Homotopy Colimits

by

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Submitted to the Department of Mathematics
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Abstract

The colimit functor does not take weakly equivalent diagrams to weakly equivalent spaces. We explain why this is a difficulty for homotopy theorists and explain some of the reasons one might be interested in a functor similar to the colimit with slightly different properties. We show how for diagrams which are cofibrant in the model category structure on diagrams the colimit functor is well-behaved with respect to homotopies, and we describe how this leads to the definition of the homotopy colimit functor. We show that the homotopy colimit is the total left derived functor of the colimit functor on the category of diagrams of spaces. We also prove that just as the colimit functor is adjoint to the constant diagram functor in the category of diagrams, the homotopy colimit functor is adjoint to the constant diagram in the homotopy category of diagrams.

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1. What is the homotopy colimit and why should you care anyway?

The first question is always, why should you read this paper? Why would one consider the concept of homotopy colimits at all? The name is certainly suggestive: the homotopy colimit is a version of the colimit that is somehow "right" for homotopy theory. So then the question becomes, what is wrong with the usual colimit, from the point of view of homotopy theory?

The basic problem is that colimits are not homotopy invariant. Precisely, if $\boldsymbol{D}$ and $\boldsymbol{E}$ are two diagrams and $f\colon \boldsymbol{D} \to \boldsymbol{E}$ is a weak equivalence on each space in the diagrams, it is not necessarily true that the colimit of $\boldsymbol{D}$ is weakly equivalent to the colimit of $\boldsymbol{E}$. Here is an example: consider the diagrams

$$\begin{array}{ccc} S^1 & \xrightarrow{\ i\ } & D^2 \\ {\scriptstyle i}\big\downarrow & & \\ D^2 & & \end{array} \qquad \text{and} \qquad \begin{array}{ccc} S^1 & \xrightarrow{\ i\ } & * \\ {\scriptstyle i}\big\downarrow & & \\ * & & \end{array}$$

The colimit of the first diagram will be S^2 whereas the colimit of the second will be a point. Those are not hard to see; in the first diagram, you are gluing the two disks together along the image of the circle in each (their boundaries), so you get a sphere; in the second diagram, you are gluing the two points together along the image of the circle in each (the points themselves), so you get a single point. Clearly a point and a 2-sphere are not weakly equivalent.

From the point of view of homotopy theory, this is clearly a big problem. The first requirement for pretty much any concept in homotopy theory is that it be homotopy invariant, because we generally want to be working in the homotopy category[1] of whatever model category we're dealing with. In this paper, we will deal in particular with the model categories of pointed and unpointed topological spaces and pointed and unpointed simplicial sets.

Notation 1.1. The symbol $\mathcal{S}pc$ will denote any of the following four model categories: pointed topological spaces, unpointed topological spaces, pointed simplicial sets, and unpointed simplicial sets.

We can restate the problem that colim is not homotopy invariant in terms of the homotopy category by saying that there is no functor

[1] If you are not reasonably familiar with model categories, you might want to go read [1], an excellent introduction. It also includes a discussion of homotopy pushouts and homotopy pullbacks in a similar vein to the discussion here.

that can fill in the bottom part of this square and make the diagram commute:

$$
\begin{array}{ccc}
\mathcal{S}pc^{\mathcal{I}} & \xrightarrow{\text{colim}} & \mathcal{S}pc \\
\downarrow{\scriptstyle \gamma} & & \downarrow{\scriptstyle \gamma} \\
\mathrm{Ho}(\mathcal{S}pc^{\mathcal{I}}) & \overset{?}{\dashrightarrow} & \mathrm{Ho}\,\mathcal{S}pc
\end{array}
\tag{1.2}
$$

But hold on a minute! We are talking about the homotopy category of diagrams $\mathrm{Ho}(\mathcal{S}pc^{\mathcal{I}})$ when we don't even know what the model category structure on diagrams is, or for that matter if there even is one. Happily, it turns out that there is one (in fact, there is more than one, for different sorts of diagrams). The structure that will be useful in this context is for the category of diagrams of fixed but arbitrary shape in topological spaces or simplicial sets.

Theorem 1.3. *Let $\mathcal{I}$ be a small category. Then there is a model category structure on the category $\mathcal{S}pc^{\mathcal{I}}$ of diagrams $\mathcal{D}\colon \mathcal{I} \to \mathcal{S}pc$ in which:*

1. *A weak equivalence is a map $\boldsymbol{D} \to \boldsymbol{E}$ in which $\boldsymbol{D}_\alpha \to \boldsymbol{E}_\alpha$ is a weak equivalence in $\mathcal{S}pc$ for every $\alpha \in \mathrm{Ob}(\mathcal{I})$.*
2. *A fibration is a map $\boldsymbol{D} \to \boldsymbol{E}$ in which $\boldsymbol{D}_\alpha \to \boldsymbol{E}_\alpha$ is a fibration in $\mathcal{S}pc$ for every $\alpha \in \mathrm{Ob}(\mathcal{I})$.*
3. *A cofibration is a map that has the left lifting property with respect to the trivial fibrations.*

We won't prove this theorem in this paper; see [2].

