REALIZABILITY CONDITIONS FOR NEW CLASSES
OF THREE-ELEMENT-KIND NETWORKS

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REALIZABILITY CONDITIONS FOR NEW CLASSES OF THREE-ELEMENT-KIND NETWORKS

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Abstract

The assumption of ideal elements places a practical limitation on the theory of Modern Network Synthesis. To extend the theory to cope with more realistic elements, several new classes of three-element-kind networks have been studied. Realizability conditions have been derived for driving-point impedances of these networks. This is achieved through physical arguments leading to a property of driving-point functions which we call the difference function. It is shown that the positiveness of the difference function over an appropriate portion of the s-plane represents a necessary and sufficient condition for a rational function to be realizable as the driving-point impedance of a given network class.

The most notable contribution is perhaps the establishment of a general theory for networks composed of resistances and reactances with semiuniform loss (i.e., all inductors have one Q; all capacitors have another Q). This theory is almost as comprehensive as that now available for RLCT networks.
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I. INTRODUCTION

1.1 STATEMENT OF THE PROBLEM

RLC networks are fairly well understood, because of the formulation of the necessary and sufficient PR conditions on their driving-point immittances. Very little has been done, however, to extend this theory to networks composed of more general lumped, bilateral and linear circuit elements.

The objective of this report is to establish necessary and sufficient conditions for the realizability of certain general classes of three-element-kind networks and to determine when the well-known RLC realization procedures can be extended to handle these new circuit elements.

The motivation for this work is twofold.

(i) A general three-element-kind synthesis theory would enable one to better predict the effects of parasitics in RLC networks, and therefore to better compensate for them.

(ii) Materials technology is now producing many new circuit elements. A general three-element-kind synthesis theory would enable one to estimate the potentialities and limitations of networks employing the new elements.

1.2 HISTORY OF THE PROBLEM

The problem of synthesizing two-element-kind networks was first successfully attacked, in 1924, by Foster, who established the realizability conditions for LC networks. Bode showed subsequently that any two-element-kind network could be obtained from LC networks by simple frequency transformations and demonstrated thereby that the problem of synthesizing with two arbitrary building blocks was equivalent to that of synthesizing with inductances and capacitances.

The problem of synthesizing three-element-kind networks was first successfully treated, in 1931, by Brune, who established the realizability conditions for RLC networks. In 1939, Darlington showed that a simple frequency transformation enables one to establish realizability conditions for networks composed of resistances and uniformly lossy reactive elements (that is, all inductors and all capacitors have the same loss factor, or $Q$). Darlington also showed how one could account for losses of semi-uniform type (all inductors have one loss factor; all capacitors have another loss factor) in the insertion loss theory of filter design. In 1962, Foster identified the triplets of building blocks for which realizability conditions can be derived from the RLC theory by means of frequency transformations.

At the present time, no general theory exists for synthesis with three or more arbitrary components.
1.3 SUMMARY AND RESULTS

Two problems are considered in this report.

(i) The establishment of a theory of synthesis for networks composed of ideal resistances and reactances having losses of the semi-uniform type.

(ii) The establishment of a theory of synthesis for networks composed of three arbitrary linear building blocks.

The first problem is of interest for two reasons. First, a solution to the problem would enable engineers to cope, in a more realistic manner, with the inevitable losses associated with reactances. Second, the problem represents the simplest one of the second type that is still unsolved.

The second problem is also of interest in connection with evaluating the new circuit elements now being produced by materials technology.

The main results of this research are the following.

(i) A complete solution of the first problem is obtained. This is achieved by introducing a property of driving-point functions which is called the difference function. The difference function plays a role in the first problem analogous to that played by the real part function in the normal (RLC) case. It is shown that the positiveness of the difference function over an appropriate portion of the s-plane represents a necessary and sufficient condition for a rational function to be realizable as the driving-point function of a network incorporating semi-uniform losses. Most of the familiar RLC synthesis procedures are extended to cover semi-uniform losses by employing the difference function in much the same manner that one employs the real part in the normal (RLC) case.

(ii) A partial solution of the second problem is found. This is achieved by extending the concept of the difference function. It is shown that for several new classes of networks the positiveness of the difference function over appropriate regions of the s-plane constitutes both a necessary and sufficient condition for the realizability of their driving point functions. For the general case (i.e., for networks composed of any three linear circuit elements), however, it is shown that the positiveness of the difference functions is not sufficient to describe the driving point functions of the network class.

1.4 NOTATION

Networks composed of ideal inductances, ideal capacitances, resistances and ideal transformers are sometimes referred to as "RLC networks"; we prefer to indicate the transformer explicitly and will therefore use the notation "RLCT networks". More generally, networks composed of multiples of three linear building blocks, \( z_1(s), z_2(s), z_3(s), \) and ideal transformers, will be called \( Z_1Z_2Z_3T \) networks.

The class of networks composed of multiples of lossy inductors \( (z_1 = s + \alpha_L) \), lossy capacitors \( (z_2 = (s+\alpha_C)^{-1}) \), resistors \( (z_3 = 1) \), and ideal transformers, will be
called RL'C'T networks. The purpose of the prime and the double prime is to indicate that the dissipation ratios \( \alpha_L \) and \( \alpha_C \) in general, are, nonzero and unequal (semi-uniform losses).

Instead of working with the parameters \( \alpha_L \) and \( \alpha_C \) in the RL'C'T case, it is convenient to introduce the related parameters

\[
\alpha = \frac{1}{2} (\alpha_L - \alpha_C) \tag{1}
\]

and

\[
\delta = \frac{1}{2} (\alpha_L + \alpha_C). \tag{2}
\]

Furthermore, without loss of generality, we shall assume throughout this report that the loss ratio of the inductors exceeds the loss ratio of the capacitors so that

\[
\alpha > 0. \tag{3}
\]
II. PROPERTIES OF $Z_1Z_2T$ NETWORKS

Before we discuss the three-element-kind case, it is worth having a good understanding of the two-element-kind case, which will be considered here.

It should be made clear that our objective is not to develop realization procedures for generalized reactance networks ($Z_1Z_2T$ networks), as such procedures already exist. The purpose is to help the reader focus on properties of these networks that will become important later.

Although these results apply to reactance networks in general, the development is illustrated by means of $L'C''T$ networks. Analogies between the ordinary $LCT$ case and the $L'C''T$ case have been made throughout to help clarify the material.

2.1 NATURAL FREQUENCIES

The locus of open-circuit and short-circuit natural frequencies of $Z_1Z_2T$ networks depends on the nature of the circuit elements $Z_1$ and $Z_2$ alone. In other words, any $Z_1Z_2T$ network, irrespective of topology and element values, must have driving-point impedances with poles and zeros located along specific contours of the $s$-plane, determined solely by the circuit elements.

In the $LCT$ case it is well known that natural frequencies must be located along the $j\omega$-axis. To see how this comes about and simultaneously derive the corresponding locus in the $L'C''T$ case, we shall use an approach suggested by Thornton\textsuperscript{4} based on conservation of complex power.

With reference to the experiment shown in Fig. 1, we can write

$$V I^* = Z[I]^2 = z_1 \sum_{all \, z_1} |I_1|^2 + z_2 \sum_{all \, z_2} |I_2|^2 + \sum_{all \, V_T} V_T I_T^*$$  \hspace{1cm} (4)

or

$$Z(s) = k_1(s)z_1(s) + k_2(s)z_2(s),$$  \hspace{1cm} (5)

where $k_1(s)$ and $k_2(s)$ are non-negative and real for any $s$.

![Fig. 1. General $Z_1Z_2T$ driving-point impedance, $Z$.](image)
It follows that the zeros of $Z(s)$ are restricted to the locus described by

$$k_1 z_1 + k_2 z_2 = 0; \text{ for } k_1 \text{ and } k_2 \text{ non-negative and real.} \quad (6)$$

A corresponding argument on the admittance basis leads to the following locus for the zeros of $Y(s)$ or the poles of $Z(s)$:

$$k_1'y_1 + k_2'y_2 = 0; \text{ for } k_1' \text{ and } k_2' \text{ non-negative and real.} \quad (7)$$

That the loci described by Eqs. 6 and 7 are the same is easily seen by multiplying Eq. 7 by $z_1 z_2$.

The problem of determining the pole-zero locus can be made more familiar by rewriting Eq. 6 as follows

$$1 + k \frac{z_1}{z_2} = 0 \text{ for } k > 0. \quad (8)$$

This is just the well-known root locus problem of control theory, and techniques for determining this contour are described by nearly every textbook in this field. The details of obtaining the root loci, used as examples throughout this report, will therefore not be included.

The pole-zero contours of driving-point functions of the ordinary LCT case and the L'C''T case are shown in Fig. 2.

![Fig. 2. Pole-zero contour of driving-point impedances for (a) LCT networks ($z_1 = s$ and $z_2 = \frac{1}{s}$); (b) L'C''T networks ($z_1 = s + \alpha_L$ and $z_2 = \frac{1}{s + \alpha_C}$).](image)

2.2 PHASE RELATIONS BETWEEN $Z_1$ AND $Z_2$ ALONG THE POLE-ZERO CONTOUR

The root locus problem of the previous section is clearly equivalent to finding all
points of the s-plane where \( z_1(s) \) and \( z_2(s) \) are 180 degrees out of phase. This vector interpretation is illustrated in Fig. 3 for the LCT case and the L'C"T case.

![Fig. 3. Illustrating the 180° phase difference of the \( z_1 \) and \( z_2 \) vectors of (a) LCT networks; (b) L'C"T networks.](image)

If we denote the pole-zero contours shown in Fig. 2a and 2b by \( C_{LC} \) and \( C_{L'C''} \), respectively, then

\[
\frac{\text{Re} \ z_1(s)}{\text{Im} \ z_1(s)} = \frac{\text{Re} \ z_2(s)}{\text{Im} \ z_2(s)} = 0 \text{ for } s \in C_{LC} \text{ and } \omega \neq 0
\]

and

\[
\frac{\text{Re} \ z_1(s)}{\text{Im} \ z_1(s)} = \frac{\text{Re} \ z_2(s)}{\text{Im} \ z_2(s)} = \frac{\alpha}{\omega} \text{ for } s \in C_{L'C''} \text{ and } \omega \neq 0.
\]

The unique relationship between the real and imaginary parts of \( z_1 \) and \( z_2 \) along the pole-zero contour will be important in the development of later chapters.

2.3 GENERALIZED ALTERNATION PROPERTY

The alternation property of poles and zeros of driving-point immittances of ordinary reactance networks is well known. An analogous property will now be derived for the driving-point impedance \( Z(s) \) of the \( Z_1Z_2T \) network of Fig. 4a. Our strategy will be a combination of impedance multiplication and frequency transformation.

It is clear that multiplying each branch impedance of the \( Z_1Z_2T \) network of Fig. 4a by \( z_2^{-1} \) leads to the situation shown in Fig. 4b. Figure 4c illustrates the network obtained by replacing each \( \frac{z_1}{z_2} \) -element of Fig. 4b by a 1-henry inductance.

