



THE MUTUAL INTERACTIONS OF PLASMA ELECTRONS

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## Abstract

The scattering of electrons by the fields of other electrons in a plasma has been investigated, and a method devised for handling it. It has been found that this process has an important effect on the velocity distribution under certain conditions, namely: when  $\frac{E}{N Q}$  is small, and  $\frac{n}{N Q}$  large. Such conditions actually occur experimentally and for them this theory predicts that the distribution will be nearly maxwellian.

The scattering at small angles is more important than that at large angles. Or the velocity distribution is maintained principally by many small changes in energy rather than by less frequent large changes. The exact size,  $\alpha$ , chosen for the potential hill used in computing the scattering cross section is of relatively small importance. This approximation, then, is not critical.

Fairly satisfactory distribution functions have been found for discharges in which inelastic collisions are unimportant and also for some in which ionization cannot be neglected.

## Table of Contents

Section	Page
I Introduction	1
II General Equations	4
III Constant Cross Section	14
IV Calculation of Cross Sec.	20
V Integration over Angles	23
VI Special Cases	32
VII Conclusions	50

Figure	After Page
1	8
2	14
3	26
4	35
5,6,7	35
8	39
<b>9,10</b>	41
11,12	47
13, 14	48
15	49
Tables	36
I	42
II	48
<b>III</b>	38
<b>IV</b>	

## I - Introduction

Many measurements made with probes in gas discharges have shown the existence of electronic velocity distributions of Maxwell - Boltzmann form, with very high average energies. Such results are, in particular, found when the electron density is high, and the net space charge (electrons plus positive ions) is low, as in the plasma of a low pressure arc. This form of distribution requires a very high rate of exchange of energy among the electrons.

In theoretical studies, this electronic interaction has been considered only by Langmuir (14) and Gabor (9). Of the two, Gabor's account is the more complete. It is not, however, satisfactory. He takes into account only the electrostatic phenomena, and is thus unable to compare their relative importance with that of collision processes and the macroscopic field. In his calculations, Gabor considers the electrons to be scattered by the fields of partially shielded, stationary positive ions. Because of the great mass of the positive ions, this cannot lead to as rapid a change in the velocity distribution as will the scattering of electrons by electrons. Furthermore, his methods of averaging are somewhat obscure.

At low electron densities, the electronic interactions can be neglected, as is shown by the quite good agreement of theory with experimental data. (7,2/)

Since these interactions increase with the square of the density, while the number of collisions with atoms increases with the first power, at some density the two effects will be equally important.

The object of this thesis is first, to find a means of calculating the mutual interactions of the electrons; second, to discover the exact conditions under which they become important; third to find approximately the form of the velocity distribution in the cases in which neither process can be neglected.

The discussion will be limited to discharges between large, plane-parallel electrodes, so that only one direction in space can be singled out. It will be assumed that the positive ion and electron densities are equal, and that the drift current is small compared to the random electron current. It will be assumed further that the discharge is homogeneous in space, so that diffusion can be neglected.

The simplest mode of exchange of energy among the electrons is that between individual pairs, or by "collisions". Among more complicated processes is that suggested by Langmuir of "plasma oscillations". It is the mutual collisions which will be investigated in this paper.

The problem is simplified by considering only the following processes:

- a acceleration by the field,
- b elastic collisions with atoms,
- c inelastic collisions with atoms, (excitation and ionization)
- d mutual collision.

Collisions between electrons will be infrequent compared to collisions with atoms, but they are important. When two electrons collide, each is likely to suffer a considerable change in energy, while when one collides with an atom elastically, it loses on the average, only a fraction  $\frac{m}{M}$  of its energy. Collisions with ions will be as infrequent as those with other electrons, and will result in as little an energy change as a collision with an atom. Accordingly, electron-ion collisions will be neglected.

Let  $\eta$  be the electron density, and  $\eta f d\gamma$  the number of electrons per  $\text{cm}^3$  in an element  $d\gamma$  of velocity space.  $f$  will depend on only the absolute velocity  $u$ , and the angle  $\omega$  that this velocity makes with the field.

$$f = f(u, \cos\omega)$$

As  $f$  is nearly spherically symmetrical, it can be developed in terms of spherical harmonics, the first two terms giving:

$$f = f_0(u^2) + f_1(u^2) \cos\omega$$

and  $f_1$  will be much smaller than  $f_0$ .

Following Boltzmann and Lorentz, the usual procedure to determine the distribution function would be to find

the number of electrons entering and leaving each small element of velocity space per second on account of each type of process, and to set the sum equal to zero. In our problem, this involves too many successive integrations of the unknown function  $f$ , and is unmanageable.

Two equations are necessary to determine  $f_0$  and  $f_1$ . These will be an energy balance and a momentum balance equation. The first can be obtained by requiring the net number of electrons leaving each central sphere of velocity space to be zero. The second is found by placing a similar requirement on the number leaving a region on one side of a plane perpendicular to the direction of the field. The first will mean that there will be just as many electrons having an initial velocity  $u$ , less than any given value  $w$ , which acquire in unit time a velocity greater than  $w$ , as there are having initial velocity greater than  $w$  and final velocity less than  $w$ . The second equation expresses a similar balance of the changes in velocity component  $u_x$  parallel to the field.

## II - The General Equations

Consider the energy balance equation, involving processes a, b, c, and d.



a Electric field  $E$ ,

In time  $dt$ , an electron moving in a field  $E$  increases its velocity by an amount  $\frac{eE}{m} dt \cos\omega$ . The number per sec.

that leave the central sphere of radius  $w$  will be:

$$2\pi \frac{eE}{m} N_0 \int_0^\pi (f_0 + f_1 \cos\omega) \cos\omega \sin\omega \, d\omega$$

or

$$\frac{4\pi n e E}{3m} w^2 f (w^2)$$

b Elastic Collisions,

An electron of velocity  $u$ , colliding elastically with an atom at rest, loses speed in amount  $\delta u =$

$\frac{m}{M} u (1 - \cos\theta)$ , where  $\theta$  is the angle between the initial and final velocity directions.

Let:

$N$  = atomic density

$\omega$  = angle  $u$  makes with field  $E$

$\psi$  = corresponding azimuthal angle

$\sigma_a(u, \theta)$  = atomic cross section for elastic scattering at angle  $\theta$

Then the number per sec. of electrons originally having a vectorial velocity in the element  $u^2 \sin\omega \, d\omega \, d\psi$  which are scattered by angles between  $\theta$  and  $\theta + d\theta$  will be:

$$N \eta u^3 f \sin \omega \, d\omega \, d\psi \cdot 2\pi u \sigma_a(u, \theta) \sin \theta \, d\theta$$

The only electrons able to cross the sphere  $w$  due to elastic collisions will be those for which  $w < u < w + \delta u$ . The number per sec. entering such a sphere will be:

$$2\pi N \eta \int_0^\pi w^3 f \sin \omega \, d\omega \cdot 2\pi \int_0^\pi \delta u \sigma_a \sin \theta \, d\theta$$

If  $Q = 2\pi \int_0^\pi \sigma_a (1 - \cos \theta) \sin \theta \, d\theta$ ,

this is

$$\frac{2\pi N \eta m}{M} w^4 Q \int_0^\pi (f_0 + f_1 \cos \omega) \sin \omega \, d\omega$$

Or, finally,

$$\frac{4\pi N \eta m}{M} w^4 Q f_0$$

e Inelastic collisions,

Let:

$$\frac{1}{2} m u_e^2 = \text{excitation energy}$$

$$\frac{1}{2} m u_i^2 = \text{ionization energy}$$

$$Q_i(u) = \text{cross section for ionization}$$

$$Q_e(u) = \text{cross section for excitation}$$

It is assumed that when an electron excites an atom by collision, it loses just an energy  $\frac{1}{2} m u_e^2$ . Electrons with velocity greater than  $\sqrt{w^2 + u_e^2}$  will not enter the sphere  $w$  in this way. It is also assumed that there is just one ionization energy, and that the surplus kinetic

energy in an ionizing process is equally divided between the two resulting electrons. An electron will make, on the average,  $N u Q_e$  exciting, and  $N u Q_i$  ionizing collisions per sec. Then the number per sec. entering the sphere  $w$  is,

$$\begin{aligned}
 & 2\pi N \mathcal{N} \int_w^{\sqrt{w^2 + u_e^2}} Q_e(u) u^3 du \int_0^\pi (f_0 + f_1 \cos\omega) \sin\omega d\omega \\
 & + 2\pi N \mathcal{N} \int_w^{\sqrt{w^2 + u_i^2}} 2Q_i(u) u^3 du \int_0^\pi (f_0 + f_1 \cos\omega) \sin\omega d\omega \\
 & = 4\pi N \mathcal{N} \left\{ \int_w^{\sqrt{w^2 + u_e^2}} Q_e(u) f_0 u^3 du + 2 \int_w^{\sqrt{w^2 + u_i^2}} Q_i f_1^2 u^3 du \right\}
 \end{aligned}$$

Let  $J(w)$  be defined by setting the above expression equal to  $4\pi N \mathcal{N} J(w)$ .

#### d Mutual collisions,

The mutual collisions of electrons are much more complicated.

Let:

- $u$  = initial velocity of electron considered,
- $v$  = " " " scattering electron,
- $U$  = velocity of center of gravity,
- $\bar{u}$  = velocity of first electron relative to C.G.
- $u', \bar{u}', v'$  = corresponding quantities after collision,
- $\theta$  = scattering angle referred to C.G.
- $\phi$  = corresponding azimuthal angle,
- $\sigma(\bar{u}, \theta)$  = mutual scattering cross section for angle  $\theta$ ,
- $\chi$  = angle between directions of  $u$  and  $v$ ,

$\eta$  = corresponding azimuthal angle,  
 $\delta$  = angle between directions of  $U$  and  $\bar{u}$ ,  
 $\lambda$  = " " " "  $U$  and  $\bar{u}$ ;  
 $\mu$  = corresponding azimuthal angle,  
 $\nu$  = angle  $v$  makes with the field  $E_0$ .

Fig. 1 shows the geometry of a collision of an electron of velocity  $u$  with one of velocity  $v$ . Their relative velocity is  $2\bar{u}$ . Vectorially:

$$U = \frac{1}{2}(u + v), \quad \bar{u} = \frac{1}{2}(u - v)$$

$$u = U + \bar{u}$$

Algebraically:

$$\underline{1} \quad 4\bar{u}^2 = u^2 + v^2 - 2uv \cos \chi$$

$$\underline{2} \quad 4U^2 = u^2 + v^2 + 2uv \cos \chi$$

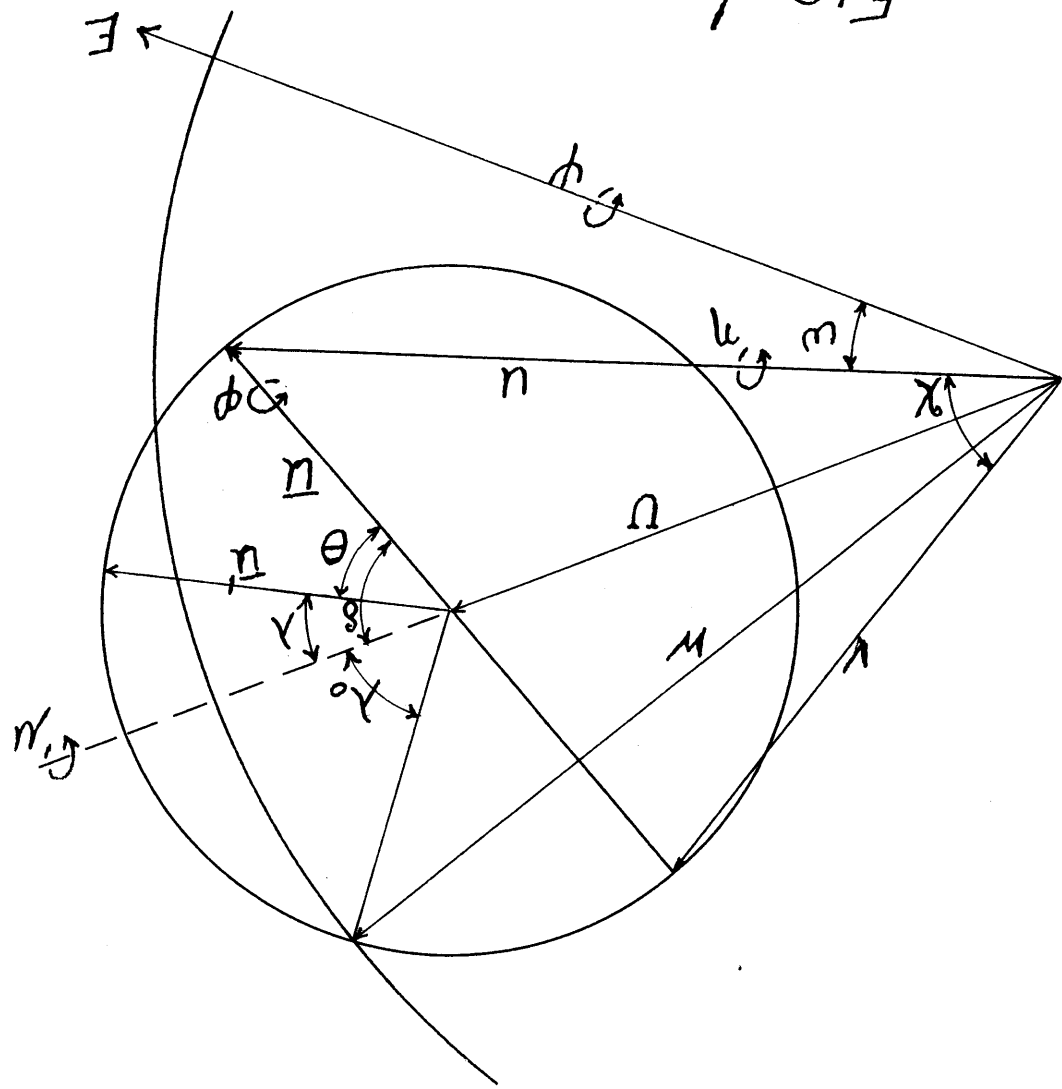
After collision the velocity of the first particle is:  $u' = U + \bar{u}'$ .  $\bar{u} = \bar{u}'$ , or  $u, u', v, v'$  all lie on the same sphere of radius  $\bar{u}$ .  $\theta$  and  $\phi$  are the polar angles of  $\bar{u}'$  about  $U$  as an axis, and  $\lambda$  and  $\mu$  are the polar angles of the same velocity about the axis  $U$ . The polar angles of  $v$  about the axis  $u$  are  $\chi$  and  $\eta$ .  $\sigma$  depends on the relative velocity.

If  $u$  lies in the element of velocity space  $d\gamma_u$ , and  $v$  in the element  $d\gamma_v$ , the number of such collisions per sec. is:

$$2\bar{u}\sigma(\bar{u}, \theta) \sin\theta d\theta d\phi \eta^2 f(u, \cos\omega) f(v, \cos\nu) d\gamma_u d\gamma_v$$

To find the rate at which electrons of velocity  $u < w$

Fig. 1



are given a velocity  $u > w$  by mutual collision, this expression must be integrated: first, over all values of  $\theta$  and  $\phi$  on that portion of the sphere of radius  $\bar{u}$  which lies outside the sphere of radius  $w$ ; second, over a domain of  $\gamma_v$  (velocity space of second electron) such that the sphere  $\bar{u}$  cuts the sphere  $w$ ; third, over a domain of  $\gamma_u$  with  $u < w$ . The rate at which the reverse process takes place is found by integrating over  $\theta$  and  $\phi$  on that part of the sphere  $\bar{u}$  inside the sphere  $w$ ; over a part of the  $\gamma_v$ ; and over a part of  $\gamma_u$  with  $u > w$ .

Because of symmetry, it is more convenient to use  $\lambda$  and  $\mu$  than  $\theta$  and  $\phi$ .

$$\text{3} \quad \cos \theta = \cos \delta \cos \lambda - \sin \delta \sin \lambda \cos \mu$$

For the purposes of integration,  $\sin \theta d\theta d\phi$  can be replaced by  $\sin \lambda d\lambda d\mu$ . Then  $\mu$  will run from  $-\pi$  to  $\pi$ , and  $\lambda$  from 0 to  $\lambda_0$ , where  $\lambda_0$  is the value of  $\lambda$  on the circle of intersection of the spheres  $\bar{u}$  and  $w$ . On this circle, vectorially:  $\bar{u} + U = w$ , or  $\bar{u}^2 + U^2 + 2\bar{u}U \cos \chi$

$$= w^2$$

$$\text{4} \quad \cos \lambda_0 = \frac{2w^2 - u^2 - v^2}{4U\bar{u}}$$

Also,

$$\bar{u}^2 + U^2 + 2\bar{u}U \cos \delta = u^2$$

whence,

$$\underline{5} \quad \cos \delta = \frac{u^2 - v^2}{4U \bar{u}}$$

In the integration over  $\gamma$ ,  $v$ ,  $\chi$ , and  $\eta$  can be used as variables, if  $\cos \nu$  in  $f(v, \cos \nu)$  is replaced by  $\cos \delta \cos \chi + \sin \delta \sin \chi \cos \eta$ .

Since  $-1 \leq \cos \lambda_0 \leq 1$ , and since, through  $U$  and  $\bar{u}$ ,  $\cos \lambda_0$  depends on  $\chi$ ,  $\chi$  must be restricted. Expanding expression 4 for  $\cos \lambda_0$ ,

$$u^4 + v^4 - 4u^2v^2 \cos^2 \chi \geq 4w^4 + u^4 + v^4 - 4w^2(u^2 + v^2)$$

$$\underline{6} \quad \cos^2 \chi \leq \frac{w^2(u^2 + v^2 - w^2)}{u^2 v^2}$$

Because  $u^2 + v^2 \geq w^2$ ,  $\cos^2 \chi \geq 0$ . When  $u$  and  $v$  are on the same side of sphere  $w$ , (that is, when  $u$  and  $v$  are both greater or both less than  $w$ ) the fraction is less than one, and it follows that  $\chi_0 \leq \chi < \pi - \chi_0$ . When  $u$  and  $v$  are on opposite sides of sphere  $w$ , there is no restriction on  $\chi$ , as the fraction is greater than one. In this case it is possible to let  $\chi_0 = 0$ . Then the limits are always  $\chi_0$  and  $\pi - \chi_0$ .

The number of electrons leaving the sphere because of mutual collisions is:

$$I_1 = \eta^2 \int u^2 du \int_0^\pi f(u, \cos \omega) \sin \omega d\omega \int_0^{2\pi} d\psi \int v^2 dv \int_{\chi_0}^{\pi - \chi_0} \sin \chi d\chi \\ \cdot \int_{-\pi}^{\pi} f(v, \cos \nu) d\eta \cdot 2\bar{u} \int_0^{\lambda_0} \sin \lambda d\lambda \int_{-\pi}^{\pi} \sigma(\bar{u}, \theta) d\mu$$

Let:

$$B_1 = \int_0^{\lambda_0} \sin \lambda d\lambda \int_{-\pi}^{\pi} \sigma[\bar{u}, \theta(\delta, \lambda, \mu)] d\mu$$

The order of integration can be changed:

$$I_1 = 2\eta^2 \int u^2 du \int v^2 dv \int_{\chi_0}^{\pi - \chi_0} \bar{u} B_1 \sin \chi d\chi \int_0^\pi f(u, \cos \omega) \sin \omega d\omega \\ \cdot \int_0^{2\pi} f(v, \cos \nu) d\eta \int_0^{2\pi} d\psi$$

Expanding  $f$  and substituting for  $\cos \nu$

$$\int_0^{2\pi} f(v, \cos \nu) d\eta = \int_0^{2\pi} \left[ f_0(v^2) + f_1(v^2) (\cos \chi \cos \omega + \sin \chi \sin \omega \cos \eta) \right] d\eta \\ = 2\pi \left( f_0(v^2) + f_1(v^2) \cos \chi \cos \omega \right)$$

Taking the product of this result with  $f(u, \cos \omega)$ ,

$$\int_0^\pi \left[ f_0(u^2) + f_1(u^2) \cos \omega \right] \cdot \left[ f_0(v^2) + f_1(v^2) \cos \chi \cos \omega \right] \sin \omega d\omega \\ = 2f_0(u^2)f_0(v^2) + \frac{2}{3}f_1(u^2)f_1(v^2)$$

$f_1$  is much smaller than  $f_0$ , so the product  $f_1 f_1$  is negligible.



Then: 8

$$I_1 = 16\pi^2 \eta^2 \int f_0(u^2) u^3 du \int f_0(v^2) v^3 dv \int_{\chi_0}^{\pi-\chi_0} \bar{u} B_1 \sin \chi d\chi$$

Turning to the electrons entering the sphere due to mutual collisions, let:

$$9 \quad B_2 = \int_0^\pi \sin \lambda d\lambda \int_{-\pi}^\pi (\bar{u}, \theta) d\mu$$

The number entering is: 10

$$I_2 = 16\pi^2 \eta^2 \int f_0(u^2) u^3 du \int f_0(v^2) v^3 dv \int_{\chi_0}^{\pi-\chi_0} \bar{u} B_2 \sin \chi d\chi$$

The four processes a, b, c, and d have now been calculated, and the energy equation can be written symbolically:

$$a + b + c + d = 0, \quad \text{or:}$$

$$11 \quad \frac{4\pi Q E_w}{3m} f_1 - \frac{4\pi m N Q w^2}{M} f_0 + I_1 - I_2 - 4\pi N J = 0$$

This must be true for all values of  $w$ .

If the mutual scattering terms are omitted, and the inelastic terms  $J$  considered constant, one can derive from this equation, by one differentiation, Eq. 8 of Morse, Allis and Lamar (with diffusion left out).

$$\frac{eE}{2w} \frac{d}{dw} (w^2 f_1) - \frac{m^2 N}{M} \frac{d}{2v} \frac{d}{dw} (w^2 Q f_0) = 0$$

$4\pi N \eta J$  is the number of electrons entering the sphere due to inelastic impacts.

The mutual collisions of electrons will not be as important in the momentum balance equation as in the energy relation. In collisions with atoms, electrons of energy  $H$  undergo a change of energy of the order  $\frac{m}{M}H$ . In corresponding encounters with other electrons, the change is of the order  $\frac{H}{2}$ . On the other hand, this is not true of the changes in momentum. Electrons are scattered by atoms nearly isotropically, and lose almost all of their directed momentum. The pair interactions of electrons will be negligible for the purposes of this equation, because they are so much less frequent than collisions with atoms. The inelastic impacts, which are also much less frequent than the elastic, will be neglected too.

An equation corresponding to eq.11 can be set up for momentum balance. This leads to:

$$\frac{12}{m} \frac{\partial E}{\partial w} f_0 + NQwf_1 = 0$$

which is equation 7 of Morse, Allis, and Lamar. Its derivation from the integral form is given in the appendix.

So far, those integrations have been performed which involve only the geometry of the mutual scattering process. It is now necessary to consider the cross-section function  $\sigma$ . The true form of this is difficult to handle; so it is worthwhile to postpone its treatment, and to consider

first the assumption that  $\sigma$  is constant. This would be true if the electrons behaved like hard spheres. The difficulties arising from the form of  $\sigma$  will be avoided, and it will be possible to show clearly the peculiarities of mutual collisions, as well as the method of handling them.

First, the energy and momentum equations can be combined to eliminate  $f_1$ .

$$\frac{13}{M} \frac{-4\pi N Q \eta_w}{M} f_0 - \frac{4\pi}{3NQ} \left(\frac{eE}{m}\right)^2 w \frac{d}{dw} f_0 + (I_1 - I_2) - 4\pi N J = 0$$

The four terms come from the elastic collisions, the field, mutual collisions, and inelastic collisions respectively.

### III - Constant Cross Section

It is convenient to represent mutual collisions on a diagram of the  $v^2, u^2$  plane. (Fig.2) Each point stands for the velocities of a pair of particles. Because of the conservation of energy, every point P, representing the result of a collision between a pair P, will lie on a  $45^\circ$  diagonal line through P. Points in the shaded region, for which  $u^2 + v^2 < w^2$ , cannot lead to collisions in which a particle crosses the sphere  $w$ . The lines  $u = w$ , and  $v = w$  divide the plane into four quadrants, each of which must be treated separately. In 2 ( $u < w, v > w$ ) and 4 ( $u > w$  and  $v < w$ ), the angle  $\chi_0 = 0$ , because  $u$  and  $v$  lie on op-

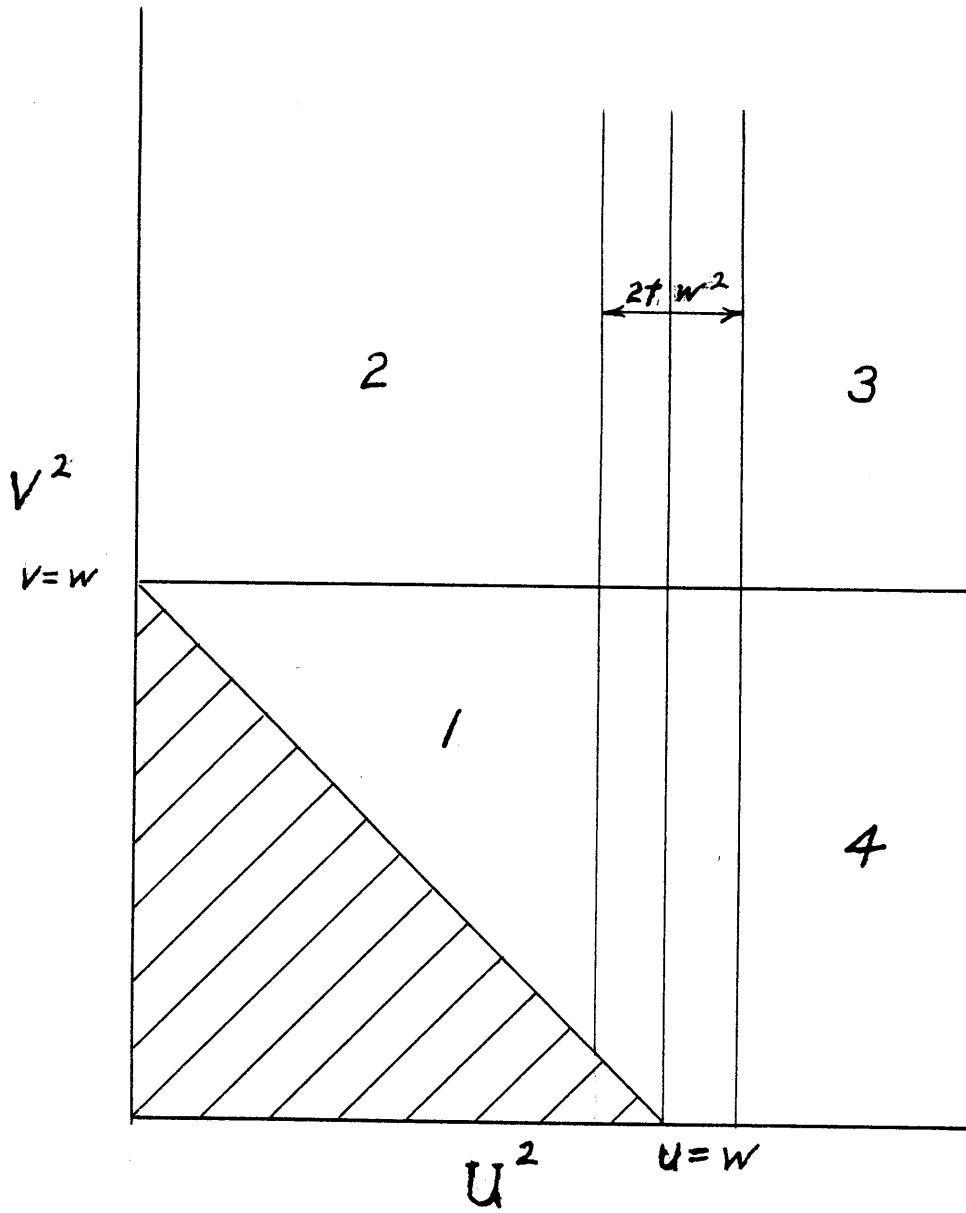


Fig 2

posite sides of sphere  $w$ . In 1 ( $u < w, v < w$ ) and 3 ( $u > w, v > w$ ),  $0 < \chi_0 < \frac{\pi}{2}$ .

