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TECHNICAL REPORT 468

DECEMBER 31, 1968

MASSACHUSETTS INSTITUTE OF TECHNOLOGY
RESEARCH LABORATORY OF ELECTRONICS
CAMBRIDGE, MASSACHUSETTS 02139

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The research reported in this document was made possible in part by support extended the Massachusetts Institute of Technology, Research Laboratory of Electronics, by the JOINT SERVICES ELECTRONICS PROGRAMS (U. S. Army, U. S. Navy, and U.S. Air Force) under Contract No. DA 28-043-AMC-02536(E), and by the National Aeronautics and Space Administration (Grant NGL 22-009-013).

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Submitted to the Department of Electrical Engineering, M. I. T., August 19, 1968, in partial fulfillment of the requirements for the degree of Doctor of Science.

(Manuscript received September 25, 1968)

Abstract

We are concerned with the detection of a set of M messages that are transmitted over a channel disturbed by chaotic thermal noise when quantum effects in the communication systems are taken into account. Our attention was restricted to the special case in which the density operators specifying the states of the received field are commutative. From quantum-mechanical description of the noise and signal fields, the structure and performance of the quantum-mechanical optimum receiver are found.

Two special communication systems have been studied: (i) a system in which signals have known classical amplitudes but unknown absolute phases, and the signal field is in coherent states; (ii) a system in which the classical amplitudes of the signal field are Gaussian random processes, and the received field in the absence of noise is in completely incoherent states. Bounds on the probability of error in these systems are derived. For both systems, the minimum attainable error probability is expressed in the form $\exp[-\tau CE(R)]$, where E(R) is the system reliability function which is a function of the information rate R of the system, τ is the time allotted for the transmission of a message, and C is the capacity of the system. For these two types of systems, the expressions for C and E(R) are derived.

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I. INTRODUCTION

The concept of quantization of electromagnetic radiations in communication channels and the notion of quantum noise were introduced, in 1952, by Gabor.[1] According to Shannon's derivation, the capacity of a continuous Gaussian channel with bandwidth W, average signal power S, and additive thermal noise power N, is $W \ln \left(1 + \frac{S}{N}\right)$. Gabor noted that such an expression for the channel capacity is unsatisfactory, since it approaches infinity as the noise power N approaches zero. Shannon's result is an intuitively expected one, however, as long as we assume that the received signal can be measured with increasing accuracy when the noise power decreases and the electromagnetic waves can be described classically. In actual communication systems, quantum effects become noticeable at low noise energy levels, and often become more important than the additive thermal noise. In this case, classical physics no longer adequately describes the electromagnetic fields of signal and noise.

In the domain of quantum mechanics, even an ideal observation introduces unavoidable perturbations to the system. Moreover, quantum theory limits the precision with which the fields can be measured. The uncertainty in the observed values of the dynamical variables of the signal field (for example, its amplitude and phase) can be considered an additional source of noise in the system which is sometimes called the "quantum noise." The energy associated with the quantum noise at frequency f has been shown to be of the order of hf, where h is the Planck's constant. At temperature T°K, it becomes larger than the average thermal noise energy when the ratio kT/hf ($\approx 0.7\lambda T$) becomes of order unity, where k is Boltzmann's constant, and λ is the wavelength of the transmitted electromagnetic wave in centimeters. For example, at 273°K and in the visible light frequency range, the quantum-noise energy is approximately 10^6 times larger than the average thermal-noise energy. Hence, in this frequency range, one can no longer ignore quantum noise in the derivation of channel capacity.

Interest in quantum effects in communication systems has been stimulated more recently by the development of communication systems using laser beams. The frequencies of the transmitted signals in these systems are in the optical frequency range. We shall briefly review related work done in this area.

1.1 RELATED WORK

Most of the previous work is concerned with the derivation of the capacities of channels that are quantum-mechanical models of communication systems. Emphasis was placed on the dependence of the channel capacity on the receiver at the output of the channel. Therefore, in these models, specific receivers are included as parts of the channels.

Stern,[2,3] Gordon,[4] and others[5-7] investigated the effects of Planck's quantization of radiation energy on the maximum entropy of an electromagnetic wave and the information capacity of a communication channel. In their studies, a continuous channel is replaced by an equivalent discrete one usually called a "photon channel." In a photon channel, the transmitter transmits an exact number of photons in each unit of time. The receiver measures the number of photons arriving at the output of the channel. For photon channels with various probability distributions of the additive noise and propagation losses, expressions for the channel capacities have been derived. As one would expect, the channel capacities remain finite as the thermal noise approaches zero in the model. But the concept of a photon channel is nonphysical. It has been noted[8] that there is no known physical system that is able to generate a precise number of photons. Furthermore, it is not possible to treat the photons as classical particles and use their number as desired symbols of an information source.

Gordon[4] also investigated the effect of quantization on the information capacities of communication systems using receiver systems other than photon counters. Quantum-mechanical limitations on the accuracy of measurements on the signal field were taken into account by modeling the receivers as noisy ones. For receiving systems, such as linear amplifiers, and heterodyne and homodyne converters, quantum noise in the receiver is found to be additive and Gaussian with energy proportional to hf. Therefore, the energy of the effective noise in the system (thermal noise and quantum noise) is nonzero at absolute zero temperature. Capacities of channels that include specific types of receivers and in which the thermal noise is white and Gaussian have been found to be finite at zero additive noise level. She[9] has also considered the fluctuations caused by repeated measurements that are found to be dependent on each other. In She's study, the capacity of a channel including a receiver that measures the complex electric field was found to be qualitatively the same as that obtained by the others.

The statistical signal detection problem that includes quantum-mechanical considerations was formulated by Helstrom.[10-12] The problem of deciding whether a signal has been transmitted becomes that of selecting one of the two possible density operators that gives a better description of the state of the field inside of the receiving cavity after it is exposed to a field in which a transmitted signal may or may not be present. It is found that the optimum receiver measures a dynamical variable that is represented by a projection operator whose null space is the subspace spanned by the eigenvectors corresponding to the positive eigenvalues of the operator $\rho_1 - d\rho_2$. Here, ρ_1 and ρ_2 are the density operators specifying the states of the field when the signal field is present and when the signal field is not present, respectively; and d is a constant depending on the cost function to be minimized. For equiprobable messages, d is equal to one when the cost function is the probability of error. Unfortunately, the optimum receiver cannot be found readily when the signal is coherent, the type usually generated by a laser. In the weak-signal case, when a threshold detector is used to approximate the optimum detector, Helstrom has shown that the effective signal-to-noise ratio does not become infinite when the thermal-noise energy approaches zero as in the classical case.

1.2 OBJECTIVES

We shall be concerned with the detection of a set of M messages that are transmitted through a channel disturbed by chaotic thermal noise. Quantum-mechanical limitations on the precision of measurements made by the receiver are taken into account. From quantum-mechanical descriptions of the noise field and the signal field, the dynamical variables of the fields to be measured by the optimum receiver and the decision procedure used by the receiver are determined. Bounds on the minimum attainable error probability in the form $\exp[-\tau CE(R)]$ are derived, where $E(R)$ is the system reliability function that is a function of the information rate R of the system, τ is the time allotted for the transmission of a message, and C is the capacity of the system. We shall be primarily concerned with the derivation of the system reliability functions of two communication systems: (i) a system in which the signal field is in known coherent states, but absolute phases of the signals are unknown; and (ii) a system in which the signal field is in completely incoherent states.

In Section II, a quantum-mechanical model of a communication system is described. Constraints imposed on the communication systems are discussed in detail.

In Section III, the structures of quantum-mechanical optimum receivers for a set of M signals that are represented by commutative density operators are specified. The minimum error probability attainable by the optimum receivers is expressed in terms of eigenvalues of the density operators. These results will be used in subsequent sections when specific communication systems are studied.

The reliability function for orthogonal signals in coherent states with random phases is derived in Section IV. (In the classical limit, this is the case of orthogonal signals with random phases but known amplitudes.) An expression for the channel capacity C is also derived. General behaviors of the system reliability functions are shown. The optimal performance for a system in which signals are in completely incoherent states is determined in Section V. As we shall see, this is a quantum-mechanical model of a fading channel in which classical amplitudes of signals are sample functions of Gaussian random processes.

In Section VI, possible extensions of our study and related research problems will also be discussed.

II. QUANTUM-MECHANICAL COMMUNICATION SYSTEM

We shall describe a model of a communication system that takes into account the quantum nature of electromagnetic radiation, as well as the quantum-mechanical limitation on the precision of measurements. The general characteristics of this model are presented in section 2.2. As in classical communication systems, a received signal is represented by an attenuated electromagnetic wave that propagates in the midst of randomly fluctuating background radiation of thermal origin. Quantum-mechanical descriptions of the received electromagnetic field will be briefly summarized. The detailed nature of the electromagnetic field generated by specific signal sources are discussed in Sections IV and V.

2.1 Quantum Description of the Electromagnetic Field

Let $\vec{\epsilon}_c(\underline{r}, t)$ denote a real classical wave function characterizing a time-limited electric field at the point $\underline{r}$ and time t. (Our discussions are confined to time-limited signal fields unless otherwise specified.) The average instantaneous electromagnetic power associated with such a field is zero outside of a time interval of length τ. Without loss of generality, let us suppose that this time interval is $(0, \tau)$.

According to the sampling theorem, such an electric field can be specified by a discrete spectrum in the frequency domain. In particular, the function $\vec{\epsilon}_c(\underline{r}, t)$ can be expanded in terms of the normal vector mode functions, $\vec{U}_k(\underline{r})$, of an appropriately chosen spatial volume of finite size L^3. The mode functions $\vec{U}_k(\underline{r})$ form a complete set. Moreover, they satisfy the orthonormality condition

$$\int_{L^3} \vec{U}_k(\underline{r}) \cdot \vec{U}_n(\underline{r})\, d^3\underline{r} = \delta_{kn}; \qquad \text{for all } k \text{ and } n$$

and the transversality condition

$$\nabla \cdot \vec{U}_k(\underline{r}) = 0; \qquad \text{for all } k.$$

A particular set of mode functions suitable for our purpose is the plane traveling-wave mode functions. That is,

$$\vec{U}_{\underline{k}}(\underline{r}) = L^{-3/2}\, \vec{e}_{\underline{k}} \exp(-i\underline{k}\cdot r),$$

where $\vec{e}_{\underline{k}}$ is a unit polarization vector perpendicular to the propagation vector $\underline{k}$. The possible values of $\underline{k}$ are given by

$$\underline{k} = \frac{2\pi}{L}\,(k_x\vec{e}_x + k_y\vec{e}_y + k_z\vec{e}_z),$$

where k_x, k_y, and k_z are integers from $-\infty$ to ∞. Expanding the function $\vec{\epsilon}_c(\underline{r}, t)$ in terms of these $\vec{U}_k(\underline{r})$, we obtain

$$\vec{\epsilon}_c(\underline{r}, t) = i \sum_{\underline{k}=0}^{\infty} \sqrt{\frac{\hbar\omega_k}{2L^3}}\, \vec{e}_{\underline{k}} \left\{C_k \exp[i(\underline{k}\cdot\underline{r} - \omega_k t)] - C_k^* \exp[-i(\underline{k}\cdot\underline{r} - \omega_k t]\right\} \quad 0 \leq t \leq \tau. \tag{1}$$

In this equation, the coefficients C_k and C_k^* are dimensionless constants. The quantities ω_k are equal to

$$\frac{2\pi n}{\tau}; \quad \text{for } n = 0, 1, 2, \ldots .$$

Since it is necessary that the condition

$$k = |\underline{k}| = \frac{\omega_k}{c}$$

be satisfied for all $\underline{k}$, we can choose L to be

$$L = c\tau, \tag{2}$$

where c is the velocity of the light.

Throughout this study, we shall be concerned with just one of the polarization components of the electric field, which can be adequately described by a scalar electric field $\epsilon_c(\underline{r}, t)$. Without loss of generality, we assume that the field propagates in the z direction. Thus, the expression for $\epsilon_c(\underline{r}, t)$ simplifies to

$$\epsilon_c(\underline{r}, t) = i \sum_{k=0}^{\infty} \sqrt{\frac{\hbar\omega_k}{2L}} \left\{C_k \exp\left[i\omega_k\left(\frac{z}{c} - t\right)\right] - C_k^* \exp\left[-i\omega_k\left(\frac{z}{c} - t\right)\right]\right\}. \tag{3}$$

For the purpose of quantizing the field, let us rewrite Eq. 3 as

$$E(\underline{r}, t) = i \sum_{k=0}^{\infty} \sqrt{\frac{\hbar\omega_k}{2L}} \left\{a_k \exp\left[i\omega_k\left(\frac{z}{c} - t\right)\right] - a_k^+ \exp\left[-i\omega_k\left(\frac{z}{c} - t\right)\right]\right\}. \tag{4}$$

When the field is quantized, the coefficients a_k and a_k^+ are regarded as operators satisfying the commutation relation[14]:

$$[a_k, a_n] = \left[a_k^+, a_n^+\right] = 0; \quad \text{for all } k \text{ and } n$$

$$\left[a_k, a_n^+\right] = \delta_{kn},$$

where $[x, y] = xy - yx$. Hence $E(\underline{r}, t)$ in Eq. 4 is a Hermitian operator. The Hamiltonian of the field is

$$H = \sum_{k=0}^{\infty} \left(a_k^+ a_k + \frac{1}{2}\right)\hbar\omega_k.$$

The normally ordered product $a_k^+ a_k$ is just the number operator N_k of the k^{th} mode, whose eigenvalues are all non-negative integers. That is,

$$N_k |n_k\rangle = n_k |n_k\rangle; \quad n_k = 0, 1, 2\ldots$$

for all k. Operators a_k and a_k^+ are usually called the annihilation and creation operators of the k^{th} mode, respectively. When operators a_k and a_k^+ are applied to the state $|n_k\rangle$, we have

$$a_k |n_k\rangle = \sqrt{n_k}\, |n_k - 1\rangle, \quad a_k |0\rangle = 0 \tag{5}$$

$$a_k^+ |n_k\rangle = \sqrt{n_k + 1}\, |n_k + 1\rangle.$$

Let us define an operator $E^{(+)}(\underline{r}, t)$ as

$$E^{(+)}(\underline{r}, t) = i \sum_{k=0}^{\infty} \sqrt{\frac{\hbar\omega_k}{2L}}\, a_k \exp\left[i\omega_k \left(\frac{z}{c} - t\right)\right]. \tag{6}$$

The electric field operator $E(\underline{r}, t)$ can be expressed in terms of $E^{(+)}(\underline{r}, t)$:

$$E(\underline{r}, t) = E^{(+)}(\underline{r}, t) + E^{(-)}(\underline{r}, t), \tag{7}$$

where $E^{(-)}(\underline{r}, t)$ is the Hermitian adjoint of the operator $E^{(+)}(\underline{r}, t)$. A state of the field is said to be a coherent state when the state vector $|\{\alpha_k\}\rangle$ is a right eigenvector of the operator $E^{(+)}(\underline{r}, t)$. That is,

$$E^{(+)}(\underline{r}, t)\ |\{\alpha_k\}\rangle = \epsilon(\underline{r}, t)\ |\{\alpha_k\}\rangle. \tag{8}$$

We write the eigenvalue

$$\epsilon(\underline{r}, t) = i \sum_{k=0}^{\infty} \sqrt{\frac{\hbar\omega_k}{2L}}\, \alpha_k \exp\left[i\omega_k \left(\frac{z}{c} - t\right)\right]. \tag{9}$$

Equations 8 and 9 imply that the k^{th} normal mode of the field is in the coherent state $|\alpha_k\rangle$ which satisfies the relation

$$a_k |\alpha_k\rangle = \alpha_k |\alpha_k\rangle.$$

The coherent-state vector $|\{\alpha_k\}\rangle$ of the entire field is just the direct product

$$\prod_{k=0}^{\infty} |\alpha_k\rangle = |\alpha_o, \alpha_1, \ldots \alpha_k, \ldots\rangle.$$

From Eq. 5, one can easily show that the coherent-state vector $|\alpha_k\rangle$ of the k^{th} normal mode can also be expressed as

$$|\alpha_k\rangle = \exp\left[a_k^+\alpha_k - \alpha_k^* a_k\right] |0\rangle. \tag{10}$$

The set of coherent-state vectors $|\{\alpha_k\}\rangle$ has been shown[15, 13] to form a complete set. That is,

$$\int |\{\alpha_k\}\rangle \langle\{\alpha_k\}| \prod_{k=0}^{\infty} \frac{d^2\alpha_k}{\pi} = \underline{1},$$

where $\underline{1}$ is the identity operator. Therefore, any state vector of the field and any linear operator can be expanded in terms of the coherent-state vectors $|\{\alpha_k\}\rangle$. Such an expansion, together with other properties of coherent states, have been discussed extensively by Glauber.[8, 15, 13]

In all cases that we shall consider, the field is not in a pure state. Rather, it is in a statistical mixture of states specified by a density operator ρ whose eigenvalues are non-negative. The density operator ρ is defined in such a way that the trace $\mathrm{tr}(\rho)$ equals one, and the trace $\mathrm{tr}(\rho X)$ equals the expected outcome when the observable corresponding to the operator X is measured. In all cases that will be considered, the density operator of the field can be expanded in terms of the coherent-state vector in the form

$$\rho = \int p(\{\alpha_k\}) \; |\{\alpha_k\}\rangle\langle\{\alpha_k\}| \prod_{k=0}^{\infty} d^2\alpha_k, \tag{11}$$

where $d^2\alpha_k$ stands for $(d\,\mathrm{Re}\,[\alpha_k])(d\,\mathrm{Im}\,[\alpha_k])$. (Re [ ] and Im [ ] denote the real and imaginary parts of the complex quantity in brackets.) Such a representation of the density operator is called the P-representation. The function $p(\{\alpha_k\})$ in Eq. 11 is called the weight function.

It is easy to show (see Appendix A) that the expansion of $E^{(+)}(r,t)$ in Eq. 6 is not unique. In particular, one may expand $E^{(+)}(r,t)$ as

$$E^{(+)}(\underline{r}, t) = i\sum_{j=0}^{\infty} \sqrt{\frac{\hbar\omega_j'}{2L}}\, b_j V_j(\underline{r}, t) \exp\left[i\omega_j'\left(\frac{z}{c} - t\right)\right], \tag{12}$$

where the set of new normal mode functions $V_j(\underline{r}, t)$ are given by Eq. A. 2 in general. The operators b_j are related to the a_k by the equation

$$b_j = \sum_{k=0}^{\infty} V_{jk} a_k. \tag{13}$$

V_{jk}'s are elements of unitary matrix V.

As shown in Appendix A, the operators b_j satisfy the same commutative relations that are satisfied by the operators a_k. Hence they can be considered as annihilation operators associated with the set of new normal modes in terms of which $E^{(+)}(\underline{r}, t)$ is expanded. Let $\{|\beta_k\rangle\}$ denote the set of right eigenvectors of the operators in the set $\{b_j\}$. Clearly,

$$\beta_j = \sum_k V_{jk}\alpha_k.$$

The density operator of the field can also be expanded in terms of coherent states $|\{\beta_k\}\rangle$ instead of $|\{\alpha_k\}\rangle$. From Eq. A. 6, we have

$$\rho = \int p'(\{\beta_k\})\,|\{\beta_k\}\rangle\langle\{\beta_k\}|\prod_k d^2\beta_k. \tag{14a}$$

The new weight function $p'(\{\beta_k\})$ is obtained by substituting the relation

$$\alpha_k = \sum_j \beta_j V^+_{kj} \tag{14b}$$

in the weight function $p(\{\alpha_k\})$.

The electric field at the receiver is usually generated by two independent sources. Let us assume that the first source alone brings the field to a state described by the density operator

$$\rho_1 = \int p_1(\{\alpha_k\})\,|\{\alpha_k\}\rangle\langle\{\alpha_k\}|\prod_k d^2\alpha_k,$$

and the second source acting alone generates a field in a state described by the density operator

$$\rho_2 = \int p_2(\{\alpha_k\})\,|\{\alpha_k\}\rangle\langle\{\alpha_k\}|\prod_k d^2\alpha_k.$$

Then the resultant field generated by these two independent sources is a state specified by the density operator

$$\rho = \int p(\{\alpha_k\})\,|\{\alpha_k\}\rangle\langle\{\alpha_k\}|\prod_k d^2\alpha_k,$$

where the weight function $p(\{\alpha_k\})$ is[8, 16]

$$p(\{\alpha_k\}) = \int p_1(\{\beta_k\})\,p_2(\{\alpha_k-\beta_k\})\prod_k d^2\beta_k.$$

2.2 QUANTUM-MECHANICAL COMMUNICATION SYSTEM

We are concerned with a communication system shown in Fig. 1. The input to the system is a sequence of M-ary symbols denoted $m_1, m_2, \ldots m_M$. For simplicity, it

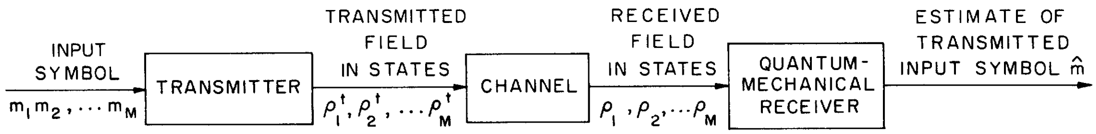

Fig. 1. Quantum-mechanical model of a communication system.

is assumed that the M symbols occur with equal probability, and successive symbols in the sequence are statistically independent. If one input symbol occurs at every τ seconds, the information rate of the input sequence is

$$R = \frac{\ln M}{\tau} \qquad \text{(nats/sec)}.$$

The output of the system is also an M-ary sequence whose information rate is R nats/sec. The output sequence is to reproduce the input sequence within some specified degree of fidelity. For our purpose, this fidelity is measured by an error probability $P(\epsilon)$ which will be defined later.

An input symbol m_j is represented by a transmitted electromagnetic field whose state is specified by a density operator ρ_j^t. (By state, we mean either a pure state or a mixture of pure states. For our purpose, there is no need to make such a distinction.) Therefore, in any particular time interval of length τ, the transmitted electromagnetic field is in one of the M states given by the density operators ρ_1^t, ρ_2^t, ... ρ_M^t corresponding to the input symbol m_1, m_2, ... m_M, respectively. The electromagnetic field at the output of the channel (the received field) in the corresponding time interval of length τ is in one of the M states specified by density operators ρ_1, ρ_2, ... ρ_M. Because of inverse-square loss, disturbance of additive background noise in the channel, and so forth, the density operators ρ_1, ρ_2, ... ρ_M usually are not the same as those specifying the transmitted states ρ_1^t, ρ_2^t, ... ρ_M^t.

For simplicity, the channel is assumed to be memoryless. It will also be assumed that there is no intersymbol interference. Since the input symbols are statistically independent, the states of the electromagnetic field at the receiver in any two time intervals are also statistically independent. Moreover, we shall only be concerned with systems that do not employ coding. In such a system, an error probability suitable as a measurement of the system performance is the baud error probability $P(\epsilon)$, which is defined to be

$$P(\epsilon) = \Pr(\hat{m} \neq m), \tag{15}$$

where $\hat{m}$ is the estimation in the output sequence of a particular symbol m in the input sequence. Our objective is to design the receiving system so that the probability of error $P(\epsilon)$ is minimized. It has been shown that the receiving system designed to minimize $P(\epsilon)$ makes independent estimations on successive symbols. Hereafter, therefore, one only need be concerned with the problem of making an optimum estimation of a single transmitted input symbol. In the rest of this section, we shall discuss the characteristics of additive noise field and some of the constraints imposed on the transmitted signal field which apply to both types of signals considered in Sections IV and V.

2.2.1 Transmitted Signal Field

We assume that the transmitted signal field is linearly polarized, that is, no polarization modulation is utilized in the system. In many practical systems, depolarization effect on the field during transmission is negligible and additive thermal noise field has statistically independent polarization components. Therefore, one can describe the relevant component of the field at the receiver by a scalar electric field operator.

It will also be assumed that the transmitted electromagnetic field is not modulated spatially. In the absence of additive noise, the field over the receiving aperture is taken to be a plane wave whose angle of arrival is known. Hence, the electric field at the receiver in the absence of noise field can be described by the electric field operator in Eq. 4. Both of these constraints (no polarization modulation and no spatial modulations in the signal field) impose no real restriction and can be generalized. We shall not consider these generalizations here.

We shall also restrict ourselves to the case in which the electric fields generated by the signal source corresponding to different input symbols are narrow-band and orthogonal. (The field generated by the signal source is strictly time-limited. It is approximately bandlimited in the sense that its energy outside of a finite bandwidth is essentially zero.[16, 17]) By orthogonal signal fields, we mean the following: Let $S_jV_j(\underline{r}, t)$ denote the classical complex amplitude of the electric field at the receiver and time t in the absence of additive background noise, when input symbol m_j is transmitted. That is,

$$\mathrm{Tr}\left[\rho_j^t E(\underline{r}, t)\right] = S_jV_j(\underline{r}, t) + S_j^*V_j^*(\underline{r}, t); \qquad j = 1, 2, \dots M. \tag{16}$$

(The parameter S_j may either be a known constant or a random variable.) We say that two signal fields corresponding to the input symbols m_j and $m_{j'}$ are orthogonal if

$$\int_{L^3} \int V_j(\underline{r}, t)\, V_{j'}^*(\underline{r}, t)\, d^3\underline{r}dt = \delta_{jj'}, \tag{17}$$

where the integration over time is carried out over a signaling interval. If we write $V_j(\underline{r}, t)$ as

$$V_j(\underline{r}, t) = \sum_{k=0}^{\infty} V_{jk} \exp\left[-i \frac{2\pi k}{\tau} \left(\frac{z}{c} - t\right)\right]$$

by Parseval's theorem, Eq. 17 is equivalent to the condition

$$\sum_{k=0}^{\infty} V_{jk} V^{*}_{j'k} = \delta_{jj'}. \tag{18}$$

A particular set of orthogonal signals that are studied in Sections IV and V is that for frequency position modulation. This set of signal waveforms satisfies also the condition

$$V_{jk} V^{*}_{j'k} = |V_{jk}|^2 \delta_{jj'} \tag{19}$$

approximately.

It is easy to see that the orthogonality condition is equivalent to the requirement that the normal modes of the electric field excited by the signal source corresponding to different input symbols will be different. (The $E^{(+)}(\underline{r}, t)$ operator and mode functions are given by Eqs. A. 7 and A. 8, respectively, for example.) We shall explore the implications of the orthogonality condition further in Sections IV and V when expressions of density operators for specific signals satisfying this constraint are derived.

Just as in a classical communication system, the orthogonality constraint on signal fields is a stringent one when the available bandwidth is limited. When the bandwidth for transmission is unlimited, however, orthogonal waveforms yield the best possible performance in classical systems. In our study, this constraint is imposed so that the analysis in Section V will be tractable. For the type of signals studied in Section IV, it has been possible to specify the optimum receiver and to evaluate the optimum performance only when the signal fields corresponding to different input symbols are orthogonal. Fortunately, this condition is often satisfied in practical systems.

2.2.2 Noise Field

Four types of noise are present in a quantum-mechanical communication channel, the source quantization noise, partition noise, quantum noise, and additive thermal noise. When the received electromagnetic field is described quantum mechanically, one need not be concerned with the source quantization noise. Partition noise is associated with the inverse-square law attenuation present in practical communication systems. The effect of such loss on information capacity of a communication channel has been studied by Hagfors[5] and Gordon.[4] "Quantum noise" is sometimes introduced to account for the limitations of the receiving system attributable to quantum-mechanical considerations (see Gordon,[4] She[9]). These types of noises will also emerge naturally from our formulation of the problem; thus, there is no need to consider them separately.

By additive thermal noise, we mean the chaotic background radiation of a thermal nature which is also present at the receiver. The classical amplitude of the electric field associated with the thermal noise is known to be a zero-mean stationary Gaussian random process. The source of this additive thermal noise field can be considered as a large collection of independent stationary sources. At thermal equilibrium, the noise field is in a state specified by a density operator, $\rho^{(n)}$, which in the P-representation is given by

$$\rho^{(n)} = \int \prod_k \frac{1}{\pi\langle n_k\rangle} \exp\left(-\frac{|\alpha_k|^2}{\langle n_k\rangle}\right) |\alpha_k\rangle\langle\alpha_k|\, d^2\alpha_k. \tag{20}$$

Such a density operator specifies a completely incoherent state. That this is so follows from the same kind of argument that leads to a Gaussian distribution for the sum of a large number of independent random variables.[13] $\langle n_k\rangle$ in Eq. 20 is the average number of photons in the k^{th} normal mode in the noise field (per unit volume). At background temperature T, it is given by

$$\langle n_k\rangle = \frac{1}{\exp\left(\frac{\hbar\omega_k}{kT}\right) - 1}. \tag{21}$$

Similarly to the white-noise assumption, we shall assume that the density operator $\rho^{(n)}$ of the noise field in the P-representation has a weight function of the form

$$p_n(\{\alpha_k\}) = \prod_k \frac{1}{\pi\langle n_k\rangle} \exp\left(-\frac{|\alpha_k|^2}{\langle n_k\rangle}\right)$$

independent of the choices of normal mode functions. That is, the noise modes are uncorrelated:

$$\mathrm{Tr}\left[\rho^{(n)} a_k^+ a_{k'}\right] = \langle n_k\rangle \delta_{kk'}.$$

(Classically, this restriction can be interpreted as follows: When a sample function of the random process of the noise electric field, $n(\underline{r}, t)$, is expanded in terms of any arbitrary orthonormal set $\{V_k(\underline{r}, t)\}$

$$n(\underline{r}, t) = \sum_k n_k V_k(\underline{r}, t),$$

the expansion coefficients n_k are statistically independent Gaussian random variables. This condition is satisfied by the additive white Gaussian noise.) To achieve analytical tractability in later sections, we shall further assume that the number of photons in all relevant modes are equal. That is,

$$\mathrm{Tr}\left[\rho^{(n)} a_k^+ a_k\right] = \langle n_k\rangle = \langle n\rangle.$$

It can be shown (see Appendix B) that the outcomes of measurements of any dynamical variables associated with different modes of the field are statistically independent. Thus, when the signal fields are linearly polarized plane waves as described in section 2.2.1, the relevant component of the total received field (signal field and noise field) can also be described by the scalar electric field operator given by Eq. 4. When the transmitted symbol is m_j, the received electric field is in a state specified by the density operator

$$\rho_j^t = \int p_j^t(\{\alpha_k\}) \, |\{\alpha_k\}\rangle \langle \{\alpha_k\}| \prod_k d^2\alpha_k$$

in the absence of noise field. In the presence of the thermal noise field, the density operator for all relevant modes of the total field at the receiver is given by

$$\rho_j = \int p_j(\{\alpha_k\}) \prod_k |\{\alpha_k\}\rangle \langle \{\alpha_k\}| \, d^2\alpha_k$$

$$p_j(\{\alpha_k\}) = \int p_j^t(\{\beta_k\}) \exp\left(-\sum_k \frac{|\alpha_k - \beta_k|^2}{\langle n\rangle}\right) \prod_k \frac{d^2\alpha_k}{\pi\langle n\rangle}. \tag{22}$$

III. GENERAL SPECIFICATION OF THE OPTIMUM RECEIVER

In the communication system shown in Fig. 1, the M equiprobable input symbols $m_1, m_2, \ldots, m_M$ are represented by the states of the transmitted electromagnetic wave. The state of the electromagnetic field at the receiver is specified by the density operator ρ_j when the input symbol is m_j ($j = 1, 2, \ldots, M$). Now, the structure and performance of the optimum receiving system will be found for a system in which the set of density operators are commutative. That is,

$$\rho_j \rho_k = \rho_k \rho_j; \quad \text{for all } j,\ k = 1, 2, \ldots M \tag{23}$$

By optimum receiver, we mean a receiving system that estimates the transmitted symbol in such a way that the error probability defined in Eq. 15 is minimal. At this point, other characteristics of the density operators ρ_j are purposely not specified, since discussions throughout this section apply to any communication system wherein density operators are commutative.