So now that we know more about what Diagram 1.2 means, let's think about it just a bit. What does it tell us about what the homotopy colimit might be? Since the colimit is not homotopy-invariant, we can't hope to find a functor making the diagram commute, but we'd like the homotopy colimit to be something as close to that as possible. Instead of asking for it to commute on the nose, let's say that there should be a natural transformation from the composite in one direction to the composite in the other direction. But we'd like some sense of it being as close as possible, so we'd like to add in some kind of universal property; we probably want a requirement about other such functors with natural transformations factoring in some way. The "total derived functor" is exactly the concept we are looking for.

Definition 1.4. Let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor between model categories. Let $\gamma_{\mathcal{C}}\colon \mathcal{C} \to \mathrm{Ho}(\mathcal{C})$ and $\gamma_{\mathcal{D}}\colon \mathcal{D} \to \mathrm{Ho}(\mathcal{D})$ denote the natural maps from the model categories $\mathcal{C}$ and $\mathcal{D}$ to their homotopy categories. Then the *total left derived functor* of F is a functor $LF\colon \mathrm{Ho}(C) \to \mathrm{Ho}(D)$ together with a natural transformation $t\colon LF \circ \gamma_{\mathcal{C}} \to \gamma_{\mathcal{D}} \circ F$ with the following universal property: Suppose $G\colon \mathrm{Ho}(\mathcal{C}) \to \mathrm{Ho}(\mathcal{D})$ is another functor and $s\colon G \circ \gamma_{\mathcal{C}} \to \gamma_{\mathcal{D}} \circ F$ is a natural transformation.

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{\;F\;} & \mathcal{D} \\ {\scriptstyle \gamma_{\mathcal{C}}}\big\downarrow & & \big\downarrow{\scriptstyle \gamma_{\mathcal{D}}} \\ \mathrm{Ho}(\mathcal{C}) & \overset{LF}{\underset{G}{\rightrightarrows}} & \mathrm{Ho}(\mathcal{D}) \end{array}$$

Then there exists a unique natural transformation $s'\colon G \to LF$ such that the composite natural transformation

$$G \circ \gamma_{\mathcal{C}} \xrightarrow{s' \circ \gamma_{\mathcal{C}}} (LF) \circ \gamma_{\mathcal{C}} \xrightarrow{\;t\;} \gamma_{\mathcal{D}} \circ F \tag{1.5}$$

is the natural transformation s.

If a total left derived functor exists then by the usual argument it is unique, but one does not know that a random functor between model categories will have a total left derived. We will show that the total derived functor of the colimit does exist by giving an explicit construction of it, and we will then define the homotopy colimit to be exactly that functor (see Section 4). But before we embark on the project of constructing the homotopy colimit, we will look first at some other motivations for it.

2. Another reason for homotopy colimits

One of the remarkable things about mathematics is the way that seemingly unrelated issues connect. There are other ways that a homotopy theorist might have come up with the idea that something related to the colimit but a little bit different would be handy. If you play around in homotopy theory enough, you will find that there are situations when colimits behave nicely if things are somehow "cofibrant enough," but not otherwise. For example, the cofiber of a map $f\colon A \to X$ can be cooked up by taking the colimit of the diagram $* \leftarrow A \xrightarrow{f} X$, *if f is a cofibration.* Even if f isn't a cofibration, the colimit of that diagram will be X/A, but f must be a cofibration to guarantee that X/A will be weakly equivalent to the cone on X by f, $X \cup_f CA$. An example of this (which has to be thought through with some care) is to take f to be the embedding of a half-open line segment into the plane along the closed topologist's sine curve.

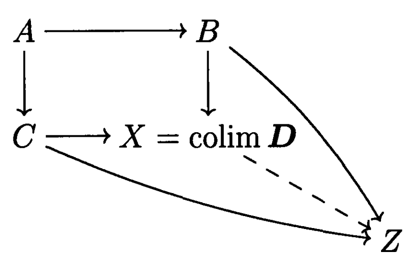
It turns out that solving the problem about colimit failing to be homotopy-invariant solves this problem too, and it's yet another reason that homotopy colimits come in handy. So, in the specific example above, as long as the spaces in the diagram are themselves cofibrant then the homotopy colimit of the diagram will be the cofiber (see Section 4)

3. Adjointness

There is yet another approach to homotopy colimits that is worth thinking about. It leads to a different universal property that we would like the homotopy colimit to have. In another example of the remarkable way that mathematics ties together, it will turn out that we can define homotopy colimits in a way that satisfies *both* universal properties.

Colimit is defined in terms of a universal property: given a diagram $\boldsymbol{D}$, the colimit is defined to be a space X, together with maps from each space in $\boldsymbol{D}$ to X making all triangles commute, such that, given another space Z and maps from each space in $\boldsymbol{D}$ to Z making all triangles commute, there exists a unique map from X to Z making everything commute. For example, let's consider the case that $\boldsymbol{D}$ is a pushout diagram $C \leftarrow A \rightarrow B$. Then for any map from $\boldsymbol{D}$ to a space Z there exists a unique dotted arrow in the following diagram making the diagram commute:

(3.1)



This can be expressed equivalently by the following adjointness relation:

$$\mathrm{Hom}_{\mathcal{S}pc}(\mathrm{colim}\,\boldsymbol{D}, Z) = \mathrm{Hom}_{\mathcal{S}pc^{\mathcal{J}}}(\boldsymbol{D}, \Delta Z)$$

where ΔZ is the diagram having the same shape as $\boldsymbol{D}$, with every space in the diagram equal to Z and every map equal to the identity map (Δ is known as the "constant diagram functor"). It would be worthwhile to stop and think this through if this formulation is unfamiliar to you.