Integration over quadrants 1 and 2 gives the electron flux across the line  $u = w$  in an increasing direction; over 3 and 4, the flux in a decreasing direction.

Let:

$$B'_1 = \int_{\chi_0}^{\pi - \chi_0} \bar{u} B_1 \sin \chi \, d\chi \quad \text{Quad. 1}$$

$$B'_2 = \int_0^{\pi} \bar{u} B_1 \sin \chi \, d\chi \quad 2$$

14

$$B'_3 = \int_{\chi_0}^{\pi - \chi_0} \bar{u} B_3 \sin \chi \, d\chi \quad 3$$

$$B'_4 = \int_0^{\pi} \bar{u} B_3 \sin \chi \, d\chi \quad 4$$

From eq's. 7 and 9,

$$B_1 = \int_0^{\lambda_0} \sin \lambda \, d\lambda \int_{-\pi}^{\pi} \sigma \, d\mu$$

$$= 2\pi\sigma(1 - \cos \lambda_0)$$

$$B_3 = \int_{\lambda_0}^{\pi} \sin \lambda \, d\lambda \int_{-\pi}^{\pi} \sigma \, d\mu = 2\pi\sigma(1 + \cos \lambda_0)$$

Repeating eq's. 1 and 2,

$$4\bar{u}^2 = u^2 + v^2 - 2uv \cos \chi$$

$$4U^2 = u^2 + v^2 + 2uv \cos \chi$$

Eliminating  $\cos \chi$ ,

$$4U^2 = 2u^2 + 2v^2 - 4\bar{u}^2$$

15

When  $u$  and  $v$  are held constant,

$$4\bar{u}d\bar{u} = uv \sin\chi d\chi$$

$$\sin\chi d\chi = \frac{4\bar{u} d\bar{u}}{u v}$$

Substituting for  $\cos\chi_0$  from eq.4,

$$B_1 = 2\pi\sigma \left[ 1 - \frac{2w^2 - u^2 - v^2}{4U\bar{u}} \right]$$

$$B_2 = 2\pi\sigma \left[ 1 + \frac{2w^2 - u^2 - v^2}{4U\bar{u}} \right]$$

Thus the integration over  $\chi$  can be replaced by one over  $\bar{u}$ , if the proper limits are found. Let the limits on  $\bar{u}$  be  $\bar{u}_1$  and  $\bar{u}_2$ ,  $\bar{u}_1 < \bar{u}_2$ , so that when,

$$\chi = \chi_0, \quad \bar{u} = \bar{u}_1$$

$$\chi = \pi - \chi_0, \quad \bar{u} = \bar{u}_2$$

For Quad's 2 and 4,  $\chi_0 = 0$ , and thus:

$$4\bar{u}_1^2 = (u - v)^2, \quad 4\bar{u}_2^2 = (u + v)^2$$

Since  $\bar{u}$  is essentially positive,

Quad.	2	4
$\bar{u}_1 =$	$(v - u)1/2$ ,	$(u - v)1/2$
$\bar{u}_2 =$	$(v + u)1/2$ ,	$(u + v)1/2$

For quad's 1 and 3, from inequality 6,

$$\cos^2 \chi_0 = \frac{(u^2 + v^2 - w^2)}{u^2 v^2}$$

Using this value for  $\cos \chi_0$ ,

$$4 \bar{u}_1^2 = u^2 + v^2 - 2w \sqrt{u^2 + v^2 - w^2}$$

$$4 \bar{u}_2^2 = u^2 + v^2 + 2w \sqrt{u^2 + v^2 - w^2}$$

Let  $s^2 = u^2 + v^2 - w^2$ .  $1/2 ms^2$  is the energy of the two particles in excess of that necessary to produce a collision in which one electron crosses the level  $w$ . In terms of  $s$ ,

$$4 \bar{u}_1^2 = (w - s)^2$$

$$4 \bar{u}_2^2 = (w + s)^2$$

This gives:

	Quad.	1	3
$\bar{u}_1$	=	$(w - s)1/2$	$(s - w)1/2$
$\bar{u}_2$	=	$(w + s)1/2$	$(s + w)1/2$

The integrals of equations 14(over  $\chi$ ) can now be thrown into the forms:

$$B'_1 = \frac{8\pi\sigma}{uv} \int_{(w-s)1/2}^{(w+s)1/2} \bar{u} \left[ 1 - \frac{w^2 - s^2}{2\bar{u} \sqrt{2u^2 + 2v^2 - 4\bar{u}^2}} \right] \bar{u} d\bar{u}$$

$$B'_1 = \frac{8\pi\sigma}{uv} \left[ \frac{\bar{u}^3}{3} + \frac{(w^2 - s^2) \sqrt{2u^2 + 2v^2 - 4\bar{u}^2}}{8} \right]_{(w-s)1/2}^{(w+s)1/2}$$

16a  $B'_1 = \frac{8\pi\sigma}{uv} s^3$

Similarly:

$$B'_3 = \frac{8\pi\sigma}{uv} \int_{(s-w)1/2}^{(s+w)1/2} \bar{u} \left[ 1 + \frac{w^2 - s^2}{2\bar{u} \sqrt{2u^2 + 2v^2 - 4\bar{u}^2}} \right] \bar{u} d\bar{u}$$

$$\underline{16b} \quad B'_3 = \frac{8\pi\sigma}{3uv} w^3$$

$$B'_2 = \frac{8\pi\sigma}{uv} \int_{(v-u)1/2}^{(v+u)1/2} \bar{u} \left[ 1 - \frac{w^3 - s^3}{2\bar{u} \sqrt{2u^3 + 2v^3 - 4\bar{u}^3}} \right] \bar{u} d\bar{u}$$

$$\underline{16c'} \quad B'_2 = \frac{4\pi\sigma}{3uv} u(3v^3 - 3w^3 + 2u^3)$$

$$B'_4 = \frac{8\pi\sigma}{vu} \int_{(u-v)1/2}^{(u+v)1/2} \bar{u} \left[ 1 + \frac{w^3 - s^3}{2\bar{u} \sqrt{2u^3 + 2v^3 - 4\bar{u}^3}} \right] \bar{u} d\bar{u}$$

$$\underline{16d'} \quad B'_4 = \frac{4\pi\sigma}{3uv} v(3w^3 - v^3)$$

$B'_2$  and  $B'_4$  can be simplified by taking advantage of the symmetry in  $u$  and  $v$ . They appear eq.13 as terms of  $I_1 - I_2$ , say  $16\pi^3 \eta^3 I'$ , where

$$I' = \int_0^w f_0(u^3) u^3 du \int_w^\infty f_0(v^3) v^3 dv B'_2 \\ - \int_w^\infty f_0(u^3) u^3 du \int_0^w f_0(v^3) v^3 dv B'_4$$

Substituting  $B'_2$  and  $B'_4$  from eq's.16',

$$I' = \frac{4\pi\sigma}{3uv} \int_0^w f_0(u^3) u du \int_w^\infty f_0(v^3) v dv u(3v^3 - 3w^3 + 2u^3) \\ - \frac{4\pi\sigma}{3} \int_w^\infty f_0(u^3) u du \int_0^w f_0(v^3) v dv v(3w^3 - v^3)$$



Because of the symmetry of  $I'$  in  $u$  and  $v$ ,  $2u^2$  can be subtracted from the bracket in the first integrand, if  $2v^2$  is subtracted from that in the second.

$$I' = 4\pi\sigma \int_0^W f_0(u^2)u^2 du \int_W^\infty f_0(v^2)v^2 dv \frac{(v^2 - W^2)}{v} \\ - 4\pi\sigma \int_W^\infty f_0(u^2)u^2 du \int_0^W f_0(v^2)v^2 dv \frac{(W^2 - v^2)}{u}$$

But this is equivalent to taking:

$$\underline{16c} \quad B'_s = \frac{4\pi\sigma}{v} (v^2 - W^2)$$

$$\underline{16d} \quad B'_u = \frac{4\pi\sigma}{u} (W^2 - v^2)$$

The functions  $B'_j$  derived here are much simpler than the ones to be calculated later for the shielded Coulomb field. They do show, however, the same symmetry properties in  $u$  and  $v$ .

If all processes except the mutual interactions are neglected, the solution of the balance equations will be a Maxwellian distribution.

$$f_0 = A e^{-u^2/W^2}$$

This fact can be used to check the calculations. The  $u^2, v^2$  plane can be divided into strips between the diagonal lines  $u^2 + v^2 = c^2$  and  $u^2 + v^2 = c^2 + \Delta c^2$ . The first integration may be carried out along the length of these strips, instead

of along lines parallel to the  $u^2$  or  $v^2$  axis. The maxwellian distribution function has the unique property of being constant along such a diagonal line. In other words, for this function and this path of integration, the  $f$ 's disappear from the inner integral. It turns out that the  $B_j$  functions can then be integrated. The result is zero for every value of  $c^2$ , showing that the mutual collisions do not change this distribution, This was a valued check on the correctness complicated expressions derived ~~from~~ from the assumption of a shielded potential hole.

#### IV - Calculation of Cross-Section

The force between two electrons in the plasma is  $\frac{e^2}{r^2}$  when they are very close, where  $r$  is their distance of separation. When  $r$  is larger, this force is reduced. Because the space is macroscopically neutral, at very large distances  $r$  the force between electrons will be zero. There is a space charge of positive ions, which, as far as the faster moving electrons are concerned, is nearly uniformly distributed. Near each electron there will be, on average, a deficiency of other electrons, leaving an average net positive density which "shields" the electrons from each other.

A suitable approximation to the interaction potential is:

$$V = \frac{e^2}{r} e^{-r/d}$$

This defines a potential "hill" whose radius is measured

by  $\alpha$ . The value to be assigned to  $\alpha$  is uncertain. A lower limit is the mean distance of separation of the particles,  $\eta^{-\frac{1}{3}}$ . This may be of the order  $10^{-4}$  cm. An obvious, but certainly excessive upper limit is the size of the apparatus, say 1cm. It turns out that this enormous range for  $\alpha$  is reflected in the results by an altogether smaller effect.  $\alpha$  cannot, however, be infinite, for this would give the Coulomb potential, and with it, an infinite collision cross section. No value will be specified for this parameter at present.

The cross section  $\sigma(\bar{u}, \theta)$  for this potential hill is found with the aid of quantum mechanics. The Born approximation can be used. (29)

Let the wave function  $\Psi$  of the two electrons be a product of two functions, one representing the motion of the center of gravity, and the other,  $\psi$ , representing relative motion. If  $\psi$  be approximated by the sum of a plane wave,  $\psi_0$ , and a spherically scattered wave,  $\psi_1$ , then

$$\sigma = r^2 \frac{|\psi_1|^2}{|\psi_0|^2}$$

Let:  $\psi_0 = e^{ikx_1 - ikx_2}$ , where  $x_1$  and  $x_2$  are the distances of the two electrons from the center of gravity, and

$$k = \frac{2\pi m \bar{u}}{h}$$

Then the first approximation to the solution of Schrödinger's

equation, with  $V = \frac{e^2}{r} e^{-r/\alpha}$ , gives:

$$\psi' = \frac{4\pi^2 m e^2}{h^2} \cdot \frac{1}{4k^2 \sin^2 \frac{\theta}{2} + \frac{1}{\alpha^2}}$$

Or since  $|\psi_0|^2 = 1$ ,

$$\sigma = \left( \frac{4\pi^2 m e^2}{h^2} \right)^2 \frac{1}{\left( 4k^2 \sin^2 \frac{\theta}{2} + \frac{1}{\alpha^2} \right)^2}$$

This can be written:

$$\underline{17} \quad \sigma(\bar{u}, \theta) = \frac{e^4}{16\pi^2 \bar{u}^4} \cdot \frac{1}{\left( \sin^2 \frac{\theta}{2} + \beta^2 \right)^2}$$

where,

$$\underline{18} \quad \beta^2 = \frac{h^2}{(4\pi\alpha)^2 m^2 \bar{u}^2} = \left( \frac{\bar{\lambda}}{2\pi\alpha} \right)^2$$

$1/\beta$  measures the size of the hill  $\alpha$  in terms of  $\bar{\lambda}$ , the wavelength corresponding to the relative velocity  $2\bar{u}$ .

$\beta^2$  is a very small number. Measured in volts,

$$\frac{1}{2} m \bar{u}^2 = \frac{e V_r}{900}, \quad \beta^2 = \frac{150 h^2}{16 \pi^2 \alpha^2 m e V_r}$$

$$\beta^2 = \frac{0.95 \cdot 10^{-16}}{V_r \alpha^2}$$

If the electronic density is  $10^{12}$  per cc., and  $\alpha$  is set equal to  $\eta^{-\frac{1}{3}}$

$$\beta^2 \approx \frac{10^{-8}}{V_r}$$

and if  $\alpha^2$  is larger,  $\beta^2$  is

by so much the smaller.

It is this term  $\beta^2$  in the denominator of  $\sigma$ , that prevents the cross section from becoming infinite at small angles, as it would for a Coulomb field. It will be the controlling part for distant encounters. Since it is so small, it will scarcely affect  $\sigma$  for large angles. These correspond to close encounters, and scattering by a nearly pure Coulomb field.

#### V - Integration over the Angles

In this section, the functions B and B' are calculated. Equation 17 can also be written:

$$\sigma = \frac{e^4}{4m^2 \bar{u}^4} \frac{1}{(1+2\beta^2 - \cos\theta)^2}$$

By equation 2 for  $\cos\theta$ , this becomes

$$\sigma = \frac{e^4}{4m^2 \bar{u}^4} \frac{1}{(1+2\beta^2 - \cos\delta\cos\lambda + \sin\delta\sin\lambda\cos\mu)^2}$$

When this is integrated over  $\mu$  and  $\lambda$ , we obtain the functions  $B_1$  and  $B_2$  which are given on page 24. The details of this and the following calculations are given in the appendix.

~~xxxxxxxxxxxxxxxxxxxxxxxx~~

These functions B are to be integrated over  $\chi$ , the results are the  $B_j'$  functions. Exactly as was done in the case of constant cross section, an integration over  $\bar{u}$  is substituted for that over  $\chi$ .

$$\sin\chi d\chi = \frac{4\bar{u} d\bar{u}}{uv}$$

$$B_1 = \frac{2\pi e^4}{4m^2\bar{\mu}^4} \left[ 1 - \frac{(2\beta^2+1) \cos \lambda_0 - \cos \alpha}{4(\beta^4+\beta^2) \{4\beta^4+4\beta^2-2(2\beta^2+1) \cos \alpha \cos \lambda_0 + \cos^2 \alpha + \cos^2 \lambda_0\}^{\frac{1}{2}}} \right]$$

$$B_2 = \frac{2\pi e^4}{4m^2\bar{\mu}^4} \left[ 1 + \frac{(2\beta^2+1) \cos \lambda_0 - \cos \alpha}{4(\beta^4+\beta^2) \{4\beta^4+4\beta^2-2(2\beta^2+1) \cos \alpha \cos \lambda_0 + \cos^2 \alpha + \cos^2 \lambda_0\}^{\frac{1}{2}}} \right]$$

A new symbol "a" is defined,

$$a = \frac{2h}{4\pi\alpha m w} = \frac{2\bar{u}\beta}{w} = \frac{\lambda_w}{2\pi\alpha}$$

where  $\lambda_w$  is the wavelength corresponding to the velocity  $w$ . Since  $\beta$  is extremely small, "a" also is very small.

It is found upon evaluating the integrals over  $\bar{u}$ , that:

$$\underline{19a} \quad B'_1 = \frac{2\pi e^4}{m^2 w^2 a^2 uv} \left[ s - \frac{w^2 - u^2 + w^2 a^2}{2wa} \tan^{-1} \frac{2was}{w^2 - u^2 + w^2 a^2} \right]$$

$$\underline{19c} \quad B'_2 = \frac{2\pi e^4}{m^2 w^2 a^2 uv} \left[ u - \frac{w^2 - u^2 - w^2 a^2}{2wa} \tan^{-1} \frac{2wau}{w^2 - u^2 + w^2 a^2} \right]$$

$$\underline{19b} \quad B'_3 = \frac{2\pi e^4}{m^2 w^2 a^2 uv} \left[ w - \frac{u^2 - w^2 + w^2 a^2}{2wa} \tan^{-1} \frac{2waw}{u^2 - w^2 + w^2 a^2} \right]$$

$$\underline{19d} \quad B'_4 = \frac{2\pi e^4}{m w^2 a^2 uv} \left[ v - \frac{u^2 - w^2 - w^2 a^2}{2wa} \tan^{-1} \frac{2wav}{u^2 - w^2 + w^2 a^2} \right]$$

These may be compared with eq's. 16, which list the corresponding quantities for  $\sigma = \text{constant}$ .

The functions  $B'_j$  are large near the line  $u = w$ , and decrease rapidly away from it. In quadrants 2 and 3, they are independent of  $v$ . In quadrants 3 and 4 they are asymptotic to zero as  $u$  becomes large, and  $B'_4$  is zero when  $v=0$ .  $B'_2$  is zero at  $u = 0$ , and  $B'_1 = 0$  along the line  $u^2 + v^2 = w^2$

These functions are plotted in fig.3. In this figure, the value 0.1 was used for "a", so that the peaks are not so sharp.

The  $B_j^1$  functions would, when multiplied by the distribution function, be very difficult to integrate. Therefore approximations are made. Over most of the plane, it is satisfactory to expand the arc-tangent functions, retaining only terms through the order  $a^2$ . But this expansion breaks down for  $|u-w| < w^2 a$ . Hence, in a narrow strip about the  $u = w$  line, some other method must be used. Over the width of this strip, the product  $f(u^2)f(v^2)$  does not change anywhere nearly as rapidly as  $B_j^1$ . So the distribution product is expanded, and the exact form of  $B_j^1$  is retained.

When  $w^2 - u^2 \gg w^2 a$ ,

$$\underline{20a} \quad B_1^1 = \frac{2\pi e^4}{m^2 uv} \cdot \frac{4s^3}{3(w^2 - u^2)^2}$$

$$\underline{20c} \quad B_2^1 = \frac{2\pi e^4}{m^2 uv} \left( \frac{4u^3}{3(w^2 - u^2)^2} + \frac{2u}{w^2 - u^2} \right)$$

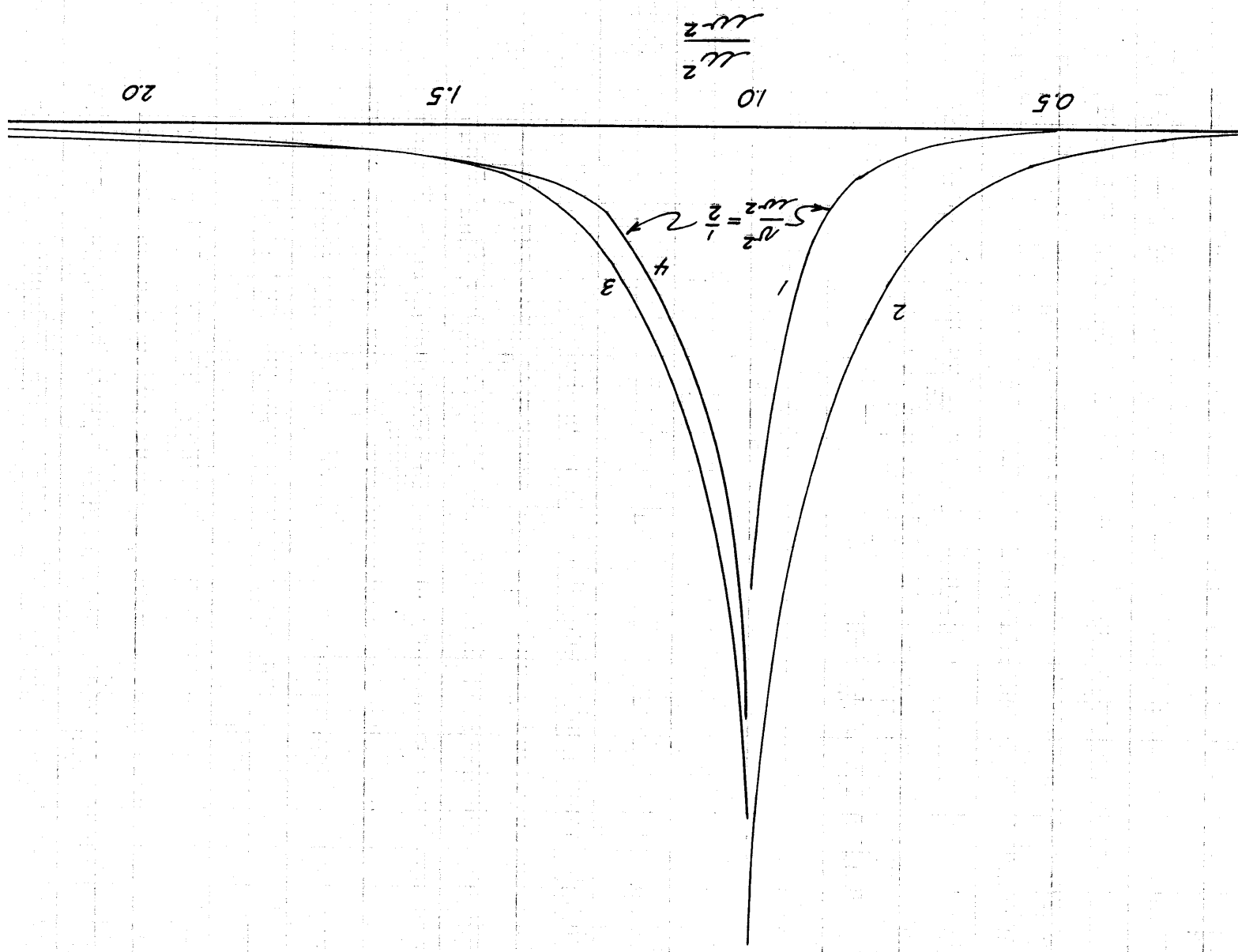
$$\underline{20b} \quad B_3^1 = \frac{2\pi e^4}{m^2 uv} \cdot \frac{4w^3}{3(u^2 - w^2)^2}$$

$$\underline{20d} \quad B_4^1 = \frac{2\pi e^4}{m^2 uv} \left( \frac{4v}{3(u^2 - w^2)^2} + \frac{2v}{u^2 - w^2} \right)$$

These expressions no longer containing "a", which is associated with small angle scattering. They represent that part of the flux across the level  $u = w$ , which is caused by collisions in which there is a comparatively large exchange



Fig. 3



of energy. They become infinite at  $u = w$ , while the more exact expressions do not. Outside of a strip of width  $tw^a$  (fig.2) on each side of this line, they are very good approximations. ( $t \gg a$ ) Inside this strip the other approximation is used. The number "t" must be so chosen that on the one hand,  $t \gg a$ , and on the other, so that the expansion of  $f_0(u^a)f_0(v^a)$  is sufficiently precise inside the strip. It is shown later that this choice is not critical.

In order that the approximation in the interior of the strip may give a correct result ~~for~~ <sup>under</sup> conditions giving rise to a Maxwellian distribution, the product  $f_0(u^a)f_0(v^a)$  is expanded along lines  $u^a + v^a = \text{constant}$ . For this purpose the variables  $x$  and  $y$  are introduced.

$$\begin{aligned} \underline{21} \quad x &= \frac{u^a + v^a}{\sqrt{2}} & u^a &= \frac{x + y}{\sqrt{2}} \\ y &= \frac{u^a - v^a}{\sqrt{2}} & v^a &= \frac{x - y}{\sqrt{2}} \end{aligned}$$

Let

$$f_0(u^a)f_0(v^a) = \phi(x, y)$$

$$\text{Then } \phi(x, y) \simeq \phi(x, y_0) + \left[ \frac{\partial}{\partial y} \phi(x, y) \right]_{y_0} y_1$$

where  $y_1 = y - y_0$ , and  $y_0$  is to be so chosen that the point  $(x, y_0)$  lies on the line  $u = w$ .

