NETWORKS OF GAUSSIAN CHANNELS WITH APPLICATIONS TO FEEDBACK SYSTEMS

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Abstract

This paper discusses networks (directed graphs) having one input node, one output node, and an arbitrary number of intermediate nodes, whose branches are noisy communications channels, in which the input to each channel appears at its output corrupted by additive Gaussian noise. Each branch is labeled by a non-negative real parameter which specified how noisy it is. A branch originating at a node has as input a linear combination of the outputs of the branches terminating at that node.

The channel capacity of such a network is defined. Its value is bounded in terms of branch parameter values and procedures for computing values for general networks are described. Explicit solutions are given for the class $D_0$ which includes series-parallel and simple bridge networks and all other networks having $r$ paths, $b$ branches, and $v$ nodes with $r = b - v + 2$, and for the class $D_1$ of networks which is inductively defined to include $D_0$ and all networks obtained by replacing a branch of a network in $D_1$ by a network in $D_1$.

The general results are applied to the particular networks which arise from the decomposition of a simple feedback system into successive forward and reverse (feedback) channels. When the feedback channels are noiseless, the capacities of the forward channels are shown to add. Some explicit expressions and some bounds are given for the case of noisy feedback channels.
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Introduction

The min-cut max-flow theorem\textsuperscript{11–13} gives the capacity of a network made up of branches of given capacity. It applies to networks of noisy communications channels if the assumption is made that arbitrarily large delays and arbitrarily complex encoding and decoding operations may take place at each interior node.

This paper presents the theory of networks of another kind of channel—a channel with additive Gaussian noise, for which the only operation which takes place at a node is linear combination of the arriving signal and noise voltages, with no significant delay and no decoding or recoding.

The Problem

Consider the Class $D$ of two-terminal networks like that shown in Fig. 1, in which there are no cycles, each of the $b$ branches $B_i$ is directed, and each branch lies on one of the $r$ paths $R_i$ which go from the input terminal on the left to the output terminal on the right. A signal voltage $e_i$ of mean-square value $P_i$ (the signal power) is applied to the input terminal, node $V_1$, at the left. At each interior node, the output (signal plus noise) of each branch $B_i$, arriving at the node is given a (positive or negative) weight, the branch transmission $t_i$, and the resulting linear combination of signal and noise voltages is supplied as input to all the branches leaving that node.

Each branch $B_i$ adds to its input voltage $e_i$, a Gaussian noise voltage $n_i$ whose mean-square value (the noise power) is a constant $N_i$ (the noise-to-signal power ratio), also called the parameter of the branch) times the mean-square value $P_i$ (the input power) of its input voltage. The noise voltage in each branch is statistically independent of the noise voltages in other branches and of the signal voltage:

\[
\overline{e_i^2} = P_i, \quad 0 \leq i \leq b; \quad \overline{n_i^2} = N_iP_i, \quad 1 \leq i \leq b
\]

Since the branch input voltage and its noise are uncorrelated, the mean-square value of the branch output voltage (the output power) is just

\[
\overline{(e_i + n_i)^2} = \overline{e_i^2} + \overline{n_i^2} = P_i + N_iP_i = P_i(1 + N_i).
\]

The power output of each branch generator depends on the power level at its input, and thus on the power level of the signal and of all other noise generators which affect its input power, as well as on the values of the branch transmissions. However, once the power levels of the signal and of all noises and the values of the branch transmissions are fixed, the network is linear. The final output at the right-hand output terminal $V_r$ is a linear combination of the $b$ branch noise generator voltages and the signal voltage $e_0$. We constrain the values of the branch transmissions $t_i$ by requiring that the coefficient of $e_0$ in this sum be unity.

The network is equivalent to a single branch (noisy channel) of the same kind as the component branches,
since the linear combination of the $b$ branch noise voltages which appears in the output is a Gaussian noise voltage, and the overall action of the two-terminal network is to receive an input signal and to produce at its output the input signal plus an independent Gaussian noise. The ratio of output noise power to signal power, $N_{b+1}$, is a function of the branch transmissions as well as the parameters $N_i$ of the network branches. The optimum noise-to-signal power of the network, $N_{opt}$, is defined as the minimum value of $N_{b+1}$ which can be obtained by varying the branch transmissions.

The problem is to find $N_{opt}$ as a function of the given $N_i$.

**Series and Parallel Networks**

To express the results most simply in important special cases it is convenient to associate with each branch, not only the parameter $N_i$, but the signal-to-noise ratio,

$$S_i = 1/N_i,$$

and the capacity per use of the channel,

$$C_i = \frac{1}{2} \log (1 + S_i).$$

Equivalent quantities are defined for the network: $S_{opt}$ is the maximum signal-to-noise ratio attainable by varying the branch transmissions, and $C_{opt}$ is the largest channel capacity so attainable.

We can then state three results.