It is natural to ask if we might not be able to give a description of the homotopy colimit that is analogous to this definition of the colimit. The answer is yes.

The first approach that I would think of to finding a universal property similar to that of colim that hocolim might satisfy would be to essentially replace every map in the diagram by a homotopy class of maps. Then, by adding "up to homotopy" anywhere it makes sense, this would lead us to the idea that the homotopy colimit of a diagram $\boldsymbol{D}$ would be a space X together with maps to it from the spaces in $\boldsymbol{D}$, commuting up to homotopy, such that the following property holds:

(3.2) For any space Z with maps from each space in $\boldsymbol{D}$ to Z that commute up to homotopy, there is a map from X to Z, unique up to homotopy, that makes everything commute up to homotopy.

Unfortunately, I would be wrong. To see what is going on, let's rephrase things in terms of adjointness. If $\operatorname{hocolim} \boldsymbol{D}$ and $\boldsymbol{D}$ are cofibrant and Z is fibrant, then (3.2) can be expressed by the adjointness relation

$$\operatorname{Hom}_{\operatorname{Ho} \mathcal{S}pc}(\operatorname{hocolim}(\boldsymbol{D}), Z) = \operatorname{Hom}_{(\operatorname{Ho} \mathcal{S}pc)^{\mathcal{J}}}\big(\boldsymbol{D}, \Delta(Z)\big) \tag{3.3}$$

Let's trace through this carefully to see that it is equivalent in that case.

Note that the right hand Hom is taken in the category $(\operatorname{Ho} \mathcal{S}pc)^{\mathcal{J}}$. The first thing we need to do to decode (3.3) is to understand this category. The objects of $(\operatorname{Ho} \mathcal{S}pc)^{\mathcal{J}}$ are diagrams in which the maps are zigzags, i.e., composites of actual maps going forward and weak equivalences going backward. In the case where the domain is cofibrant and the target is fibrant, we can think of a zigzag as a homotopy class of maps. So if we consider the special case of an object of $(\operatorname{Ho} \mathcal{S}pc)^{\mathcal{J}}$ in which the spaces which are domains are cofibrant and the spaces which are targets are fibrant, the maps in the diagram will be homotopy classes of maps. Okay, now that we understand the objects, let's consider the maps. A map between two objects $\boldsymbol{D}, \boldsymbol{E} \in (\operatorname{Ho} \mathcal{S}pc)^{\mathcal{J}}$ is, for each space in $\boldsymbol{D}$, a map in $\operatorname{Ho} \mathcal{S}pc$ to the corresponding space in $\boldsymbol{E}$. In general, a map in $\operatorname{Ho} \mathcal{S}pc$ will be a zigzag, but if the domain is cofibrant and the target is fibrant, it will be a homotopy class of maps. So we can think of a map between $\boldsymbol{D}$ and $\boldsymbol{E}$ in $(\operatorname{Ho} \mathcal{S}pc)^{\mathcal{J}}$ as a bunch of little homotopy classes of maps between the spaces.

So now let's see why (3.3) is equivalent to (3.2) in the case that $\boldsymbol{D}$ and $\operatorname{hocolim} \boldsymbol{D}$ are cofibrant and Z is fibrant. The left hand side of (3.3) is straightforward enough: it is just homotopy classes of maps between $\operatorname{hocolim} \boldsymbol{D}$ and Z. For the right hand side, first note if Z is fibrant, then ΔZ will be fibrant as well (that falls straight out of the definition). So the right hand side will be exactly maps from the spaces in $\boldsymbol{D}$ to Z that commute up to homotopy. So (3.3) tells us that if we have maps from the spaces in $\boldsymbol{D}$ to Z commuting up to homotopy, we can get a map from $\operatorname{hocolim} \boldsymbol{D}$ to Z that is unique up to homotopy. That is exactly (3.2).

Note that (3.2) is exactly the property that the colimit in the homotopy category would have. As it turns out, there is no colimit functor in the homotopy category, so it is certainly a good thing that that is not what the homotopy colimit is defined to be!

But I promised you at the beginning of the section that the homotopy colimit does satisfy an adjointness property similar to that of the colimit. I just chose the wrong one. The property hocolim does have is

$$\mathrm{Hom}_{\mathrm{Ho}\,\mathcal{S}pc}(\mathrm{hocolim}(\boldsymbol{D}), Z) = \mathrm{Hom}_{\mathrm{Ho}(\mathcal{S}pc^{\mathcal{I}})}\big(\boldsymbol{D}, \Delta(Z)\big) \tag{3.4}$$

Note that this time, the right hand Hom is taken in the category $\mathrm{Ho}(\mathcal{S}pc^{\mathcal{I}})$ rather than $(\mathrm{Ho}\,\mathcal{S}pc)^{\mathcal{I}}$. We'll prove that the homotopy colimit satisfies this later on, after we've defined hocolim, but for now let's just try to figure out what this new property means.