Let  $S_j$  represent the strip integrals of the  $j$ th quadrant. Below, symbols  $H_{j,k}$  ( $j=1,2,3,4$ ;  $k=1,2$ ) are used temporarily to show more clearly the structure of the  $S_j$ 's.  $(B'_{juv})$  will be carried along as a single function, of either

$u^2$  and  $v^2$ , or  $x$  and  $y$ , depending on which set is in use at the moment.

$$S_j = 4\pi^2 \eta^2 \int f_0(u^2) f_0(v^2) B_j' du^2 dv^2 UV$$

$$= 4\pi^2 \eta^2 \int \left( \int B_j'(uv) \left\{ \phi(x, y_0) + \left[ \frac{\partial \phi(x, y)}{\partial y} \right]_{y_0} \right\} dy_1 \right) dx$$

For the first integration,  $x$  is held constant, and therefore  $\phi(x, y_0)$  and  $\left[ \frac{\partial \phi(x, y)}{\partial y} \right]_{y_0}$  are also constant.

Let

$$H_{j,1} = \int (B_j'(uv)) dy_1$$

$$H_{j,2} = \int (B_j'(uv)) y_1 dy_1$$

$$\text{Then } S_j = 4\pi^2 \eta^2 \left\{ \int \phi(x, y_0) H_{j,1} dx + \int \left[ \frac{\partial \phi(x, y)}{\partial y} \right]_{y_0} H_{j,2} dx \right\}$$

Once having obtained this for  $S_j$ , it is more convenient to evaluate  $H_{j,k}$  and  $S_j$  by returning to the variables  $u^2$  and  $v^2$ .

$$y = \sqrt{2}u^2 - x, \quad y_0 = \sqrt{2}w^2 - x$$

$$y_1 = y - y_0 = \sqrt{2}(u^2 - w^2)$$

In the first integration,

$$dy_1 = \sqrt{2} du^2$$

$$\underline{22a} \quad H_{j,1} = \sqrt{2} \int (B_j'(uv)) du^2$$

$$\underline{22b} \quad H_{j,2} = 2 \int (B_j'(uv)) (u^2 - w^2) du^2$$

The limits on  $u^2$  are: for the first and second quadrants,  $w^2(1-t)$  and  $w^2$ ; for the third and fourth,  $w^2$  and  $w^2(1+t)$ .

Also,

$$\phi(x, y_0) = f_0(w^2) f_0(v^2), \quad \text{and}$$

$$\sqrt{2} \left[ \frac{\partial \phi(x, y)}{\partial y} \right]_{y_0} = f_0'(w^2) f_0(v^2) - f_0(w^2) f_0'(v^2)$$

where

$$f_0'(w^2) = \frac{d}{dw^2} f_0(w^2), \quad f_0'(v^2) = \frac{d}{dv^2} f_0(v^2)$$

As  $u^2 = w^2$  when  $y = y_0$ , throughout the second integration,

$$x = \frac{w^2 + y^2}{\sqrt{2}}, \quad dx = \frac{dw^2}{\sqrt{2}}$$

substituting all this in the expression for  $S_j$ , 23

$$S_j = 4\pi^2 \eta^2 f(w^2) \int \frac{f(v^2) H_{j,1}}{\sqrt{2}} dv^2 + \int \frac{f'(w^2)f(v^2) - f(w^2)f'(v^2)}{2} \cdot \frac{H_{j,2}}{2} dv^2$$

Neglecting terms of higher order than the second in  $a$ ,

$$\begin{aligned} \frac{24}{\sqrt{2}} H_{1,1} &\approx \frac{2\pi e^4}{m^2} \frac{\pi w^2}{2wa} - v - \frac{4v^3 v^3}{3w^2 t} \\ \frac{1}{2} H_{1,2} &\approx \frac{2\pi e^4}{m^2} \cdot (-1) \left[ \frac{1}{3} + \eta \frac{wt}{2av} \right] \frac{4v^3}{3} \\ \frac{1}{\sqrt{2}} H_{2,1} &\approx \frac{2\pi e^4}{m^2} \cdot \left[ \frac{\pi w}{2} + w - \frac{4w}{3t} \right] \\ \frac{1}{2} H_{2,2} &\approx \frac{-2\pi e^4}{m^2} \left[ \frac{1}{3} + \eta \frac{t}{2a} \right] \frac{4w^3}{3} \\ \frac{1}{\sqrt{2}} H_{3,1} &\approx \frac{2\pi e^4}{m^2} \left[ \frac{\pi w}{2} - w - \frac{4w}{3t} \right] \\ \frac{1}{2} H_{3,2} &\approx \frac{2\pi e^4}{m^2} \left[ \frac{1}{3} + \eta \frac{t}{2a} \right] \frac{4w^3}{3} \\ \frac{1}{\sqrt{2}} H_{4,1} &\approx \frac{2\pi e^4}{m^2} \left[ \frac{\pi v^2}{2wa} + v - \frac{4v^3}{3w^2 t} \right] \\ \frac{1}{2} H_{4,2} &\approx \frac{2\pi e^4}{m^2} \left[ \frac{1}{3} + \eta \frac{tw}{2av} \right] \frac{4v^3}{3} \end{aligned}$$

The explicit calculations are in the appendix.

Because the integrals from quadrants 3 and 4 are subtracted from those of quadrants 1 and 2 to find the net flux, all terms will disappear which contain  $a$  and  $t$  otherwise than in the logarithm alone.

Assembling all terms from both approximations, the total contribution to the balance equation 13 is written out explicitly on page 31, eq. 25. That form was used as the frame of the numerical computations.

In order to verify that errors have not crept into our rather complicated calculations, proof is given in the appendix that both the values of  $B_j'$  listed in eq's. 19, and the approximate forms, permit an equilibrium solution of Maxwell-Boltzmann form.

$$\text{Let } I_1 - I_2 = \frac{8\pi^3 e^4 \eta^2}{m^2} w^3 K$$

where  $K$  is a factor of dimensions  $[w^6]$ , representing the integrals of eq. 25. The balance equation 13 is then:

$$-\frac{NQm}{M} w^4 f_0 - \frac{1}{3NQ} \left(\frac{Ee}{m}\right)^2 w \frac{d}{dw} f_0 - NJ + \frac{2\pi^3 e^4 \eta^2 w^3}{m^2} K = 0$$

Letting

$$i = \frac{\pi e^4 \eta}{m^2 NQ}, \quad \mathcal{E} = \frac{Ee}{mNQ}, \quad \text{this becomes:}$$

$$\underline{26} \quad -\frac{m}{M} w^4 f_0 - \frac{1}{3} \mathcal{E}^2 w \frac{d}{dw} f_0 - \frac{J}{Q} + 2\pi i w^3 K = 0$$

$\mathcal{E}$  corresponds to the parameter  $\frac{E}{p}$  of the Townsend theory. "i" is a measure of the intensity of ionization.

The function  $K$  can be separated into two parts: those terms containing the factor  $\ln \frac{t}{2a}$ ; and the remainder. The

25

$$\begin{aligned}
I_1 - I_2 &= \frac{8\pi e^4 \eta^2}{m^2} \left\{ \int_0^{w^2} f_0(v^2) dv^2 \int_{w^2-v^2}^{w^2(1-t)} \frac{4}{3} f_0(u^2) \frac{s^3}{(w^2-u^2)^2} du^2 \right. \\
&\quad + \int_{w^2}^{\infty} f_0(v^2) dv^2 \int_0^{w^2(1-t)} f_0(u^2) \left[ \frac{4}{3} \frac{u^3}{(w^2-u^2)^2} + \frac{2u}{w^2-u^2} \right] du^2 \\
&\quad - \int_{w^2}^{\infty} f_0(v^2) dv^2 \int_{w^2(1+t)}^{\infty} f_0(u^2) \frac{4u^3}{3(u^2-w^2)^2} du^2 \\
&\quad - \int_0^{w^2} f_0(v^2) dv^2 \int_{w^2(1+t)}^{\infty} f_0(u^2) \left[ \frac{4u^3}{3(u^2-w^2)^2} + \frac{2u}{u^2-w^2} \right] du^2 \\
&\quad + 2f_0(w^2) \left[ w \int_{w^2}^{\infty} f_0(v^2) dv^2 - \int_0^{w^2} v f_0(v^2) dv^2 \right] \\
&\quad + \frac{8w^3}{3} \left[ \frac{1}{3} + \ln \frac{t}{2a} \right] \left[ -f_0'(w^2) \int_{w^2}^{\infty} f_0(v^2) dv^2 + f_0(w^2) \int_{w^2}^{\infty} f_0'(v^2) dv^2 \right] \\
&\quad + \frac{8}{3} \int_0^{w^2} \left[ \frac{1}{3} + \ln \frac{tw}{2av} \right] \left[ -f_0'(w^2) f_0(v^2) + f_0(w^2) f_0'(v^2) \right] v^3 dv^2 \left. \right\}
\end{aligned}$$

former come from the narrow strip, and correspond roughly to smallangle collisions, in which little energy is transferred. The latter come mostly from the regions of the  $u^2, v^2$  plane outside of the strip of width  $tw^2$ , and represent close collisions with large energy exchanges.

K contains a number of rather complicated integrals, and a solution of equation 26 cannot be found directly. It is necessary to select a ~~form~~ function having the properties to be expected in a solution, and to test it by substituting it in the equation. For most choices, the integrals will have to be evaluated by numerical quadrature.

## VI - Special Cases

### A

The first special case investigated was that in which the effects of the field, the elastic collisions, and mutual collisions were included but the effects of inelastic impacts neglected. Then  $J = 0$ .

It is known that if the mutual interaction is also neglected, the distribution function has the form:

$$f_0 = A e^{-w^2/2w_0^2}$$

When only the pair collisions are considered, the distribution is maxwellian:

$$f_0 = A e^{-w^2/w_0^2}$$

The function chosen as a trial solution is:

$$f_0 = \frac{A}{2\pi w_0^2} e^{-(gw^2/w_0^2) - (1-g)(w^2/2w_0^2)}$$

$$27 \quad f = \frac{A}{2\pi w_0^3} e^{-g(w^2/w_0^2) - (1-g)(w^4/2w_0^4)} = \frac{A}{2\pi w_0^3} e^{-\rho}$$

with  $0 \leq g \leq 1$ .

At  $g = 1$  or  $0$ , this reduces to one or the other of those above.  $g$  and  $w_0$  are parameters which are to be determined so as to give the best approximation to a solution of eq 26.  $g$  is a measure of the relative importance of electron-electron and electron-atom collisions. The number of electrons having speeds between  $w$  and  $w + dw$  is:  $4\pi w^2 f(w^2) dw$ . Taking the derivative,

$$\begin{aligned} \frac{d}{dw} w^2 f(w^2) &= \frac{d}{dw} w^2 e^{-\rho} \\ &= 2 \left( w - g \frac{w^3}{w_0^2} - (1-g) \frac{w^5}{w_0^4} \right) \end{aligned}$$

The derivative is zero at

$$1 - g \frac{w^3}{w_0^2} - (1-g) \frac{w^5}{w_0^4} = 0, \quad \text{or when } w^2 = w_0^2.$$

$w_0$  is therefore the most probable velocity. "A" is a normalization constant such that

$$1 = \frac{2A}{w_0^3} \int_0^\infty w^2 e^{-\rho} dw$$

Letting

$$z = \frac{w^2}{w_0^2}, \quad 1 = A \int_0^\infty z e^{-gz - (1-g)z^2} dz.$$

And  $A$  is a function of  $g$  alone.

Define:

$$\frac{\ln t}{2a} = \ln \frac{t}{2a_0} + \ln \frac{w}{w_0}, \quad \text{or: } a_0 = \frac{2h}{4\pi\alpha m w_0} = \frac{\lambda_0}{2\pi\alpha}$$



From pp. 22 and 25,

$$a_0 = \frac{2 \cdot 0.975 \cdot 10^{-8}}{\alpha \sqrt{V_0}}$$

$V_0$  is the energy  $1/2 \cdot m v_0^2$  expressed in electron volts. If  $\alpha$  is of the order  $10^{-4}$ , and  $V_0$  is a few volts,  $a_0$  is of the order  $1 \cdot 10^{-4}$ .  $a_0$  is introduced to separate the dependence of  $K$  on the parameters  $\alpha$ ,  $g$ , and  $w_0$  from that on  $w$ .

Define: 
$$K' = \frac{4\pi^2 w_0^6}{A^2} \frac{w}{w_0} \frac{K}{e^{-\rho}}$$

Then:

$$\frac{A^2}{4\pi^2 w_0^6} K' = \frac{w}{w_0} K e^{\rho}$$

$K'$  is a dimensionless function of only  $\ln \frac{t}{2a_0}$ ,  $g$ , and  $w/w_0$ .

Substituting in equation 26, and dividing by  $\frac{w^2}{w_0^2} A e^{-\rho}$ ,

$$-\frac{m}{M} \frac{w^2}{w_0^2} \frac{w^2}{w_0^2} + \frac{2}{3} \mathcal{E}^2 \left[ g + (1-g) \frac{w^2}{w_0^2} \right] + A i K' = 0$$

Rearranging:

28

$$i A K' = -\frac{2}{3} \mathcal{E}^2 g + \frac{w^2}{w_0^2} \left[ \frac{2}{3} \mathcal{E}^2 (1-g) \right]$$

Because  $\alpha$  and  $t$  only appear in the equation through their logarithms, the results are not critically dependent on the exact values given them. In the numerical calculations,  $t$  was chosen equal to  $\frac{1}{25}$ . This is safely larger than  $a_0$ , and is small enough so that the error in the expansion of  $f(u^2)f(v^2)$  will be small.

It is seen that  $A i K'$  is linear in  $w^*/w_0^2$ , or

$$A K' = k_1 + k_2 w^*/w_0^2$$

This is an condition that  $K'$  must satisfy in order that  $f_0$  may be a solution. In fig. 4 ~~xxxxx~~  $K_8$  is plotted as a function of  $w^*/w_0^2$ , for  $g = 0$ , and  $g = 0.732$ . These curves are seen to be fairly straight over the range containing most of the electrons. ( $w^* < 4w_0^2$ ) The agreement is best for the larger values of  $g$ . The amount of agreement measures the correctness of the function  $f_0$  which has been assumed. The deviation of the curves at high velocities indicates that the function gives too many high speed particles.

The slopes and intercepts of the straight portions of the curves similar to fig. 4 (but containing the factor  $A$ ) are plotted in fig's. 5 and 6 as the functions  $k_1$  and  $k_2$  of  $g$  and  $\ln a_0$ .

From eq. 28 follow equations 29 and 29'.

$$\underline{29} \quad i k_1 = - \frac{2}{3} e^{2g}$$

$$\underline{29'} \quad i k_2 = - \frac{2}{3} e^{2g} (1-g) + \frac{m}{M} w_0^2$$

These equations can be solved for  $g$  and  $w_0^2$  in terms of the parameters  $i$ ,  $e^{2g}$ , and  $\ln a_0$ . In particular,  $g$  and  $w_0^2/e^{2g}$  are functions of  $\log a_0$  and the ratio  $e^{2g}/i$  only. This is shown in fig. 7, where lines of constant  $g$  and constant  $w_0^2/e^{2g}$  are plotted on the  $\log a_0, e^{2g}/i$  plane.

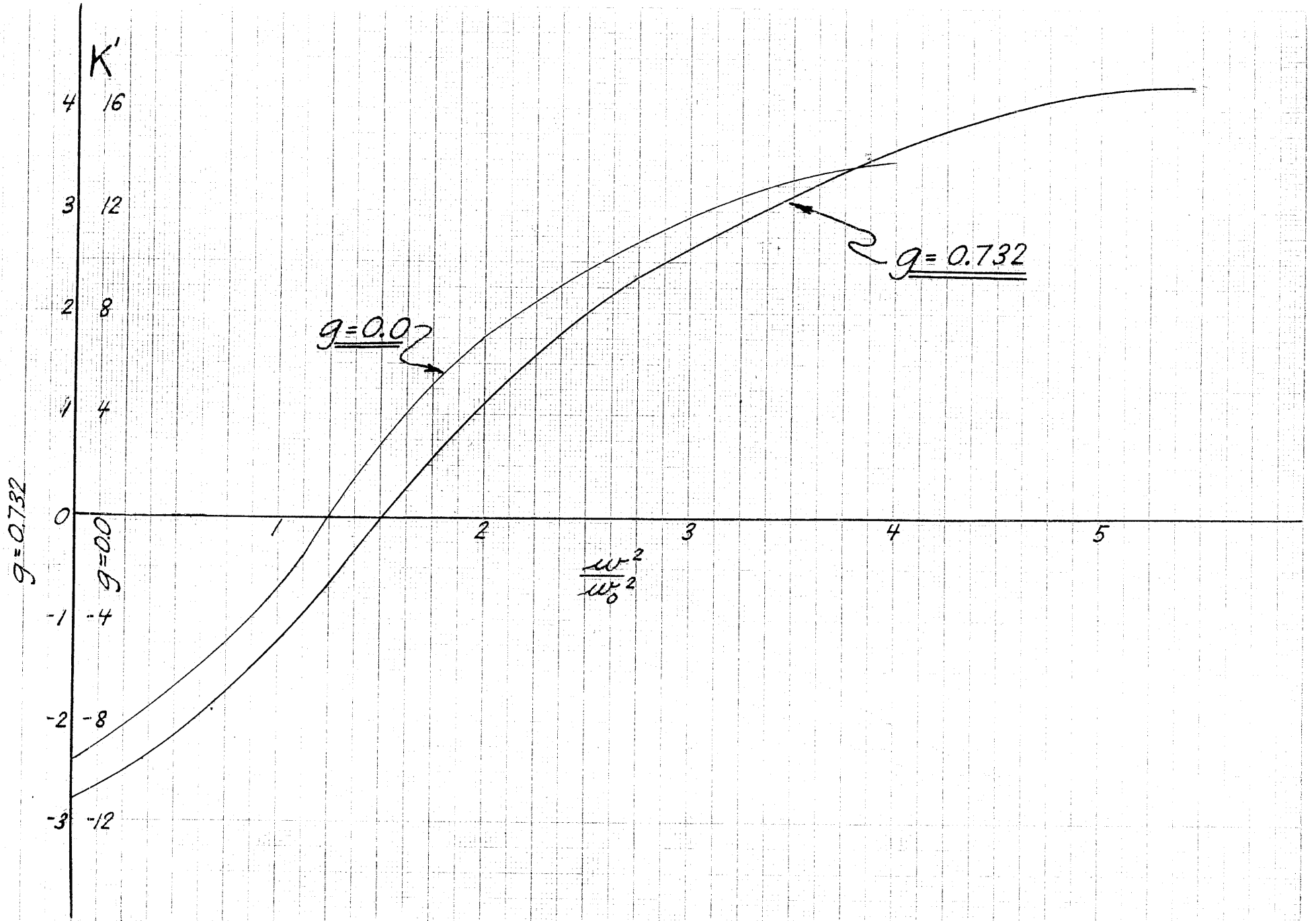


Fig 4

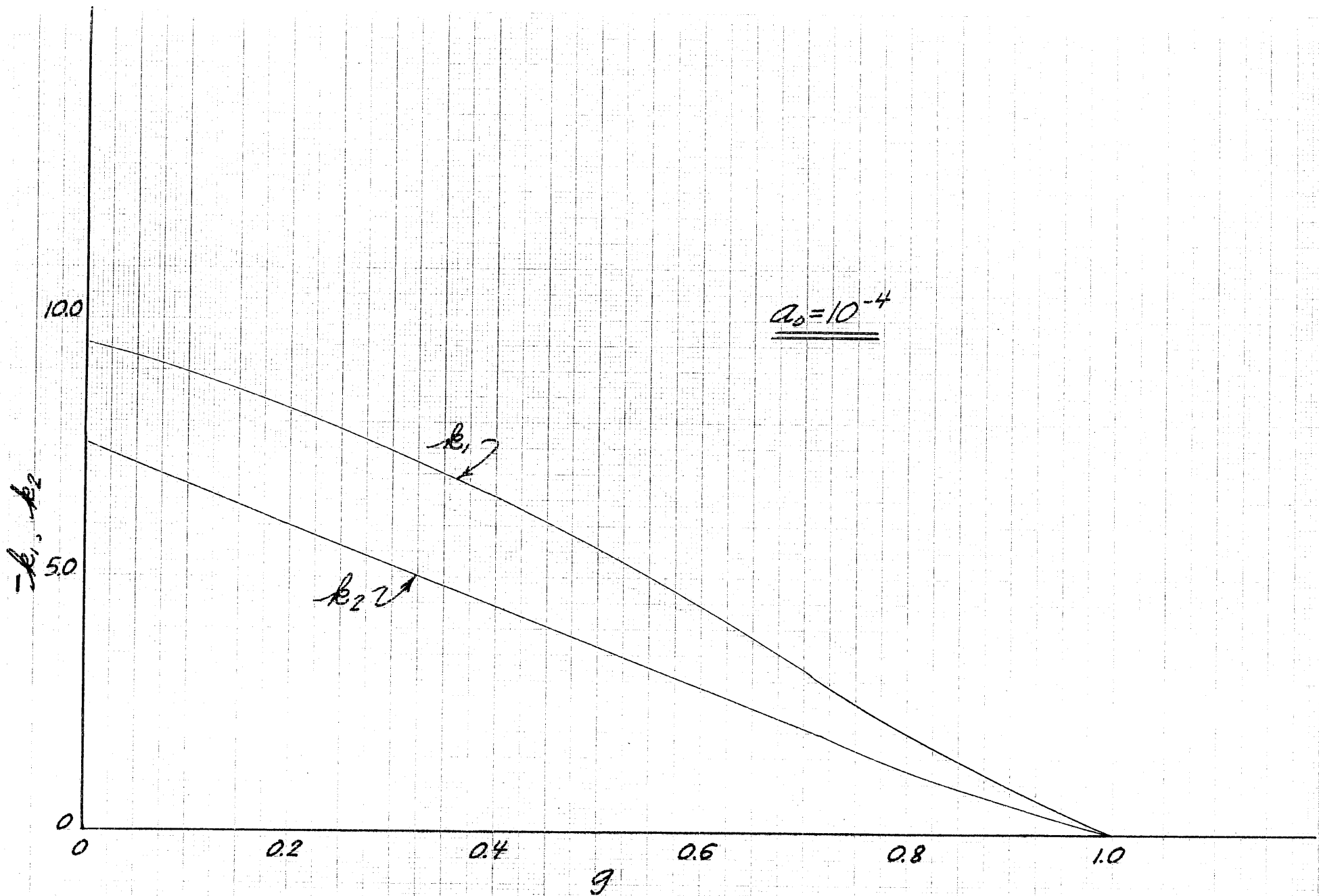


Fig. 5

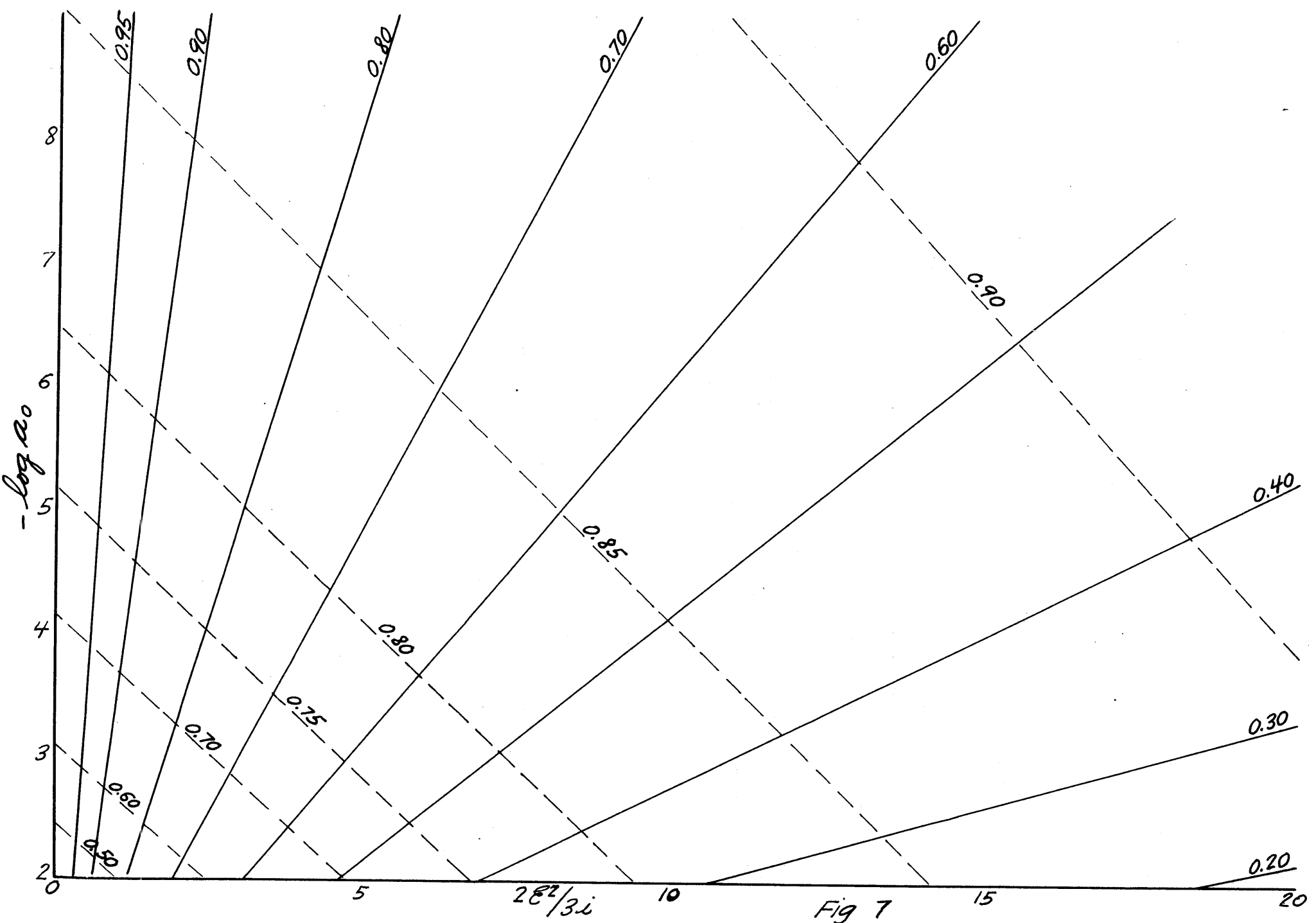


Fig 7

Empirical equations for  $g$  and  $w_0$  are:

$$\underline{30} \quad g = \frac{0.26 + 2.46 \log a_0}{0.26 + 2.46 \log a_0 - \frac{2}{3} e^{2/1}}$$

$$\underline{30'} \quad \frac{w_0}{e^{2/1}} = \frac{M}{m} \left[ \frac{-2.92 - 239 \log a_0}{e^{2/1}} + \frac{2}{3} \right] (1-g)$$

It has been mentioned that  $K$  consists of two parts. One does not contain the parameter  $a_0$ . That is, it is independent of the size of the potential hill. It corresponds mainly to collisions involving a large change of energy. The second part, that containing the factor  $\ln t/2a_0$ , corresponds to glancing collisions with small energy exchange. These glancing collisions are much more numerous than the first kind, and it is interesting to see which type is the more important. Table I gives data for two values of  $a_0$  that fall in the experimental range. Column 2 gives that part of the function  $\frac{w_0}{e^{2/1}} K' e^{-\rho}$ , which is independent of  $\ln t/2a_0$ . Columns 3 and 5 list that part of the same function containing the factor. Columns 4 and 6 give the sums of 2 with 3 and 5 respectively.

