)$ contacts. Specifically, we show that incrementally $\epsilon$-blocking partial concentration networks can be built with $O(N(\log \log 1/\epsilon)^V)$ contacts if and only if incrementally nonblocking partial concentration networks can be built with $O(N(\log N)^V)$ contacts.

3.1 AN UPPER BOUND OF $O(N)$ FOR REARRANGEABLY NONBLOCKING NETWORKS

Theorem 1.

For all $a$ and $b$ such that $1 < a < b$ there exists $c$ such that for all sufficiently large $N$, rearrangeably nonblocking concentration networks can be constructed with $A = aN$ inputs, $B = bN$ outputs, at most $cN$ contacts, and capacity $N$. Furthermore, the coefficient $c$ need be at most $O(b\log b/\log a)$ as $a$ tends to unity with $b$ fixed or $b$ tends to infinity with $a$ fixed.

Proof: Consider the set $E(A, B, K)$ of all bipartite switching graphs with $A$ inputs, $B$ outputs and $K$ edges incident with each output (there are $\binom{A}{K}^B$ such graphs). Such a switching graph can be considered to be a switching network if each path of length 1 from an input to an output is considered to be a fan. We show that for all $a$ and $b$ such that $1 < a < b$ there exists a $K$ such that for all sufficiently large $N$, $E(aN, bN, K)$ contains a rearrangeably nonblocking concentration network with capacity $N$. Furthermore, we show that $K$ need be at most $O(\log b/\log a)$ as $a$ tends to unity with $b$ fixed or $b$ tends to infinity with a fixed.

A graph in $E(aN, bN, K)$ will correspond to a rearrangeably nonblocking concentration network with capacity $N$ if every subgraph obtained by considering only $N$ of the
outputs has a matching of size $N$ (that is, a set of $N$ edges such that no two have a vertex in common). By Hall's theorem a bipartite graph can fail to have a matching of size $N$ only if, for some $M \leq N$, there exists a set $Y$ of $M$ outputs and a set $X$ of $M-1$ inputs such that every input which is adjacent to some output in $Y$ appears in $X$. We shall obtain a bound on the fraction of networks in $E(AN, BN, K)$ for which, for some $M \leq N$, there exists a set $Y$ of $M$ outputs and a set $X$ of $M$ inputs such that every input that is adjacent to some output in $Y$ appears in $X$. This bound will be shown to be less than one for all sufficiently large $N$, thereby establishing the existence of a rearrangeably nonblocking concentration network with capacity $N$ in $E(AN, BN, K)$.

Consider a set $Y$ of $M$ of the $B$ outputs and a set $X$ of $M$ of the $A$ inputs. The fraction of the networks in $E(A, B, K)$ for which every input which is adjacent to some output in $Y$ appears in $X$ is

$$\left(\frac{\binom{M}{K}}{\binom{A}{K}}\right)^M,$$

which is at most $(M/A)^{KM}$. Since there are $\binom{A}{M}$ ways of choosing $X$ and $\binom{B}{M}$ ways of choosing $Y$, the fraction of networks in $E(A, B, K)$ for which there exists a set $Y$ of $M$ outputs and a set $X$ of $M$ inputs with the stated property is at most

$$\binom{A}{M} \binom{B}{M} (M/A)^{KM}.$$

Thus the fraction of networks in $E(A, B, K)$ for which, for some $M \leq N$, there exists a set $Y$ of $M$ outputs and a set $X$ of $M$ inputs with the stated property is at most

$$\sum_{1 \leq M \leq N} \binom{A}{M} \binom{B}{M} (M/A)^{KM}.$$

We now need only show that for all $a$ and $b$ such that $1 < a < b$, we can choose $K$ so that this sum will be less than one for all sufficiently large $N$.

The ratio of successive terms in the sum is $(A-M)/(M+1)((B-M)/(M+1))((M+1)/A)^K ((M+1)/M)^{KM}$. The last factor is an increasing function of $M$ for $M \geq 1$ (which in fact converges to $e^K$). Differentiation shows that the product of the first three factors will also be an increasing function of $M$ if $K > 2 + (A-M)/(M+1)$, which will hold for all sufficiently large $N$ if $K > 2 + 1/(a-1) + 1/(b-1)$. This bound is $O(1/\log a)$ as $a$ tends to unity with $b$ fixed or $b$ tends to infinity with $a$ fixed. When the ratio of successive terms in the sum is an increasing function of $M$, the general term in the sum must assume its maximum either for $M = 1$ or for $M = N$. In the first case the sum is at most
\[ N\left(\frac{A}{N}\right)\left(\frac{B}{N}\right)\frac{1}{A^K} \]

and will become arbitrarily small as \( N \) increases if \( K > 3 \). In the second case, the sum is at most

\[ N\left(\frac{A}{N}\right)\left(\frac{B}{N}\right)\frac{1}{A^K}N^K. \]

The logarithm of this quantity is asymptotic to \((aH(1/a) + bH(1/b) - K \log a) N\), where \( H(x) = -x \log x - (1-x) \log (1-x) \) (see Peterson). Thus the sum will become arbitrarily small as \( N \) increases if \( K > (aH(1/a) + bH(1/b))/\log a \). This bound is \( O(\log b/\log a) \) as \( a \) tends to unity with \( b \) fixed or \( b \) tends to infinity with \( a \) fixed. This completes the proof.

3.2 A LOWER BOUND OF \( O(N \log N) \) FOR INCREMENTALLY NONBLOCKING NETWORKS

**Theorem 2.**

Any incrementally nonblocking concentration network having fewer inputs than outputs and capacity \( N \) must have at least \( O(N \log N) \) contacts.

**Proof:** Let \( G \) be an incrementally nonblocking concentration network with fewer inputs than outputs and capacity \( N \). The states of \( G \), being sets of routes, may be ordered by inclusion and it is this order to which we refer throughout this proof. Let \( S \) be a maximal state. Since there are fewer inputs than outputs but as many busy inputs as busy outputs, there must be some output, say \( y \), which is idle in \( S \). Since \( S \) is maximal, no admissible route in \( S \) satisfies the request \( y \).

Now consider the set of all states less than or equal to \( S \) for which no admissible route satisfies the request \( y \). This set is not empty, since \( S \) itself is such a state. Let \( T \) be a state that is minimal in this set. There must be at least \( N \) routes in \( T \), say \( Q^M_M \) (\( 1 \leq M \leq N \)), since the request \( y \) is blocked in \( T \). Since \( T \) is minimal, deleting any route, say \( Q^M_M \), results in a state \( T-Q^M_M \) in which the request \( y \) can be satisfied by some admissible route, say \( R^M_M \). By the definition of admissibility, \( Q^M_M \) and \( R^M_M \) must have some vertex in common, say \( z^M_M \). Since the \( Q^M_M \) (\( 1 \leq M \leq N \)) form a state, no two of them can have a vertex in common. Thus the \( z^M_M \) (\( 1 \leq M \leq N \)), and hence the \( R^M_M \) (\( 1 \leq M \leq N \)), must all be distinct. Thus in any incrementally nonblocking concentration network with fewer inputs than outputs and capacity \( N \), there is a request that is satisfied by at least \( N \) distinct routes.

For each \( 1 \leq M \leq N \), the network \( G-R^M_M \) obtained from the network \( G \) by deleting all of the vertices of \( R^M_M \) and all of the edges incident with them is an incrementally nonblocking concentration network with fewer inputs than outputs and capacity \( N-1 \). For if \( S \)
were a blocked state of G-\( R_M \) of order less than N-1, then the state S+\( R_{M'} \) obtained from the state S by adding the route \( R_{M'} \) would be a blocked state of G of order less than N, contradicting the hypothesis that G is an incrementally nonblocking concentration network with capacity N.

Let \( E(N) \) denote the minimum number of states in any incrementally nonblocking concentration network with fewer inputs than outputs and capacity N. For each \( 1 \leq M \leq N \), G-\( R_M \) has at least \( E(N-1) \) states. Thus G has at least \( N \cdot E(N-1) \) states of the form S+\( R_{M'} \) where \( 1 \leq M \leq N \), and S ranges over the states of G-\( R_M \). Since \( E(1) \geq 1 \), iteration of this inequality gives \( E(N) \geq N! \). By the argument used in the derivation of the information-theoretic lower bound, it follows that G must have \( O(N \log N) \) contacts.

This completes the proof.

For incrementally nonblocking networks, Benes\(^{15} \) has introduced the distinction between strictly nonblocking and wide-sense nonblocking. A strictly nonblocking network is an incrementally nonblocking network as we have defined it; a network is wide-sense nonblocking with capacity N if there exists a set of states with the following properties.

1. For any state in the set with fewer than N routes, and any request in this state, an admissible route that satisfies this request can be added to this state to yield another state in the set;
2. From any state in the set, any route may be deleted to yield another state in the set.

Benes\(^{15} \) shows that a wide-sense nonblocking network must have at least as many states as a strictly nonblocking network. (He considers only connection networks, but his proof applies with only trivial changes to concentration networks.) Since our bound on the number of contacts in incrementally nonblocking concentration networks is obtained from a bound on the number of states in these networks, it applies to wide-sense nonblocking networks as well as to those that are strictly nonblocking.

### 3.3 The \( \epsilon \)-Dependence of the Complexity of Incrementally \( \epsilon \)-Blocking Networks

We shall now prove the first of three theorems concerning the \( \epsilon \)-dependence of the complexity of \( \epsilon \)-blocking networks. Each of these theorems establishes a reciprocal relationship between the complexity of \( \epsilon \)-blocking networks and that of nonblocking networks. In the first, we show that incrementally \( \epsilon \)-blocking partial concentration networks can be built with \( O(N \log \log 1/\epsilon)^V \) contacts if and only if incrementally nonblocking partial concentration networks can be built with \( O(N \log N)^V \) contacts. Analogous theorems for connection and distribution networks will be presented in other sections.

We shall use the following probabilistic and structural assumptions. We need (i) a probability distribution on the states, and (ii) a probability distribution on the requests in a given state. Assumption (i) may be decomposed as follows: probability distribution (ia)
for the order of the state; (ib) for the assignment realized by the state, given
the order of the state; and (ic) for the state, given the assignment realized by the
state. In effect, we make no restriction on the probability distribution for the order
of the state, but assume that (ia) the order of the state assumes only the value mN
(for some $0 \leq m \leq 1$). From this we derive a bound on the blocking probability that
is uniform in m. This bound applies also to any probability distribution for which
the order of the state assumes only values not exceeding N, since any such distribu-
tion can be expressed as a convex combination of distributions for which the order
of the state assumes only the value mN (for some $0 \leq m \leq 1$). Thus this assumption
entails no real restriction.

In proving the direct part of Theorem 3 (that is, that the existence of nonblocking
networks with $O(N(\log N)^v)$ contacts implies the existence of $\epsilon$-blocking networks with
$O(N(\log \log 1/\epsilon)^v)$ contacts), we shall use the following assumptions: (ia) the order of
the state assumes only the value mN (for some $0 < m < 1$), (ib) given the order of the
state, the probability distribution for the assignment realized by the state is uniform,
and (ii) given the state, the probability distribution on the requests in that state is uni-
form. These assumptions are very natural; they are essentially symmetry hypotheses
regarding the inputs and outputs.

In proving the converse part of Theorem 3 (that is, that the existence of $\epsilon$-blocking
networks implies the existence of nonblocking networks), stronger assumptions are
needed, but several alternatives are available. The simplest is to assume that given the
order of the state, the probability distribution for the state is uniform, and that given the
state, the probability distribution on the requests in that state is uniform. The first of
these assumptions is, in fact, derived as a consequence of the "maximum entropy postu-
late" in the thermodynamic model of Benes. For networks in which different assignments
may be realized by different numbers of states, however, it will be incompatible with the
assumptions used for the direct part of the theorem. We base our proof, therefore, on a
more complicated set of assumptions that has the advantage of including the assumptions
used in the direct part. Specifically, we assume (ia) the order of the state assumes only
the value mN (for some $0 \leq m \leq 1$), (ib) given the order of the state, the probability distri-
bution for the assignment realized by the state is uniform, (ic) given the assignment
realized by the state, the probability distribution for the state is uniform, and (ii) given
the state, the probability distribution on the requests in that state is uniform. These
assumptions, together with the assumption that the networks involved are uniform, will
enable us to obtain a bound on the ratio between the probability of the most probable state
and that of the least probable state. This bound will serve as well for our purposes as
the assumption that all states that realize a given assignment are equiprobable.

In summary, the assumptions we use for the direct part are very natural, while
those we use for the converse part are less so. But while the particular assumptions we
use for the converse part may not be met in practice, the fact that alternative assump-
tions will also serve indicates that unless they are violated in a dramatic way the
converse part will be valid, so we should not expect a better upper bound than that provided by the direct part.