Let's again start by carefully sorting out what category $\mathrm{Ho}(\mathcal{S}pc^{\mathcal{I}})$ is. The objects are the same as the objects of $\mathcal{S}pc^{\mathcal{I}}$ (normal diagrams with actual maps which commute on the nose). The morphisms are zigzags, which, when the domain is cofibrant and the target is fibrant, we can think of as homotopy classes of maps of diagrams. So let's pause and think about what a homotopy class of maps between diagrams actually is. We define a homotopy in the obvious way. First, we define $\boldsymbol{D} \times I$ to be the diagram obtained from $\boldsymbol{D}$ by crossing all the objects with I; then, if $\boldsymbol{D}$ and $\boldsymbol{E}$ are two diagrams, we define two maps $f, g\colon \boldsymbol{D} \to \boldsymbol{E}$ to be homotopic if there is a map $H\colon \boldsymbol{D} \times I \to \boldsymbol{E}$ that restricts appropriately to f and g; finally, we use that definition of homotopy to divide up our maps into homotopy classes. So a homotopy class of maps between diagrams is a sort of coherent homotopy class between the individual maps between spaces.

Now let's go back to (3.4) and start with the special case where $\boldsymbol{D}$ and $\mathrm{hocolim}\,\boldsymbol{D}$ are cofibrant and Z is fibrant so that our Homs all become homotopy classes of maps (recall that if Z is fibrant then ΔZ will be too). Then, denoting homotopy classes of maps from X to Y by $[X, Y]$, the property becomes

$$[\mathrm{hocolim}\,\boldsymbol{D}, Z]_{\mathcal{S}pc} = [\boldsymbol{D}, \Delta(Z)]_{\mathcal{S}pc^{\mathcal{I}}} \tag{3.5}$$

Now this looks just like the property that colim satisfies, except that we have homotopy classes of maps instead of honest-to-God maps. A natural thing to wonder about, then, is the behavior of colim with respect to homotopies — does the colimit preserve homotopies itself, i.e., is the following equation true for the colimit as well?

$$[\mathrm{colim}\,\boldsymbol{D}, Z]_{\mathcal{S}pc} = [\boldsymbol{D}, \Delta(Z)]_{\mathcal{S}pc^{\mathcal{I}}} \tag{3.6}$$

If not, how does it fail? We know that it is not homotopy-invariant, but that is a different issue.

Surprisingly enough, it turns out that the colimit does preserve homotopies.

Proposition 3.7. *Let* $[\ ,\ ]$ *denote homotopy classes of maps. Then for any cofibrant diagram* $\boldsymbol{D} \in \mathcal{S}pc^{\mathcal{I}}$ *and any fibrant space* Z*,*

$$[\operatorname{colim} \boldsymbol{D}, Z]_{\mathcal{S}pc} = [\boldsymbol{D}, \Delta(Z)]_{\mathcal{S}pc^{\mathcal{I}}} \tag{3.8}$$

Before we prove this, we will need to prove a couple of lemmas. The first thing we will need to do is to show that (3.6) makes sense. Homotopy is an equivalence relation only if the domain is cofibrant and the target is fibrant. We know that $\boldsymbol{D}$ is cofibrant and Z is fibrant, but we need to know that $\operatorname{colim} \boldsymbol{D}$ is cofibrant as well. Fortunately, this is true, and not too hard to prove.

Lemma 3.9. *If* $\boldsymbol{D}$ *is a cofibrant diagram, then* $\operatorname{colim} \boldsymbol{D}$ *is a cofibrant space.*

Proof. Since the constant diagram functor Δ preserves trivial fibrations, this follows from statement 2 in Lemma 4.3. □

We will also need the following lemma.

Lemma 3.10. *For any diagram* $\boldsymbol{D} \in \mathcal{S}pc^{\mathcal{I}}$*,*

$$\operatorname{colim}(\boldsymbol{D} \times I) \cong (\operatorname{colim} \boldsymbol{D}) \times I$$

Proof. It suffices to show that

$$\operatorname{Hom}_{\mathcal{S}pc}(\operatorname{colim}(\boldsymbol{D} \times I), Z) = \operatorname{Hom}_{\mathcal{S}pc}(\operatorname{colim}(\boldsymbol{D}) \times I, Z)$$

for all $Z \in \mathcal{S}pc$. This is true basically because colim and the functor taking X to $X \times I$ are both left adjoints:

$$\begin{aligned}
\operatorname{Hom}_{\mathcal{S}pc}(\operatorname{colim}(\boldsymbol{D} \times I), Z) &= \operatorname{Hom}_{\mathcal{S}pc^{\mathcal{I}}}(\boldsymbol{D} \times I, \Delta Z) \\
&= \operatorname{Hom}_{\mathcal{S}pc^{\mathcal{I}}}(\boldsymbol{D}, (\Delta Z)^I) \\
&= \operatorname{Hom}_{\mathcal{S}pc^{\mathcal{I}}}(\boldsymbol{D}, \Delta(Z^I)) \\
&= \operatorname{Hom}_{\mathcal{S}pc}(\operatorname{colim}(\boldsymbol{D}), Z^I) \\
&= \operatorname{Hom}_{\mathcal{S}pc}(\operatorname{colim}(\boldsymbol{D}) \times I, Z) \qquad (3.11)
\end{aligned}$$

□

Now we are in a position to prove our proposition.