*The approach illustrated by Fig. 4 was suggested by Professor Harry B. Lee.
By comparing the last two figures, it is readily seen that

\[
\frac{1}{z_2} Z(s) = Z_{RL}(\frac{z_1}{z_2}) ,
\]  

(11)

or

\[
Z(s) = z_2 \cdot Z_{RL}(\frac{z_1}{z_2}) = Kz_2 \frac{\left(\frac{z_1}{z_2} + n_1\right) \left(\frac{z_1}{z_2} + n_2\right) \ldots}{\left(\frac{z_1}{z_2} + d_1\right) \left(\frac{z_1}{z_2} + d_2\right) \ldots} ,
\]  

(12)

where \(0 < n_1 < d_1 < n_2 < d_2 \ldots\), and \(K > 0\). [This last result is also implied by a theorem due to Bode.]
To see how this result implies the alternation of poles and zeros in the L'C"T case, let us determine the roots of

\[ \frac{z_1}{z_2} + m = 0 \quad \text{for} \quad z_1 = s + \alpha_L, \quad z_2 = \frac{1}{s + \alpha_C}, \quad \text{and} \quad m \geq 0. \tag{13} \]

Equation 13 can be written

\[ s^2 + (\alpha_L + \alpha_C)s + \alpha_L\alpha_C + m = 0. \tag{14} \]

The roots of this equation are

\[ s_{1,2} = -\delta \pm \sqrt{\alpha^2 - m}, \tag{15} \]

where \( \delta = \frac{1}{2} (\alpha_L + \alpha_C) \), and \( \alpha = \frac{1}{2} (\alpha_L - \alpha_C) \).

Setting \( m = n_1, d_1, n_2, d_2, \ldots \) etc., in expression (15) leads to pole-zero patterns of the type shown in Fig. 5 which are representative illustrations of the generalized alternation property for L'C"T networks. Notice how the driving-point impedance must have either a pole at \( s = -\alpha_C \) or a zero at \( s = -\alpha_L \).

Similar properties can be derived for driving-point admittances by a dual approach.

### 2.4 FOSTER AND CAUER-LIKE L'C" NETWORKS

Certain necessary conditions on driving-point impedances of L'C"T networks have been established. We have found that in order for a driving-point impedance \( Z(s) \) to belong to this function class, it must be a rational function, have a positive constant multiplier, and have alternating poles and zeros as shown in Fig. 5. Let us call such a function an L'C" function.

Having established this, it seems logical to ask whether the converse statement is true. To show that this is the case is the objective of this section. In particular, it will be shown that both Foster-like and Cauer-like realizations can be used to realize any L'C" function.

The starting point for making a Foster-like realization is well known; it is always a partial fraction expansion. For the case that we are now considering a partial fraction expansion will be of the form

\[ Z(s) = k_L(s+\alpha_L) + \frac{k_C}{s + \alpha_C} + \sum_{i=1}^{m} \frac{k_{i-}}{s + \sigma_{i}} + \frac{k_{i+}}{s + \sigma_{i}^{*}} \tag{16} \]
The various residues are most easily determined by the graphical interpretation based on the formula

$$k_v = [(s - s_v) Z(s)]_{s = s_v}. \quad (17)$$

As an example, consider the evaluation of the residues associated with a pair of real poles symmetrically located with respect to $s = -\delta$ for the function represented by Fig. 6.

It is readily seen that both residues must be real, the residue of the pole to the right of $s = -\delta$ being positive, the residue to the left being negative. Furthermore, we see that if the residue at $-\sigma_1$ is equal to $k_1$, then the residue at $-\sigma_1'$ must equal

$$k_1' = \frac{\sigma_1 - \alpha_C}{\sigma_1' - \alpha_C} k_1. \quad (18)$$
The two terms corresponding to these poles in the partial fraction expansion can therefore be combined to give

\[
z_1(s) = \frac{k_1}{s + \sigma_1} + \frac{k_1'}{s + \sigma_1'} = \frac{k(s + \sigma_L)}{s^2 + (\sigma_1 + \sigma_1')s + \sigma_1 \sigma_1'},
\]

where

\[
k = k_1 \frac{\sigma_1' - \sigma_1}{\sigma_1' - \sigma_C} > 0,
\]

and

\[
\sigma_1 + \sigma_1' = \sigma_L + \sigma_C.
\]

A similar approach can be used to determine the residues along the \(\delta\)-axis. In this connection it is more convenient to evaluate the residues of \(\frac{Z(s)}{s + \sigma_L}\), rather than \(Z(s)\),
because of the resulting symmetry of the pole-zero pattern with respect to the \( \delta \)-axis. The residues of all \( \delta \)-axis poles of \( j\omega \frac{Z(s)}{s + \alpha_L} \) are readily seen to be positive.

Now, combining the terms in the partial fraction expansion (19), one obtains

\[
Z(s) = k_L(s + \alpha_L) + \frac{k_C}{s + \alpha_C} + \sum_i \frac{k_i(s + \alpha_L)}{s^2 + (\alpha_L + \alpha_C)s + \alpha_L \alpha_C + m_i},
\]  

(20)

where \( k_L, k_C, k_i, \) and \( m_i \) are all non-negative.

It follows immediately that a Foster-like realization of the form shown in Fig. 7 can be used to realize any \( L' C'' \) function.

Of course, the canonical form shown in Fig. 7 is not the only possible realization of an \( L' C'' \) function. It is easily shown that any of the well-known canonical LC networks can be extended to the \( L' C'' \) case. An example of a Cauer-like realization is shown in Fig. 8.

![Fig. 7. Foster-like realization of a general \( L' C'' \) driving-point impedance.](image)

![Fig. 8. Cauer-like realization of a general \( L' C'' \) driving-point impedance.](image)

The driving-point impedance of the ladder realization shown in Fig. 8 can be represented in the familiar continued fraction expansion as follows:

\[
Z(s) = L_1(s + \alpha_L) + \frac{1}{C_2(s + \alpha_C) + \frac{1}{L_3(s + \alpha_L)}} + \cdots + \frac{1}{L_{n-1}(s + \alpha_L) + \frac{1}{C_n(s + \alpha_C)}},
\]  

(21)

Therefore, in order to arrive at a ladder network by the well-known division and inversion process, it is easily seen that one must first factor the impedance function as shown below:

\[
Z(s) = \frac{A_{2n-1}(s + \alpha_C)^n - 1(s + \alpha_L)^n + A_{2n-3}(s + \alpha_C)^{n-2}(s + \alpha_L)^{n-1} + \cdots + A_3(s + \alpha_C)(s + \alpha_L)^2}{B_{2n}(s + \alpha_C)^n + B_{2n-2}(s + \alpha_C)^{n-1}(s + \alpha_L)^{n-1} + \cdots + B_2(s + \alpha_C)(s + \alpha_L) + B_0},
\]  

(22)
provided that \( Z(s) \) has a zero at \( s = \alpha_L' \), and

\[
Z(s) = \frac{A_{2n}(s+\alpha_L)^n(s+\alpha_C)^n + A_{2n-2}(s+\alpha_L)^{n-1} + \ldots + A_2(s+\alpha_L)(s+\alpha_C) + A_0}{B_{2n-1}(s+\alpha_L)^n + B_{2n-3}(s+\alpha_C)^n + \ldots + B_1(s+\alpha_C)^2(s+\alpha_L) + B_0(s+\alpha_C)^2},
\] (23)

provided that \( Z(s) \) has a pole at \( s = -\alpha_C \).

Notice that from the factored form of \( Z(s) \) an alternative test for determining whether \( Z(s) \) is an \( L'C'' \) function suggests itself. Namely, \( Z(s) \) is an \( L'C'' \) function if and only if

\[
Z'(s) = Z(s) \bigg|_{\alpha_L' = \alpha_C'} = 0
\]

represents a reactance function.

2.5 \( L'C''T \) TWO-PORTS

Necessary conditions in order for a set of open-circuit impedance parameters to correspond to an \( L'C''T \) two-port are readily obtained by the standard arguments used in connection with the experiment shown in Fig. 9. The results are the following:

(a) **Along the \( \delta \)-axis:**

Poles must be simple with residues satisfying the conditions

\[
\text{Arg } k_{11}(v) = \text{Arg } k_{22}(v) = \text{Arg } k_{12}(v) = -\tan^{-1} \frac{\alpha}{\omega};
\]

Furthermore,

\[
\text{Re } k_{11}(v) > 0, \text{ Re } k_{22}(v) > 0 \quad \text{and} \quad [\text{Re } k_{11}(v)]^2 - [\text{Re } k_{12}(v)]^2 > 0.
\] (24)

The multipliers associated with a pole at infinity must satisfy these conditions in the limit.

(b) **Along the real axis:**

Poles in the interval

\[-\alpha_C > \sigma > -\delta\] (25)

must be simple with residues satisfying the conditions

\[
k_{11}(p) > 0, \quad k_{22}(p) > 0
\]

\[
k_{11}(p)k_{22}(p) - k_{12}(p)^2 > 0.
\] (26)

From Eq. 18 it follows that poles in the interval
must be simple with residues satisfying the conditions

$$k_{11}(p) \leq 0, \quad k_{22}(p) \leq 0$$

$$k_{11}(p)k_{22}(p) - (k_{12}(p))^2 \geq 0.$$

(c) At the point $s = -\delta$:

Poles must be of double order and correspond to open circuit impedances of the form

$$z(s) = \frac{k_2}{(s + \delta)^2} + \frac{k_1}{s + \delta} + R(s),$$

where the $k_2$'s satisfy (26), and the corresponding $k_1$'s have the property

$$k_1 = \frac{k_2}{\alpha}.$$

Fig. 9. Circuit relevant to the establishment of the residue conditions associated with L'C'T two-ports.

To show that these conditions are also sufficient, in order to describe an L'C'T two-port, we shall outline a realization procedure based on an extension of the Foster development.

Typical partial fraction expansions of the open-circuit parameters in which complex conjugate terms are combined, will read as follows:

$$z_{11}(s) = k_{11}^{11}(s + \alpha_L) + \frac{k_{11}^{11}}{s + \alpha_C} + \sum_{\nu=1}^{n} \frac{2\text{Re} k_{\nu}^{11}(s + \alpha_L)}{s^2 + (\alpha_L + \alpha_C) + \delta^2 + \omega_{\nu}^2} + \sum_{\gamma=1}^{m} \frac{k_{\gamma}^{11}(s + \alpha_L)}{s^2 + (\sigma + \sigma') \gamma + \sigma \gamma},$$

$$z_{12}(s) = k_{12}^{12}(s + \alpha_L) + \frac{k_{12}^{12}}{s + \alpha_C} + \sum_{\nu=1}^{n} \frac{2\text{Re} k_{\nu}^{12}(s + \alpha_L)}{s^2 + (\alpha_L + \alpha_C) + \delta^2 + \omega_{\nu}^2} + \sum_{\gamma=1}^{m} \frac{k_{\gamma}^{12}(s + \alpha_L)}{s^2 + (\sigma + \sigma') \gamma + \sigma \gamma},$$

$$z_{21}(s) = k_{21}^{21}(s + \alpha_L) + \frac{k_{21}^{21}}{s + \alpha_C} + \sum_{\nu=1}^{n} \frac{2\text{Re} k_{\nu}^{21}(s + \alpha_L)}{s^2 + (\alpha_L + \alpha_C) + \delta^2 + \omega_{\nu}^2} + \sum_{\gamma=1}^{m} \frac{k_{\gamma}^{21}(s + \alpha_L)}{s^2 + (\sigma + \sigma') \gamma + \sigma \gamma},$$

$$z_{22}(s) = k_{22}^{22}(s + \alpha_L) + \frac{k_{22}^{22}}{s + \alpha_C} + \sum_{\nu=1}^{n} \frac{2\text{Re} k_{\nu}^{22}(s + \alpha_L)}{s^2 + (\alpha_L + \alpha_C) + \delta^2 + \omega_{\nu}^2} + \sum_{\gamma=1}^{m} \frac{k_{\gamma}^{22}(s + \alpha_L)}{s^2 + (\sigma + \sigma') \gamma + \sigma \gamma}.$$
\[
z_{22}(s) = k_{L}^{22}(s+\alpha_{L}) + \frac{k_{C}^{22}}{s+\alpha_{C}} + \sum_{\nu=1}^{n} \frac{2\text{Re} \ k_{\nu}^{22}(s+\alpha_{L})}{s^{2}+(\alpha_{L}+\alpha_{C})s+\delta^{2}+\omega_{\nu}^{2}} + \sum_{\gamma=1}^{m} \frac{k_{\gamma}^{22}(s+\alpha_{L})}{s^{2}+(\sigma_{\gamma}+\sigma'_{\gamma})s+\sigma_{\gamma}\sigma'_{\gamma}}
\]

\[
z_{12}(s) = k_{L}^{12}(s+\alpha_{L}) + \frac{k_{C}^{12}}{s+\alpha_{C}} + \sum_{\nu=1}^{n} \frac{2\text{Re} \ k_{\nu}^{12}(s+\alpha_{L})}{s^{2}+(\alpha_{L}+\alpha_{C})s+\delta^{2}+\omega_{\nu}^{2}} + \sum_{\gamma=1}^{m} \frac{k_{\gamma}^{12}(s+\alpha_{L})}{s^{2}+(\sigma_{\gamma}+\sigma'_{\gamma})s+\sigma_{\gamma}\sigma'_{\gamma}}
\]

By following the standard procedure\textsuperscript{6} the open-circuit parameters can be readily implemented by an L'C'T two-port, provided the residue conditions are satisfied.
III. PROPERTIES OF RL'C''T NETWORKS

Necessary and sufficient conditions for a function to be the driving-point impedance of an RLCT network were established, in 1931, by O. Brune. We shall now establish a similar set of conditions for driving-point functions of RL'C''T networks.