$\frac{W^2}{W_0^2}$	$a_0 = 10^{-4}$		$a_0 = 10^{-5}$		
	1	2	3	4	5
0.342	-0.255	-0.373	-0.628	-0.530	-0.785
0.683	-0.381	-0.461	-0.742	-0.656	-0.937
1.366	-0.0091	-0.0695	-0.0786	-0.0989	-0.1080
2.732	+0.102	+0.222	+0.334	+0.316	+0.418
5.464	+0.00261	+0.00513	+0.00774	+0.00729	+0.00990

Table I.

It is seen that the larger part is that containing  $\ln a_0$ , or that representing small angle collisions.

It is interesting to see if conditions can exist for which this theory would predict a large value of  $g$ .

A search through the literature shows that there are very few papers on the subject of gaseous conduction which present enough data to determine all three parameters. The required data are:

- a the gas used
- b ~~the~~ atomic density
- c the field strength
- d the electronic density
- e an estimate of the average energy

Among the few papers giving all this data are, one by T. J. Killian (11), and one by A.H. van Gorcum (8). In Killian's work, the gas pressure was so low (a few bars) that the mean free path of the electrons was comparable to the tube diameter or larger. Plainly, this theory is inapplicable to such a case.

Van Gorcum reports work done on neon at a pressure of 4.7 mm.Hg. at 0°C. In the well developed plasma, he found by probe measurements that the electrons seemed to have very nearly a Maxwell distribution. The values found for the quantities listed above clustered about those in table IV.

1.67 · 10<sup>17</sup> cm<sup>-3</sup>      9.5 · 10<sup>10</sup> cm<sup>-3</sup>      0.93 volt/cm      2.7 e.v.



N	$\eta$	E	$V_{av}$
$1.67 \cdot 10^{17} \text{ cm}^{-3}$	$9.5 \cdot 10^{-3} \text{ cm}^{-3}$	0.93 volts/cm	2.7 e.v.

Table IV

For neon,  $Q \approx 0.28 \text{ cm}^3$

If  $\frac{eV_0}{300} = \frac{1}{2} mw_0^2$ , then for a maxwellian distribution,

$$V_0 = \frac{2}{3} V_{av}$$

For this case, taking  $\alpha = \eta^{-1/3} = 2.19 \cdot 10^{-4}$  we have;

$$\frac{2\mathcal{E}^2}{31} = 1.93, \quad a_0 = 6.6 \cdot 10^{-5}, \quad \log a_0 = -4.18$$

then from equation 30,  $g = 0.84$ .

This is quite a large value for  $g$ , and hence the mutual collisions are very important. If  $w_0$  is calculated from eq. 30', and  $V_{0,calc} = \frac{1}{2} mw_{0,calc}^2$ ,

$$\underline{V_{0,calc} = 1.39 \text{ e. v.}}$$

From the experiment,

$$\underline{V_0 = 1.8 \text{ e.v.}}$$

These values are in fairly good agreement, considering the roughness of the approximations, and the fact that ionization has been neglected in the computations.

If  $\alpha$  is chosen to equal 1 cm, it is found that:

$$g = 0.91, \text{ and } V_{\text{calc}} = 1.45 \text{ e.v.}$$

The agreement between these two theoretical results is astonishing in view of the violently differing values of  $\alpha$  that were used.

In discussing measurements made with probes in a plasma, it is usual to plot the logarithm of the electron current to the probe against the retarding voltage. For a Maxwellian distribution, the resulting curve is a straight line:

$$\log i_p = c + V/V_0$$

In the general case, the current to the probe can be calculated from the equation: (5)

$$i_p = \int_0^{\infty} \frac{f(w^2) (w^2 - \frac{2eV}{m}) w dw}{\sqrt{2eV/m}}$$

It may be mentioned here that probe measurements are notoriously treacherous.

In fig. 8,  $\log i_p$  is plotted against  $V$  for the functions obtained by setting  $g$  equal to 0.5, 0.8, 1.0. The corresponding curves given by van Gorcum are quite a bit straighter than the one given here for  $g = 0.8$ . This would indicate that the mutual interactions are of even greater importance than is predicted by this theory. On the other hand, the conditions in his tube were much more complicated than those assumed

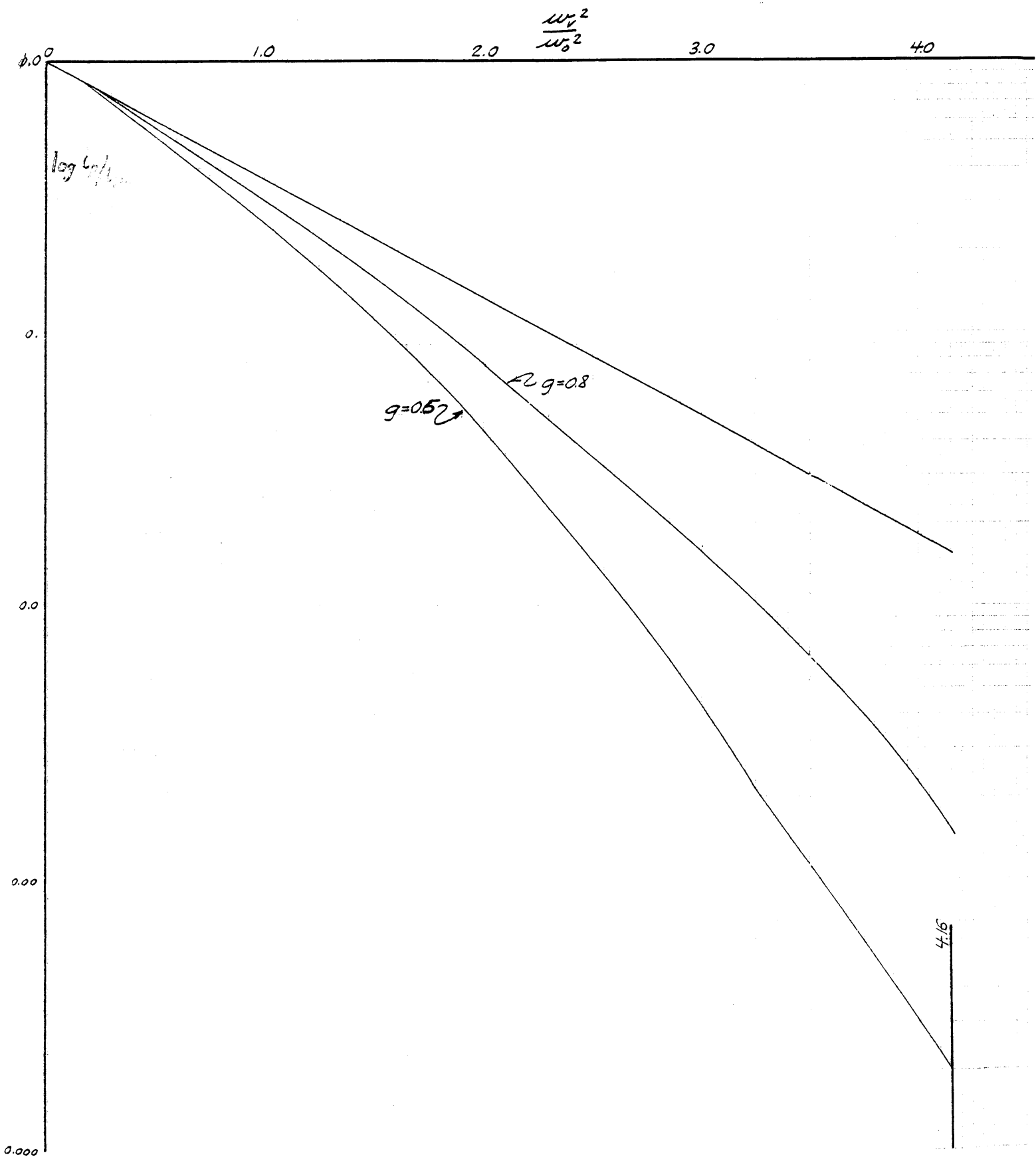


Fig 8

more complicated than those assumed here. This fact may account for part of the divergence.

B.

Inelastic collisions can be taken into account by assuming that every electron which acquires a velocity greater than a fixed value  $w_1$  suffers such a collision and loses all its energy. This assumption is equivalent to setting  $f(w^2) = 0$  for  $w > w_1$ , and  $\frac{J(w^2)}{Q} = \text{constant} = j$  for  $w < w_1$ . Eq. 26 takes the form

$$32 \quad \frac{-m w^2}{M} f_0 - \frac{1}{3} e^{2w} \frac{d}{dw} f_0 + 2\pi i w^3 K = j \quad (w < w_1)$$

The assumption may be expected to be good except near  $w = w_1$  and near  $w = 0$ .  $f_0$  must be sharply cut off at  $w_1$ . In an actual discharge, it is not zero at this point, but only very small, and decreasing very rapidly for  $w > w_1$ . Near the origin the function may be expected to be large, for the assumption that the electrons undergoing an inelastic collision lose all their energy gives an excessive piling up there.

Two forms of distribution function were tested:

$$a \quad f_0 = \frac{A}{2\pi w_0^3} \ln \frac{w^2}{w_1^2} e^{-w^2/w_0^2}$$

$$b \quad f = \frac{A}{2\pi w_0^3} \left(1 - \frac{w^2}{w_1^2}\right) e^{-w^2/w_0^2}$$

Consider function a. Let

$$\frac{w^2}{w_0^2} = z, \quad \frac{w_1^2}{w_0^2} = \lambda$$

Then

$$\frac{1}{A} = \int_0^{\infty} z^{\frac{1}{2}} \ln \frac{1}{\lambda z} e^{-z} dz$$

Equation 32 becomes

$$\begin{aligned} \frac{m}{M} A w_0 z^{\frac{1}{2}} \ln \frac{1}{\lambda z} e^{-z} + \frac{2}{3} \frac{C^2}{w_0^2} A (z \ln \frac{1}{\lambda z} + 1) e^{-z} \\ + \frac{A^2 C}{w_0^2} K_2 = 2\pi j \end{aligned}$$

Where  $K_2 = \frac{w_0^2 K}{A^2}$ , and  $K_2$  is dimensionless.

This is of the form

$$-Cz^{\frac{1}{2}} \ln \frac{1}{\lambda z} e^{-z} + F(z \ln \frac{1}{\lambda z} + 1) e^{-z} + iK_2 = 2\pi j$$

The terms on the left represent collisions with atoms, acceleration by the field, and mutual collisions, respectively.

$K_2(z, \lambda, \ln a_0)$  was evaluated by numerical integration. Values of  $\lambda$  used were 1/2.08 and 1/4.16. (i.e.,  $w_0/w^2 = 2.08$  or 4.16) Curves of  $K_2(z, \lambda)$ ,  $(z^{\frac{1}{2}} \ln \frac{1}{\lambda z}) e^{-z}$  and  $(z \ln \frac{1}{\lambda z} + 1) e^{-z}$  are shown in fig's. 9 and 10.  $K_2$  is finite at  $z = 0$ , but is very large for small values of  $z$ . From the form of the curves, it is plain that no choice of the parameters  $C$ ,  $F$ , and  $K_2$  can satisfy the equation. This function is not, then, a satisfactory solution.

It is, however, interesting, to compare the relative

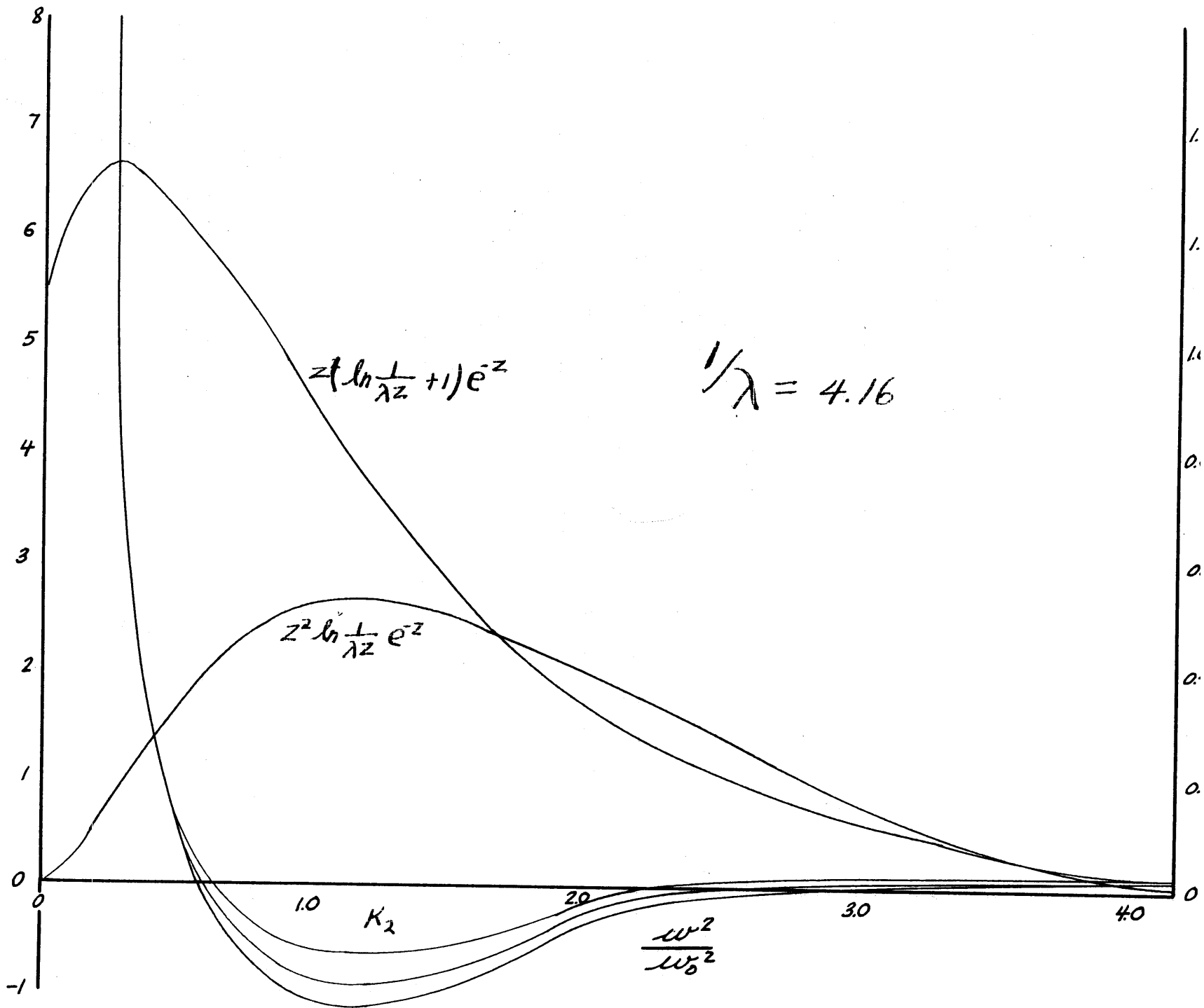


Fig 9

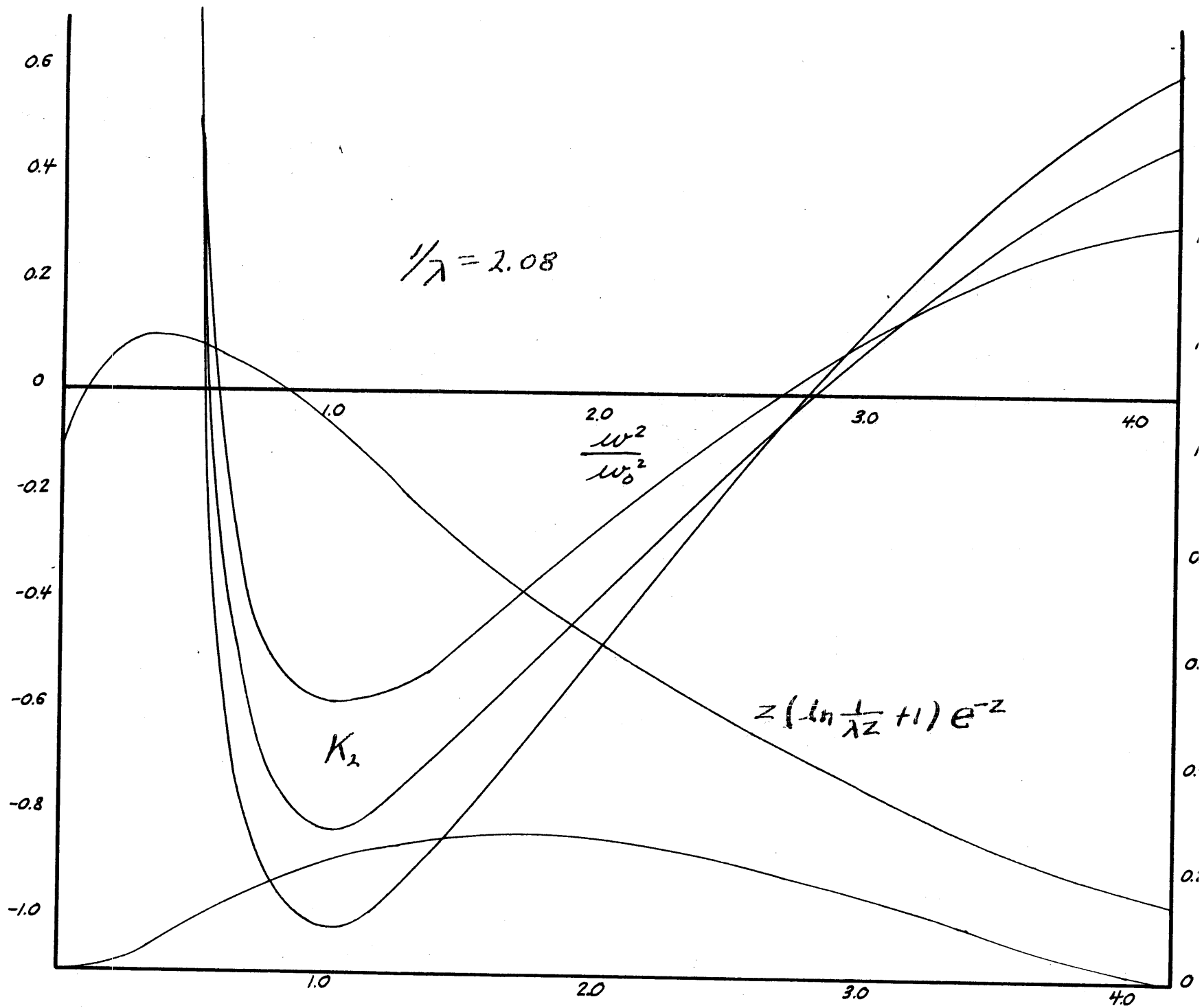


Fig 10

magnitudes of those terms of  $K$  which contain the factor  $\ln t/2a_0$  and those which do not. These values are given in tables II ~~and~~ . The actual factor used was

$$\frac{8}{9} + \frac{8}{3} \ln \frac{t}{2a_0}$$

Column 2 contains the value of the terms independent of this quantity. Columns 3 and 5, the values of the terms containing it, and Col's. the sums.

Here, as in the case ~~of~~ where ionization is neglected, the terms representing the effect of small angle collisions are the more important.



$w^2/w_0^2$		$8/9 + 8/3 \ln t/2a_0$						
		10		15		20		
1	2	3	4	5	6	7	8	
$1/\lambda = 2.03$	0.25	+2.276	+2.612	+4.888	+4.214	+6.490	+5.817	+8.093
	0.50	+0.750	-0.1110	+0.639	-0.172	+0.578	-0.233	+0.517
	1.00	-0.0248	-0.5713	-0.596	-0.857	-0.882	-1.143	-1.168
	2.00	-0.00639	-0.1569	-0.1632	-0.329	-0.235	-0.301	-0.307
	4.00	+0.00693	+0.0471	+0.0540	+0.0670	+0.0739	+0.0868	+0.0937
$1/\lambda = 4.16$	0.50	-0.146	-0.437	-0.583	-0.678	-0.824	-0.919	-1.065
	1.00	-0.0079	-0.244	-0.252	-0.366	-0.374	-0.487	-0.495
	2.00	+0.032	+0.276	+0.303	+0.403	+0.435	+0.530	+0.562

Table II

C

It has been seen that the most important of the mutual collisions are the glancing ones. The integrals are greatly simplified if the others are neglected, that is, if only the terms containing the factor  $\ln t/2a$  are considered.

Now:

$$\ln \frac{t}{2a} = \ln \frac{t}{2a_i} + 2 \ln \frac{w}{w_i}$$

$$\text{where } a_i = \frac{2h}{4\pi\alpha m w_i}$$

Except very near  $w = 0$ , the term  $2 \ln w/w_i$  will be small compared to  $\ln t/2a_i$ . It will be neglected.

Then (eq.25)

$$I_1 - I_2 = \frac{8\pi^3 e^4 \eta^2}{m^2} w^3 \cdot \frac{8}{3} \ln \frac{t}{2a_i} \left\{ -f_0'(w^2) \int_{w^2}^{\infty} f_0(v^2) dv^2 \right. \\ \left. + f_0(w^2) \int_{w^2}^{\infty} f_0'(v^2) dv^2 - f_0'(w^2) \int_0^{w^2} f_0(v^2) \frac{v^3}{w^3} dv^2 \right. \\ \left. + f_0(w^2) \int_0^{w^2} \frac{v^3}{w^3} f_0'(v^2) dv^2 \right\}$$

The expression in brackets is

$$\int_0^{\infty} \left\{ \left[ \frac{\partial}{\partial u^2} - \frac{\partial}{\partial v^2} \right]_{u=w} \left[ f_0(u^2) f_0(v^2) \right] \right\} dv^2 \\ + \int_0^{\infty} \left\{ \left[ \frac{\partial}{\partial u^2} - \frac{\partial}{\partial v^2} \right]_{u=w} \left[ f_0(u^2) f_0(v^2) \right] \right\} \frac{v^3}{w^3} dv^2$$

We can now verify that the Maxwell distribution is the equilibrium solution with the mutual interaction abbreviated in this manner. For, let

$$f_0(u^2) f_0(v^2) = A^2 e^{-\frac{u^2+v^2}{w^2}}$$

Then

$$\left[ \frac{\partial}{\partial u^2} - \frac{\partial}{\partial v^2} \right] \left[ f_0(u^2) f_0(v^2) \right] = 0$$

$$\text{or } I_1 - I_2 = 0 .$$


---

Returning to general form

$$f_0'(w^2) \int_{w^2}^{\infty} f_0'(v^2) dv^2 + f_0(w^2) \int_0^{w^2} f_0'(v^2) \frac{v^3}{w^3} dv^2$$

By an integration by parts

$$= f_0(w^2) \left\{ \left[ f_0(v^2) \right]_{w^2}^{\infty} + \left[ \frac{v^3}{w^3} f_0(v^2) \right]_0^{w^2} - \frac{3}{2} \int_0^{w^2} \frac{v}{w^3} f_0(v^2) dv^2 \right\}$$

$$= f_0(w^2) \cdot \left(-\frac{3}{2}\right) \int_0^{w^2} \frac{v}{w^3} f_0(v^2) dv^2$$

Substituting this  
in  $I_1 - I_2$ ,

$$\frac{33}{I_1 - I_2} = \frac{8\pi^3 e^4 \eta^2 w^3}{m^2} \frac{8}{3} \ln \frac{t}{2a_i} \left\{ -f_0'(w^2) \int_{w^2}^{\infty} f_0(v^2) dv^2 \right.$$

$$\left. - f_0'(w^2) \int_0^{w^2} \frac{v^3}{w^3} f_0(v^2) dv^2 - \frac{3}{2} f_0(w^2) \int_0^{w^2} \frac{v}{w^3} f_0(v^2) dv^2 \right\}$$

We now make an assumption about  $f_0$ . Let:

$$f_0 = \frac{A}{2\pi w_0^3} \left( 1 - \frac{w^2}{w_1^2} \right) e^{-w^2/w_0^2}$$

Defining:  $z = \frac{w^2}{w_1^2}$ ,  $\lambda = \frac{w_0^2}{w_1^2}$

In terms of  $z$ ,

$$f_0(w^2) = f_0(w_0^2 z) = \frac{A}{2\pi w_0^3} (1 - \lambda z) e^{-z}$$

$$\frac{1}{A} = \int_0^{1/\lambda} (1 - \lambda z) z e^{-z} dz$$

The upper limit is  $1/\lambda$ , since  $f_0 = 0$  when  $z > 1/\lambda$ .