Proof of Proposition 3.7. By the universal property of the colimit, we know that maps $f, g\colon \boldsymbol{D} \to \Delta Z$ yield maps $\tilde{f}, \tilde{g}\colon \operatorname{colim} \boldsymbol{D} \to Z$ and, going the other direction, that maps $p, q\colon \operatorname{colim} \boldsymbol{D} \to Z$ yield maps $\hat{p}, \hat{q}\colon \boldsymbol{D} \to \Delta Z$. Then the questions at hand are these: if f and g are homotopic, will $\tilde{f}$ and $\tilde{g}$ be homotopic, and if if p and q are homotopic, will $\hat{p}$ and $\hat{q}$ be homotopic?

So suppose that $H\colon \boldsymbol{D}\times I \to \Delta Z$ is a homotopy between $f, g\colon \boldsymbol{D} \to \Delta Z$. We are looking for a map from $(\operatorname{colim} \boldsymbol{D}) \times I$ to ΔZ that restricts appropriately to $\tilde{f}$ and $\tilde{g}$. The obvious first thing to think about is the colimit of H, $\tilde{H}\colon \operatorname{colim}(\boldsymbol{D}\times I) \to Z$. By the preceding lemma, $\operatorname{colim}(\boldsymbol{D}\times I) = (\operatorname{colim}\boldsymbol{D})\times I$, so $\tilde{H}$ is also a map from $(\operatorname{colim}\boldsymbol{D})\times I$ to Z. The reader can verify that $\tilde{H}$ restricts appropriately to f and g by tracing through the isomorphism between $(\operatorname{colim}\boldsymbol{D})\times I$ and $\operatorname{colim}(\boldsymbol{D}\times I)$ outlined in the lemma above.

The proof that homotopic maps $p, q\colon \operatorname{colim}\boldsymbol{D} \to Z$ yield homotopic maps $\hat{p}, \hat{q}\colon \boldsymbol{D} \to \Delta Z$ is similar. □

Hmm. What's happening here? I have claimed that the homotopy colimit will satisfy the adjointness property (3.4) once we have defined it. Furthermore, we know that (3.4) is equivalent to the adjointness property (3.5) when $\boldsymbol{D}$ and $\operatorname{hocolim}\boldsymbol{D}$ are cofibrant and Z is fibrant. And we know that in that case colimit itself satisfies (3.6). This sure makes it seem like hocolim and colim are pretty closely related on cofibrant diagrams. Actually, if we put all of this information together correctly, we can conclude that they must be *equal* on cofibrant diagrams.

So let's see, in order to show that $\operatorname{hocolim}(\boldsymbol{D}) = \operatorname{colim}(\boldsymbol{D})$ as elements in the homotopy category, it suffices to show that

$$\operatorname{Hom}_{\operatorname{Ho}\mathcal{S}pc}(\operatorname{hocolim}\boldsymbol{D}, Z) = \operatorname{Hom}_{\operatorname{Ho}\mathcal{S}pc}(\operatorname{colim}\boldsymbol{D}, Z)$$

for all spaces Z. Letting Z' be a fibrant approximation for Z, we have

$$\begin{aligned}
\operatorname{Hom}_{\operatorname{Ho}\mathcal{S}pc}(\operatorname{hocolim}\boldsymbol{D}, Z) &= \operatorname{Hom}_{\operatorname{Ho}\mathcal{S}pc}(\operatorname{hocolim}\boldsymbol{D}, Z') \\
&= \operatorname{Hom}_{\operatorname{Ho}(\mathcal{S}pc^{\mathcal{J}})}(\boldsymbol{D}, \Delta Z') && \text{by (3.4)} \\
&= [\boldsymbol{D}, \Delta Z']_{\mathcal{S}pc^{\mathcal{J}}} \\
&= [\operatorname{colim}\boldsymbol{D}, Z']_{\mathcal{S}pc} && \text{by (3.6)} \\
&= \operatorname{Hom}_{\operatorname{Ho}\mathcal{S}pc}(\operatorname{colim}\boldsymbol{D}, Z)
\end{aligned}$$

So this tells us that when we define the homotopy colimit, we'd better define it to be equal to the colimit on cofibrant diagrams with cofibrant homotopy colimits. But knowing that, we actually know how to define it everywhere, since we want the homotopy colimit of weakly equivalent diagrams to be weakly equivalent. So we can define the homotopy colimit of an arbitrary diagram $\boldsymbol{D}$ to be the colimit of a cofibrant approximation $\widetilde{\boldsymbol{D}}$. That is exactly what we will do in the next section.

This is actually a pretty reasonable thing for the definition to be. After all, we commented in Section 2 that colim often behaves nicely on cofibrant things anyway.