In classical communication theory, for M equiprobable input messages, the receiver that minimizes the probability of error is a maximum-likelihood receiver. Such a receiver measures the magnitudes of all variables $X_1, X_2, \ldots X_K$ of the received electromagnetic field that are relevant to the determination of the transmitted input symbol. The receiver estimates that the input symbol is m_j when the M inequalities

$$P(x_1, x_2, \ldots x_K/m_j) \geq P(x_1, x_2, \ldots x_K/m_i); \quad i = 1, 2, \ldots M$$

are satisfied, where $P(x_1, x_2, \ldots x_K/m_j)$ is the conditional probability distribution of the outcomes of the measurement of the variables $X_1, X_2, \ldots X_K$, given that the transmitted input symbol is m_j.

The derivation of such a classical optimum receiver is based on the assumption that all field variables can be measured simultaneously with arbitrary accuracy. In fact, only a subset of all dynamical variables of the field can be measured simultaneously with arbitrary precision. In general, the minimum attainable probability of error of a receiver using the maximum-likelihood decision procedure depends on the choice of variables observed simultaneously by the receiver. Therefore, before designing an optimum decision procedure for the receiver, one should first choose the subset of variables measured simultaneously by the receiver.

3.1 QUANTUM-MECHANICAL MODEL OF THE RECEIVING SYSTEM

As described in Section II, within the time interval of length τ when the signal field is expected to arrive, the electromagnetic field at the receiver is in a state specified by one of the density operators $\rho_1, \rho_2, \ldots \rho_M$. For convenience, we now idealize the receiving system as a cavity (or cavities). Since most optical frequency receivers admit radiation through an aperture into a system that occupies a finite region of space, a cavity is a natural idealization of these practical systems. The physical implementation of specific receivers is discussed in Sections IV and V, wherein it is shown that no generality is lost in such an idealization. (The idealization is also useful because it permits us to speak of receiver measurements on the field which do not affect the behaviors of the signal source and the noise source.)

We assume that the normal-mode functions of the cavity are such that the signal field can be expanded in terms of them. The cavity, initially empty, is exposed to the signal source through an aperture for the time interval within which the signal is expected to arrive. At the end of this time interval, the aperture is closed. The electromagnetic field inside of the cavity is then in a state specified by one of the density operators ρ_1, ρ_2, ... ρ_M. We assume that the time interval τ is long in comparison with the period of any Fourier component of the signal field. Moreover, opening and closing of the aperture do not disturb the electromagnetic field outside of the cavity. After the aperture is closed, measurements are made on the field inside of the cavity.

Let X_1, X_2, ... X_K denote the dynamical variables measured simultaneously by the receiver. These variables are represented by their corresponding linear operators which will be denoted X_1, X_2, ... X_K. For simplicity, let us assume that these operators are Hermitian and are commutative. Therefore their eigenvalues are real. Moreover, their eigenstates form a complete orthogonal set.

We shall assume that the measurement of any observable X_i is instantaneous. The possible outcomes of such a measurement are the eigenvalues x_{i1}, x_{i2}, ... x_{ik}, ... of the operator X_i. If the observed value of the dynamical variable X_i is x_{ik}, the field is in an eigenstate of X_i associated with the eigenvalue x_{ik} immediately after the measurement.

Since the operators X_1, X_2, ... X_K are commutative, there exists a complete set of eigenstates that are simultaneously eigenstates of these operators. In this case, it is meaningful to speak of the outcome of a simultaneous observation of these dynamical variables as an ordered K-tuple $\left(x_{1i_1}, x_{2i_2}, \dots x_{Ki_K}\right)$, and of the probability of a particular outcome being observed. To simplify following discussions, let us suppose that X_1, X_2, ..., X_K forms a complete set of commutative operators. Therefore, there is a unique simultaneous eigenstate of X_1, X_2, ... X_k associated with each K-tuple of eigenvalues $\left(x_{1i_1}, x_{2i_2}, \dots x_{Ki_K}\right)$. (It will be shown eventually that this condition is not necessary.) In the following discussion let us denote the set of K operators X_1, X_2, ... X_K by $\underline{X}$ and the K-tuple $\left(x_{1i_1}, x_{2i_2}, \dots x_{Ki_K}\right)$ by $x_{\underline{k}}$ whose associated eigenstate is $|x_{\underline{k}}\rangle$.

The conditional probability distribution of the outcome $x_{\underline{k}}$, given that the received field is in the state specified by the density operator ρ_j before the measurement, is

$$P(x_{\underline{k}}/\rho_j) = \langle x_{\underline{k}}|\rho_j|x_{\underline{k}}\rangle. \tag{24}$$

It is clear that the probability distribution of the outcome of any subsequent measurement depends only on the outcome of the first measurement, and not on the initial state of the system. Therefore, according to the theorem of irrelevance,[20] the subsequent measurements can be discarded in our consideration. Thus far, for simplicity, we have assumed that operators X_1, X_2, ... X_K have discrete spectra. It is easy to see that this assumption imposes no restrictions, since all of our discussions can be generalized by replacing sums with integrals and probability distributions with their corresponding probability density functions.

Unfortunately, the class of receivers that measure a set of dynamical variables corresponding to commutative Hermitian operators as described above does not include such devices as laser amplifiers. Since the field at the output of a laser amplifier is classical, it has precisely measurable amplitude and phase. Hence, conceptually, one may consider that the amplifier performs a simultaneous measurement of the conjugate variables, the amplitude and the phase of the input field. (It has been shown[18] that an additive Gaussian noise injected by the amplifier accounts for the inevitable error in the measurement imposed by the uncertainty principle.) One can generalize the class of receivers considered here to include those devices that make noisy simultaneous measurements of field variables corresponding to noncommutative Hermitian operators. In general, such measurements of noncommutative operators can be characterized by a complete set of states $\{|x_{\underline{k}}\rangle\}$.[19] The states $|x_{\underline{k}}\rangle$ are no longer required to be orthogonal as in the case when the measured variables are representable by commutative Hermitian operators. When the field is in a state specified by the density operator ρ_j, the probability that the outcome of a measurement associated with the complete set $\{|\chi_{\underline{k}}\rangle\}$ is $\chi_{\underline{k}}$ is again given by Eq. 24. It will become obvious in Section 3.2 that when the set of density operators $\{\rho_j\}$ is commutative, the optimum performance can be achieved by using a receiver that measures a set of dynamical variables corresponding to commutative Hermitian operators. Hence such a generalization of the receiver model as described above is not necessary, as long as we confine our attention to commutative density operators.

Let us assume momentarily that the receiver uses a randomized strategy to estimate the transmitted input symbol as m_j with probability $p_{j\underline{k}}$ when the outcome of the measurement of $\underline{X}$ is $x_{\underline{k}}$. Clearly,

$$\sum_{j=1}^{M} p_{j\underline{k}} = 1; \quad \text{for all } k \text{ [that is, for all K-tuples } (x_{1i_1}, \ldots x_{Ki_K})].$$

It will be shown that the optimum receiver, which minimizes the error probability $P(\epsilon)$, uses the maximum-likelihood decision procedure. That is, $p_{j\underline{k}}$ is equal to one when

$$P(x_{\underline{k}}/m_j) \geq p(x_{\underline{k}}/m_i); \quad \text{for } i,\ j = 1, 2, \ldots M$$

and is equal to zero otherwise.

The conditional probability that a correct estimation is made, given that the transmitted input symbol is m_j, is

$$P(C/m_j) = \sum_{\underline{k}} p_{j\underline{k}} \langle x_{\underline{k}} | \rho_j | x_{\underline{k}} \rangle.$$

It follows that the conditional probability of error, given that the transmitted input symbol is m_j, is

$$P(\epsilon/m_j) = 1 - \sum_{\underline{k}} p_{j\underline{k}} \langle x_{\underline{k}} | \rho_j | x_{\underline{k}} \rangle.$$

Hence, the error probability is

$$P(\epsilon) = 1 - \frac{1}{M} \sum_{j=1}^{M} \sum_{\underline{k}} p_{j\underline{k}} \langle x_{\underline{k}} | \rho_j | x_{\underline{k}} \rangle. \tag{25}$$

In general, it is difficult to find the set of eigenvectors $\{|x_{\underline{k}}\rangle\}$ and the set of probabilities $\{p_{j\underline{k}}\}$ that minimize the right-hand side of Eq. 25. In this report, we shall consider only the special case when the ρ_j are commutative. For this special case, we are able to find both the structure and the performance of the optimum receiver.

3.2 OPTIMUM RECEIVER FOR EQUIPROBABLE INPUT SYMBOLS REPRESENTED BY COMMUTATIVE DENSITY OPERATORS

Let us consider the special case in which the received field representing the transmitted input symbol m_1, or m_2, ..., or m_M is in a state specified by the density operator ρ_1, of ρ_2, ..., or ρ_M, respectively, and

$$\rho_i \rho_j = \rho_j \rho_i \quad \text{for all } i, \ j = 1, 2, \ldots M.$$

For such a set of density operators, there exists a complete orthogonal set of eigenstates that are simultaneously eigenstates of all of the density operators in the set. Let these eigenstates be represented by the kets $|r_k\rangle$. Then, the density operator ρ_j can be written

$$\rho_j = \sum_k r_{jk} |r_k\rangle \langle r_k|; \qquad j = 1, 2, \ldots M, \tag{26}$$

where r_{jk} is the eigenvalue of the density operator ρ_j corresponding to the eigenstate $|r_k\rangle$. Again, without loss of generality, the eigenspectra of the density operators are supposed to be discrete.

3.2.1 Performance of the Optimum Receiver

Substituting the expressions for the ρ_j's in Eq. 26 in Eq. 25, we obtain

$$P(\epsilon) = 1 - \frac{1}{M} \sum_{j=1}^{M} \sum_{i} \sum_{\underline{k}} p_{j\underline{k}} r_{ji} |\langle x_{\underline{k}} | r_i \rangle|^2. \tag{27}$$

Let $\left(\max_{1 \leq j \leq M} r_{ji}\right)$ denote the largest of the M eigenvalues $r_{1i}, r_{2i}, \ldots, r_{Mi}$ for a given value of i. Then, it is clear that

$$P(\epsilon) \geq 1 - \frac{1}{M} \sum_{j=1}^{M} \sum_{i} \sum_{\underline{k}} p_{j\underline{k}} \left(\max_{1 \leq j \leq M} r_{ji}\right) |\langle x_{\underline{k}} | r_i \rangle|^2. \tag{28}$$

Because

$$\sum_{j=1}^{M} p_{j\underline{k}} = 1; \quad \text{for all } \underline{k} \tag{28a}$$

and

$$\sum_{\underline{k}} |\langle x_{\underline{k}} | r_i \rangle|^2 = 1; \quad \text{for all } i. \tag{28b}$$

Eq. 28 can be simplified to

$$P(\epsilon) \geq 1 - \frac{1}{M} \sum_{i} \left(\max_{1 \leq j \leq M} r_{ji}\right). \tag{29}$$

(Note that Eq. 28b is true as long as the set of states $\{|x_{\underline{k}}\rangle\}$ is complete. It is not necessary for the set to be orthogonal.) The right-hand side of Eq. 29 gives a lower bound to the error probability in the reception of signals specified by the density operators $\rho_1, \rho_2, \ldots, \rho_M$.

3.2.2 Specifications of the Optimum Receiver

The lower bound to the error probability in (29) can be achieved by a receiver that measures a dynamical variable represented by a linear operator X whose set of eigenstates coincides with the set $\{|r_k\rangle\}$ (or a set of operators whose simultaneous eigenstates are $|r_k\rangle$). Let x_k denote the eigenvalue of the operator X corresponding to the eigenstate $|r_k\rangle$. For the moment, let us suppose that all x_k are distinct; that is,

$$x_k \neq x_i \qquad \text{for all } i \neq k.$$

Since

$$\langle r_k | x_i \rangle = \langle r_k | r_i \rangle = \delta_{ki},$$

the right-hand side of (27) can be simplified to

$$1 - \frac{1}{M} \sum_{j=1}^{M} \sum_{k} p_{jk} r_{jk}. \tag{30}$$

By choosing the p_{jk} in such a way that for all k, p_{jk} is equal to one if

$$r_{jk} \geq r_{ik}; \qquad \text{for } i, \ j = 1, 2, \ldots . M \tag{31}$$

and p_{ik} is equal to zero for all other $i \neq j$, the right-hand side of Eq. 30 becomes

$$1 - \frac{1}{M} \sum_{k} \left(\max_{1 \leq j \leq M} r_{jk} \right),$$

which is the minimum attainable error probability. (This rule becomes ambiguous whenever Eq. 31 is satisfied for more than one value of j. The ambiguity can be resolved, however (for example, by letting $p_{jk} = 1$ for the smallest value of j satisfying (31) and setting all other p_{ik} equal to zero). Clearly, the minimum value of $P(\epsilon)$ is not affected by the way this ambiguity is resolved.) Since

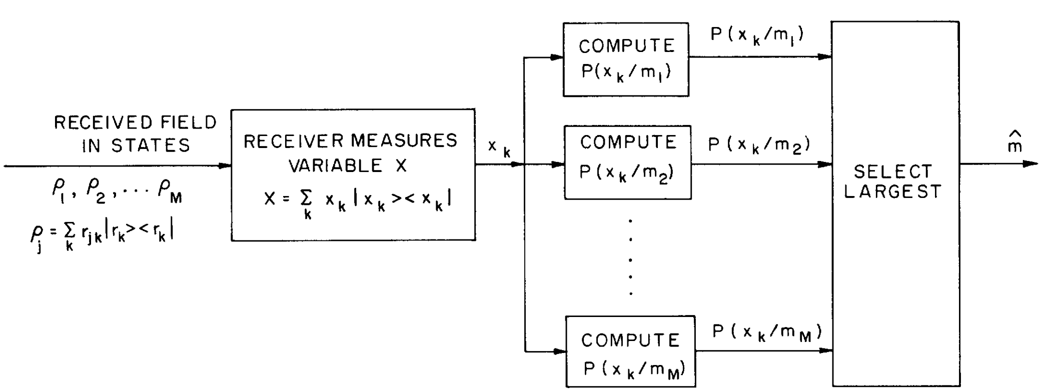

Fig. 2. Quantum-mechanical optimum receiver for signals represented by commutative density operators.

$$r_{jk} = \langle r_k | \rho_j | r_k \rangle = P(x_k/m_j),$$

it is evident that the strategy described above is a maximum-likelihood decision rule. The optimum receiver is shown schematically in Fig. 2.

From our discussion, it is evident that the operator X that represents the variable measured by the optimum receiver is Hermitian. Moreover, it is not unique. In practical situations, one may choose a dynamical variable among a class of variables representable by Hermitian operators which can be measured most conveniently.

First, one can show that it is not necessary for the set of eigenstates $\{|x_k\rangle\}$ of the operator X to be the set $\{|r_k\rangle\}$ in order that the receiver measuring the variable X will achieve the minimum attainable error probability. To show this, let us denote by R_j a subset of eigenstates of the density operators. An eigenstate $|r_k\rangle$ is in R_j if the eigenvalue of ρ_i associated with $|r_k\rangle$, r_{jk}, is larger than that of all other density operators. That is, $|r_k\rangle$ is in R_j if

$$r_{jk} > r_{ik} \quad \text{for all } i = 1, 2, \ldots . (j-1)$$

$$r_{jk} \geq r_{ik} \quad \text{for all } i = (j+1), \ldots\ldots, M.$$

Clearly, the sets R_j are disjoint and their union $\bigcup_{j=1}^{M} R_j$ is the entire set of eigenstates $\{|r_k\rangle\}$. Let us also denote by R_j' the subset of eigenstates of X which are not orthogonal to some eigenstate $|r_i\rangle$ in R_j. That is, $|x_k\rangle$ is in R_j' if

$$\langle x_k | r_i \rangle \neq 0 \quad \text{for} \quad \text{some } |r_i\rangle \text{ in } R_j,\ j = 1, 2, \ldots M.$$

We claim that the receiver measuring the variable X can achieve the minimum attainable error probability if and only if the eigenstates of X satisfy the following conditions: (i) The set of eigenstates of the operator X relates to the set $\{|r_k\rangle\}$ by a unitary transformation. That is,

$$\sum_k |\langle x_k | r_i \rangle|^2 = \sum_i |\langle x_k | r_i \rangle|^2 = 1$$

and R_j' is nonempty if R_j is nonempty. (ii) The subsets R_j', j = 1, 2, ... M are disjoint. Let us rewrite the error probability as follows:

$$P(\epsilon) = 1 - \frac{1}{M} \sum_{j=1}^{M} \left\{ \sum_{\{k/|x_k\rangle \in R_j'\}} \sum_{\{i/|r_i\rangle \in R_j\}} (p_{1k}r_{1i}+p_{2k}r_{2i}+\ldots p_{Mk}p_{Mi}) |\langle x_k | r_i \rangle|^2 \right\} \tag{32}$$

According to the definition of R_j', the right-hand side of (32) is minimized by letting $p_{jk} = 1$ for $|x_k\rangle \in R_j'$. It follows that if conditions (i) and (ii) are satisfied, (32) becomes

$$P(\epsilon) = 1 - \frac{1}{M}\left\{\sum_{j=1}^{M} \sum_{\{i/|r_i\rangle \in R_j\}} r_{ji}\right\} \tag{33}$$

which is the minimum attainable value of $P(\epsilon)$.

On the other hand, if condition (i) is not satisfied, the minimum value of $P(\epsilon)$ cannot be achieved. For example, suppose that the set R_1' is empty while the set R_1 is non-empty. It is clear that the right-hand side of (32) is larger than or equal to

$$1 - \frac{1}{M} \sum_{\{i/|r_i\rangle \notin R_1\}} \left(\max_{1 \leq j \leq M} r_{ji}\right),$$

which is larger than the value given by Eq. 33.

To show that the minimum attainable value of $P(\epsilon)$ cannot be achieved when condition (ii) is not satisfied, let us suppose that the sets R_1' and R_2' are not disjoint. Let C denote the set intersection of R_1' and R_2', $R_1' \cap R_2'$. Equation 32 can be rewritten

$$P(\epsilon) \geq 1 - \frac{1}{M}\left\{\sum_{\{i/|r_i\rangle \in R_1\}} r_{1i}\left[\sum_{\{k/|x_k\rangle \in R_1'-C\}} |\langle x_k|r_i\rangle|^2 + \sum_{\{k/|x_k\rangle \in C\}} p_{1k}|\langle x_k|r_i\rangle|^2\right]\right.$$

$$+ \sum_{\{i/|r_i\rangle \in R_2\}} r_{2i}\left[\sum_{\{k/|x_k\rangle \in R_2'-C\}} |\langle x_k|r_i\rangle|^2 + \sum_{\{k/|x_k\rangle \in C\}} p_{2k}|\langle x_k|r_i\rangle|^2\right]$$

$$\left. + \sum_{j=3}^{M} \sum_{\{i/|r_i\rangle \in R_j\}} r_{ji}\right\}.$$

Since

$$\sum_{\{k/|x_k\rangle \in R_j'\}} |\langle x_k|r_i\rangle|^2 = \sum_k |\langle x_k|r_i\rangle|^2 = 1; \quad \text{for all } |r_i\rangle \in R_j,\ j = 1, 2, \ldots M$$

and

$$p_{1k} + p_{2k} = 1; \quad \text{for } |x_k\rangle \in C,$$

we cannot make both terms

$$\sum_{\{k/|x_k\rangle \in R_1'-C\}} |\langle x_k|r_i\rangle|^2 + \sum_{\{k/|x_k\rangle \in C\}} p_{1k}|\langle x_k|r_i\rangle|^2$$

$$\sum_{\{k/|x_k\rangle \in R_2'-C\}} |\langle x_k|r_i\rangle|^2 + \sum_{\{k/|x_k\rangle \in C\}} p_{2k}|\langle x_k|r_i\rangle|^2$$

equal to one unless the set C is empty.

Second, we note that the eigenspectrum of the operator X need not be simple. It is obvious that the eigenvalues associated with the eigenstates of X in any subset R_j' need not be distinct. On the other hand, it is necessary that the eigenvalue associated with an eigenstate $|x_k\rangle$ in R_j' is not equal to that associated with an eigenstate $|x_{k'}\rangle$ in R_i' whenever $i \neq j$.

Sometimes, it is more convenient for the receiver to measure not just one dynamical variable of the field but a set of simultaneously measurable variables represented by commuting operators $X_1, X_2, \ldots. X_K$. The outcomes of such a measurement are the K-tuples $x_{\underline{k}}$ with corresponding eigenstates $|x_{\underline{k}}\rangle$. Clearly, the discussion above applies directly to this case.

Thus far, the time at which the receiver makes the measurement on the received field has not been specified. Here, we shall show that the optimum performance of the system is independent of the time at which the observation is made. The choice of the dynamical variable (or variables) measured by the optimum receiver does depend, however, on the time of the observation.

According to the causal law, in the Schrödinger picture, the density operator of a system at any time $t \geq t_o$ is

$$\rho_j(t) = U(t, t_o)\, \rho_j(t_o)\, U^+(t, t_o)$$

if the state of the field at t_o is specified by the density operator $\rho_j(t_o)$, and the system is undisturbed in the time interval (t, t_o). The time evolution operator $U(t, t_o)$, since it is unitary, satisfies

$$U(t_o, t_o) = 1 \qquad U^+(t, t_o)\, U(t, t_o) = 1.$$

When the density operators are expanded in terms of their simultaneous eigenstates,

$$\rho_j(t) = \sum_i r_{ji} |r_i(t)\rangle \langle r_i(t)|$$

$$= \sum_i r_{ji} U(t, t_o) |r_i(t_o)\rangle \langle r_i(t_o)|\, U^+(t, t_o).$$

Thus, letting $P(\epsilon, t)$ be the error probability when the measurement is made at time t, we obtain

$$P(\epsilon,t) = 1 - \frac{1}{M} \sum_{j=1}^{M} \sum_{k} \sum_{i} r_{ji} \langle x_k(t) |\, U(t, t_o)\, | r_i(t_o) \rangle \langle r_i(t_o) |\, U^+(t, t_o)\, | x_k(t) \rangle\, p_{jk}$$

$$\geq 1 - \frac{1}{M} \sum_{j=1}^{M} \sum_{k} \sum_{i} \left(\max_{1 \leq j \leq M} r_{ji} \right) | \langle x_k(t) |\, U(t, t_o)\, | r_i(t_o) \rangle |^2\, p_{jk}$$

$$= 1 - \frac{1}{M} \sum_{k} \left(\max_{1 \leq j \leq M} r_{jk} \right),$$

which turns out to be independent of t. In general, if the optimum receiver measures the operator $X(t_o)$ at time t_o, the dynamical variable that should be measured at time t is

$$X(t) = U(t, t_o)\, X(t_o)\, U^+(t, t_o).$$

IV. ORTHOGONAL SIGNALS WITH KNOWN AMPLITUDE AND RANDOM PHASE

A communication system that has been studied extensively in classical communication theory is one in which the receiver knows all characteristics of the received signal field except its absolute phase. When the signal field is not modulated spatially, its classical amplitude at the receiver can be described by the function

$$2\ \mathrm{Re}\left[S(t)\ \exp\left\{i\omega\left(\frac{z}{c}-t\right)+i\phi\right\}\right], \tag{34}$$

where S(t) is one of a set of time functions $\{S_j(t)\}$ depending on the transmitted input symbol, and ϕ is a random variable with a uniform probability density function

$$p_\phi(x) = \begin{cases} \dfrac{1}{2\pi}, & 0 \leqslant x \leqslant 2\pi \\ 0, & \text{elsewhere.} \end{cases}$$

The randomness of the parameter ϕ corresponds to the uncertainty in the phase of the received signal. Such an uncertainty is caused by fluctuations such as slow oscillator drift, small random variations in the propagation time between the transmitter and the receiver, and so forth. When signal fields are in the optical-frequency range, one is often ignorant of the exact signal phase.

We shall now find the structure of the optimum receiver and the performance of this communication system in the quantum limit. To describe the received signal field quantum-mechanically, let us expand the electric field operator as in Eq. 4:

$$E(\underline{r},t) = i\sum_k \sqrt{\frac{\hbar\omega_k}{2L}}\left\{a_k \exp\left[i\omega_k\left(\frac{z}{c}-t\right)\right] - a_k^+ \exp\left[-i\omega_k\left(\frac{z}{c}-t\right)\right]\right\}.$$

When the transmitted symbol is m_j, the field present at the receiver in the absence of the additive thermal-noise field is in a state specified by the density operator

$$\rho_j^t = \int_0^{2\pi} \frac{d\phi}{2\pi}\,|\sigma_{j1}e^{i\phi}, \sigma_{j2}e^{i\phi}, \dots \sigma_{jk}e^{i\phi}, \dots\rangle\langle \dots \sigma_{jk}e^{i\phi}, \dots \sigma_{j2}e^{i\phi}, \sigma_{j1}e^{i\phi}| \tag{35}$$

during a time interval of length τ within which the signal is expected to arrive at the receiver. In Eq. 35, $\sigma_{j1}, \sigma_{j2}, \dots \sigma_{jk}, \dots$ are complex quantities known to the receiver, while ϕ is a random variable evenly distributed over the interval $(0, 2\pi)$. That is, the signal field at the receiver is in one of the coherent states $|\sigma_{j1}e^{i\phi}, \sigma_{j2}e^{i\phi}, \dots \sigma_{jk}e^{i\phi}, \dots\rangle$. It should be pointed out that in classical communication theory, a signal with a known amplitude but an unknown absolute phase is sometimes called an <u>incoherent</u> signal. In the quantum mechanical limit, such signal fields are in <u>coherent</u> states. Later, in

Section V, we shall show that when the classical signal waveforms are sample functions from Gaussian random processes, the corresponding quantum signal fields are in completely incoherent states. Throughout this section, our discussion is confined to the type of signal fields in states specified by density operators of the form ρ_j^t in Eq. 35, and we shall use the quantum-mechanical meaning of "coherence."

It has been shown[15] that an electromagnetic field in a coherent state is generated by a classical current source. By a classical current source, we mean one whose reaction with the process of radiation is either negligible or at least predictable in principle. Such a model of the signal source is an excellent approximation of most macroscopic sources, since the number of atoms in such sources is so large that variations in the total current vector become statistically predictable.

When the amplitude of the electric field in the state specified by ρ_j^t in Eq. 35 is measured, the outcome at time t for a given ϕ is a sample function of a Gaussian random process with the mean-value function

$$\mathrm{Tr}\left[\rho_j^t E(\underline{r},t)\right] = 2\,\mathrm{Re}\left\{\sum_k \sqrt{\frac{\hbar\omega_k}{2L}}\, i\sigma_{jk} \exp\left[i\omega_k\left(\frac{z}{c}-t\right)+i\phi\right]\right\}. \tag{36}$$

Its associated covariance function approaches zero in the classical limit ($\hbar \to 0$). Hence, if the σ_{jk} and ω_k are so chosen that

$$S_j(t) = \sum_k \sqrt{\frac{\hbar\omega_k}{2L}}\, i\sigma_{jk} \exp\left[i(\omega_k-\omega)\left(\frac{z}{c}-t\right)\right]; \qquad j = 1, 2, \ldots M,$$

the set of signals characterized by the density operators ρ_j^t ($j = 1, 2, \ldots M$) is just the set of signals given by (34) in the classical limit.

As we have pointed out, we are concerned with the particular set of M orthogonal signals corresponding to frequency position modulation. If $\mathcal{S}_j(\omega)$ is the Fourier spectrum of the time function $S_j(t)$ in the classical signal field (Eq. 34), then the $\mathcal{S}_j(\omega)$ satisfy the condition

$$\mathcal{S}_j(\omega)\,\mathcal{S}^*_{j'}(\omega) = |\mathcal{S}_j(\omega)|^2\,\delta_{jj'}. \tag{37a}$$

(Again, the $S_j(t)$ are strictly time-limited signals; they cannot be strictly frequency-limited. They are, however, approximately bandlimited, in the sense that the energy associated with each of the $S_j(t)$ outside of a frequency range of the order of $\frac{1}{\tau}$ Hz is essentially zero; hence, Eq. 37a can be satisfied approximately.) Similarly, in the corresponding quantum-mechanical system, the set of complex quantities σ_{jk} in the density operators ρ_j^t satisfy the constraint in Eq. 19. That is,

$$\sigma_{jk}\sigma^*_{j'k} = |\sigma_{jk}|^2\,\delta_{jj'}. \tag{37b}$$

We shall now derive the structure of the optimum receiver and bounds on its performance for a set of M unmodulated signals. (For each transmitted input symbol, the signal source excites only a single mode of oscillation.) It will be shown that these results also apply to the case of arbitrary narrow-band orthogonal coherent signals with a common random phase.

4.1 UNMODULATED ORTHOGONAL SIGNAL WITH RANDOM PHASE

Let us consider the communication system shown in Fig. 1. When the transmitted input symbol is m_j, the signal component of the received field during the time interval $(0, \tau)$ when the signal is expected to arrive at the receiver is described by the density operator

$$\rho_j^t = \frac{1}{2\pi}\int_0^{2\pi} d\phi \left|0, 0, \dots \sigma_j e^{i\phi}, \dots 0\right\rangle\left\langle 0, 0, \dots \sigma_j e^{i\phi}, 0, \dots\right|; \quad j = 1, 2, \dots M. \tag{38}$$

That is, when the transmitted input symbol is m_j, only the j^{th} mode of the field is excited, while the other modes remain in their vacuum states. For simplicity, the relevant modes of the field are denoted as the 1st, 2nd, ... Mth. When the background thermal-noise field is also present, the received field is in a state specified by the density operator ρ_j.

$$\rho_j = \int_0^{2\pi} \frac{d\phi}{2\pi} \int \exp\left[-\frac{\left|a_j - \sigma_j e^{i\phi}\right|^2}{\langle n\rangle}\right] |a_j\rangle\langle a_j| \frac{d^2 a_j}{\pi\langle n\rangle} \left\{\prod_{k\neq j} \int \exp\left[-\frac{|a_k|^2}{\langle n\rangle}\right] |a_k\rangle\langle a_k| \frac{d^2 a_k}{\pi\langle n\rangle}\right\}$$

$$= \int I_o\left(\frac{2|a_j||\sigma_j|}{\langle n\rangle}\right) \exp\left[-\frac{|\sigma_j|^2}{\langle n\rangle} - \sum_k \frac{|a_k|^2}{\langle n\rangle}\right] \prod_k |a_k\rangle\langle a_k| \frac{d^2 a_k}{\pi\langle n\rangle}.$$

(The state of the additive thermal-noise field is specified by the density operator $\rho^{(n)}$ in Eq. 20.)

It is evident that the states of modes of the received field other than the 1st, 2nd, ... Mth ones are independent of the transmitted input symbol. Since the outcomes in measuring dynamical variables of any two different modes are statistically independent, as shown in Appendix B, we need only consider the first M modes. The density operator ρ_j specifying the states of the relevant modes is given by

$$\rho_j = \left(\frac{1}{\pi\langle n\rangle}\right)^M \left\{\exp\left[-\frac{|\sigma_j|^2}{\langle n\rangle}\right]\right\} \int \exp\left[-\sum_{k=1}^M \frac{|a_k|^2}{\langle n\rangle}\right] I_o\left(\frac{2|a_j||\sigma_j|}{\langle n\rangle}\right) \prod_{k=1}^M |a_k\rangle\langle a_k|\, d^2 a_k;$$

$$j = 1, 2, \dots M \tag{39}$$

4.1.1 Structure of the Optimum Receiver

It is obvious from Eq. 39 that all ρ_j are diagonalized in the number representation. When the density operator ρ_j is expanded in terms of the simultaneous eigenvectors of the number operators $N_1, N_2, \ldots N_M$ of the individual modes, $|n_1, n_2, \ldots n_M\rangle$, we obtain

$$\rho_j = \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \cdots \sum_{n_M=0}^{\infty} \left(\frac{1}{1+\langle n\rangle}\right)^M \left(\frac{\langle n\rangle}{1+\langle n\rangle}\right)^{n_1+n_2+\ldots n_M} \left\{\sum_{r=0}^{n_j} \binom{n_j}{r} \frac{1}{r!} \left[\frac{|\sigma_j|^2}{\langle n\rangle(1+\langle n\rangle)}\right]^r\right\}$$

$$\exp\left(-\frac{|\sigma_j|^2}{1+\langle n\rangle}\right) |n_1, n_2, \ldots n_M\rangle \langle n_1 \ldots n_M| \qquad \text{for } j = 1, 2, \ldots M.$$

Therefore, according to the discussion in section 3.2, the optimum receiver for the reception of this set of M signals measures the number of photons in each of the M modes simultaneously.