**Series Networks**

A network in $D$ in which all $b$ branches are in series has $N_{opt}$ given by

$$1 + N_{opt} = \prod_{i=1}^{b} (1 + N_i).$$

**Parallel Networks**

A network in $D$ in which all $b$ branches are in parallel has $S_{opt}$ given by

$$S_{opt} = \sum_{i=1}^{b} S_i.$$

**Duality**

Given two channels of capacities $C_1$ and $C_2$. Let the optimum capacity of the network consisting of the two channels in series be $C_s$. Let the optimum capacity of the two channels connected in parallel be $C_p$. Then

$$C_1 + C_2 = C_s + C_p.$$

The result on series networks expressed by (5) does not seem to have been published. The result for parallel branches expressed by (6) is known as optimum diversity combining, or the ratio squared, and was discovered independently of the general theory. Both follow directly from the general results following. The duality relationship of (7) follows directly from (4), (5), and (6), and also seems not to have been published. We have

$$C_s = \frac{1}{2} \log \left(1 + \frac{1}{N_i}\right)$$

$$= \frac{1}{2} \log \left(1 + \frac{1}{(1 + N_1)(1 + N_2) - 1}\right)$$

$$= \frac{1}{2} \log \left(\frac{(1 + N_1)(1 + N_2)}{N_1 + N_2 + N_1N_2}\right)$$

$$= \frac{1}{2} \log \left(\frac{(1 + S_1)(1 + S_2)}{1 + S_1 + S_2}\right)$$

$$= \frac{1}{2} \log (1 + S_1)(1 + S_2) - \frac{1}{2} \log (1 + S_1 + S_2)$$

$$= C_1 + C_2 - C_p.$$
arrive at a given time over different paths. This can be accomplished for any network in \( F \), or indeed in \( D \), if an initial set of delay values \( d_i \) are given for the branches \( B_i \), by increasing some of them in the following fashion. Assign a delay value to each node \( V_j \) equal to the maximum delay obtained by adding the delay values of the branches along each path from \( V_i \) to \( V_j \). Then assign to \( B_i \) the new delay value \( d_i' \), which is the difference between the delay values of its terminal and initial nodes, \( d_i' \geq d_i \).

We will henceforth assume that this process has been carried out for all networks in \( F \) or \( D \), and that all have the uniform delay property. It is then not necessary to keep track of the delay values of networks or branches. We now state results for feedback networks.

**Noiseless Feedback:** For a network in \( F \), if all feedback branches are noiseless, and the \( k \) forward branches have capacities \( C_{2j-1} \), \( 1 \leq j \leq k \), then the optimum capacity of the network is given by

\[
C_{\text{opt}} = \sum_{i=1}^{k} C_{2i-1}
\]

and the optimum signal-to-noise ratio \( S_{\text{opt}} \) by

\[
1 + S_{\text{opt}} = \prod_{i=1}^{k} (1 + S_{2i-1}).
\]

In particular, if

\[
S = \sum_{i=1}^{k} S_{2i-1}
\]

is fixed, but an arbitrarily large \( k \) is available, we have

\[
1 + S_{\text{opt}} = \lim_{k \to \infty} \prod_{i=1}^{k} \left( 1 + \frac{S}{k} \right) = e^S,
\]

\[
S_{\text{opt}} = e^S - 1.
\]

**Noisy Feedback, \( k = 2 \):** For a network in \( F \) with two noisy forward channels \( B_1 \) and \( B_2 \), and one noisy feedback channel \( B_3 \), the optimum signal-to-noise ratio is

\[
S_{\text{opt}} = S_1 + S_2 + \frac{S_1 S_2 S_3}{(1 + S_1)(1 + S_2) + S_3}.
\]

Unfortunately, a general formula like (11) for a network in \( F \) with \( k > 2 \) is not available, although the computation of \( S_{\text{opt}} \) for any particular case is a straightforward numerical analysis problem. However, we do have some inequalities which hold for all nets in \( F \) and which provide some insight.

**Noiseless Feedback, General Case:** For a network in \( F \) with \( k \) noisy forward branches \( B_{2j-1} \), \( 1 \leq j \leq k \), and \( k - 1 \) noisy feedback branches \( B_{3j} \), \( 1 \leq j \leq k - 1 \), the optimum signal-to-noise ratio \( S_{\text{opt}} \) is bounded by

\[
S_{\text{opt}} \geq \sum_{i=1}^{k} S_{2i-1}
\]

\[
1 + S_{\text{opt}} \leq \prod_{i=1}^{k} (1 + S_{2i-1})
\]

\[
S_{\text{opt}} \leq \sum_{i=1}^{2k-1} S_i.
\]

If signal-to-noise ratio costs \( c_f \) per unit for forward channels and \( c_s \) per unit for feedback channels, so that the total cost for a network in \( F \) is

\[
c = c_f \sum_{i=1}^{k} S_{2i-1} + c_s \sum_{i=1}^{k-1} S_{2i},
\]

then for sufficiently large \( S_{\text{opt}} \), the cost per unit of \( S_{\text{opt}} \) may be made arbitrarily close to \( c_s \):

\[
\frac{c}{S_{\text{opt}}} \leq c_s (1 + \delta).
\]