The arguments we use to prove the direct and converse parts of Theorem 3 are similar to those that will be used in the analogous theorems for connection and distribution networks. To prove the direct part, we consider networks containing many nonblocking networks of fixed size and show that the blocking probability decreases exponentially as the size of the nonblocking networks increases. To prove the converse part, we consider $\epsilon$ to be a rapidly decreasing function of $N$ and show that our probabilistic assumptions assign a probability to the least probable state that is greater than the blocking probability, which means there can be no blocking states.

Theorem 3.

For all $v > 1$, the following propositions are equivalent.

1. For all $a$ and $b$ such that $1 < a < b$ there exists $c$ such that for all sufficiently large $N$, uniform incrementally nonblocking concentration networks can be built with $aN$ inputs, $bN$ outputs, $cN(\log_2 N)^v$ contacts, and capacity $N$.

2. For all $a$ and $b$ such that $1 < a < b$ there exists $c$ such that for all sufficiently small $\epsilon > 0$ and all sufficiently large $N$, uniform incrementally $\epsilon$-blocking concentration networks can be built with $aN$ inputs, $bN$ outputs, $cN(\log_2 \log_2 \frac{1}{\epsilon})^v$ contacts, and capacity $N$.

Proof (1. implies 2.): Given $a$ and $b$ such that $1 < a < b$, we wish to build uniform incrementally $\epsilon$-blocking concentration networks with $aN$ inputs, $bN$ outputs, and capacity $N$. Choose $r$ such that $1 < ar < a$. By 1., there exists $c$ such that for all sufficiently large $n$, uniform incrementally nonblocking concentration networks can be built with $ar^n$ inputs, $br^n$ outputs, $cn(\log_2 n)^v$ contacts, and capacity $n$. By taking $R = N/r^n$ copies of such networks, we obtain uniform networks having $aN$ inputs, $bN$ outputs, and $(c/r)N(\log_2 n)^v$ contacts. We now need only show that there exists $h$ such that when $n = h \log_2 1/\epsilon$, these networks are incrementally $\epsilon$-blocking with capacity $N$. To do this we shall examine the blocking probability of these networks and show that it decreases exponentially as $n$ increases, at a rate uniform in $R$ (and hence in $N$).

By assumption (ia), the network is in a state with $mN$ busy outputs (for some $0 \leq m \leq 1$). The realizable assignments with $mN$ busy outputs are those with no more than $n$ busy outputs on any subnetwork, and by assumption (ib) these are equiprobable. A request in such a state can be satisfied if the subnetwork on which it appears has fewer than $n$ busy outputs. By assumption (ii), a randomly chosen request is equally likely to be any one of the $(b-m)N$ idle outputs, and thus is more likely to appear on a subnetwork with fewer than $n$ busy outputs than on one with $n$ busy outputs. We may therefore assume, without undue optimism, that a randomly chosen request is equally likely to appear on any of the $R$ subnetworks. If we focus attention on the subnetwork on which the request appears, the blocking probability is simply the probability that this
subnetwork has \( n \) busy outputs, and this is \( \binom{brn}{n} F((mR-1)n, R-1, brn, n)/F(mrn, R, brn, n) \), where \( F(p, q, s, t) \) is the number of ways in which \( p \) busy outputs can be distributed over the outputs of \( q \) subnetworks having \( s \) outputs each, in such a way that no subnetwork has more than \( t \) busy outputs. To obtain an upper bound on this expression, we use an upper bound for the application of \( F \) in the numerator and a lower bound for the application of \( F \) in the denominator.

\[
F(p, q, s, t) = \sum_{C} \prod_{1 \leq i \leq q} \binom{s}{C_i},
\]

where the sum is over all compositions \( p = C_1 + C_2 + \ldots + C_q \) such that \( 0 \leq C_i \leq t \) for all \( 1 \leq i \leq q \). There are at most \((t+1)^q\) terms in this sum and it is easy to show that the largest term is the one in which all of the \( C_i \) for \( 1 \leq i \leq q \) are equal, so that an upper bound is \( F(p, q, s, t) \leq (t+1)^q \binom{s}{p/q}^q \). Since the sum is at least as large as one of its terms, a lower bound is \( F(p, q, s, t) \geq \binom{s}{p/q}^q \). Thus we find that the blocking probability is at most

\[
\binom{brn}{n} (n+1)^R \binom{brn}{((mR-1)/(R-1))n}^{R-1} / \binom{mrn}{rn}^R,
\]

and we need now only show that this quantity decreases exponentially as \( n \) increases, at a rate uniform in \( R \). The logarithm of this quantity is asymptotic to \( (H(1/br) + (R-1)H((mR-1)/(br(R-1))) - R H(m/b)) brn \), where \( H(x) = -x \log x - (1-x) \log (1-x) \). Differentiation shows that \( H''(x) < 0 \) for \( 0 < x < 1 \), so that \( H \) is strictly concave. The quantity in parentheses is negative for all \( R \) and all \( 0 \leq m \leq 1 \) because the function \( H \), being strictly concave, lies above any of its chords. The limit of this quantity as \( R \) tends to infinity is \( H(1/br) + (mR-1)/br)H'(m/b) - H(m/b) \), which is negative for all \( 0 \leq m \leq 1 \) because the function \( H \), being strictly concave, lies below any of its tangents. This completes the proof.

Proof (2. implies 1.): Given \( a \) and \( b \) such that \( 1 < a < b \), we wish to construct uniform incrementally nonblocking concentration networks with \( aN \) inputs, \( bN \) outputs, and capacity \( N \). By 2., there exists \( c \) such that for all sufficiently small \( \epsilon > 0 \) and all sufficiently large \( N \), uniform incrementally \( \epsilon \)-blocking concentration networks can be constructed with \( aN \) inputs, \( bN \) outputs, \( cN(\log_2 \log_2 1/\epsilon)^V \) contacts, and capacity \( N \). Choose \( d > c \). By taking \( \epsilon = \exp_2 -3dN(\log_2 N)^V \), we obtain, for all sufficiently large \( N \), uniform networks having \( aN \) inputs, \( bN \) outputs, and at most \( dN(\log_2 N)^V \) contacts. We now need only show that, for all sufficiently large \( N \), these networks are incrementally nonblocking with capacity \( N \). By assumption (ia), the number of routes in the state assumes only the value \( mN \) (for some \( 0 \leq m \leq 1 \)). We shall show that no state has probability less than \( \exp_2 -2dN(\log_2 N)^V \). Thus, by assumption (ii), the smallest positive blocking
probability would be \((\exp^2 - 2dN(\log_2 N)^V)/bN\). Since, however, the blocking probability of these networks is at most \(\exp^2 - 3dN(\log_2 N)^V\), they must be, for all sufficiently large \(N\), incrementally nonblocking with capacity \(N\).

Since the networks have at most \(dN(\log_2 N)^V\) contacts, they have at most \(\exp^2 dN(\log_2 N)^V\) states. Since in a network with capacity \(N\) each stage must have at least \(N\) contacts, they have at most \(d(\log_2 N)^V\) stages. Since branching occurs at most two ways at each stage, the number of routes satisfying a given request is at most \(\exp^2 d(\log_2 N)^V\). Thus the number of states realizing a given assignment is at most \(\exp^2 dN(\log_2 N)^V\). Since, by assumption (ib), all of the realizable assignments are equiprobable, and since, by assumption (ic), all of the states realizing a given assignment are equiprobable, the probability of each state is inversely proportional to the number of states that realize the same assignment. Since each realizable assignment is realized by at least one state and at most \(\exp^2 dN(\log_2 N)^V\) states, the ratio of the probability of the most probable state to that of the least probable state is at most \(\exp^2 dN(\log_2 N)^V\). It follows that no state can have probability less than \(\exp^2 - 2dN(\log_2 N)^V\), for if the probability of the least probable state were less than this, the probability of the most probable state would be less than \(\exp^2 - dN(\log_2 N)^V\) and the sum of the probabilities of all of the states would be less than unity, a contradiction. This completes the proof.

**Corollary 1.** Since incrementally nonblocking concentration networks cannot be built with fewer than \(O(N\log N)\) contacts, incrementally \(\epsilon\)-blocking concentration networks cannot be built with fewer than \(O(N\log \log 1/\epsilon)\) contacts.

**Corollary 2.** Since the nonconstructive upper bound of \(O(N\log N)\) contacts\(^7\) and the construction with \(O(N(\log N)^2)\) contacts\(^8\) for incrementally nonblocking connection networks apply also to incrementally nonblocking concentration networks, we have nonconstructive and constructive upper bounds of \(O(N\log \log 1/\epsilon)\) and \(O(N(\log \log 1/\epsilon)^2)\) contacts, respectively, for incrementally \(\epsilon\)-blocking concentration networks.
IV. CONNECTION NETWORKS

The connection problem is the oldest of the three problems with which we deal and has received the most attention. Rearrangeably nonblocking connection networks were first studied by Slepian\(^3\) and their incrementally nonblocking counterparts by Clos.\(^4\) In each case, it was shown that three-stage networks with \(O(N^{3/2})\) contacts could be constructed. Recursive substitution of Slepian's construction yields rearrangeably nonblocking networks with \(O(N \log N \log^{3/2} N)\) contacts, while from Clos's construction we obtain incrementally nonblocking networks with \(O(N \log^6 N / \log^2 N)\) contacts. We now know that both of these problems can be solved with a number of contacts having the same order of growth as the information-theoretic lower bound. Benes\(^1\) has given an \(O(N \log N)\) construction for rearrangeably nonblocking networks which has been refined by Joel\(^17\) and by Waksman.\(^18\) Bassalygo and Pinsker\(^7\) have given a nonconstructive upper bound of \(O(N \log N)\) contacts for incrementally nonblocking networks, although the best known construction, that of Cantor,\(^8\) requires \(O(N \log^2 N)\) contacts.

Recently, some attention has been given to the problem of computing a state that realizes a given assignment in a rearrangeably nonblocking network. Tsao-Wu and Opferman\(^12\) have found an algorithm requiring \(O(N \log N)\) arithmetic operations for the network of Joel and Waksman. We shall study the problem of computing an admissible route satisfying a given request in an incrementally nonblocking network. One characteristic of efficient incrementally nonblocking networks is the large number of routes satisfying a given request. In Cantor's network,\(^8\) for example, there are \(O(N \log N)\) routes satisfying a given request, each route having length \(O(\log N)\). In the language of automata theory, we may say that an admissible route satisfying a given request may be computed nondeterministically with \(O(\log N)\) operations (the number of operations needed to check whether a given route is admissible), while there is no obvious deterministic algorithm requiring less than \(O(N \log^2 N)\) operation (the number of operations needed to search exhaustively over all routes for an admissible one). We shall attempt to reduce this gap. Specifically, we shall derive a deterministic algorithm requiring \(O(\log^2 N)\) arithmetic operations for Cantor's network. The same ideas can be used to obtain algorithms for any of the incrementally nonblocking connection networks for which explicit constructions have been given.

This section will conclude with some results on incrementally \(\epsilon\)-blocking connection networks. The first problem to be considered here is that of obtaining estimates of, or bounds on, the blocking probability for a given network. Lee\(^10\) proposed an approximate probabilistic model on which such computations could be based. Ikeno\(^9\) used this model to show that for any positive blocking probability and a capacity proportional to the number of inputs and outputs (with any coefficient of proportionality less than one), incrementally \(\epsilon\)-blocking networks can be constructed with a number of contacts asymptotic to \(4eN \log_e N\) (where \(e = 2.718 \ldots\) is the base of the natural logarithm). We give an improved version of this result in which no approximations are used and in which the
number of contacts is asymptotic to $6N \log_2 N$. Marcus studied the $\epsilon$-dependence of the number of contacts and showed that incrementally $\epsilon$-blocking networks with $N$ input and outputs and capacity $N$ could be constructed with $O(N \log N) + O(N \log 1/\epsilon)$ contacts. We reduce the second term by giving nonconstructive and constructive upper bounds of $O(N \log \log 1/\epsilon)$ and $O(N (\log \log 1/\epsilon)^2)$ contacts, respectively, and, in effect, by giving a lower bound of $O(N \log \log 1/\epsilon)$ contacts. Specifically, we show that incrementally $\epsilon$-blocking connection networks can be built with $O(N \log N) + O(N (\log \log 1/\epsilon)^V)$ contacts if and only if incrementally nonblocking connection networks can be built with $O(N (\log N)^V)$ contacts.

4.1 AN $O((\log N)^2)$ ALGORITHM FOR ROUTING IN CANTOR'S NETWORK

Our first theorem concerns the problem of finding an admissible route satisfying a given request. We shall perform the computation in several phases, in such a way that a fixed fraction of the remaining routes are excluded in each phase. The first step is to label all of the routes satisfying a given request with natural numbers (more precisely, with their representations in a mixed-radix notation). Next, we must maintain a representation of the network state that allows the successive digits of the label of an admissible route to be computed efficiently. This will be possible if we assign the blocking of one route by another to the rank in which their common vertex appears, and represent the blocking in each rank separately. When this is done, the successive digits can be computed by simple arithmetic operations. Finally, we must be able to update the representation of the network state efficiently as routes are added and deleted. This will be possible if we exploit the symmetry of the network in order to obtain a nearly irredundant representation. When this is done, it is found that the number of bits in the representation is proportional to the number of contacts in the network. These basic ideas can be applied to any of the incrementally nonblocking connection networks for which explicit constructions have been given. We derive the algorithm for Cantor's network in detail, since this is the most complicated case.