Given that m_j is transmitted, the joint conditional probability distribution of the observed numbers of photons $n_1, n_2, \ldots n_M$ in the modes $1, 2, \ldots M$, respectively, is

$$P(n_1, n_2, \ldots n_M/m_j) = \langle n_1, n_2, \ldots n_M| \rho_j |n_1, n_2, \ldots n_M\rangle$$

$$= \left(\frac{1}{1+\langle n\rangle}\right)^M \left(\frac{\langle n\rangle}{1+\langle n\rangle}\right)^{n_1+n_2+\ldots n_M} \left\{\sum_{r=0}^{n_j} \binom{n_j}{r} \frac{1}{r!} \left[\frac{|\sigma_j|^2}{\langle n\rangle(1+\langle n\rangle)}\right]^r\right\}$$

$$\exp\left(-\frac{|\sigma_j|^2}{1+\langle n\rangle}\right); \qquad j = 1, 2, \ldots M. \tag{40}$$

The receiver estimates that the transmitted input symbol is m_j if the conditional probability $P(n_1, n_2, \ldots n_M/m_j)$ is maximum over all j. (Again, when more than one j maximizes the quantity $P(n_1, \ldots n_M/m_j)$, the decision rule becomes ambiguous. But the performance of the receiver is independent of the manner in which the ambiguity is resolved.) According to (40), the maximization of $P(n_1, n_2, \ldots n_M/m_j)$ amounts to the maximization of the quantity

$$f_j = \sum_{r=0}^{n_j} \binom{n_j}{r} \left[\frac{|\sigma_j|^2}{\langle n\rangle(1+\langle n\rangle)}\right]^r \frac{1}{r!} \exp\left(-\frac{|\sigma_j|^2}{1+\langle n\rangle}\right) \tag{41}$$

A block diagram of the optimum receiver is shown in Fig. 3. As discussed in

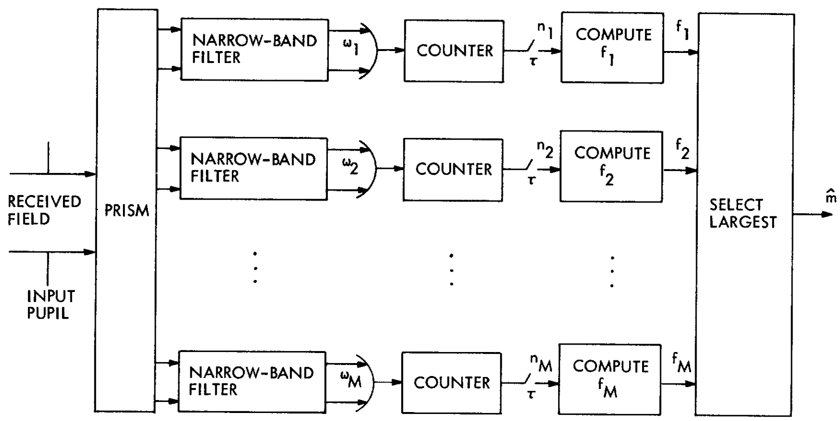

Fig. 3. Optimum receiver for unmodulated signals with random phases.

Appendix C, the photomultiplier tube and counter combinations indeed measure the dynamical variables N_k, $k = 1, 2, \ldots M$.

For the sake of simplicity, we shall assume that the complex quantities σ_j are all equal; that is,

$$\sigma_j = \sigma; \qquad j = 1, 2, \ldots M. \tag{42}$$

Let us note that when input symbol m_j is transmitted, the average number of photons in the j^{th} mode of the received field in the absence of additive noise is given by

$$\mathrm{Tr}\left[\rho_j^t a_j^+ a_j\right] = \int_0^{2\pi} \langle 0, 0, \ldots \sigma_j e^{i\phi}, \ldots 0 |\, a_j^+ a_j \,| 0, 0, \ldots \sigma_j e^{i\phi}, \ldots 0 \rangle \frac{d\phi}{2\pi}$$

$$= |\sigma_j|^2.$$

On the other hand, the average number of photons in the k^{th} mode of the received signal field is

$$\mathrm{Tr}\left[\rho_j^t a_k^+ a_k\right] = 0$$

for all $k \neq j$. Hence, Eq. 42 implies that the average numbers of photons in the signal field used to transmit all input symbols are equal. Furthermore, if

$$\omega_k \approx \omega; \qquad \text{for all } k = 1, 2, \ldots M, \tag{43}$$

Eq. 42 also implies that the average energy in the signal field is independent of the transmitted input symbol because when input symbol m_j is transmitted, the average energy in the received signal field is

$$E_j = \sum_{k=1}^{M} \mathrm{Tr}\left[\rho_j a_k^{+} a_k\right] \hbar\omega_k$$

$$= |\sigma_j|^2 \hbar\omega_j. \tag{44}$$

For signals satisfying the additional constraint in Eq. 42, the inequality

$$P(n_1, n_2, \dots n_M/m_j) \geq P(n_1, n_2, \dots n_M/m_i); \qquad i, j = 1, 2, \dots M$$

is satisfied if and only if

$$n_j \geq n_i \qquad i, j = 1, 2, \dots M.$$

Hence, the receiver estimates that the transmitted symbol is m_j when the number of photons in the j^{th} mode, n_j, is observed to be larger than all other n_i. Such a receiver can be implemented as shown in Fig. 4.

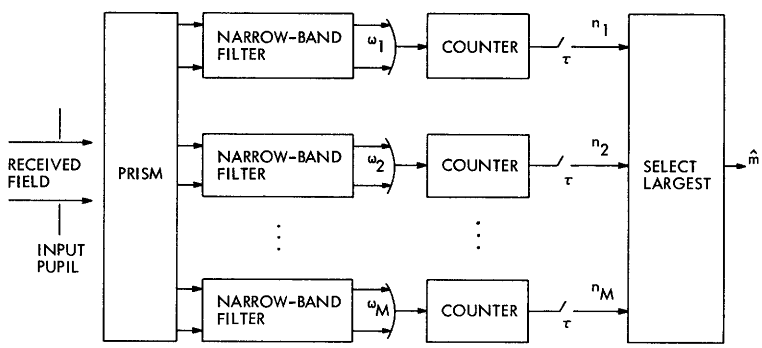

Fig. 4. Optimum receiver for equal-strength unmodulated signals with random phases.

4.1.2 Performance of the Optimum Receiver

The conditional probability of error, given that m_1 is transmitted, satisfies the following inequalities

$$P(\epsilon/m_1) \leq \Pr\,(n_1 \leq n_2, \text{ or } n_1 \leq n_3, \dots, \text{ or } n_1 \leq n_M/m_1) \tag{45a}$$

$$P(\epsilon/m_1) \geq \Pr\,(n_1 < n_2, \text{ or } n_1 < n_3, \dots, \text{ or } n_1 < n_M/m_1). \tag{45b}$$

The error probability $P(\epsilon)$ is

$$P(\epsilon) = P(\epsilon/m_1). \tag{46}$$

Since

$$\Pr\,(n_1<n_2,\text{ or } n_1<n_3,\ldots \text{ or } n_1<n_M/m_1) = 1 - \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{n_1}\cdots\sum_{n_M=0}^{n_1} P(n_1,n_2,\ldots n_M/m_1)$$

and

$$\Pr\,(n_1\leqslant n_2,\text{ or } n_1\leqslant n_3,\ldots \text{ or } n_1\leqslant n_M/m_1) = 1 - \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{n_1-1}\cdots\sum_{n_M=0}^{n_1-1} P(n_1,n_2,\ldots n_M/m_1),$$

by substituting Eqs. D.3 and D.4 (see Appendix D) in Eqs. 45 and 46, we can bound the error probability $P(\epsilon)$ as

$$P(\epsilon) \geqslant \sum_{i=1}^{M-1}\binom{M-1}{i}(-1)^{i+1}\frac{\langle n\rangle^i}{(1+\langle n\rangle)^{i+1}-\langle n\rangle^{i+1}}\exp\left\{-\frac{|\sigma|^2[(1+\langle n\rangle)^i-\langle n\rangle^i]}{(1+\langle n\rangle)^{i+1}-\langle n\rangle^{i+1}}\right\} \tag{47a}$$

$$P(\epsilon) \leqslant \sum_{i=1}^{M-1}\binom{M-1}{i}(-1)^{i+1}\frac{(1+\langle n\rangle)^i}{(1+\langle n\rangle)^{i+1}-\langle n\rangle^{i+1}}\exp\left\{-\frac{|\sigma|^2[(1+\langle n\rangle)^i-\langle n\rangle^i]}{(1+\langle n\rangle)^{i+1}-\langle n\rangle^{i+1}}\right\}. \tag{47b}$$

Unfortunately, these bounds cannot be expressed in closed forms. In the limit of high noise energy levels, $\langle n\rangle \gg 1$ (the classical limit), both the upper bound and the lower bound of $P(\epsilon)$ in (47) can be approximated by

$$P(\epsilon) \approx \sum_{i=1}^{M-1}(-1)^{i+1}\binom{M-1}{i}\frac{1}{i+1}\exp\left[\frac{-i|\sigma|^2}{(1+i)\langle n\rangle}\right]$$

$$= \sum_{i=1}^{M-1}(-1)^{i+1}\binom{M-1}{i}\frac{1}{i+1}\exp\left[\frac{-iE_r}{(1+i)\eta_o}\right]. \tag{48}$$

In this equation, $E_r = |\sigma|^2\,\hbar\omega$ is the average received signal energy, and $\eta_o = \langle n\rangle\hbar\omega$ is the average thermal noise energy in each mode of the noise field. This expression is identical to that of the minimum attainable error probability for a set of M classical orthogonal signals whose absolute phase is unknown.[20]

In the following derivation we shall express the bounds to the error probability $P(\epsilon)$ in terms of three parameters: the information rate R, the channel capacity C, and the time constraint length τ. As defined in Section II, τ is the time allotted to the transmission of a single-input symbol. When the number of equiprobable input symbols is M, the information rate of the system is

$$R = \frac{\ln M}{\tau} \text{ nats/sec.} \tag{49}$$

We shall find that the channel capacity C is given by the expression

$$C = p \ln \frac{1 + \langle n \rangle}{\langle n \rangle}, \tag{50a}$$

where

$$p = \frac{|\sigma|^2}{\tau} \tag{50b}$$

is the average number of photons in the signal field at the receiver per unit time. We note that C can also be expressed as

$$C = \frac{P}{2\eta_z} \ln \frac{2\eta_z + \eta_o}{\eta_z}, \tag{51}$$

where $P = \frac{|\sigma|^2 \hbar\omega}{\tau}$ is the average received signal power in the system, and $\eta_z = \frac{1}{2}\hbar\omega$ is usually called the zero-point fluctuation noise of the field. As before, $\eta_o = \langle n \rangle \hbar\omega$.

Our objective is to find bounds to $P(\epsilon)$ of the following form:

$$P(\epsilon) \geq K_1 \exp[-\tau C E(R)] \tag{52a}$$

$$P(\epsilon) \leq K_2 \exp[-\tau C E(R)], \tag{52b}$$

where K_1 and K_2 are not exponential functions of τ. The exponential factor E(R) in these expressions is the system reliability function. We shall be primarily concerned with the exponential term in the bounds of $P(\epsilon)$, rather than with the coefficients K_1 and K_2. This is justified by the fact that neither coefficient K_1 nor K_2 depends exponentially on τ. That is,

$$\lim_{C\tau\to\infty} \frac{\ln P(\epsilon)}{\tau C} = \lim_{C\tau\to\infty} [-E(R)] \tag{53}$$

An upper bound to $P(\epsilon)$ that can be obtained easily is the union bound. It is not an exponentially tight bound at high rates. Let $P_2(i,j)$ denote the probability of error when there are only two transmitted input symbols, m_i and m_j. With M = 2, Eq. 47b gives an upper bound of $P_2(i,j)$:

$$P_2(i,j) \leq \frac{1}{2}\left(1 + \frac{1}{2\langle n \rangle + 1}\right) \exp\left(\frac{-|\sigma|^2}{1 + 2\langle n \rangle}\right) \qquad \text{for all } i \neq j.$$

Therefore, for M equiprobable input symbols, the conditional probability of error, given that m_1 is transmitted, is

$$P(\epsilon/m_1) = \Pr \text{ (the estimate } \hat{m} \text{ is set to be } m_2 \text{ or } m_3 \ldots \text{ or } m_M/m_1)$$

$$\leq \sum_{k=2}^{M} P_2(1,k)$$

$$< M\left\{\frac{1}{2}\left(1 + \frac{1}{1+2\langle n\rangle}\right)\right\} \exp\left\{-\frac{|\sigma|^2}{1+2\langle n\rangle}\right\}.$$

Substituting $M = \exp(R\tau)$ in the expression above, we obtain

$$P(\epsilon) = P(\epsilon/m_1)$$

$$\leq \frac{1}{2}\left(1 + \frac{1}{1+2\langle n\rangle}\right) \exp\left\{-\tau\left[\frac{p}{1+2\langle n\rangle} - R\right]\right\}$$

$$= \frac{1}{2}\left(1 + \frac{\eta_z}{\eta_o + \eta_z}\right) \exp\left\{-\tau\left[\frac{p}{2(\eta_o+\eta_2)} - R\right]\right\}. \tag{54}$$

<u>Upper Bound to $P(\epsilon)$</u>

To find an upper bound to the error probability $P(\epsilon)$ that is exponentially tighter than the union bound at high rates, let us denote the probability

$$\sum_{\substack{k=1\\k\neq j}}^{M} \sum_{n_k=0}^{\infty} P(n_1, n_2, \ldots n_M/m_i)$$

by $p_i(n_j)$ (for $i, j = 1, 2, \ldots M$). According to Eq. 40,

$$p_j(n_j) = \left(\frac{1}{1+\langle n\rangle}\right)\left(\frac{\langle n\rangle}{1+\langle n\rangle}\right)^{n_j} \sum_{r=0}^{n_j} \binom{n_j}{r} \frac{1}{r!}\left[\frac{|\sigma|^2}{\langle n\rangle(1+\langle n\rangle)}\right]^r \exp\left[\frac{-|\sigma|^2}{1+\langle n\rangle}\right] \qquad j = 1, 2, \ldots M \tag{55a}$$

$$p_j(n_i) = \frac{1}{1+\langle n\rangle}\left(\frac{\langle n\rangle}{1+\langle n\rangle}\right)^{n_i} \qquad \begin{array}{l} i \neq j \\ i, j = 1, 2, \ldots M \end{array} \tag{55b}$$

The upper bound in Eq. 45 can be rewritten

$$P(\epsilon) = P(\epsilon/m_1)$$

$$\leq \sum_{n_1=0}^{\infty} p_1(n_1) \Pr(n_1 \leq n_2, \text{ or } n_1 \leq n_3, \ldots \text{ or } n_1 \leq n_M/m_1)$$

$$\leq \sum_{n_1=0}^{\infty} p_1(n_1) [\Pr(n_1 \leq n_2, \text{ or } n_1 \leq n_3, \ldots \text{ or } n_1 \leq n_M/m_1)]^{\delta}$$

for any value of δ in the interval $0 \leq \delta \leq 1$. Since

$$\Pr(n_1 \leq n_2, \text{ or } n_1 \leq n_3, \ldots \text{ or } n_1 \leq n_M/m_1) \leq (M-1) \Pr(n_1 \leq n_2/m_1),$$

the expression in the right-hand member of the previous equation can be further upper-bounded:

$$P(\epsilon) < \sum_{n_1=0}^{\infty} p_1(n_1) M^{\delta} \left[\sum_{n_2=n_1}^{\infty} p_1(n_2) \right]^{\delta}. \tag{56}$$

Substituting Eqs. 55 and D. 5 (see Appendix D) in Eq. 56, and expressing the upper bound in the form of Eq. 52b, we obtain

$$P(\epsilon) \leq \frac{(1+\langle n\rangle)^{\delta}}{(1+\langle n\rangle)^{1+\delta} - \langle n\rangle^{1+\delta}} \exp\left\{-\tau C\left[\frac{p}{C} \frac{(1+\langle n\rangle)^{\delta} - \langle n\rangle^{\delta}}{(1+\langle n\rangle)^{1+\delta} - \langle n\rangle^{1+\delta}} - \delta \frac{R}{C}\right]\right\} \qquad 0 \leq \delta \leq 1, \tag{57}$$

where p and C are defined as in Eqs. 50a and 50b, respectively.

Since (57) is valid for any value of δ within the interval (0, 1), the best asymptotic bound is obtained when the exponent is maximized over the value of δ in the range $0 \leq \delta \leq 1$. The coefficient

$$K_2 = \frac{(1+\langle n\rangle)^{\delta}}{(1+\langle n\rangle)^{1+\delta} - \langle n\rangle^{1+\delta}} \tag{58}$$

is a monotonically decreasing function of δ. Therefore, it is not minimized by the choice of δ that maximizes the exponent.

It is clear, however, that

$$\lim_{C\tau\to\infty} \frac{1}{\tau C} \ln K_2 = 0.$$

Hence, (53) is satisfied for all values of δ.

Let us denote the exponent in (57) by

$$e_1(R,\delta) = e_o(\delta) - \delta \frac{R}{C},$$

where

$$e_o(\delta) = \frac{p}{C} \frac{(1+\langle n\rangle)^{\delta} - \langle n\rangle^{\delta}}{(1+\langle n\rangle)^{1+\delta} - \langle n\rangle^{1+\delta}}.$$

Clearly, the best lower bound, $E_L(R)$, of the reliability function $E(R)$ is given by

$$E_L(R) = \max_{0\leqslant\delta\leqslant 1} \left[e_o(\delta) - \delta \frac{R}{C}\right].$$

Since

$$\frac{de_o}{d\delta} = \frac{\left(\dfrac{\langle n\rangle}{1+\langle n\rangle}\right)^{\delta} \left(\dfrac{1}{1+\langle n\rangle}\right)^{2}}{\left[1 - \left(\dfrac{\langle n\rangle}{1+\langle n\rangle}\right)^{\delta+1}\right]^{2}} \tag{59}$$

and the second derivative of $e_1(R,\delta)$ is negative for all $\langle n\rangle$ not equal to zero, the equation

$$\frac{\partial e_1(R,\delta)}{\partial\delta} = \frac{de_o}{d\delta} - \frac{R}{C} = 0$$

gives the value of δ that maximizes the exponent factor $e_1(R,\delta)$ for rates R within the range

$$\left.\frac{de_o}{d\delta}\right|_{\delta=1} C \leqslant R \leqslant \left.\frac{de_o}{d\delta}\right|_{\delta=0} C.$$

Let us denote by R_c the rate $\left.\frac{de_o}{d\delta}\right|_{\delta=1} C$. From (59), it is evident that

$$R_c = \frac{\langle n\rangle(1+\langle n\rangle)}{(1+2\langle n\rangle)^2} C \tag{60}$$

and

$$\left.\frac{de_o}{d\delta}\right|_{\delta=0} = 1.$$

Hence the value of δ that maximizes $e_1(R,\delta)$ is given by

$$\frac{R}{C} = \frac{\left(\frac{1}{1+\langle n\rangle}\right)^2 \left(\frac{\langle n\rangle}{1+\langle n\rangle}\right)^{\delta}}{\left[1 - \left(\frac{\langle n\rangle}{1+\langle n\rangle}\right)^{\delta+1}\right]^2}; \quad \text{for } R_c \leqslant R \leqslant C.$$

Solving this equation for δ and recalling that $C = p \ln \frac{1+\langle n\rangle}{\langle n\rangle}$, we obtain

$$E_L(R) = \frac{1}{\langle n\rangle \ln \frac{1+\langle n\rangle}{\langle n\rangle}} \left\{ \frac{1 + 2\langle n\rangle \frac{R}{C} - \sqrt{1 + 4\frac{R}{C}\langle n\rangle(1+\langle n\rangle)}}{1 - \sqrt{1 + 4\frac{R}{C}\langle n\rangle(1+\langle n\rangle)}} \right\}$$

$$- \frac{R}{C} \frac{1}{\ln \frac{1+\langle n\rangle}{\langle n\rangle}} \ln \left\{ \frac{1 + 2\frac{R}{C}\langle n\rangle(1+\langle n\rangle) + \sqrt{1 + 4\frac{R}{C}\langle n\rangle(1+\langle n\rangle)}}{2(1+\langle n\rangle)^2 \frac{R}{C}} \right\};$$

$$\text{for } R_c \leqslant R \leqslant C. \qquad (61a)$$

For rates less than R_c, the value of δ that maximizes the exponent function $e_1(R, \delta)$ is 1. Therefore, within the range $0 \leqslant R \leqslant R_c$, the function $E_L(R)$ is identical to the exponent in the union bound. That is,

$$E_L(R) = \frac{1}{(1+2\langle n\rangle) \ln \frac{1+\langle n\rangle}{\langle n\rangle}} - \frac{R}{C}; \quad 0 \leqslant R \leqslant R_c. \qquad (61b)$$

We shall discuss the general behavior of $E_L(R)$ in section 4.3 after showing that it is equal to the reliability function for $R_c \leqslant R \leqslant C$.

Lower Bound to $P(\epsilon)$

We are primarily interested in finding a lower bound for $P(\epsilon)$ which has the same exponential behavior as the upper bound that has been derived for the range of R, $R_c \leqslant R \leqslant C$. (The details of the derivation are discussed in Appendix E.)

It is clear that the lower bound in Eq. 45b can be rewritten

$$P(\epsilon) \geqslant \sum_{n_1=0}^{\infty} p_1(n_1) \left\{1 - [\Pr(n_1 \geqslant n_2/m_1)]^{(M-1)}\right\}. \qquad (62)$$

The right-hand side of this inequality can be further lower-bounded (see Appendix E):

$$P(\epsilon) \geq \frac{M}{4} \Pr\,(n_2 > d/m_1)\ \Pr\,(n_1 \leq d/m_1), \tag{63a}$$

provided that the parameter d is chosen so that

$$M \Pr\,(n_2 > d/m_1) \leq 1. \tag{63b}$$

When input symbol m_1 is transmitted, the conditional probability distributions of random variables n_1 and n_2 are given by Eqs. 55a and 55b, respectively. Hence

$$\Pr\,(n_2 > d/m_1) = \left(\frac{\langle n \rangle}{1 + \langle n \rangle}\right)^{[d]+1},$$

where [d] denotes the integral part of the parameter d. Clearly, the condition in (63b) is satisfied if the parameter d is chosen to be

$$d = \frac{R\tau}{\ln \frac{1 + \langle n \rangle}{\langle n \rangle}}. \tag{64}$$

From Eqs. E.11, E.13, and E.15 (see Appendix E), we obtain the following lower bound to the error probability $P(\epsilon)$:

$$P(\epsilon) \geq K_1 \exp[-\tau C\, e'(R, \delta_1)]. \tag{65a}$$

(Note that similarly to the coefficient in the upper bound of $P(\epsilon)$, the coefficient K_1 is also a function of δ_1 which is not maximized for the value of δ_1 that minimizes $e'(R, \delta_1)$. It is sufficient for us to know that $\lim\limits_{\tau C \to \infty} \frac{\ln K_1}{\tau C} \to 0$.) The exponential function in Eq. 65a is

$$e'(R, \delta_1) = \frac{1}{\tau C}\left\{d \ln \frac{1 + \langle n \rangle}{\langle n \rangle} - \delta_1 d \ln \frac{|\sigma|^2}{\langle n \rangle (1 + \langle n \rangle)} + \frac{|\sigma|^2}{1 + \langle n \rangle} \right.$$
$$\left. + d \ln (1 - \delta_1) - \delta_1 d \ln \frac{1 - \delta_1}{\delta_1} + \delta_1 d \ln \delta_1 d - \delta_1 d \right\}, \tag{65b}$$

where δ_1 is within the range $0 \leq \delta_1 \leq 1$. Since the inequality in (65a) is satisfied for all values of δ_1 in the interval (0, 1), the best asymptotic lower bound of $P(\epsilon)$ is obtained by minimizing $e'(R, \delta_1)$ over δ_1, for $0 \leq \delta_1 \leq 1$. Thus, one obtains an upper bound, $E_u(R)$, of the reliability function $E(R)$:

$$E_u(R) = \min_{0 \leq \delta_1 \leq 1} e'(R, \delta_1).$$

It will be shown (see Appendix E), that the value of δ_1 that minimizes the function

$e'(R, \delta_1)$ in (65b) is

$$\delta_1^o = \frac{1}{2\frac{R}{C}\langle n\rangle(1+\langle n\rangle)} + \sqrt{\frac{1}{\langle n\rangle(1+\langle n\rangle)\frac{R}{C}} + \frac{1}{4\langle n\rangle^2(1+\langle n\rangle)^2\left(\frac{R}{C}\right)^2}}. \tag{66}$$

At $\delta_1 = \delta_1^o$,

$$E_u(R) = e'\left(R, \delta_1^o\right)$$

$$= \frac{1}{\langle n\rangle \ln \frac{1+\langle n\rangle}{\langle n\rangle}} \frac{1+2\langle n\rangle\frac{R}{C} - \sqrt{1+4\frac{R}{C}\langle n\rangle(1+\langle n\rangle)}}{1-\sqrt{1+4\frac{R}{C}\langle n\rangle(1+\langle n\rangle)}}$$

$$-\frac{R}{C}\frac{1}{\ln\frac{1+\langle n\rangle}{\langle n\rangle}} \ln\left\{\frac{1+2\frac{R}{C}\langle n\rangle(1+\langle n\rangle)+\sqrt{1+4\frac{R}{C}\langle n\rangle(1+\langle n\rangle)}}{2\frac{R}{C}(1+\langle n\rangle)^2}\right\}. \tag{67}$$

It is obvious that the function $E_u(R)$ is identically equal to $E_L(R)$, the lower bound of the system reliability function given in Eq. 61a, for the range of R, $R_c \leqslant R \leqslant C$.

It will now be shown that the bounds that we have just derived apply to all systems in which signals are narrow-band and orthogonal.

4.2 NARROW-BAND ORTHOGONAL SIGNALS WITH RANDOM PHASE

We are concerned with the case in which the received electromagnetic field excited by the signal source is narrow-band. As in section 4.1, the exact phase of the received field is unknown to the receiver. When the electric field operator $E(\underline{r}, t)$ at the receiver is expanded in terms of its Fourier components as in Eq. 4, and when the transmitted input symbol is m_j, the state of the signal component of the received field is described by the density operator

$$\rho_j^t = \int_0^{2\pi} \prod_k |\sigma_{jk}e^{i\phi}\rangle\langle\sigma_{jk}e^{i\phi}| \frac{d\phi}{2\pi}; \qquad j = 1, 2, \ldots M. \tag{68}$$

The set of complex numbers σ_{jk} satisfies the constraint of Eq. 19,

$$\sigma_{jk}^*\sigma_{j'k} = |\sigma_{jk}|^2\,\delta_{jj'},$$

when the M signals are orthogonal, as in a frequency position modulation system.

For example, the general form of the frequency spectrum of the signal field is shown in Fig. 5.

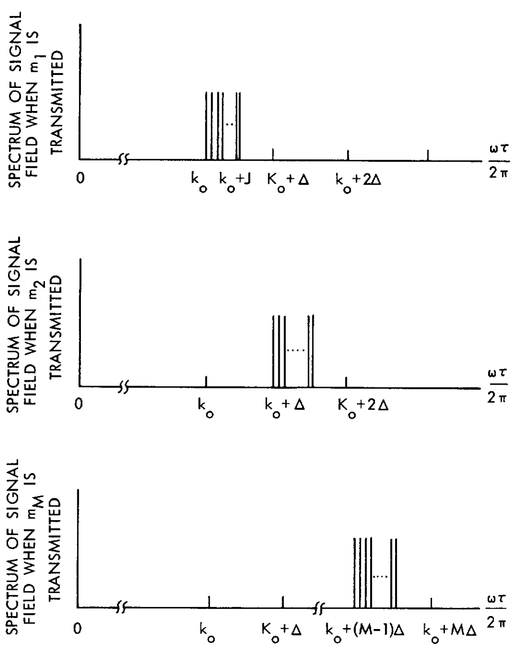

Fig. 5. Frequency spectra of M-ary orthogonal narrow-band signals.

In the presence of additive thermal-noise field which is in a state specified by the density operator $\rho^{(n)}$ in Eq. 20, the received field is in the state given by

$$\rho_j = \int_0^{2\pi} \frac{d\phi}{2\pi} \prod_k \exp\left[-\frac{|\alpha_k-\sigma_{jk}e^{i\phi}|^2}{\langle n\rangle}\right] |\alpha_k\rangle\langle\alpha_k| \frac{d^2\alpha_k}{\pi\langle n\rangle}. \tag{69}$$

In Appendix F, these density operators are shown to be commutative.

We shall show that the minimum attainable error probability in the reception of signals specified by the density operators in (69) is identical to that of the signal set discussed in section 4.1, if

$$|\sigma|^2 = \sum_k |\sigma_{jk}|^2; \qquad \text{for all } j = 1, 2, \ldots M. \tag{70}$$

Intuitively, this conclusion is to be expected. As discussed in section 2.1 and Appendix A, the received electric field operator can also be expanded in terms of the orthogonal mode functions

$$V_j(\underline{r}, t) = \frac{1}{|\sigma|} \sum_k \sigma_{jk} \exp\left[-i(\omega_j'-\omega_k)\left(\frac{z}{c}-t\right)\right]; \qquad j = 1, 2, \ldots M \tag{71}$$

with the ω_j' chosen so that

$$\int d^3\underline{r} \int V_{j'}(\underline{r}, t)\, V_j^*(\underline{r}, t)\, dt = \delta_{jj'}; \quad \text{for all } j \text{ and } j'.$$

Thus

$$E(\underline{r}, t) = i \sum_j \sqrt{\frac{\hbar\omega_j'}{2L}} \left\{ b_j V_j(\underline{r}, t) \exp\left[i\omega_j'\left(\frac{z}{c} - t\right)\right] - b^+ V_j^*(\underline{r}, t) \exp\left[-i\omega_j'\left(\frac{z}{c} - t\right)\right]\right\}.$$

The operators b_j are given by Eq. 13. It is obvious that when a signal field is in the state specified by the density operator ρ_j^t in (68), only the mode with the normal mode function $V_j(\underline{r}, t)$ is in an excited state, while all other modes remain in their ground state. This is exactly the same condition satisfied by the signal set in section 4.1.