Calculating the field term in eq. 32,

$$\begin{aligned} w \frac{d}{dw} f_0(w^2) &= \frac{2A}{2\pi w_0^3} \left( \frac{-w^2}{w_0^2} * \frac{w^2}{w_0^2 w_1^2} - \frac{w^2}{w_1^2} \right) e^{-w^2/w_0^2} \\ &= \frac{-2A}{2\pi w_0^3} (1 + \lambda - \lambda z) z e^{-z} \end{aligned}$$

Likewise,

$$\begin{aligned} f_0' &= \frac{-A}{2\pi w_0^3} \left( \frac{1}{w_0^2} - \frac{w^2}{w_0^2 w_1^2} + \frac{1}{w_1^2} \right) e^{-w^2/w_0^2} \\ &= \frac{-A}{2\pi w_0^3} \cdot \frac{1}{w_0^2} (1 + \lambda - \lambda z) e^{-z} \end{aligned}$$

Consider the integrals in  $I_1 - I_3$ .

$$\begin{aligned} \int_{w^2}^{\infty} f_0(v^2) dv^2 &= \frac{A}{2\pi w^3} \int_{w^2}^{w^2} \left(1 - \frac{v^2}{w^2}\right) e^{-v^2/w^2} dv^2 \\ &= \frac{A}{2\pi w^3} \int_z^{\infty} (1 - \lambda y) e^{-y} dy \end{aligned}$$

when we set  $y = v^2/w^2$

Under the same substitution,

$$\int_0^{w^2} \frac{v^3}{w^3} f_0(v^2) dv^2 = \frac{A}{2\pi w^3 z} \int_0^z y^{3/2} (1 - \lambda y) e^{-y} dy$$

and

$$\int_0^{w^2} \frac{v}{w^2} f_0(v^2) dv^2 = \frac{A}{2\pi w^3 z} \int_0^z y^{1/2} (1 - \lambda y) e^{-y} dy$$

Evaluating these three integrals,

$$\int_{w^2}^{\infty} f_0(v^2) dv^2 = \frac{A}{2\pi w^3} \left[ (1 - \lambda - \lambda z) e^{-z} + e^{-1/\lambda} \right]$$

$$\int_0^{w^2} \frac{v^3}{w^3} f_0(v^2) dv^2 = \frac{A}{2\pi w^3 z^{3/2}} \left[ \Gamma_z(5/2) - \Gamma_z(7/2) \right]$$

$$\int_0^{w^2} \frac{v}{w^2} f_0(v^2) dv^2 = \frac{A}{2\pi w^3 z^{1/2}} \left[ \Gamma_z(3/2) - \Gamma_z(5/2) \right]$$

Substituting these in eq. 33 for  $I_1 - I_3$ ,

$$33' \quad I_1 - I_2 = \frac{8\pi^3 e^4 \eta^2 A^2}{m^2 \omega_0^3 4\pi^2} \frac{8}{3} \ln \frac{t}{2a_i} z^{3/2}$$

$$\cdot \left\{ \begin{aligned} & (1+\lambda-\lambda z) e^{-z} [(1-\lambda-\lambda z)e^{-z} + \lambda e^{-\lambda}] \\ & + \frac{1}{z^{3/2}} (1+\lambda-\lambda z) e^{-z} [\Gamma_z(5/2) - \Gamma_z(7/2)] \\ & - \frac{3}{2} \frac{1}{z^{3/2}} (1-\lambda z) e^{-z} [\Gamma_z(3/2) - \Gamma_z(5/2)] \end{aligned} \right\}$$

36 The balance equation 32 then becomes:

$$-\frac{m}{M} \omega_0 A z^2 (1-\lambda z) e^{-z} + \frac{2}{3} \frac{A}{\omega_0} e^{\lambda} z (1+\lambda-\lambda z) e^{-z}$$

$$+ \frac{A^2 i}{\omega_0^3} \cdot \frac{8}{3} \ln \frac{t}{2a_i} K_3 = 2\pi j$$

$$\text{where } \frac{8}{3} \frac{A^2 i}{\omega_0^3} \ln \frac{t}{2a_i} K_3 = \frac{I_1 - I_2}{4\pi N Q} = 2\pi i \omega_0^3 K$$

$K_3$  is  $z^{3/2}$  times the expression in braces, and is dimensionless.

Eq. 36 is of the form

$$37 \quad -C z^2 (1-\lambda z) e^{-z} + F z (1+\lambda-\lambda z) e^{-z} + m K_3 = 2\pi j$$

The functions  $z^2(1-\lambda z)e^{-z}$ ,  $z(1+\lambda-\lambda z)e^{-z}$ , and  $K_3$  are plotted in fig's. 11 and 12

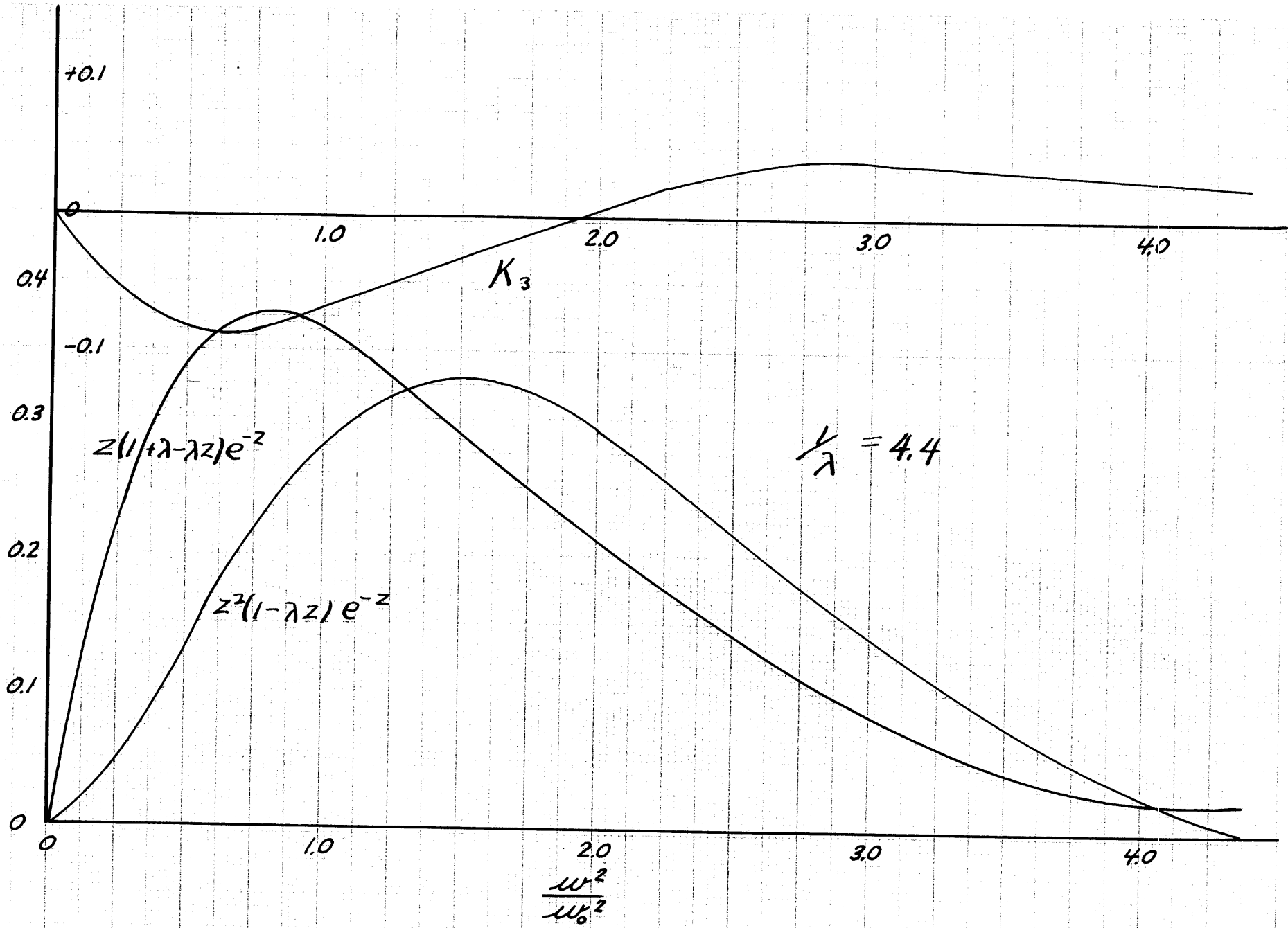


Fig. 11

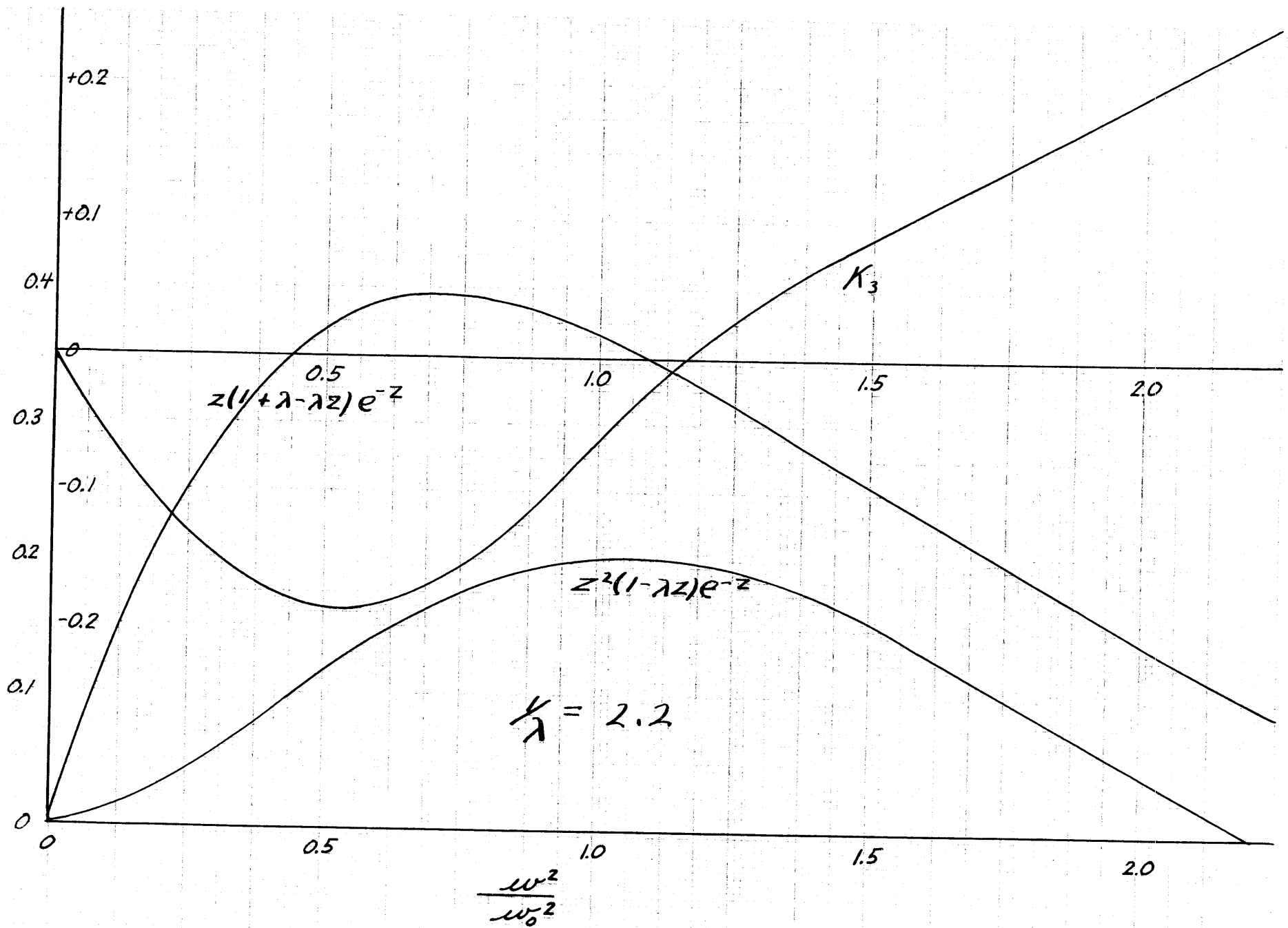


Fig. 12



Suppose that the values of  $N, \eta, E,$  and  $a_0$  are given. If the function  $A (1 - w^2/w_0^2) e^{-w^2/w_0^2}$  is the correct distribution function for this discharge, equation 36 will be satisfied for all values of  $z$  (or  $w^2/w_0^2$ ). But the function was assumed ad hoc. Hence the problem must be considered from the opposite point of view. We ask, "Are there any values of the parameters  $N, \eta, a_0, E$  for which equation 36 is satisfied for all or nearly all values of  $z$ ?" If such a set be found we will have a problem to which our  $f_0$  is a solution. There are really only three independent parameters in eq. 36, as is seen when it is written as eq. 37. They are  $\lambda, C/F,$  and  $M/F$ . The problem is to find what, if any, values of these parameters will make the sum of the terms on the left of 36 nearly constant -- and positive. We can pick  $\lambda$  arbitrarily, and then may be able to find, by a least squares method, a reasonably satisfactory pair of values for  $C/F$  and  $M/F$ . This procedure was followed. In figs. <sup>13 & 14</sup> the values of this sum are plotted for the sets  $\lambda, C/F, M/F$  tabulated below.

	$1/\lambda$	2.2	4.4
$C/F$		0.63	0.42
$M/F$		6.56	33.0

Table III

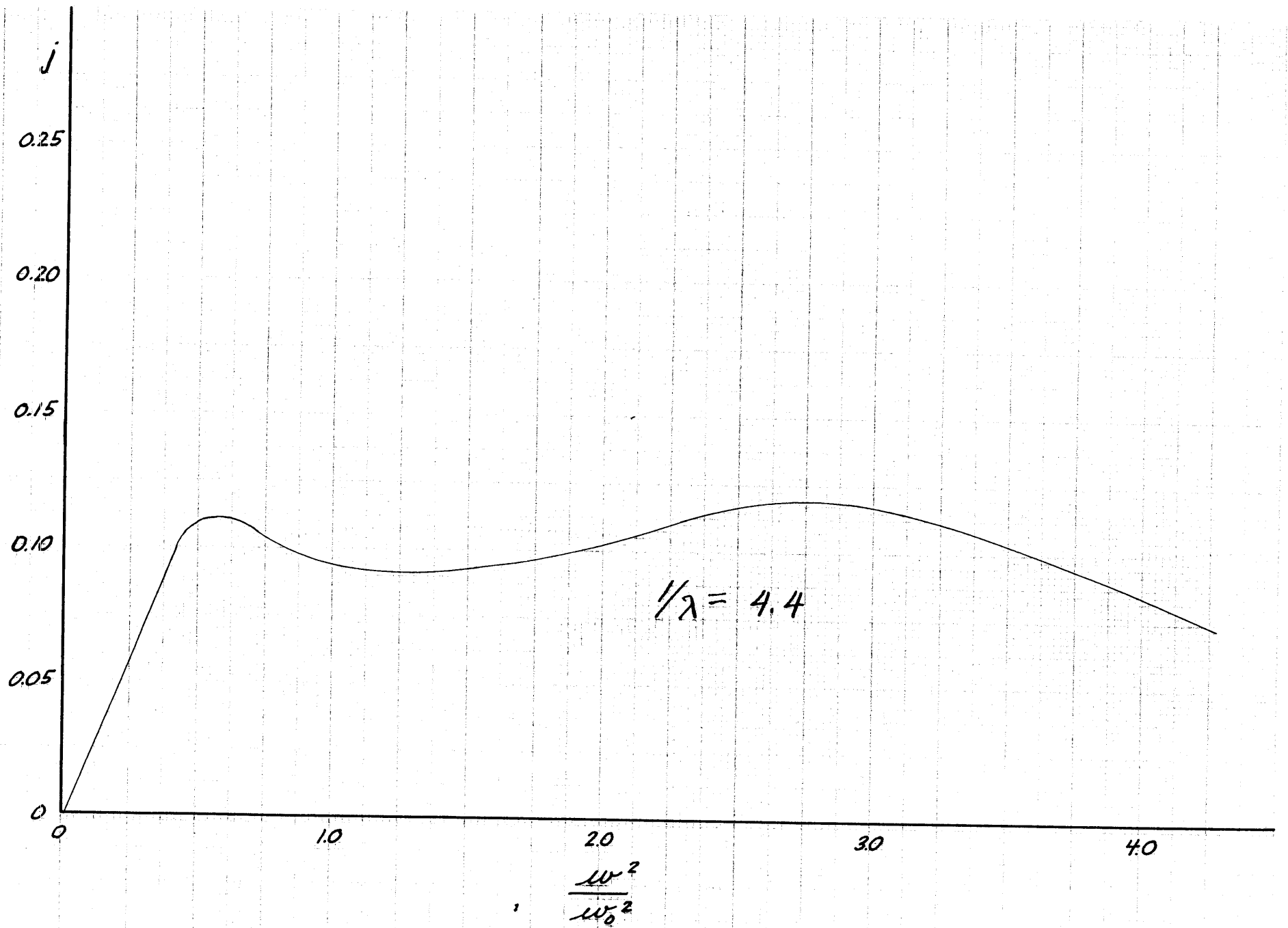


Fig. 13

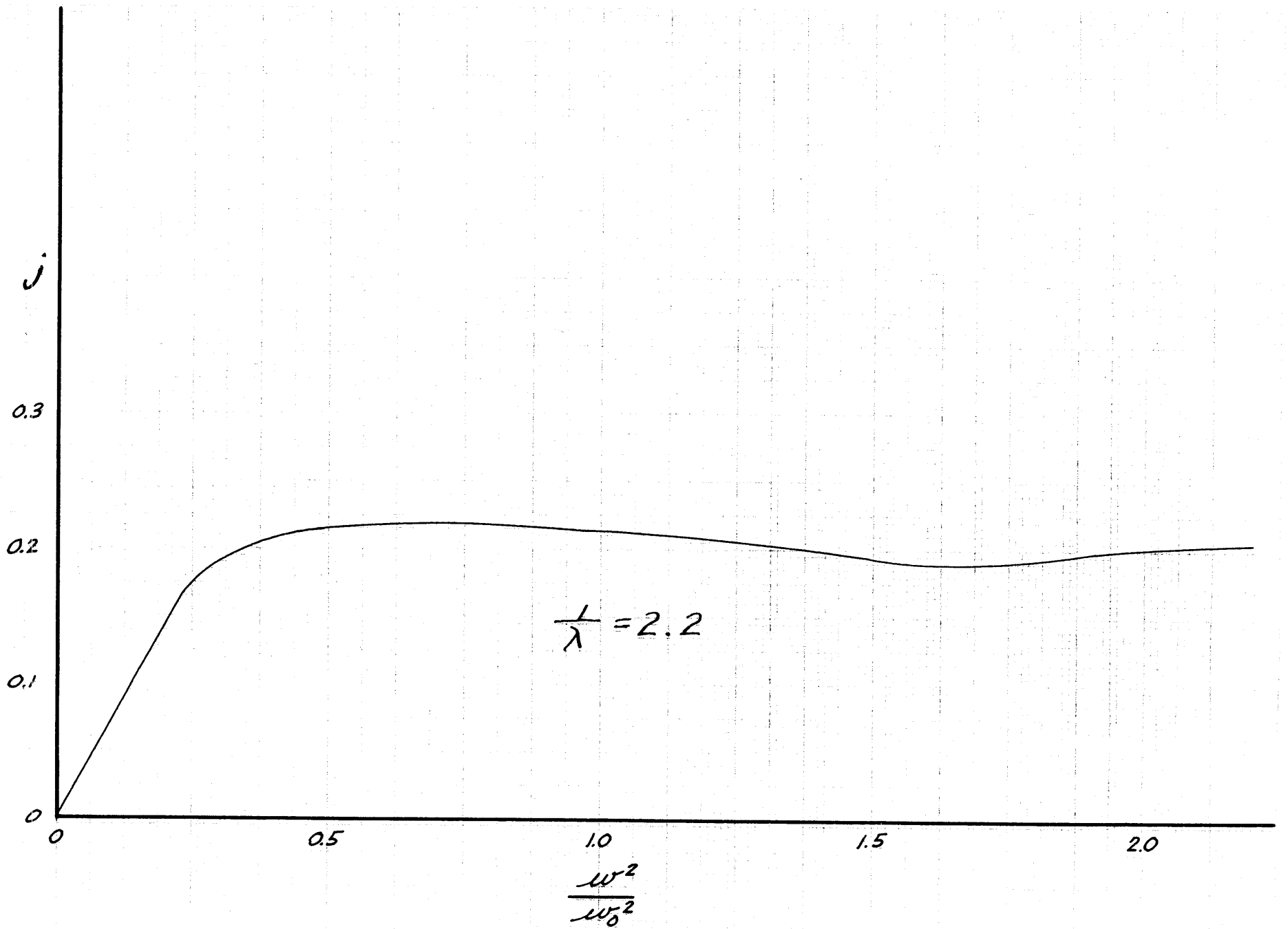
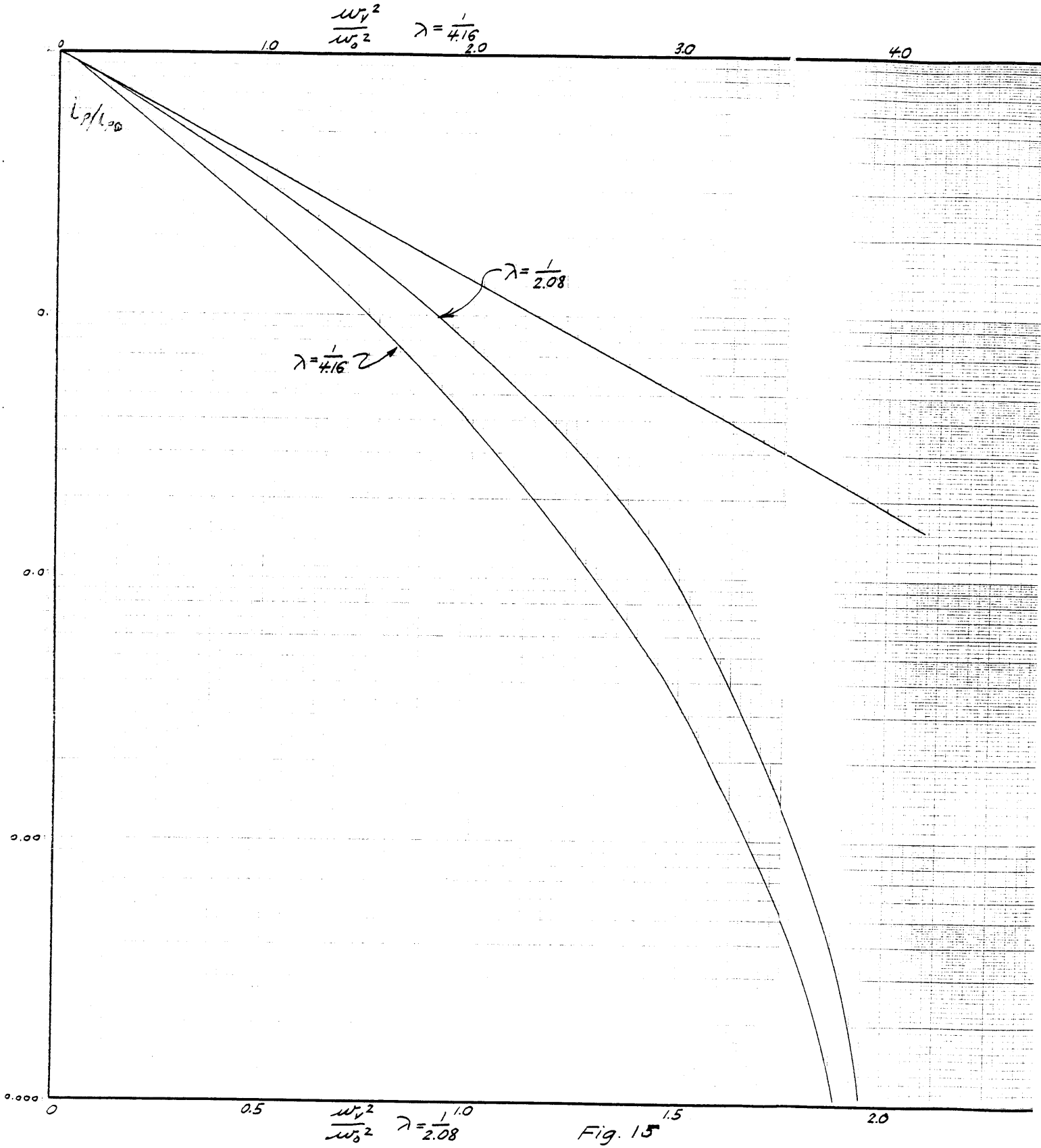


Fig. 14

The curves are fairly level except at the lower end. The degree to which they approximate a horizontal line is a measure of the correctness of the chosen function. It is seen that the fit is reasonably good, except the distribution function gives too few slow electrons.

From the tabulated values of the ratios, the relations among the parameters of the discharge can be found.

Semi- logarithmic curves of probe current vs voltage are plotted for this type of distribution. (fig./5).



## VII Conclusions

We have considered some of the processes taking place in a simplified model of the plasma or positive column region of an electrical discharge through gas at low pressure. The most difficult problem arose when the collisions of particles of equal mass were considered.

The first part of this paper deals with this question in a general way. We have been interested primarily in the velocity distribution function of the electrons. Two balance equations in this unknown function were set up. One, expressing conservation of energy, required that the number of electrons acquiring in unit time a speed greater than any value,  $w$ , be equal to the number losing energy and falling below this speed. The second expressed balance of the momentum parallel to the field. From these, one non-linear integral equation was derived.

The geometry of the mutual scattering process was examined and the necessary integrations performed insofar as these depended only on this geometry. This much of the work applies to any type of cross section function irrespective of its dependence on relative velocity or scattering angle. As an example, the calculation for

the case of elastic spheres was carried out up to the point where integrations over the distribution function,  $f$ , became necessary.

In the second part of this thesis the cross section function  $\sigma$  is derived for electron-electron collisions. This was done by means of a wave-mechanical approach. For the interaction potential a shielded Coulomb field, 
$$v = \frac{e^2}{r} e^{-r/a}$$
 was used. Next, those integrations depending on  $\sigma$  but not on  $f$  were carried out. In view of the relative complexity of the resulting expressions, two different approximations were made. One was valid when the velocity  $v$  of the particle was very nearly equal the velocity  $w$ . This corresponded to grazing collisions or small angle scattering. The other was used when  $v$  was appreciably different from  $w$ , and corresponded to close collisions, or large angle scattering by a pure Coulomb field.