We assume that the machine on which the algorithm is to be implemented has random-access storage of words that are large enough to represent any natural number not exceeding $N$ and we assume that the instruction set of the machine includes the usual arithmetic and logical operations on such words. Furthermore, we assume that the addresses of operands may be computed, by indexing or indirect addressing, for example. By counting arithmetic operations on natural numbers not exceeding $N$, we are suppressing an extra factor of $O(\log N)$ which would appear if we counted logical operations on bits (or symbols from any other finite alphabet).

Theorem 4.

For Cantor's network, an admissible route satisfying a given request can be computed with $O((\log N)^2)$ arithmetic operations (on natural numbers not exceeding $N$).
Proof: Because our algorithm must refer to the vertices and edges of the network with which it deals, we shall employ a specific embodiment of the network in which these vertices and edges have names. Cantor's network is \( G = (H, C_{z,2}, H') \), where \( H \) is obtained from \( C_{z,2} \) by deleting one input and the edges incident upon it, \( H' \) is the transpose of \( H \), and \( \phi \) is an arbitrary bijection from the inputs of \( H' \) to the outputs of \( H \) (see sections 2.2.1, 2.2.2, 2.2.4 and 2.2.5). To avoid exceptional cases, our algorithm will deal with a variant of this network in which no vertices or edges are deleted in forming \( H \). The algorithm will work correctly as long as at least one input on each copy of \( H \) (and one output on each copy of \( H' \) ) is idle.

Our specific embodiment of \( H \) will have \( k+1 \) ranks of vertices (numbered 0, 1, \ldots, \( k \) ), with edges joining vertices in successive ranks. The 0th rank (the inputs) contains \( 2^k \) vertices which we assume are labeled with the strings in \( E_2 \) (where \( E_2 = \{0, 1\} \) is the binary alphabet) and the remaining \( k \) ranks (including the \( k \)th, the outputs) each contain \( 2^k \) vertices which we assume are labeled with the strings in \( E_{2k}E_2^{k-1} \) (where \( E_{2k} = \{0, 1, \ldots, 2k-1\} \) is the 2k-ary alphabet). For all \( 1 \leq j \leq k \), there is an edge between a vertex in the \( j \)-1st rank and one in the \( j \)th rank if and only if their labels agree or their labels differ only in the \( j \)th digit (see Fig. 3). (The vertices within each rank have been ordered lexicographically by the reversals of their labels, rather than by their labels themselves, in order to reduce the crossing of edges.)

Our specific embodiment of \( G = (H, C_{z,2}, H') \phi \) is obtained by taking two copies of (our specific embodiment of) \( H \) and two copies of its transpose \( H' \), and by adding an edge between each output of a copy of \( H \) and each input of a copy of \( H' \) having the same label (see Fig. 4). This corresponds to taking \( \phi \) to be the identity map from the inputs of \( H' \) to the outputs of \( H \). For each string in \( E_{2k}E_2^{k-1} \), the four edges interconnecting the outputs of \( H \) and the inputs of \( H' \) having this label constitute a copy of \( C_{z,2} \). Thus the copies of \( C_{z,2} \) (hereafter called junctors) are labeled with the strings in \( E_{2k}E_2^{k-1} \) in a natural way. We shall often identify a vertex with its label (when the copy of \( H \) or \( H' \) and the rank in which it appears are clear from context) or a junctor with its label. This completes the description of our specific embodiment of the network.

If \((x, y)\) is a request to be satisfied, we shall let \( H_{x,y} \) denote the copy of \( H \) on which the input \( x \) appears and \( H'_{y} \), the copy of \( H' \) on which the output \( y \) appears. Let \( s \) be the label of the input \( x \) on \( H_{x,y} \) and let \( t \) be the label of the output \( y \) on \( H'_{y} \). The routes that satisfy this request are in one-to-one correspondence with the junctors. If we find a junctor \( u \) such that the input \( s \) has access to the output \( u \) in \( H_{x,y} \), and the input \( u \) has access to the output \( t \) in \( H'_{y} \), then \( u \) will label a junctor through which an admissible route from \( x \) to \( y \) will pass. Indeed, given \( u \), it is computationally trivial to list the vertices of this route. The first \( k+1 \) (numbered 0, 1, \ldots, \( k \) ) appear in \( H_{x,y} \); for \( 0 \leq j \leq k \), the \( j \)th vertex is the one in the \( j \)th rank whose label is the first \( j \) digits of \( u \) followed by the last \( k-j \) digits of \( s \) (thus the 0th is \( s \) and the \( k \)th is \( u \)). The last \( k+1 \) (numbered \( k+1, k+2, \ldots, 2k+1 \) ) appear in \( H'_{y} \); for \( 0 \leq j \leq k \), the \( k+1+j \)th vertex is the one in the \( k-j \)th rank whose label is the first \( k-j \) digits of \( u \) followed by the last \( j \) digits of \( t \) (thus
Fig. 3. Network H for \( k = 3 \).

Fig. 4. Network \( G = (H, C_{2^k}, H') \). Lines between boxes represent edges.
the \( k+1 \)st is \( u \) and the \( 2k+1 \)st is \( t \). In the first example the route from \( s = 000 \) to \( t = 101 \) through \( u = 510 \) has the following vertices.

<table>
<thead>
<tr>
<th>label</th>
<th>rank</th>
</tr>
</thead>
<tbody>
<tr>
<td>000</td>
<td>( 0^{th} )</td>
</tr>
<tr>
<td>500</td>
<td>( 1^{st} )</td>
</tr>
<tr>
<td>510</td>
<td>( 2^{nd} )</td>
</tr>
<tr>
<td>510</td>
<td>( 3^{rd} )</td>
</tr>
<tr>
<td>511</td>
<td>( 2^{nd} )</td>
</tr>
<tr>
<td>501</td>
<td>( 1^{st} )</td>
</tr>
<tr>
<td>101</td>
<td>( 0^{th} )</td>
</tr>
</tbody>
</table>

Thus the problem of finding an admissible route satisfying a given request reduces to that of finding a junctor accessible to both the input and the output of this request.

There are many possible representations that could be used for the state of the network. The most obvious representation is a list of the routes in the state. Since a route in the network is uniquely determined by specifying its origin, termination, and the junctor through which it passes, \( O(k) = O(\log N) \) bits are sufficient to represent a route. Since there are at most \( 2^k = N \) routes in a state, \( O(k^2) = O(N \log N) \) bits are sufficient to represent a state of the network (they are also necessary, of course, since many bits are needed to specify a permutation of \( 2^k = N \) elements). This representation, while efficient in terms of storage, is not well suited to the task of searching for admissible routes. Therefore, we shall add some redundant information (specifically, the number of busy vertices in certain subsets of the vertices) which will facilitate this task. This additional information will take \( O(k^2 2^k) = O(N \log N)^2 \) bits. This is more than that required by the list of routes, but is proportional to the number of contacts in the network.

For each copy \( H_x \) of \( H \) and for every \( s' \) in \( E_{2^k} \) and \( u' \) in \( E_{2^k} \) such that \( \ell(s') \geq 0 \), \( \ell(u') \geq 1 \), and \( \ell(s') + \ell(u') \leq k \) (where \( \ell(v) \) denotes the length of \( v \)), we define \( Q_{x',s',u'} \) to be the number of busy vertices of the form \( u'v\) \( s' \) (for some \( v \) in \( E_{2^k} \)) in the \( k-\ell(s')^{th} \) rank of \( H_x \). Since there are \( 2^k - \ell(s') \) vertices of the form \( u'v \) \( s' \), we have \( 0 \leq Q_{x',s',u'} \leq 2^k - \ell(s') + 1 \), so that \( Q_{x',s',u'} \) takes \( k - \ell(s') + 1 \) bits for its representation. Summing over the indicated range of \( s' \) and \( u' \) (there are \( 2^k \) values of \( s' \) such that \( \ell(s') = i \) and \( k2^{i-1} \) values of \( u' \) such that \( \ell(u') = j \)), we find that \( Q \) takes \( O(k^2 2^k) = O(N \log N)^2 \) bits for its representation. Similarly, for each copy \( H'_y \) of \( H' \) and for every \( t' \) in \( E_{2^k} \) and \( u' \) in \( E_{2^k} \) such that \( \ell(t') \geq 0 \), \( \ell(u') \geq 1 \), and \( \ell(t') + \ell(u') \leq k \), we define \( Q_{y',t',u'} \) to be the number of busy vertices of the form \( u'v\) \( t' \) (for some \( v \)
in $E^{*}_{2}$ in the $k^{-\ell(t')}^{th}$ rank of $H^x_{y}$. Clearly, $Q'$ will also take $O(k^{2}2^{k}) = O(N(\log N)^{2})$ bits for its representation. In the example, the set of vertices whose busy vertices are counted by $Q_x^{s'},s',u'$ is as indicated for the following typical values of $s'$ and $u'$ ($\wedge$ denotes the null string).

<table>
<thead>
<tr>
<th>$s'$</th>
<th>$u'$</th>
<th>rank</th>
<th>set of vertices</th>
</tr>
</thead>
<tbody>
<tr>
<td>00</td>
<td>5</td>
<td>1$^{st}$</td>
<td>${500}$</td>
</tr>
<tr>
<td>0</td>
<td>5</td>
<td>2$^{nd}$</td>
<td>${500, 510}$</td>
</tr>
<tr>
<td>$\wedge$</td>
<td>5</td>
<td>3$^{rd}$</td>
<td>${500, 510, 501, 511}$</td>
</tr>
<tr>
<td>0</td>
<td>51</td>
<td>2$^{nd}$</td>
<td>${510}$</td>
</tr>
<tr>
<td>$\wedge$</td>
<td>51</td>
<td>3$^{rd}$</td>
<td>${510, 511}$</td>
</tr>
<tr>
<td>$\wedge$</td>
<td>510</td>
<td>3$^{rd}$</td>
<td>${510}$</td>
</tr>
</tbody>
</table>

The arrays $Q$ and $Q'$, together with the list of routes described above, will constitute our representation of the state of the network.

To find an admissible route satisfying a given request, we use the values of $Q$ and $Q'$ to compute the successive digits of the label of a junctor through which the route will pass. As we do this we increment the appropriate values of $Q$ and $Q'$ to indicate that the vertices of this route are now busy. Finally, we add the route to the list of routes in the state. To delete a route from the state of the network, we use the list of routes to find the junctor through which the route passes. We then decrement the appropriate values of $Q$ and $Q'$ to indicate that the vertices of this route are now idle and delete the route from the list of routes in the state.

Suppose that, for some $1 \leq j \leq k$, we have computed the first $j-1$ digits $u'$ of a junctor corresponding to an admissible route and we wish to compute the $j^{th}$ digit $d$. For each $d$, there are $2^{k-j}$ junctors whose labels begin with $u'd$. If we find a $d$ such that the sum of the number of such junctors to which $s$ is denied access and the number to which $t$ is denied access is less than $2^{k-j}$, then there must be at least one such junctor to which neither $s$ nor $t$ is denied access and through which an admissible route may pass. Thus $u'd$ will be the first $j$ digits of a junctor corresponding to an admissible route.

Let $u'$ be the first $j-1$ digits of a junctor corresponding to an admissible route and let $s'$ be a suffix of $s$ such that $0 \leq \ell(s') \leq k-\ell(u')-1$. If the vertex labeled $u'dv's'$ in the $k-\ell(s')^{th}$ rank of $H^y_{x}$ is busy, then $s$ will be denied access to all junctors whose labels begin with $u'dv$, since the route from $s$ to any of these junctors passes through this vertex. Thus for each busy vertex of the form $u'dv's'$ in the $k-\ell(s')^{th}$ rank of $H^y_{x}$, there will be $2^{\ell(s')}^{th}$ junctors whose labels begin with $u'dv$ to which $s$ is denied access, since that is the number of junctors whose labels begin with $u'dv$. Since $s$ is denied access to a
junctor only if its route to that junctor is blocked in this way for some suffix \(s'\) of \(s\), the number of junctors whose labels begin with \(u'd\) to which \(s\) is denied access is at most

\[
R_{x',s,u'd} = \sum_{s'} 2^{\ell(s')} Q_{x',s',u'd'}
\]

where \(s'\) runs over all the suffixes of \(s\) such that \(0 \leq \ell(s') \leq k-\ell(u'd)\). (There may be fewer such junctors to which \(s\) is denied access, since busy vertices in different ranks may deny access to the same junctor.) Similarly, the number of junctors whose labels begin with \(u'd\) to which \(t\) is denied access is at most

\[
R_{y',t,u'd} = \sum_{t'} 2^{\ell(t')} Q_{y',t',u'd'}
\]

where \(t'\) runs over all suffixes of \(t\) such that \(0 \leq \ell(t') \leq k-\ell(u'd)\). Thus if we can find a \(d\) such that \(R_{x',s,u'd} + R_{y',t,u'd} < 2^{k-j}\), there will be at least one junctor whose label begins with \(u'd\) corresponding to an admissible route. We shall prove by induction on \(j\) that we can always find such a \(d\).