To establish the claim made in the last paragraph rigorously, let A_j denote the set of indices k that are such that

$$\sigma_{jk} \neq 0, \qquad j = 1, 2, \ldots M.$$

Then, the density operators in (69) can be written

$$\rho_j = \int_0^{2\pi} \frac{d\phi}{2\pi} \int \exp\left[-\sum_{k \notin A_j} \frac{|\alpha_k|^2}{\langle n\rangle} - \sum_{k \in A_j} \frac{|\alpha_k - \sigma_{jk} e^{i\phi}|^2}{\langle n\rangle}\right] \prod_k |\alpha_k\rangle\langle\alpha_k| \frac{d^2\alpha_k}{\pi\langle n\rangle}$$

$$j = 1, 2, \ldots M. \qquad (72)$$

To find a representation in which the density operators $\rho_1, \rho_2, \ldots \rho_M$ are diagonalized, let us consider a unitary transformation V which relates a set of new annihilation operators $\{b_k\}$ with the set of annihilation operators $\{a_k\}$:

$$b_j = \sum_k V_{jk} a_k.$$

Let us, in particular, choose the elements of V so that , for all j = 1, 2, ... M,

$$\begin{aligned} &V_{k(j), m} = \sigma_j^{-1} \sigma_{jk}^*; \quad V_{m, k(j)}^+ = \sigma_j^{-1} \sigma_{jk}; && \text{for all } m \in A_j \\ &V_{k(j), m} = 0; && \text{for all } m \notin A_j \end{aligned} \qquad (73)$$

where k(j) is an index in the set A_j, and σ_j is given by the normalization condition

$$\sigma_j = \left\{ \sum_m |\sigma_{jm}|^2 \right\}^{1/2}.$$

The rows and columns of the matrix V defined in (73) obviously satisfy the condition

$$\sum_{k} V_{mk} V_{kn}^{+} = \sum_{n} V_{mk}^{+} V_{kn} = \delta_{mn}. \tag{74}$$

The other rows of the matrix V can be chosen arbitrarily, provided that the condition in (74) is satisfied. For example, for the specific set of signals shown in Fig. 5, σ_{jk} is nonzero only for $k = k_o + (j-1)\Delta, k_o + (j-1)\Delta + 1, \ldots, k_o + j\Delta - 1$. The unitary matrix V is in the form given by Eq. A-8b, with the submatrices U_j chosen so that

$$U_{ln}^{(j)} = \sigma_{j,(k_o+(j-1)\Delta+n)}; \qquad j = 1, 2, \ldots M.$$

According to Eqs. 14, the density operator ρ_j in Eq. 72, expanded in terms of the right eigenvectors $|\{\beta_j\}\rangle$ of the operators b_j, is given by

$$\rho_j = \int_0^{2\pi} \frac{d\phi}{2\pi} \int \exp\left[\frac{-|\beta_{k(j)} - \sigma_j e^{i\phi}|^2}{\langle n \rangle} - \sum_{k \neq k(j)} \frac{|\beta_k|^2}{\langle n \rangle}\right] \prod_k |\beta_k\rangle \langle \beta_k| \frac{d^2\beta_k}{\pi \langle n \rangle}.$$

$$j = 1, 2, \ldots M \tag{75}$$

From this expression, it is clear that modes other than k(1), k(2), ... k(M) do not contain relevant information and can be exempt from further consideration. For simplicity, let us rename the modes k(1), k(2), ... k(M) as the 1st, 2nd, ... Mth modes, respectively. Obviously, the density operator ρ_j specifying the states of the relevant modes is identical to that given in Eq. 39. Therefore, the optimum performance in the reception of this set of signals is identical to that of the set discussed in section 4.1 (the unmodulated signals), if it is also assumed that

$$\sigma_j = \sigma; \qquad j = 1, 2, \ldots M.$$

The structure of the optimum receiver for the reception of narrow-band orthogonal signals differs from that for unmodulated signals, however. Clearly, the density operator ρ_j in Eq. 69 is diagonalized when it is expanded in terms of the eigenvectors of the operators $b_j^+ b_j$; $j = 1, 2, \ldots M$. Hence, according to section 3.2, the optimum receiver measures the dynamical variables of the field corresponding to the linear operators $b_j^+ b_j$. That is,

$$b_j^+ b_j = \frac{1}{|\sigma|^2} \sum_m \sum_n \sigma_{jm} \sigma_{jn}^* a_m^+ a_n.$$

It can be seen from (75) that the joint probability distribution of the outcomes $n_1, n_2, \ldots n_M$ is also given by Eq. 40 when $b_1^+ b_1, b_2^+ b_2, \ldots b_M^+ b_M$ are measured simultaneously and the transmitted input symbol is known to be m_j. Therefore, the optimum receiver sets the estimation of the transmitted input symbol m to m_j when

$$n_j \geq n_k \qquad j, k = 1, 2, \ldots M.$$

Hereafter, it is only necessary to study the implementation of that part of the receiver which makes the measurements of the variables $b_1^+b_1, b_2^+b_2, \ldots b_M^+b_M$. (In other words, how do we implement the mathematical operation of applying the unitary transformation V to the set of annihilation operators $\{a_k\}$?)

Let us digress for a moment to consider the corresponding classical system in which the complex amplitude of the signal field when m_j is transmitted is

$$S_j(t) = \sum_k \sigma_{jk} \exp[i(\omega_j'-\omega_k)t]\{\exp[-i\omega_j't+i\phi]\}.$$

The complex amplitude of the received electric field in the presence of the additive noise field can be written

$$\epsilon_c(\underline{r}, t) = \sum_k \epsilon_k \exp\left[-i(\omega_j'-\omega_k)\left(\frac{z}{c}-t\right)\right] \exp\left[i\omega_j'\left(\frac{z}{c}-t\right)+i\phi_j\right].$$

When the phases ϕ_j are random variables evenly distributed over the interval $(0, 2\pi)$, one way to implement the classical optimum receiver is that shown in Fig. 6. That is, the classical optimum receiver measures the quantities

$$f_j = \left| \int_0^T \left\{ \int_{\underline{r}} \epsilon_c(\underline{r}, t)\, d\underline{r} \right\} S_j^*(t)\, dt \right|^2; \qquad j = 1, 2, \ldots M,$$

which according to Parseval's theorem are equal to

$$f_j = \left| \sum_k \sigma_{jk}^* \epsilon_k \right|^2; \qquad j = 1, 2, \ldots M. \tag{76}$$

The estimation $\hat{m}$ is set to m_j if

$$f_j \geq f_k \qquad \text{for all } k = 1, 2, \ldots M.$$

An equivalent implementation of the classical optimum receiver is shown in Fig. 7. When quantum effects in the system are taken into account, a semiclassical analysis shows[21] that the effective noise level increases from $\eta_o/2$ (the classical value) to $\frac{1}{2}\left(\eta_o + \frac{\hbar\omega}{2}\right)$, while the f_j remain as Rayleigh random variables. Hence the system reliability function is the same as that for the classical system, except that the channel capacity C is decreased to the value $\frac{P}{\eta_o + \eta_z}$. Clearly, the system shown in Fig. 7 is not a quantum-mechanical optimum receiver.

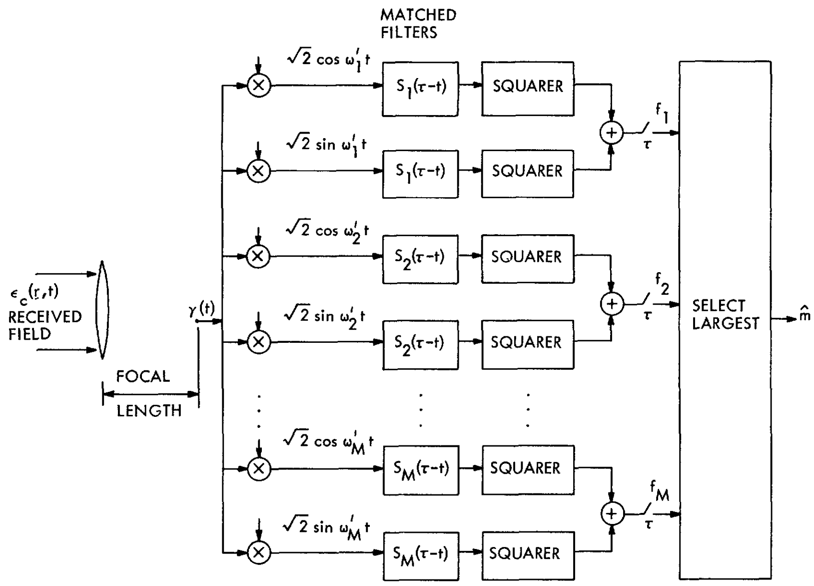

Fig. 6. Classical optimum receiver for orthogonal signals with random phases (I).

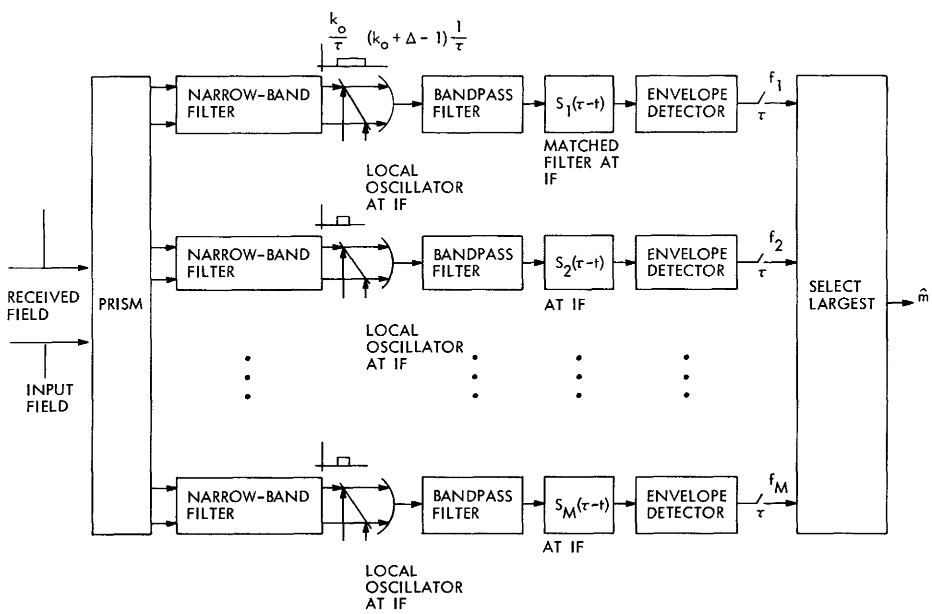

Fig. 7. Classical optimum receiver for orthogonal signals with random phases (II).

It can be seen that the optimum quantum-mechanical receiver measuring the variables $b_1^+b_1$, $b_2^+b_2$, ... $b_M^+b_M$ reduces to these classical optimum receivers in the classical limit, however. This is because, classically, σ_{jk} can be interpreted as the complex amplitudes of the Fourier components of the transmitted signal field for input symbol m_j. The operators a_k can be interpreted as that of the total received field. Therefore, the variables $b_j^+b_j$ (where b_j is defined to be $\sum_k \sigma_{jk}^* a_k$) are the quantities f_j in Eq. 76 in the classical limit.

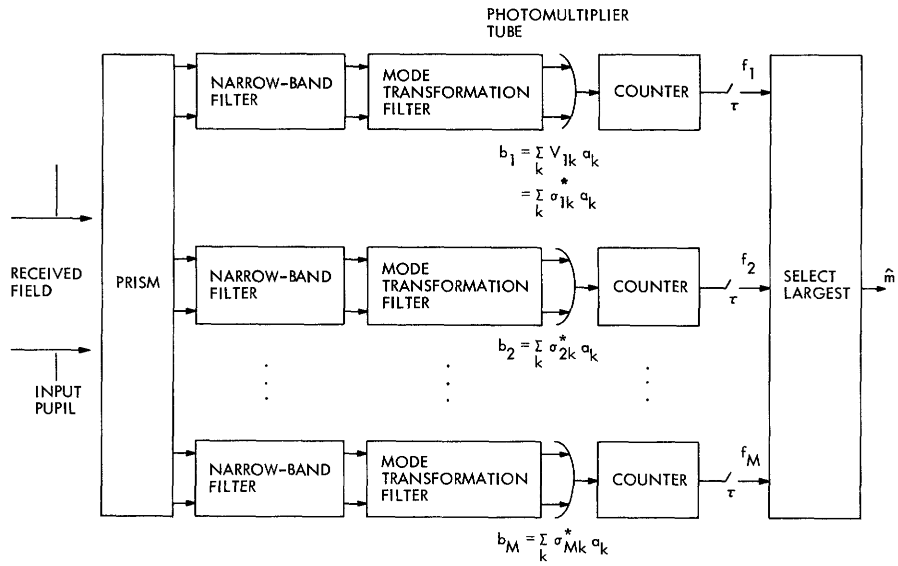

Fig. 8. Quantum-mechanical optimum receiver for orthogonal signals in coherent states with random phases.

An idealized quantum optimum receiver is shown in Fig. 8. The input aperture of each of the mode transformation filters is opened within the time interval $(0, \tau)$ when the transmitted signals are expected to arrive at the receiver. It can be shown (Appendix G) that by adjusting the coupling coefficients between the field at the output of the narrow-band filter and the field inside of the mode transformation filter, the annihilation operator b_j associated with the only normal mode of the j^{th} filter is related to the operators a_k by

$$b_j = \sum_k \sigma_{jk}^* a_k$$

after a transient period that is short compared with τ.

4.3 SUMMARY AND DISCUSSION

In sections 4.1 and 4.2, we have obtained asymptotic bounds of the minimum attainable error probability $P(\epsilon)$ in the reception of M input symbols represented by electromagnetic waves in coherent states. The absolute phase of the signal field is assumed to be a random variable distributed uniformly over the interval $(0, 2\pi)$. The lower bound and the upper bound of $P(\epsilon)$ can be written in the form given by Eqs. 52a and 52b, respectively. The system reliability function $E(R)$ in the exponent is given by Eq. 61a when the information rate R is within the range $R_c \leq R \leq C$.

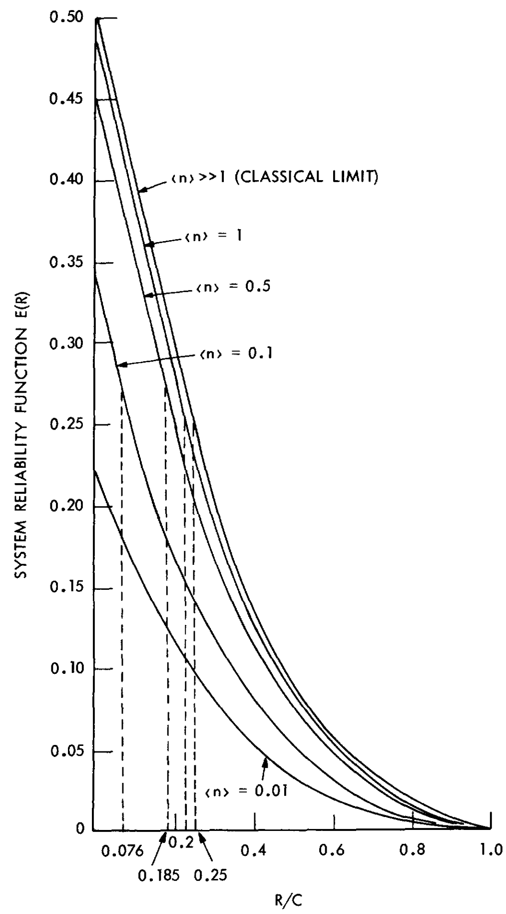

Fig. 9. System reliability function (for orthogonal signals in coherent states).

A quick computation shows that the reliability function is positive for all values of $R < C$ when $\langle n\rangle \neq 0$. Its general behavior for several different values of $\langle n\rangle$ is shown in Fig. 9. In contrast to the corresponding classical channel (the additive white Gaussian channel), the system reliability function $E(R)$ depends not only on R/C, but also on the average noise level $\langle n\rangle$. We note that the channel capacity $C = p \ln \frac{1+\langle n\rangle}{\langle n\rangle}$ is approximately equal to $C_c = \frac{p}{\langle n\rangle} = \frac{P}{\eta_o}$ for large $\langle n\rangle$, where C_c is the capacity of the classical white Gaussian channel.[20] For other values of $\langle n\rangle$, we have

$$C < p\left\{\frac{1+\langle n\rangle}{\langle n\rangle} - 1\right\} = \frac{p}{\langle n\rangle}.$$

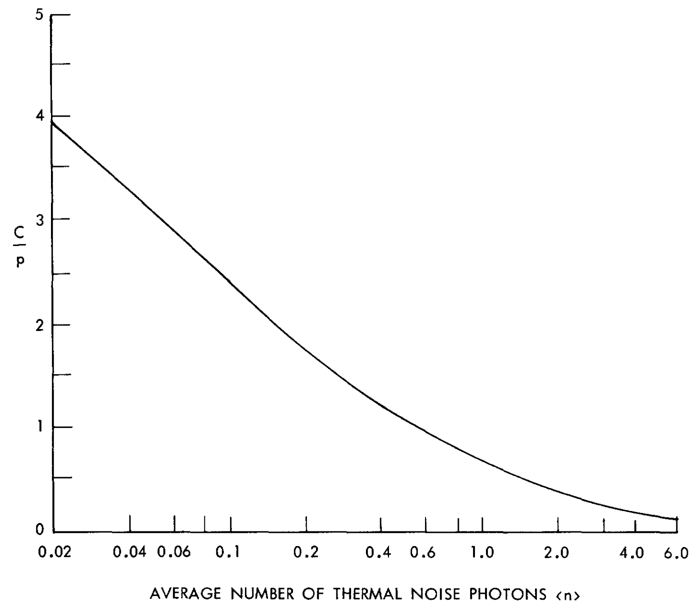

Fig. 10. Channel capacity as a function of noise levels.

Figure 10 shows the value of C per unit average signal photon as a function of $\langle n \rangle$, the average number of noise photons. Also, for large $\langle n \rangle$, E(R) approaches as a limit the classical reliability function of the white Gaussian channel with infinite bandwidth. That is, for large $\langle n \rangle$,

$$E(R) \rightarrow E_c(R) = \begin{cases} \frac{1}{2} - \frac{R}{C_c} & 0 \leqslant \frac{R}{C_c} \leqslant \frac{1}{4} \\ \left(1 - \sqrt{\frac{R}{C_c}}\right)^2; & \frac{1}{4} \leqslant \frac{R}{C_c} \leqslant 1 \end{cases} \tag{77}$$

In Fig. 11, the exponent factor CE(R) is plotted as a function of R for several values of p and $\langle n \rangle$; the corresponding values of $\frac{p}{\langle n \rangle} E_c(R)$ are also shown. The exponential factor $\frac{p}{\langle n \rangle} E_c(R)$ is obtained when quantum effects are completely ignored. When the "quantum noise" in the system is also taken into account as in section 4.2, the performance of the classical optimum receiver is inferior to that of the quantum-mechanical optimum receiver, as expected (see Fig. 11).

When $\langle n \rangle = 0$, the capacity C becomes infinite. One is tempted to conclude from this fact that an arbitrarily small probability of error can be achieved at any information rate R by increasing the values of the parameters τ and M while keeping the average signal power fixed, whenever $\langle n \rangle = 0$. Such a conclusion is not valid, since for very small $\langle n \rangle$

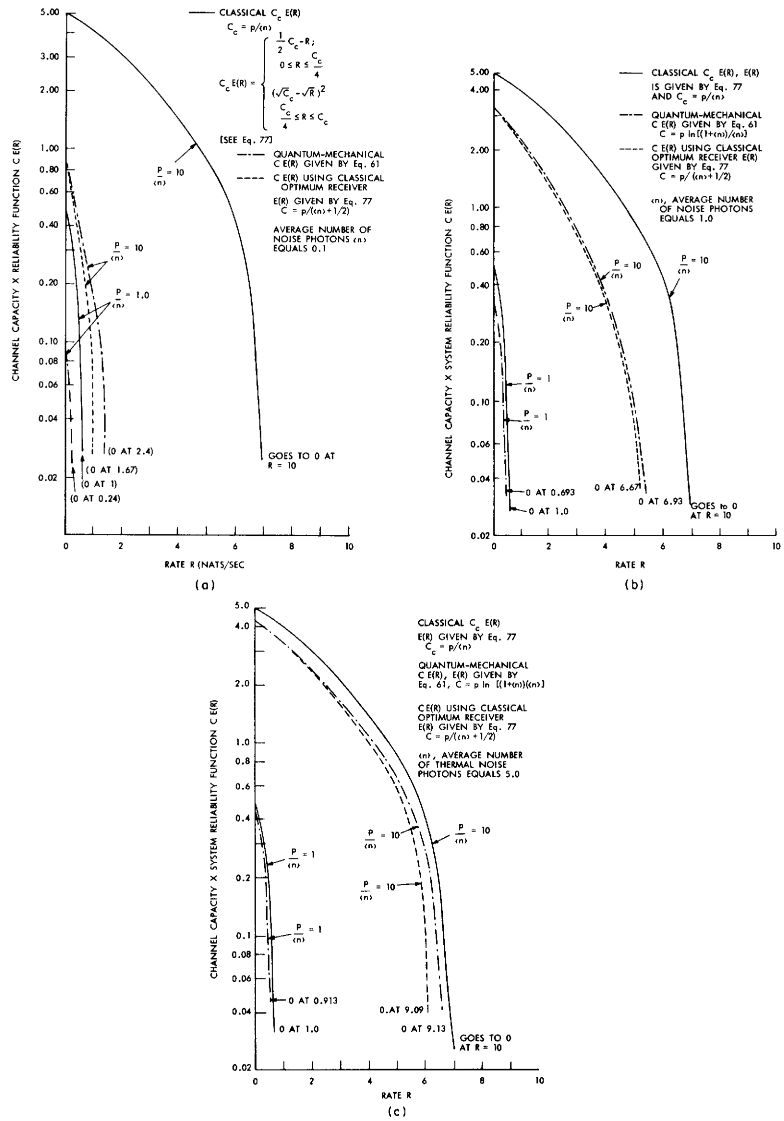

Fig. 11. (a) System reliability function × channel capacity ($\langle n \rangle = 0.1$).
(b) System reliability function × channel capacity ($\langle n \rangle = 1.0$).
(c) System reliability function × channel capacity ($\langle n \rangle = 5.0$).

$$E(R) \approx \frac{1}{-\ln \langle n \rangle} \left\{ 1 - \frac{R}{C} - \frac{R}{C} \ln \left(\frac{C}{R} + 2\langle n \rangle \right) \right\}; \quad \langle n \rangle \ll 1. \tag{78}$$

As $\langle n \rangle$ approaches zero, this expression also approaches zero as a limit.

As a matter of fact, when the average number of noise photons $\langle n \rangle$ is equal to zero, the exponent in the error probability becomes τp. That is,

$$P(\epsilon) = \exp(-\tau p).$$

(The coefficients K_1 and K_2 are equal to one when M is large because, with no noise present, an error occurs only when no photon is detected which happens with probability $\exp(-|\sigma|^2)$. The probability that an error occurs, given that no photon is detected, is $(M-1)/M$.) The fact that $P(\epsilon)$ is independent of the number of input symbols, M, when $\langle n \rangle$ equals zero implies that an arbitrarily small probability of error can still be achieved at an arbitrarily large information rate for the transmission of a fixed number of photons per second in the signal field, p. Since the signals are orthogonal, however, the total bandwidth for all input symbols can no longer be regarded as narrow. That is, Eq. 43 is no longer valid. The fact that the average numbers of photons per second in the signal fields representing the input symbols are all equal implies that the average power in the signal fields grows with the number of input symbols M, because of increasing energy per photon. Hence, the small error probability is achieved only by an accompanied increment in the power of the transmitted signals.

In practice, one usually holds the average signal power (or peak power) fixed. Therefore it is more meaningful to derive the system reliability functions under the assumption that the energies in the signal fields representing different input symbols are equal. That is,

$$|\sigma_j|^2 \hbar\omega_j = E_j = E; \qquad j = 1, 2, \ldots M.$$

Unfortunately, we are not able to obtain any analytic result in this case. It is quite clear that the values of n_i and n_j satisfying the condition

$$f_j \geq f_i$$

for f_j in Eq. 41 can only be determined numerically.

In the absence of noise, one can determine the error probability for equal-energy signals in an alternative way. When m_j is transmitted, the mode whose natural frequency is ω_j is excited to a coherent state $|\sigma_j e^{i\phi}\rangle$, while other modes remain in ground states. Then

$$P(\epsilon/m_j) = \exp\left(-|\sigma_j|^2\right)$$
$$= \exp\left(-\frac{E}{\hbar\omega_j}\right)$$

and

$$P(\epsilon) = \frac{1}{M} \sum_{j=1}^{M} \exp\left(-\frac{E}{\hbar\omega_j}\right). \tag{79}$$

When the signals are orthogonal in frequency, one can express the frequency ω_j in terms of some fixed frequency ω_o:

$$\omega_j = \omega_o\left(1 + j\,\frac{2\pi}{\tau\omega_o}\right),$$

where τ is the length of the signaling time interval. Substituting this expression in Eq. 79, we obtain

$$P(\epsilon) = \sum_{j=1}^{M} \left\{\frac{1}{M} \exp\left[\frac{-E}{\hbar\omega_o\left(1 + j\,\frac{2\pi}{\tau\omega_o}\right)}\right]\right\}.$$

For large value of τ and ω_o (optical frequency), the right-hand side of the last equation can be approximated by

$$P(\epsilon) \approx \sum_{j=1}^{M} \frac{1}{M} \exp\left(-\frac{E}{\hbar\omega_o}\right) \exp\left(\frac{2\pi E j}{\hbar\tau\omega_o^2}\right)$$

$$= \exp\left[-\tau\left(\frac{p}{\hbar\omega_o} + R\right)\right] \frac{\exp\left[\frac{M2\pi E}{\hbar\tau\omega_o^2}\right] - 1}{\exp\left[\frac{2\pi E}{\hbar\tau\omega_o^2}\right] - 1}.$$

In most practical cases, $\frac{2\pi E}{\hbar\tau\omega_o} \ll 1$. We can further simplify the last expression to

$$P(\epsilon) \sim \frac{\hbar\omega_o^2}{2\pi p} \left\{\exp\left[-\tau\left(\frac{p}{\hbar\omega_o} + R\right)\right]\right\}\left\{\exp\left[\frac{2\pi p\, e^{R\tau}}{\hbar\omega_o^2\tau}\right] - 1\right\}.$$

It is clear from this expression that $P(\epsilon)$ does not decrease exponentially with τ for any arbitrary value of R, as in the case of infinite channel capacity.

V. ORTHOGONAL SIGNALS IN COMPLETELY INCOHERENT STATES

We have considered the reception of signals, each of which is generated by a classical source of known amplitude but whose absolute phase is unknown to the receiver. The received field is in a coherent state when there is no thermal-noise field present at the receiver. We shall now find the structure and performance of the optimum receiver when the received field, in the absence of an additive thermal-noise field, is also in completely incoherent states. (Note that we are still using "coherence" in its quantum-mechanical sense.)

Let us again expand the electric-field operator in terms of its Fourier components and plane-wave mode functions as in Eq. 4. When the transmitted input symbol is m_j, in the time interval $(0,\tau)$ during which the signal is expected to arrive at the receiver, the signal field is in a state specified by the density operator

$$\rho_j^{(t)} = \frac{1}{|\det K_j^{(t)}|} \int \cdots \int \exp\left[-\sum_m \sum_n \alpha_m^* \left[K_j^{(t)}\right]_{mn}^{-1} \alpha_n\right] \prod_m |\alpha_m\rangle\langle\alpha_m| \frac{d^2\alpha_m}{\pi}$$

$$j = 1, 2, \ldots M. \tag{80a}$$

In this expression, $K_j^{(t)}$ is the mode correlation matrix whose elements are

$$\left[K_j^{(t)}\right]_{mn} = \operatorname{Tr}\left[\rho_j^{(t)} a_m^+ a_n\right]. \tag{80b}$$

In the presence of a thermal radiation field that is in the state specified by the density operator $\rho^{(n)}$ in Eq. 20, the received field is in a state specified by the density operator

$$\rho_j = \frac{1}{|\det K_j|} \int \cdots \int \exp\left[-\sum_m \sum_n \alpha_m^* [K_j]_{mn}^{-1} \alpha_n\right] \prod_m |\alpha_m\rangle\langle\alpha_m| \frac{d^2\alpha_m}{\pi}. \tag{81}$$

Since the signal source and the noise source are independent, the mode correlation matrix K_j is

$$K_j = K_j^{(t)} + K^{(n)}.$$

That is,

$$[K_j]_{mn} = \left[K_j^{(t)}\right]_{mn} + \langle n\rangle \delta_{mn}. \tag{82}$$

When the mode correlation matrix K_j is diagonalized, the element $[K_j]_{kk}$ is equal to the average number of photons in the k^{th} mode of the received field going through the receiving aperture in the time interval $(0,\tau)$ at thermal equilibrium.

As in Section IV, we shall only consider the case in which density operators are pairwise commutative. The necessary and sufficient condition for the density operator in (81) to be commutative is that the mode correlation matrices K_1, K_2, ... K_M are commutative. The last condition is satisfied whenever the signal fields are orthogonal (that is, they satisfy the condition in Eq. 19). The commutativity of the mode correlation matrices K_1, K_2, ... K_M implies the existence of a unitary transformation matrix V that is such that the matrices

$$R_j = V^+K_jV; \qquad j = 1, 2, \dots M$$

are diagonalized. When the elements of K_j are given by (82), the kk^{th} element of the matrix R_j can be written

$$[R_j]_{kk} = \langle n\rangle + s_{jk}, \tag{83a}$$

where

$$s_{jk} = \left[V^+K_j^{(t)}V\right]_{kk}. \tag{83b}$$

Since Eq. 81 can be rewritten

$$\rho_j = \frac{1}{|\det R_j|} \int \exp\left[-\underline{a}^+K_j^{-1}\underline{a}\right] \prod_k |a_k\rangle\langle a_k| \frac{d^2a_k}{\pi},$$

where $\underline{a}$ is the column matrix whose elements are a_k, we have

$$\rho_j = \frac{1}{|\det R_j|} \int \exp\left[-\underline{a}^+V^+VK_j^{-1}V^+V\underline{a}\right] \prod_k |\beta_k\rangle\langle\beta_k| \frac{d^2\beta_k}{\pi}.$$

In this expression, $|\beta_k\rangle = \sum_n V_{kn}|a_n\rangle$ is the right eigenstate of the operator $b_k = \sum_n V_{kn}a_n$. Writing the right-hand side of the last equation in terms of the β_k, we obtain

$$\rho_j = \int \exp\left[-\sum_k \frac{|\beta_k|^2}{\langle n\rangle + s_{jk}}\right] \prod_k |\beta_k\rangle\langle\beta_k| \frac{d^2\beta_k}{\pi(\langle n\rangle + s_{jk})}. \tag{84}$$

The density operator ρ_j in (84) specifies the state of the received field, when the electric-field operator is expanded in terms of the b_k and their adjoints. The associated normal mode functions (for narrow-band signals) are

$$V_k(\underline{r}, t) = \sum_n V_{kn} \exp\left[-i(\omega_k'-\omega_n)\left(\frac{z}{c}-t\right)\right].$$

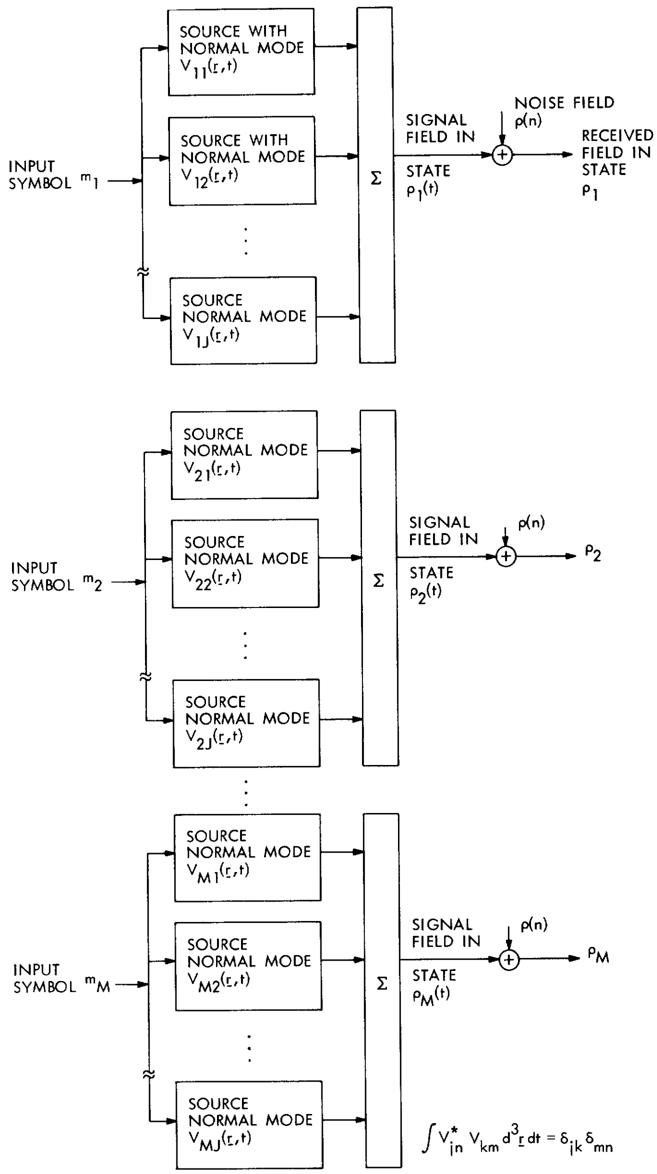

Fig. 12. A diversity system.