4. Definition and proof of properties of homotopy colimits

We now know that we need to define the homotopy colimit of a diagram $\boldsymbol{D}$ to be the colimit of a cofibrant approximation $\widetilde{\boldsymbol{D}}$. However, there is a technical difficulty with this definition: we must know that it doesn't matter which cofibrant approximation we choose. The following proposition takes care of that difficulty for us.

Proposition 4.1. *The colimit functor preserves weak equivalences between cofibrant diagrams, i.e., if $f\colon \boldsymbol{D} \to \boldsymbol{E}$ is a weak equivalence of cofibrant diagrams, then $f_*\colon \operatorname{colim} \boldsymbol{D} \to \operatorname{colim} \boldsymbol{E}$ is also a weak equivalence.*

We will need two general lemmas about model categories:

Lemma 4.2. *(K. Brown) Let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor between model categories. If F carries acyclic cofibrations between cofibrant objects to weak equivalences, then F preserves all weak equivalences between cofibrant objects.*

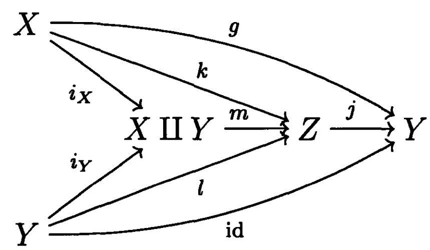
Proof. Let $g\colon X \to Y$ be a weak equivalence between cofibrant objects in $\mathcal{C}$. We will find a factorization of g as $g = kj$ where j is a trivial cofibration and k is a trivial fibration that has a right inverse l that is a trivial cofibration. Then by the two-out-of-three axiom for a model category, $F(k)$ will be a weak equivalence, since $\mathrm{id} = k \circ l$ and id and $F(l)$ will each be weak equivalences. So then $F(g)$ will be a weak equivalence as well.

Consider the map $g \amalg \mathrm{id}\colon X \amalg Y \to Y$ that is g on X and the identity on Y. We can factor this map as

$$X \amalg Y \xrightarrow{m} Z \xrightarrow{j} Y$$

where m is a cofibration and j is a trivial fibration by MC5, the factorization axiom for model categories. We can use this factorization to get a factorization of g as $g = j \circ m \circ i_X$, where i_X is just the injection of X into $X \amalg Y$.

The factorization of g that we are looking for will be $j \circ k$ where $k = m \circ i_X$. The map that we will want as a right inverse to j is the map $l\colon Y \to Z$ that is the composition $Y \xrightarrow{i_Y} X \amalg Y \xrightarrow{m} Z$. It is easy to see that it is a right inverse, since $j \circ l = j \circ m \circ i_Y = (g \amalg \mathrm{id}) \circ i_Y = \mathrm{id}$.



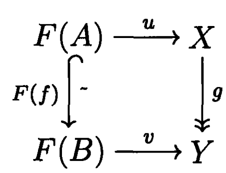
We need only show now that k and l are trivial cofibrations. Since X and Y are cofibrant, i_X and i_Y are cofibrations. Since m is a cofibration by assumption, then $k = m \circ i_X$ and $l = m \circ i_Y$ are themselves cofibrations. Since $g = j \circ k$ and g and j are already known to be weak equivalences, k is a weak equivalence. Similarly, since $\mathrm{id} = j \circ l$ and id and j are weak equivalences, l is a weak equivalence.

□

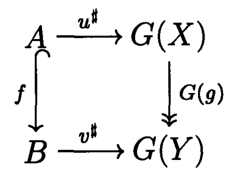
Lemma 4.3. *Suppose that $F\colon \mathcal{C} \to \mathcal{D}$ and $G\colon \mathcal{D} \to \mathcal{C}$ are an adjoint pair, so that $\mathrm{Hom}_{\mathcal{D}}(F(X), Y) = \mathrm{Hom}_{\mathcal{C}}(X, G(Y))$ for all $X \in \mathcal{C}$ and $Y \in \mathcal{D}$. Then*

1. *If G preserves fibrations, F preserves trivial cofibrations.*
2. *Similarly, if G preserves trivial fibrations, then F preserves cofibrations.*

Proof. Suppose G preserves fibrations. We want to show that F preserves trivial cofibrations, so we want to show that if $f\colon A \to B$ is a trivial cofibration in $\mathcal{C}$, then $F(f)\colon F(A) \to F(B)$ is a trivial cofibration in $\mathcal{D}$, i.e., that $F(f)\colon F(A) \to F(B)$ has the left lifting property with respect to all fibrations in $\mathcal{D}$. So let $g\colon X \to Y$ be a fibration in $\mathcal{D}$. Suppose we are given the commutative diagram on the left, and consider also the adjoint diagram on the right:



and



Since G preserves fibrations, we can get a lift $w\colon B \to G(X)$ in the left-hand diagram. Thus $w^\sharp\colon F(B) \to X$ is a lift in the right-hand diagram.

The second statement follows from a similar argument. □

Proof of Proposition 4.1. The constant diagram functor Δ preserves fibrations. Thus, by Lemma 4.3, the colimit functor preserves trivial cofibrations, and therefore by Lemma 4.2, it preserves all weak equivalences between cofibrant objects. □

So with that fact proven, we can (finally!) define the homotopy colimit.