3.1 COMPLEX POWER CONSIDERATIONS

The following physical considerations yield valuable insight into the property of RLCT impedances which asserts that

\[ \text{Re} \left[ Z(j \omega) \right] \geq 0. \]  \hspace{1cm} (34)

Consider the experiment shown in Fig. 10. Clearly no power is dissipated in the lossless coupling network for \( s = j \omega \). Therefore all of the power that is supplied at the input must end up in the load. That is, we have

\[ |I|^2 \text{Re} \left[ Z(j \omega) \right] = \frac{|V_2|^2}{1} \]  \hspace{1cm} (35)

or equivalently

\[ \text{Re} \left[ Z(j \omega) \right] = \frac{|V_2|^2}{|I_1|^2}. \]  \hspace{1cm} (36)

![Fig. 10. Resistance-terminated lossless network relevant to the physical power considerations of Section 3.1.](image)

Thus \( \text{Re} \left[ Z(j \omega) \right] \) corresponds to the power dissipated in the resistor for a one ampere drive and, therefore, is non-negative.

Similar insight into RL'C''T impedances can be gained by considering the corresponding experiment shown in Fig. 11. With reference to this experiment one can write

\[ \text{Re} \left[ P_{IN} \right] = \text{Re} \left[ P_N \right] + \text{Re} \left[ P_{OUT} \right] \]  \hspace{1cm} (37)

or equivalently

\[ \text{Re} \left[ P_{IN} \right] - \text{Re} \left[ P_N \right] = \text{Re} \left[ P_{OUT} \right]. \]  \hspace{1cm} (38)

where
$P_{IN} = \text{the applied complex power}$

$P_N = \text{the complex power dissipated in the coupling network}$

$P_{OUT} = \text{the complex power delivered to the 1Ω load resistance.}$

Fig. 11. Resistance-terminated semi-uniformly lossy network relevant to the physical power considerations of Section 3.1.

Clearly, we have

$$P_{IN} = |I|^2 Z$$

$$P_{OUT} = \frac{|V_2|^2}{1}. \quad (39)$$

Moreover, the following observations\(^9\) are pertinent for excitation frequencies on the $\delta$-axis \[ s = -\frac{1}{2} (\alpha_L + \alpha_C) \];

(i) the imaginary part of $P_{IN}$ is dissipated entirely within the coupling network.

(ii) the real and the imaginary parts of the complex power $P_i$ dissipated in each branch of the coupling network satisfy the relationship (see Section 2.2)

$$\frac{\text{Re} \{ P_i \}}{\text{Im} \{ P_i \}} = \frac{\text{Re} \{ z_{L,i}(-\delta+j\omega) \}}{\text{Im} \{ z_{L,i}(-\delta+j\omega) \}} = \frac{\text{Re} \{ z_{C,i}(-\delta+j\omega) \}}{\text{Im} \{ z_{C,i}(-\delta+j\omega) \}} = \frac{-\alpha}{\omega}. \quad (41)$$

Hence

$$\text{Re} \{ P_i \} = \frac{-\alpha}{\omega} \text{Im} \{ P_i \}$$

so that the real and imaginary parts of $P_N$ satisfy

$$\text{Re} \{ P_N \} = \frac{\alpha}{\omega} \text{Im} \{ P_N \}. \quad (42)$$

Consequently, we have

$$\text{Re} \{ P_N \} = \frac{\alpha}{\omega} \text{Im} \{ P_N \} = \frac{\alpha}{\omega} \text{Im} \{|I|^2 Z\} = \frac{\alpha}{\omega} |I|^2 \text{Im} \{ Z \}. \quad (43)$$

Substitution of Eqs. 39-41 in Eq. 38 yields

$$\left\{ \text{Re} \{ Z(-\delta+j\omega) \} - \frac{\alpha}{\omega} \text{Im} \{ Z(-\delta+j\omega) \} \right\} |I|^2 = P_{OUT}. \quad (44)$$
Thus the function

\[ D_Z = \text{Re}[Z(-\delta+j\omega)] - \frac{\partial}{\partial \omega} \text{Im}[Z(-\delta+j\omega)] \]  

(45)

is non-negative, since it represents the power dissipated in the 1-Ω resistor.

Just as one can generalize the result of Eq. 36 to RLCT networks that contain an arbitrary number of resistors, so one can also extend the result of Eq. 45 to RL'C''T networks that contain an arbitrary number of resistors. The extension is completely straightforward; accordingly we omit the details.

Because the \( D_Z \)-function which appears in (45) represents the difference between the power supplied at the driving-point terminals of Fig. 11 and that dissipated in the coupling network, we refer to it as the impedance derived difference function. For frequencies not on the \( \delta \)-axis we shall define \( D_Z \) as follows

\[ D_Z(s) = \text{Re}[Z(s)] - \frac{\partial}{\partial \omega} \text{Im}[Z(s)] \]  

(46)

We have thus obtained the following result: The impedance derived difference function for RL'C''T networks must be non-negative along the \( \delta \)-axis, that is,

\[ D_Z(-\delta+j\omega) \geq 0. \]  

(47)

Note that on the real axis the \( D_Z \)-function is defined as follows:

\[ D_Z(-\delta+j0) = \lim_{\omega \to 0} \frac{D_Z(-\delta+j\omega)}{\omega} = \text{Re}[Z(-\delta)] - a \frac{\partial}{\partial \omega} \left\{ \text{Im}[Z(-\delta)] \right\}. \]

In the sequel it will be seen that the impedance derived difference function plays a role in the theory of RL'C''T networks fully as important as that played by the impedance real part in the theory of RLCT networks.

3.2 PROPERTIES OF THE IMPEDANCE DERIVED DIFFERENCE FUNCTION

In the RLCT case one recalls that driving-point impedances have a positive real part not only on the \( j \)-axis, but also in the entire region of the \( s \)-plane to the right of the \( j \)-axis. We next show that a similar situation obtains in the RL'C''T case. Specifically, we show that the impedance derived difference function is not only positive along the \( \delta \)-axis, but also in the entire region of the \( s \)-plane to the right of the \( \delta \)-axis.

The property in question can easily be derived from the familiar energy expression for the driving-point impedance,

\[ Z(s) = z_L \frac{\sum |I_L|^2}{|I|^2} + z_C \frac{\sum |I_C|^2}{|I|^2} + z_R \frac{\sum |I_R|^2}{|I|^2} \]  

(48)
where
\[ z_{L_1}(s) = s + \alpha_L \]
\[ z_{C_1}(s) = \frac{1}{s + \alpha_C} \]
\[ z_R(s) = 1. \]

By using this expression, the impedance derived difference function can be expressed in terms of the \( D_z \)-functions of the circuit elements as follows:

\[
D_Z(s) = \frac{\sum |I_{L_1}|^2}{|I|^2} D_{Z_{L_1}} + \frac{\sum |I_{C_1}|^2}{|I|^2} D_{Z_{C_1}} + \frac{\sum |I_R|^2}{|I|^2},
\]

(49)

where we have defined

\[
D_{Z_{L_1}}(s) = \text{Re} \, z_{L_1}(s) - \frac{\alpha}{\omega} \text{Im} \, z_{L_1}(s) = \sigma + \frac{1}{2} (\alpha_L + \alpha_C), \quad \text{and} \quad (50)
\]

\[
D_{Z_{C_1}}(s) = \text{Re} \, z_{C_1}(s) - \frac{\alpha}{\omega} \text{Im} \, z_{C_1}(s) = \frac{1}{(\sigma + \alpha_C)^2 + \frac{\omega^2}{\omega}} \left[ \sigma + \frac{1}{2} (\alpha_L + \alpha_C) \right].
\]

(51)

It is readily seen that both \( D_{Z_{L_1}}(s) \) and \( D_{Z_{C_1}}(s) \) are positive for \( \sigma > -\frac{1}{2} (\alpha_L + \alpha_C) \) and \( \omega \pm 0 \). It follows that

\[
D_Z(s) > 0 \quad \text{for} \quad \text{Re} \, s > -\delta \quad \text{and} \quad \text{Im} \, s \neq 0,
\]

(52)

which represents a generalization of \( \text{Re} \, Z(s) > 0 \) for \( \text{Re} \, s > 0 \) in the ordinary RLCT case where \( \alpha = \delta = 0 \).

An additional property of RL'C"T impedances can be deduced from the observation that all three circuit elements are real and positive along the finite real axis to the right of \(-\alpha_C\). This yields the condition

\[
Z(\sigma) > 0 \quad \text{for} \quad \sigma > -\alpha_C.
\]

(53)

[Strictly speaking we should exclude the point at infinity, since \( \lim_{\sigma \to \infty} Z(\sigma) > 0 \).]

Finally, we add the obvious condition that \( Z(s) \) must be real for real \( s \).

To summarize, we have found that driving-point impedances of RL'C"T networks must have the following properties:

\[ Z(s) = \text{real} \quad \text{for} \quad s = \text{real} \]
\[ D_Z(s) \geq 0 \quad \text{for} \quad \text{Re} \, s = -\frac{1}{2} (\alpha_L + \alpha_C), \quad \omega \neq 0 \]
\[ D_Z(s) > 0 \quad \text{for} \quad \text{Re} \, s > -\frac{1}{2} (\alpha_L + \alpha_C), \quad \omega \neq 0 \]
\[ Z(\sigma) > 0 \quad \text{for} \quad \sigma > -\alpha_C. \]
A function with these properties will hereafter be called a PRD$_Z$ function to indicate that the $D_Z$-function is PR (positive and real) to the right of the $\delta$-axis.

By pursuing a dual line of reasoning, one can derive analogous properties of RL'C"T driving-point admittances. The properties in question are most easily stated in terms of the following quantity, which we shall refer to as the admittance derived difference function,

$$D_Y(s) = \text{Re } Y(s) + \frac{\alpha}{\omega} \text{Im } Y(s).$$

(54)

The conditions are as follows:

- $Y(s) = \text{real}$ for $s = \text{real}$
- $D_Y(s) \geq 0$ for $\text{Re } s = -\frac{1}{2} (\alpha_L + \alpha_C), \omega \neq 0$
- $D_Y(s) > 0$ for $\text{Re } s > -\frac{1}{2} (\alpha_L + \alpha_C), \omega \neq 0$
- $Y(s) > 0$ for $\sigma > -\alpha_C$.