In the following discussion we shall consider only frequency orthogonal signals. For the sake of clarity, we denote by $|\beta_{jk}\rangle$ and $V_{jk}(\underline{r},t)$ the eigenstate $|\beta_{(j-1)\Delta+k+k_o}\rangle$ and mode functions $V_{(j-1)\Delta+k+k_o}(\underline{r},t)$, respectively, for $j = 1,2,\ldots M$ and $k = 1,2,\ldots J$, where $\Delta \geqslant J$. (Δ, J, and k_o are positive integers.) In this case, the set of normal mode functions are chosen to be that given by Eq. A.8 and the electric-field operator is expanded as in Eq. A. 7. The density operator, ρ_j, specifying the states of all relevant modes of the received field when m_j is transmitted can be written

$$\rho_j = \left\{\int \exp\left[-\sum_{k=1}^{J} \frac{|\beta_{jk}|^2}{\langle n\rangle + s_{jk}}\right] \prod_{k=1}^{J} |\beta_{jk}\rangle\langle\beta_{jk}| \frac{d^2\beta_{jk}}{\pi(\langle n\rangle + s_{jk})}\right\}$$
$$\left\{\int \exp\left[-\sum_{\substack{j'=1\\ j'\neq j}}^{M}\sum_{k=1}^{J} \frac{|\beta_{j'k}|^2}{\langle n\rangle}\right] \prod_{k=1}^{J}\prod_{\substack{j'=1\\ j'\neq j}}^{M} |\beta_{j'k}\rangle\langle\beta_{j'k}| \frac{d^2\beta_{j'k}}{\pi\langle n\rangle}\right\}. \tag{85}$$

That is, the signal field is in the state specified by

$$\rho_j^{(t)} = \left\{\int \exp\left[-\sum_{k=1}^{J} \frac{|\beta_{jk}|^2}{s_{jk}}\right] \prod_{k=1}^{J} |\beta_{jk}\rangle\langle\beta_{jk}| \frac{d^2\beta_{jk}}{\pi s_{jk}}\right\}$$
$$\left\{\int \prod_{\substack{j'=1\\ j'\neq j}}^{M}\prod_{k=1}^{J} \delta^{(2)}(\beta_{j'k})|\beta_{j'k}\rangle\langle\beta_{j'k}|\, d^2\beta_{j'k}\right\}. \tag{86}$$

For simplicity, it is assumed that only J modes of the field are excited by the signal source at a time. (The J modes excited by the signal source when m_j is transmitted are denoted as the $(j-1)\Delta + k_o + 1^{st}$, $(j-1)\Delta + k_o + 2^{nd}$, ..., $(j-1)\Delta + k_o + J^{th}$ modes in Eqs. 85 and 86.) Eventually, it will become evident that there is no loss of generality in making this assumption. The block diagram of the transmitter and the channel is shown in Fig. 12.

Before proceeding to find the optimum receiver and to investigate its performance for such a set of signals, we digress for a moment to discuss the corresponding classical channel.

5.1 CORRESPONDING CLASSICAL COMMUNICATION SYSTEM

The type of signals characterized by the density operators in Eq. 85 can be generated by all natural light sources that can be considered to be made up of a very large number of independently radiating atoms. Such random and chaotic excitation is characteristic of most incoherent macroscopic sources, for example, gas discharges, incandescent radiators, and so forth.

The density operators $\rho_j^{(t)}$ in Eq. 86 also describe the state of the received field in a Rayleigh fading channel, in the absence of an additive noise field. In such a channel, there is a large number of isolated point scatterers located at random points along the propagation path. The scatterers are moving at a speed so slow that their movement in the time interval of length τ, and hence the Doppler shift, may be neglected.

When the transmitted input symbol is m_j, the transmitted electric field has a classical waveform in a frequency position modulation system

$$\epsilon_{cj}^{(t)}(\underline{r},t) = \mathrm{Re}\left[s_j v_j(\underline{r},t)\exp\left[-i\omega_j\left(\frac{z}{c}-t\right)\right]\right] \qquad j = 1,2,\ldots M, \tag{87}$$

where

$$\omega_j = \frac{2\pi}{\tau}[k_o+(j-1)\Delta].$$

The $v_j(\underline{r},t)$, the complex envelopes of the $\epsilon_{cj}^{(t)}(\underline{r},t)$, constitute a set of orthonormal waveforms. The waveform of the received electric field in the absence of additive noise is

$$\epsilon_{cj}(\underline{r},t) = \mathrm{Re}\left[x(\underline{r},t)s_j v_j(\underline{r},t)\exp\left[-i\omega_j\left(\frac{z}{c}-t\right)-i\phi(\underline{r},t)\right]\right], \tag{88}$$

where $x(\underline{r},t)$ and $\phi(\underline{r},t)$ are sample functions of random processes that are such that at any time t and point $\underline{r}$, $x(\underline{r},t)$ and $\phi(\underline{r},t)$ are Rayleigh-distributed and uniformly distributed random variables, respectively. For simplicity, we suppose that the duration of the signal τ is very short compared with the rate of variation of $x(\underline{r},t)$ and $\phi(\underline{r},t)$; that is, these processes are relatively constant over the signaling time interval. Similarly, we suppose that they can be considered as constants in space over the beam width, too. Although these assumptions are not always realistic, they lead to illustrative results which can be generalized easily to account for time and spatial variations in $x(\underline{r},t)$ and $\phi(\underline{r},t)$. In this case, the received electric field is again a plane wave over the receiving aperture. Equation 88 simplifies to

$$\epsilon_{cj}(\underline{r},t) = \mathrm{Re}\left[xs_j v_j(\underline{r},t)\exp\left[-i\omega_j\left(\frac{z}{c}-t\right)-i\phi\right]\right], \tag{89}$$

where the probability density functions of x and ϕ are

$$p_x(u) = \frac{u}{B}\exp\left(-\frac{u^2}{2B}\right); \qquad u \geq 0 \tag{90a}$$

$$p_\phi(u) = \begin{cases} \frac{1}{2\pi} & 0 \leq u \leq 2\pi \\ 0 & \text{elsewhere} \end{cases} \tag{90b}$$

In the quantum-mechanical limit, the state of the received field with electric field amplitude given by (89) is specified by the density operator $\rho_j^{(t)}$ in Eq. 86, with $J = 1$.

In classical communication theory, it has been shown that when additive white Gaussian noise is also present, the minimum attainable error probability in transmitting one of the M equally likely orthogonal signals over a Rayleigh-fading channel decreases only inversely with the transmitted energy. The reason for this inferior performance is that it is highly probable that the actual received energy on any given transmission will be small, even when the average received energy is high. That is, the probability of a "deep fade" is appreciable.

One way to reduce the error probability is to circumvent the high probability of a deep fade on a single transmission by means of diversity transmission. In a diversity system, several transmissions are made for each input symbol. These transmissions are spaced either in time, space, or frequency in such a way that the fading experienced by each transmission is statistically independent. This is possible, since in all practical scattering channels the scatterers move randomly with respect to one another as time goes on.

Without loss of generality, we shall confine our discussion to frequency diversity systems. Let J denote the number of diversity transmissions. In the absence of additive noise, the transmitted waveform and the received waveform are given respectively by

$$\epsilon_{cj}^{(t)}(\underline{r},t) = \mathrm{Re}\left[\sum_{k=1}^{J} s_{jk}v_{jk}(\underline{r},t)\,\exp\left[-i\omega_j\left(\frac{z}{c}-t\right)\right]\right]$$

$$\epsilon_{cj}(\underline{r},t) = \mathrm{Re}\left[\sum_{k=1}^{J} x_{jk}s_{jk}v_{jk}(\underline{r},t)\,\exp\left[-i\omega_j\left(\frac{z}{c}-t\right)-i\phi_{jk}\right]\right],$$

where

$$\omega_j = \frac{2\pi}{T}\left[k_o+(j-1)\Delta\right]$$

when m_j is transmitted. The x_{jk} and the ϕ_{jk} are statistically independent Rayleigh-distributed and uniformly distributed random variables, respectively. The complex envelopes $v_{jk}(\underline{r},t)$ are given in general by Eq. A.8a. The waveforms $\epsilon_{cj}(\underline{r},t)$ are outputs of a system shown in Fig. 13. It can be seen that in the quantum-mechanical limit, the relevant modes of the received field are in the state given by the density operator in (86) when no noise is present and m_j is transmitted. The coherent states $|\beta_{jk}\rangle$ are right eigenstates of the operators b_{jk}. The matrix elements V_{jk} are given by Eq. A.8b.

Although in the present discussion our attention was confined to the classical diversity system, it is evident that the density operators in (86) specify the states of the received field in the more general fading dispersive channel. This is due to the fact that a fading dispersive channel can be represented canonically as a classical diversity system[22] as shown in Fig. 13.

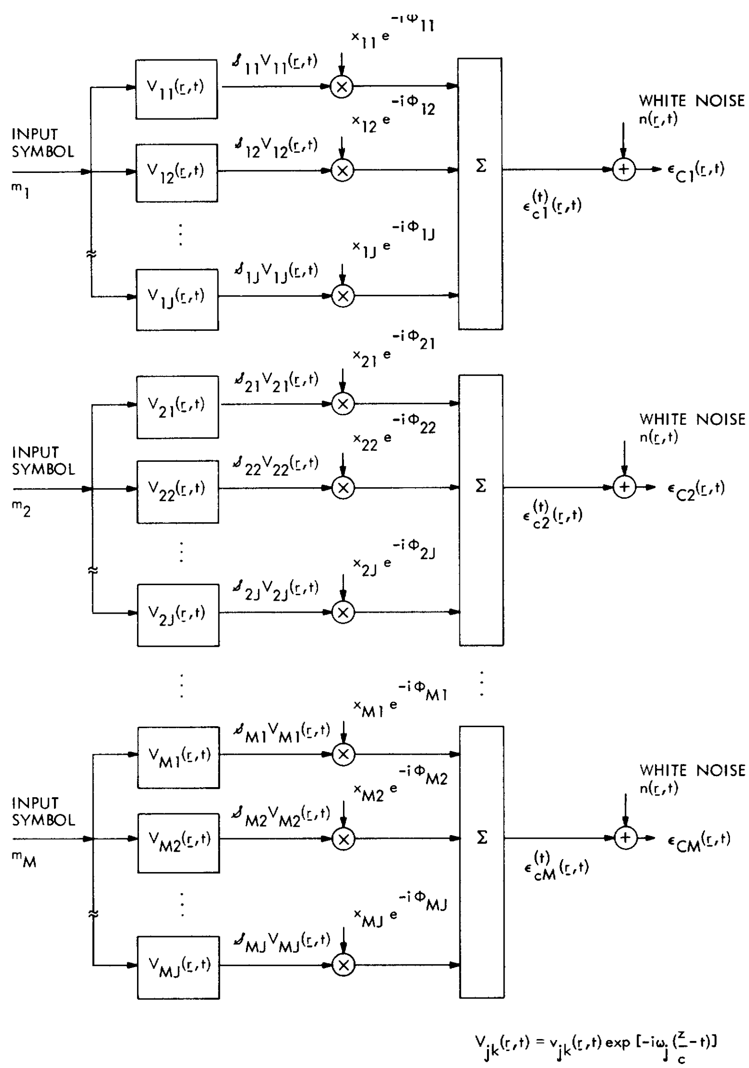

Fig. 13. A classical diversity system.

5.2 OPTIMUM RECEIVER AND ITS PERFORMANCE

Let us again consider the simple case in which the mode correlation matrices K_j in Eq. 81 are diagonalized. When the number of diversities in the system is J, the density operator ρ_j which specifies the states of the relevant modes of the received field is simply

$$\rho_j = \left\{ \int \exp\left[-\sum_{k=1}^{J} \frac{|\alpha_{jk}|^2}{\langle n\rangle + s_{jk}}\right] \prod_{k=1}^{J} |\alpha_{jk}\rangle\langle\alpha_{jk}| \frac{d^2\alpha_{jk}}{\pi(\langle n\rangle + s_{jk})} \right\}$$

$$\left\{ \int \exp\left[-\sum_{\substack{j'=1\\ j'\neq j}}^{M} \sum_{k=1}^{J} \frac{|\alpha_{j'k}|^2}{\langle n\rangle}\right] \prod_{k=1}^{J} \prod_{\substack{j'\neq j\\ j'=1}}^{M} |\alpha_{j'k}\rangle\langle\alpha_{j'k}| \frac{d^2\alpha_{j'k}}{\pi\langle n\rangle} \right\} \quad j = 1, 2, \ldots M, \tag{91}$$

while the electric-field operator is expanded in terms of the plane-wave normal mode functions and its Fourier components

$$E(\underline{r}, t) = \sum_{j=1}^{M} \sum_{k=1}^{J} \sqrt{\frac{\hbar\omega_{jk}}{2L}} \left\{a_{jk} \exp\left[i\omega_{jk}\left(\frac{z}{c} - t\right)\right] - a_{jk}^{+} \exp\left[-i\omega_{jk}\left(\frac{z}{c} - t\right)\right]\right\} + \text{irrelevant terms.}$$

(For the sake of clarity, we denote by ω_{jk}, a_{jk}, and $|\alpha_{jk}\rangle$ the frequency $\omega_{(j-1)\Delta+k+k_o}$, the operator $a_{(j-1)\Delta+k+k_o}$, and the coherent state $\left|\alpha_{(j-1)\Delta+k_o+k}\right\rangle$, respectively, for some integer k_o and $\Delta \geq J$.) When m_j is transmitted, the general form of the frequency spectrum of the signal field is as shown in Fig. 14.

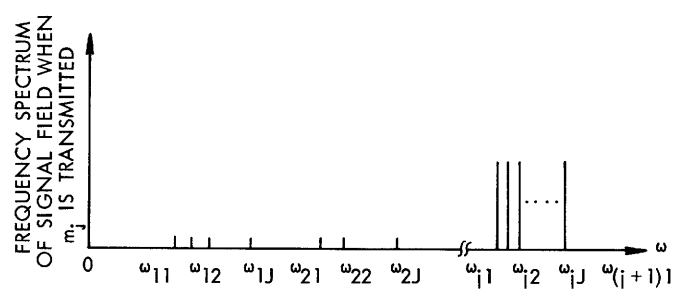

Fig. 14. Frequency spectrum of the signal field.

Eventually, we shall modify the receiver structure so that it will be optimum for the reception of any arbitrary narrow-band orthogonal signals. It will become evident that for narrow-band signals, the optimum performance in transmitting M equally likely input symbols in a J-fold diversity system using orthogonal waveforms does not depend on the specific choice of the orthogonal set of waveforms.

5.2.1 Structure of the Optimum Receiver

It is obvious that the density operator ρ_j given by Eq. 91 is diagonalized in the number representation. Let $\prod_{j=1}^{M} \prod_{k=1}^{J} |n_{jk}\rangle$ denote the simultaneous eigenstate of the operators $a_{11}^{+}a_{11}, a_{12}^{+}a_{12}, \ldots a_{MJ}^{+}a_{MJ}$ corresponding to eigenvalues $n_{11}, n_{12}, \ldots n_{MJ}$. In terms of these eigenstates, the density operator ρ_j can be expanded as

$$\rho_j = \left(\frac{1}{1+\langle n\rangle}\right)^{(M-1)J} \sum_{n_{11}=0}^{\infty} \sum_{n_{12}=0}^{\infty} \cdots \sum_{n_{MJ}=0}^{\infty} \prod_{k=1}^{J} \left\{ \prod_{\substack{j'\neq j \\ j'=1}}^{M} \left(\frac{\langle n\rangle}{1+\langle n\rangle}\right)^{n_{j'k}} \left(\frac{1}{1+s_{jk}+\langle n\rangle}\right)\left(\frac{\langle n\rangle + s_{jk}}{1+\langle n\rangle + s_{jk}}\right)^{n_{jk}} \right\} \left\{ \prod_{j=1}^{M} |n_{jk}\rangle\langle n_{jk}| \right\}.$$

Therefore the optimum receiver measures simultaneously the dynamical variables $a_{11}^{+}, a_{11}, a_{12}^{+}a_{12}, \ldots, a_{MJ}^{+}a_{MJ}$ (see Section III).

When the variables $a_{jk}^{+}a_{jk}$ are measured simultaneously, the probability that the distribution of the outcome will be the MJ-tuple $\underline{n} = (n_{11}, n_{12}, \ldots n_{1J}, \ldots n_{MJ})$, given that the density operator of the received field is ρ_j, is

$$P(\underline{n}/m_j) = \prod_{k=1}^{J} \left\{ \prod_{\substack{j'=1 \\ j'\neq j}}^{M} \left(\frac{1}{1+\langle n\rangle}\right)^{(M-1)J} \left(\frac{\langle n\rangle}{1+\langle n\rangle}\right)^{n_{j'k}} \frac{1}{1+\langle n\rangle + s_{jk}} \left(\frac{\langle n\rangle + s_{jk}}{1+\langle n\rangle + s_{jk}}\right)^{n_{jk}} \right\}. \tag{92a}$$

To simplify our notation, let

$$q_{jk} = \frac{1}{1+\langle n\rangle + s_{jk}}; \qquad j = 1, 2, \ldots M$$

$$q_o = \frac{1}{1+\langle n\rangle}.$$

Then

$$P(\underline{n}/m_j) = q_o^{(M-1)J} \prod_{k=1}^{J} \left\{ q_{jk}(1-q_{jk})^{n_{jk}} \prod_{\substack{j'=1 \\ j'\neq j}}^{M} (1-q_o)^{n_{j'k}} \right\}. \tag{92b}$$

Since the inequality

$$P(\underline{n}/m_j) \geq P(\underline{n}/m_i)$$

is satisfied if and only if

$$\prod_{k=1}^{J} q_{jk}\left(\frac{1-q_{jk}}{1-q_o}\right)^{n_{jk}} \geq \prod_{k=1}^{J} q_{ik}\left(\frac{1-q_{ik}}{1-q_o}\right)^{n_{ik}},$$

the value of j which maximizes $P(\underline{n}/m_j)$ also maximizes the quantity

$$f_j = \sum_{k=1}^{J} \left\{ n_{jk} \ln \left(\frac{1 - q_{jk}}{1 - q_o} \right) + \ln q_{jk} \right\}. \tag{93}$$

Hence, the optimum receiver needs only to evaluate the quantities f_j. The transmitted input symbol is estimated as m_j if

$$f_j \geq f_i; \quad i \neq j, \quad i, j = 1, 2, \ldots . M. \tag{94}$$

Again, such a decision rule will yield ambiguous results whenever (94) is satisfied for more than one value of j. The error probability of the system will not be affected,

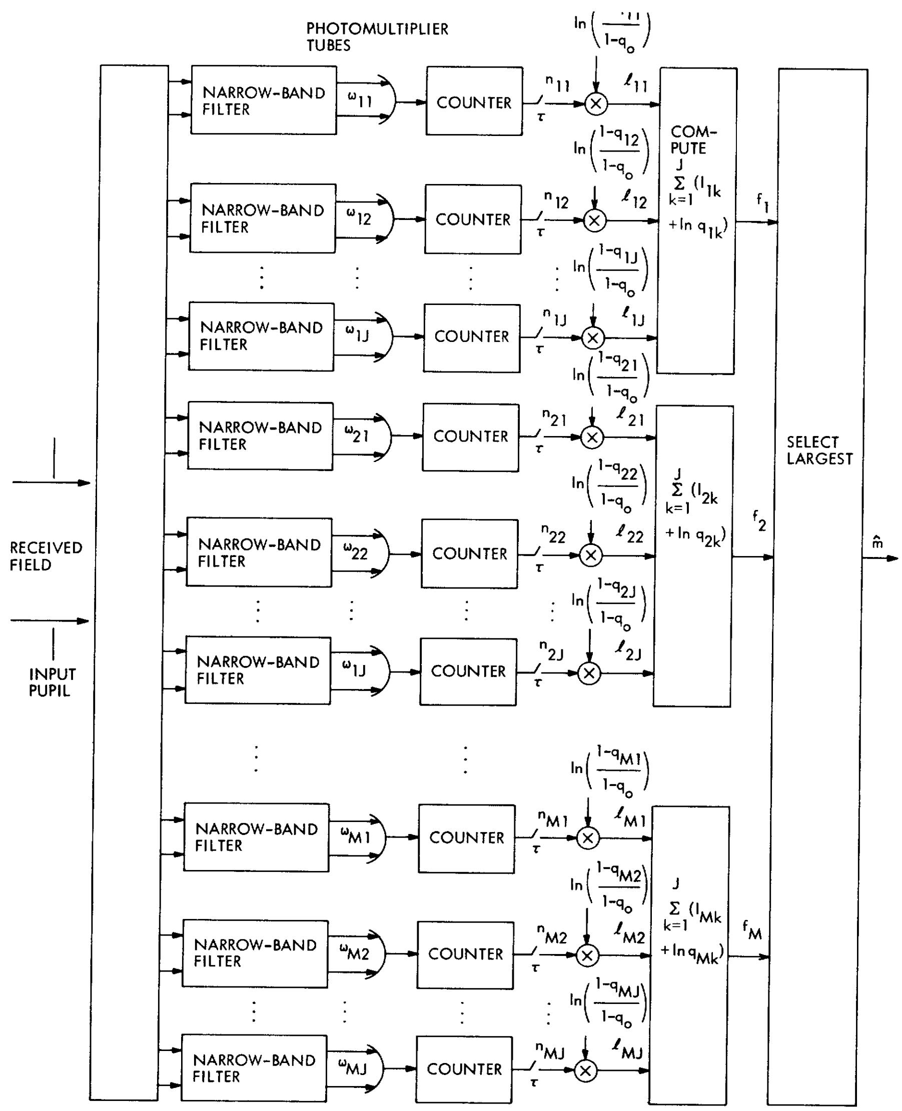

Fig. 15. Optimum receiver for frequency orthogonal signals in completely incoherent state.

however, by the way in which such ambiguities are resolved. The portion of the optimum receiver which makes observations on the field and computes the quantities f_j is shown in Fig. 15.

It is difficult to evaluate the performance of such a system. In the following discussion, we shall consider only the case in which the numbers of signal photons s_{jk} are equal. That is,

$$s_{jk} = s \quad j = 1, 2, \ldots M; \quad k = 1, 2, \ldots . \tag{95}$$

For the narrow-band signals considered here, (95) also implies that the energies in all of the J modes (J diversity paths) of the signal field are equal as in equal-strength diversity systems. It has been shown[23] classically that the system performance is optimized when the energies in the diversity paths are equal. We shall not now try to prove the optimality of the equal-strength diversity system. Rather, the assumption in Eq. 95 is made for simplicity. For this special case, the structure of the optimum receiver simplifies to that shown in Fig. 16.

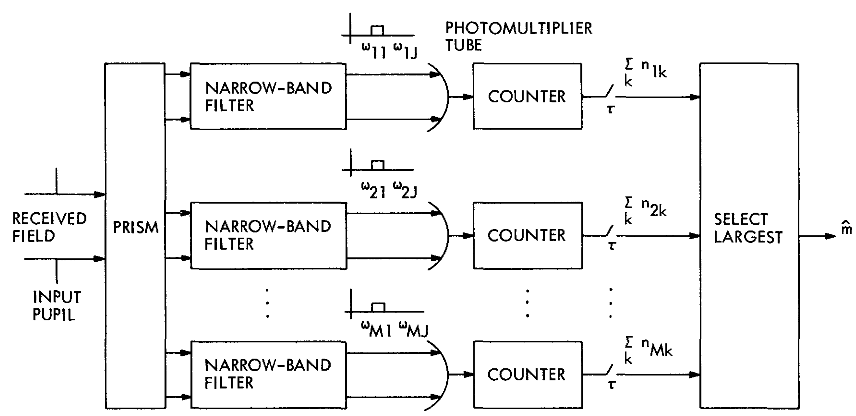

Fig. 16. Optimum receiver for equal-strength orthogonal signals in completely incoherent states.

One can easily modify the structure of the optimum receiver for the reception of arbitrary narrow-band signals specified by the density operators in Eqs. 85 and 86. In this case, the density operators are diagonalized in the $\prod_{j=1}^{M} \prod_{k=1}^{J} b_{jk}^{+} b_{jk}$ representation, where $b_{jk} = \sum_n V_{(j-1)\Delta+k_0+k,\,(j-1)\Delta+n+k_0}\, a_{jn}$. The quantities V_{kn} are elements of the matrix V that diagonalizes the mode correlation matrices $K_1, K_2, \ldots$ and K_M. The normal modes excited by the signal source have mode functions given by Eq. A.8a when the signals are frequency-orthogonal. For this set of signals, the optimum receiver is shown in Fig. 17.

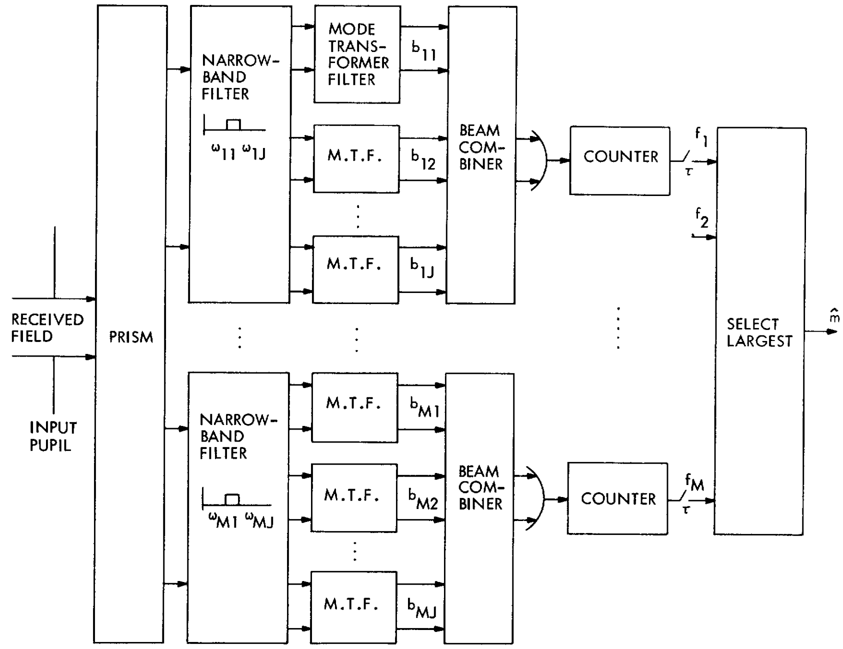

Fig. 17. Optimum receiver for equal-strength orthogonal signals in completely incoherent states.

5.2.2 Performance of the Optimum Receiver

We have described the forms of the optimum receiver in a quantum-mechanical Gaussian-Gaussian channel, wherein the received signal field is also in completely incoherent states. Now we shall evaluate the performance of this system. Again, it is appropriate to describe the performance of the system by the probability of any incorrect M-ary decision, $P(\epsilon)$, since we shall only be concerned with systems that do not utilize any coding. Furthermore, the channel is assumed to be memoryless and there is no intersymbol interference.

Just as in Section IV, the bounds to the error probability $P(\epsilon)$ will be expressed as

$$K_1 \exp[-\tau CE(R)] \leq P(\epsilon) \leq K_2 \exp[-\tau CE(R)],$$

where τ is the time allotted to the transmission of a single input symbol. The quantity

$$C = p \ln \frac{1 + \langle n \rangle}{\langle n \rangle}$$

is the capacity of the system in which the noise field is in a completely incoherent state, but the signal field is in a coherent state with an unknown phase factor (Eq. 50a). In the expression of C, $p = \frac{S}{\tau}$, where S is the total number of photons transmitted through J diversity paths. Hence $S = Js$. Our attention will be focused upon the reliability function $E(R)$ which again can be used to characterize the system.

The system reliability function can be expressed in terms of a set of parametric equations, as well as in terms of the solution of a maximization problem. We shall first find a lower bound $E_L(R)$ of the reliability function. Then an upper bound $E_u(R)$ of the reliability function will be found which turns out to be identically equal to $E_L(R)$.

Upper Bound to $P(\epsilon)$

Let $p_c(n)$ denote the probability distribution of the random variable $f_j = \sum_{k=1}^{J} n_{jk}$ when the transmitted input symbol is m_j. Let $p_i(n)$ denote the probability distribution of the random variable $f_i = \sum_{k=1}^{J} n_{ik}$, when the transmitted input symbol is m_j and the value of i is not equal to j. The moment-generating functions corresponding to the probability distributions $p_c(n)$ and $p_i(n)$ are denoted by $g_c(t)$ and $g_i(t)$, respectively. That is,

$$g_c(t) = \sum_{n=0}^{\infty} p_c(n) \exp(tn) \tag{96a}$$

$$g_i(t) = \sum_{n=0}^{\infty} p_i(n) \exp(tn). \tag{96b}$$

Since the n_{jk} are statistically independent, for all $k = 1, 2, \ldots J$ and $j = 1, 2, \ldots M$, and

$$P(n_{jk}/m_j) = q_s(1-q_s)^{n_{jk}}$$

$$P(n_{ik}/m_j, i \neq j) = q_o(1-q_o)^{n_{ik}},$$

where

$$q_s = \frac{1}{1 + \langle n \rangle + s},$$

it follows that

$$g_c(t) = \left[\frac{q_s}{1-(1-q_s)\, e^t}\right]^J \tag{97a}$$

$$g_i(t) = \left[\frac{q_o}{1-(1-q_o)\, e^t}\right]^J. \tag{97b}$$

The corresponding semi-invariant moment-generating functions denoted by $\gamma_c(t)$ and $\gamma_i(t)$, respectively, are

$$\gamma_c(t) = \ln g_c(t) = J \ln \frac{q_s}{1-(1-q_s)\,e^t} \tag{98a}$$

$$\gamma_i(t) = \ln g_i(t) = J \ln \frac{q_o}{1-(1-q_o)\,e^t}. \tag{98b}$$

With these preliminaries taken care of, we now proceed to derive an exponentially tight bound on the error probability, $P(\epsilon)$. Since much of the derivation of this upper bound is similar to that presented in Section IV, we shall now only outline the derivation.