The results for noiseless feedback and for noisy feedback with \( k = 2 \) were published by the author some time ago. Schalkwijk and Kailath have recently investigated the noiseless case from the point of view of error probability for the transmission of discrete messages. Turin has also dealt with a closely related question. The noiseless feedback results of (8) and (9) are remarkable, since they permit the transmission of a continuous signal of fixed bandwidth over a noisy channel at a rate equal to channel capacity, no matter how large the bandwidth of the forward channel. No coding or decoding is needed, provided that a noiseless feedback channel is available. Furthermore, they do so without introducing any of the discontinuities which must occur when a continuous signal is mapped onto a space of higher dimensionality—discontinuities which were pointed out by Shannon and Kotelnikov and have recently been discussed by Wozencraft and Jacobs. Equation (10) implies that a signal-to-noise ratio of 10 in bandwidth \( W \) is equivalent to a signal-to-noise ratio of \( e^{10} - 1 \), or about 22 000 if the available forward channel is wideband and has white noise, and a noiseless feedback channel is available (see the literature for further discussion).

The inequality (12) follows from the parallel network result of (6). The result of setting all feedback channel transmissions at zero and using the forward channels in parallel gives the right side of (12). The optimum choice of branch transmissions must do at least as well. The second inequality, (13), says that noise in the feedback
channels does not help; the right side is just the noiseless feedback result of (9). It is a consequence of a more general result which will be given, and which shows that increasing \( N_i \) in any branch cannot decrease \( N_{opt} \). The third result, (14), says that, given a choice, it is better to use signal-to-noise ratio in the forward rather than the feedback channels. The total \( S_{opt} \) attainable by feedback is less than would be attained by taking all of the feedback channels, turning them around, and using them in parallel with the forward channels, which gives the right side of (14) by (6). This will also be derived later. The final result, (15), shows why feedback is interesting even if it does not do as well as the same amount of signal-to-noise ratio in the forward direction, by (14). Signal-to-noise ratio in the feedback direction may be cheaper, as when a satellite is communicating to Earth, and if it is, it is possible by means of feedback to buy forward signal-to-noise ratio at the same cost, if one wants enough of it. Equation (15) is a direct consequence of (11). It is necessary only to choose \( S_i \) equal to \( S_o \); and \( S_o \) so large that it is possible to have \( S_i < S_o \) and \( S_o' > S_2 \) at the same time. For \( k \geq 2 \) the result will be of the same character, but better, i.e., a smaller \( \delta \) will do. Or a smaller amount of \( S_{opt} \) can be bought at the same unit cost—but the absence of a formula makes the demonstration harder.

**General Results**

To state and prove the theorem from which the above results follow we need some further definitions. For each pair of paths \( R_i, R_j \) from \( V_i \) to \( V_j \), in a network in \( D \), we define \( G_{ij} \) as a product which contains one factor \((1 + N_i)\) for each branch \( B_i \), which lies in both paths; if \( R_i \) and \( R_j \) share no branches, \( G_{ij} = 1 \). Formally, if we treat the symbol \( R_i \) as denoting the set of branches which are contained in the \( i \)th path, then \( R_i \cap R_j \) is the set of branches which the two paths have in common, and

\[
G_{ij} = \prod_{i \in R_i, j \in R_j} (1 + N_i)
\]

\[
= 1 \quad \text{for} \quad R_i \cap R_j \text{ empty.} \tag{16}
\]

We also define the *path transmission* \( T_i \) of path \( R_i \) as the product of the branch transmissions \( t_i \) for those branches which lie on \( R_i \):

\[
T_i = \prod_{i \in R_i} t_i. \tag{17}
\]

The *network transmission* \( T_{o,k+1} \) is the sum of all path transmissions. By the assumption made in the discussion following (2), the branch transmissions \( t_i \) are constrained so that the network transmission, which is the coefficient of the signal voltage \( e_o \) in the output, is unity.

\[
T_{o,k+1} = \sum_{i=1}^{r} T_i = 1. \tag{18}
\]

**Theorem**

For any network in \( D \), we have

\[
1 + N_{opt} = \min_{t_i} \left\{ \sum_{i=1}^{r} \sum_{j=1}^{s} G_{ij} T_i T_j \right\} \geq 1 / \left\{ \sum_{i=1}^{r} \sum_{j=1}^{s} G_{ij}^{-1} \right\}
\]

and

\[
S_{opt} = 1 / \min_{t_i} \left\{ \sum_{i=1}^{r} \sum_{j=1}^{s} (G_{ij} - 1) T_i T_j \right\}
\]

\[
\leq \sum_{i=1}^{r} \sum_{j=1}^{s} [G_{ij} - 1]^{-1}, \tag{20}
\]

where the \( T_i \) are given in terms of the \( t_i \) by (17) and are subject to the constraint (18), and \( G_{ij}^{-1} \) and \([G_{ij} - 1]^{-1}\) are elements of the inverses of the matrices \([G_{ij}]\) and \([G_{ij} - 1]\). The inverses of \([G_{ij}]\) and \([G_{ij} - 1]\) always exist unless there is at least one noiseless path from input to output, so that some \( G_{ii} = 1 \). In this case \( N_{opt} = 0 \) and \( S_{opt} = \infty \). These values are attained by setting \( T_i = 1 \) and all other \( T_j = 0, j \neq i \).