Suppose that \(j = 1\). The input \(s\) may be denied access to \(2^{k-1}\) junctors by a busy vertex in the 1\(^{st}\) rank, \(2^{k-2}\) junctors by each of 2 busy vertices in the 2\(^{nd}\) rank, and in general \(2^{k-i}\) junctors by each of \(2\) busy vertices in the \(i^{th}\) rank. If at least one input (other than \(s\)) on \(H_{x'}\) is idle, \(s\) will be denied access to less than \(k-1\) junctors. Similarly, if some output (other than \(t\)) on \(H_{y'}\) is idle, \(t\) will be denied access to less than \(k-1\) junctors. Since each junctor begins with some \(d\) in \(E\),

\[
\sum_{d \in E} (R_{x',s,d} + R_{y',t,d}) < k^k.
\]

Thus, for some \(d\) in \(E\), \(R_{x',s,d} + R_{y',t,d} < 2^{k-1}\), as was to be shown.

Now suppose that \(j > 1\). By inductive hypothesis, \(R_{x',s,u'} + R_{y',t,u'} < 2^{k-j+1}\). Since each junctor which begins with \(u'\) begins with \(u'd\) for some \(d\) in \(E\),

\[
\sum_{d \in E} (R_{x',s,u'd} + R_{y',t,u'd}) < 2^{k-j+1}.
\]

Thus, for some \(d\) in \(E\), \(R_{x',s,u'd} + R_{y',t,u'd} < 2^{k-j}\), as was to be shown. Thus from \(Q\) and \(Q'\) we can compute the successive digits of a junctor corresponding to an admissible route.

The two vertices of a route which appear in the \(j^{th}\) ranks of \(H_{x'}\) and \(H_{y'}\) are determined by \(s\), \(t\), and the first \(j\) digits \(u'\) of the junctor through which this route will pass. Thus after computing each digit of \(u\), two busy vertices are determined, and we must
update $Q$ and $Q'$ to reflect them. The values of $Q$ and $Q'$ corresponding to sets of vertices in which these two busy vertices appear are $Q_{x',s'',u''}$ and $Q'_{y',t'',u''}$, where $s''$ is the suffix of $s$ of length $k-j$, $t''$ is the suffix of $t$ of length $k-j$, and $u''$ runs over the set of all prefixes of $u'$ such that $1 \leq \ell(u'') \leq j$. These values of $Q$ and $Q'$ must be incremented when the route is added to the state of the network, and decremented when it is deleted.

The following algorithm will update the representation of the state of $G$ to reflect the addition of an admissible route satisfying the request $(x, y)$.

1. let $H_{x'}$ be the copy of $H$ on which the input $x$ appears;
2. let $H'_{y'}$ be the copy of $H'$ on which the output $y$ appears;
3. let $s$ be the label of $x$ on $H_{x'}$;
4. let $t$ be the label of $y$ on $H'_{y'}$;
5. $u' = \Lambda$;
6. for $j$ from 1 through $k$ do
   begin
   d = 0;
   until $R_{x',s,u'd} + R'_{y',t,u'd} < 2^{k-j}$ do $d = d + 1$;
   $u' = u'd$;
   let $s''$ be the suffix of $s$ of length $k-j$;
7. let $t''$ be the suffix of $t$ of length $k-j$;
8. let $U$ be the set of prefixes $u''$ of $u'$ such that $1 \leq \ell(u'') \leq j$;
9. for all $u''$ in $U$ do increment $Q_{x',s'',u''}$;
10. for all $u''$ in $U$ do increment $Q'_{y',t'',u''}$;
end;
11. $u = u'$

From the considerations preceding the algorithm, it should be clear that for $j = 1$ the "until ... do ..." statement terminates for some $d < 2k$, and that for $j > 1$ it terminates for some $d < 2$. Thus the predicate of the "until ... do ..." statement is evaluated at most $4k$ times. Since each value of $R$ and $R'$ can be computed from the values of $Q$ and $Q'$ with $O(k) = O(\log N)$ arithmetic operations, we easily verify that $O(k^2) = O((\log N)^2)$ arithmetic operations are required for the entire algorithm. At the conclusion of the algorithm, $u$ is assigned the label of a junctor through which an admissible route from $x$ to $y$ may pass. The vertices of the route can be computed from $x$, $y$, and $u'$ as indicated earlier.

The following algorithm will update the representation of the state of $G$ to reflect the
deletion of the route from x to y through the junctor u.

let $H'_x$ be the copy of $H$ on which the input x appears;
let $H'_y$ be the copy of $H'$ on which the output y appears;
let s be the label of x on $H'_x$;
let t be the label of y on $H'_y$;
for j from 1 through k do
begin
let $s''$ be the suffix of s of length $k-j$;
let $t''$ be the suffix of t of length $k-j$;
let $U$ be the set of prefixes $u''$ of u such that $l(u'') \leq j$;
for all $u''$ in U do decrement $Q_{x',s'',u''}$;
for all $u''$ in U do decrement $Q_{y',t'',u''}$
end

The route-deletion algorithm simply reverses the actions taken by the route-addition algorithm, and again we verify easily that $O(k) = O((\log N)^2)$ arithmetic operations are required. This completes the proof.

4.2 AN UPPER BOUND ASYMPTOTIC TO $6N \log_2 N$ FOR INCREMENTALLY $\epsilon$-BLOCKING NETWORKS

We turn now to the problem of obtaining an upper bound on the number of contacts required in incrementally $\epsilon$-blocking connection networks. Our result is that for any $\epsilon > 0$ incrementally $\epsilon$-blocking connection networks with a capacity $N$ equal to the number of inputs and outputs can be built with a number of contacts asymptotic to $6N \log_2 N$. In Theorem 5 we obtain this result for one particular value of $\epsilon$ and a capacity equal to one-half the number of inputs and outputs; in the Corollary this result is extended to arbitrary $\epsilon > 0$ and capacity equal to the number of inputs and outputs. In each case, we give an explicit construction for the networks.

We shall consider graded networks having the following form. There are several stages, each containing an equal number of square crossbar switches of some uniform size $n$. The inputs of the switches in the first stage are the network inputs, the outputs of the switches in each stage but the last are identified according to some one-to-one correspondence with the inputs of the switches in the following stage, and the outputs of the switches in the last stage are the network outputs.

The probabilistic assumptions that we use to evaluate blocking probabilities are the following. We assume that each network input is idle with probability $q$ (and hence busy with probability $1-q$) and that these probabilities are independent. As a consequence, the
number of busy network inputs in any subset of the network inputs is binomially dis-
tributed. We also assume random routing of the traffic through the network. That is, we
assume that each switch effects a random permutation of its inputs to its outputs (and
that the permutations effected by different switches are independent), so that if some
number of inputs on a switch are busy, each subset of that number of outputs of the
switch is equally likely to be the set of busy outputs. As a consequence, the number of
busy links in any subset of the links within a given rank (or the number of busy network
outputs in any subset of the network outputs) is binomially distributed.

The assumption previously used to attack this problem was that each vertex in the
network was busy with equal probability and that these probabilities were independent.
A strong objection to this assumption is that it assigns nonzero probabilities to config-
urations of busy and idle vertices which do not correspond to any state of the network,
that is, which do not correspond to any set of routes from inputs to outputs. Indeed,
most of the probability is assigned to such configurations. In contrast, our assumptions
assign nonzero probabilities only to states of the network. The uniform network struc-
ture to which we restrict our attention allows us to deduce the independence of the links
within a rank (or of the outputs) from the assumed independence of the inputs.

We begin with a heuristic derivation of our result. A network of the form described
above with capacity \( N \) will have \( N/(1-q) \) inputs and outputs and hence \( nN/(1-q) \)
contacts per stage. If a particular switch has one input that is known to be idle and
\( n-1 \) others that are idle independently with probability \( q \), then the expected number
of idle outputs will be \( 1+(n-1)q \). Therefore a network input will have an average of
\( 1+(n-1)q \) paths to outputs of switches in the first stage, \( (1+(n-1)q)^2 \) paths to outputs
of switches in the second stage and, in general, \( (1+(n-1)q)^k \) paths to outputs of switches
in the \( k^{th} \) stage. Thus, the number of stages necessary to make the average number
of paths equal to the number of outputs is the logarithm of \( N/(1-q) \) to the base
\( 1+(n-l)q \). With this number of stages, the total number of contacts in the net-
work is

\[
(n/\log(1+(n-1)q))(N/(1-q)) \log (N/(1-q)).
\]

To minimize the coefficient of \( N \log_2 N \), we must minimize \( n/(1-q) \log_2 (1+(n-1)q) \).
Equating the partial derivatives of this expression to zero, we obtain \( q/(1-q) = \log \left( (1-q)^2 / (1-2q) \right) \), and \( (n-1)/n = \log \left( n^2 / (2n-1) \right) \). The substitutions \( x = q/(1-q) \) and \( x = (n-1)/n \) reduce these to the common equation \( x = \log 1/(1-x^2) \), which has the solution
\( x = 0.714566... \), which implies \( q = 0.416759... \), and \( n = 3.503314... \). For these
values, the coefficient of \( N \log_2 N \) is 5.826670... Since we must choose an integral
value of \( n \), it is pleasant to discover that \( q = 1/2 \) and \( n = 3 \) result in a coefficient of 6,
which is only 3% more than the minimum that we found. Two interesting comparisons
can be made. First, the best known rearrangeably nonblocking connection networks with
capacity \( N \) have a number of contacts asymptotic to \( 6N \log_3 N \); our incrementally \( \epsilon \)-
blocking networks come within \( (\log 3)/(\log 2) = 1.585... \) of this. Second, the number of
bits required to specify a permutation of \(N\) elements is asymptotic to \(N \log_2 N\); our networks have a number of contacts asymptotic to \(6N \log_2 N\) and for each copy of \(C_{3,3}\) with 9 contacts there are 6 states, so the number of bits required to specify the state of such a network is within \((2 \log 6)/(3 \log 2) = 1.723\ldots\) of the minimum.

The argument just made is not rigorous, of course, since the average number of paths to outputs tells us very little about the average number of accessible outputs and the average number of accessible outputs tells us very little about the probability that few of the outputs will be accessible. It may be surprising, therefore, to find that a rigorous argument leading to the same result can be offered.

**Theorem 5.**

For \(\epsilon = 59/75\) incrementally \(\epsilon\)-blocking connection networks with a capacity \(N\) equal to one-half the number of inputs and outputs can be built with a number of contacts asymptotic to \(6N \log_2 N\).

**Proof:** Let \(F_v = C_{n,n}^V\) (see sections 2.2.1 and 2.2.4), and let \(Q_v\) be the linking probability (the complement of the blocking probability) of \(F_v\). Incrementing \(v\) by one increases the number of stages by one, multiplies the number of inputs and outputs by \(n\), and decreases the linking probability (since there is just one route satisfying each request and the length of this route increases linearly with \(v\)). We see in Lemma 1 that \(Q_v = (f'(1)/n)^{V-1}\), where \(f\) is a probability generating function such that \(f'(1) = 1+(n-1)q\) and \(f''(1) = q(n-1)(2+q(n-2))\). Let \(G_{0,v} = F_v\) and let \(G_{w+1,v} = C_{n,n}^V G_{w,v} C_{n,n}^V\phi\) where \(\phi\) is the identity map from the inputs of \(C'_{n,n}\) to the outputs of \(C_{n,n}\) (see sections 2.2.2 and 2.2.5), and let \(P_{w,v}\) be the blocking probability of \(G_{w,v}\). Incrementing \(w\) by one increases the number of stages by two, multiplies the number of inputs and outputs by \(n\), and (when \(n\) and \(q\) are large enough) decreases the blocking probability (since the expected number of routes satisfying a given request increases exponentially with \(w\)). We show in Lemma 2 that \(P_{w,v} = g_w(1-Q_v)\), where \(g\) is a probability generating function such that \(g'(1) = (f'(1))^2/n = (1+(n-1)q)^2/n\), \(g''(1) = (f''(1))^2/n(n-1) = q^2(n-1)(2+q(n-2))^2/n\), \(g_0(z) = z\), and \(g_{w+1}(z) = g(g_w(z))\). The network \(G_{w,v}\) has \(n^{w+v}\) inputs and outputs, \((2w+v)n^{w+v+1}\) contacts, capacity \((1-q)n^{w+v}\), and blocking probability \(g_w(1-(f'(1)/n)^{V-1})\). We want to minimize \(2w+v\) while holding \(w+v\) constant, subject to the constraint that the blocking probability remain bounded below one. We see in Lemma 4 that if \(g'(1) > 1\) and \(m = \log (n/f'(1))/\log g'(1)\), then \(g_{mv}(1-(f'(1)/n)^{V-1}) \leq 1 - g'(1)(g'(1)-1)/2g''(1) < 1\) for all sufficiently large \(v\). The network \(G_{mv,v}\) has \(n^{(m+1)v}\) inputs and outputs, \((2m+1)n^{(m+1)+v+1}\) contacts, and capacity \((1-q)n^{(m+1)v}\). Such a network with capacity \(N\) will have \(((2m+1)/(m+1))(n/\log n)(N/(1-q)) \log (N/(1-q))\) contacts. Substituting the expressions for \(f'(1)\) and \(g'(1)\) in that for \(m\), we get \((n/\log (1+(n-1)q)) (N/(1-q)) \log (N/(1-q))\), which is exactly the expression we obtained from our heuristic argument. Again setting \(n = 3\) and \(q = 1/2\), we find that \(g'(1) = 4/3\), which justifies our assumption that \(g'(1) > 1\), and that \(g^*(1) = 25/24\), so that \(g'(1)(g'(1)-1)/2g''(1) = 59/75.\)
Thus this sequence of networks has a blocking probability of at most $59/75$, a capacity $N$ equal to one-half the number of inputs and outputs, and a number of contacts asymptotic to $6N \log_2 N$. This completes the proof (under the assumption of Lemmas 1, 2, and 4).