A function with these properties will hereafter be called a PRD$_Y$ function.

It can readily be shown that if a function is PRD$_Z$, its inverse must be PRD$_Y$ and vice versa. Although a given function is not necessarily both PRD$_Z$ and PRD$_Y$, it is convenient to introduce the notation PRD to describe functions that are PRD$_Z$ either "right side up" or "upside down".

### 3.3 ADDITIONAL PROPERTIES OF PRD FUNCTIONS

The reader will recall that PR functions can have neither poles nor zeros in the right-half of the $s$-plane. A similar property of PRD functions will now be established.

The conditions

- $D_Z(s) > 0$ for $\text{Re } s > -\delta$ and $\text{Im } s \neq 0$,

and

- $Z(\sigma) > 0$ for $\sigma > -\alpha_C$

imply that the real and imaginary parts of a PRD$_Z$ function cannot simultaneously vanish, in the finite $s$-plane to the right of the $\delta$-axis, excluding the interval $-\delta < \sigma < -\alpha_C$; $Z(s)$ can therefore not have any zeros in this region.

A similar statement can be made for the reciprocal of $Z(s)$, which must be a PRD$_Y$ function. Thus PRD functions can have neither poles nor zeros within the shaded region $R$ shown in Fig. 12.

It is logical at this point, to inquire whether PRD functions may have poles on the boundary of the region $R$. In order to answer this question it is convenient to break the contour up into the three parts shown in Fig. 13.
Assume first that \( Z(s) \) has a pole of order \( n \) at \( s_v \) on \( C_{L'C''} \) as shown in Fig. 14. In the vicinity of \( s_v \) one can then write the following approximate expression

\[
Z(s) \approx \frac{k_v}{(s - s_v)^n}. \tag{55}
\]

Furthermore, for points on the semicircular arc \( C \), shown in Fig. 14, Eq. 55 takes the form

\[
Z(s) \bigg|_{s \in C} \approx \frac{|k_v| e^{j(n\theta)}}{\rho^n} \left[ \cos(n\theta - \phi) - j \sin(n\theta - \phi) \right], \tag{56}
\]

where

\[
k_v = \frac{|k_v| e^{j\phi}}{\rho^n}.
\]

Thus the impedance derived difference function becomes

\[
D_Z(s) \bigg|_{s \in C} \approx \frac{\frac{|k_v|}{\rho^n}}{1 + \left(\frac{\alpha}{\omega}\right)^2} \cos \left[ n\theta - \phi - \tan^{-1} \frac{\alpha}{\omega} \right], \tag{57}
\]

which clearly remains positive only if \( n = 1 \) and the following conditions are satisfied

(i) \( \text{Re } k_v > 0 \)

(ii) \( \text{Arg } k_v = -\tan^{-1} \frac{\alpha}{\omega} \).

Assume next that \( Z(s) \) has a pole of order \( n \) at \( s_v \) on \( C_{R'} \), as shown in Fig. 15. An approximate expression for the impedance derived difference function on the semicircular arc \( C' \), in this case, is

\[
D_Z(s) \bigg|_{s \in C'} \approx \frac{|k_v|}{\rho^n} \sin (n\theta - \phi). \tag{58}
\]

Clearly, \( D_Z \) remains positive on \( C' \) only if \( n = 1 \) and \( k_v \) is real and positive.

Now, assume that \( Z(s) \) has an \( n^{th} \)-order pole at \( C_{I'} \), as shown in Fig. 16. From the first two terms of a Laurent expansion around this pole, one obtains the following approximate expression for the impedance derived difference function on the semicontour \( C'' \) of Fig. 16.

\[
D_Z(s) \bigg|_{s \in C''} \approx \frac{|k_{-n}|}{\rho^n} \left\{ \cos \left[ n\theta - \phi - n \right] + \frac{\alpha}{\omega} \sin \left[ n\theta - \phi - n \right] \right\}
\]

\[
+ \frac{|k_{-n+1}|}{\rho^{n-1}} \left\{ \cos \left[ (n-1)\theta - \phi - (n+1) \right] + \frac{\alpha}{\omega} \sin \left[ (n-1)\theta - \phi - (n+1) \right] \right\}. \tag{59}
\]
Fig. 12. Region $R$ where PRD functions can have neither poles nor zeros.

Fig. 13. Identification of the various parts of the contour bounding the analytic region, $R$, of PRD functions.

Fig. 14. Modified $C_{L'}C''$ contour pertinent to the residue evaluation of a pole at $s = s_{\nu}$.

Fig. 15. Semicircular contour, $C'$, is relevant to the residue evaluation of a pole at $s = s_{\nu}$.

Fig. 16. Semicircular contour, $C''$, is relevant to the residue evaluation of a pole at $s = -\delta$. 
Clearly, only the values \( n = 1 \) and \( n = 2 \) are consistent with the positiveness of \( D_z \) on \( C'' \). In the event \( n = 1 \), then \( k_{-1} \) must be real and positive. If \( n = 2 \), then \( k_{-2} \) must be positive; in addition, \( Z \) must have a simple pole at \( s = -6 \) with a positive real residue

\[
k_{-1} \geq \frac{k_{-2}}{\alpha},
\]

since

\[
D_z(-6+j\omega) = -\frac{k_{-2}}{\omega^2} + \frac{\alpha}{\omega} \frac{k_{-1}}{\omega}
\]

must remain non-negative as \( \omega \to 0 \).

Finally, assume that \( Z(s) \) has a pole of order \( n \) at infinity. In this case

\[
Z(s) \approx A s^n = A \rho^n \cos(n\theta + j \sin n\theta)
\]

and

\[
D_z(s) \approx A \rho^n \left| \sqrt{1 + \left(\frac{\alpha}{\omega}\right)^2} \cos \left[n\theta + \tan^{-1} \left(\frac{\alpha}{\omega}\right)\right]\right|
\]

for large \( s \). The positiveness of \( D_z \) in \( R \) is thus seen to imply that \( Z(s) \) can, at most, have a first-order pole at infinity with a (real) positive constant multiplier.

To summarize, poles on the boundary of the shaded region \( R \) of Fig. 13 must satisfy the following conditions:

(i) On \( C_L' \), \( C'' \) poles must be simple and have residues satisfying the conditions \( \Re k_v > 0 \) and \( \Arg k_v = -\tan^{-1} \frac{\alpha}{\omega} \).

(ii) On \( C_R \) poles must be simple and have positive real residues.

(iii) At \( C_I \) the Laurent expansion of \( Z(s) \) must be of the form

\[
Z(s) = \frac{k_{-2}}{(s+\delta)^2} + \frac{k_{-1}}{s+\delta} + \sum_{i=0}^{\infty} k_i (s+\delta)^i,
\]

where

\[
k_{-1} \geq \frac{k_{-2}}{\alpha} > 0.
\]

(iv) At infinity poles must be simple and have a positive real multiplier.

If we call these conditions the PRD\( Z \) residue conditions, then we can state the following necessary conditions on any PRD\( Z \) function:

A. It is analytic in \( R \).

B. Its impedance derived difference function is non-negative on the \( \delta \)-axis.

C. It satisfies the PRD\( Z \) residue conditions.
These conditions clearly represent a generalization of the ABC conditions of PR functions. Hereafter, we shall refer to these conditions as the ABC\textsubscript{Z} conditions of PRD\textsubscript{Z} functions.

By pursuing a dual line of reasoning, a similar set of conditions for any PRD\textsubscript{Y} function can be readily established. The conditions in question are as follows:

A. It is analytic in $\mathbb{R}$.

B. (i) Its admittance derived difference function is non-negative on the $\delta$-axis, and

(ii) $Y(-\alpha C) > 0$.

C. (i) Poles on $C_L, C''$ (including the point at infinity) must be simple and have residues that satisfy the following conditions: $\text{Re} \ k_Y > 0$ and $\text{Arg} \ k_Y = \tan^{-1} \omega$;

(ii) Poles on $C_R$ must be simple and have a negative real residue; and

(iii) Poles at $C_I$ can be of either first or second order. For simple poles the residue must be negative. For second-order poles the Laurent expansion of $Y(s)$ must be of the form

$$Y(s) = \frac{k_{-2}}{(s+\delta)^2} + \frac{k_{-1}}{s+\delta} + \sum_{i=0}^{\infty} k_i (s+\delta)^i,$$

where

$$k_1 \geq \frac{-k_{-2}}{\alpha} > 0.$$

These conditions will hereafter be referred to as the ABC\textsubscript{Y} conditions of PRD\textsubscript{Y} functions.

Condition B(ii) should be noted, since no analogous condition is necessary in the ABC\textsubscript{Z} conditions to completely describe PRD\textsubscript{Z} functions, as we shall show.

3.4 ABC CONDITIONS $\leftrightarrow$ PRD CONDITIONS

We have shown that the ABC conditions are a necessary set of conditions for the PRD character of a function. Now we shall show that the ABC conditions also represent a sufficient set of conditions for a function to be PRD.

THEOREM 1. Any rational function with real coefficients that satisfies the ABC\textsubscript{Z} conditions must be a PRD\textsubscript{Z} function.

Proof:

We begin by observing that if a pole satisfies the residue condition on the boundary of the analytic region $R$ (Fig. 12) this will guarantee the positiveness of the $D_{L}^{-}$ function on a small semicircular detour into the analytic region $R$. So in the presence of such poles, we shall modify the boundary (as shown in Fig. 17) with the assurance that
\[ D_Z(s) > 0 \text{ for } s \in C' (\omega \neq 0) \tag{60} \]

where \( C' \) represents the modified boundary.

To show that
\[ D_Z(s) > 0 \text{ for } s \in R (\omega \neq 0) \tag{61} \]

we introduce two auxiliary functions, \( \frac{Z(s)}{s+\alpha_L} \) and \( -(s+\alpha_L) Z(s) \). These are both analytic in \( R \) and have imaginary parts related to \( D_Z(s) \). By applying the maximum and minimum value theorem to their imaginary parts, the validity of (61) is readily established as follows.

First we show that the imaginary part of
\[ A_1(s) = \frac{Z(s)}{s+\alpha_L} \tag{62} \]
is non-negative along the closed contour \( C_U \) shown in Fig. 18.

To this end, we write
\[
\text{Im} A_1(s) = \frac{1}{(\sigma + \alpha_L^2 + \omega^2)} \left\{ (\sigma + \alpha_L) \text{Im} Z(s) - \omega \text{Re} Z(s) \right\}. \tag{63}
\]

On the real axis (excluding singular points), both \( \omega \) and \( \text{Im} Z(s) \) are zero, so it follows that
\[ \text{Im} A_1(s) = 0. \tag{64} \]

To determine the behavior of \( \text{Im} A_1(s) \) on the \( \delta \)-axis (again excluding singular points) and on the semicircular arcs, we use the non-negativeness of the \( D_Z \)-function, that is,
\[ D_Z(s) = \text{Re} Z(s) - \frac{\alpha}{\omega} \text{Im} Z(s) \geq 0 \tag{65} \]

for \( s \in [\delta \text{-axis and the semicircular arcs}] \).

For \( \omega > 0 \) this inequality can be written
\[ \alpha \text{Im} Z(s) - \omega \text{Re} Z(s) \leq 0 \]

which immediately yields
\[ \text{Im} A_1(-\delta+j\omega) \leq 0 \quad \text{for } \omega > 0 \tag{66} \]
on the analytic portion of the \( \delta \)-axis. Furthermore,
\[ \sigma + \alpha_L \approx \alpha \tag{67} \]
on the semicircular contours along the \( \delta \)-axis, so it follows that the imaginary part of \( A_1(s) \) must be negative on the semicircular arcs off the \( \delta \)-axis.
For the semicircular contours along the real axis, we observe that any function of the form

\[ \text{Re } Z(s) - \frac{K}{\omega} \text{ Im } Z(s) \text{ for } K > 0, \]  

which is strictly positive on these contours, implies a positive real residue and vice versa. Therefore, if we set

\[ \sigma + \alpha_L \approx K \]  

at each pole on the real axis, it follows that the imaginary parts of \( A_1(s) \) must be negative along semicircular arcs along the real axis.