The conditional probability of error, $P(\epsilon/m_1)$, given that the input symbol transmitted is m_1, is

$$P(\epsilon/m_1) \leq \sum_{n=0}^{\infty} p_c(n)\ \Pr\,(n\leq f_2,\ \text{or } n\leq f_3,\ldots,\ \text{or } n\leq f_M/m_1).$$

For some parameter δ in the range $(0,1)$, we have

$$P(\epsilon/m_1) = \sum_{n=0}^{\infty} p_c(n)\ \{\Pr\,(n\leq f_2,\ \text{or } n\leq f_3,\ldots,\ \text{or } n\leq f_M/m_1)\}^{\delta}$$

$$< M^{\delta} \sum_{n=0}^{\infty} p_c(n) \left[\sum_{m=n}^{\infty} p_i(m)\right]^{\delta}$$

$$= M^{\delta} \sum_{n=0}^{\infty} p_c(n) \left[\sum_{m=n}^{\infty} \exp(tm-tm)\ p_i(m)\right]^{\delta}.$$

If the parameter t is non-negative, we have

$$P(\epsilon/m_1) \leq M^{\delta}\left[\sum_{n=0}^{\infty} p_c(n)\ \exp(-t\delta n)\right]\left[\sum_{m=0}^{\infty} p_i(m)\ \exp(tm)\right]^{\delta}.$$

By substituting the expressions

$$M = \exp(R\tau)$$

$$\sum_{n=0}^{\infty} p_c(n)\ \exp(-t\delta n) = \exp[\gamma_c(-\delta t)]$$

$$\sum_{m=0}^{\infty} p_i(n)\ \exp(tm) = \exp[\gamma_i(t)]$$

in the last equation, it becomes

$$P(\epsilon/m_1) \leqslant \exp[-\{-\delta R\tau-\gamma_c(-\delta t)-\delta\gamma_i(t)\}]; \qquad \text{for } 0 \leqslant \delta \leqslant 1, \text{ and } t \geqslant 0.$$

It is clear that $P(\epsilon/m_1)$ is independent of the transmitted input symbol. Therefore,

$$P(\epsilon) = \exp\{-[-\delta R\tau-\gamma_c(-\delta t)-\delta\gamma_i(t)]\}; \qquad \text{for } 0 \leqslant \delta \leqslant 1, \ t \geqslant 0. \tag{99}$$

Equation 99 is satisfied for all values of $t \geqslant 0$ and $0 \leqslant \delta \leqslant 1$. In particular, it is satisfied for the values of t and δ which maximize the function

$$E_o(\delta, t, R) = -\frac{\delta R}{C} - \frac{1}{\tau C}\left[\gamma_c(-\delta t)+\delta\gamma_i(t)\right]. \tag{100}$$

A lower bound of the reliability function is, therefore, given by

$$E_L(R) = \max_{0\leqslant\delta\leqslant 1} \ \max_{t\geqslant 0} \ E_o(\delta, t, R). \tag{101}$$

Substituting the expressions for $\gamma_i(t)$ and $\gamma_c(t)$ in Eqs. 98a and 98b in Eq. 100, we obtain

$$E_o(\delta, t, R) = -\delta\frac{R}{C} + \frac{1}{\tau C}\left\{\delta J\ln\left[1-(1-q_o)\,e^{t}\right] + J\ln\left[1-(1-q_s)\,e^{-\delta t}\right]\right\} - \frac{1}{\tau C}\left\{\delta J\ln q_o + J\ln q_s\right\}. \tag{102}$$

The right-hand side of Eq. 102 is maximized with respect to t when the value of t is given by

$$t = \ln\left(\frac{1-q_s}{1-q_o}\right)^{\frac{1}{1+\delta}}. \tag{103}$$

(Since q_s is always smaller than q_o, the right-hand side of (103) is always positive.) Substituting (103) in (100) and (101), we obtain

$$E_1(\delta) - \delta\frac{R}{C} = \max_{t\geqslant 0} E_o(\delta, t, R)$$

$$= \frac{(1+\delta)J}{\tau C}\left\{\ln\left[1-(1-q_o)^{\frac{\delta}{1+\delta}}(1-q_s)^{\frac{1}{1+\delta}}\right] - \ln q_o^{\frac{\delta}{1+\delta}} q_s^{\frac{1}{1+\delta}}\right\} - \frac{\delta R}{C}. \tag{104a}$$

The function $E_1(\delta)$ can also be expressed in terms of $\langle n\rangle$ and s.

$$E_1(\delta) = \frac{(1+\delta)J}{\tau C} \ln \left\{ \frac{1 - \left(\frac{\langle n\rangle}{1+\langle n\rangle}\right)^{\frac{\delta}{1+\delta}} \left(\frac{\langle n\rangle + s}{1+\langle n\rangle + s}\right)^{\frac{1}{1+\delta}}}{\left(\frac{1}{1+\langle n\rangle}\right)^{\frac{\delta}{1+\delta}} \left(\frac{1}{1+\langle n\rangle + s}\right)^{\frac{1}{1+\delta}}} \right\}$$

$$= \frac{(1+\delta)J}{\tau C} \ln \left\{ 1 + (\langle n\rangle + s) \left[1 - \left(\frac{\langle n\rangle}{1+\langle n\rangle} \ \frac{1+\langle n\rangle + s}{\langle n\rangle + s} \right)^{\frac{\delta}{1+\delta}} \right] \right\}$$

$$- \frac{\delta J}{\tau C} \ln \left\{ 1 + \frac{s}{1+\langle n\rangle} \right\}. \qquad (104b)$$

The lower bound $E_L(R)$ of the reliability function $E(R)$ is obtained by maximizing $E_1(\delta) - \delta \frac{R}{C}$ over all values of δ that is such that $0 \leqslant \delta \leqslant 1$. The partial derivative $E_1(\delta)$ with respect to δ is

$$\frac{\partial E_1(\delta)}{\partial \delta} = \frac{J}{\tau C} \left\{ \ln \left[(1+\langle n\rangle) \left[1 - \left(\frac{\langle n\rangle}{1+\langle n\rangle}\right)^{\frac{\delta}{1+\delta}} \left(\frac{\langle n\rangle + s}{1+\langle n\rangle + s}\right)^{\frac{1}{1+\delta}} \right] \right] \right.$$

$$\left. - \frac{1}{1+\delta} \ \frac{\left(\frac{\langle n\rangle}{1+\langle n\rangle}\right)^{\frac{\delta}{1+\delta}} \left(\frac{\langle n\rangle + s}{1+\langle n\rangle + s}\right)^{\frac{1}{1+\delta}} \ln \left(\frac{\langle n\rangle}{1+\langle n\rangle} \ \frac{1+\langle n\rangle + s}{\langle n\rangle + s} \right)}{1 - \left(\frac{\langle n\rangle}{1+\langle n\rangle}\right)^{\frac{\delta}{1+\delta}} \left(\frac{\langle n\rangle + s}{1+\langle n\rangle + s}\right)^{\frac{1}{1+\delta}}} \right\}.$$

Let us denote by R_c the quantity

$$R_c = \left.\frac{\partial E_1(\delta)}{\partial \delta}\right|_{\delta=1} C$$

$$= \frac{J}{\tau} \left\{ \ln \left[(1+\langle n\rangle) \left(1 - \sqrt{\frac{\langle n\rangle}{1+\langle n\rangle} \left(\frac{\langle n\rangle + s}{1+\langle n\rangle + s}\right)} \right) \right] - \frac{\sqrt{\frac{\langle n\rangle}{1+\langle n\rangle} \ \frac{\langle n\rangle + s}{1+\langle n\rangle + s}} \ \ln \sqrt{\frac{\langle n\rangle}{1+\langle n\rangle} \left(\frac{\langle n\rangle + s + 1}{\langle n\rangle + s}\right)}}{1 - \sqrt{\frac{\langle n\rangle}{1+\langle n\rangle} \ \frac{\langle n\rangle + s}{1+\langle n\rangle + s}}} \right\}$$

(105)

and by C_g the quantity

$$C_g = \left.\frac{\partial E_1(\delta)}{\partial \delta}\right|_{\delta=0} C$$

$$= \frac{J}{\tau}\left\{(\langle n\rangle + s)\ \ln\left(1 + \frac{s}{\langle n\rangle}\right) - (1 + \langle n\rangle + s)\ \ln\left(1 + \frac{s}{1+\langle n\rangle}\right)\right\}. \tag{106}$$

Since $\frac{\partial^2 E_1}{\partial \delta} > 0$, setting

$$\frac{\partial E_1(\delta)}{\partial \delta} - \frac{R}{C} = 0 \tag{107}$$

will yield the value of δ that maximizes the function $E_1(\delta) - \delta\frac{R}{C}$ with respect to δ for rates $R_c \leq R \leq C_g$. Hence, the best lower bound $E_L(R)$ of the reliability function is given by

$$E_L(R) = \frac{J(1+\delta)}{\tau C} \ln\left\{1 + (\langle n\rangle + s)\left[1 - \left(\frac{\langle n\rangle}{1+\langle n\rangle}\ \frac{1+\langle n\rangle + s}{\langle n\rangle + s}\right)^{\frac{\delta}{1+\delta}}\right]\right\}$$

$$- \frac{J\delta}{\tau C} \ln\left(\frac{1+\langle n\rangle + s}{1+\langle n\rangle}\right) - \delta\frac{R}{C} \tag{108a}$$

and

$$R = \frac{J}{\tau}\left\{\ln\left[(1+\langle n\rangle)\left(1 - \left(\frac{\langle n\rangle + s}{1+\langle n\rangle + s}\right)^{\frac{1}{1+\delta}}\left(\frac{\langle n\rangle}{1+\langle n\rangle}\right)^{\frac{\delta}{1+\delta}}\right)\right]\right.$$

$$\left. - \frac{1}{1+\delta}\ \frac{\left(\frac{\langle n\rangle}{1+\langle n\rangle}\right)^{\frac{\delta}{1+\delta}}\left(\frac{\langle n\rangle + s}{1+\langle n\rangle + s}\right)^{\frac{1}{1+\delta}} \ln\left(\frac{\langle n\rangle}{1+\langle n\rangle}\ \frac{1+\langle n\rangle + s}{\langle n\rangle + s}\right)}{1 - \left(\frac{\langle n\rangle + s}{1+\langle n\rangle + s}\right)^{\frac{1}{1+\delta}}\left(\frac{\langle n\rangle}{1+\langle n\rangle}\right)^{\frac{\delta}{1+\delta}}}\right\} \tag{108b}$$

for rates R in the range (R_c, C_g). For rates R in the range $(0, R_c)$, the value of δ that maximizes the function $E_1(\delta) - \delta\frac{R}{C}$ is $\delta = 1$. Thus, we have

$$E_L(R) = \frac{2J}{\tau C} \ln\left\{\frac{1 - \sqrt{\frac{\langle n\rangle}{1+\langle n\rangle}\ \frac{\langle n\rangle + s}{1+\langle n\rangle + s}}}{\sqrt{\frac{1}{1+\langle n\rangle}\ \frac{1}{1+\langle n\rangle + s}}}\right\} - \frac{R}{C}. \tag{109}$$

The general behavior of the function $E_L(R)$ will be discussed in section 5.3. Now, we show that the upper bound of $P(\epsilon)$ derived above is an exponentially tight one. To this end, let us express $E_L(R)$ parametrically.

From Eqs. 100 and 103, it is clear that the function $E_1(\delta)$ can be expressed in terms of $\gamma_c(\ )$ and $\gamma_i(\ )$

$$E_1(\delta) = -\frac{1}{\tau C}\left\{\gamma_c\left(\frac{\delta}{1+\delta}\ln\frac{1-q_o}{1-q_s}\right) + \delta\gamma_i\left(\frac{1}{1+\delta}\ln\frac{1-q_s}{1-q_o}\right)\right\} - \frac{\delta R}{C}.$$

Let us define a parameter

$$\mathscr{s} = \frac{\delta}{1+\delta}\ln\frac{1-q_o}{1-q_s}. \tag{110}$$

Since $0 \leq \delta \leq 1$, the value of $\mathscr{s}$ lies in the range

$$\frac{1}{2}\ln\frac{1-q_o}{1-q_s} \leq \mathscr{s} \leq 0.$$

The exponent function $E_L(R)$, where

$$E_L(R) = \max_{0\leq\delta\leq 1}\left\{E_1(\delta) - \delta\frac{R}{C}\right\}$$

is also given parametrically by two equations, one of which is obtained from Eq. 108b.

$$\begin{aligned}\frac{R}{C} = \frac{1}{\tau C}\Bigg\{&\left[\left(\frac{1}{1+\delta}\right)^2\ln\frac{1-q_s}{1-q_o}\right]\gamma_c'(\mathscr{s}) \\ &+\left[\frac{\delta}{(1+\delta)^2}\ln\frac{1-q_s}{1-q_o}\right]\gamma_i'\left(\frac{1}{1+\delta}\ln\frac{1-q_s}{1-q_o}\right) - \gamma_i\left[\frac{1}{1+\delta}\ln\frac{1-q_s}{1-q_o}\right]\Bigg\}\end{aligned} \tag{111}$$

for $R_c \leq R \leq C_g$. Since

$$\frac{1}{1+\delta}\ln\frac{1-q_s}{1-q_o} = \mathscr{s} + \ln\frac{1-q_s}{1-q_o},$$

we have

$$\gamma_i\left(\frac{1}{1+\delta}\ln\frac{1-q_s}{1-q_o}\right) = \gamma_c(\mathscr{s}) + J\ln\frac{q_o}{q_s}$$

$$\gamma_i'\left(\frac{1}{1+\delta}\ln\frac{1-q_s}{1-q_o}\right) = \gamma_c'(\mathscr{s}).$$

The right-hand side of (111) simplifies to

$$\frac{R}{C} = \frac{1}{\tau C}\left\{\left(s + \ln\frac{1-q_s}{1-q_o}\right)\gamma_c'(s) - J\ln\frac{q_o}{q_s} - \gamma_c(s)\right\}. \tag{112a}$$

Thus for $\frac{1}{2}\ln\frac{1-q_o}{1-q_s} \leq s \leq 0$, that is, $R_c \leq R \leq C_g$, the exponent function $E_L(R)$ is given by

$$E_L(R) = \frac{1}{\tau C}\left[s\gamma_c'(s) - \gamma_c(s)\right]; \qquad R_c \leq R \leq C_g. \tag{112b}$$

For $0 \leq R \leq R_c$, $E_L(R)$ is obtained by setting s to $\frac{1}{2}\ln\frac{1-q_o}{1-q_s}$.

$$E_L(R) = \frac{2J}{\tau C}\ln\frac{1-\sqrt{(1-q_s)(1-q_o)}}{\sqrt{q_s q_o}} - \frac{R}{C}. \tag{112c}$$

It will be shown that the exponent function of a lower bound of $P(\epsilon)$ is equal to $E_L(R)$ in (112).

Lower Bound to $P(\epsilon)$

As shown in Appendix E, a lower bound of the probability of error is

$$P(\epsilon) \geq \frac{M}{4}\Pr\left(f_j \leq d/m_j\right)\Pr\left(f_i > d/m_j\right), \tag{113}$$

provided the parameter d is so chosen that the inequality

$$M \Pr\left(f_i > d/m_j\right) \leq 1 \tag{114}$$

is satisfied. Using the Chernov bound for $\Pr(f_i > d/m_j)$, we can rewrite the condition in (114) as

$$M \exp\left[-t\gamma_i'(t) + \gamma_i(t)\right] \leq 1; \qquad t \geq 0. \tag{115}$$

It has been shown[23] that

$$\Pr\left(f_i > d/m_j\right) \geq B_1 \exp\left[-t\gamma_i'(t) + \gamma_i(t)\right]; \qquad t \geq 0 \tag{116a}$$

and

$$\Pr\left(f_j \leq d/m_j\right) \geq B_2 \exp\left[-s\gamma_c'(s) + \gamma_i(s)\right]; \qquad s \leq 0, \tag{116b}$$

where the parameter d is related to t and s by the equation

$$d = \gamma_i'(t) = \gamma_c'(s). \tag{117}$$

For our purpose, it is sufficient to know that the coefficients B_1 and B_2 in these equations are not exponential functions of τ. It is clear from (117) that the relation between the parameters t and s is

$$\frac{(1-q_o)\, e^t}{1-(1-q_o)\, e^t} = \frac{(1-q_s)\, e^{s}}{1-(1-q_s)\, e^{s}}.$$

Or, equivalently,

$$t = \ln \frac{1-q_s}{1-q_o} + s. \tag{118}$$

Since $t \geq 0$, the parameter s lies in the range $\left(\ln \frac{1-q_o}{1-q_s},\ 0\right)$.

Substituting Eqs. 116a and 116b in Eq. 113, we obtain

$$P(\epsilon) \geq \frac{B_1B_2}{4} M \exp[-s\gamma_c'(s)+\gamma_c(s)+\gamma_i(t)-t\gamma_i'(t)]; \qquad s \leq 0,\ t \geq 0. \tag{119}$$

If this lower bound is written

$$P(\epsilon) \geq K_1 \exp[-\tau C E_u(R)],$$

the exponent $E_u(R)$ is given by

$$E_u(R) = \min_{\ln \frac{1-q_o}{1-q_s} \leq s \leq 0} E_2(s, R),$$

where

$$E_2(s, R) = \frac{1}{\tau C}\left\{\left(2s+\ln\frac{1-q_s}{1-q_o}\right)\gamma_c'(s)-\gamma_c(s)-\gamma_i\left(s+\ln\frac{1-q_s}{1-q_o}\right)\right\}-\frac{R}{C}.$$

Since

$$\gamma_i\left(s+\ln\frac{1-q_s}{1-q_o}\right) = J\ln\frac{q_o}{q_s} + \gamma_c(s),$$

the expression of $E_2(s, R)$ can be further simplified to

$$E_2(s, R) = \frac{1}{\tau C}\left\{\left(2s + \ln\frac{1-q_s}{1-q_o}\right)\gamma_c'(s) - 2\gamma_c(s) - J\ln\frac{q_o}{q_s}\right\} - \frac{R}{C}. \tag{120}$$

Substituting (118) in (119) and expressing $\gamma_i(t)$ in terms of $\gamma_c(s)$, we obtain the following inequality

$$R \leq \frac{1}{\tau}\left\{\left(s + \ln\frac{1-q_s}{1-q_o}\right)\gamma_c'(s) - \gamma_c(s) - J\ln\frac{q_o}{q_s}\right\}. \tag{121}$$

To determine the value of s that minimizes the right-hand side of (120), let us note that

$$\frac{\partial E_2}{\partial s} = \left(2s + \ln\frac{1-q_s}{1-q_o}\right)\gamma_c''(s)$$

and $\gamma_c''(s) \geq 0$. That is, the function $E_2(s, R)$ decreases with s for $s \leq \frac{1}{2}\ln\frac{1-q_o}{1-q_s}$, but increases with s for $\frac{1}{2}\ln\frac{1-q_o}{1-q_s} \leq s \leq 0$. Let us also note that the value of the right-hand side of (121), when s is equal to $\frac{1}{2}\ln\frac{1-q_o}{1-q_s}$, is equal to

$$\frac{1}{2\tau}\frac{J\sqrt{(1-q_o)(1-q_s)}}{1-\sqrt{(1-q_o)(1-q_s)}}\ln\frac{1-q_s}{1-q_o} - \frac{J}{\tau}\ln\frac{q_s}{1-\sqrt{(1-q_s)(1-q_o)}} - \frac{J}{\tau}\ln\frac{q_o}{q_s}.$$

But this quantity is equal to R_c as defined in Eq. 105. Therefore, it follows that for R within the range $0 \leq R \leq R_c$, the function $E_2(s, R)$ is minimized by letting

$$s = \frac{1}{2}\ln\frac{1-q_o}{1-q_s}.$$

Then

$$E_u(R) = \frac{2J}{\tau C}\left\{\ln\left[1-\sqrt{(1-q_s)(1-q_o)}\right] - \ln\sqrt{q_s q_o}\right\} - \frac{R}{C}.$$

Note that $E_u(R)$ is identically equal to $E_L(R)$ as given by Eq. 109, when the former is expressed in terms of $\langle n\rangle$ and s. For $R_c \leq R$, we note that the right-hand side of (121) is an increasing function of s. At $s = 0$, it becomes

$$\frac{1}{\tau}\left\{\frac{J(1-q_s)}{1-(1-q_s)}\ln\frac{1-q_s}{1-q_o} - J\ln\frac{q_o}{q_s}\right\},$$

which is equal to the quantity C_g defined in Eq. 106. Therefore, for $C_g \geq R \geq R_c$, the exponent is also given by the parametric equations (112).

To summarize, we have found the reliability function E(R) of the system given by

(i) For $0 \leqslant R \leqslant R_c$ (Eq. 109):

$$E(R) = \frac{2J}{\tau C}\left\{\ln\left(1 - \sqrt{\frac{\langle n\rangle}{1+\langle n\rangle}\,\frac{\langle n\rangle+s}{1+\langle n\rangle+s}}\right) + \ln\sqrt{(1+\langle n\rangle)(1+\langle n\rangle+s)}\right\} - \frac{R}{C}.$$

(ii) For $R_c \leqslant R \leqslant C_g$, the reliability function E(R) satisfies the parametric equations (Eqs. 112a and 112b):

$$\frac{R}{C} = \frac{J}{C\tau}\left\{\left[\mathscr{s} + \ln\left(\frac{\langle n\rangle+1}{\langle n\rangle}\,\frac{\langle n\rangle+s}{1+\langle n\rangle+s}\right)\right]\frac{(\langle n\rangle+s)\,e^{\mathscr{s}}}{1+(\langle n\rangle+s)(1-e^{\mathscr{s}})} - \ln\frac{\dfrac{1+\langle n\rangle+s}{1+\langle n\rangle}}{1+(\langle n\rangle+s)(1-e^{\mathscr{s}})}\right\}$$

$$E(R) = \frac{J}{C\tau}\left\{\frac{\mathscr{s}(\langle n\rangle+s)\,e^{\mathscr{s}}}{1+(\langle n\rangle+s)(1-e^{\mathscr{s}})} + \ln\left[1+(\langle n\rangle+s)(1-e^{\mathscr{s}})\right]\right\},$$

where R_c and C_g are given by (105) and (106), respectively. Alternatively, the reliability function is also the solution to the following maximization problem

$$E(R) = \max_{0\leqslant\delta\leqslant 1}\left\{E_1(\delta) - \delta\frac{R}{C}\right\} \tag{122a}$$

$$E_1(\delta) = \frac{J(1+\delta)}{C\tau}\ln\left\{1+(\langle n\rangle+s)\left[1-\left(\frac{\langle n\rangle}{1+\langle n\rangle}\,\frac{1+\langle n\rangle+s}{\langle n\rangle+s}\right)^{\frac{\delta}{1+\delta}}\right]\right\} - \frac{\delta J}{C\tau}\ln\left(1+\frac{s}{1+\langle n\rangle}\right). \tag{122b}$$

5.3 DISCUSSION

It can be shown from Eq. 122 that the reliability function of the system, in which orthogonal signal fields in completely incoherent states are transmitted in the midst of an additive thermal-noise field, approaches the reliability function of a Rayleigh fading channel in the limit of large $\langle n\rangle$ (the classical limit). Since, for large $\langle n\rangle$,

$$1+\langle n\rangle \cong \langle n\rangle$$

$$\left(\frac{\langle n\rangle}{1+\langle n\rangle}\right)^{\frac{\delta}{1+\delta}} \approx 1 - \frac{\delta}{1+\delta}\,\frac{1}{\langle n\rangle}$$

$$\left(\frac{s+\langle n\rangle}{1+\langle n\rangle+s}\right)^{\frac{\delta}{1+\delta}} \approx 1 - \frac{\delta}{1+\delta}\,\frac{1}{s+\langle n\rangle},$$

the right-hand side of (122) becomes

$$E(R) \approx \max_{0\leq\delta\leq 1} \left\{\frac{J(\delta+1)}{\tau C_c} \ln\left(1+\frac{\delta}{1+\delta}\,\frac{s}{\langle n\rangle}\right) - \frac{J\delta}{\tau C_c} \ln\left(1+\frac{s}{\langle n\rangle}\right) - \delta\frac{R}{C}\right\}.$$

Substituting $s = \frac{S}{J}$ and $\tau C_c = \frac{S}{\langle n\rangle}$ in the equation above, we obtain

$$E(R) = \max_{0\leq\delta\leq 1} \left\{\frac{J(1+\delta)}{\frac{S}{\langle n\rangle}} \ln\left(1+\frac{\delta}{1+\delta}\,\frac{S}{\langle n\rangle J}\right) - \frac{J\delta}{\frac{S}{\langle n\rangle}} \ln\left(1+\frac{S}{J\langle n\rangle}\right) - \frac{\delta R}{\frac{S}{\langle n\rangle}}\right\}. \tag{123}$$

Equation 123 is just the reliability function for a classical fading channel when the number of equal strength diversity paths is J.[22]

As in the case of signals that are in coherent states as discussed in Section IV, the reliability function depends not only on the signal-to-noise ratio $\frac{S}{\langle n\rangle}$, but also on the noise level $\langle n\rangle$ and the number of diversity paths J. The optimum reliability function, $E^o(R)$, is obtained by maximizing the function E(R) in (122) with respect to J, or alternatively, with respect to s. That is,

$$E^o(R) = \max_{s\geq 0}\left[\max_{0\leq\delta\leq 1}\left\{E_1(\delta,s) - \delta\frac{R}{C}\right\}\right],$$

where

$$E_1(\delta,s) = \frac{1+\delta}{s\ln\frac{1+\langle n\rangle}{\langle n\rangle}} \ln\left\{1+(\langle n\rangle+s)\left[1-\left(\frac{\langle n\rangle}{1+\langle n\rangle}\;\frac{1+\langle n\rangle+s}{s+\langle n\rangle}\right)^{\frac{\delta}{1+\delta}}\right]\right\}$$

$$-\frac{\delta}{s\ln\frac{1+\langle n\rangle}{\langle n\rangle}}\ln\left[1+\frac{s}{1+\langle n\rangle}\right]. \tag{124}$$

Let s^o denote the value of s that maximizes the function $E_1(\delta,s)$. Then

$$E^o(R) = \max_{0\leq\delta\leq 1}\left[E_1(\delta,s^o) - \delta\frac{R}{C}\right] \tag{125a}$$

if the value of s^o does not exceed S. When s^o is larger than S, we have

$$E^o(R) = \max_{0\leq\delta\leq 1}\left[E_1(\delta,S) - \delta\frac{R}{C}\right]. \tag{125b}$$

Our task is to determine the values of E^o and s^o as functions of R/C and $\langle n\rangle$.

For a fixed value of δ, $E_1(\delta, s)$ is maximized with respect to s at $s = s^o$, where s^o is given by

$$\left.\frac{\partial E_1(\delta, s)}{\partial s}\right|_{s=s^o} = 0. \tag{126}$$

Since

$$\frac{\partial E_1(\delta, s)}{\partial s} = -\frac{1}{s} E_1(\delta, s) - \frac{\delta}{s \ln \frac{1+\langle n\rangle}{\langle n\rangle}} \frac{1}{1+\langle n\rangle + s}$$

$$+ \frac{1+\delta}{s \ln \frac{1+\langle n\rangle}{\langle n\rangle}} \frac{1 - \left(\frac{\langle n\rangle}{1+\langle n\rangle} \frac{1+\langle n\rangle+s}{s+\langle n\rangle}\right)^{\frac{\delta}{1+\delta}} \left(1 - \frac{\delta}{1+\delta} \frac{1}{1+\langle n\rangle+s}\right)}{1 + (\langle n\rangle + s)\left[1 - \left(\frac{\langle n\rangle}{1+\langle n\rangle} \frac{1+\langle n\rangle+s}{s+\langle n\rangle}\right)^{\frac{\delta}{1+\delta}}\right]},$$

Eq. 126 becomes

$$E_1(\delta, s^o) = \frac{1 - \left(\frac{\langle n\rangle}{1+\langle n\rangle} \frac{1+\langle n\rangle+s^o}{s^o+\langle n\rangle}\right)^{\frac{\delta}{1+\delta}}}{\left\{1 + (\langle n\rangle + s^o)\left[1 - \left(\frac{\langle n\rangle}{1+\langle n\rangle} \frac{1+\langle n\rangle+s^o}{s^o+\langle n\rangle}\right)^{\frac{\delta}{1+\delta}}\right]\right\} \ln \frac{1+\langle n\rangle}{\langle n\rangle}}. \tag{127}$$

Equation 127 gives the value of s^o as a function of δ. To find $E^o(R)$, we compute the partial derivative of the quantity $E_1(\delta, s^o) - \delta \frac{R}{C}$ with respect to δ.

$$\frac{\partial}{\partial \delta}\left[E_1(\delta, s^o) - \delta \frac{R}{C}\right] = \frac{\partial E_1(\delta, s^o)}{\partial \delta} - \frac{R}{C}.$$

Substituting (124) in the preceding expression, we have

$$\frac{\partial E_1}{\partial \delta} - \frac{R}{C} = \frac{1}{s^o \ln \frac{1+\langle n\rangle}{\langle n\rangle}} \left\{ \ln \left[(1+\langle n\rangle)\left[1 - \left(\frac{\langle n\rangle}{1+\langle n\rangle}\right)^{\frac{\delta}{1+\delta}} \left(\frac{\langle n\rangle + s^o}{1+\langle n\rangle+s^o}\right)^{\frac{1}{1+\delta}}\right]\right] \right.$$

$$\left. + \frac{\left(\frac{\langle n\rangle}{1+\langle n\rangle}\right)^{\frac{\delta}{1+\delta}} \left(\frac{\langle n\rangle+s^o}{1+\langle n\rangle+s^o}\right)^{\frac{1}{1+\delta}} \ln \frac{1+\langle n\rangle}{\langle n\rangle} \frac{s^o+\langle n\rangle}{1+s^o+\langle n\rangle}}{(1+\delta)\left[1 - \left(\frac{\langle n\rangle}{1+\langle n\rangle}\right)^{\frac{\delta}{1+\delta}} \left(\frac{\langle n\rangle+s^o}{1+\langle n\rangle+s^o}\right)^{\frac{1}{1+\delta}}\right]} \right\} - \frac{R}{C}. \tag{128}$$

Therefore, the optimum reliability function $E^o(R)$ is given by the following equation: For $R \leq R_c^o$, where

$$\frac{R_c^o}{C} = \frac{1}{s^o \ln \frac{1+\langle n\rangle}{\langle n\rangle}} \left\{ \ln \left[(1+\langle n\rangle)\left(1 - \sqrt{\frac{\langle n\rangle}{1+\langle n\rangle}\,\frac{\langle n\rangle + s^o}{1+\langle n\rangle+s^o}}\right)\right] \right.$$

$$\left. + \frac{\sqrt{\frac{\langle n\rangle}{1+\langle n\rangle}\,\frac{\langle n\rangle+s^o}{1+\langle n\rangle+s^o}} \ln \sqrt{\frac{1+\langle n\rangle}{\langle n\rangle}\,\frac{s^o+\langle n\rangle}{1+s^o+\langle n\rangle}}}{1 - \sqrt{\frac{\langle n\rangle}{1+\langle n\rangle}\,\frac{\langle n\rangle+s^o}{1+\langle n\rangle+s^o}}} \right\}, \tag{129}$$

$$E^o(R) = \frac{1 - \sqrt{\frac{\langle n\rangle}{1+\langle n\rangle}\,\frac{1+s^o+\langle n\rangle}{s^o+\langle n\rangle}}}{\left[1+(\langle n\rangle+s^o)\left(1 - \sqrt{\frac{\langle n\rangle}{1+\langle n\rangle}\,\frac{1+\langle n\rangle+s^o}{s^o+\langle n\rangle}}\right)\right] \ln \frac{1+\langle n\rangle}{\langle n\rangle}} - \frac{R}{C}. \tag{130}$$

The value of s^o in Eqs. 129 and 130 is obtained by setting $\delta = 1$ in (127). For the range R that is such that $R_c^o \leq R \leq C_g^o$, where

$$\frac{C_g^o}{C} = \frac{1}{s^o \ln \frac{1+\langle n\rangle}{\langle n\rangle}} \left\{ (\langle n\rangle + s^o) \ln \left(1 + \frac{s^o}{\langle n\rangle}\right) - (1+\langle n\rangle+s^o) \ln \left(1+\frac{s^o}{1+\langle n\rangle}\right)\right\}, \tag{131}$$

$$E^o(R) = E_1(\delta^o, s^o) - \delta^o \frac{R}{C}. \tag{132a}$$

The values of δ^o and s^o are given by

$$\frac{R}{C} = \frac{1}{s^o \ln \frac{1+\langle n\rangle}{\langle n\rangle}} \left\{ \ln \left[(1+\langle n\rangle) \left[1 - \left(\frac{\langle n\rangle}{1+\langle n\rangle} \right)^{\frac{\delta^o}{1+\delta^o}} \left(\frac{\langle n\rangle + s^o}{1+\langle n\rangle + s^o} \right)^{\frac{1}{1+\delta^o}} \right] \right] \right.$$

$$\left. + \frac{1}{1+\delta} \frac{\left(\frac{\langle n\rangle}{1+\langle n\rangle} \right)^{\frac{\delta^o}{1+\delta^o}} \left(\frac{\langle n\rangle + s^o}{1+\langle n\rangle + s^o} \right)^{\frac{1}{1+\delta^o}} \ln \frac{1+\langle n\rangle}{\langle n\rangle} \frac{s^o + \langle n\rangle}{1 + s^o + \langle n\rangle}}{1 - \left(\frac{\langle n\rangle}{1+\langle n\rangle} \right)^{\frac{\delta^o}{1+\delta^o}} \left(\frac{\langle n\rangle + s^o}{1+\langle n\rangle + s^o} \right)^{\frac{1}{1+\delta^o}}} \right\} \tag{132b}$$

and

$$E_1(\delta^o, s^o) = \frac{1 - \left(\frac{\langle n\rangle}{1+\langle n\rangle} \frac{1+\langle n\rangle + s^o}{\langle n\rangle + s^o} \right)^{\frac{\delta^o}{1+\delta^o}}}{\left\{ 1 + (\langle n\rangle + s^o) \left[1 - \left(\frac{\langle n\rangle}{1+\langle n\rangle} \frac{1+\langle n\rangle + s^o}{\langle n\rangle + s^o} \right)^{\frac{\delta^o}{1+\delta^o}} \right] \right\} \ln \frac{1+\langle n\rangle}{\langle n\rangle}}, \tag{132c}$$

with $0 \leq \delta^o \leq 1$ and $s^o \geq 1$.