In the third part we attempt to find functions which are solutions of the problem. When the term representing inelastic impacts is disregarded in the equation, it is found that the function

$$f_0 = \frac{A}{2\pi w_0^3} e^{-g w^2/w_0^2 - (1-g) w^4/2w_0^4}$$

is a fairly good approximate solution. Here the parameter

$g$  is a measure of the relative importance of the mutual collisions. When the inelastic term is large the function

$$f_0 = \frac{A}{2\pi w_0^3} \left(1 - \frac{w^2}{w_i^2}\right) e^{-w^2/w_0^2}$$

was acceptable, though it gave too few low velocity electrons.

These trial functions were not expected to be exact solutions but they served to test the assumptions and to indicate the relative importance of the various processes.

The result of the calculations on mutual collisions can be clearly divided into two parts. The first is a true collision process with large angle scattering by a Coulomb field. It is independent of the approximations, and of the size,  $\alpha$ , of the potential hill. The second part represents small angle scattering and is of a diffraction nature. It contains the factor  $\frac{\pi t d}{\lambda_w}$ . Here  $t$  is the width of the strip in the  $u^2, v^2$  plane, and  $\lambda_w$  is the electron wavelength. Thus this depends both on the approximation involving  $t$ , and on the assumptions concerning  $\alpha$ . The dependence is not critical, however. A change in the value of  $\alpha$  from  $10^{-4}$  to 1 produced only a relatively small change in the final result. It turns out that this second part is the largest over most of the



range of velocities. In other words, the velocity distribution is maintained mainly by many exchanges of small amounts of energy rather than by less frequent large changes.

The mutual interaction of electrons becomes important when  $\frac{n}{N Q}$  is large and  $\frac{E}{N Q}$  small, where  $n$  is the electron density,  $E$  the field strength,  $N$  the atomic density, and  $Q$  the atomic elastic cross section for electrons. This can be pictured as follows. The electron-electron collisions tend to set up a Maxwell distribution. This is disturbed by the collisions with atoms. The number of encounters of a given electron with others is proportional to  $n$ , and the number of collisions with atoms to  $NQ$ . The first fraction measures the relative frequency of the two types of collision. The field also tends to disturb the equilibrium distribution. It adds energy and imposes a drift on the electrons. A measure of its effect is the field strength times the mean free path. This is  $\frac{E}{N Q}$ .

The Townsend discharge, with very small  $n$ , is an example of the cases where this mutual interaction is certainly negligible.

In the plasma of an arc, on the other hand, the conditions are such that this mutual collision process is

very important indeed. The calculations show that it very largely determines the form of the distribution. A comparison with one of the rare complete sets of experimental data showed a fairly good check.

A single electron-electron collision cannot change the mean energy of the distribution. But the aggregate of such encounters does change the form of the distribution and, in particular, alters the most probable velocity. In this way the rate is changed at which other processes take place and the mutual encounters may very well lead to a different mean energy.

When inelastic impacts were taken into account in finding an approximate distribution function, it was considered that their effect was merely to cause electrons acquiring a velocity greater than some value  $w$ , to lose all their energy. A more exact treatment is desirable. This would be possible using the complete balance equation 13 with approximate ionization and excitation probabilities. In this way the form of the distribution above the critical potentials could be found and the intensity of ionization calculated. An examination by power series of the abridged equation 31 shows that  $f$  must have a logarithmic singularity at the origin. This is caused by the piling up of low velocity electrons, which results from

the crude way in which the inelastic collisions are handled. The more correct treatment would remove this difficulty.

List of Symbols

First ap-  
pearing on page:

A	Normalization factor	--
a	= $\lambda/2\pi\alpha$	25
$a_0; a_1$	a at $w = w_0, w_1$	34, 43
$B_1 B_2$	Integrals	11, 12
$B'_j, (j = 1, 2, 3, 4)$		15
C		47
E	Electric field strength	5
$\mathcal{E}$	= $E e/m NQ$	30
e	base of natural logarithms	--
$e$	charge of electron	5
f	distribution function	3
$f_0, f_1$	parts of f	3
F		47
g	parameter in $f_0$	32
H	energy	13
$H_j, k$	Integrals, ( $j = 1, 2, 3, 4; k = 1, 2$ )	27
h	Planck's constant	21
i	degree of ionization	30
J	Ionization integral	7
j	" constant	40
K	Sum of integrals	30
$K_1$	" " "	34
$K_2$	" " "	41
$K_3$	" " "	47

k	wave number	21
$k_1, k_2$	defined on	35
M	mass of atom	3
$m$		47
m	mass of electron	3
N	atomic density	5
$\eta$	electronic "	3
$\mathcal{Q}$	elastic cross section	6
$\mathcal{Q}_e, \mathcal{Q}_i$	excitation, ionization cross section	6
r	distance	20
$S_j$	integrals	27
s	velocity, $s^2 = u^2 + N^2 - w^2$	17
t	$2t w^2 =$ width of strip	27
dt	element of time	5
U	velocity of center of gravity	7
u	velocity	3
$\bar{u}, \bar{u}'$	half relative velocity	7
$u', v, v'$	velocities	7
$u_x$	velocity parallel field	4
$u_e, u_i$	excitation, ionization velocities	6
V	potential	20
V	retarding potential, probe	39
$V_o$	energy (volts)	38
$V_r$	energy of relative motion (volts)	22
w, or $\mu$	velocity	4

$w_0$	velocity, parameter in $f_0$	32
$w_1$	" " " $f_0$	40
$K$	$(u^2 + v^2)/\sqrt{2}$	27
$y$	$(u^2 - v^2)/\sqrt{2}$	27
$y$	dummy variable of integration, = $v^2 / w_0^2$	46
$z$	$w^2 / w_0^2$	33, 41
$\alpha$	radius of potential hole	30
$\beta$	= $h/4\pi\alpha$ $m\bar{u} = \bar{\lambda}/2\pi\alpha$	22
$d\gamma$	element of velocity space	3
$\delta$	angle between $U$ and $\bar{u}$	8
$\chi, \eta$	polar angles about $U$	7, 8
$\theta, \phi$	" " " $\bar{u}$	7
$\lambda, \mu$	" " " $U$	8
$\nu$	angle	8
$\lambda$	$w_0^2/w_1^2$	40
$\bar{\lambda}, \lambda_w, \lambda_0$	electron wavelengths	22, 25, 23
$\Psi, \psi, \psi_0, \psi_1$	wave functions	21
$\rho$	defined on	33
$\sigma$	mutual cross section	7
$\sigma_a$	elastic partial cross section	5
$\chi_0$	limit of $\chi$	10
$\omega, \psi$	polar angles about $E$ (refer to $u$ )	3

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## APPENDIX I

$$\sigma(\bar{u}, \theta) = \frac{e^4}{4m^2\bar{u}^4} \frac{1}{(1 + 2\beta^2 - \cos\delta \cos\lambda + \sin\delta \cos\lambda \cos\mu)^2} \quad (50)$$

$$B_1 = \int_0^{\lambda_0} \sin\lambda \, d\lambda \int_{-\pi}^{\pi} \sigma \, d\mu$$

$$B_2 = \int_{\lambda_0}^{\pi} \sin\lambda \, d\lambda \int_{-\pi}^{\pi} \sigma \, d\mu$$

$$\text{Let } L = \int_{-\pi}^{\pi} \sigma \, d\mu$$

Setting  $\gamma^2 = 1 + 2\beta^2$ , the integral is of the form:

$$L = \frac{e^4}{4m^2\bar{u}^4} \int_{-\pi}^{\pi} \frac{d\mu}{(p + q \cos\mu)^2}$$

with  $p = \gamma^2 - \cos\delta \cos\lambda$ ,  $q = \sin\delta \sin\lambda$

Then, by Pierce's Short Table of Integrals #'s 308, 300:

$$L = \frac{e^4}{4m^2\bar{u}^4} \cdot \frac{2p}{(p^2 - q^2)^{3/2}} \left[ \tan^{-1} \frac{\sqrt{p^2 - q^2} \tan \frac{1}{2}\mu}{p - q} \right]_{-\pi}^{\pi}$$

$$L = \frac{2\pi e^4}{4m^2\bar{u}^4} \cdot \frac{p}{(p^2 - q^2)^{3/2}}$$

$$\text{Now, } p^2 - q^2$$

$$= \gamma^4 - 2\gamma^2 \cos\delta \cos\lambda + \cos^2\delta \cos^2\lambda - \sin^2\lambda \sin^2\delta$$

$$= \gamma^4 - \sin^2\delta - 2\gamma^2 \cos\delta \cos\lambda + \cos^2\lambda$$

$$L = \frac{2\pi e^4}{4m^2\bar{u}^4} \frac{\gamma^2 - \cos\delta \cos\lambda}{\{\gamma^4 - \sin^2\delta - 2\gamma^2 \cos\delta \cos\lambda + \cos^2\lambda\}^{3/2}}$$

But  $B_1 = \int_0^{\lambda_0} L \sin \lambda d\lambda$ , and setting  $x = \cos \lambda$

$$B_1 = \frac{2\pi e^4}{4m^2 \bar{u}^4} \int_{\cos \lambda_0}^1 \frac{(\gamma^2 - x \cos \delta) dx}{(p' + q'x + x^2)^{3/2}}$$

where  $p' = \gamma^4 - \sin^2 \delta$ ,  $q' = -2\gamma^2 \cos \delta$ .

By Pierce's 162, 170

$$\begin{aligned} B_1 &= \frac{2\pi e^4}{4m^2 \bar{u}^4} \left[ \frac{\gamma^2 \cdot 2(2x + q') + 2\cos \delta (q'x + 2p')}{(4p' - q'^2)(p' + q'x + x^2)^{1/2}} \right]_{\cos \lambda_0}^1 \\ &= \frac{2\pi e^4}{4m^2 \bar{u}^4} \left[ \frac{2\gamma^2(2x - 2\gamma^2 \cos \delta) + 2\cos \delta(-2\gamma^2 x \cos \delta + 2\gamma^4 - 2\sin^2 \delta)}{(4\gamma^4 - 4\sin^2 \delta - 4\gamma^4 \cos^2 \delta)(\gamma^4 - \sin^2 \delta - 2\gamma^2 x \cos \delta + x^2)^{1/2}} \right]_{\cos \lambda_0}^1 \\ &= \frac{2\pi e^4}{4m^2 \bar{u}^4} \left[ \frac{\gamma^2 x - \cos \delta}{(\gamma^4 - 1)(\gamma^4 - 1 + \cos^2 \delta - 2\gamma^2 x \cos \delta + x^2)^{1/2}} \right]_{\cos \lambda_0}^1 \\ &= \frac{2\pi e^4}{4m^2 \bar{u}^4 (\gamma^4 - 1)} \left[ 1 - \frac{\gamma^2 \cos \lambda_0 - \cos \delta}{(\gamma^4 - 1 + \cos^2 \delta - 2\gamma^2 \cos \delta \cos \lambda_0 + \cos^2 \lambda_0)^{1/2}} \right] \end{aligned}$$

$B_2$  is obtained the same way using  $\lambda_0$  and  $\pi$  as the limits of  $x$ .

$$B_2 = \frac{2\pi e^4}{4m^2 \bar{u}^4 (\gamma^4 - 1)} \left[ 1 + \frac{\gamma^2 \cos \lambda_0 - \cos \delta}{(\gamma^4 - 1 + \cos^2 \delta - 2\gamma^2 \cos \delta \cos \lambda_0 + \cos^2 \lambda_0)^{1/2}} \right]$$

Remembering that (eqs. 4 and 5)

$$\cos \lambda_0 = \frac{2w^2 - u^2 - v^2}{4U\bar{u}}, \quad \cos \delta = \frac{u^2 - v^2}{4U\bar{u}},$$

$$\gamma^2 = 2\beta^2 + 1 = \frac{w^2 a^2}{2\bar{u}^2} + 1,$$

and letting  $b^2 = \bar{u}^2 \beta^2 = \frac{\omega^2 a^2}{4}$ ,

$$B_{1,2} = \frac{2\pi e^4}{16m^2 b^2 (b^2 + \bar{u}^2)} \left\{ 1 \mp \frac{\left( \frac{2b^2}{\bar{u}^2} + 1 \right) \frac{2\omega^2 - u^2 - v^2}{4U\bar{u}} - \frac{u^2 - v^2}{4U\bar{u}}}{\left[ \frac{4b^2}{\bar{u}^4} (b^2 + \bar{u}^2) - \left( \frac{4b^2}{\bar{u}^2} + 2 \right) \left( \frac{2\omega^2 - u^2 - v^2}{16U^2 \bar{u}^2} \right) + \frac{(2\omega^2 - u^2 - v^2)^2 + (u^2 - v^2)^2}{16U^2 \bar{u}^2} \right]^{1/2}} \right\}$$

The upper sign is used for  $B_1$ , the lower for  $B_2$ .

$$(2\omega^2 - u^2 - v^2)^2 - 2(2\omega^2 - u^2 - v^2)(u^2 - v^2) + (u^2 - v^2)^2 = 4(\omega^2 - u^2)^2$$

$$\text{and } 4U^2 = 2u^2 + 2v^2 - 4\bar{u}^2$$

$$B_{1,2} = \frac{2\pi e^4}{16m^2 b^2 (b^2 + \bar{u}^2)} \left\{ 1 \mp \frac{b^2(2\omega^2 - u^2 - v^2) + \bar{u}^2(\omega^2 - u^2)}{\bar{u} \left[ 8b^4(u^2 + v^2) - b^2(2\omega^2 - u^2 - v^2)(u^2 - v^2) + \left\{ 8b^4(u^2 + v^2) - 16b^4 + (\omega^2 - u^2)^2 \right\} \bar{u}^2 - 16b^2 \bar{u}^4 \right]^{1/2}} \right\}$$

$$\text{Let: } g = 8b^4(u^2 + v^2) - b^2(2\omega^2 - u^2 - v^2)(u^2 - v^2)$$

$$h = 8b^2(u^2 + v^2) - 16b^4 + (\omega^2 - u^2)^2$$

$$k = 16b^2.$$

Then:

$$B_{1,2} = \frac{2\pi e^4}{16m^2 b^2} \left\{ 1 \mp \frac{b^2(2\omega^2 - u^2 - v^2) + \bar{u}^2(\omega^2 - u^2)}{\bar{u} (g + h\bar{u}^2 - k\bar{u}^4)^{1/2}} \right\}$$

Now:

$$B'_i = \int_{\chi_0}^{\pi - \chi_0} \bar{u} B_j \sin \chi \, d\chi = \frac{1}{uv} \int_{\bar{u}_1}^{\bar{u}_2} \bar{u}_j^2 B \, d\bar{u},$$

where for

$i = 1, j = 1,$	$2\bar{u}_1 = w - s,$	$2\bar{u}_2 = w + s$	Sign	-
$i = 2, j = 1,$	$2\bar{u}_1 = v - u,$	$2\bar{u}_2 = v + u$		-
$i = 3, j = 2,$	$2\bar{u}_1 = s - w,$	$2\bar{u}_2 = s + w$		+
$i = 4, j = 2,$	$2\bar{u}_1 = u - v,$	$2\bar{u}_2 = u + v$		+

$$B'_i = \frac{2\pi e^4}{4m^2 \ell^2 uv} \int_{\bar{u}_1}^{\bar{u}_2} \left\{ \frac{\bar{u}^2}{\ell^2 + \bar{u}^2} \mp \frac{\ell^2 \bar{u} (2w^2 - u^2 - v^2) + \bar{u}^3 (w^2 - u^2)}{(\ell^2 + \bar{u}^2) [g + h\bar{u}^2 - k\bar{u}^4]^{\frac{1}{2}}} \right\} d\bar{u}$$

Setting  $z = \bar{u}^2,$

$$B'_i = \frac{2\pi e^4}{4m^2 \ell^2 uv} \left\{ \int_{\bar{u}_1}^{\bar{u}_2} d\bar{u} - \int_{\bar{u}_1}^{\bar{u}_2} \frac{\ell^2 d\bar{u}}{\ell^2 + \bar{u}^2} \mp \int_{z_1}^{z_2} \frac{z(w^2 - u^2) dz}{2\sqrt{z} \sqrt{g + hz - kz^2}} \mp \frac{1}{2} \int_{z_1}^{z_2} \frac{\ell^2 (w^2 - v^2) dz}{z(\ell^2 + z) \sqrt{g + hz - kz^2}} \right\}$$

By Pierce's 49, 161

$$B'_i = \frac{2\pi e^4}{4m^2 \ell^2 uv} \left\{ \bar{u}_2 - \bar{u}_1 - \left[ \ell \tan^{-1} \frac{\bar{u}}{\ell} \right]_{\bar{u}_1}^{\bar{u}_2} \pm \left[ \frac{w^2 - v^2}{2\sqrt{k}} \tan^{-1} \frac{h - 2kz}{2\sqrt{k} \sqrt{g + hz - kz^2}} \right]_{z_1}^{z_2} \right. \\ \left. \pm \left[ \frac{\ell^2 (w^2 - v^2)}{2\sqrt{\ell^2 h - g + k\ell^4}} \tan^{-1} \frac{2g - \ell^2 h + (h + 2\ell^2 k)z}{2\sqrt{\ell^2 h - g + k\ell^4} \sqrt{g + hz - kz^2}} \right]_{z_1}^{z_2} \right\} \quad (51)$$

Examine terms in order and by quadrant:

Quadrant 1	$\bar{u}_2 - \bar{u}_1 = s$	(52)
" 2	" = u	
" 3	" = w	
" 4	" = v	

The first arc tangent term is

$$\tan \frac{\bar{u}_2}{\ell} - \tan \frac{\bar{u}_1}{\ell} = \tan \frac{(\bar{u}_2 - \bar{u}_1)\ell}{\bar{u}_2 \bar{u}_1 + \ell^2}$$

Then this term becomes, by quadrants,

$$\left. \begin{array}{l}
 \text{Quadrant 1} \quad - b \tan^{-1} \frac{4bs}{w^2 - s^2 + 4b^2} \\
 \text{" 2} \quad - b \tan^{-1} \frac{4bu}{v^2 - u^2 + 4b^2} \\
 \text{" 3} \quad - b \tan^{-1} \frac{4bw}{s^2 - w^2 + 4b^2} \\
 \text{" 4} \quad - b \tan^{-1} \frac{4bv}{u^2 - v^2 + 4b^2}
 \end{array} \right\} \quad (53)$$

Consider the second arc tangent term:

$$\sqrt{k} = 4b$$

$$h - 2kz = 8b^2(u^2 + v^2) - 16b^4 + (w^2 - u^2)^2 - 32b^2\bar{u}^2$$

Inserting the appropriate values of  $\bar{u}$ , and  $\bar{u}_2$

	$h - 2kz_1$	$h - 2kz_2$
Quadr. 1 or 3	$(w^2 - u^2)^2 - 16b^4 + 16b^2ws$	$(w^2 - u^2)^2 - 16b^4 - 16b^2ws$
" 2 or 4	$(w^2 - u^2)^2 - 16b^4 + 16b^2uv$	$(w^2 - u^2)^2 - 16b^4 - 16b^2uv$

Also, the radical is

$$\begin{aligned}
 g + hz - kz^2 &= 8b^4(u^2 + v^2) - b^2(2w^2 - u^2 - v^2)(u^2 - v^2) \\
 &\quad + \{8b^2(u^2 + v^2) - 16b^4 + (w^2 - u^2)^2\}\bar{u}^2 - 16b^2\bar{u}^4
 \end{aligned}$$

Inserting  $\bar{u} = \bar{u}_2 = \frac{1}{2}(w+s)$ , for quadr. 1 this becomes

$$\begin{aligned}
 &b^4 \{8(u^2 + v^2) - 4(w+s)^2\} + b^2 \{-(w+s)^4 + 2(u^2 + v^2)(w+s)^2 - (2w^2 - u^2 - v^2)(u^2 - v^2)\} \\
 &+ \frac{(w^2 - u^2)^2}{4}(w+s)^2
 \end{aligned}$$

Now,

$$\begin{aligned}
 8(u^2 + v^2) - 4(w+s)^2 &= 8u^2 + 8v^2 - 4w^2 - 8ws - 4u^2 - 4v^2 + 4w^2 \\
 &= 4w^2 - 8ws + 4s^2 = 4(w-s)^2
 \end{aligned}$$

And

$$\begin{aligned}
 & -(w+s)^4 + 2(u^2+v^2)(w+s)^2 - (2w^2-u^2-v^2)(u^2-v^2) \\
 &= (w+s)^2 [2(u^2+v^2) - (w^2+2ws+u^2+v^2-w^2)] - (w^2-s^2)(u^2-v^2) \\
 &= (w+s)^2(w-s)^2 - (w^2-s^2)(u^2-v^2) \\
 &= (w^2-s^2)[(w^2-s^2) - u^2+v^2] = 2(w^2-s^2)(w^2-u^2) \\
 \text{or } [g + hz_2 - kz_2^2]_1 & \\
 &= \frac{(w^2-u^2)(w+s)^2}{4} + 2b^2(w^2-s^2)(w^2-u^2) + 4b^4(w-s)^2 \\
 &= \frac{1}{4} \left\{ (w^2-u^2)(w+s) + 4b^2(w-s) \right\}^2
 \end{aligned}$$

for the first quadrant.

Similarly, for the lower limit  $\bar{u}_1 = \frac{1}{2}(w-s)$ ,

$$[g + hz_1 - kz_1^2]_1 = \frac{1}{4} \left\{ (w^2-u^2)(w-s) + 4b^2(w+s) \right\}^2.$$

In quadrant 2

$$\begin{aligned}
 [g + hz_2 - kz_2^2]_2 &= 4b^4 \left\{ 2(u^2+v^2) - (v+u)^2 \right\} \\
 &+ b^2 \left\{ -(v+u)^4 + 2(u^2+v^2)(v+u)^2 - (2w^2-u^2-v^2)(u^2-v^2) \right\} + \frac{(w^2-u^2)^2(v+u)^2}{4}.
 \end{aligned}$$

$$\text{But } 2(u^2+v^2) - (v+u)^2 = (v-u)^2$$

$$\begin{aligned}
 \text{and } & -(v+u)^4 + 2(u^2+v^2)(v+u)^2 - (2w^2-u^2-v^2)(u^2-v^2) \\
 &= (v+u)^2(v-u)^2 + (v^2-u^2)(2w^2-u^2-v^2) \\
 &= (v^2-u^2) \left\{ v^2u^2 + 2w^2 - u^2 - v^2 \right\} = 2(v^2-u^2)(w^2-u^2)
 \end{aligned}$$

Therefore,  $[g + hz_2 - kz_2^2]_2$

$$\begin{aligned}
 &= \frac{(w^2-u^2)^2(v+u)^2}{4} + 2b^2(w^2-u^2)(v^2-u^2) + 4b^4(v-u)^2 \\
 &= \frac{1}{4} \left\{ (w^2-u^2)(v+u) + 4b^2(v-u) \right\}^2.
 \end{aligned}$$

Likewise, for the same quadrant,  
 $[g + hz_1 - kz_1^2]_2 = \frac{1}{4} \{ (w^2 - u^2)(v - u) + 4b^2(v + u) \}^2$

The expressions for quadrant 3 are the same as for quadrant 1; and for quadrant 4 as for 2, since  $\bar{u}_1$  and  $\bar{u}_2$  enter only in their squares and fourth powers. But they can be written:

$$[g + hz_2 - kz_2^2]_3 = \frac{1}{4} \{ (u^2 - w^2)(s + w) + 4b^2(s - w) \}^2$$

$$[g + hz_1 - kz_1^2]_3 = \frac{1}{4} \{ (u^2 - w^2)(s - w) + 4b^2(s + w) \}^2$$

$$[g + hz_2 - kz_2^2]_4 = \frac{1}{4} \{ (u^2 - w^2)(u + v) + 4b^2(u - v) \}^2$$

$$[g + hz_1 - kz_1^2]_4 = \frac{1}{4} \{ (u^2 - w^2)(u - v) + 4b^2(u + v) \}^2$$

$$\text{Let } a_1^2 = a_2^2 = -a_3^2 = -a_4^2 = w^2 - u^2.$$

The expressions above are of the form

$$\frac{1}{4} (a^2 \cdot 2\bar{u}_a + 4b^2 \cdot 2\bar{u}_b)^2$$

$$\text{where when } \begin{array}{ll} a = 1, & b = 2 \\ a = 2, & b = 1 \end{array}$$

Then the second arc-tangent terms in  $B_j'$  become, including the ambiguous sign

Quad.

$$\textcircled{1} + \frac{\alpha_1^2}{8b} \left\{ \tan^{-1} \frac{\alpha_1^4 - 16b^4 - 16b^2ws}{4b[\alpha_1^2(w+s) + 4b^2(w-s)]^{\frac{1}{2}}} - \tan^{-1} \frac{\alpha_1^4 - 16b^4 + 16b^2ws}{4b[\alpha_1^2(w-s) + 4b^2(w+s)]^{\frac{1}{2}}} \right\}$$

$$\textcircled{2} + \frac{\alpha_2^2}{8b} \left\{ \tan^{-1} \frac{\alpha_2^4 - 16b^4 - 16b^2uv}{4b[\alpha_2^2(v+u) + 4b^2(v-u)]^{\frac{1}{2}}} - \tan^{-1} \frac{\alpha_2^4 - 16b^4 + 16b^2uv}{4b[\alpha_2^2(v-u) + 4b^2(v+u)]^{\frac{1}{2}}} \right\}$$



Quad.

$$(3) -\frac{a_3^2}{8b} \left\{ \tan^{-1} \frac{a_3^4 - 16b^4 - 16b^2ws}{4b[a_3^2(s+w) + 4b^2(s-w)]^{\frac{1}{2}}} - \tan^{-1} \frac{a_3^4 - 16b^4 + 16b^2ws}{4b[a_3^2(s-w) + 4b^2(s+w)]^{\frac{1}{2}}} \right\}$$

$$(4) +\frac{a_4^2}{8b} \left\{ \tan^{-1} \frac{a_4^4 - 16b^4 - 16b^2uv}{4b[a_4^2(u+v) + 4b^2(u-v)]^{\frac{1}{2}}} - \tan^{-1} \frac{a_4^4 - 16b^4 + 16b^2uv}{4b[a_4^2(u-v) + 4b^2(u+v)]^{\frac{1}{2}}} \right\}$$