Equality holds on the right in (19) and (20) for networks in the set \( D_o \) which includes any network in \( D \) with \( r \) paths, \( b \) branches, and \( v \) nodes for which

\[
r = b - v + 2, \tag{21}
\]

and for networks in the set \( D_i \) which includes the networks in \( D_o \) and, inductively, any network which is constructed from a network in \( D_i \) by replacing any branch by another network in \( D_i \).

Note that \( D_0 \) contains simple series networks, for which \( r = 1 \) and \( b = v - 1 \), and simple parallel networks, for which \( r = b \) and \( b = 2 \). \( D_i \) therefore contains all series-parallel networks, but it contains others as well—for example, the (topologically equivalent) networks of Figs. 1 and 2, for which \( b = 5, v = 4, \) and \( r = 3 \), but not the network of Fig. 3, for which \( b = 11, v = 6, \) and \( r = 8 \), or any network in \( F \) with \( k > 2 \).

*Proof:* For the proof we need one more definition. \( T_{ij} \), the transmission from branch \( i \) to branch \( j \), is just the network transmission as defined in (18) for the subnetwork consisting of branch \( i \) and all other branches which lie on some directed path which goes through branch \( i \) to the initial node of branch \( j \). (Thus, \( B_i \) is included in the subnetwork, but \( B_j \) is not; and \( t_i \) is a common factor of all of the terms in the sum \( T_{ij} \).) If there are no paths through \( B_i \) and \( B_j \), or if \( B_i \) precedes \( B_j \) on such a path, then \( T_{ij} = 0 \). \( T_{o,i} \) is the transmission of a subnetwork with input node \( V_i \) and output node the initial node of \( B_i \); and \( T_{i,i+1} \) is the transmission of the subnetwork of paths through branch \( i \) to the output node \( V_r \).

We now derive an expression for \( P_{o,i} \), the output power of the network. By the statistical independence of the noise voltage generators from one another and from the signal source, the output power at the right-hand node is the sum of the powers transmitted to that node by these
b + 1 separate sources. The source in branch i contributes an amount of power equal to its generated power $P_i N_i$ times the square of the transmission from $B_i$ to the output. Thus,

$$P_{b+1} = \sum_{i=0}^{b} P_i N_i T_{i,b+1}^2 = P_0(1 + N_{b+1}), \quad (22)$$

where the right-most equality follows from the fact that by the constraint of (18), (2) holds for $i = b + 1$, and where the signal power contributed to the output is represented in the sum by the term for $i = 0$, with $N_0 = 1$ and $T_{0,b+1} = 1$.

Similarly the input power to any branch $B_i$ may be expressed as the sum of the contributions of the generators which lie to its left:

$$P_i = \sum_{k=0}^{i-1} P_k N_k T_k^2. \quad (23)$$

Here we have assumed that the branches are numbered in an order such that if $B_i$ precedes $B_j$ on some directed path, $i < j$.

By successive substitution of (23) into (22) and in the resulting expressions, the subscripts on the $P$'s appearing on the right can all be reduced to zero. The result is a sum of terms, all of which have $P_0$ as a factor. There is one term for each of the $2^i$ subsets $W_m$ of the $b$ branches which has the property that all of the branches in $W_m$ are included in a path from input to output, i.e., that there is an integer $f$ with $R_f \supseteq W_m$. If $W_m$ is such a subset, say $W_m = (B_i, B_j, B_k)$ with $i < j < k$, then the corresponding term is

$$P_{b+1} T_{i,j}^2 N_i N_j N_k T_{k,b+1}^2 = P_0(T_0 T_i T_j T_k T_{k,b+1}^2) N_i N_j N_k, \quad (24)$$

The product of the transmission terms which appears on the right is just the sum of the transmissions of all paths from input to output which include all three of the branches $B_i, B_j, B_k$. If there are no such paths, then one or more of the $T_{ij}$ in (24) will vanish. Thus the output power is expressed in terms of the path transmissions $T_i$ and the branch parameters $N_i$. Dividing through by $P_0$ gives an expression for $1 + N_{b+1}$,

$$1 + N_{b+1} = \sum_{k=0}^{2^b-1} \left\{ \sum_{i,j:k \in W_k} T_{i,j} \right\}^2 \prod_{j=1}^{k} N_j, \quad (25)$$

where $W_0$ is the null set, for which the product is taken to be 1. The sum is also 1 for $k = 0$, since it is just the square of the network transmission of (18). Thus excluding the term for $k = 0$ gives an expression for $N_{b+1}$ as a sum of products of positive terms, which is monotone nondecreasing in each $N_j$. We thus have proved Lemma 1.

**Lemma 1**

For any given set of path transmissions $T_i$, the network noise-to-signal ratio $N_{b+1}$ is a monotone nondecreasing function of each branch noise-to-signal ratio $N_i$.

This lemma provides the proof of (13), which was referred to previously.

We have also proved that $N_{b+1}$ can vanish for a non-vanishing set of path transmissions only if there is some path $R_i$ along which every branch is noiseless, so that setting $T_i = 1$ and $T_j = 0, j \neq i$ gives a right-hand side in (25) in which only the term for $W_0$ remains. The matrix $||G_{ij} - I||$ will be singular if, and only if, there is such a noiseless path since it will then map the transmission vector $T$ with $T_i = 1$ and $T_j = 0, j \neq i$ into the null vector. The matrix $||G_{ij}||$ can be singular only under the same circumstances, but may not be even when noiseless paths exist.