Equations 129-132 have been solved numerically. The results are shown in Fig. 18, where the optimum average number of signal photons per diversity path, s^o, is plotted as a function of R/C for $\langle n\rangle = 0.1$, $\langle n\rangle = 1.0$, and $\langle n\rangle = 10$. Also shown in Fig. 18 is the value of s^o in the classical limit (13). The values of $\frac{s^o}{\langle n\rangle}$ for rates R less than R_c^o are independent of R/C, but they are functions of the noise level $\langle n\rangle$. It is interesting to note, however, that if the effective noise in the system is taken to be $\left(\langle n\rangle + \frac{1}{2}\right)$, the optimum ratio, $\frac{s^o}{\langle n\rangle + \frac{1}{2}}$, is roughly 3 for $R \leq R_c^o$ independent of the value of $\langle n\rangle$.

For rates greater than R_c^o, the value of s^o increases rapidly with R/C. That is, for a fixed value of S, the optimum number of equal-strength diversity paths decreases at higher information rate. From Fig. 18, it is clear that, for increasing R/C, s^o increases without bound. Hence, when the average number of transmitted photons is fixed at S, a point where the value of s^o is equal to S will eventually be reached. That is, the optimum value of J is equal to one. The rate at which $S = s^o$ is called

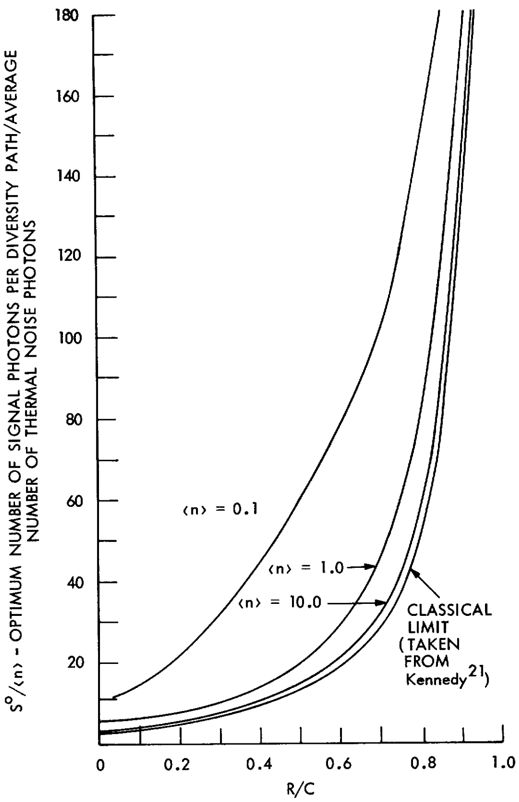

Fig. 18. Optimum number of signal photons per diversity path vs R/C.

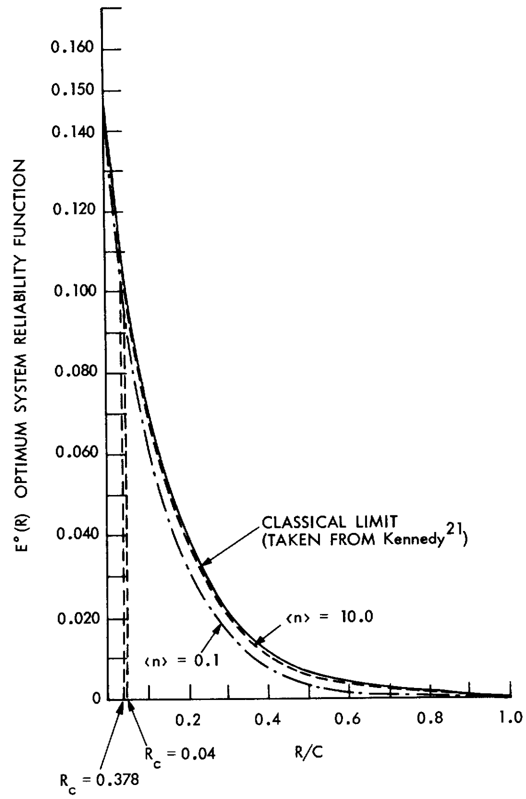

Fig. 19. Optimum system reliability function.

the threshold rate for the given value of S.

Let us assume, for the present, that, for any given value of R/C, S is extremely large so that S/s^o is larger than 1. In this limiting case, the optimum value of $E^o(R)$ as a function of R/C is given by Eqs. 130 and 132. The general behavior of $E^o(R)$ at rates above the threshold is given by Fig. 19 for different thermal-noise levels. The reliability function for the optimum classical fading channel is also shown for comparison.

When the value of S is finite and fixed, the threshold effects should be considered. At rates such that $S \geq s^o$, the optimum value of J is equal to S/s^o. At rates such that $s^o \geq S$, the optimum value of J is equal to 1. Hence, for rates less than threshold rate the optimum reliability function $E^o(R)$ is identical to that derived under the supposition that S is extremely large. For rates greater than the threshold, the optimum value of E as a function of s is given by Eq. 125b.

VI. CONCLUSION

6.1 SUMMARY OF RESEARCH

The problem of communicating a set of M input symbols through a channel disturbed by additive thermal noise has been studied. The quantum-mechanical model for such communication systems has been described, and the characteristics of the thermal-noise field have also been discussed. Our attention was restricted to the special case in which the density operators specifying the states of the electromagnetic field at the receiver are commutative.

A quantum-mechanical receiving system was modeled as a system that measures a set of dynamical variables. It was found that when the density operators are commutative, the optimum receiver will measure those dynamical variables that are represented by Hermitian operators whose eigenstates are the simultaneous eigenstates of the density operators. On the basis of the outcome of the measurement, the estimation of the transmitted input symbol is made with the aid of the maximum-likelihood decision rule, as for the classical optimum receiver. The optimum performance of the system was derived and was expressed in terms of the eigenvalues of the density operators. The results were used to study two specific communication systems.

The signal field representing a given input symbol is in a coherent state and has an absolute phase unknown to the receiver. In this case, the classical waveform of the electric field has a known amplitude but a random phase. It was found that when the signal fields representing different input symbols are orthogonal (the corresponding classical electric fields have orthogonal waveforms), the density operators specifying the states of the received field for all input symbols are commutative. The optimum receiver for this set of signals transmitted in the midst of additive thermal noise measures simultaneously the number of photons in each of the relevant normal modes of the received field. (The normal-mode functions are chosen to be the classical waveforms representing the input symbols.) When the average rates of photons in the signal fields representing all input symbols are equal (p photons/sec) and the average number of thermal-noise photons in each mode is $\langle n\rangle$, the channel capacity was found to be $p \ln \frac{1+\langle n\rangle}{\langle n\rangle}$. Furthermore, the system reliability function was found. Its general behavior was shown in Fig. 9.

The structure of the optimum receiver for a signal field that is also in completely incoherent states and its performance have been studied. In the classical limit, the electric field waveforms are sample functions of Gaussian random processes. Hence, this system models quantum mechanically a Rayleigh fading channel. For equal-strength diversity systems, the system reliability function and the channel capacity were found in terms of the number of diversity paths J, the average number of signal photons per diversity path and the average number of thermal-noise photons. Moreover, the optimum signal strength per diversity path, expressed in terms of the average number of photons in the signal field transmitted through each diversity path, was found numerically as a function of $\frac{R}{C}$, and was shown in Fig. 18.

6.2 FUTURE RESEARCH PROBLEMS

It appears that the area of quantum-mechanical communication theory is wide open. The problems studied in this research can be generalized in several ways. For example, instead of completely incoherent or completely coherent signal fields, one can also consider the case in which signals are partially coherent. As another example, one can study the effect of utilizing coding for the type of signals studied in Section V.

Moreover, there are two major areas that warrant further research effort. One is to study communication systems utilizing a broader class of signals. In particular, it would be most interesting to investigate the structure of the optimum receiver when the density operators of the received fields for different input symbols are noncommutative. In many communication systems of practical interest, the states of the received fields are specified by noncommutative density operators, for example, when signal fields are in known coherent states (corresponding to the case of known signals in the classical limit), or when signal fields are in coherent states with random phase but are not orthogonal, or when the average thermal photons in each mode of the received field are not equal, and so forth. Unfortunately, the problem of specifying the structure of the optimum receiver when the density operators of the received field are noncommutative appears to be extremely difficult. (For binary signals, Helstrom[11, 12] has studied the structure of the optimum receiver; however, the generalization of his results to M-ary signals is by no means straightforward.)

The other area of great interest is the problem of modeling quantum-mechanical receivers. In the present work, we assumed that a receiver estimates the transmitted input symbols on the basis of the outcomes of measurements of some dynamical variables of the received field. It is not clear that all receivers can be so modeled mathematically. There are various ways in which one can model quantum-mechanical communication systems. For example, one can describe a receiver by its interacting Hamiltonian with the received field. Such modeling problems are indeed important and challenging.

APPENDIX A

Expansions of the Electric Field Operator

We shall show that the expansion of the operator $E^{(+)}(\underline{r}, t)$ in Eq. 6 is not unique. To show this, we introduce unitary matrix V with elements V_{jk}. Since $V^{+}V = VV^{+} = 1$,

$$\sum_{j=0}^{\infty} V^{+}_{mj} V_{jn} = \sum_{j=0}^{\infty} V^{*}_{jm} V_{jn} = \delta_{mn}. \tag{A. 1}$$

The right-hand side of Eq. 6 can be rewritten

$$E^{(+)}(\underline{r}, t) = i \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} \sum_{j=0}^{\infty} V^{*}_{jm} V_{jk} a_k \sqrt{\frac{\hbar\omega_m}{2L}} \exp\left[i\omega_m \left(\frac{z}{c} - t\right)\right].$$

Let

$$V_j(\underline{r}, t) = \sum_{k=0}^{\infty} \sqrt{\frac{\omega_k}{\omega'_j}} V^{*}_{jk} \exp\left[-i(\omega'_j - \omega_k)\left(\frac{z}{c} - t\right)\right] \tag{A. 2}$$

$$b_j = \sum_{k=0}^{\infty} V_{jk} a_k \tag{A. 3}$$

$$b^{+}_j = \sum_{k=0}^{\infty} V^{+}_{kj} a^{+}_k. \tag{A. 4}$$

The expansion of $E^{(+)}(\underline{r}, t)$ simplifies to

$$E^{(+)}(\underline{r}, t) = i \sum_{j=0}^{\infty} \sqrt{\frac{\hbar\omega'_j}{2L}} b_j V_j(\underline{r}, t) \exp\left[i\omega'_j \left(\frac{z}{c} - t\right)\right]. \tag{A. 5}$$

The operators b_j satisfy all commutative relations satisfied by the operators a_k, since

$$\begin{aligned} b_k b^{+}_j - b^{+}_j b_k &= \left(\sum_m V_{km} a_m\right)\left(\sum_n a^{+}_n V^{+}_{nj}\right) - \left(\sum_n a^{+}_n V^{+}_{nj}\right)\left(\sum_m V_{km} a_m\right) \\ &= \sum_m \sum_n V_{km} V^{+}_{nj} \left(a_m a^{+}_n - a^{+}_n a_m\right) \\ &= \delta_{kj} \qquad \text{for all } k \text{ and } j. \end{aligned}$$

Similarly,

$$b_k b_j - b_j b_k = b_j^+ b_k^+ - b_k^+ b_j^+ = 0 \qquad \text{for all k and j.}$$

Hence, the operators b_k can be considered as the annihilation operators of a set of new normal modes in terms of which $E^{(+)}(\underline{r}, t)$ is expanded.

Let $\{|\beta_k\rangle\}$ denote the set of right eigenstates of the operators in the set $\{b_k\}$.

$$b_k |\{\beta_k\}\rangle = \beta_k |\{\beta_k\}\rangle.$$

These states are also coherent states. Therefore, the density operators may be expanded in terms of coherent states $|\beta_k\rangle$, instead of the set $\{|\alpha_k\rangle\}$. Since

$$\prod_k |\alpha_k\rangle = \exp\left[\sum_k \left(a_k^+\alpha_k - \alpha_k^* a_k\right)\right] |0\rangle$$

$$= \exp\left[\sum_k \left(b_k^+\beta_k - \beta_k^* b_k\right)\right] |0\rangle$$

$$= \prod_k |\beta_k\rangle$$

and

$$\prod_k d^2\alpha_k = \prod_k d^2\beta_k,$$

the density operator in Eq. 11 can be rewritten

$$\rho = \int p'(\{\beta_k\}) |\{\beta_k\}\rangle\langle\{\beta_k\}| \prod_k d^2\beta_k. \tag{A. 6a}$$

The new weight function $p'(\{\beta_k\})$ is obtained by substituting the relation

$$\alpha_k = \sum_j \beta_j V_{kj}^+ \tag{A. 6b}$$

in the weight function $p(\{\alpha_k\})$.

A set of mode functions $\{V_j(\underline{r}, t)\}$, which will be especially useful, is the one for which the $V_j(\underline{r}, t)$ are narrow-band functions of time. That is, the matrix elements V_{jk} are so chosen that for any given value of j, both V_{jk} and $V_{jk'}$ are nonzero only if ω_k is approximately equal to $\omega_{k'}$. Moreover, to expand the relevant component of the received electric field in frequency position modulation systems in Sections IV and V, the functions $V_j(\underline{r}, t)$ are chosen as follows.

We denote by $V_{jk}(\underline{r}, t)$ and b_{jk} the function $V_{(j-1)\Delta+k+k_o}(\underline{r}, t)$ and the annihilation operator $b_{(j-1)\Delta+k+k_o}$ for $k = 1, 2, \ldots \Delta$, where Δ and k_o are positive integers. The orthonormality condition becomes

$$\int_{L^3} \int V_{jn}^*(\underline{r}, t) \cdot V_{km}(\underline{r}, t)\, dt d\underline{r} = \delta_{jk}\delta_{nm}.$$

In terms of this set of normal mode functions, the operator $E^{(+)}(\underline{r}, t)$ can be written

$$E^{(+)}(\underline{r}, t) = i \sum_{j=1}^{M} \sqrt{\frac{\hbar\omega_j}{2L}} \sum_{k=0}^{\Delta-1} V_{jk}(\underline{r}, t) \exp\left[i\omega_j\left(\frac{z}{c} - t\right)\right] b_{jk} + \text{other terms}, \tag{A. 7}$$

where

$$\omega_j = \frac{2\pi}{\tau}(k_o + (j-1)\Delta).$$

The mode functions $V_{jk}(\underline{r}, t)$ are approximately bandlimited functions. The functions $V_{jk}(\underline{r}, t)$ are approximately bandlimited lowpass signals, in the sense that their energy outside of the frequency bandwidth $\frac{\Delta}{\tau}$ (Hz) is essentially zero.[16] The quantities Δ and k_o are chosen so that

$$k_o + \Delta(j-1) \approx k_o + j\Delta.$$

If the Fourier expansion of the function $V_{jk}(\underline{r}, t)$ is

$$V_{jk}(\underline{r}, t) = \sum_{n=0}^{\Delta-1} U_{kn}^{(j)*} \exp\left[-i\,\frac{2\pi n}{\tau}\left(\frac{z}{c} - t\right)\right] \qquad \begin{array}{l} j = 1, 2, \ldots M \\ k = 1, 2, \ldots \Delta \end{array} \tag{A. 8a}$$

it is clear that we must have

$$\sum_{k=0}^{\Delta-1} U_{kn}^{(j)*} U_{km}^{(j)} = \delta_{nm} \qquad \begin{array}{l} \text{for all } m, n = 1, \ldots \Delta \\ j = 1, 2, \ldots M \end{array}$$

Let U_j denote the matrix whose elements are $U_{kn}^{(j)}$. The unitary matrix V is given by

$$V = \left|\begin{array}{c|c|c} A & 0 & 0 \\ \hline 0 & \begin{array}{c|c|c|c} U_1 & 0 & \cdots & 0 \\ \hline 0 & U_2 & \cdots & 0 \\ \hline \vdots & \vdots & \cdots & \vdots \\ \hline 0 & 0 & \cdots & U_M \end{array} & 0 \\ \hline 0 & 0 & B \end{array}\right| \tag{A. 8b}$$

The general form of the frequency spectrum of the function $V_{jk}(\underline{r}, t) \exp[-i\omega_j t]$ is shown in Fig. A-1.

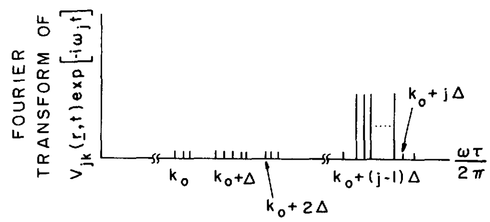

Fig. A-1. Frequency spectrum of the function $V_{jk}(\underline{r}, t) \exp[-i\omega_j t]$.

APPENDIX B

Statistical Independence of Observed Values of Dynamical Variables Associated with Different Modes

We shall prove that when dynamical variables associated with different modes of the field are measured simultaneously, the outcomes are statistically independent random variables if the field is in a state specified by a density operator ρ of the form

$$\rho = \prod_k \rho_k. \tag{B.1}$$

In this expression, ρ_k is the density operator describing the state of the k^{th} mode of the field alone. Let the weight function in the P-representation of ρ_k be denoted $p_k(\alpha_k)$. The density operator ρ in P-representation is just

$$\rho = \int \prod_k p_k(\alpha_k) \, |\alpha_k\rangle \langle \alpha_k| \, d^2\alpha_k. \tag{B.2}$$

Examples of density operators satisfying Eq. B.1 are given by Eqs. 38 and 85.

Let X_k denote a dynamical variable of the k^{th} mode of the field. (For example, X_k is the amplitude of the k^{th} mode or any other function of the annihilation and creation operators a_k and a_k^+ of the k^{th} mode.) Clearly,

$$X_k X_{k'} = X_{k'} X_k \qquad \text{for all } k \text{ and } k'. \tag{B.3}$$

When X_k is measured independently of other variables associated with other modes of the field, the possible outcome is a random variable denoted x_k. Let $M_{\underline{x}}(i\underline{v})$ denote the characteristic function associated with the joint probability distribution (or density function) of the random variables $x_1, x_2, \dots x_k, \dots$. From Section III, it is given by

$$\begin{aligned} M_{\underline{x}}(i\underline{v}) &= M_{x_1, x_2, \dots x_k, \dots}(iv_1, iv_2, \dots iv_k, \dots) \\ &= \mathrm{Tr}\left[\rho \exp\left[\sum_k iv_k X_k\right]\right]. \end{aligned}$$

When the density operator ρ can be written as a product, as in Eq. B.1, the characteristic function $M_{\underline{x}}(i\underline{v})$ can be expressed as

$$\begin{aligned} M_{\underline{x}}(i\underline{v}) &= \mathrm{Tr}\left[\prod_k \rho_k \exp(iv_k X_k)\right] \\ &= \prod_k \int p_k(\alpha_k) \langle \alpha_k| \exp(iv_k X_k) \, |\alpha_k\rangle \, d^2\alpha_k. \end{aligned}$$

Since the characteristic function of the variable x_k is

$$M_{x_k}(iv_k) = \int p_k(\alpha_k) \langle \alpha_k | \exp(iv_k X_k) | \alpha_k \rangle \, d^2\alpha_k,$$

the characteristic function $M_{\underline{x}}(i\underline{v})$ factors into a product

$$M_{\underline{x}}(i\underline{v}) = \prod_k M_{x_k}(iv_k).$$

That is, the random variables x_k are statistically independent.

APPENDIX C

Photon Counting Statistics

The optimum receivers found in both Sections IV and V measure the number of operators N_k of the received field simultaneously. We shall now show that the measurements of these dynamical variables can be achieved by using photon counters.

Let us consider the case in which the field is in a state specified by the density operator

$$\rho = \int p(\{\alpha_k\}) \prod_k |\alpha_k\rangle \langle\alpha_k| \, d^2\alpha_k. \tag{C. 1}$$

According to Section III, when the number operators $N_1, N_2 \ldots, N_k \ldots$ are measured, the probability that the outcomes are $n_1, n_2, \ldots n_k, \ldots$ is

$$\begin{aligned} P(n_1, n_2, \ldots n_k, \ldots) &= \int p(\{\alpha_k\}) \prod_k |\langle n_k|\alpha_k\rangle|^2 \, d^2\alpha_k \\ &= \int p(\{\alpha_k\}) \prod_k \frac{|\alpha_k|^{2n_k}}{n_k!} \exp\left(-|\alpha_k|^2\right) d^2\alpha_k. \end{aligned} \tag{C. 2}$$

The moment-generating function associated with this probability distribution is defined to be

$$Q_i(\underline{\lambda}) = \overline{\prod_k (1-\lambda_k)^{n_k}}.$$

From Eq. C. 2, we have

$$\begin{aligned} Q_i(\underline{\lambda}) &= \int p(\{\alpha_k\}) \prod_k \sum_{n_k=0}^{\infty} (1-\lambda_k)^{n_k} \frac{|\alpha_k|^{2n_k}}{n_k!} \exp\left(-|\alpha_k|^2\right) d^2\alpha_k \\ &= \int p(\{\alpha_k\}) \prod_k \exp\left(-\lambda_k|\alpha_k|^2\right) d^2\alpha_k. \end{aligned} \tag{C. 3}$$

If one measures the operator $\sum_k N_k$, the outcomes are integers $0, 1, 2, \ldots,$ with probability distribution

$$P(n) = \sum_{\{n_k\}/\sum_k n_k = n} \int p(\{\alpha_k\}) \prod_k |\langle n_k|\alpha_k\rangle|^2 \, d^2\alpha_k.$$

The corresponding moment-generating function $Q_i(\lambda)$ is given by

$$Q_i(\lambda) = \int p(\{a_k\}) \prod_k \exp\left(-\lambda |a_k|^2\right) d^2 a_k. \tag{C.4}$$

We shall show that the statistics of the total photon counts recorded in an interval of length t by a photomultiplier tube and counter combination shown in Fig. C-1 can be characterized by the same moment-generating functions. It has been shown that when the

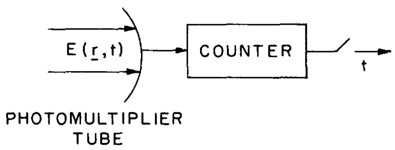

Fig. C-1. Photomultiplier tube and counter combination.

field at the photomultiplier tube is in the state ρ, the moment-generating function associated with the total number of photon counts recorded in any time interval of length τ is given by

$$Q_P(\lambda, t) = \int p(\{a_k\}) \left(\prod_k d^2 a_k\right) \exp\left[-\lambda s \int_{t_o}^{t_o+t} dt' \int_{\text{Vol of PMT}} \sigma(r)\, \epsilon^*(\underline{r}, t', \underline{a})\, \epsilon(\underline{r}, t', \underline{a})\, d^3 r\right]. \tag{C.5}$$

In this expression, $\epsilon(\underline{r}, t, \underline{a})$ is

$$\epsilon(\underline{r}, t, \underline{a}) = i \sum_k \sqrt{\frac{\hbar\omega_k}{2L}} \exp[-i\omega_k t]\, a_k U_k(\underline{r}), \tag{C.6}$$

where s is a constant characterizing the sensitivity of the photomultiplier tube which is assumed to be independent of the frequency over the bandwidth of the field present. The function $\sigma(\underline{r})$ specifies the number of photosensitive elements per unit volume of the photocathode.

Let us assume that the sensitive region of the photocathode is a very thin layer of elements lying in the plane perpendicular to the z axis. The function $\sigma(\underline{r})$, therefore, is approximately a delta function of the z coordinate. This assumption is often satisfied in practice. Since the electromagnetic field that is present consists of plane waves travelling in the positive z direction, the spatial integration in Eq. C.5 becomes trivial. Substituting Eq. C.6 in Eq. C.5, we obtain

$$Q_P(\lambda, t) = \int p(\{a_k\}) \left(\prod_k d^2 a_k\right) \exp\left[-\lambda\eta' \sum_k \sum_{k'} \sqrt{\frac{\hbar\omega_k\omega_{k'}}{4L^2}} \int_{t_o}^{t+t_o} a_k a_{k'}^*\, e^{i(\omega_k - \omega_{k'})t'}\, dt'\right]. \tag{C.7}$$

Without loss of generality, let us assume that the signal field arrives at the photon counter at $t = 0$. If the duration of the counting interval is chosen to be τ, the time integral in Eq. C.7 becomes

$$I = \int_0^\tau \exp[i(\omega_{k'} - \omega_k)t]\, dt \cong \delta_{kk'}$$

and

$$Q_P(\lambda, \tau) = \int p(\{a_k\}) \exp\left[-\lambda\eta|a_k|^2\right] \prod_k d^2 a_k. \tag{C.8}$$

It is clear that except for the parameter η, which can often be made approximately equal to one in practice, the generating function $Q_P(\lambda, \tau)$ is equal to $Q_i(\lambda)$ in Eq. C.4. That is, by choosing the duration of the counting interval to be τ, the measurement of the operator $\sum_k N_k$ can be accomplished by using a photomultiplier tube and counter combination.

When, instead of the operator $\sum_k N_k$, one measures $N_1, N_2, \ldots N_k, \ldots$ simultaneously by using the system shown in Fig. 9, only one single mode of the field reaches the photomultiplier tube. The moment-generating function associated with the photon count n_k is

$$Q_P(\lambda_k, \tau) = \int p_k(a_k)\, d^2 a_k \exp\left(-\lambda_k \eta |a_k|^2\right).$$

Since the n_k are statistically independent, and

$$p(\{a_k\}) = \prod_k p_k(a_k),$$

the moment-generating function associated with the joint probability distribution of $n_1, n_2, \ldots n_k, \ldots$ is

$$Q_P(\underline{\lambda}, \tau) = \int p(\{a_k\}) \prod_k d^2 a_k \exp\left(-\lambda_k \eta_k |a_k|^2\right)$$

which can be made equal to $Q_i(\underline{\lambda})$ in Eq. C.3 approximately.

Again, it should be evident that the results above remain valid independently of the choice of the normal mode functions used in the expansion of the function $\epsilon(\underline{r}, t, \underline{a})$ (see Section II). In general, one can write $\epsilon(\underline{r}, t, \underline{a})$ as

$$\epsilon(\underline{r}, t, \underline{a}) = i \sum_k \sqrt{\frac{\hbar\omega_k}{2L}}\, V_k(\underline{r}, t)\, \beta_k \exp(-i\omega_k t).$$

The counting interval has length τ, since the functions $V_k(\underline{r}, t)$ are such that

$$\int_0^\tau V_k(\underline{r}, t)\, V^*_{k'}(\underline{r}, t) \exp[-i(\omega_k - \omega_{k'})t]\, dt = \delta_{kk'}.$$

APPENDIX D

Evaluation of the Sum $\Sigma_{n_1} \ldots\ldots \Sigma_{n_M} P(n_1, n_2, \ldots n_M/m_j)$

We shall evaluate the sum

$$S_1 = \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{n_1-1} \cdots \sum_{n_M=0}^{n_1-1} P(n_1, n_2, \ldots n_M/m_1).$$

The probability $P(n_1, n_2, \ldots n_M/m_1)$ is given by Eq. 40:

$$S_1 = \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{n_1-1} \cdots \sum_{n_M=0}^{n_1-1} \left(\frac{1}{1+\langle n\rangle}\right)^M \left(\frac{\langle n\rangle}{1+\langle n\rangle}\right)^{n_1+n_2+\ldots n_M} \left\{\sum_{r=0}^{n_1} \binom{n_1}{r} \frac{1}{r!} \left[\frac{|\sigma|^2}{\langle n\rangle (1+\langle n\rangle)}\right]^r\right\} \exp\left(\frac{-|\sigma|^2}{1+\langle n\rangle}\right)$$

$$= \left\{\frac{1}{1+\langle n\rangle} \exp \frac{-|\sigma|^2}{1+\langle n\rangle}\right\} \sum_{n_1=0}^{\infty} \sum_{r=0}^{n_1} \binom{n_1}{r} \frac{1}{r!} \left(\frac{\langle n\rangle}{1+\langle n\rangle}\right)^{n_1} \left[1 - \left(\frac{\langle n\rangle}{1+\langle n\rangle}\right)^{n_1}\right]^{(M-1)} \cdot \left[\frac{|\sigma|^2}{\langle n\rangle (1+\langle n\rangle)}\right]^r$$

$$= \left\{\frac{1}{1+\langle n\rangle} \exp \frac{-|\sigma|^2}{1+\langle n\rangle}\right\} \left\{\sum_{i=0}^{M-1} \binom{M-1}{i} \sum_{n_1=0}^{\infty} \sum_{r=0}^{n_1} \frac{1}{r!} \left[\frac{|\sigma|^2}{\langle n\rangle (1+\langle n\rangle)}\right]^r \binom{n_1}{r} (-1)^i \cdot \left[\left(\frac{\langle n\rangle}{1+\langle n\rangle}\right)^{i+1}\right]^{n_1}\right\}$$

$$= \left\{\frac{1}{1+\langle n\rangle} \exp \frac{-|\sigma|^2}{1+\langle n\rangle}\right\} \left\{\sum_{i=0}^{M-1} \binom{M-1}{i} (-1)^i \sum_{r=0}^{\infty} \frac{1}{r!} \left[\frac{|\sigma|^2}{\langle n\rangle (1+\langle n\rangle)}\right]^r \sum_{n_1=r}^{\infty} \binom{n_1}{r} \cdot \left[\left(\frac{\langle n\rangle}{1+\langle n\rangle}\right)^{i+1}\right]^{n_1}\right\}.$$

The last sum on the right-hand side of this equation is of the form

$$\sum_{n_1=r}^{\infty} \binom{n_1}{r} q^{n_1} = \binom{r}{r} q^r + \binom{r+1}{r} q^{r+1} + \dots \binom{j}{r} q^j + \dots,$$

where $q < 1$. It is easy to see that this sum is the coefficient of the x^r term in the power series expansion of the function

$$[(1+x)q]^r + [(1+x)q]^{r+1} + \dots\dots = \frac{(1+x)^r q^r}{(1-q) - xq}, \tag{D.1}$$

where x is a formal variable of arbitrarily small value that is such that the series in the left-hand side of Eq. D. 1 converges. The power series expansion of the right-hand side of Eq. D. 1 is

$$q^r \left[\sum_{j=1}^{r} \binom{r}{j} x^j\right] \left\{\frac{1}{1-q}\left[1 + x\,\frac{q}{1-q} + x^2\,\frac{q^2}{1-q} + \dots\right]\right\}.$$

The coefficient of the x^r term in this expansion is equal to

$$\sum_{n=r}^{\infty} \binom{n}{r} q^r = \frac{q^r}{1-q}\left[1 + \binom{r}{1}\frac{q}{1-q} + \dots\dots \binom{r}{r}\left(\frac{q}{1-q}\right)^r\right]$$

$$= \frac{q^r}{(1-q)^{r+1}}. \tag{D.2}$$

Therefore, one obtains

$$S_1 = \left\{\frac{1}{1+\langle n\rangle} \exp\frac{-|\sigma|^2}{1+\langle n\rangle}\right\} \sum_{i=0}^{M-1} \binom{M-1}{i} (-1)^i \sum_{r=0}^{\infty} \frac{1}{r!}\left[\frac{|\sigma|^2}{\langle n\rangle(1+\langle n\rangle)}\right]^r \frac{\left[\left(\frac{\langle n\rangle}{1+\langle n\rangle}\right)^{i+1}\right]^r}{\left[1 - \left(\frac{\langle n\rangle}{1+\langle n\rangle}\right)^{i+1}\right]^{r+1}}$$

$$= \sum_{i=0}^{M-1} \binom{M-1}{i} (-1)^i \frac{(1+\langle n\rangle)^i}{(1+\langle n\rangle)^{i+1} - \langle n\rangle^{i+1}} \exp\left\{\frac{-|\sigma|^2[(\langle n\rangle+1)^i - \langle n\rangle^i]}{(1+\langle n\rangle)^{i+1} - \langle n\rangle^{i+1}}\right\}. \tag{D.3}$$