(Refer to eq. 51, table 54, and the list for  $g+hz-kz^2$ )

These can be reduced to

Quad.

$$\left. \begin{array}{l} 1 \quad + \frac{a_1^2}{4b} \tan^{-1} \frac{-4bs}{a_1^2 + 4b^2} \\ 2 \quad \frac{a_2^2}{4b} \tan^{-1} \frac{-4bu}{a_2^2 + 4b^2} \\ 3 \quad \frac{a_3^2}{4b} \tan^{-1} \frac{-4bw}{a_3^2 + 4b^2} \\ 4 \quad \frac{a_4^2}{4b} \tan^{-1} \frac{-4bv}{a_4^2 + 4b^2} \end{array} \right\} (55)$$

Consider now the third arc-tangent term

$$\begin{aligned} b^2h - g + b^4k &= b^2 \{ (w^2 - u^2)^2 + (2w^2 - u^2 - v^2)(u^2 - v^2) \} \\ &= b^2 (w^4 - 2w^2u^2 + u^4 + 2w^2u^2 - 2w^2v^2 - u^4 + v^4) \\ &= b^2 (w^2 - v^2)^2. \end{aligned}$$

$$\begin{aligned} \text{Also: } 2g - b^2h + (h + 2b^2k)z &= 16b^4(u^2 + v^2) - 2b^2(2w^2 - u^2 - v^2)(u^2 - v^2) \\ &\quad - 8b^4(u^2 + v^2) + 16b^6 - b^2(w^2 - u^2)^2 + [8b^2(u^2 + v^2) - 16b^4 + (w^2 - u^2)^2 + 32b^4] \bar{u}^2 \\ &= \frac{1}{4} \{ 64b^6 + 16b^4 [2(u^2 + v^2) + 4\bar{u}^2] + 4b^2 [2(2w^2 - u^2 - v^2)(v^2 - u^2) - (w^2 - u^2)^2 + 8(v^2 + u^2)\bar{u}^2] \\ &\quad + 4(w^2 - u^2)^2 \bar{u}^2 \}. \end{aligned}$$

If now we let

$$\beta_1^2 = -\beta_2^2 = -\beta_3^2 = \beta_4^2 = w^2 - v^2$$

then

$$-(2w^2 - u^2 - v^2)(u^2 - v^2) = (\alpha^2 + \beta^2)(\alpha^2 - \beta^2) = \alpha^4 - \beta^4$$

for all quadrants.

$$2g - \ell^2 h + (h + 2\ell^2 k)z$$

$$= \frac{1}{4} \left\{ 64\ell^6 + 16\ell^4 [2(u^2 + v^2) + 4\bar{u}^2] + 4\ell^2 [2(\alpha^4 - \beta^4) - \alpha^4 + 8(v^2 + u^2)\bar{u}^2] + 4\alpha^4 \bar{u}^2 \right\}$$

The expression  $\sqrt{\ell^2 h - g + k\ell^4}$  appears both outside and under the  $\tan^{-1}$  operator, and can be replaced in both places by  $\ell\beta^2 = \pm \ell(w^2 - v^2)$ ; also remembering that  $\sqrt{g + hz - z^2}$  has the form

$$\frac{1}{2} (\alpha^2 \cdot 2\bar{u}_\alpha + 4\ell^2 \cdot 2\bar{u}_\beta),$$

the third arc-tangent terms become

$$\pm \frac{\ell}{2} \left\{ \tan^{-1} \frac{\frac{1}{4} \{ 64\ell^6 + 16\ell^4 [2(v^2 + u^2) + 4\bar{u}_2^2] + 4\ell^2 [2(\alpha^4 - \beta^4) - \alpha^4 + 8(v^2 + u^2)\bar{u}_2^2] + 4\alpha^4 \bar{u}_2^2 \}}{2\ell\beta^2 \cdot \frac{1}{2} (\alpha^2 \cdot 2\bar{u}_2 + 4\ell^2 \cdot 2\bar{u}_1)} \right. \\ \left. - \tan^{-1} \frac{\frac{1}{4} \{ 64\ell^6 + 16\ell^4 [2(v^2 + u^2) + 4\bar{u}_1^2] + 4\ell^2 [2(\alpha^4 - \beta^4) - \alpha^4 + 8(v^2 + u^2)\bar{u}_1^2] + 4\alpha^4 \bar{u}_1^2 \}}{2\ell\beta^2 \cdot \frac{1}{2} (\alpha^2 \cdot 2\bar{u}_1 + 4\ell^2 \cdot 2\bar{u}_2)} \right\},$$

where the + sign now applies to quadrants 1 and 3, and the - sign to quadrants 2 and 4.

These can be reduced to

$$\mp \ell \tan^{-1} \frac{4\ell\beta^2 (\bar{u}_2 - \bar{u}_1)}{(\alpha^2 + 4\ell^2)(\alpha^2 + \beta^2 + 4\ell^2) + 16\ell^2 (\bar{u}_2 - \bar{u}_1)^2}$$

$$= \mp b \left[ \tan^{-1} \frac{4b(\bar{u}_2 - \bar{u}_1)}{a^2 + 4b^2} - \tan^{-1} \frac{4b(\bar{u}_2 - \bar{u}_1)}{a^2 + \beta^2 + 4b^2} \right] \quad (56)$$

Then, combining all terms making up the  $B_j$ 's (52, 53, 55, 56)

$$B_1' = \frac{2\pi e^4}{4m^2 b^2 uv} \left\{ s - b \tan^{-1} \frac{4bs}{a_1^2 + \beta_1^2 + 4b^2} - \frac{a_1^2}{4b} \tan^{-1} \frac{4bs}{a_1^2 + 4b^2} - b \tan^{-1} \frac{4bs}{a_1^2 + 4b^2} + b \tan^{-1} \frac{4bs}{a_1^2 + \beta_1^2 + 4b^2} \right\}$$

$$B_2' = \frac{2\pi e^4}{4m^2 b^2 uv} \left\{ u - b \tan^{-1} \frac{4bu}{a_2^2 + \beta_2^2 + 4b^2} - \frac{a_2^2}{4b} \tan^{-1} \frac{4bu}{a_2^2 + 4b^2} - b \tan^{-1} \frac{4bu}{a_2^2 + 4b^2} - b \tan^{-1} \frac{4bu}{a_2^2 + \beta_2^2 + 4b^2} \right\}$$

$$B_3' = \frac{2\pi e^4}{4m^2 b^2 uv} \left\{ w - b \tan^{-1} \frac{4bw}{a_3^2 + \beta_3^2 + 4b^2} - \frac{a_3^2}{4b} \tan^{-1} \frac{4bw}{a_3^2 + 4b^2} - b \tan^{-1} \frac{4bw}{a_3^2 + 4b^2} + b \tan^{-1} \frac{4bw}{a_3^2 + \beta_3^2 + 4b^2} \right\}$$

$$B_4' = \frac{2\pi e^4}{4m^2 b^2 uv} \left\{ v - b \tan^{-1} \frac{4bv}{a_4^2 + \beta_4^2 + 4b^2} - \frac{a_4^2}{4b} \tan^{-1} \frac{4bv}{a_4^2 + 4b^2} + b \tan^{-1} \frac{4bv}{a_4^2 + 4b^2} - b \tan^{-1} \frac{4bv}{a_4^2 + \beta_4^2 + 4b^2} \right\}$$

Now

$$\frac{b}{uv} \tan^{-1} \frac{4bu}{a_2^2 + \beta_2^2 + 4b^2} = \frac{b}{uv} \tan^{-1} \frac{4bu}{v^2 - u^2 + 4b^2}$$

$$\frac{b}{uv} \tan^{-1} \frac{4bv}{a_4^2 + \beta_4^2 + 4b^2} = \frac{b}{uv} \tan^{-1} \frac{4bv}{u^2 - v^2 + 4b^2}$$

These are symmetrical in  $u$  and  $v$ , and, as in the case where  $\sigma = \text{constant}$ , we are permitted to subtract the first from  $B_2'$  if we also subtract the second from  $B_4'$ .

Then finally:

$$B'_1 = \frac{2\pi e^4}{4m^2 b^2 uv} \left[ s - \frac{w^2 - u^2 + 4b^2}{4b} \tan^{-1} \frac{4bs}{w^2 - u^2 + 4b^2} \right]$$

$$B'_2 = \frac{2\pi e^4}{4m^2 b^2 uv} \left[ u - \frac{w^2 - u^2 - 4b^2}{4b} \tan^{-1} \frac{4bu}{w^2 - u^2 + 4b^2} \right]$$

$$B'_3 = \frac{2\pi e^4}{4m^2 b^2 uv} \left[ w - \frac{u^2 - w^2 + 4b^2}{4b} \tan^{-1} \frac{4bw}{u^2 - w^2 + 4b^2} \right]$$

$$B'_4 = \frac{2\pi e^4}{4m^2 b^2 uv} \left[ v - \frac{u^2 - w^2 - 4b^2}{4b} \tan^{-1} \frac{4bv}{u^2 - w^2 + 4b^2} \right]$$

Since  $2b = wa$ , these are the same as equations 19 of the main text.

## APPENDIX II

### STRIP INTEGRALS

The functions to be calculated are the  $H_{j,k}$ .  
By their definition, and eqs. 22

$$\frac{1}{\sqrt{2}} H_{j,1} = \int (B'_j uv) du^2$$

$$\frac{1}{2} H_{j,2} = \int (B'_j uv)(u^2 - w^2) du^2$$

Limits on  $u^2$  are

Quadrants 1 & 2	$w^2(1-t)$	to	$w^2$
" 3 & 4	$w^2$	to	$w^2(1+t)$

and the integration is carried out along the  
line  $u^2 + v^2 = (c^2 + 1)w^2$ .

Replace  $u^2$  by the dimensionless variable

$$z = \frac{u^2}{w^2} - 1.$$

Then along the path of integration,

$$v^2 = (c^2 + 1)w^2 - u^2 = w^2(c^2 - z)$$

$$s^2 = u^2 + v^2 - w^2 = w^2 c^2$$

$$du^2 = w^2 dz.$$

Inserting these in equations 19 for  $B'_j$ ,

$$B'_1 = \frac{2\pi e^4}{m^2 w^2 a^2 uv} \left[ wc - \frac{(-z + a^2)w}{2a} \tan^{-1} \frac{2ac}{-z + a^2} \right]$$

or

$$uvB'_1 = \frac{2\pi e^4}{m^2 \omega a^2} \left[ c - \frac{(-z+a^2)}{2a} \tan^{-1} \frac{2ac}{-z+a^2} \right]$$

$$uvB'_2 = \frac{2\pi e^4}{m^2 \omega a^2} \left[ \sqrt{z+1} - \frac{(-z-a^2)}{2a} \tan^{-1} \frac{2a\sqrt{z+1}}{-z+a^2} \right]$$

$$uvB'_3 = \frac{2\pi e^4}{m^2 \omega a^2} \left[ 1 - \frac{(z+a^2)}{2a} \tan^{-1} \frac{2a}{z+a^2} \right]$$

$$uvB'_4 = \frac{2\pi e^4}{m^2 \omega a^2} \left[ \sqrt{c^2-z} - \frac{(z-a^2)}{2a} \tan^{-1} \frac{2a\sqrt{c^2-z}}{z+a^2} \right]$$

Then, writing

$$\left. \begin{aligned} \frac{2\pi e^4 \omega}{m^2 a^2} \mathcal{H}_{j,1} &= \frac{1}{\sqrt{2}} H_{j,1} \\ \frac{2\pi e^4 \omega^3}{m^2 a^2} \mathcal{H}_{j,2} &= \frac{1}{2} H_{j,2} \end{aligned} \right\} \quad (60) \quad ;$$

and also replacing  $z$  by  $z' = -z$  for quadrants 1 and 2, we have:

$$\mathcal{H}_{1,1} = \int_0^t \left[ c - \frac{z'+a^2}{2a} \tan^{-1} \frac{2ac}{z'+a^2} \right] dz'$$

$$\mathcal{H}_{1,2} = - \int_0^t \left[ c - \frac{z'+a^2}{2a} \tan^{-1} \frac{2ac}{z'+a^2} \right] z' dz'$$

$$\mathcal{H}_{2,1} = \int_0^t \left[ \sqrt{1-z'} - \frac{z'-a^2}{2a} \tan^{-1} \frac{2a\sqrt{1-z'}}{z'+a^2} \right] dz'$$

$$\mathcal{H}_{2,2} = - \int_0^t \left[ \sqrt{1-z'} - \frac{z'-a^2}{2a} \tan^{-1} \frac{2a\sqrt{1-z'}}{z'+a^2} \right] z' dz'$$

(61)

$$\begin{aligned}
 \mathcal{H}_{3,1} &= \int_0^t \left[ 1 - \frac{z+a^2}{2a} \tan^{-1} \frac{2a}{z+a^2} \right] dz \\
 \mathcal{H}_{3,2} &= \int_0^t \left[ 1 - \frac{z+a^2}{2a} \tan^{-1} \frac{2a}{z+a^2} \right] z dz \\
 \mathcal{H}_{4,1} &= \int_0^t \left[ \sqrt{c^2-z} - \frac{z-a^2}{2a} \tan^{-1} \frac{2a\sqrt{c^2-z}}{z+a^2} \right] dz \\
 \mathcal{H}_{4,2} &= \int_0^t \left[ \sqrt{c^2-z} - \frac{z-a^2}{2a} \tan^{-1} \frac{2a\sqrt{c^2-z}}{z+a^2} \right] z dz
 \end{aligned}
 \tag{61}$$

Since  $z'$  is only a dummy variable, the prime can be dropped.

Integrating by parts:

$$\begin{aligned}
 &\int_0^t \left[ c - \frac{z+a^2}{2a} \tan^{-1} \frac{2ac}{z+a^2} \right] dz \\
 &= \left[ cz - \frac{(z+a^2)^2}{4a} \tan^{-1} \frac{2ac}{z+a^2} + \frac{1}{4a} \int \frac{(z+a^2)^2 (-2ac) dz}{(z+a^2)^2 + 4a^2c^2} \right]_0^t \\
 &= \left[ cz - \frac{(z+a^2)^2}{4a} \tan^{-1} \frac{2ac}{z+a^2} - \frac{cz}{2} - ac^2 \tan^{-1} \frac{2ac}{z+a^2} \right]_0^t
 \end{aligned}$$

At  $z=0$  this expression is:

$$\left( -\frac{a^3}{4} - ac^2 \right) \tan^{-1} \frac{2c}{a} \simeq -ac^2 \left( \frac{\pi}{2} - \frac{a}{2c} \right),$$

since  $a$  is very small.

If  $t \gg 2ac$ , the arc tangent terms can be

expanded when  $z = t$ . The upper limit gives approximately:

$$\frac{ct}{2} - \frac{c(t+a^2)}{2} + \frac{2a^2c^3}{3(t+a^2)} - \frac{2a^2c^3}{t+a^2} = -\frac{a^2c}{2} - \frac{4a^2c^3}{3t}$$

Retaining only second order terms in  $a$ ,

$$\mathcal{H}_{1,1} \approx \frac{\pi ac^2}{2} - a^2c - \frac{4a^2c^3}{3t}$$

Now at  $u = w$ ,  $w^2c^2 = v^2$ ; and in the second integration we obtain  $S_j$ ; we replace  $c$  by  $\frac{v}{w}$ :

$$\mathcal{H}_{1,1} \approx \frac{\pi av^2}{2w^2} - \frac{a^2v}{w} - \frac{4a^2v^3}{3w^3t} \quad (62)$$

$\mathcal{H}_{3,1}$  may be found from this by setting  $c = 1$  or  $v = w$ .

$$\mathcal{H}_{3,1} \approx \frac{\pi a}{2w^2} - a^2 - \frac{4a^2}{3t} \quad (63)$$

$$\begin{aligned} -\mathcal{H}_{1,2} &= \int_0^t \left[ c - \frac{z+a^2}{2a} \tan^{-1} \frac{2ac}{z+a^2} \right] z dz \\ &= \left[ \frac{cz^2}{2} - \frac{2z^3+3a^2z^2}{12a} \tan^{-1} \frac{2ac}{z+a^2} \right]_0^t + \frac{1}{12a} \int_0^t \frac{(2z^3+3z^2a^2)(-2ac) dz}{(z+a^2)^2+4a^2c^2} \\ &\approx \left[ \frac{cz^2}{2} - \frac{2z^3+3a^2z^2}{12a} \tan^{-1} \frac{2ac}{z+a^2} \right]_0^t - \int_0^t \left[ \frac{cz}{3} - \frac{a^2c}{6} - \frac{1}{3} \frac{4a^2c^3z}{(z+a^2)^2+4a^2c^2} \right] dz \end{aligned}$$

Again retaining only terms through second order of  $a$ ,

$$-\mathcal{H}_{1,2} \approx \left\{ \frac{cz^2}{2} - \frac{2z^3+3z^2a^2}{12a} \tan^{-1} \frac{2ac}{z+a^2} - \frac{cz^2}{6} + \frac{a^2cz}{6} + \frac{4a^2c^3}{6} \ln \left[ \frac{(z+a^2)^2+4a^2c^2}{(z+a^2)^2+4a^2c^2} \right] \right\}_0^t$$



Now

$$-\frac{2t^3+3t^2a^2}{12a} \tan^{-1} \frac{2ac}{t+a^2} \approx \left[ \frac{2ac}{t+a^2} - \frac{8a^3c^3}{3(t+a^2)^3} \right] \frac{2t^3+3t^2a^2}{12a}$$

$$\approx -\frac{ct^2}{3} - \frac{a^2ct}{6} + \frac{4a^2c^3}{9}$$

and

$$\frac{4a^2c^3}{6} \ln[(t+a^2)^2 + 4a^2c^2] \approx \frac{4a^2c^3}{3} \ln t$$

At  $z=0$ , the logarithm term is  $\frac{4a^2c^3}{3} \ln 2ac$ ;

$$\therefore -\mathcal{H}_{1,2} \approx \frac{4a^2c^3}{3} \left[ \frac{1}{3} + \ln \frac{t}{2ac} \right] \approx \frac{4a^2v^3}{3w^3} \left[ \frac{1}{3} + \ln \frac{tw}{2ac} \right] \quad (64)$$

Likewise,

$$\mathcal{H}_{3,2} = \frac{4a^2}{3} \left[ \frac{1}{3} + \ln \frac{t}{2a} \right] \quad (65)$$

Consider

$$\mathcal{H}_{4,1} = \int_0^t \left[ \sqrt{c^2-z} - \frac{z-a^2}{2a} \tan^{-1} \frac{2a\sqrt{c^2-z}}{z+a^2} \right] dz$$

$$\text{Let } cr = \sqrt{c^2-z}, \quad \rho = \frac{a}{c}$$

$$z = c^2(1-r^2), \quad dz = -2c^2rdr$$

$$\mathcal{H}_{4,1} = \frac{2}{3} \left[ c^3 - (c^2-t)^{3/2} \right] - \int_1^{\frac{\sqrt{c^2-t}}{c}} \frac{c^3}{2\rho} (1-r^2-\rho^2) \tan^{-1} \frac{2pr}{1-r^2+\rho^2} (-2rdr)$$

Integrating by parts:

$$\mathcal{H}_{4,1} = \frac{2}{3} \left[ c^3 - (c^2-t)^{3/2} \right] - \frac{c^3}{4\rho} \left[ (1-r^2-\rho^2)^2 \tan^{-1} \frac{2pr}{1-r^2+\rho^2} \right]_1^{\frac{\sqrt{c^2-t}}{c}}$$

$$+ \frac{c^3}{4\rho} \int_1^{\frac{\sqrt{c^2-t}}{c}} \frac{(1-r^2-\rho^2)^2 \cdot 2\rho(1+r^2+\rho^2)}{(1-r^2+\rho^2)^2 + 4r^2\rho^2} dr$$

Call this integral term  $G$ . Then

$$\begin{aligned} G &= \frac{c^3}{4\rho} \int_1^{\frac{\sqrt{c^2-t}}{c}} \left[ 2(1+r^2+\rho^2)\rho - 4\rho^2 \frac{2\rho(1+r^2+\rho^2)}{(1-r^2+\rho^2)^2 + 4\rho^2 r^2} \right] dr \\ &= \frac{c^3}{4\rho} \left\{ 2\rho \left( r + \frac{r^3}{3} + \rho^2 r \right) - 4\rho^2 \tan^{-1} \frac{2\rho r}{1-r^2+\rho^2} \right\}_1^{\frac{\sqrt{c^2-t}}{c}} \end{aligned}$$

Substituting  $z$  back,

$$\begin{aligned} \mathcal{H}_{4,1} &= \frac{2}{3} \left[ c^3 - (c^2-t)^{3/2} \right] + \left[ -\frac{(z-a^2)^2}{4a} \tan^{-1} \frac{2a\sqrt{c^2-z}}{z+a^2} \right. \\ &\quad \left. + \frac{c^2\sqrt{c^2-z}}{2} + \frac{(c^2-z)^{3/2}}{6} + \frac{a^2\sqrt{c^2-z}}{2} - ac^2 \tan^{-1} \frac{2a\sqrt{c^2-z}}{z+a^2} \right]_0^t \\ &= -\frac{(c^2-t)^{3/2}}{2} + \frac{c^2\sqrt{c^2-t}}{2} + \frac{a^2\sqrt{c^2-t}}{2} - \frac{a^2c}{2} \\ &\quad - \left[ \frac{(t-a^2)^2}{4a} + ac^2 \right] \tan^{-1} \frac{2a\sqrt{c^2-t}}{t+a^2} + \left[ \frac{a^3}{4} + ac^2 \right] \tan^{-1} \frac{2c}{a} \end{aligned}$$

Expanding the arc tangents:

$$\begin{aligned} \mathcal{H}_{4,1} &\approx -\frac{(c^2-t)^{3/2}}{2} + \frac{c^2\sqrt{c^2-t}}{2} + \frac{a^2\sqrt{c^2-t}}{2} - \frac{a^2c}{2} - \frac{t\sqrt{c^2-t}}{2} \\ &\quad + \frac{3a^2\sqrt{c^2-t}}{2} + \frac{4a^2(c^2-t)^{3/2}}{6t} - \frac{2a^2c^2\sqrt{c^2-t}}{t} + ac^2 \left[ \frac{\pi}{2} - \frac{a}{2c} \right] \end{aligned}$$

To second order in  $a$ ,

$$\mathcal{H}_{4,1} \approx -\frac{4a^2(c^2-t)^{\frac{3}{2}}}{3t} - \frac{a^2c}{2} - \frac{a^2c^2}{2\sqrt{c^2}} + \frac{\pi ac^2}{2}$$

If  $t \ll c^2$ ,

$$\frac{4(c^2-t)^{\frac{3}{2}}}{3t} \approx \frac{4c^3}{3t} - 2c$$

Then

$$\mathcal{H}_{4,1} \approx \frac{\pi ac^2}{2} + a^2c - \frac{4a^2c^3}{3t}$$

or

$$\mathcal{H}_{4,1} \approx \frac{\pi av^2}{2w^2} + \frac{a^2v}{w} - \frac{4a^2v^3}{3w^3t} \quad (66)$$

Likewise, setting  $c = 1$ ,

$$\mathcal{H}_{2,1} \approx \frac{\pi a}{2} + \frac{a^2}{w} - \frac{4a^2}{3t} \quad (67)$$

Now take

$$\mathcal{H}_{4,2} = \int_0^t z\sqrt{c^2-z} dz - \frac{c^5}{2\rho} \int_1^{\frac{\sqrt{c^2-t}}{c}} \left[ \frac{(1-r^2)^3}{3} - \frac{\rho^2(1-r^2)^2}{2} \right] \tan^{-1} \frac{2rp}{1-r^2+\rho^2} (-2rdr)$$

Integrating by parts,

$$\begin{aligned} \mathcal{H}_{4,2} = & -\frac{2}{3}t(c^2-t)^{\frac{3}{2}} - \frac{4}{15}(c^2-t)^{\frac{5}{2}} + \frac{4}{15}c^3 - \frac{c^5}{2\rho} \left\{ \left[ \frac{(1-r^2)^3}{3} - \frac{\rho^2(1-r^2)^2}{2} \right] \tan^{-1} \frac{2rp}{1-r^2+\rho^2} \right\}_{\frac{\sqrt{c^2-t}}{c}} \\ & + \frac{c^5}{2\rho} \int_1^{\frac{\sqrt{c^2-t}}{c}} \left[ \frac{(1-r^2)^3}{3} - \frac{\rho^2(1-r^2)^2}{2} \right] \frac{(1+r^2+\rho^2) 2\rho dr}{(1-r^2+\rho^2)^2 + 4\rho^2 r^2} \end{aligned}$$

Call the integral  $G_2$ , then to second order in  $\rho$ ,

$$G_2 \approx \frac{c^5}{2\rho} \int_1^{\sqrt{c^2-t}} \left[ \frac{2\rho}{3} (-r^4 + \rho^2 r^2 + 1 + 7\rho^2) - \rho^3 (r^2 + 1 - \rho^2) + \frac{16}{3} \frac{\rho^3 (r^2 - 1)}{(1 - r^2 + \rho^2)^2 + 4\rho^2} \right] dr$$

$$\approx \frac{c^5}{2} \left\{ -\frac{2r^5}{15} + \frac{2\rho^2 r^3}{9} + \frac{2r}{3} + \frac{14\rho^2 r}{3} - \frac{\rho^2 r^3}{3} - \rho^2 r \right\} \frac{\sqrt{c^2-t}}{c} + \frac{8c^5 \rho^2}{3} \int_1^{\sqrt{c^2-t}} \frac{\sqrt{c^2-t} (r^2 - 1) dr}{r^4 - 2(1 - \rho^2)r^2 + (1 + \rho^2)^2}$$