We next show the equivalence of the right side of (25) to the quadratic form:

$$\sum_{i=1}^{b+1} \sum_{j=i}^{b+1} G_{ij} T_i T_j, \quad (26)$$

where the $T_i$ are still subject to the constraint (18). Substituting into (26) the definition (16) of $G_{ij}$ gives

$$\sum_{i=1}^{b+1} \sum_{j=i}^{b+1} T_i T_j \prod_{m:a_i \in R_f \cap R_j} (1 + N_m). \quad (27)$$

Expanding the product gives

$$\sum_{i=1}^{b+1} \sum_{j=i}^{b+1} T_i T_j \prod_{k:m:b_i \cap W_k} N_m. \quad (28)$$

Inverting the order of summation to sum over all $W_k$,

$$\sum_{k=0}^{2^b-1} \sum_{i,j:k \in W_k} T_i T_j \prod_{m:a_i \cap W_k} N_m. \quad (29)$$

We then recognize that the parentheses enclose a term which is just the square of the sum of $T_i$ over the $i$ for which $W_k$ is included in $R_i$;

$$\sum_{k=0}^{2^b-1} \left\{ \sum_{i:k \in W_k} T_i \right\}^2 \prod_{m:a_i \cap W_k} N_m, \quad (30)$$

which is just the right side of (25).

We have thus proved that for $T_i$ constrained by (18),

$$1 + N_{b+1} = \sum_{i=1}^{b+1} \sum_{j=i}^{b+1} G_{ij} T_i T_j. \quad (31)$$

Squaring (18) gives

$$1 = 1^2 = \left\{ \sum_{i=1}^{b} T_i \right\}^2 = \sum_{i=1}^{b} \sum_{j=1}^{b} T_i T_j, \quad (32)$$

and subtracting (32) from (31) gives

$$N_{b+1} = \sum_{i=1}^{b} \sum_{j=i}^{b} (G_{ij} - 1) T_i T_j, \quad (33)$$

or

$$S_{b+1} = 1 - \sum_{i=1}^{b} \sum_{j=i}^{b} (G_{ij} - 1) T_i T_j. \quad (34)$$

Now $N_{opt}$, by definition, is the minimum value of $N_{b+1}$ as the branch transmissions are varied, and $S_{opt}$...
is its reciprocal. We have therefore proved the first part of the theorem: namely, the equalities on the left in (19) and (20).

To obtain the inequalities on the right in (19) and (20), we minimize (31) and (33) by varying the path transmissions $T_i$ independently, subject only to the constraint imposed by (18). The additional constraints imposed by the topology of the network and by (17), which expresses the $T_i$, in terms of the real independent variables $t_k$, are ignored. The results are lower bounds to the minima which (31) and (33) can actually attain in the network.

Using a Lagrange multiplier $2M$, we set the derivative of

$$
\sum_{i=1}^{r} \sum_{j=1}^{i} G_{ij} T_j T_i - 2M \sum_{i=1}^{r} T_i
$$

with respect to $T_i$, equal to zero. This gives

$$
\sum_{j=1}^{i} G_{ij} T_j = M, \quad 1 \leq j \leq r.
$$

Using the minimizing $T_i$, which satisfy (36), we multiply by $T_i$ and sum, using the constraint of (18) and attaining a lower bound to $1 + N_{opt}$:

$$
\sum_{i=1}^{r} \sum_{j=1}^{i} G_{ij} T_j T_i = M \sum_{i=1}^{r} T_i = M \leq 1 + N_{opt}.
$$

Solving (36) for the minimizing $T_i$ gives

$$
T_i = M \sum_{i=1}^{r} G_{ij} T_j.
$$

Summing on $j$ and using (18),

$$
1 = \sum_{i=1}^{r} T_i = M \sum_{i=1}^{r} \sum_{j=1}^{i} G_{ij} T_j,
$$

or from (37),

$$
1 + N_{opt} \geq M = 1 \left\{ \sum_{i=1}^{r} \sum_{j=1}^{i} G_{ij} T_j \right\}.
$$

This completes the proof of (19) in the theorem. The derivation of (20) is strictly parallel and will be omitted. It remains only to prove the assertions made for networks in $D_0$ and in $D_\lambda$. To prove that equality holds on the right in (19) and (20) for networks in $D_0$, it is necessary to show that for such networks it is possible to vary path transmissions independently by varying branch transmissions. In fact we prove a stronger result.

**Lemma 2**

A network in $D$ which has $r = b - v + 2$ has a cutset of $r$ branches each of which is included in just one path. Removal of this cutset divides the network into two parts: a tree connected to $V_1$ (which may reduce to $V_1$ alone), and a tree connected to $V_*$ (which may reduce to $V_*$ alone).

Given Lemma 2, we can set the $r$ transmissions of the branches in the cutset as the $r$ desired path transmissions and set the transmissions of all other branches equal to unity.