Finally, along the quarter-circle at infinity, it is readily seen that the imaginary part of \( A_1(s) \) vanishes. If \( Z(s) \) has a pole at infinity, one can write

\[ Z(s) = k_\infty (s + \alpha_L) + Z(s) \text{ where } k_\infty > 0 \]  

and it follows immediately that

\[ \text{Im } A_1(s) = \text{Im } \frac{Z_1(s)}{s + \alpha_L} = 0. \]  

Putting the various pieces together, we have shown that

\[ \text{Im } A_1(s) < 0 \text{ for } s \in C_U' \]  

Using the maximum value theorem on the imaginary part of \( A_1(s) \) on the boundary \( C_U' \) of the region \( R_U \), we obtain

\[ \text{Im } A_1(s) < 0 \text{ for } s \in R_U', \]  

which we can write

\[ \text{Re } Z(s) - \frac{\sigma + \alpha_L}{\omega} \text{ Im } Z(s) > 0 \text{ for } s \in R_U'. \]  

By repeating the same line of reasoning for the behavior of \( A_1(s) \) on \( C_L' \) of Fig. 19, we obtain the analogous result

\[ \text{Re } Z(s) - \frac{\sigma + \alpha_L}{\omega} \text{ Im } Z(s) > 0 \text{ for } s \in R_L'. \]  

Conditions (72) and (73) together show that

\[ \text{Re } Z(s) - \frac{\sigma + \alpha_L}{\omega} \text{ Im } Z(s) > 0 \text{ for } \text{Re } s > -\delta \text{ and Im } s \neq 0. \]
Next, we consider the auxiliary function

$$A_2(s) = -(s + \alpha_C) Z(s)$$  \hspace{1cm} (77)

and repeat the same line of reasoning.

Without including the details, we arrive at the following analogous result

$$\text{Re } Z(s) + \frac{\sigma + \alpha_C}{\omega} \text{Im } Z(s) > 0 \text{ for } \text{Re } s > -\delta \text{ and } \text{Im } s \neq 0.$$  \hspace{1cm} (78)

Adding Eqs. 76 and 78, we obtain

$$\text{Re } Z(s) - \frac{\alpha}{\omega} \text{ Im } Z(s) = D_z(s) > 0 \text{ for } \text{Re } s > -\delta \text{ and } \text{Im } s \neq 0,$$  \hspace{1cm} (79)

which completes a major portion of the proof.

One more thing remains to be shown before we know that $Z(s)$ is a PRD$_Z$ function, namely, the positive character of the impedance on the finite real axis for $\sigma > -\alpha_C$.

Knowing that $Z(s)$ must be a rational function with real coefficients and have a positive real multiplier (this follows from the positiveness of the $D_z$-function in the right half-plane, which was just established), it follows that

$$Z(\sigma) > 0 \text{ for large } \sigma.$$  \hspace{1cm} (80)

$Z(\sigma)$ will therefore be strictly positive for $\infty > \sigma > -\alpha_C$, provided $Z(s)$ has no zeros in this interval. This is readily shown to be the case by reductio ad absurdum.

Assume that $Z(s)$ has a zero in this interval, say, at $s = \sigma_0$. Then since the sign of $D_z(s)$ and $D_y(s)$ must be the same in the region R (Fig. 12), it follows that the corresponding pole of $Y(s)$ must be simple and have a negative real residue. This in turn describes the behavior of $\text{Re } Z(\sigma)$ versus $\sigma$ at this point, since the residue equals

$$\frac{\partial}{\partial \sigma} \text{Re } Z(\sigma) + j \frac{\partial}{\partial \sigma} \text{Im } Z(\sigma) = \frac{\partial}{\partial \sigma} \text{Re } Z(\sigma) < 0.$$  \hspace{1cm} (81)

The behavior of $Z(\sigma) = \text{Re } Z(\sigma)$ versus $\sigma$ in the vicinity of $\sigma_0$ must therefore be as shown in Fig. 20. Since $Z(\sigma) > 0$ for large $\sigma$, it follows that there must be at least one more zero crossing this time with

$$\frac{\partial}{\partial \sigma} \text{Re } Z(\sigma) > 0.$$  \hspace{1cm} (82)

But this is impossible; $Z(s)$ can therefore not have any zeros on the finite real axis to the right of $-\alpha_C$. This completes the proof of Theorem 1.

By pursuing a dual argument, the equivalence of the ABC$_Y$ conditions and the PRD$_Y$ conditions can be established.
Fig. 17. Modified boundary on which the $D_z$ function must be non-negative.

Fig. 18. Contour used to show that the $D_z$ function must be positive for $s \in R_U$.

Fig. 19. Contour used to show that the $D_z$ function must be positive for $s \in R_L$.

Fig. 20. Contradictory behavior of an assumed $PRD_z$ function in the vicinity of a zero to the right of $-\alpha_c$ on the real axis.
3.5 PRD\textsubscript{Z} CONDITIONS IN POLAR FORM

We are not going to pursue this topic at length, since what we might call the rectangular form of the PRD\textsubscript{Z} conditions of section 3.3, or the equivalent ABC\textsubscript{Z} conditions, will be our starting point in all future developments throughout this report. The insight that is obtained, however, by an awareness of the constraints on the angle of PRD\textsubscript{Z} functions, entirely justifies the inclusion of this section. For this reason, we state these conditions by means of a representative, self-explanatory figure. This figure follows from an appropriate vector interpretation of Eq. 48, which for convenience, is repeated below.

\begin{equation}
Z(s) = z_{L,1}(s) \frac{\sum |I_{L,i}|^2}{|I|^2} + z_{C,1}(s) \frac{\sum |I_{C,i}|^2}{|I|^2} + \frac{\sum |I_R|^2}{|I|^2}. \tag{83}
\end{equation}

It is clear that \(Z(s)\) can be thought of as the vector sum of linear combinations of the vectors \(z_{L,1}(s)\), \(z_{C,1}(s)\), and \(z_R(s)\), where

\begin{align*}
z_{L,1}(s) &= s + \alpha_L \\
z_{C,1}(s) &= \frac{1}{s+\alpha_C} \\
z_R(s) &= 1.
\end{align*}

Since the multiplier of each of these vectors, at a given frequency, is a non-negative real number, it follows that the directions of the first two component vectors will suffice in determining permissible angles of \(Z(s)\).

With these points in mind, the shaded areas in Fig. 21 are seen to represent regions of allowable end points for a \(Z\)-vector originating at the points \(s = s_v\).

We see that

\begin{equation}
\text{Arg}(s+\alpha_L) \geq \text{Arg} Z(s) \geq \text{Arg}(s+\alpha_C)^{\circ} \quad \text{for Re } s \geq -\delta \text{ and Im } s \geq 0 \tag{84}
\end{equation}

and

\begin{equation}
\text{Arg}(s+\alpha_C)^{\circ} \geq \text{Arg} Z(s) \geq \text{Arg}(s+\alpha_L) \quad \text{for Re } s \geq -\delta \text{ and Im } s \leq 0. \tag{85}
\end{equation}

It can be shown that the constraints on the angle of \(Z(s)\) as expressed by (84) and (85) imply the PRD\textsubscript{Z} property, and vice versa. These conditions could therefore be called the Polar Form of the PRD\textsubscript{Z} conditions.

[The vice versa can be established by first showing that any PRD\textsubscript{Z} function is realizable as the driving point impedance of an RL'C'T network (Sec. IV), and then invoke the energy argument leading to (83).]
3.6 ANALOGIES WITH THE RLCT CASE

Previously, the analogy between the PR case and the PRD case has been striking. It is possible to place this analogy even more in evidence by using a simple polynomial factorization of PRD functions. Such a factorization was made in section 2.4 in order to perform a continued fraction expansion of lossy reactance functions. Let us now elaborate on this idea and thus obtain some strong symmetries between the RLCT and the RL'C"T case. These analogies will facilitate several of our future developments.

We have already observed the analogy between Re Z(jw) in the PR case and DZ(-6+jw) in the PRDZ case. In order to make the similarities between these even more evident, let us make a simple frequency translation and express the PRDZ function, Z(s), in terms of λ, where

\[ s = \lambda - \delta \quad \text{and} \quad \lambda = \nu + j\xi. \]

We may then write

\[ Z(s) \bigg|_{s = -\delta + j\omega} = \tilde{Z}(\lambda) \bigg|_{\lambda = j\xi}, \]

since \( \omega = \xi \). It follows that the \( D_Z \)-function of \( Z(s) \) along the \( \delta \)-axis in the \( s \)-plane must equal the \( D_Z \)-function of \( \tilde{Z}(\lambda) \), \( D_Z(\lambda) \), along the \( j\xi \)-axis in the \( \lambda \)-plane, that is,

\[ D_Z(s) \bigg|_{s = -\delta + j\omega} = D_Z(\lambda) \bigg|_{\lambda = j\xi} = \text{Re} \tilde{Z}(j\xi) - \frac{\alpha}{\xi} \text{Im} \tilde{Z}(j\xi). \]

Fig. 21. Illustrating the PRDZ conditions in polar form. Shaded areas represent regions of allowable end points for a Z-vector originating at \( s = s_3 \).
To enhance the analogy between \( \text{Re } Z(j\omega) \) and \( D_Z(j\xi) \), let us represent the PRDZ function, \( \tilde{Z}(\lambda) \), in factored form as follows:

\[
\tilde{Z}(\lambda) = \frac{A_{2n+1}(\lambda+\alpha)^{n+1}(\lambda-\alpha)^n + A_{2n}(\lambda+\alpha)^n(\lambda-\alpha)^n + \ldots + A_1(\lambda+\alpha) + A_0}{B_{2m+1}(\lambda-\alpha)^{m+1}(\lambda+\alpha)^m + B_{2m}(\lambda-\alpha)^m(\lambda+\alpha)^m + \ldots + B_1(\lambda-\alpha) + B_0}.
\]  

Furthermore, let us define the "even" part of the factored polynomial as the sum of the terms having coefficients with even subscripts and the "odd" part as the sum having coefficients with odd subscripts. We can then write

\[
\frac{\tilde{Z}(\lambda)}{Q(\lambda)} = \frac{M_1 + N_1}{M_2 + N_2},
\]

where \( M \) and \( N \) are used to denote the "even" and "odd" parts, respectively.

Now, it is true that

\[
D_Z(j\xi) = \frac{M_1M_2 - N_1N_2}{|M_2 + N_2|^2} |_{\lambda = j\xi}
\]

which has a rather striking resemblance to the well-known expression

\[
\text{Re } Z(j\omega) = \frac{m_1m_2 - n_1n_2}{|m_2 + n_2|^2} |_{s = j\omega}
\]

in the RLCT case.

The validity of expression (91) can be readily shown; accordingly, we omit the details.

A pertinent question arises from the representation (90) of \( \tilde{Z}(\lambda) \). Do the various ratios of "even" over "odd" parts and "odd" over "even" parts represent lossy reactance or susceptance functions, and if so, which? In this connection let us restrict the polynomials \( P(\lambda) \) and \( Q(\lambda) \) to be Hurwitz polynomials. Then we can formulate the following theorems.