Call the last integral  $G_3$ . Then, setting

$$r^4 - 2(1 - \rho^2)r^2 + (1 + \rho^2)^2 = (r^2 - p_1^2)(r^2 - p_2^2),$$

$$G_3 = \frac{8c^5 \rho^2}{3} \int_1^{\sqrt{c^2-t}} \frac{\sqrt{c^2-t} (r^2 - 1) dr}{(r^2 - p_1^2)(r^2 - p_2^2)} = \frac{8c^5 \rho^2}{p_1^2 - p_2^2} \int_1^{\sqrt{c^2-t}} \left[ \frac{p_1^2 - 1}{r^2 - p_1^2} - \frac{p_2^2 - 1}{r^2 - p_2^2} \right] dr$$

$$= \frac{8c^5 \rho^2}{3(p_1^2 - p_2^2)} \left\{ \frac{p_1^2 - 1}{-2p_1} \ln \frac{p_1 + r}{p_1 - r} - \frac{p_2^2 - 1}{-2p_2} \ln \frac{p_2 + r}{p_2 - r} \right\}$$

Now,  $p_1^2 p_2^2 = (1 + \rho^2)$ , and if  $p_1^2 = A e^{i\theta}$ ,  $p_2^2 = A e^{-i\theta}$ , with  $A$  real, then  $p_1^2 = (1 + \rho^2) e^{i\theta}$ ,  $p_2^2 = (1 + \rho^2) e^{-i\theta}$ ;

also  $p_1^2 + p_2^2 = 2(1 - \rho^2)^2$ , or  $(1 + \rho^2) 2 \cos \theta = 2(1 - \rho^2)$ ,

$$\cos \theta = \frac{1 - \rho^2}{1 + \rho^2}, \quad \sin \theta = \frac{2\rho}{1 + \rho^2}$$

$$p_1^2 - p_2^2 = (1 + \rho^2) 2i \sin \theta = 4i\rho$$

$$p_1 = \sqrt{1 + \rho^2} e^{\frac{i\theta}{2}}, \quad p_2 = \sqrt{1 + \rho^2} e^{-\frac{i\theta}{2}}$$

$$\cos \frac{\theta}{2} = \frac{1}{\sqrt{1 + \rho^2}}, \quad \sin \frac{\theta}{2} = \frac{\rho}{\sqrt{1 + \rho^2}}$$

$$p_1 = 1 + \rho i; \quad p_1^2 - 1 = -\rho^2 + 2\rho i; \quad \frac{p_1^2 - 1}{-p_1} = -\rho \frac{\rho + i(2 + \rho)}{1 + \rho^2}$$

$$p_2 = 1 - pi; \quad p_2^2 - 1 = -\rho^2 - 2pi; \quad \frac{p_2^2 - 1}{-p_2} = -\rho \frac{\rho - i(2 + \rho)}{1 + \rho^2}$$

$$\frac{p_{1+r}}{p_{1-r}} = \frac{1+r+ip}{1-r+ip} = \sqrt{\frac{(1+r)^2 + \rho^2}{(1-r)^2 + \rho^2}} e^{-i \tan^{-1} \frac{2pr}{1-r^2+\rho^2}}$$

$$\frac{p_{2+r}}{p_{2-r}} = \frac{1+r-ip}{1-r-ip} = \sqrt{\frac{(1+r)^2 + \rho^2}{(1-r)^2 + \rho^2}} e^{i \tan^{-1} \frac{2pr}{1-r^2+\rho^2}}$$

$$\frac{p_1^2 - 1}{-p_1} \ln \frac{p_{1+r}}{p_{1-r}} = -\frac{\rho}{1+\rho^2} [\rho + i(2+\rho)] \left[ \frac{1}{2} \ln \frac{(1+r)^2 + \rho^2}{(1-r)^2 + \rho^2} - i \tan^{-1} \frac{2pr}{1-r^2+\rho^2} \right]$$

$$\frac{p_2^2 - 1}{-p_2} \ln \frac{p_{2+r}}{p_{2-r}} = \frac{-\rho}{1+\rho^2} [\rho - i(2+\rho)] \left[ \frac{1}{2} \ln \frac{(1+r)^2 + \rho^2}{(1-r)^2 + \rho^2} + i \tan^{-1} \frac{2pr}{1-r^2+\rho^2} \right]$$

Then, dropping terms of higher order in  $\rho$

$$G_3 = -\frac{8c^5 \rho^2}{3 \cdot 4pi} pi \ln \frac{(1+r)^2 + \rho^2}{(1-r)^2 + \rho^2}$$

Assembling the terms of  $\mathcal{H}_{4,2}$ ,

$$\mathcal{H}_{4,2} \approx \frac{-2t(c^2-t)^{\frac{3}{2}}}{3} - \frac{4(c^2-t)^{\frac{5}{2}}}{15} + \frac{4c^3}{15} - \frac{c^5}{2\rho} \left\{ \frac{(1-r^2)^3}{3} - \frac{\rho^2(1-r^2)^2}{2} \tan^{-1} \frac{2rp}{1-r^2+\rho^2} \right\} \frac{\sqrt{c^2-t}}{c}$$

$$+ \frac{c^5}{2} \left\{ -\frac{2r^5}{15} + \frac{2\rho^2 r^3}{9} + \frac{2r}{3} + \frac{14\rho^2 r}{3} - \frac{\rho^2 r^3}{3} - \rho^2 r \right\} \frac{\sqrt{c^2-t}}{t} - \frac{2c^5 \rho^2}{3} \ln \frac{(1+r)^2 + \rho^2}{(1-r)^2 + \rho^2}$$

In substituting in the limits, and using  $\rho = \frac{c}{a}$

$$\mathcal{H}_{4,2} \approx \frac{-2t(c^2-t)^{\frac{3}{2}}}{3} - \frac{4(c^2-t)^{\frac{5}{2}}}{15} + \frac{4c^3}{15} \left( \frac{t^3}{6a} - \frac{at^2}{4} \right) \tan^{-1} \frac{2a\sqrt{c^2-t}}{t+a^2} - \frac{(c^2-t)^{\frac{5}{2}}}{15}$$

$$\begin{aligned}
& + \frac{c^5}{15} + \frac{a^2(c^2-t)^{\frac{3}{2}}}{9} - \frac{a^2c^3}{9} + \frac{c^4\sqrt{c^2-t}}{3} - \frac{c^5}{3} + \frac{7a^2c^2\sqrt{c^2-t}}{3} - \frac{7a^2c^3}{3} - \frac{a^2(c^2-t)^{\frac{3}{2}}}{6} \\
& + \frac{a^2c^3}{6} - \frac{a^2c^2\sqrt{c^2-t}}{2} + \frac{a^2c^3}{2} - \frac{2a^2c^3}{3} \ln \frac{(c+\sqrt{c^2-t})^2+a^2}{(c-\sqrt{c^2-t})^2+a^2} + \frac{2a^2c^3}{3} \ln \frac{4c^2+a^2}{a^2}
\end{aligned}$$

Since  $c^2 \gg t \gg a^2$ ,

$$\ln \frac{(c+\sqrt{c^2-t})^2+a^2}{(c-\sqrt{c^2-t})^2+a^2} \approx \ln \frac{16c^4}{t^2}; \quad \ln \frac{4c^2+a^2}{a^2} \approx \ln \frac{4c^2}{a^2}$$

Expanding the arc-tangent,

$$\begin{aligned}
\left(\frac{t^3}{6a} - \frac{at^2}{4}\right) \tan^{-1} \frac{2a\sqrt{c^2-t}}{t+a^2} & \approx \left(\frac{t^3}{6a} - \frac{at^2}{4}\right) \left[ \frac{2a\sqrt{c^2-t}}{t+a^2} - \frac{8a^3(c^2-t)^{\frac{3}{2}}}{3(t+a^2)^3} \right] \\
& \approx \frac{t^2\sqrt{c^2-t}}{3} - \frac{a^2t\sqrt{c^2-t}}{3} - \frac{a^2t\sqrt{c^2-t}}{2} - \frac{4a^2(c^2-t)^{\frac{3}{2}}}{9}
\end{aligned}$$

Collecting terms

$$\begin{aligned}
\mathcal{H}_{4,2} & \approx -\frac{16a^2c^3}{9} - \frac{4a^2(c^2-t)^{\frac{3}{2}}}{9} + \frac{8a^2c^2}{3}\sqrt{c^2-t} + \frac{4a^2c^3}{3} \ln \frac{t}{2ac} \\
& \approx \frac{4a^2c^3}{9} + \frac{4a^2c^3}{3} \ln \frac{t}{2ac} \\
& \approx \frac{4a^2v^3}{3w^3} \left( \frac{1}{3} + \ln \frac{tw}{2av} \right) \quad (68)
\end{aligned}$$

Setting  $c=1$ ,

$$-\mathcal{H}_{2,2} \approx \frac{4a^2}{3} \left( \frac{1}{3} + \ln \frac{t}{2a} \right) \quad (69)$$

In view of the definitions of  $\mathcal{H}_{j,k}$  (eqs. 60, 61), the equations 62 through 69 are equivalent to eqs. 24 of the main text.

## APPENDIX III

## MOMENTUM EQUATION

The number of electrons per c.c., having speeds in the range  $u + du$  and velocities parallel to the field in the range  $\xi + d\xi$  is

$$2\pi n \left[ f_0(u^2) + \frac{\xi}{u} f_1(u^2) \right] u du d\xi.$$

In time  $dt$ , the field increases the velocity of each particle parallel to the field an amount  $\frac{eE}{m} dt$ . Then, per unit time, the number of particles for which this component of velocity increases from less than  $y$  to greater than  $y$  is

$$\begin{aligned} & \frac{2\pi n}{dt} \int_{|y|}^{\infty} u du \int_{y - \frac{Ee}{m} dt}^y (f_0 + \frac{\xi}{u} f_1) d\xi \\ &= \frac{2\pi n}{dt} \int_{|y|}^{\infty} u du \left[ f_0 \frac{Ee}{m} dt + \frac{f_1}{u} \left( \frac{eE}{m} y dt - \frac{E^2 e^2}{m^2} dt^2 \right) \right] \\ &= \frac{2\pi n E e}{m} \int_{|y|}^{\infty} (u f_0 + y f_1) du. \quad (70) \end{aligned}$$

In the case of elastic collisions with atoms, we neglect the energy loss, and consider only the scattering.

Let  $\theta, \phi$  be polar angles about  $u$   
 $\omega, \psi$  " " " "  $E$ , of  $u$   
 $\lambda, \mu$  " " " "  $E$ , of  $u'$ .

The number of electrons scattered per c.c. per second at

angles in range  $\sin\theta d\theta d\phi$  out of a velocity space element  $u^2 \sin\omega d\omega d\psi du$  is

$$N\eta(f_0 + f_1 \cos\omega) u^3 \sigma_a(u, \theta) \sin\theta d\theta d\phi \sin\omega d\omega d\psi.$$

The velocities are scattered over a sphere of radius  $u_0$ . The number of electrons leaving in this way the region of velocity space to the left of plane  $y$  is found by integrating with respect to  $\theta$  and  $\phi$  for that part of the sphere to the right of the plane, and with respect to  $\omega$ ,  $\psi$ , and  $u$  for such values as make the initial velocity lie to the left of the plane, i.e.,

$$N\eta \int_{|y|}^{\infty} u^3 du \int_{\cos^{-1} \frac{y}{u}}^{\pi} (f_0 + f_1 \cos\omega) \sin\omega d\omega \int_0^{2\pi} d\psi \int \sigma_a(u, \theta) \sin\theta d\theta \int d\phi$$

Now  $\sin\theta d\theta d\phi$  can be replaced by  $\sin\lambda d\lambda d\mu$  for purposes of integration, if it is remembered that

$$\cos\theta = \cos\omega \cos\lambda + \sin\omega \sin\lambda \sin\mu$$

Then limits of  $\lambda$  and  $\mu$  are simpler than those of  $\theta$  and  $\phi$ :

$$\mu, \quad 0 \text{ to } 2\pi$$

$$\lambda, \quad \lambda_0 \text{ to } \pi, \quad \text{where } \cos\lambda_0 = \frac{y}{u}.$$

Now if we let

$$x = \cos\omega, \quad z = \cos\lambda,$$

and write  $\theta = \theta(x, z; \mu)$ ,

our expression becomes:

$$N\eta \int_{|y|}^{\infty} u^3 du \int_{-1}^{\frac{y}{u}} (f_0 + f_1 x) dx \int_0^{2\pi} d\psi \int_{\frac{y}{u}}^{\pi} \sigma_a\{u, \theta(x, z; \mu)\} dz \int_0^{2\pi} d\mu.$$



Similarly, the number scattered into the region to the left of the plane is

$$N\eta \int_{|y|}^{\infty} u^3 du \int_{\frac{y}{u}}' (f_0 + f_1, x) dx \int_0^{2\pi} d\psi \int_{-1}^{\frac{y}{u}} \sigma_a \{u, \theta(x, z; \mu)\} dz \int_0^{2\pi} d\mu.$$

The net number leaving by collision is the difference of these two expressions. These differ only in the interchange of the limits for  $x$ , and  $z$ , since  $\theta(x, z; \mu)$  is symmetrical in  $x$  and  $z$ . Therefore the  $f_0$  term disappears. Requiring the total number leaving due to both field and collisions to be zero, we obtain

$$\begin{aligned} \frac{2\pi\eta eE}{m} \int_{|y|}^{\infty} (uf_0 + yf_1) du + 2\pi N\eta \int_{|y|}^{\infty} u^3 f_1 du \left\{ \int_{-1}^{\frac{y}{u}} x dx \int_{\frac{y}{u}}' dz \int_0^{2\pi} \sigma_a \{u, \theta(x, z; \mu)\} d\mu \right. \\ \left. - \int_{\frac{y}{u}}' x dx \int_{-1}^{\frac{y}{u}} dz \int_0^{2\pi} \sigma_a \{u, \theta(x, z; \mu)\} d\mu \right\} = 0. \end{aligned}$$

Now, a similar equation can be written for the plane at  $-y$ ,

$$\begin{aligned} \frac{2\pi\eta eE}{m} \int_{|y|}^{\infty} (uf_0 - yf_1) du + 2\pi N\eta \int_{|y|}^{\infty} u^3 f_1 du \left\{ \int_{-1}^{-\frac{y}{u}} x dx \int_{-\frac{y}{u}}' dz \int_0^{2\pi} \sigma_a \{u, \theta(x, z; \mu)\} d\mu \right. \\ \left. - \int_{-\frac{y}{u}}' x dx \int_{-1}^{-\frac{y}{u}} dz \int_0^{2\pi} \sigma_a \{u, \theta(x, z; \mu)\} d\mu \right\} = 0 \end{aligned}$$

But, since

$$\sigma_a(-x, -z; \mu) = \sigma_a(x, z; \mu)$$

we can replace  $x$  by  $-x$ , and  $z$  by  $-z$  in the integrals, and the collision term then becomes identical with the corresponding term in the equation for  $+y$ .

Adding the two equations,

$$\frac{4\pi\eta eE}{m} \int_y^\infty u f_0 du + 4\pi N\eta \int_y^\infty u^3 f_1 du \left\{ \int_{-1}^{\frac{y}{u}} x dx \int_{\frac{y}{u}}^1 dz \int_0^{2\pi} \sigma_a \{u, \theta(x, z; \mu)\} d\mu \right. \\ \left. - \int_{\frac{y}{u}}^1 x dx \int_{-1}^{\frac{y}{u}} dz \int_0^{2\pi} \sigma_a \{u, \theta(x, z; \mu)\} d\mu \right\} = 0$$

The absolute value signs can now be dropped, by making the restriction  $y \geq 0$ .

Divide by  $4\pi\eta$ , and differentiate with respect to  $y$ .

$$\frac{eE}{m} y f_0'(y^2) + N \int_y^\infty u^3 f_1 du \left\{ \frac{y}{u^2} \int_{\frac{y}{u}}^1 dz \int_0^{2\pi} \sigma_a \left\{ u, \theta\left(\frac{y}{u}, z; \mu\right) \right\} d\mu \right. \\ \left. - \frac{1}{u} \int_{-1}^{\frac{y}{u}} x dx \int_0^{2\pi} \sigma_a \left\{ u, \theta\left(x, \frac{y}{u}; \mu\right) \right\} d\mu - \frac{y}{u^2} \int_{\frac{y}{u}}^1 dz \int_0^{2\pi} \sigma_a \left\{ u, \theta\left(\frac{y}{u}, z; \mu\right) \right\} d\mu \right. \\ \left. + \frac{1}{u} \int_{-1}^{\frac{y}{u}} x dx \int_0^{2\pi} \sigma_a \left\{ u, \theta\left(x, \frac{y}{u}; \mu\right) \right\} d\mu \right\} = 0,$$

or

$$\frac{eE}{m} y f_0'(y^2) + N \int_y^\infty u^3 f_1 du \int_{-1}^{\left(\frac{y}{u^2} - \frac{x}{u}\right)} dx \int_0^{2\pi} \sigma_a \left\{ u, \theta\left(x, \frac{y}{u}; \mu\right) \right\} d\mu = 0.$$

Now the integrals in  $x$  and  $\mu$  cover the whole sphere. Therefore they can be replaced by integrals in  $\theta$  and  $\phi$ .

$$dx dy = -\sin\theta d\theta d\phi.$$

Since  $\theta(\cos\omega, \cos\lambda, \mu)$  has become  $\theta(\cos\omega, \cos\lambda_0; \mu)$ ,

the transformation to  $\theta, \phi$  is

$$x = \cos\omega = \frac{y}{u} \cos\theta - \sin\lambda_0 \sin\theta \cos\phi;$$

$$\begin{aligned}
& \int_{-1}^1 \left( \frac{y}{u^2} - \frac{x}{u} \right) dx \int_0^{2\pi} \sigma_a \left\{ u, \theta \left( x, \frac{y}{u}; \mu \right) \right\} d\mu \\
&= \int_0^\pi \sin \theta d\theta \int_0^{2\pi} \left( \frac{y}{u^2} - \frac{y}{u^2} \cos \theta + \frac{\sin \lambda_0}{u} \sin \theta \cos \phi \right) \sigma_a(u, \theta) d\phi \\
&= \frac{2\pi y}{u^2} \int_0^\pi (1 - \cos \theta) \sigma_a(u, \theta) d\mu = \frac{y}{u^2} Q(u).
\end{aligned}$$

The equation then is

$$\frac{eE}{m} y f_0'(y^2) + Ny \int_y^\infty u f_0'(u^2) Q(u) du = 0$$

Divide by  $y$ , and differentiate again:

$$\frac{eE}{m} \frac{d}{dy} f_0'(y^2) + NQ(y) y f_0'(y^2) = 0.$$

This is equation            quoted in the main paper.

If the equations for  $+y$  and  $-y$  are subtracted instead of added, we obtain:

$$\frac{4\pi eE}{m} \int_y^\infty f_1(u^2) y du = 0.$$

It is impossible to satisfy this as well as the previous equation and the energy equation. It enters because  $f$  was approximated by  $f_0 + f_1 \cos \omega$ , and shows just what has been neglected.

## APPENDIX IV

## THE MAXWELLIAN AS A CHECK SOLUTION

We wish to show that the  $B_j'$  functions integrate to zero along a path  $u^2 + v^2 = (c^2 + 1)w^2$ .

There are two cases,  $c^2 > 1$ , and  $c^2 < 1$ .

Along this path

$$v^2 = (c^2 + 1)w^2 - u^2, \quad u^2 + v^2 - w^2 = c^2w^2.$$

$$\text{When } v^2 = w^2, \quad u^2 = c^2w^2.$$

Dropping the constant factor  $\frac{2\pi e^4}{m^2 w^2 a^2}$ , the expressions to be evaluated are:

Case 1.  $c^2 > 1$ :

$$\begin{aligned} R = & \int_0^{c^2 w^2} \left( u - \frac{w^2 - u^2 - a^2 w^2}{2wa} \tan^{-1} \frac{2wau}{w^2 - u^2 + a^2} \right) du^2 - \int_1^{c^2 w^2} \left( w - \frac{u^2 - w^2 + a^2 w^2}{2wa} \tan^{-1} \frac{2wa^2}{u^2 - w^2 + a^2} \right) du^2 \\ & - \int_{c^2 w^2}^{(c^2 + 1)w^2} \left( \sqrt{c^2 + 1 - u^2} - \frac{u^2 - w^2 - a^2 w^2}{2wa} \tan^{-1} \frac{2wa \sqrt{c^2 + 1 - u^2}}{u^2 - w^2 + a^2} \right) du^2. \end{aligned}$$

Case 2.  $c^2 < 1$ :

$$\begin{aligned} R' = & \int_0^{c^2 w^2} \left( u - \frac{w^2 - u^2 - a^2 w^2}{2wa} \tan^{-1} \frac{2wau}{w^2 - u^2 + a^2} \right) du^2 + \int_{c^2 w^2}^{w^2} \left( c - \frac{w^2 - u^2 + a^2 w^2}{2wa} \tan^{-1} \frac{2wac}{w^2 - u^2 + a^2} \right) du^2 \\ & - \int_w^{(c^2 + 1)w^2} \left( \sqrt{c^2 + 1 - u^2} - \frac{w^2 - u^2 + a^2 w^2}{2wa} \tan^{-1} \frac{2wa \sqrt{c^2 + 1 - u^2}}{u^2 - w^2 + a^2} \right) du^2. \end{aligned}$$

Let  $z = \frac{u^2}{w^2} - 1$  and  $z' = -z$ , as in Appendix II. Then

$$\frac{R}{w^3} = \int_0^1 \left( \sqrt{1-z'} - \frac{z'-a^2}{2a} \tan^{-1} \frac{2a\sqrt{1-z'}}{z'+a^2} \right) dz' - \int_0^{c^2-1} \left( 1 - \frac{z+a^2}{2a} \tan^{-1} \frac{2a}{z+a^2} \right) dz$$

$$- \int_{c^2-1}^{c^2} \left( \sqrt{c^2-z} - \frac{z-a^2}{2a} \tan^{-1} \frac{2a\sqrt{c^2-z}}{z+a^2} \right) dz,$$

and

$$\frac{R'}{w^3} = \int_{1-c^2}^1 \left( \sqrt{1-z'} - \frac{z'-a^2}{2a} \tan^{-1} \frac{2a\sqrt{1-z'}}{z'+a^2} \right) dz' + \int_0^{1-c^2} \left( c - \frac{z'+a^2}{2a} \tan^{-1} \frac{2ac}{z'+a^2} \right) dz'$$

$$- \int_0^{c^2} \left( \sqrt{c^2-z} - \frac{z-a^2}{2a} \tan^{-1} \frac{2a\sqrt{c^2-z}}{z+a^2} \right) dz.$$

Using results from Appendix II:

$$\frac{R}{w^3} = \left[ -\frac{2}{3}(1-z')^{\frac{3}{2}} - \frac{(z'-a^2)^2}{4a} \tan^{-1} \frac{2a\sqrt{1-z'}}{z'+a^2} + \frac{\sqrt{1-z'}}{2} + \frac{(1-z')^{\frac{3}{2}}}{6} + \frac{a^2\sqrt{1-z'}}{2} \right. \\ \left. - a \tan^{-1} \frac{2a\sqrt{1-z'}}{z'+a^2} \right]_0^1 - \left[ \frac{z}{2} - \left\{ \frac{(z+a^2)^2}{2a} + a \right\} \tan^{-1} \frac{2a}{z+a^2} \right]_0^{c^2-1}$$

$$- \left[ \frac{z}{2} \sqrt{c^2-z} - \left\{ \frac{(z-a^2)^2}{4a} + ac^2 \right\} \tan^{-1} \frac{2a\sqrt{c^2-z}}{z+a^2} + \frac{a^2}{2} \sqrt{c^2-z} \right]_{c^2-1}^{c^2},$$

or

$$\frac{R}{w^3} = \left[ \left( \frac{a^3}{4} + a \right) \tan^{-1} \frac{2}{a} - \frac{a^2}{2} \right] - \left[ \frac{c^2-1}{2} - \left\{ \frac{(c-1+a^2)^2}{4a} + a \right\} \tan^{-1} \frac{2a}{c^2-1+a^2} \right. \\ \left. + \left( \frac{a^3}{4} + a \right) \tan^{-1} \frac{2}{a} \right] - \left[ -\frac{c^2-1}{2} + \left\{ \frac{(c^2-1-a^2)^2}{4a} + ac^2 \right\} \tan^{-1} \frac{2a}{c^2-1+a^2} - \frac{a^2}{2} \right] = 0.$$

$$\begin{aligned}
\frac{R'}{w^3} &= \left[ \frac{z\sqrt{1-z'}}{2} - \left\{ \frac{(z'-a^2)^2}{4a} + a \right\} \tan^{-1} \frac{2a\sqrt{1-z'} + \frac{a^2\sqrt{1-z'}}{2}}{z'+a^2} \right]'_{1-c^2} + \left[ \frac{cz'}{2} \right. \\
&\quad \left. - \left\{ \frac{(z+a^2)^2}{4a} + ac^2 \right\} \tan^{-1} \frac{2ac}{z'+a^2} \right]_{1-c^2} - \left[ \frac{z\sqrt{c^2-z}}{2} + \frac{a^2\sqrt{c^2-z}}{2} \right. \\
&\quad \left. - \left\{ \frac{(z-a^2)^2}{4a} + ac^2 \right\} \tan^{-1} \frac{2a\sqrt{c^2-z}}{z+a^2} \right]_{c^2} \\
&= \left[ -\frac{c(1-c^2)}{2} + \left\{ \frac{(1-c^2-a^2)^2}{4a} + a \right\} \tan^{-1} \frac{2ac}{1-c^2+a^2} - \frac{a^2c}{2} \right] \\
&\quad + \left[ \frac{c(1-c^2)}{2} - \left\{ \frac{(1-c^2+a^2)^2}{4a} + ac^2 \right\} \tan^{-1} \frac{2ac}{1-c^2+a^2} + \left( \frac{a^3}{4} + ac^2 \right) \tan^{-1} \frac{2c}{a} \right] \\
&\quad - \left[ -\frac{a^2c}{2} + \left( \frac{a^3}{4} + ac^2 \right) \tan^{-1} \frac{2c}{a} \right] = 0.
\end{aligned}$$