To prove Lemma 2, assign weights to nodes and branches from the left, assigning weight 1 to node $V_1$ and then assigning to each branch the weight of its initial node and to each node the sum of the weights of its incoming branches. With this assignment the weight of a node or a branch is clearly the number of routes from the input node $V_1$ to that node or branch.

Choose from each of the $r$ paths the right-most branch of weight 1. This set of branches, $c$ in number, is a cutset, since it interrupts each path. We have $c \leq r$: $c = r$ if, and only if, no branch is selected more than once.

Deleting the cutset of $c$ branches divides the network into two parts, $M_1$ connected to $V_1$ and $M_2$ connected to $V_*$, $M_1$, which contains $b_1$ branches and $v_1$ nodes, is a tree, since it is connected and since all of its nodes are of weight 1, so that there is only one path from $V_1$ to each node. Thus $b_1 = v_1 - 1$, as for any tree.

$M_2$ is connected to $V_*$ and thus includes at least one tree. Let one of the trees included in $M_2$ have $b_2$ branches and $v_2$ nodes, with $b_2 = v_2 - 1$. Then there are two possible situations. i) $M_2$ is a tree. In that case $R = b - v + 2$. Or ii) $M_2$ is larger than a tree, and includes $b_2$ branches beyond the $b_2$ branches in a tree which it includes. In that case $r > b - v + 2$. We will prove the labelled statements.

i) If $M_2$ is a tree, then $b = c + b_1 + b_2 = c + (v_1 - 1) + (v_2 - 1) = c + v - 2$, or $c = b - v + 2$. Since each branch in the cutset connects two trees, it completes just one path, so the number of paths $r = c$, and $r = b - v + 2$.

Q.E.D.;

ii) If $M_2$ contains $b_2$ branches beyond those contained in a tree, then $b = c + b_1 + b_2 + b_2 = c + (v_1 - 1) + (v_2 - 1) + b_2 = c + v - 2 + b_2$, or

$$
c = b - v + 2 - b_2.
$$

(41)

Now each branch among the $b_2$ has weight $\geq 2$ by construction, so it lies on at least two paths. Without these $b_2$ branches, $V_*$ has weight at least $c$, since the $c$ branches in the cutset have weight 1 each and are connected to $V_*$. Adding each of the $b_2$ additional branches adds a weight $\geq 2$ to $V_*$, since each of them is connected to $V_*$ through the tree included in $M_2$. Thus the total weight $r$ of $V_*$ is $r \geq c + 2b_2$. Combining this with (41) gives

$$
r \geq c + 2b_2 = b - v + 2 + b_2 > b - v + 2.
$$

(42)

Q.E.D.;

For a network $M$ which is in $D$ but not in $D_0$, $r > b - v + 2$; and it is impossible to independently vary the path transmissions. For $b - v + 2$ is the cyclomatic number of the graph $M'$ obtained from $M$ by adding a branch $B_{k+1}$ directed from $V_1$ to $V_2$, and is thus the maximum number of linearly independent cycles in a graph-theoretic sense. Thus the set of $r$ cycles in $M'$, each of which consists of a path $R_i$ from $V_1$ to $V_2$, followed by the branch $B_{k+1}$ from $V_1$ to $V_2$, are linearly dependent in the graph-theoretic sense. Therefore, so is the set of the paths themselves in $M$. 

IEEE TRANSACTIONS ON INFORMATION THEORY, JULY 1967
The linear dependence of the $R_p$ implies, by taking logarithms in (17), one or more linear relations between the logarithms of the path transmissions $\log T_i$, leading to constraints of the form

$$\log T_i + \log T_j = \log T_n + \log T_k, \quad \text{or} \quad T_i T_j = T_n T_k. \quad (43)$$

and no selection of values for the branch transmissions $t_k$ can provide independent control of all path transmissions.

It may still be possible to achieve equality in (19) and (20) for a network in $D$ which is not in $D_0$, however, if the optimizing values of the path transmissions happen to satisfy the additional constraints of the form (43) imposed by the network topology. This happens in particular for the networks which are in $D_1$ but not in $D_0$.

**Lemma 3**

Given a network $M$ in $D$, and a network $M'$ in $D_0$. Let $M''$ be constructed by replacing branch $B_i$ in $M$ by the network $M'$. Then the value of the parameter $N_{opt}$ of $M''$ will be the same as the value of the parameter $N_{opt}$ of $M$ if the latter is evaluated using the parameter value $N_{opt}$ of $M'$ for $B_i$. The path transmissions obtained in computing $N'_{opt}$ will lead to the same set of transmissions for the subnetwork $M'$ as are obtained directly in the computation of $N'_{opt}$.

The network $M''$ is equivalent to the network $M$ with some value of the parameter $N_i$ for branch $B_i$ by the argument following (2), i.e., the subnetwork $M'$ is equivalent to some noisy branch $B_i$, and the only question is what its parameter value is. The optimum set of path transmissions for $M''$ must lead to the same transmissions inside $M'$ as does the direct optimization of $M'$. Any other choice would give a larger value to the parameter of $M''$ by Lemma 1.