**THEOREM 2.** If \( Q(\lambda) \) is a Hurwitz polynomial, then it can be written in the following form:

\[
Q(\lambda) = B_{2m+1}(\lambda-\alpha)^{m+1}(\lambda+\alpha)^m + B_{2m}(\lambda-\alpha)^m(\lambda+\alpha)^m + \ldots + \ldots
\]

\[
+ B_1(\lambda-\alpha) + B_0,
\]

where all the coefficients are non-negative and no intermediate coefficient is missing.
Let the "even" and "odd" parts of $Q(\lambda)$ be written

$$M_B = B_{2m}(\lambda-\alpha)^m(\lambda+\alpha)^m + \ldots + B_0$$

and

$$N_B = B_{2m+1}(\lambda-\alpha)^{m+1}(\lambda+\alpha)^m + \ldots + B_1(\lambda-\alpha)$$

respectively. Then the ratio $M_B/N_B$ is a PRD$_Z$ reactance function.

**Proof:**

To prove this theorem let us express the "even" and "odd" parts, $M_B$ and $N_B$, in terms of the true even and odd parts of $Q(\lambda)$, respectively. We can then write

$$Q(\lambda) = m_B + n_B = \left[ m_B + \frac{\alpha n_B}{\lambda} \right] + (\lambda-\alpha)\frac{n_B}{\lambda} = M_B + N_B,$$

where

$$M_B = m_B + \frac{\alpha n_B}{\lambda}$$

and

$$N_B = (\lambda-\alpha)\frac{n_B}{\lambda}.$$

The ratio $M_B/N_B$ must be a PR reactance function in the $\lambda$-plane, so

$$\Psi_B(\lambda) = \frac{m_B}{n_B} = \frac{\lambda - \alpha}{\lambda} \frac{M_B}{N_B} - \frac{\alpha}{\lambda}$$

must have a pole-zero pattern of the form shown in Fig. 22a.

The pole-zero pattern of

$$\Psi'_B(\lambda) = \frac{\lambda}{\lambda - \alpha} \Psi_B(\lambda) = \frac{M_B}{N_B} - \frac{\alpha}{\lambda - \alpha}$$

must therefore be of the form shown in Fig. 22b. $\Psi'_B(\lambda)$ is therefore seen to be a PRD$_Z$ reactance function, and so is

$$\frac{M_B}{N_B} = \Psi'_B(\lambda) + \frac{\alpha}{\lambda - \alpha},$$

since a sum of PRD$_Z$ reactance functions must be a PRD$_Z$ reactance function.
Finally, since $\frac{M_B}{N_B}$ is a PRD$_Z$ reactance function, a continued fraction expansion must yield only positive coefficients; this can only be the case if all of the B's are positive and no intermediate coefficient is missing. Q.E.D.

THEOREM 3. If $P(\lambda)$ is a Hurwitz polynomial with an even number of zeros to the right of $\lambda = -\alpha$ along the real axis, then it can be written in the following form:

$$P(\lambda) = A_{2n+1}(\lambda+\alpha)^{n+1}(\lambda-\alpha)^n + A_{2n}(\lambda+\alpha)^n(\lambda-\alpha)^n + \ldots + A_1(\lambda+\alpha)^n + A_0,$$  \hspace{1cm} (102)

where all the coefficients are non-negative and no intermediate coefficient is missing.

Let the "even" and "odd" parts of $P(\lambda)$ be written

$$M_A = A_{2n}(\lambda+\alpha)^n(\lambda-\alpha)^n + \ldots + A_0$$ \hspace{1cm} (103)

and

$$N_A = A_{2n+1}(\lambda+\alpha)^{n+1}(\lambda-\alpha)^n + \ldots + A_1(\lambda+\alpha)$$

$$= (\lambda+\alpha) \left[ A_{2n+1}(\lambda+\alpha)^n(\lambda-\alpha)^n + \ldots + A_1 \right], \text{ respectively.} \hspace{1cm} (104)$$

Then the ratio $M_A/N_A$ is a PRD$_Y$ susceptance function.

Proof:

In order to prove that $N_A/M_A$ is a PRD$_Y$ susceptance function, we need to know that $A_0$ and $A_1$ are non-negative. This is established in Appendix A.

Let us express the "even" and "odd" parts, $M_A$ and $N_A$, in terms of the true even and odd parts of $P(\lambda)$, $m_A$ and $n_A$, respectively. We can then write

$$P(\lambda) = m_A + n_A = \left[ m_A - \frac{\alpha n_A}{\lambda} \right] + (\lambda+\alpha) \frac{n_A}{\lambda} = M_A + N_A,$$ \hspace{1cm} (105)

where

$$M_A = m_A - \frac{\alpha n_A}{\lambda}$$ \hspace{1cm} (106)
and

$$N_A = (\lambda + \alpha) \frac{n_A}{\lambda}. \tag{107}$$

The ratio $m_A / n_A$ must be a PR reactance function in the $\lambda$-plane, so

$$\psi_A(\lambda) = \frac{m_A}{n_A} = \frac{\lambda + \alpha}{\lambda} \frac{M_A}{N_A} + \frac{\alpha}{\lambda} \tag{108}$$

must have a pole-zero pattern, as shown in Fig. 23a.

The pole-zero pattern of

$$\psi'_A(\lambda) = \frac{\lambda}{\lambda + \alpha} \psi_A(\lambda) = \frac{M_A}{N_A} + \frac{\alpha}{\lambda + \alpha} \tag{109}$$

must therefore be as shown in Fig. 23b.

---

**Fig. 23. Pole-zero pattern of (a) a reactance function; (b) a modified reactance function.**

$\psi'_A(\lambda)$ is therefore seen to be a PRD$_Y$ susceptance function. The real question, however, is whether or not

$$\frac{M_A}{N_A} = \psi'_A(\lambda) - \frac{\alpha}{\lambda + \alpha} \tag{110}$$

is a PRD$_Y$ susceptance function. The answer is clearly in the affirmative if we can show that the residue of $M_A / N_A$ at $\lambda = -\alpha$ is positive.

We have

$$\text{Res} \left[ \frac{M_A}{N_A} \right]_{\lambda = -\alpha} = \frac{A_0}{A_1} > 0, \tag{111}$$

since both $A_0$ and $A_1$ are positive (Appendix A). Q.E.D.

Finally, it should be noted that $P(\lambda)$ cannot have an odd number of zeros in the region $-\alpha < \nu < 0$ if all the coefficients ($A_i$'s) are to be positive.
IV. SYNTHESIS OF RL'C''T DRIVING-POINT IMPEDANCES

Having established that the driving-point impedance of any RL'C''T network must be a PRD\textsubscript{Z} function, it seems logical to ask whether the converse statement is true. We shall show that this is indeed the case.

In particular, it will be shown that any PRD\textsubscript{Z} function can be realized as the driving-point impedance of an RL'C''T network by a proper modification of the Brune procedure.

4.1 REACTANCE REDUCTION AND RESISTANCE REDUCTION OF PRD FUNCTIONS

It is well known that certain PR functions are realizable by the simple procedure of pole removals along the j\omega axis (reactance reductions) combined with appropriate resistance removals. This development requires nothing beyond the realization procedures applicable in the two-element-kind case, and is therefore sometimes referred to as the Foster preamble.

A similar situation exists in the RL'C''T case, and the analogous procedure for dissipative reactance reductions and resistance removals from PRD functions will now be developed.

Consider a PRD\textsubscript{Z} function; in general such a function may possess poles along the \delta-axis and at the point s = -\alpha\textsubscript{C}. It may therefore be written in the form

\[
Z(s) = k_L(s + \alpha_L) + \frac{k_C}{s + \alpha_C} + \sum_{i=1}^{N} 2\text{Re}k_i \frac{s + \alpha_L}{s^2 + 2\delta s + \delta^2 + \omega_i^2} + Z_R(s) \tag{112}
\]

where

- \(k_L \geq 0\),
- \(k_C \geq 0\),
- \(\text{Re}k_i \geq 0\),

and

- \(\omega_i \geq 0\).

The first N+2 terms in this expansion are clearly realizable by simple dissipative reactance networks as shown in Fig. 7. Furthermore, the removal of these terms must leave a PRD\textsubscript{Z} remainder function, \(Z_R(s)\). This follows since the removal does not affect either the analyticity in the region R of Fig. 12 nor the behavior of the impedance derived difference function along the \delta-axis. Each of the removed component impedances corresponds to a difference function that is identically zero along the \delta-axis, so \(D_Z(-\delta + j\omega)\) must be the same for \(Z(s)\) and \(Z_R(s)\).
A dual argument leads to analogous conclusions for \( \delta \)-axis poles of a PRD_{Y} function. [In this connection, it should be noted that the admittance of a series connection of a lossy inductor and a lossy capacitor evaluated at \( s = -\alpha_{C} \), is equal to zero. The pole removal does therefore not affect the value of the admittance at \( s = -\alpha_{C} \).] The situation at \( s = -\alpha_{L} \) is not, however, analogous to the situation at \( s = -\alpha_{C} \) in the PRD_{Z} case. The residue \( k_{-\alpha} \) of a simple pole of a PRD_{Y} function \( Y(s) \) at \( s = -\alpha_{L} \), can be either positive or negative, and only if the residue is positive \( Y(-\alpha_{C}) > 0 \) (such that the remainder admittance is non-negative at \( s = -\alpha_{C} \)) can a simple susceptance removal of the form \( \frac{k_{-\alpha}}{s+\alpha_{L}} \) be made.

To summarize the situation; poles along the \( \delta \)-axis and a pole at \( s = -\alpha_{C} \) of a PRD_{Z} function are always removable and lead to a simpler remainder function which is guaranteed to be PRD_{Z}; a similar statement is true for PRD_{Y} functions if we add the requirement that in order for a pole at \( s = -\alpha_{L} \) to be removable, it must have a positive residue and contribute to the positive value of the admittance at \( s = -\alpha_{C} \), by no more than the total value of the admittance at this point.

It is possible that the removal of \( \delta \)-axis poles and a pole at \( s = -\alpha_{C} \) of a PRD_{Z} function may lead to a remainder function that has \( \delta \)-axis zeros or a zero at \( s = -\alpha_{L} \). Turning the remainder function upside down, we can then proceed to remove the corresponding poles of the PRD_{Y} function (except possibly at \( s = -\alpha_{L} \)) which in turn may lead to a remainder function with zeros along the \( \delta \)-axis and at \( s = -\alpha_{C} \). This process will eventually come to a halt (assuming that the original function represents a true three-element-kind network). At this point the remainder function takes on one of the following two forms:

(i) a constant

(ii) a rational function representing a PRD_{Z} function, having neither poles nor zeros along the \( \delta \)-axis or at the points \( s = -\alpha_{C} \) and \( s = -\alpha_{L} \). [The function may, of course, have a nonremovable zero at \( s = -\alpha_{L} \).]

This reduced PRD_{Z} function will be called minimum reactive in analogy with the ordinary RLCT terminology, although it should not be inferred from this that no further reactance removals are possible without destroying its PRD_{Z}-character. (We shall show that it is possible to remove a reactance function from a minimum reactive PRD_{Z} function and leave a PRD_{Z} remainder function provided the function has poles along the real axis in the interval \(-\delta < \sigma < -\alpha_{C}\). This, however, is not a degree-reducing operation.) Similarly, the corresponding PRD_{Y} function will be called minimum susceptive.

If the minimum reactive remainder function is just a constant (Case i), then a complete realization has essentially been achieved by reactance removals alone. The total realization takes the form of a dissipative ladder structure terminated in a resistance.
For the second case it might be possible to proceed in analogy with resistance removals of ordinary PR functions. A resistance removal in the ordinary case implies removing the minimum value of the even part along the $j\omega$-axis. In the RL'C''T case this means removing a resistance equal to the minimum value of the $D_z$-function along the $\delta$-axis. Although this does not in general reduce the order of the $PRD_z$-function, it does leave a $PRD_z$ remainder function that may or may not have a zero where the $D_z$-function is zero. If it has a zero at this point the degree reducing reactance removals suggested by the modified Foster preamble are reapplied. Thus repeating the entire cycle will either realize the function completely or leave a remainder function of the second form suggested on page 35, which in addition has a $D_z$-function that vanishes somewhere along the $\delta$-axis. At this point no further reduction is possible without changing the strategy.