**Lemma 1.**

Let $F_v = C_{n,n}^V$ and $Q_v$ be the linking probability of $F_v$. Then $Q_v = (f'(1)/n)^{V-1}$, where $f$ is a probability generating function such that $f'(1) = 1+(n-1)q$ and $f''(1) = q(n-1) (2+q(n-2))$.

**Proof:** We proceed by induction on $v$. Since $F_1 = C_{n,n}$ is nonblocking, $Q_1 = 1 = (f'(1)/n)^0$. We now assume that $Q_v = (f'(1)/n)^{V-1}$ and show that $Q_{v+1} = (f'(1)/n)^V$. Let $(x,y)$ be the request to be satisfied. In the factorization $F_{v+1} = C_{n,n} x F_v$ let $x$ be the input $s$ on the copy $C_{n,n,x}$ of $C_{n,n}$ and $y$ be the output $t$ on the copy $F_v y$ of $F_v$. There is a unique route from $x$ to $y$ in $F_{v+1}$. This route will be admissible if and only if the output $y'$ of $C_{n,n,x}$ (which is the input $x'$ of $F_v y$) is idle, the request $(s,y')$ in $C_{n,n,x}$ can be satisfied (given that $y'$ is idle), and the request $(x',t)$ in $F_v y'$ can be satisfied (given that $x'$ is idle). The switch $C_{n,n,x}$ has one input $s$ that is known to be idle and $n-1$ others that are independently idle with probability $q$, so that the generating function for the number of idle outputs is

$$f(z) = \sum_{1 \leq i \leq n} \binom{n-1}{i-1} q^{i-1} (1-q)^{n-i} z^i.$$ 

Differentiating and using the binomial theorem, we obtain $f'(1) = 1+(n-1)q$ and $f''(1) = q(n-1) (2+q(n-2))$. By our random-routing assumption, the probability that the output $y'$ is idle is simply the average number of idle outputs divided by the total number of outputs, $f'(1)/n$. Given that $y'$ is idle, the request $(s,y')$ in $C_{n,n,x}$ can be satisfied, since this network is nonblocking. Given that $x'$ is idle, the request $(x',t)$ in $F_v y'$ can be satisfied with probability $(f'(1)/n)^{V-1}$ by inductive hypothesis, since by our probabilistic assumptions all of the inputs of this network other than $x'$ are idle independently with probability $q$. Thus the probability that the request $(x,y)$ in $F_{v+1}$ can be satisfied is $Q_{v+1} = (f'(1)/n) (f'(1)/n)^{V-1} = (f'(1)/n)^V$. This completes the proof.

**Lemma 2.**

Let $G_{0,v} = F_v$ and $G_{w+1,v} = (C_{n,n}, G_{w,v}, C_{n,n})$, where $\phi$ is the identity map from the inputs of $C_{n,n}$ to the outputs of $C_{n,n}$. (The networks $C_{n,n}$ and $C_{n,n}$ are isomorphic, but we use this notation so that we can distinguish between "copies of $C_{n,n}$" and "copies of $C_{n,n}'"). Let $P_{w,v}$ be the blocking probability of $G_{w,v}$. Then $P_{w,v} = g_w(1-Q_v)$, where $g$ is a probability generating function such that $g'(1) = (f'(1))^2/n = (1+(n-1)q)^2/n$, $g''(1) = (f''(1))^2/n(n-1) = q^2(n-1)(2+q(n-2))^2/n$, $g_0(z) = z$ and $g_{w+1}(z) = g(g_w(z))$.

**Proof:** We proceed by induction on $w$. Since $G_{0,v} = F_v$, $P_{0,v} = 1-Q_v = g_0(1-Q_v)$. We now assume that $P_{w,v} = g_w(1-Q_v)$ and show that $P_{w+1,v} = g_{w+1}(1-Q_v)$. To do this,
we need only show that \( P_{w+1,v} = g(P_{w,v}) \), by inductive hypothesis. Let \( (x, y) \) be the request to be satisfied. In the factorization \( G_{w+1,v} = (C_{n,n}, G_{w,v} C'_{n,n})' \) let \( x \) be the input \( s \) on the copy \( C_{n,n,x'} \) of \( C_{n,n} \) and \( y \) be the output \( t \) on the copy \( C'_{n,n,y'} \) of \( C'_{n,n} \). There are \( n \) routes from \( x \) to \( y \) in \( G_{w+1,v} \), one corresponding to each copy of \( G_{w,v} \). The route corresponding to the copy \( G_{w,v,u} \) of \( G_{w,v} \) will be admissible if and only if the output \( u \) of \( C_{n,n,x'} \) (which is the input \( x' \) of \( G_{w,v,u} \)) is idle and the input \( u \) of \( C'_{n,n,y'} \) (which is the output \( y' \) of \( G_{w,v,u} \)) is idle, the request \( (s, u) \) in \( C_{n,n,x} \) and the request \( (u, t) \) in \( C'_{n,n,y} \) can be satisfied (given that the links \( u \) are idle), and the request \( (x', y') \) in \( G_{w,v,u} \) can be satisfied (given that the links \( x' \) and \( y' \) are idle).

Let \( X' \) be the set of \( u \) for which the input \( x' \) of \( G_{w,v,u} \) is idle and \( Y' \) be the set of \( u \) for which the output \( y' \) of \( G_{w,v,u} \) is idle. Let \( I, J, \) and \( K \) denote the cardinalities of \( X' \), \( Y' \), and \( X' \cap Y' \), respectively, and let \( g \) denote the generating function of \( K \). We shall show that \( g'(1) = (f'(1))^2/n = (1+(n-1)q)^2/n \) and \( g''(1) = (f''(1))^2 /n(n-1) = q^2(n-1)(2+q(n-2))^2/n \). The random variables \( X' \) and \( Y' \) are not independent, but we see in Lemma 3 that their dependence is such as to produce a lower blocking probability than independence would. Let \( h_{i,j} \) be the generating function of \( K \), given that \( I = i \) and \( J = j \). From the independence of \( X' \) and \( Y' \) (and hence of \( I \) and \( J \)) we have

\[
g(z) = \sum_{1 \leq i \leq n} \Pr (I = i) \sum_{1 \leq j \leq n} \Pr (J = j) h_{i,j}(z).
\]

Differentiating, we obtain

\[
g'(1) = \left( \sum_{1 \leq i \leq n} \Pr (I = i) i \right) \left( \sum_{1 \leq j \leq n} \Pr (J = j) j \right) / n
\]

\[
= (f'(1))^2 /n = (1+(n-1)q)^2/n
\]

and

\[
g''(1) = \left( \sum_{1 \leq i \leq n} \Pr (I = i) i(i-1) \right) \left( \sum_{1 \leq j \leq n} \Pr (J = j) j(j-1) \right) / n(n-1)
\]

\[
= (f''(1))^2 /n(n-1) = q^2(n-1)(2+q(n-2))^2/n,
\]

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as was to be shown.

If \( K = k \), there are \( k \) copies of \( G_{w,v} \) on which \( x' \) and \( y' \) are idle and through which an admissible route from \( x \) to \( y \) might pass. Given that the links \( u \) are idle, the request \((s, u)\) in \( C_{n,n,x'} \) and the request \((u, t)\) in \( C_{n,n,y'} \) can be satisfied, since these networks are nonblocking. Given that \( x' \) and \( y' \) are idle on a copy of \( G_{w,v}' \) the probability that the request \((x', y')\) can be satisfied is \( P_{w,v} \), and these probabilities are independent, since by our probabilistic assumptions all of the inputs of these networks other than the \( x' \) are idle independently with probability \( q \). Thus, since \( g \) is the generating function for \( K \), \( P_{w+1,v} = g(P_{w,v}) \). This completes the proof (under the assumption of Lemma 3).

**Lemma 3.**

The dependence between \( X' \) and \( Y' \) (as defined in Lemma 2) is such as to produce a lower blocking probability than independence would. Specifically, we shall show that the generating function \( g \) computed in Lemma 2 is an upper bound on the true generating function.

**Proof:** We use symbols without underlines to denote true probabilities and symbols with underlines to denote the corresponding probabilities computed under the assumption of independence. Consider one of the \( n \) copies of \( G_{w,v} \) and the input \( x' \) and output \( y' \) on it. Let \( \Pr(x') \) denote the probability that \( x' \) is idle, and \( \Pr(x') \) the probability that \( x' \) is busy. We have \( \Pr(x') = \Pr(x') \) and \( \Pr(x') = \Pr(x') \). Since the copy of \( G_{w,v} \) under consideration has as many busy outputs as busy inputs, and we assume random routing, we have \( \Pr(x'| y') \geq \Pr(x') \) and \( \Pr(x'| y') \geq \Pr(x') \). Thus we have \( \Pr(x' \land y') \geq \Pr(x') \Pr(y') = \Pr(x' \land y') \), and \( \Pr(x' \land y') \geq \Pr(x') \Pr(y') = \Pr(x' \land y') \).

Consider \( m \) of the \( n \) copies of \( G_{w,v} \) and the inputs \( x' \) and outputs \( y' \) on them. Let the random variable \( I \) denote the number of copies on which the input \( x' \) is idle, the random variable \( I \) denote the number of copies on which the input \( x' \) is idle, the random variable \( J \) denote the number on which the output \( y' \) is idle, and the random variable \( K \) denote the number on which both \( x' \) and \( y' \) are idle. For \( 0 \leq i \leq m \) and \( 0 \leq j \leq m \), let \( A(i, j, k, m) = \Pr(K \geq k \mid I = i \land J = j) \). We have \( A(i, j, k, m) \geq A(i, j, k, m) \geq A(i, j, k, m) \). We shall show that \( A(i, j, k, m) \geq A(i, j, k, m) \). If any of the four conditions \( i = 0, i = m, j = 0, \) or \( j = m \) holds, the result is trivial. For \( 0 < i < m \) and \( 0 < j < m \), we use induction on \( m \). If \( m = 0 \), the result is trivial. If \( m > 0 \), we consider one of the \( m \) copies of \( G_{w,v} \) and its input \( x' \) and output \( y' \), and write

\[
A(i, j, k, m) = \Pr(x' \land y') A(i-1, j-1, k-1, m-1) \\
+ \Pr(x' \land y') A(i-1, j, k, m-1) \\
+ \Pr(x' \land y') A(i, j-1, k, m-1) \\
+ \Pr(x' \land y') A(i, j, k, m-1).
\]

By inductive hypothesis, we have
This may be rewritten

\[ A(i, j, k, m) \geq \text{Pr} \left( x' \land y' \right) \left( A(i-1, j-1, k, m-1) - A(i-1, j, k, m-1) \right) + \text{Pr} (x') A(i-1, j, k, m-1) + \text{Pr} (x') A(i, j-1, k, m-1) + \text{Pr} (x' \land y') (A(i, j, k, m) - A(i, j-1, k, m-1)). \]

Since the quantities in parentheses are non-negative, we have

\[ A(i, j, k, m) > \text{Pr} \left( x' \land y' \right) (A(i-1, j-1, k, m-1) - A(i-1, j, k, m-1)) + \text{Pr} (x') A(i-1, j, k, m-1) + \text{Pr} (x') A(i, j-1, k, m-1) + \text{Pr} (x' \land y') (A(i, j, k, m) - A(i, j-1, k, m-1)). \]

Thus \( A(i, j, k, m) \geq A(i, j, k, m). \)

Let \( m = n \) and \( B(i, j) = \text{Pr} (I \geq i \land J \geq j) \). Since \( \text{Pr} (x' \land y') \geq \text{Pr} (y') \) for each copy of \( G_{w, y'} \), we have \( B(i, j) \geq \text{Pr} (I \geq i | J \geq j) \geq \text{Pr} (I \geq i) \). Thus \( B(i, j) = \text{Pr} (I \geq i | J \geq j) \) \( \geq \text{Pr} (I \geq i) \) \( \geq \text{Pr} (I \geq i) \) \( \geq B(i, j) = B(i, j) \).