Lemma 3 completes the proof of the theorem. Lemma 2 covers networks in $D_0$ and Lemma 3 justifies the extension of the results to networks in $D_1$. More practically, it permits the solution of network problems of large order by local reductions—the combining of series or parallel branches, etc.—which greatly reduces the computation. Unfortunately the other tool used for the local reduction of resistive networks—the star-mesh transformation—cannot be used for Gaussian channels, since it leads to transformed branches which have correlated generators. This takes us outside of our present model. Networks with correlated noise present problems which are discussed briefly in a later section.

**Proof of Earlier Results**

The result of (5) follows from the theorem by noting that for a series network $r = 1$, and $||G_i|| = ||G_n||$. Thus,

$$G_{ii} = \prod_{k=1}^{i} (1 + N_k) = 1/G_{ii}^\prime. \quad (44)$$

Equation (6) follows by noting that for a parallel network, $r = b$ and $||G_{ii} - 1||$ is diagonal with elements $G_{ii} = N_i$, so that

$$[G_{ii} - 1]^{-1} = S_i, \quad \sum_{i=1}^{b} \sum_{j=1}^{b} [G_{ii} - 1]^{-1} = \sum_{i=1}^{b} S_i. \quad (45)$$

Equation (11) follows from the evaluation of (20) for the network of Fig. 1. Equation (9) follows by letting $S_i$ approach infinity in (11), for $k = 2$. For larger $k$, the first three branches are combined into an equivalent forward branch of capacity $C_1 + C_2$ and it is combined with the next noisy forward branch and the next noiseless feedback branch in the same way, etc.

Equations (12) and (13) have already been justified. Equation (14) follows by throwing away all but the linear terms, i.e., terms having a single $N_i$ as a factor, in (33). By (25) this reduces the right side and provides a lower bound to $N_{opt}$ or an upper bound to $S_{opt}$. The resulting equations are those for a set of resistors—the noisy branches—with resistance $= N_i$, all in parallel—both the forward and the feedback branches—with the noiseless branches acting as short circuits at the two ends and the conductances $S_i = 1/N_i$ adding.

**Reduction of Another Problem to the Above**

A more general problem concerning networks of Gaussian channels can be reduced to the previous results. Consider the class of two-terminal networks as in $D$ (mentioned previously), but in which each node may supply a different linear combination of the voltages on its incoming branches to each outgoing branch. This model still leaves the operation at the node simple and linear, and provides an increased number of independently controllable path transmissions. Thus it enlarges the class of networks for which explicit solution is possible and for which equality holds in (19) and (20).

As an example, the network shown in Fig. 4 consisting of five vertices connected by four branches forming a directed path from $V_1$ to $V_2$ to $V_3$ to $V_4$ to $V_5$, with three additional branches from $V_1$ to $V_5$, $V_3$ to $V_6$, and $V_2$ to $V_4$ has $b = 7$, $v = 5$, and $r = 5$, and is thus not in $D_0$; it has no two-terminal subnetworks, and is thus not in $D_1$.

The reduction to the former case replaces each node $V_i$ which has $I_i > 1$ incoming branches and $O_i > 1$ outgoing branches by $I_i$ nodes at each of which one of the incoming branches arrives and $O_i$ nodes from each of which one of the outgoing branches leaves, together with $I_i O_i$ noiseless branches connecting each of the $I_i$ arrival nodes to each of the $O_i$ departure nodes. The added noiseless branches permit the formation of the desired different linear combinations of input branch voltages for each output branch. In the case of the five-node network already described, replacing $V_5$ by 4 nodes and 4 branches, as shown in Fig. 5, adds 3 nodes, 4 branches and no paths. Thus $b - v + 2 = 7 + 4 - (5 + 3) + 2 = 5 = r$, and the resulting net is in $D_0$. 


Each accepts inputs of power to a branch. No analog to be advantageous to use less than the maximum possible input power. This is always valid as a model of a physical channel so long as the channel is always used and that they are free to allocate their limited resources between the different forward and feedback channels in the way which maximizes the resulting $S_{\text{opt}}$ of $F$. This freedom may even extend to deciding how large $k$ should be, if the available forward and feedback channels have infinite bandwidth.

In the case of noiseless feedback $k = \infty$ is best and gives the result of (10). When the feedback is noisy, evaluating what $S_{\text{opt}}$ is the best division of limited power gives and how $S_{\text{opt}}$ depends on $k$ involves a great deal of numerical solutions of linear equations subject to constraints of the form of (43). Even evaluating the upper bound to $S_{\text{opt}}$ of (20) is not easy. Lower bounds to $S_{\text{opt}}$ which are more meaningful than that of (12) can be computed, however, by making use of iteration of networks for which $k = 2$, as shown in Fig. 7.