Definition 4.4. Let $\boldsymbol{D}$ be a diagram. Then $\operatorname{hocolim}(\boldsymbol{D})$ is defined to be the colimit of $\widetilde{\boldsymbol{D}}$, where $\widetilde{\boldsymbol{D}}$ is a functorial cofibrant approximation to $\boldsymbol{D}$.

Recall that any model category has functorial cofibrant approximations. We need the approximation to be functorial to make hocolim itself functorial.

So now we have a definition. But to know that it is a reasonable one, we need to know that it satisfies a bunch of conditions. Above all, it had better be homotopy invariant. It had also better be the total derived functor. We also want to know that it satisfies our adjointness condition. Fortunately, all these things are easy to check.

Proposition 4.5. *The homotopy colimit is homotopy invariant, i.e., if $\boldsymbol{D}$ is weakly equivalent to $\boldsymbol{E}$, then* $\operatorname{hocolim} \boldsymbol{D} \cong \operatorname{hocolim} \boldsymbol{E}$.

Proof. This falls right out of the definition. If $\boldsymbol{D}$ is weakly equivalent to $\boldsymbol{E}$, then

$$\operatorname{hocolim}(\boldsymbol{D}) = \operatorname{colim}(\widetilde{\boldsymbol{D}}) \xrightarrow{f^*} \operatorname{colim}(\widetilde{\boldsymbol{E}}) = \operatorname{hocolim}(\boldsymbol{E})$$

and we know from Proposition 4.1 that f^* is a weak equivalence. □

The next thing we would like to show is that the homotopy colimit is the total left derived functor of the colimit. But recall that a total left derived functor is not just a functor but a pair consisting of a functor and a natural transformation. In our case, we want a natural transformation $t : \operatorname{hocolim} \circ \gamma_{\mathcal{S}pc^{\mathcal{I}}} \to \gamma_{\mathcal{S}pc} \circ \operatorname{colim}$, where $\gamma_{\mathcal{S}pc}$ and $\gamma_{\mathcal{S}pc^I}$ are the natural functors from $\mathcal{S}pc$ to $\operatorname{Ho}(\mathcal{S}pc)$. Thus $t_{\boldsymbol{D}}$ will be a morphism in the homotopy category from $\operatorname{hocolim}(\boldsymbol{D})$ to $\operatorname{colim}(\boldsymbol{D})$. Since $\operatorname{hocolim}(\boldsymbol{D}) = \operatorname{colim}(\widetilde{\boldsymbol{D}})$, we can define $t_{\boldsymbol{D}} = \operatorname{colim}(p_{\boldsymbol{D}})$, where $p_{\boldsymbol{D}} \colon \widetilde{\boldsymbol{D}} \to \boldsymbol{D}$ is the cofibrant approximation map.

Proposition 4.6. *The homotopy colimit is the total left derived functor of the colimit.*

Proof. We need to show that hocolim and t have the universal property for the total left derived functor (see Definition 1.4). So let

$G\colon \operatorname{Ho}(\mathcal{S}pc^{\mathcal{I}}) \to \operatorname{Ho}(\mathcal{S}pc)$ be another functor with a natural transformation $s\colon G \circ \gamma_{\mathcal{S}pc^{\mathcal{I}}} \to \gamma_{\mathcal{S}pc} \circ \operatorname{colim}$.

$$\begin{array}{ccc} \mathcal{S}pc^{\mathcal{I}} & \xrightarrow{\operatorname{colim}} & \mathcal{S}pc \\ {\scriptstyle \gamma_{\mathcal{S}pc^{\mathcal{I}}}}\downarrow & & \downarrow{\scriptstyle \gamma_{\mathcal{S}pc}} \\ \operatorname{Ho}(\mathcal{S}pc^{\mathcal{I}}) & \overset{\operatorname{hocolim}}{\underset{G}{\rightrightarrows}} & \operatorname{Ho}(\mathcal{S}pc) \end{array}$$

Define a natural transformation $s'\colon G \to \operatorname{hocolim}$ by $s'_{\boldsymbol{D}} = s_{\widetilde{\boldsymbol{D}}} G(p_{\boldsymbol{D}}^{-1})$, where $p_{\boldsymbol{D}}\colon \widetilde{\boldsymbol{D}} \to \boldsymbol{D}$ is the cofibrant approximation map. We must now check that $t \circ s' \circ \gamma_{\mathcal{S}pc^{\mathcal{I}}} = s$. Let $\boldsymbol{D}$ be an element of $\mathcal{S}pc^{\mathcal{I}}$. Then we must show that $t_{\boldsymbol{D}} \circ s'_{\gamma_{\mathcal{S}pc^{\mathcal{I}}}(\boldsymbol{D})} = s_{\boldsymbol{D}}$ (watch out — $s' \circ \gamma_{\mathcal{S}pc^{\mathcal{I}}}$ is not the composite of two natural transformations; it is defined by $(s' \circ \gamma_{\mathcal{S}pc^{\mathcal{I}}})_{\boldsymbol{D}} = s'_{\gamma_{\mathcal{S}pc^{\mathcal{I}}}(\boldsymbol{D})}$). Now

$$t_{\boldsymbol{D}} \circ s'_{\gamma_{\mathcal{S}pc^{\mathcal{I}}}(\boldsymbol{D})} = \operatorname{colim}(p_{\boldsymbol{D}}) \circ s_{\widetilde{\boldsymbol{D}}} \circ G(p_{\boldsymbol{D}}^{-1})$$