Let \( C(k) = \text{Pr} (K \geq k) \), and \( b(i, j) = \text{Pr} (I = i \land J = j) \). Then we have

\[ C(k) = \sum_{0 < i < n} \sum_{0 < j < n} A(i, j, k, n) b(i, j) \geq \sum_{0 < i < n} \sum_{0 < j < n} A(i, j, k, n) b(i, j). \]

Since \( i' \geq i \) and \( j' \geq j \) imply \( A(i', j', k, n) \geq A(i, j, k, n) \), and \( B(i, j) \geq B(i, j) \), we have

\[ C(k) \geq \sum_{0 < i < n} \sum_{0 < j < n} A(i, j, k, n) b(i, j). \]

Thus \( C(k) \geq C(k) \).

Let \( c(k) = \text{Pr} (K = k) \). Then the true generating function for \( K \) is

\[ g(z) = \sum_{0 \leq k \leq n} c(k) z^k. \]
Since \( k' \geq k \) and \( 0 \leq z \leq 1 \) imply \( z^{k'} \leq z^k \), and \( C(k) \geq C(k) \), we have

\[
g(z) = \sum_{0 \leq k \leq n} C(k) z^k,
\]

so that the generating function computed under the assumption of independence is an upper bound on the true generating function. This completes the proof.

**Lemma 4.**

If \( g \) is a generating function such that \( g'(1) > 1 \), and \( m = (\log n/f'(1))/\log g'(1) \), then

\[
g_{mv}(1-(f'(1)/n)^{V-1}) \leq 1 - g'(1)(g'(1)-1)/2g^n(1) < 1, \text{ for all sufficiently large } v.
\]

**Proof:** Since \( g \) is a probability generating function, \( g(1) = 1 \) and \( g'(1), g''(1), \ldots \) are all non-negative. From the definition of \( g_w \) we can readily compute

\[
g_w'(1) = (g'(1))^w
\]

and

\[
g_w^n(1) = g^n(1)(g'(1))^{w-1} ((g'(1))^w-1)/(g'(1)-1).
\]

Since \( g^n(1) - g'(1)(g'(1)-1) \) is the variance of the random variable of which \( g \) is the generating function, we have

\[
g^n(1) \geq g'(1)(g'(1)-1).
\]

From this and the expressions for \( g_w'(1) \) and \( g_w^n(1) \) we deduce

\[
g_w^n(1) \geq g_w'(1)(g_w'(1)-1).
\]

Now let \( d_v = g_{mv}'(1)/g_{mv}^n(1) \). By the inequality just derived, \( d_v \leq 1/(g_{mv}'(1)-1) \). If \( m = (\log n/f'(1))/\log g'(1) \), then for all sufficiently large \( v \), \( 1/(g_{mv}'(1)-1) \leq (f'(1)/n)^{V-1} \). Thus, for all sufficiently large \( v \), \( d_v \leq (f'(1)/n)^{V-1} \leq 1-d_v \) and, since \( g \) is an increasing function, \( g_{mv}(1-(f'(1)/n)^{V-1}) \leq g_{mv}(1-d_v) \). Thus we need only show that, for all sufficiently large \( v \), \( g_{mv}(1-d_v) \leq 1 - g'(1)(g'(1)-1)/2g^n(1) \). For all sufficiently large \( v \), \( d_v \) will be sufficiently small that

\[
g_{mv}(1-d_v) \leq 1 - g_{mv}'(1)d_v + g_{mv}^n(1)d_v^2/2.
\]

Substituting the definition of \( d_v \), we find

\[
g_{mv}(1-d_v) \leq 1 - (g_{mv}'(1))^2/2g_{mv}^n(1).
\]

Finally, substituting the expressions for \( g_w'(1) \) and \( g_w^n(1) \), we find

\[
g_{mv}(1-d_v) \leq 1 - g'(1)(g'(1)-1)/2g^n(1).
\]
This completes the proof.

**Corollary.** For any $\epsilon > 0$, incrementally $\epsilon$-blocking connection networks can be built with a capacity $N$ equal to the number of inputs and outputs and a number of contacts asymptotic to $6N \log_2 N$.

**Proof:** Choose $M$ such that $(71/75)^M \leq \epsilon$. By Theorem 5, we can build networks $F$ with $N/M$ inputs and outputs, capacity $N/2M$, blocking probability at most $59/75$, and a number of contacts asymptotic to $(3N/M) \log_2 (N/2M)$. Consider the networks $G = (C_{M,M}, F, C_{M,M})$. By arguments similar to those used to prove Theorem 5, we may assume without undue optimism that a given request may be satisfied by the route passing through a given copy of $F$ with probability at least $(1-59/75)/4 = 4/75$ and that for different copies of $F$, these probabilities are independent. It follows that the networks $G$ have $N$ inputs and outputs, capacity $N/2$, blocking probability at most $(1-4/75)^M = (71/75)^M \leq \epsilon$, and a number of contacts asymptotic to $3N \log_2 N$. Now consider the networks $H = (C_{M,2M}, F, C_{M,2M})$ obtained by connecting two copies of $G$ in parallel. If we assign each request to one of the copies of $G$ at random, the probabilistic assumptions for each copy of $G$ will be satisfied even when all inputs of $H$ are busy. Thus the networks $H$ have $N$ inputs and outputs, capacity $N$, blocking probability at most $\epsilon$, and a number of contacts asymptotic to $6N \log_2 N$. This completes the proof.

4.3 THE $\epsilon$-DEPENDENCE OF THE COMPLEXITY OF INCREMENTALLY $\epsilon$-BLOCKING NETWORKS

Our next theorem is the analog for connection networks of Theorem 3 on partial concentration networks. We show that incrementally $\epsilon$-blocking connection networks can be built with $O(N \log N) + O(N(\log \log 1/\epsilon)^V)$ contacts if and only if incrementally non-blocking connection networks can be built with $O(N(\log N)^V)$ contacts. The probabilistic and structural assumptions that we use are the same as those for Theorem 3.

**Theorem 6.**

Suppose that for some $\epsilon < 1$ there exists $c$ such that for all sufficiently large $N$, uniform incrementally $\epsilon$-blocking connection networks can be constructed with $N$ inputs, $N$ outputs, at most $cN \log_2 N$ contacts, and capacity $N$ (see Marcus\textsuperscript{11}). Then, for all $v \geq 1$, the following propositions are equivalent.

1. There exists $d$ such that for all sufficiently large $N$ uniform incrementally non-blocking connection networks can be constructed with $N$ inputs, $N$ outputs, at most $dN(\log_2 N)^V$ contacts, and capacity $N$.

2. There exist $c$ and $d$ such that for all sufficiently small $\epsilon > 0$ and all sufficiently large $N$ uniform incrementally $\epsilon$-blocking connection networks can be constructed with $N$ inputs, $N$ outputs, at most $cN \log_2 N + dN(\log_2 \log_2 1/\epsilon)^V$ contacts, and capacity $N$. 
Proof: We wish to construct uniform incrementally $\varepsilon$-blocking connection networks with $N$ inputs, $N$ outputs, and capacity $N$. Choose $r$ such that $0 < r < 1$.

By 1., there exists $d$ such that for all sufficiently large $n$ uniform incrementally non-blocking connection networks can be constructed with $n$ inputs, $n$ outputs, at most $dn(\log_2 n)^V$ contacts, and capacity $n$. By ignoring $(1-r)n$ of the inputs on such networks, we obtain uniform incrementally non-blocking connection networks $G$ with $rn$ inputs, $n$ outputs, at most $dn(\log_2 n)^V$ contacts, and capacity $rn$. By our preliminary supposition, there exist $\delta < 1$ and $c$ such that for all sufficiently large $N$ uniform incrementally $\delta$-blocking connection networks $H$ can be constructed with $N/rn$ inputs, $N/rn$ outputs, at most $(c/rn)N \log_2 N$ contacts, and capacity $N/rn$. By taking the triple product $(G, H, G')$, where $\phi$ is an arbitrary bijection from the inputs of $G'$ to the outputs of $G$ (see sections 2.2.2 and 2.2.5), we obtain uniform networks with $N$ inputs, $N$ outputs, and at most $(c/r)N \log_2 N + (2d/r)N(\log_2 n)^V$ contacts. We need only show that there exists $h$ such that when $n = h \log_2 1/E$, these networks are incrementally $\varepsilon$-blocking with capacity $N$.

By assumption (ia) (see Section 3.3), the network is in a state with $mN$ busy inputs and outputs (for some $0 < m < 1$). Suppose that the request to be satisfied is $(x, y)$. Let us focus our attention on the copy $G_x$, of $G$ on which the input $x$ appears and the copy $G_{y'}$, of $G'$ on which the output $y$ appears. Let $X'$ be the set of $u$ for which the input $x'$ of the copy $H_u$ of $H$ is idle and let $Y'$ be the set of $u$ for which the output $y'$ of $H_u$ is idle. Let $I$, $J$, and $K$ denote the cardinalities of $X'$, $Y'$, and $X' \cap Y'$, respectively. By arguments similar to those used in the proofs of Lemmas 2 and 3 of Theorem 5, we may assume without undue optimism that $\Pr(K = k \mid I = i \land J = j) = \binom{i}{k} \binom{n-i}{j-k} / \binom{n}{j}$. If $K = k$, there are $k$ copies of $H$ on which $x'$ and $y'$ are idle and through which an admissible route from $x$ to $y$ might pass. In each such copy of $H$, the probability that the request $(x', y')$ cannot be satisfied is at most $\delta$ and these probabilities are independent, so that the probability that the request $(x, y)$ cannot be satisfied (given that $K = k$) is at most $\delta^k$, and the probability that it cannot be satisfied (given that $I = i$ and $J = j$) is at most

$$\sum_{0 \leq k \leq n} \binom{i}{k} \binom{n-i}{j-k} \delta^k / \binom{n}{j}.$$

Since there are $n+1$ terms in this sum, its value is at most $n+1$ times that of its largest term. If we show that the value of $k$ corresponding to this largest term is at least asymptotically proportional to $n$ as $n$ tends to infinity, with a positive coefficient of proportionality, it will follow that the factor $\delta^k$ in the largest term decreases exponentially as $n$ increases, and thus that the sum decreases exponentially as $n$ increases, as was to be shown.

The general term in the sum will be zero unless $0 \leq k \leq \min(i, j)$. In this range, the ratio of successive terms in the sum is $(i-k)/(j-k) \delta/(k+1)(n-i-j+k+1)$, which is
a decreasing function of \(k\). Thus the largest term of the sum occurs at that value of \(k\) which makes this ratio unity. Since this ratio is an increasing function of \(i\) and \(j\), replacing \(i\) and \(j\) by their common lower bound \((1-r)n\) gives a ratio \(((1-r)n-k)^2 \delta/(k+1)\) \(((2r-1)n+k+1)\) which is unity for a smaller or equal value of \(k\). It is easy to show that the value of \(k\) that makes this ratio unity is asymptotically proportional to \(n\) as \(n\) tends to infinity. The coefficient of proportionality \(w\) is the larger root of the equation \((1-\delta)w^2 + (1-2(1-\delta)(1-r))w - \delta(1-r)^2 = 0\), and is positive, since the first and last coefficients in this equation have opposite signs. This completes the proof.

Proof (2. implies 1.): (Essentially the same as the proof of the corresponding part of Theorem 3.)

Corollary 1. Since incrementally nonblocking connection networks cannot be built with fewer than \(O(N \log N)\) contacts, incrementally \(\epsilon\)-blocking connection networks cannot be built with fewer than \(O(N \log N) + O(N \log \log 1/\epsilon)\) contacts.

Corollary 2. From the nonconstructive upper bound \(O(N \log N)\) contacts\(^7\) and the construction with \(O(N (\log N)^2)\) contacts\(^8\) for incrementally nonblocking connection networks, we have nonconstructive and constructive upper bounds of \(O(N \log N) + O(N \log \log 1/\epsilon)\) and \(O(N \log N) + O(N (\log \log 1/\epsilon)^2)\) contacts, respectively, for incrementally \(\epsilon\)-blocking connection networks.
V. DISTRIBUTION NETWORKS

Distribution networks were first introduced by Masson and Jordan\(^5\) with the name "generalized connection networks." They consider three-stage networks (analogous to the three-stage connection networks introduced by Slepian\(^3\) and by Clos\(^4\)) and give necessary and sufficient conditions for such networks to be rearrangeably or incrementally non-blocking. Although they do not discuss the asymptotic implications of their results, it is easy to show that their conditions lead to rearrangeably and incrementally nonblocking networks with \(O(N^{5/3})\) contacts. While this is less than the obvious \(O(N^2)\), it is more than the \(O(N^{3/2})\) obtained for three-stage connection networks. Moreover, since the switches in an optimal three-stage distribution network with equally many inputs and outputs do not themselves have equally many inputs and outputs, these networks do not benefit from recursive substitution as dramatically as three-stage connection networks. Specifically, recursive substitution yields \(O(N^{3/2}\log N)\) and \(O(N^{3/2}(\log N)^{\log 6/\log 3})\) contacts for rearrangeably and incrementally nonblocking distribution networks, respectively, in contrast with the \(O(N(\log N)^{\log 3/\log 2})\) and \(O(N(\log N)^{\log 6/\log 2})\) contacts obtained for rearrangeably and incrementally nonblocking connection networks.