For the first level network, we assume that the two forward branches have equal signal-to-noise ratio, since this maximizes $S_{\text{opt}}$ in (11), for fixed $S_{\text{odd}}$. Denoting their common signal-to-noise ratio as $S_1$, the feedback branch as $S_2$, and the resulting $S_{\text{opt}}$ as $S_3$, we have from (11)

$$S_3 = 2S_1 + \frac{S_1 S_2}{(1 + S_1)^2 + S_2}. \quad (48)$$

We now consider the second-level network to consist of two forward branches of ratio $S_1$ and a feedback branch of ratio $S_2$. The resulting $S_{\text{opt}}$ is denoted by $S_3$, and we have for the $k$th level

$$S_{2k+1} = 2S_{2k} + \frac{S_{2k-1} S_{2k}}{(1 + S_{2k-1})^2 + S_{2k}},$$
$$S_{\text{odd}} = 2^k S_1,$$
$$S_{\text{even}} = S_2 + 2S_{4(k-1)} + \cdots + 2^{k-1} S_4.$$
which gives us not the best strategy, we may pick a division of bound to $S_{2k},$ can be given as an equation. Although it is of curves rather than an equation. A much weaker lower mum curves.

for, e.g., $S$ and $S_{2k+1}$ will take more of it. Choosing all combinations of values end points fixed, and fixing $S$ is optimum in the sense that by keeping the quadratic in the positive square root of each side of (51) gives a we may fix $S_1$ in $S_{2k},$ and solving it enables us to start with a desired $S_{odd}$ and to generate $S_{2k+1},$ for any $k.$ Alternatively, with more for higher-numbered branches, receives an equal amount. Seven is divided unequally, and $S$ and $S_{2k+1}$ is divided unequally, however, with more for higher-numbered branches, in the optimum case. The optimum allocation can be determined by solving (49) for $S_{2k}$:

$$S_{2k} = \frac{S_{2k+1} - 2S_{2k-1}}{1 + S_{2k+1} - S_{2k-1}}. \quad (50)$$

Now differentiating $S_{2k} + 2S_{2k-2}$ with respect to $S_{2k-1}$ for fixed $S_{2k+1}$ and $S_{2k-2}$ and setting the result equal to zero gives

$$\frac{S_{2k-2}(1 + S_{2k-3})^2}{[(1 + S_{2k-3})^2 - (1 + S_{2k-4})]^{\frac{1}{2}}} = (1 + S_{2k-1})(1 + S_{2k-3})^2 - (1 + S_{2k+1})(2 - 3S_{2k-3}) + (1 + S_{2k+1})^2. \quad (51)$$

For given $S_{2k-1}$ and $S_{2k-2},$ this equation is quadratic in $S_{2k-3}$, and solving it enables us to start with a desired $S_{1}$ and $S_{2}$ and to generate $S_{2k+1}$ for any $k.$ Alternatively, we may fix $S_{2k+1}$ and $S_{2k-1}$ and solve for $S_{2k-3}.$ Taking the positive square root of each side of (51) gives a quadratic in $S_{2k-3},$ and we can proceed from given values of $S_{2k+1}$ and $S_{2k-1}$ down to $S_1.$ In either case the resulting set of values is optimum in the sense that by keeping the end points fixed, and fixing $k,$ any other division of $S_{odd}$ will take more of it. Choosing all combinations of values for, e.g., $S_1$ and $S_3 > 2S_1$ generates the full set of optimum curves.

The result, unfortunately, must be displayed as a set of curves rather than an equation. A much weaker lower bound to $S_{opt}$ can be given as an equation. Although it is not the best strategy, we may pick a division of $S_{even}$ which gives us a fixed $c$ such that

$$1 + S_j = c(1 + S_{j-1})^2, \quad \text{odd } j. \quad (52)$$

Then from (50),

$$S_{2k} = \frac{S_{2k+1} - 2S_{2k-1}}{1 - c} \quad (53)$$

and from (49) and (52),

$$S_{even} = \frac{S_{2k+1} - 2S_{1}}{1 - c} \leq \frac{S_{opt} - S_{odd}}{1 - c}, \quad (54)$$

since $S_{2k+1} \leq S_{opt}.$ We also have from repeated application of (52)

$$c(1 + S_{opt}) \geq c(1 + S_{2k+1}) = c^2(1 + 2S_{odd})^2. \quad (55)$$

Together, (54) and (55) provide a useful analytic lower bound to $S_{opt}.$

**REFERENCES**


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Abstract—This paper discusses networks (directed graphs) having one input node, one output node, and an arbitrary number of intermediate nodes, whose branches are noisy communications channels, in which the input to each channel appears at its output corrupted by additive Gaussian noise. Each branch is labeled by a non-negative real parameter which specifies how noisy it is. A branch originating at a node has as input a linear combination of the outputs of the branches terminating at that node.

The channel capacity of such a network is defined. Its value is bounded in terms of branch parameter values and procedures for computing values for general networks are described. Explicit solutions are given for the class $D_0$ which includes series-parallel and simple bridge networks and all other networks having $r$ paths, $b$ branches, and $v$ nodes with $r = b + v - 2$, and for the class $D_1$ of networks which is inductively defined to include $D_0$ and all networks obtained by replacing a branch of a network in $D_1$ by a network in $D_1$.

The general results are applied to the particular networks which arise from the decomposition of a simple feedback system into successive forward and reverse (feedback) channels. When the feedback channels are noiseless, the capacities of the forward channels are shown to add. Some explicit expressions and some bounds are given for the case of noisy feedback channels.
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