Since G is a functor, $G(p_{\boldsymbol{D}}^{-1}) = G(p_{\boldsymbol{D}})^{-1}$, and since s is a natural transformation, $\operatorname{colim}(p_{\boldsymbol{D}}) \circ s_{\widetilde{\boldsymbol{D}}} = s_{\boldsymbol{D}} \circ G(p_{\boldsymbol{D}})$. Thus, as we had wanted,

$$t_{\boldsymbol{D}} \circ s'_{\gamma_{\mathcal{S}pc^{\mathcal{I}}}(\boldsymbol{D})} = s_{\boldsymbol{D}} \circ G(p_{\boldsymbol{D}}) \circ G(p_{\boldsymbol{D}})^{-1} = s_{\boldsymbol{D}}$$

□

Adjointness is easier.

Proposition 4.7. *For all spaces Z, the homotopy colimit satisfies the adjointness property*

$$\operatorname{Hom}_{\operatorname{Ho}(\mathcal{S}pc)}(\operatorname{hocolim} \boldsymbol{D}, Z) = \operatorname{Hom}_{\operatorname{Ho}(\mathcal{S}pc^{\mathcal{I}})}(\boldsymbol{D}, \Delta Z)$$

Proof. Let Z' be a fibrant approximation for Z.

$$\begin{aligned} \operatorname{Hom}_{\operatorname{Ho}(\mathcal{S}pc)}(\operatorname{hocolim} \boldsymbol{D}, Z) &= \operatorname{Hom}_{\operatorname{Ho}(\mathcal{S}pc)}(\operatorname{colim} \widetilde{\boldsymbol{D}}, Z) \\ &= \operatorname{Hom}_{\operatorname{Ho}(\mathcal{S}pc)}(\operatorname{colim} \widetilde{\boldsymbol{D}}, Z') \\ &= [\operatorname{colim} \widetilde{\boldsymbol{D}}, Z']_{\mathcal{S}pc} \\ &= [\widetilde{\boldsymbol{D}}, \Delta Z']_{\mathcal{S}pc^{\mathcal{I}}} \\ &= \operatorname{Hom}_{\operatorname{Ho}(\mathcal{S}pc^{\mathcal{I}})}(\widetilde{\boldsymbol{D}}, \Delta Z') \\ &= \operatorname{Hom}_{\operatorname{Ho}(\mathcal{S}pc^{\mathcal{I}})}(\boldsymbol{D}, \Delta Z') \\ &= \operatorname{Hom}_{\operatorname{Ho}(\mathcal{S}pc^{\mathcal{I}})}(\boldsymbol{D}, \Delta Z) \end{aligned}$$

□

As a final check to make sure that this definition has the properties we were looking for, let's go back to one of our motivating examples. Recall that colimits frequently work nicely when things are somehow sufficiently cofibrant, but not in general. In particular, taking the colimit of the diagram

$$\begin{array}{ccc} A & \longrightarrow & X \\ \big\downarrow & & \\ * & & \end{array} \tag{4.8}$$

reliably gives the cofiber only if the map $A \to X$ is a cofibration. I assured you then that the homotopy colimit of that diagram would give you the cofiber as long as A and X were cofibrant, regardless of whether the map is a cofibration; now that we have defined the homotopy colimit, let's verify this.

So how do we take the homotopy colimit of that diagram? Well, we take a cofibrant approximation and then take the colimit of that. So the first question is how to take a cofibrant approxiation to Diagram 4.8. We defined a cofibration in the model category on diagrams to be a map with the left lifting property with respect to acyclic fibrations. Yuck! It is completely unclear what such a thing would be.

It turns out that a diagram of the shape of Diagram 4.8 is cofibrant if the spaces themselves are cofibrant and the maps are cofibrations. This can be checked directly from the definition with some effort; there are also explicit descriptions of all the cofibrations in the model category on diagrams (see [2]) which will yield this particular case. Be warned, however, that it is not true that a diagram of arbitrary shape is cofibrant if the spaces are cofibrant and the maps are cofibrations.

So given that, we need a cofibrant approximation to the particular Diagram 4.8. Since A and X are cofibrant, the following diagram will do the trick:

$$\begin{array}{ccc} A & \longrightarrow & X \cup_f (A \times I) \\ \big\downarrow & & \\ CA & & \end{array} \tag{4.9}$$

Now we can compute the homotopy colimit of Diagram 4.8.

$$\text{hocolim(Diagram 4.8)} = \text{colim(Diagram 4.9)} = X \cup_f (CA)$$

and this is indeed the cofiber of f.

REFERENCES

[1] W. G. Dwyer and J. Spalinski, *Homotopy theories and model categories*, Handbook of Homotopy Theory (I. M. James, ed.), Elsevier Science B.V., 1996.
[2] P. S. Hirschhorn, *Localization of model categories*, in preparation.

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