Brune has suggested what to do next in the RLCT case; we shall show how to adapt his strategy to our case. But, before we proceed to discuss a general RL'C''T driving-point synthesis procedure, a few definitions seem to be in order. In the process let us summarize our findings and also show how certain minimum reactive $PRD_z$ functions can be "reduced" by reactance removals without destroying their $PRD_z$ character.

We have seen that $PRD_z$ functions in general may have both poles and zeros along the $\delta$-axis and along the real axis in the interval $-\delta < \sigma < -\alpha_c$; in addition it might have a pole at $s = -\alpha_c$. In some cases all of the natural frequencies may lie along the $\delta$-axis and along the real axis in the region $-\alpha_L < \sigma < -\alpha_c$. Examples of $PRD_z$ functions of this type were considered as driving-point impedances of L'C''T networks. It follows directly from our derivation of the impedance derived difference function that the $D_z$-function of any lossy reactance function is identically zero along the $\delta$-axis.

In this section we have used these properties of a reactance function to reduce any $PRD_z$ function to a minimum reactive $PRD_z$ function. Because of the possible occurrence of poles along the real axis in the region $-\delta < \sigma < -\alpha_c$, a minimum reactive $PRD_z$ function is not necessarily analytic to the right of the $\delta$-axis. It is convenient to make a distinction between minimum reactive $PRD_z$ functions that have poles in the region $-\delta < \sigma < -\alpha_c$ and those that do not; we shall therefore call the latter strictly minimum reactive. [Note that a strictly minimum reactive $PRD_z$ function may have a first-order pole at $s = -\delta$.] This terminology has been chosen to suggest that no lossy reactance function can be removed from such a function and leave a $PRD_z$ remainder function. This, however, is not true in the case of a minimum reactive $PRD_z$ function, as the following considerations will show.

Let $Z(s)$ be a minimum reactive $PRD_z$ function with a pole in the region $-\delta < \sigma < -\alpha_c$, i.e.
\[ Z(s) = Z_1(s) + \frac{k}{s + \sigma_i}, \]  
\[(113)\]

where \( k > 0 \) and \(-\delta < \sigma_i < -\alpha_C\). Without changing \( Z(s) \) we can then certainly write

\[ Z(s) = Z_1(s) - \frac{k'}{s + \sigma_i'} + \frac{k'}{s + \sigma_i'} + \frac{k}{s + \sigma_i}, \]  
\[(114)\]

where

\[ k' = \frac{\sigma_i - \alpha_C}{\sigma_i - \alpha_C} k < 0 \]

and

\[ \sigma_i + \sigma_i' = 2\delta. \]

By using the result of section 2.4 (Eq. 18), the last two terms of (114) are seen to correspond to a parallel \( L'C'' \) circuit. Removing this reactance function is therefore going to leave the remainder function

\[ Z_R(s) = Z_1(s) - \frac{k'}{s + \sigma_i}, \]  
\[(115)\]

which is clearly a \( PRD_Z \) function.

In effect, we have thus shown how to use a reactance removal to shift poles along the real axis in the region \(-\delta < \sigma < -\alpha_C\) into their mirror images with respect to \( s = -\delta \) and leave a \( PRD_Z \) remainder function. A minimum reactive \( PRD_Z \) function can therefore always be transformed into a strictly minimum reactive \( PRD_Z \) function. Notice, however, that this reactance "reduction" is not in general a degree reducing operation since the order of the numerator and denominator polynomials of \( Z(s) \) and \( Z_R(s) \) are clearly the same.

Similarly, a \( PRD_Y \) function without any \( \delta \)-axis poles will be called minimum susceptive or strictly minimum susceptive, depending on whether or not it has any poles along the real axis in the interval \(-\delta < \sigma < -\alpha_C\). [A strictly minimum susceptive \( PRD_Y \) function can have at most a simple pole with negative real residue at \( s = -\delta \).] Furthermore, it can be shown by the same type of reasoning as that presented above that a minimum susceptive \( PRD_Y \) function can be converted to a strictly minimum susceptive \( PRD_Y \) function by one or more susceptance removals.

Finally, a minimum resistive \( PRD_Z \) function will be used to denote a \( PRD_Z \) function whose \( D_Z \)-function vanishes at one or more points along the \( \delta \)-axis. No resistance removal can therefore be made from a minimum resistive \( PRD_Z \) function without destroying its \( PRD_Z \) character. Similarly, a minimum conductive \( PRD_Y \) function will be our definition of a \( PRD_Y \) function whose \( D_Y \)-function vanishes at one or more points along the \( \delta \)-axis.
4.2 EXTENDED BRUNE PROCEDURE

In order to carry the realization procedure beyond a minimum reactive PR function in the RLCT case, Brune has shown how to shift a pair of conjugate zeros onto the \( j\omega \)-axis and leave a PR remainder function. To do this a negative reactance removal is necessary; however, having completed one cycle by essentially the manipulations suggested in the previous section, it turns out that the negative reactance can be absorbed at the expense of an ideal transformer. We shall now show how the Brune procedure can be extended to the RL'C"T case.

For the discussion of the typical Brune cycle we will consider a minimum reactive PRD function of the form

\[
Z(s) = \frac{A_1}{s^{n+1}} + \frac{A_2}{s^m} + A_0.
\]

The first step in the procedure is to make a resistance reduction in order to make the \( D_Z \)-function vanish at some point along the \( \delta \)-axis, say at \( s = -\delta + j\omega_{\min} \). The situation at this point is illustrated in Fig. 24.

Since the \( D_Z \)-function of \( Z_1(s) \) is zero at \( s = -\delta + j\omega_{\min} \), it follows that the \( Z_1 \)-vector must be pointing either in the same direction as \( z_{L'} = s + \alpha_L \) or in the same direction as \( z_{C''} = \frac{1}{s + \alpha_C} \) at this point. (Recall that the lines of action of \( z_{L'} \) and \( z_{C''} \) are opposite all along the \( \delta \)-axis.) We can therefore write

\[
Z_1(-\delta + j\omega_{\min}) = kz_{C''}(-\delta + j\omega_{\min}),
\]

where \( k \) can be either positive or negative. (The case in which \( k = 0 \) has already been discussed in section 4.1.) Let us consider these two cases separately.

Case 1: \( k > 0 \)

The \( Z_1 \)-vector points in the direction of \( z_{C''} \), so by removing an appropriate negative lossy inductor the remainder impedance \( Z_2(s) \) can be made to have a pair of conjugate zeros at \( s = -\delta \pm j\omega_{\min} \). The value of this inductor is determined by the equation
\[ k_{L1} z_{L1}(-\delta + j\omega_{\text{min}}) = k z_{C1}(-\delta + j\omega_{\text{min}}) = Z_1(-\delta + j\omega_{\text{min}}) \]  
\[ \text{or} \]
\[ k_{L1} = -\frac{k}{\alpha^2 + \omega^2_{\text{min}}} < 0 \]

The remainder function
\[ Z_2(s) = Z_1(s) - k_{L1} z_{L1}(s), \]
which has a pair of conjugate \( \delta \)-axis zeros at \( s = -\delta \pm j\omega_{\text{min}} \), is a \( \text{PRD}_Z \) function since it is the sum of two \( \text{PRD}_Z \) functions.

At this point we have a negative inductor in the network realization; however, just as in the ordinary Brune cycle, it will be possible to absorb this element at the expense of an ideal transformer.

The next step is to remove the \( \delta \)-axis poles of \( Y_2 = Z_2^{-1} \). This is just a susceptance removal from a \( \text{PRD}_Y \) function with \( \delta \)-axis poles, and the resulting network realization, at this point, is of the form shown in Fig. 25. \( Y_3 \) and the element values of the shunt arm can be determined from the following expressions:

\[ Y_3(s) = \frac{1}{Z_1(s) - k_{L1} z_{L1}} - \frac{s + \alpha_C}{s^2 + 2\delta s + (\delta^2 + \omega^2_{\text{min}})} \]  
\[ \text{and} \]
\[ \frac{1}{k_{L2}} \frac{s + \alpha_C}{s^2 + 2\delta s + \frac{\alpha^2}{k_{C2}^2}} = 2\text{Re} k_v \frac{s + \alpha_C}{s^2 + 2\delta s + (\delta^2 + \omega^2_{\text{min}})} \]

where \( k_v \) is the residue of the pole of \( Y_2 \) at \( s = -\delta + j\omega_{\text{min}} \).

Explicit expressions for the pole of \( Y_2 \) at \( s = -\delta + j\omega_{\text{min}} \).

Explicit expressions for the element values of the shunt arm can be obtained from Eq. 122 and are as follows:

\[ k_{L2} = \frac{1}{2\text{Re} k_v} > 0 \]  
\[ \text{and} \]
\[ k_{C2} = \frac{\omega_{\text{min}}^2 + \alpha^2}{2 \text{Re} k_v}. \]

From our previous discussion it is seen that \( Y_3 \) must be a \( \text{PRD}_Y \) function. Furthermore, Eq. 121 shows that \( Y_3 \) must have a zero at infinity, since the behavior of \( Y_3 \) for large \( s \) is given by
Knowing that \( Z_3 = Y^{-1}_3 \) is a PRD function, it follows that a lossy inductor \( L_1 \) with a multiplier
\[
k_{L_3} = \frac{k_{L_1} k_{L_2}}{k_{L_1} + k_{L_2}} > 0
\]
(126)
can be removed and leave a remainder impedance \( Z_4 \) which is guaranteed to be a PRD function.

\[
Z_4(s) = Z_3(s) + \frac{k_{L_1} k_{L_2}}{k_{L_1} + k_{L_2}} s.
\]
(127)

One cycle of the extended Brune section has thus been completed, and the network realization is of the form shown in Fig. 26. In order to have made some progress, it must be true that the negative inductor \( L_1 \) can be absorbed, and that the remainder function is simpler than the original impedance \( Z(s) \).

By arguments similar to the ones in the ordinary case, it is straightforward to show that this is the case; the degrees of both numerator and denominator polynomials have been reduced by 2, and the negative inductor can be absorbed by using an ideal transformer as shown in Fig. 27.

Having shown how to complete the Brune cycle for Case 1, let us next consider the other possible case.
Case 2: \( k < 0 \)

The \( Z_1 \)-vector of Eq. 117 points in the direction of \( Z_L \) at \( s = -\delta + j\omega_{\text{min}} \). Therefore, by removing an appropriate negative lossy capacitor, a pair of conjugate zeros can be shifted onto the \( \delta \)-axis. The rest of the cycle is developed by the same type of reasoning as in the first case. This leads to a cycle of the form shown in Fig. 28, where the remainder function \( Z_4 \) is a \( \text{PRD}_Z \) function whose numerator and denominator polynomials are of degree \( n-2 \).

It is well known that several modifications of this procedure can be used to realize a PR function as the driving point impedance of an RLCT network. Analogous procedures can be carried out for the RL'C"T case.

A final point concerning poles at the intersection of the \( \delta \)-axis and the real axis is worth mentioning. It is clear that a first order pole of a \( \text{PRD}_Z \) function at \( s = -\delta \) cannot be removed without the possibility of destroying the \( \text{PRD}_Z \) character of the remainder function. This follows since the \( D_Z \)-function of an impedance of the form

\[
\frac{1}{s + \delta}
\]

is non-negative along the \( \delta \)-axis. Furthermore, a simple pole of a \( \text{PRD}_Y \) function at \( s = -\delta \) must be of the form

\[
\frac{-k}{s + \delta} \quad \text{where } k > 0;
\]

it can therefore not be removed by any RL'C"T structure.