We shall obtain upper bounds on the complexity of rearrangeably and incrementally nonblocking distribution networks in terms of the complexity of rearrangeably and incrementally nonblocking connection networks. Specifically, for rearrangeably nonblocking networks, we show that the complexity of distribution networks has the same order of growth as that of connection networks, \(O(N\log N)\) contacts. These networks are highly nonuniform, in that there are routes of length \(O(N\log N)\). (It is also possible to obtain networks with \(O(N\log N)\) contacts and routes of length at most \(O((\log N)^2)\) by using the total concentration networks of Pinsker\(^2\) in the construction of Theorem 7.) For incrementally nonblocking networks, we shall show that, for any \(v \geq 1\), an upper bound of \(O(N(\log N)^{v})\) contacts for connection networks implies an upper bound of \(O(N(\log N)^{v+1})\) contacts for distribution networks. Furthermore, an upper bound of \(O((\log N)^{w})\) arithmetic operations for routing in the concentration networks implies an upper bound of \(O((\log N)^{w+1})\) arithmetic operations for routing in the distribution networks. This, together with the best known results for connection networks, gives nonconstructive and constructive upper bounds of \(O(N(\log N)^2)\) and \(O(N(\log N)^3)\) contacts, respectively, for distribution networks, and an upper bound of \(O((\log N)^3)\) arithmetic operations for routing in the constructive distribution networks. We then turn our attention to incrementally \(\epsilon\)-blocking distribution networks. Our results here are analogous to those we obtained for connection networks: Incrementally \(\epsilon\)-blocking distribution networks can be built with \(O(N\log N) + O(N(\log \log 1/\epsilon)^{v})\) contacts if and only if incrementally nonblocking distribution networks can be built with \(O(N(\log N)^{v})\) contacts. This, together with our previous results, gives nonconstructive and constructive upper bounds of \(O(N\log N) + O(N(\log \log 1/\epsilon)^2)\) and \(O(N\log N) + O(N(\log \log 1/\epsilon)^3)\) contacts, respectively, and gives, in effect, a lower bound of \(O(N\log N) + O(N\log \log 1/\epsilon)\) contacts.
5.1 AN UPPER BOUND OF O(N log N) FOR REARRANGEABLY NONBLOCKING NETWORKS

Theorem 7 establishes a constructive upper bound of O(N log N) contacts for rearrangeably nonblocking distribution networks. The construction uses rearrangeably nonblocking connection networks in a simple feedback scheme: Of those outputs to be joined to a given input, one will be joined through a direct path, while the rest will be joined through feedback paths. The use of the feedback paths permits one input to reach many outputs.

Theorem 7.

Rearrangeably nonblocking distribution networks with N inputs, N outputs, and capacity N can be constructed with O(N log N) contacts in such a way that a state realizing a given distribution assignment can be computed with O(N log N) arithmetic operations (on natural numbers not exceeding N).

Proof: From the work of Benes, we know that rearrangeably nonblocking connection networks with N inputs, N outputs, and capacity N can be constructed with O(N log N) contacts, and from the work of Tsao-Wu and Opferman we know that a state realizing a given connection assignment can be computed with O(N log N) arithmetic operations. We shall show that a rearrangeably nonblocking distribution network can be built from two rearrangeably nonblocking connection networks, together with O(N) additional contacts, and that a state realizing a given distribution assignment can be computed by computing the states realizing two connection assignments and performing O(N) additional arithmetic operations. From this Theorem 7 will follow.

Let \( G_1 \) and \( G_2 \) be two copies of a rearrangeably nonblocking connection network with N inputs, N outputs, and capacity N. We may assume without loss of generality that the inputs and outputs of these networks are labeled with the numbers \( \{1, \ldots, N\} \), so that the identity map on this set may be used as a bijection \( \phi \) from the outputs of \( G_1 \) to the inputs of \( G_2 \) and as a bijection \( \psi \) from the outputs of \( G_2 \) to the inputs of \( G_2 \). We shall show that the iteration \( H = (G_1 \cdot G_2)_{\phi, \psi} \) (see section 2.2.6) is a rearrangeably nonblocking distribution network with N inputs, N outputs, and capacity N.

Let \( h: \{1, \ldots, N\} \rightarrow \{0, \ldots, N\} \) be an array specifying the distribution assignment to be realized by \( H \) (for all \( 1 \leq x \leq N \) and \( 1 \leq y \leq N \), \( h(y) = 0 \) if the output \( y \) is to be idle and \( h(y) = x \) if the output \( y \) is to be joined to the input \( x \) ). From \( h \) we compute the similarly defined arrays \( g_1 \) and \( g_2 \) which specify the connection assignments to be realized by \( G_1 \) and \( G_2 \), as well as an array \( c: \{1, \ldots, N\} \rightarrow \{0, 1\} \) specifying the assignments to be realized by the N copies of \( C_{1,1} \) (for all \( 1 \leq y \leq N \), \( c(y) = 0 \) if the unique contact in the copy of \( C_{1,1} \) on which the output \( y \) appears is to be open and \( c(y) = 1 \) if it is to be closed). In this computation we use an additional array \( f: \{1, \ldots, N\} \rightarrow \{0, \ldots, N\} \) as working storage. The algorithm follows.
clear $g_1$, $g_2$, $c$ and $f$ to 0's;
for $y$ from 1 through $N$ do
  let $x = h(y)$ do
    unless $x = 0$ do
      begin
        if $f(x) = 0$ then $g_1(y) = x$
        else $g_2(y) = f(x)$;
        $c(y) = 1$;
        $f(x) = y$
      end
Since whenever the statement $g_1(y) = x$ is executed $f(x)$ is subsequently set to a positive
value, this statement is executed at most once for any value of $x$, so that the map rep-
represented by $g_1$ will be injective. Similarly, since whenever the statement $g_2(y) = f(x)$
is executed $f(x)$ is subsequently set to a value larger than any previously assigned to $f$,
this statement is executed at most once for any value of $f(x)$, so that the map represented
by $g_2$ will also be injective. The arrays $g_1$ and $g_2$ will thus specify connection assign-
ments that may be realized by $G_1$ and $G_2$. Assuming this to be done, we see by induc-
tion that whenever $f(x) = y$ the output $y$ of $G_1$ will be joined to the input $x$ of $G_1$. From
this fact and the fact that $c(y) = 1$ unless $h(y) = 0$, we see that whenever $h(y) = x$ the out-
put $y$ of $H$ will be joined to the input $x$ of $H$. This completes the proof.

5.2 NONCONSTRUCTIVE AND CONSTRUCTIVE UPPER BOUNDS
OF $O(N(\log N)^2)$ AND $O(N(\log N)^3)$ FOR INCREMENTALLY
NONBLOCKING NETWORKS

Theorem 8 establishes an upper bound on the complexity of incrementally nonblocking
distribution networks in terms of the complexity of incrementally nonblocking concentra-
tion networks. The proof is based on the simple observation that if there is an equal
number of inputs and outputs, half of the outputs can be joined at any time to at most
half of the inputs; the selection of the appropriate half of the inputs can be accomplished
by a concentration network, while the joining of this half of the inputs to the appropriate
outputs can be accomplished by a distribution network of half the size.

Theorem 8.

For any $v \geq 1$, if incrementally nonblocking concentration networks with $N$ inputs,
$2N$ outputs, and capacity $N$ can be built with $O(N(\log N)^v)$ contacts, then incrementally
nonblocking distribution networks with $N$ inputs, $N$ outputs, and capacity $N$ can be built
with $O(N(\log N)^{v+1})$ contacts. Furthermore, for any $w \geq 1$, if the concentration networks
are such that an admissible route satisfying a given request can be computed with
$O((\log N)^w)$ arithmetic operations (on natural numbers not exceeding $N$), then the
distribution networks are such that an admissible route satisfying a given request can be computed with \( O((\log N)^{w+1}) \) arithmetic operations.

**Proof:** Let \( a(N) \) denote the minimum number of contacts required to build an incrementally nonblocking distribution network with \( N \) inputs, \( N \) outputs, and capacity \( N \). We shall show that an incrementally nonblocking distribution network with \( 2N \) inputs, \( 2N \) outputs, and capacity \( 2N \) can be built from two incrementally nonblocking distribution networks with \( N \) inputs, \( N \) outputs, and capacity \( N \), two incrementally nonblocking concentration networks with \( N \) inputs, \( 2N \) outputs, and capacity \( N \), and \( O(N) \) additional contacts. From the hypothesis of the theorem, this gives \( a(2N) \leq 2a(N) + O(N(\log N)^V) \). Since \( a(2) = O(1) \), iteration of this inequality gives \( a(N) \leq O(N(\log N)^{V+1}) \). Let \( b(N) \) denote the minimum number of arithmetic operations required to compute an admissible route satisfying a given request in a network of the type just described with capacity \( N \). We shall show that an admissible route satisfying a given request in such a distribution network with capacity \( 2N \) can be computed by computing an admissible route in such a distribution network with capacity \( N \), by computing an admissible route satisfying a given request in one of its constituent concentration networks with capacity \( N \), and by performing \( O(1) \) additional arithmetic operations. From the hypothesis of the theorem, this gives \( b(2N) \leq b(N) + O((\log N)^W) \). Since \( b(2) = O(1) \), iteration of this inequality gives \( b(N) \leq O((\log N)^{W+1}) \). From these results Theorem 8 will follow.

Let \( F \) be an incrementally nonblocking concentration network with \( N \) inputs, \( 2N \) outputs, and capacity \( N \) and let \( G \) be an incrementally nonblocking distribution network with \( N \) inputs, \( N \) outputs, and capacity \( N \). We may assume without loss of generality that the inputs and outputs of these networks are labeled with the natural numbers \( \{1, \ldots, N\} \) or \( \{1, \ldots, 2N\} \) as appropriate, so that the identity map on the former set may be used as a bijection \( \phi \) from the outputs of \( F' \) to the inputs of \( G \). Then \( H = D_{1,2} \times (F', G)_\phi \) (see sections 2.2.1-2.2.4) is a switching network with \( 2N \) inputs and \( 2N \) outputs (see Fig. 5). We may assume without loss of generality that the outputs of \( D_{1,2} \) are labeled with the natural numbers \( 0, N \) and that the inputs and outputs of \( H \) are labeled with the natural numbers \( \{1, \ldots, 2N\} \) in such a way that for all \( 1 \leq x \leq 2N \), the input \( x \) of \( H \) appears on a copy of \( D_{1,2} \) whose outputs are identified with the inputs \( x \) of the copies \( F'_0 \) and \( F'_N \) of \( F' \) and for all \( 1 \leq y \leq 2N \), if the output \( y \) of \( H \) is the

![Fig. 5.](image-url)
output $s$ of a copy $G_t$ of $G$, then $s + t = y$. By giving an algorithm for computing an admissible route satisfying a given request, we shall show that $H$ is an incrementally nonblocking distribution network with capacity $2N$.

Our representation of the state of $H$ includes representations of the states of $G_0$ and $G_N$, representations of the states of $F_0$ and $F_N$, two arrays $d_0$ and $d_N$: $\{1, \ldots, 2N\} \rightarrow \{0, 1\}$ specifying the states of the two $2N$ copies of $D_{1,2}$ (for all $1 \leq x \leq 2N$ and $t \in \{0, N\}$, $d_t(x) = 0$ if the contact from the unique input of the copy of $D_{1,2}$ on which the input $x$ of $H$ appears to the output $t$ of this copy is open, $d_t(x) = 1$ if it is closed), and two arrays $k_0$ and $k_N$: $\{1, \ldots, 2N\} \rightarrow \{0, \ldots, N\}$ specifying which inputs of $H$ are joined to which inputs of $G_0$ and $G_N$ (for all $1 \leq x \leq 2N$, $t \in \{0, N\}$ and $1 \leq u \leq N$, $k_t(x) = u$ if the input $x$ of $H$ is joined to the input $u$ of the copy $G_t$ of $G$ and $k_t(x) = 0$ if the input $x$ of $H$ is not joined to any input of the copy $G_t$ of $G$).

Let $f_t(x)$ denote a procedure invocation that updates the representation of the state of $F_t$ to reflect the addition of an admissible route satisfying the request $x$ and returns as its value the output of $F_t$ at which this route terminates. Let $g_t(u, s)$ denote a procedure invocation that updates the representation of the state of $G_t$ to reflect the addition of an admissible route satisfying the request $(u, s)$ and returns as its value 0 if the input $u$ of $G_t$ is not then joined to any output of $G_t$ or 1 if it is. Then the following algorithm will update the representation of the state of $H$ to reflect the addition of an admissible route satisfying the request $(x, y)$.

\[
\begin{align*}
&\text{let } s + t = y, \text{ where } 1 \leq s \leq N \text{ and } t \in \{0, N\}; \\
&\quad d_t(x) = 1; \\
&\quad \text{if } k_t(x) = 0 \text{ do } k_t(x) = f_t(x); \\
&\quad g_t(k_t(x), s)
\end{align*}
\]

The proof of the correctness of this algorithm is trivial.