Similarly,

$$S_2 = \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{n_1} \cdots \sum_{n_M=0}^{n_1} \left(\frac{1}{1+\langle n\rangle}\right)^M \left(\frac{\langle n\rangle}{1+\langle n\rangle}\right)^{n_1+n_2+\dots n_M} \left\{\sum_{r=0}^{n_1} \binom{n_1}{r} \frac{1}{r!} \left[\frac{|\sigma|^2}{\langle n\rangle(1+\langle n\rangle)}\right]^r\right\} \exp\left(\frac{-|\sigma|^2}{1+\langle n\rangle}\right)$$

$$= \sum_{i=0}^{M-1} \binom{M-1}{i} (-1)^i \frac{\langle n\rangle^i}{(1+\langle n\rangle)^{i+1} - \langle n\rangle^{i+1}} \exp\left\{\frac{-|\sigma|^2[(1+\langle n\rangle)^i - \langle n\rangle^i]}{(1+\langle n\rangle)^{i+1} - \langle n\rangle^{i+1}}\right\}. \tag{D. 4}$$

It follows that one can evaluate the sum

$$S_3 = \sum_{n_1=0}^{\infty} p_1(n_1) \left[\sum_{n_2=n_1}^{\infty} p_1(n_2)\right]^{\delta},$$

where $p_1(n_1)$ and $p_1(n_2)$ are given by Eqs. 55a and 55b, respectively.

$$S_3 = \sum_{n_1=0}^{\infty} \left(\frac{1}{1+\langle n\rangle}\right)\left(\frac{\langle n\rangle}{1+\langle n\rangle}\right)^{n_1(1+\delta)} \sum_{r=0}^{n_1} \binom{n_1}{r} \frac{1}{r!} \left[\frac{|\sigma|^2}{\langle n\rangle(1+\langle n\rangle)}\right]^r \exp\left(-\frac{|\sigma|^2}{1+\langle n\rangle}\right).$$

Substituting Eq. D. 2 in this equation, one obtains

$$S_3 = \frac{(1+\langle n\rangle)^{\delta}}{(1+\langle n\rangle)^{1+\delta} - \langle n\rangle^{\delta+1}} \exp\left\{-\frac{|\sigma|^2[(1+\langle n\rangle)^{\delta} - \langle n\rangle^{\delta}]}{(1+\langle n\rangle)^{1+\delta} - \langle n\rangle^{1+\delta}}\right\}. \tag{D. 5}$$

APPENDIX E

Lower Bound of the Error Probability for Orthogonal Signals in Coherent States

We present derivations of an asymptotic lower bound to $P(\epsilon)$ given by Eq. 62. Let us rewrite Eq. 62 as follows:

$$P(\epsilon) \geq \sum_{n_1=0}^{\infty} p_1(n_1)\left\{1-\exp[(M-1)\ln(1-\Pr(n_1<n_2/m_1))]\right\}. \tag{E. 1}$$

Since a logarithmic function satisfies the inequality

$$\ln x \leq x - 1,$$

we can further lower-bound the right-hand side of Eq. E. 1:

$$\begin{aligned} P(\epsilon) &\geq \sum_{n_1=0}^{\infty} p_1(n_1)\left\{1-\exp[-(M-1)\Pr(n_1<n_2/m_1)]\right\} \\ &\geq \sum_{n_1=0}^{[d]} p_1(n_1)\left\{1-\exp[-(M-1)\Pr(n_1<n_2/m_1)]\right\}. \end{aligned} \tag{E. 2}$$

The second inequality is justified because the expression in the braces $\{\ \}$ is always positive. The parameter d is an arbitrary positive number, and $[x]$ denotes the integer part of the quantity x. Since for all $n_1 \leq d$

$$\exp[-(M-1)\Pr(n_1<n_2/m_1)] \leq \exp[-(M-1)\Pr(d<n_2/m_1)],$$

it follows from Eq. E. 2 that

$$P(\epsilon) \geq \Pr(n_1 \leq d/m_1)\left\{1-\exp[-(M-1)\Pr(d<n_2/m_1)]\right\}.$$

Replacing the exponential function e^{-x} by its upper bound $\left(1 - x + \frac{x^2}{2}\right)$, and noting that $\frac{1}{2} \leq 1 - \frac{1}{M} \leq 1$, we obtain

$$P(\epsilon) \geq \frac{M}{4}\Pr(n_1 \leq d/m_1)\Pr(n_2 > d/m_1) \tag{E. 3}$$

if d is chosen so that

$$M\Pr(n_2 > d/m_1) \leq 1. \tag{E. 4}$$

From Eqs. 55a and 55b, we obtain

$$\Pr(n_2 > d/m_1) = \sum_{n_2=[d]+1}^{\infty} p_1(n_2)$$

$$= \left(\frac{\langle n\rangle}{1+\langle n\rangle}\right)^{[d]+1} \tag{E.5}$$

and

$$\Pr(n_1 \leqslant d/m_1) = \sum_{n_1=0}^{[d]} \sum_{r=0}^{n_1} \binom{n_1}{r} \frac{1}{r!} \frac{1}{1+\langle n\rangle} \left(\frac{\langle n\rangle}{1+\langle n\rangle}\right)^{n_1} \left[\frac{|\sigma|^2}{\langle n\rangle(1+\langle n\rangle)}\right]^r \exp\left(\frac{-|\sigma|^2}{1+\langle n\rangle}\right). \tag{E.6}$$

The condition in Eq. E.4 is simply

$$[d] + 1 \geqslant \frac{R\tau}{\ln \dfrac{1+\langle n\rangle}{\langle n\rangle}}.$$

Therefore, one can choose the parameter d to be

$$d = \frac{R\tau}{\ln \dfrac{1+\langle n\rangle}{\langle n\rangle}}.$$

For this choice of the value d, it is evident that the lower bound of $P(\epsilon)$ given by Eq. E.3 becomes

$$P(\epsilon) \geqslant K \sum_{n_1=0}^{[d]} \sum_{r=0}^{n_1} \binom{n_1}{r} \frac{1}{r!} \left(\frac{\langle n\rangle}{1+\langle n\rangle}\right)^{n_1} \left[\frac{|\sigma|^2}{\langle n\rangle(1+\langle n\rangle)}\right]^r \exp\left(\frac{-|\sigma|^2}{1+\langle n\rangle}\right)$$

$$= K \sum_{r=0}^{[d]} \frac{1}{r!} \left[\frac{|\sigma|^2}{\langle n\rangle(1+\langle n\rangle)}\right]^r \sum_{n_1=r}^{[d]} \binom{n_1}{r} \left(\frac{\langle n\rangle}{1+\langle n\rangle}\right)^{n_1} \exp\left(\frac{-|\sigma|^2}{1+\langle n\rangle}\right), \tag{E.7}$$

where the coefficient K is not an exponential function of τ.

The sum in the previous expression is certainly larger than any one term in it. In particular,

$$P(\epsilon) \geq K \frac{1}{[\delta_1 d]!} \left[\frac{|\sigma|^2}{\langle n\rangle(1+\langle n\rangle)}\right]^{[\delta_1 d]} \sum_{n_1=[\delta_1 d]}^{[d]} \binom{n_1}{[\delta_1 d]} \left(\frac{\langle n\rangle}{1+\langle n\rangle}\right)^{n_1} \exp\left(\frac{-|\sigma|^2}{1+\langle n\rangle}\right)$$

$$\geq K \frac{1}{[\delta_1 d]!} \left[\frac{|\sigma|^2}{\langle n\rangle(1+\langle n\rangle)}\right]^{[\delta_1 d]} \binom{[d]}{[\delta_1 d]} \left(\frac{\langle n\rangle}{1+\langle n\rangle}\right)^{[d]} \exp\left(\frac{-|\sigma|^2}{1+\langle n\rangle}\right) \tag{E. 8}$$

for $0 \leq \delta_1 \leq 1$. Using the Stirling's approximation

$$n! \sim n^n e^{-n},$$

we simplify the expression in Eq. E. 8 to

$$\begin{aligned} P(\epsilon) \geq K' \exp\Bigg\{&-[d] \ln \frac{1+\langle n\rangle}{\langle n\rangle} + [d] \ln [d] - [d] - 2[\delta_1 d] \ln [\delta_1 d] \\ &+ 2[\delta_1 d] - ([d]-[\delta_1 d]) \ln ([d]-[\delta_1 d]) + [d] - [\delta_1 d] \\ &+ [\delta_1 d] \ln \frac{|\sigma|^2}{\langle n\rangle(1+\langle n\rangle)} - \frac{|\sigma|^2}{1+\langle n\rangle}\Bigg\}. \end{aligned} \tag{E. 9}$$

Again, the constant K' is such that

$$\lim_{\tau\to\infty} \frac{1}{\tau} \ln K' = 0.$$

For large rates, we can approximate the integers $[d]$ and $[\delta_1 d]$ by d and $\delta_1 d$, respectively. Thus, Eq. E. 9 becomes

$$\begin{aligned} P(\epsilon) \geq K' \exp\Bigg\{&d \ln d + \delta_1 d \ln \frac{|\sigma|^2}{\langle n\rangle(1+\langle n\rangle)} - d \ln \frac{1+\langle n\rangle}{\langle n\rangle} - \frac{|\sigma|^2}{1+\langle n\rangle} \\ &-2\delta_1 d \ln \delta_1 d - (1-\delta_1)d \ln (1-\delta_1)d + \delta_1 d\Bigg\} \\ = K' \exp\Bigg\{&-d \ln \frac{1+\langle n\rangle}{\langle n\rangle} + \delta_1 d \ln \frac{|\sigma|^2}{\langle n\rangle(1+\langle n\rangle)} - \frac{|\sigma|^2}{1+\langle n\rangle} \\ &- d \ln (1-\delta_1) + \delta_1 d \ln \frac{1-\delta_1}{\delta_1} - \delta_1 d \ln \delta_1 d + \delta_1 d\Bigg\}. \end{aligned} \tag{E. 10}$$

Let us write the exponent function in Eq. E. 10 as

$$K' \exp[-\tau C e'(\delta_1, R)]. \tag{E.11}$$

Clearly, the function $e'(\delta_1, R)$ is given by

$$e'(\delta_1, R) = \frac{1}{\tau C}\left\{ d \ln \frac{1+\langle n\rangle}{\langle n\rangle} - \delta_1 d \ln \frac{|\sigma|^2}{\langle n\rangle(1+\langle n\rangle)} + \frac{|\sigma|^2}{1+\langle n\rangle} \right.$$
$$\left. -\delta_1 d \ln \frac{1-\delta_1}{\delta_1} + d \ln (1-\delta_1) + \delta_1 d \ln \delta_1 d - \delta_1 d \right\}. \tag{E.12}$$

The best upper bound of the reliability function is obtained by minimizing $e'(\delta_1, R)$ over all values of δ_1 in the interval $(0, 1)$. Let

$$E_u(R) = \min_{0 \leqslant \delta_1 \leqslant 1} e'(\delta_1, R). \tag{E.13}$$

In the following discussion, we intend to show that for $R_c \leqslant R \leqslant C$ the exponent function $E_u(R)$ is equal to the lower bound to the reliability function, $E_L(R)$, derived in Section IV.

To find the value of δ_1 that minimizes the function $e'(\delta_1, R)$, we differentiate $e'(\delta_1, R)$ with respect to δ_1

$$\frac{\partial e'}{\partial \delta_1} = -\frac{d}{\tau C} \ln \frac{|\sigma|^2}{\langle n\rangle(1+\langle n\rangle)} \frac{1-\delta_1}{\delta_1^2 d}.$$

Since

$$\frac{\partial^2 e'}{\partial \delta_1^2} = d\left\{\frac{2}{\delta_1} + \frac{1}{1-\delta_1}\right\} > 0,$$

the value of δ_1 that minimizes $e'(\delta_1, R)$, δ_1^o is given by

$$\left.\frac{\partial e'}{\partial \delta_1}\right|_{\delta_1 = \delta_1^o} = 0.$$

That is,

$$\delta_1^{o\,2} + \frac{\delta_1^o |\sigma|^2}{d\langle n\rangle(1+\langle n\rangle)} - \frac{|\sigma|^2}{d\langle n\rangle(1+\langle n\rangle)} = 0.$$

Solving this equation for δ_1^o, we obtain

$$\delta_1^o = -\frac{|\sigma|^2}{2d\langle n\rangle(1+\langle n\rangle)} + \sqrt{\frac{|\sigma|^2}{d\langle n\rangle(1+\langle n\rangle)} + \frac{|\sigma|^2}{4d^2\langle n\rangle^2(1+\langle n\rangle)^2}}. \tag{E. 14}$$

By substituting the expressions for d in Eq. E. 6 and by recalling that

$$C = \frac{|\sigma|^2}{\tau} \ln \frac{1+\langle n\rangle}{\langle n\rangle},$$

the expression for δ_1^o is simplified to

$$\delta_1^o = -\frac{1}{2\frac{R}{C}\langle n\rangle(1+\langle n\rangle)} + \sqrt{\frac{1}{\frac{R}{C}\langle n\rangle(1+\langle n\rangle)} + \frac{1}{4\left(\frac{R}{C}\right)^2\langle n\rangle^2(1+\langle n\rangle)^2}}. \tag{E. 15}$$

Therefore, we have the following equation

$$E_u(R) = \min_{0\leq\delta_1\leq 1} e_1(\delta_1, R)$$

$$= \frac{1}{\langle n\rangle \ln\frac{1+\langle n\rangle}{\langle n\rangle}} \left\{ \frac{1+2\langle n\rangle\frac{R}{C} - \sqrt{1+4\frac{R}{C}\langle n\rangle(1+\langle n\rangle)}}{1-\sqrt{1+4\frac{R}{C}\langle n\rangle(1+\langle n\rangle)}} \right.$$

$$\left. -\frac{R}{C}\frac{1}{\ln\frac{1+\langle n\rangle}{\langle n\rangle}} \ln \frac{1+2\frac{R}{C}\langle n\rangle(1+\langle n\rangle)+\sqrt{1+4\frac{R}{C}\langle n\rangle(1+\langle n\rangle)}}{2\frac{R}{C}(1+\langle n\rangle)^2}\right. . \tag{E. 16}$$

But this expression is identically equal to the function $E_L(R)$ given by Eqs. 61. Since $0 \leq \delta_1 \leq 1$,

$$1 + \frac{1}{\frac{R}{C}\langle n\rangle(1+\langle n\rangle)} \geq \sqrt{\frac{1}{\frac{R}{C}\langle n\rangle(1+\langle n\rangle)} + \frac{1}{4\left(\frac{R}{C}\right)^2\langle n\rangle^2(1+\langle n\rangle)^2}}$$

is satisfied for all values of R, and $E_u(R)$ given by Eq. E. 16 is a valid bound for all values of $R \geq 0$.

APPENDIX F

Commutativity of the Density Operators

The conditions under which the set of density operators given by Eq. 69 commute will now be investigated.

Let $R_{mn}\left(\left\{\alpha_k^*\right\},\left\{\beta_k\right\}\right)$ denote the matrix coefficient of the operator $\rho_m\rho_n$ in the coherent-state representation. That is,

$$R_{mn}\left(\left\{\alpha_k^*\right\},\left\{\beta_k\right\}\right) = \langle\{\alpha_k\}|\rho_m\rho_n|\{\beta_k\}\rangle \exp\left[\sum_k \left(\tfrac{1}{2}|\alpha_k|^2+\tfrac{1}{2}|\beta_k|^2\right)\right]$$

$$= \int_0^{2\pi}\frac{d\phi_n}{2\pi}\int_0^{2\pi}\frac{d\phi_m}{2\pi}\int\cdots\int \exp\left\{-\sum_k\left[\frac{\left|r_k-\sigma_{mk}e^{i\phi_m}\right|^2}{\langle n_k\rangle} + \frac{\left|\delta_k-\sigma_{nk}e^{i\phi_n}\right|^2}{\langle n_k\rangle}\right]\right\}\prod_k \langle\alpha_k|r_k\rangle\langle r_k|\delta_k\rangle\langle\delta_k|\beta_k\rangle\frac{d^2r_k\,d^2\delta_k}{\pi^2\langle n_k\rangle^2}.$$

By substituting the relation

$$\langle\alpha_k|r_k\rangle = \exp\left(\alpha_k^*r_k-\tfrac{1}{2}|\alpha_k|^2-\tfrac{1}{2}|r_k|^2\right)$$

in this equation, one obtains

$$R_{mn}\left(\left\{\alpha_k^*\right\},\left\{\beta_k\right\}\right) = \int_0^{2\pi}\frac{d\phi_m}{2\pi}\int_0^{2\pi}\frac{d\phi_n}{2\pi}\int \exp\left[-\sum_k\frac{\left|r_k-\sigma_{mk}e^{i\phi_m}\right|^2}{\langle n_k\rangle}+\sum_k\left(\alpha_k^*r_k-|r_k|^2\right)\right]$$

$$\cdot\prod_k\frac{d^2r_k}{\pi\langle n_k\rangle}\int\exp\left[-\sum_k\frac{\left|\delta_k-\sigma_{nk}e^{i\phi_n}\right|^2}{\langle n_k\rangle}\right]\prod_k\exp\left(r_k^*\delta_k+\delta_k^*\beta_k-|\delta_k|^2\right)\frac{d^2\delta_k}{\pi\langle n_k\rangle}$$

$$= F\int_0^{2\pi}\frac{d\phi_n}{2\pi}\exp\left\{\left(\sum_k\frac{\beta_k\sigma_{nk}^*}{1+\langle n_k\rangle}\right)e^{-i\phi_n}+\left(\sum_k\frac{\langle n_k\rangle\alpha_k^*\sigma_{nk}}{1+\langle n_k\rangle}\right)e^{i\phi_n}\right\}$$

$$\int_0^{2\pi}\frac{d\phi_m}{2\pi}\exp\left\{\left(\sum_k\frac{\alpha_k^*\sigma_{mk}}{1+\langle n_k\rangle}\right)e^{i\phi_m}+\left(\sum_k\left[\frac{\langle n_k\rangle\beta_k\sigma_{mk}^*}{1+\langle n_k\rangle}+\frac{\sigma_{nk}\sigma_{mk}^*}{(1+\langle n_k\rangle)^2}e^{i\phi_n}\right]\right)e^{-i\phi_m}\right\},$$

where

$$F = \prod_k \left(\frac{1}{1+\langle n_k\rangle}\right)^2 \exp\left[\frac{-|\sigma_{nk}|^2 + |\sigma_{mk}|^2}{1+\langle n_k\rangle} + \frac{\langle n_k\rangle^2 \alpha_k^* \beta_k}{(1+\langle n_k\rangle)^2}\right].$$

Hence

$$R_{mn}\left(\left\{\alpha_k^*\right\}, \left\{\beta_k\right\}\right) = F \sum_r \sum_p \sum_s \left(\frac{1}{r!}\right)^2 \left(\frac{1}{s!}\right)^2 \binom{r}{p} \left(\sum_k \frac{\alpha_k^* \sigma_{mk}}{1+\langle n_k\rangle}\right)^r \left(\sum_k \frac{\langle n_k\rangle \beta_k \sigma_{mk}^*}{(1+\langle n_k\rangle)^2}\right)^{r-p}$$

$$\cdot \binom{s}{\frac{s+p}{2}} \left(\sum_k \frac{\beta_k \sigma_{nk}^*}{1+\langle n_k\rangle}\right)^{\frac{s+p}{2}} \left(\sum_k \frac{\langle n_k\rangle \alpha_k^* \sigma_{nk}}{(1+\langle n_k\rangle)^2}\right)^{\frac{s-p}{2}} \left(\sum_k \frac{\sigma_{nk}\sigma_{mk}^*}{(1+\langle n_k\rangle)^2}\right)^p . \tag{F. 1}$$

The binomial coefficient $\binom{n}{x}$ is equal to zero when x is not an integer or when x is larger than n. Since

$$\rho_m \rho_n = \rho_m^+ \rho_n^+ = (\rho_n \rho_m)^+ ,$$

the matrix elements of $\rho_n \rho_m$ are

$$R_{nm}\left(\left\{\alpha_k^*\right\}, \left\{\beta_k\right\}\right) = \left\{R_{mn}\left(\left\{\beta_k^*\right\}, \left\{\alpha_k\right\}\right)\right\}^* .$$

Hence, one obtains

$$R_{nm}\left(\left\{\alpha_k^*\right\}, \left\{\beta_k\right\}\right) = F \sum_r \sum_p \sum_s \left(\frac{1}{r!}\right)^2 \left(\frac{1}{s!}\right)^2 \binom{r}{p} \left(\sum_k \frac{\alpha_k^* \sigma_{nk}}{1+\langle n_k\rangle}\right)^r \left(\sum_k \frac{\langle n_k\rangle \beta_k \sigma_{nk}^*}{(1+\langle n_k\rangle)^2}\right)^{r-p}$$

$$\cdot \binom{s}{\frac{s+p}{2}} \left(\sum_k \frac{\beta_k \sigma_{mk}^*}{1+\langle n_k\rangle}\right)^{\frac{s+p}{2}} \left(\sum_k \frac{\langle n_k\rangle \alpha_k^* \sigma_{mk}}{(1+\langle n_k\rangle)^2}\right)^{\frac{s-p}{2}} \left(\sum_k \frac{\sigma_{mk}\sigma_{nk}^*}{(1+\langle n_k\rangle)^2}\right)^p . \tag{F. 2}$$

Since $R_{mn}\left(\left\{\alpha_k^*\right\}, \left\{\beta_k\right\}\right)$ is not identically equal to $R_{nm}\left(\left\{\alpha_k^*\right\}, \left\{\beta_k\right\}\right)$, the operators ρ_n and ρ_m do not commute in general. But when

$$\sum_k \frac{\sigma_{nk}\sigma_{mk}^*}{(1+\langle n_k\rangle)^2} = \sum_k \frac{\sigma_{mk}\sigma_{nk}^*}{(1+\langle n_k\rangle)^2} = 0; \qquad m \neq n, \tag{F. 3}$$

we have

$$R_{mn}\left(\{\alpha_k^*\},\{\beta_k\}\right) = F\sum_r\sum_p\left(\frac{1}{r!}\right)^2\left(\frac{1}{p!}\right)^2\left(\sum_k\frac{\alpha_k^*\sigma_{mk}}{1+\langle n_k\rangle}\right)^r\left(\sum_k\frac{\langle n_k\rangle\beta_k\sigma_{mk}^*}{(1+\langle n_k\rangle)^2}\right)^r$$

$$\cdot\left(\sum_k\frac{\beta_k\sigma_{nk}^*}{1+\langle n_k\rangle}\right)^p\left(\sum_k\frac{\langle n_k\rangle\alpha_k^*\sigma_{nk}}{(1+\langle n_k\rangle)^2}\right)^p$$

and

$$R_{nm}\left(\{\alpha_k^*\},\{\beta_k\}\right) = F\sum_r\sum_p\left(\frac{1}{r!}\right)^2\left(\frac{1}{p!}\right)^2\left(\sum_k\frac{\alpha_k^*\sigma_{nk}}{1+\langle n_k\rangle}\right)^r\left(\sum\frac{\langle n_k\rangle\beta_k\sigma_{nk}^*}{(1+\langle n_k\rangle)^2}\right)^r$$

$$\cdot\left(\sum_k\frac{\beta_k\sigma_{mk}^*}{1+\langle n_k\rangle}\right)^p\left(\sum_k\frac{\langle n_k\rangle\alpha_k^*\sigma_{mk}}{(1+\langle n_k\rangle)^2}\right)^p.$$

One can see that these two functions are equal when $\langle n_k\rangle = n$ for all k. In such a case, Eq. F.3 is simply

$$\sum_k\sigma_{nk}\sigma_{mk}^* = \sum_k\sigma_{mk}\sigma_{nk}^* = 0; \qquad m \neq n.$$

Obviously, this condition is satisfied when

$$\sigma_{mk}\sigma_{nk}^* = |\sigma_{mk}|^2\,\delta_{mn}.$$

APPENDIX G

Physical Implementation of the Unitary Transformation

To implement the optimum receivers for the reception narrow-band orthogonal signals studied in Sections IV and V, one needs a device at the output of which only one normal mode (with annihilation operator $b = \sum_k V_k a_k$) is excited when the input is a narrow-band field. That is, a device that performs physically the unitary transformation

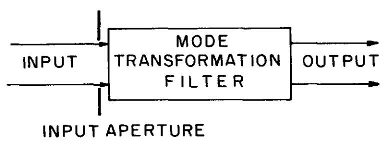

Fig. G-1. Idealized system.

V described in section 4.2. We shall prove that the idealized system shown in Fig. G-1 makes such a transformation.

When the electric field operator $E(\underline{r}, t)$ is expanded as

$$E(\underline{r}, t) = i \sum_k \sqrt{\frac{\hbar\omega_k}{2L}} \left\{ a_k \exp\left[i\omega_k \left(\frac{z}{c} - t \right) \right] - a_k^+ \exp\left[-i\omega_k \left(\frac{z}{c} - t \right) \right] \right\}$$

the electric field at the input of the filter is in the state specified by the density operator

$$\rho_i = \int \prod_k \exp\left[- \frac{|\alpha_k - V_k|^2}{\langle n \rangle} \right] |\alpha_k\rangle \langle \alpha_k| \frac{d^2\alpha_k}{\pi \langle n \rangle}.$$

At the output of the device, only one normal mode with annihilation operator

$$b = \sum_k V_k a_k$$

is excited. Moreover, it is in a state specified by the density operator

$$\rho_o = \int \exp\left[- \frac{|\beta - V|^2}{\langle n \rangle} \right] |\beta\rangle \langle \beta| \frac{d^2\beta}{\pi \langle n \rangle},$$

where

$$|V|^2 = \sum_k |V_k|^2.$$

As mentioned in section 4.2, this device is simply a matched filter in the classical limit.

Let us idealize the mode transformation filter as a cavity that has only one dominant normal mode at frequency ω. The electric field inside of the cavity is

$$E(\underline{r}, t) = i\sqrt{\frac{\hbar\omega}{2}}\, bV(\underline{r}, t)\, \exp(-i\omega t).$$

Before the aperture is opened, the field inside of the filter is not allowed to interact with the field present at the input. Hence, the total Hamiltonian of the system is

$$H_o = \sum_k \hbar\omega_k a_k^+ a_k + \hbar\omega b^+ b.$$

Clearly, $\left[a_k^+, b\right] = \left[a_k^+, b^+\right] = [a_k, b] = 0$ for all k. At $t \geq 0$, the input aperture is opened and the fields are allowed to interact, and then the Hamiltonian is given approximately by

$$H = \sum_k \hbar\omega_k a_k^+ a_k + \hbar\omega b^+ b + \sum_k \hbar\left(v_k a_k b^+ + v_k^* b a_k^+\right). \tag{G. 1}$$

Note: The term $I = \sum_k \hbar\left(v_k a_k^+ b_k^+ + v_k a_k b_k\right)$, which is also Hermitian and of the same order in the strength of coupling, is not included in Eq. G. 1. The effect of this term is small for the following reason. The terms in the right-hand side of Eq. G. 1 are approximately DC, whereas the term I is rapidly varying. Since the interaction will be in effect for many cycles of ω, the term I will average to zero compared with those in Eq. G-1.

The magnitude of the coupling coefficients v_k are assumed to be small compared with the ω_k and ω.

From the Hamiltonian and the commutation relations, we obtain the Heisenberg equations for the time variations in the operators a_k and b

$$\frac{da_k}{dt} = \frac{1}{i\hbar}[a_k, H] = -i\omega_k a_k - iv_k^* b \tag{G. 2}$$

$$\frac{db}{dt} = \frac{1}{i\hbar}[b, H] = -i\omega b - i\sum_j v_j a_j. \tag{G. 3}$$

Taking the one-sided Laplace transforms of the $a_k(t)$ and b(t), we obtain from Eqs. G. 2 and G. 3

$$(s+i\omega)\,\bar{b}(s) = b(0) - i\sum_j v_j \bar{a}_j(s) \tag{G. 4}$$

$$(s+i\omega_k)\,\bar{a}_k(s) = a_k(s) - iv_k^* \bar{b}(s). \tag{G. 5}$$

Combining Eqs. G.4 and G.5, we have

$$\bar{b}(s) = \frac{\bar{b}(0) - i \sum_k \frac{v_k a_k(0)}{s + i\omega_k}}{s + i\omega + \sum_k \frac{|v_k|^2}{s + i\omega_k}}. \tag{G.6}$$

It has been shown[6] that at time t

$$b(t) = b(0)\,\mu(t) + \sum_k v_k(t)\, a_k(0), \tag{G.7}$$

where

$$\mu(t) = \exp\left[-\frac{r}{2} t - i\omega' t\right] \tag{G.8}$$

$$v_k(t) \cong \frac{v_k \exp(-i\omega_k t) \left\{1 - \exp\left[i(\omega_k - \omega)\, t - \frac{r}{2} t\right]\right\}}{\omega_k - \omega' + i\frac{r}{2}}. \tag{G.9}$$

The parameter r is given by

$$r = 2\pi |v_k|^2 \rho(\omega_k)\Big|_{\omega_k = \omega}$$

where $\rho(\omega)$ is the density of modes at frequency ω, and

$$\omega' = \omega + \Delta\omega$$

$$\Delta\omega = \mathscr{P}\left\{\int_{-\infty}^{\infty} \frac{|v_k|^2 \rho(\omega_k)\, d\omega_k}{\omega_k - \omega}\right\}$$

is a very small frequency shift. It is clear that at $t \geqslant \frac{4}{r}$, we have

$$b(t) = \sum_k V_k a_k(0) \exp[-i\omega_k t], \tag{G.10}$$

where

$$V_k = \frac{v_k}{\omega_k - \omega' + \frac{r}{2}}.$$

Hence, when the frequency bandwidth is very small, $\omega_k \approx \omega$, the operator b at time $t \geqslant \frac{4}{r}$ is

$$b = \sum_k V_k a_k \tag{G.11}$$

in the Schrödinger picture. When it is possible to make the time constant r small compared with τ, we can achieve the desired unitary transformation by adjusting the coupling coefficient v_k's.

Acknowledgment

I wish to express my sincere gratitude to Professor Robert S. Kennedy for his guidance and encouragement. Under his supervision, the preparation of my thesis was a most valuable educational experience. I am indebted to Professor Peter Elias and Professor Hermann A. Haus who offered many helpful suggestions and comments. The support of the Research Laboratory of Electronics, M. I. T., is gratefully acknowledged. I would also like to thank my husband, Professor Chung L. Liu, who reviewed my manuscript and offered constant encouragement.

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1. ORIGINATING ACTIVITY *(Corporate author)* Research Laboratory of Electronics Massachusetts Institute of Technology Cambridge, Massachusetts 02139	2a. REPORT SECURITY CLASSIFICATION Unclassified 2b. GROUP None

3. REPORT TITLE

Reliability of Quantum-Mechanical Communication Systems

4. DESCRIPTIVE NOTES *(Type of report and inclusive dates)*

Technical Report

5. AUTHOR(S) *(First name, middle initial, last name)*

Jane W-S. Liu

6. REPORT DATE	7a. TOTAL NO. OF PAGES	7b. NO. OF REFS
December 31, 1968	113	23

8a. CONTRACT OR GRANT NO. DA 28-043-AMC-02536(E) b. PROJECT NO. 200-14501-B31F NASA Grant NGL 22-009-013	9a. ORIGINATOR'S REPORT NUMBER(S) Techncial Report 468 9b. OTHER REPORT NO(S) *(Any other numbers that may be assigned this report)* None

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13. ABSTRACT

We are concerned with the detection of a set of M messages that are transmitted over a channel disturbed by chaotic thermal noise when quantum effects in the communication systems are taken into account. Our attention was restricted to the special case in which the density operators specifying the states of the received field are commutative. From quantum-mechanical description of the noise and signal fields, the structure and performance of the quantum-mechanical optimum receiver are found.

Two special communication systems have been studied: (i) a system in which signals have known classical amplitudes but unknown absolute phases, and the signal field is in coherent states; (ii) a system in which the classical amplitudes of the signal field are Gaussian random processes, and the received field in the absence of noise is in completely incoherent states. Bounds on the probability of error in these systems are derived. For both systems, the minimum attainable error probability is expressed in the form $\exp[-\tau CE(R)]$, where E(R) is the system reliability function which is a function of the information rate R of the system, τ is the time allotted for the transmission of a message, and C is the capacity of the system. For these two types of systems, the expressions for C and E(R) are derived.

DD FORM 1473 1 NOV 65 (PAGE 1)

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quantum-mechanical detection theory						
quantum effects						
quantum-mechanical models of white Gaussian channel						
quantum-mechanical model of Rayleigh fading channel						