Let $f_t(x)$ denote a procedure invocation that updates the representation of the state of $F_t$ to reflect the deletion of the route satisfying the request $x$. Let $g_t(u, s)$ denote a procedure invocation that will update the representation of the state of $G_t$ to reflect the deletion of the route satisfying the request $(u, s)$ and returns as its value 0 if the input $u$ of $G_t$ is not then joined to any output of $G_t$ or 1 if it is. Then the following algorithm will update the representation of the state of $H$ to reflect the deletion of the route satisfying the request $(x, y)$ and will return as its value 0 if the input $x$ of $H$ is not then joined to any output of $H$ or 1 if it is.

\[
\begin{align*}
&\text{let } s + t = y, \text{ where } 1 \leq s \leq N \text{ and } t \in \{0, N\}; \\
&\quad \text{if } g_t(k_t(x), s) = 0 \text{ do } \\
&\quad \quad \text{begin} \\
&\quad \quad \quad f_t(x); \\
&\quad \quad \quad d_t(x) = 0 \\
&\quad \quad \text{end}; \\
&\quad \text{return } d_0(x) \lor d_N(x)
\end{align*}
\]
Again, the proof of correctness is trivial.

In practice, of course, these algorithms would be implemented as procedures for dealing with a distribution network determined by a parameter, so that $g_t(u,s)$ and $g_t(u,s)$ would be recursive invocations of this procedure. This completes the proof.

**Corollary.** From the nonconstructive upper bound of $O(N \log N)$ contacts and the construction with $O(N \log N)^2$ contacts for incrementally nonblocking connection networks, we have nonconstructive and constructive upper bounds of $O(N \log N)^2$ and $O(N \log N)^3$ contacts, respectively, for incrementally nonblocking distribution networks. From the $O((\log N)^2)$ algorithm for routing in Cantor’s network given in Theorem 4, we obtain an $O((\log N)^3)$ algorithm for routing in the distribution network derived from it. Since we have also seen that incrementally nonblocking concentration networks with capacity $N$ cannot be built with less than $O(N \log N)$ contacts, this approach cannot be used to build distribution networks with less than $O(N \log N)^2$ contacts, and thus cannot be used to achieve the information-theoretic lower bound.

5.3 THE $\epsilon$-DEPENDENCE OF THE COMPLEXITY OF INCREMENTALLY $\epsilon$-BLOCKING NETWORKS

Our next theorem is the analog for distribution networks of Theorem 3 on partial concentration networks. We show that incrementally $\epsilon$-blocking distribution networks can be built with $O(N \log N) + O(N \log \log 1/\epsilon)^v$ contacts if and only if incrementally nonblocking distribution networks can be built with $O(N \log N)^v$ contacts. The probabilistic and structural assumptions that we use are the same as those for Theorem 3.

**Theorem 9.**

Suppose that for some $\epsilon < 1$ there exists $c$ such that for all sufficiently large $N$ uniform incrementally $\epsilon$-blocking connection networks can be built with $N$ inputs, $N$ outputs, at most $cN \log_2 N$ contacts, and capacity $N$. Then, for all $v \geq 1$, the following propositions are equivalent.

1. There exists $d$ such that for all sufficiently large $N$ uniform incrementally nonblocking distribution networks can be built with $N$ inputs, $N$ outputs, at most $dN(\log_2 N)^v$ contacts, and capacity $N$.

2. There exist $c$ and $d$ such that for all sufficiently small $\epsilon > 0$ and all sufficiently large $N$ uniform incrementally $\epsilon$-blocking distribution networks can be built with $N$ inputs, $N$ outputs, at most $cN \log_2 N + dN(\log_2 N)^v$ contacts, and capacity $N$.

**Proof (1. implies 2.):** We wish to build uniform incrementally $\epsilon$-blocking distribution networks with $N$ inputs, $N$ outputs, and capacity $N$. Choose $r$ such that $0 < r < 1$. By 1., there exists $d$ such that for all sufficiently large $n$, uniform incrementally nonblocking distribution networks can be built with $n$ inputs, $n$ outputs, at most $dn(\log_2 n)^v$ contacts, and capacity $n$. By ignoring $(1 - r)n$ of the inputs on such networks, we obtain uniform incrementally nonblocking distribution networks $F$ with $rn$ inputs, $n$ outputs, at most...
\[ dn(\log_2 n)^V \] contacts, and capacity \( n \). By ignoring \((1-r)n\) of the outputs on such networks, we obtain uniform incrementally nonblocking distribution networks with \( n \) inputs, \( rn \) outputs, at most \( dn(\log_2 n)^V \) contacts, and capacity \( rn \). By our preliminary supposition, there exist \( \delta < 1 \) and \( c \) such that for all sufficiently large \( N \) uniform incrementally \( \delta \)-blocking connection networks \( H \) can be built with \( N/rn \) inputs, \( N/rn \) outputs, at most \( (c/rn)N\log_2 N \) contacts, and capacity \( R = N/rn \). By taking the triple product \((F, H, G)_\phi\), where \( \phi \) is an arbitrary bijection from the inputs of \( G \) to the outputs of \( F \) (see section 2.2.5), we obtain uniform networks with \( N \) inputs, \( N \) outputs, and at most \( (c/r)N\log_2 N + (2d/r)N(\log_2 n)^V \) contacts. We now need only show that there exists \( h \) such that when \( n = h \log_2 1/\epsilon \), these networks are incrementally \( \epsilon \)-blocking with capacity \( N \). To do this, we examine the blocking probability of these networks and show that it decreases exponentially as \( n \) increases, at a rate uniform in \( R \) (and thus in \( N \)).

By assumption (ia) (see Section 3.3), the network is in a state with \( mN \) busy outputs (for some \( 0 \leq m \leq 1 \)), so that the number of busy inputs will be at most \( mn \). Let \((x, y)\) be the request to be satisfied and let us focus our attention on the copy \( F_{x'} \) of \( F \) on which the input \( x \) appears and the copy \( G_{y'} \) of \( G \) on which the output \( y \) appears. Let \( X' \) be the set of \( u \) for which the input \( x' \) of the copy \( H_u \) of \( H \) is idle and \( Y' \) be the set of \( u \) for which the output \( y' \) of \( H_u \) is idle. Let \( I, J, \) and \( K \) denote the cardinalities of \( X', Y', \) and \( X' \cap Y' \), respectively. We shall obtain a conditional bound on the blocking probability, given that \( I = i \), and then obtain an unconditional bound on the blocking probability by considering the probability distribution for \( I \).

By an argument similar to that used in the proof of Theorem 6, we find that the blocking probability decreases exponentially as \( i \) increases, at a rate uniform in \( R \). Thus, for all sufficiently small \( w > 0 \), the blocking probability is at most \( \delta^w i \). We now consider the probability distribution for \( I \). Let \( B = N - I \) denote the number of busy outputs on \( F_{x'} \). If \( B = b \), in the assignment realized by the state of the network, there must be at least \( b \) outputs joined to inputs that appear on \( F_{x'} \). By assumption (ib) each busy output is joined to an input that appears on \( F_{x'} \) with the independent probability \( 1/R \). Thus we may assume, without undue optimism, that the probability that \( B = b \) is \( \binom{mrRn}{b}(1/R)^b(1-1/R)^{mrRn-b} \), so that the blocking probability is at most

\[
\sum_{0 \leq b \leq n} \binom{mrRn}{b}(1/R)^b(1-1/R)^{mrRn-b} \delta^w(n-b).
\]

Since there are \( n+1 \) terms in this sum, its value is at most \( n+1 \) times that of its largest term. The ratio of successive terms in the sum is \( (mrRn-b)/\delta^w(b+1)(R-1) \). This is a decreasing function of \( b \), so that the largest term of the sum is that corresponding to the value of \( b \) which makes this ratio unity. Solving for this value, we find that it is asymptotically proportional to \( n \) as \( n \) tends to infinity, with a coefficient of proportionality that increases to \( mr/\delta^w \) as \( R \) tends to infinity. Choosing \( w \) sufficiently small that \( mr/\delta^w < 1 \), we conclude that the factor \( \delta^w(n-b) \) in the largest term decreases
exponentially as \( n \) increases, and thus that the sum decreases exponentially as \( n \) increases, at a rate uniform in \( R \). This completes the proof.

**Proof (2. implies 1.):** (Essentially the same as the proof of the corresponding part of Theorem 3.)

**Corollary 1.** Since incrementally nonblocking distribution networks cannot be built with less than \( O(N \log N) \) contacts, incrementally \( \epsilon \)-blocking distribution networks cannot be built with less than \( O(N \log N) + O(N \log \log 1/\epsilon) \) contacts.

**Corollary 2.** From the upper bounds of Corollary 1 for incrementally nonblocking distribution networks we have nonconstructive and constructive upper bounds of \( O(N \log N) + O(N \log \log 1/\epsilon^2) \) and \( O(N \log N) + O(N \log \log 1/\epsilon^3) \) contacts, respectively, for incrementally \( \epsilon \)-blocking distribution networks.
VI. CONCLUSION

6.1 SUMMARY OF RESULTS

We have considered three network applications: partial concentration, connection, and distribution. For all three problems, we have seen that rearrangeably nonblocking networks can be built with a number of contacts that has the same order of growth as the information-theoretic lower bound. For incrementally nonblocking networks, the situation is more varied: partial concentration networks cannot be built with this order of growth, connection networks can be, and for distribution networks the question remains open (although here we have obtained an upper bound proportional to the product of the capacity and a polynomial in the logarithm of the capacity). In each case, the best upper bounds are obtained by nonconstructive means, and there is a large coefficient in the leading term of the number of contacts. For these reasons, incrementally \( \epsilon \)-blocking networks are of interest. Here again, we have obtained upper bounds with the same order of growth as the information-theoretic lower bound in all three cases. Furthermore, we have explicit constructions with a small coefficient in the leading term of the number of contacts. We have also studied the \( \epsilon \)-dependence of the number of contacts of incrementally \( \epsilon \)-blocking networks. In each case, we have obtained a reciprocal relationship between this \( \epsilon \)-dependence and the number of contacts in incrementally nonblocking networks. Finally, we have considered algorithms for controlling switching networks obtained by constructive means. For rearrangeably nonblocking networks, we have found algorithms for realizing an assignment with a number of arithmetic operations proportional to the number of contacts in the network. For incrementally nonblocking networks, we have found representations of the network state, using a number of bits proportional to the number of contacts in the network, with the property that a request can be satisfied with a number of arithmetic operations bounded by a polynomial in the logarithm of the capacity.

6.2 UNSOLVED PROBLEMS

While we have made significant progress, a number of open questions remain. The order of growth of the minimum number of contacts in incrementally nonblocking distribution networks is still not known. For partial concentration and distribution networks the order of growth is known, but the upper and lower bounds on the coefficient in the leading term of the number of contacts differ by more than an order of magnitude. The techniques used to obtain the upper bounds (even if they are made constructive) cannot be used to obtain the same coefficient as in the lower bounds, and new techniques will have to be developed to determine the minimum coefficient. This problem is especially interesting in the case of connection networks, since in this case we have a construction for incrementally \( \epsilon \)-blocking networks with a small coefficient (which is independent of \( \epsilon \).
for $\epsilon > 0$ in the leading term. This raises the question of whether the coefficient (as a function of $\epsilon$) is continuous at $\epsilon = 0$ or whether its value for $\epsilon > 0$ is bounded below its value for $\epsilon = 0$.

Acknowledgment

I would like to thank Professor Peter Elias, who supervised this research and offered sensible and constructive advice at all stages of the work. I am also grateful to Professors Michael Fischer and Vaughan Pratt, who served as readers and offered many valuable suggestions.

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We study three network applications: partial concentration, connection, and distribution, and for each application, three modes of operation: rearrangeably nonblocking, incrementally nonblocking, and incrementally ε-blocking. For all three problems we show that rearrangeably nonblocking networks can be built with a number of contacts that has the same order of growth as the information-theoretic lower bound. For incrementally nonblocking networks we show that although connection networks can be built with this order of growth, partial concentration networks cannot, and for distribution networks the question remains open. In each case, the best upper bounds are obtained by nonconstructive means, and there is a large coefficient in the leading term of the number of contacts. For these reasons, incrementally ε-blocking networks are of interest.

For rearrangeably nonblocking networks we find algorithms for realizing an assignment with a number of arithmetic operations proportional to the number of contacts in the network. For incrementally nonblocking networks, we find representations of the network state, using a number of bits proportional to the number of contacts in the network, with the property that a request can be satisfied with a number of arithmetic operations bounded by a polynomial in the logarithm of the capacity.
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