These observations lead to the following pertinent question in connection with the Brune procedure. What happens when the minimum value of the \( D_Z \)-function occurs at \( s = -\delta \). Unless a resistance removal equal to \( \text{Min } D_Z(-\delta + j\omega) = D_Z(-\delta) \) leads to a double order zero after an appropriate reactance removal, the Brune procedure will come to a halt. It is true, however, that \( D_Z(-\delta) = 0 \) implies at least a double-order zero at \( s = -\delta \). To see this, let

\[
\text{Min } D_Z(-\delta + j\omega) = D_Z(-\delta) = R.
\]

Removing the resistance \( R \) leaves a \( \text{PRD}_Z \) remainder function

\[
Z_1(s) = Z(s) = R.
\]

The difference function of \( Z_1 \) must equal zero in the limit as \( \omega \to 0 \), i.e.

\[
D_{Z_1}(-\delta) = \text{Re } Z_1(-\delta) - \alpha \frac{\partial}{\partial \omega} \text{Im } Z_1(-\delta) = 0.
\]

In order to create a zero of \( Z_1 \) at \( s = -\delta \), we subtract off an appropriate lossy inductor, \( kZ_L \), where the multiplier \( k \) is determined as follows:
Fig. 26. Typical Brune cycle.

Fig. 27. Equivalent circuits for eliminating the negative element of the T circuit of lossy inductances in the Brune cycle.

Fig. 28. Alternative form of a typical Brune cycle.
Re $Z_1(-\delta) - k\alpha = 0$

Im $Z_1(-\delta) - kw = 0$. \hspace{1cm} (133)

The second equation is satisfied for any $k$, so

$$k = \frac{\text{Re } Z_1(-\delta)}{\alpha}. \hspace{1cm} (134)$$

Let

$$Z_2(s) = Z_1(s) - kz_L = Z_1(s) - k(s + \alpha_L) \hspace{1cm} (135)$$

To show that $Z_2$ has at least a double order zero at $s = -\delta$, we will show that $Z_2$, in addition to being zero at $s = -\delta$, has a vanishing first derivative at this point, that is,

$$\frac{d}{ds} Z_2(-\delta) = 0. \hspace{1cm} (136)$$

But

$$\frac{d}{ds} Z_2(-\delta) = \frac{d}{ds} Z_1(-\delta) - k = \frac{\partial}{\partial \omega} \text{Im } Z_1(-\delta) - j \frac{\partial}{\partial \omega} \text{Re } Z_1(-\delta) - k. \hspace{1cm} (137)$$

From equations 132 and 134 it is seen that

$$\frac{\partial}{\partial \omega} \text{Im } Z_1(-\delta) = \frac{\text{Re } Z_1(-\delta)}{\alpha} = k; \hspace{1cm} (138)$$

and since $\text{Re } Z_1(-\delta + j\omega)$ is an even function of $\omega$ such that

$$\frac{\partial}{\partial \omega} \text{Re } Z_1(-\delta) = 0, \hspace{1cm} (139)$$

it follows that

$$\frac{d}{ds} Z_2(-\delta) = 0. \hspace{1cm} (140)$$

$Z_2(s)$ must therefore have at least a double order zero at $s = -\delta$, and since $Y_2 = Z_2^{-1}$ is a PRD$_Y$ function, it follows that at least a double order pole of $Y_2$ can be removed with the assurance of a PRD$_Y$ remainder function.

As an illustration of the extended Brune procedure consider the following example.

**EXAMPLE 1**

$$Z(s) = \frac{s^2 + 5s + 12}{s^2 + 6s + 9}; \hspace{1cm} \begin{cases} \alpha_L = 3 \\ \alpha_C = 1 \end{cases} \hspace{1cm} \begin{cases} \alpha = 1 \\ \delta = 2 \end{cases} \hspace{1cm} (141)$$

To test whether or not $Z(s)$ is a PRD$_Z$ function in this case, we first determine
\[ D_Z(-\delta + j\omega) = \text{Re} \left( Z(-2 + j\omega) - \frac{\alpha}{\omega} \text{Im} \left( Z(-2 + j\omega) \right) \right) \]

\[ = \frac{\omega^4 - 6\omega^2 + 17}{\omega^4 + 2\omega^2 + 1} = \frac{(\omega^2 - 3)^2 + 8}{\omega^4 + 2\omega^2 + 1} > 0 \quad \text{for all } \omega. \quad (142) \]

\[ Z(s) \text{ is clearly a PRD}_Z \text{ function and therefore realizable as the driving-point impedance of an RL'C'T network.} \]

No preamble is required in this case, so the first step is to determine the minimum value of \( D_Z(-2 + j\omega) \). It is found that this minimum occurs at \( s = -2 \pm j\sqrt{5} \) and is equal to \( \frac{1}{3} \), that is,

\[ \text{Min} \ D_Z(-2 + j\omega) = D_Z(-2 \pm j\sqrt{5}) = \frac{1}{3} = R. \quad (143) \]

Removing a resistance equal to \( R \) leaves the following PRD\(_Z\) function

\[ Z_1(s) = Z(s) - \frac{1}{3} = \frac{2}{3} \frac{s^2 + 9}{s^2 + 6s + 9} + \frac{27}{2} \]

The value of \( Z_1(s) \) at \( s = -2 + j\sqrt{5} \) determines whether this is a Case 1 or a Case 2 situation.

\[ Z_1(-2 + j\sqrt{5}) = -\frac{1}{6} (1 + j\sqrt{5}) = k_{L_1}^L Z_{L_1}(-2 + j\sqrt{5}). \quad (145) \]

It is seen to be a Case 1 situation, so we remove a lossy inductor with a multiplier equal to \( k_{L_1}^L = -\frac{1}{6} \) (Fig. 29). The remainder impedance \( Z_2 \) must be a PRD\(_Z\) function with a pair of conjugate zeros at \( s = -2 \pm j\sqrt{5} \)

\[ Z_2(s) = Z_1(s) - k_{L_1}^L Z_{L_1} = \frac{1}{6} \frac{(s + 9)}{s^2 + 6s + 9}. \quad (146) \]

Let us therefore turn \( Z_2(s) \) upside down and extract the \( \delta \)-axis poles of \( Y_2 \):

\[ Y_2(s) = 2 \frac{s + 1}{s^2 + 4s + 9} + \frac{4}{s + 9} = \frac{1}{2(s+3)} + \frac{6}{2(s+1)} + \frac{1}{4(s+3)} + \frac{3}{2}. \quad (147) \]

The corresponding network realization is placed in evidence by this expansion of \( Y_2 \) and is shown in Fig. 30.

Absorbing the negative inductor leads to the final realization shown in Fig. 31.
Fig. 29. Partially completed Brune cycle for realizing the impedance (141).

Fig. 30. Network produced by the Brune procedure for realizing the impedance (141).

Fig. 31. Final realization of the impedance (141) in Brune form.
V. AN ALTERNATE RL'C"T DRIVING-POINT REALIZATION PROCEDURE

At this point it has been established that the PRD character is both necessary and sufficient for a function to be realizable as the driving-point function of an RL'C"T network.

Having modified the Brune procedure to establish the sufficiency, one might wonder whether any of the other known RLCT driving-point realization procedures can be adapted to the RL'C"T case. This we shall now discuss.

In particular, we shall show how to carry out driving-point realizations of PRD functions by adapting the strategy suggested by Darlington for the ordinary case.

This procedure and the transformerless driving-point realizations (to be discussed in Section VI) belong to what in RLCT terminology is known as even-part realization procedures. These are based on the reconstruction of a function from its even part behavior along the jω-axis. It is logical, therefore, to begin with a section establishing some analogous properties in the RL'C"T case.

5.1 DETERMINATION OF A NETWORK FUNCTION FROM ITS DIFFERENCE FUNCTION

The one-to-one correspondence between a PR function and its real part along the jω-axis is well known. In this connection it is important to recall that only minimum reactive and minimum susceptive PR functions can be uniquely related to their respective real part functions. [This restriction can be removed by allowing jω-axis impulses in the real part function. We shall, however, not explore the corresponding possibility for the RL'C"T case.]

An analogous situation exists in the RL'C"T case. That is, the DZ-function is uniquely determined by, and determines uniquely, the corresponding Z(s), provided Z(s) is strictly minimum reactive. (A similar statement can be made for DY-functions.)

To substantiate this statement we will now show how to reconstruct the strictly minimum reactive Z(s) from its DZ-function along the δ-axis. This objective can be accomplished in one of several ways, depending on whether one chooses to proceed in analogy with the ordinary procedures suggested by Bode, Gewertz, Miyata or Pantell. We have chosen to adapt the first and the last of these schemes; the first is chosen because it is relatively straightforward and yields some valuable insight; the last is chosen because of its usefulness in connection with the Bott and Duffin realization procedure, to be developed in Section VI.

5.1.1 Extended Bode Method

Let us begin by assuming we have available a DZ-function corresponding to a strictly minimum reactive PRDZ function, Z(s), which we want to determine. The
given $D_Z$-function is of the form

$$D_Z(-\delta + j\omega) = \frac{A_{2n} \omega^{2n} + A_{2(n-1)} \omega^{2(n-1)} + \cdots + A_0}{B_{2m} \omega^{2m} + B_{2(m-1)} \omega^{2(m-1)} + \cdots + B_0}$$

where $m \geq n$. (148)

Our objective will be achieved by first reconstructing $\tilde{Z}(\lambda) = Z(\lambda - \delta)$ from (See Section 3.6).

$$D_Z(j\xi) = \frac{A_{2n} \xi^{2n} + A_{2(n-1)} \xi^{2(n-1)} + \cdots + A_0}{B_{2m} \xi^{2m} + B_{2(m-1)} \xi^{2(m-1)} + \cdots + B_0},$$

and then make a simple frequency translation to determine $Z(s)$. To accomplish this, let us write

$$D_Z(\lambda)\big|_{\lambda = j\xi} = \left[ \text{Re} \, \tilde{Z}(\lambda) - \frac{\alpha}{\xi} \text{Im} \, \tilde{Z}(\lambda) \right]_{\lambda = j\xi} = \frac{1}{2} \left[ \lambda - \frac{\alpha}{\lambda} \tilde{Z}(\lambda) \right]_{\lambda = j\xi} + \frac{1}{2} \left[ -\lambda - \frac{\alpha}{-\lambda} \tilde{Z}(-\lambda) \right]_{\lambda = j\xi}$$

$$= \frac{1}{2} \left[ F(\lambda) + F(-\lambda) \right]_{\lambda = j\xi},$$

where we have defined

$$F(\lambda) = \frac{\lambda - \alpha}{\lambda} \tilde{Z}(\lambda).$$

Thus, for $\lambda = j\xi$, $F(-\lambda)$ is just the complex conjugate of $F(\lambda)$ such that

$$D_Z(-\lambda^2)\big|_{\lambda = j\xi} = \text{Ev} \left[ F(\lambda) \right]_{\lambda = j\xi}$$

Now, if $D_Z(j\xi)$ is known, then by the standard argument of replacing $\xi^2$ by $(-\lambda^2)$ we obtain an expression for half the sum of $F(\lambda)$ and $F(-\lambda)$. Therefore, if we can separate this sum into its two constituent parts, our objective has been achieved. Since we have assumed that $\tilde{Z}(\lambda)$ is strictly minimum reactive, the $j\xi$-axis can be considered as a separation boundary, with poles belonging to $\tilde{Z}(\lambda)$ in the left-hand plane, and poles belonging to $\tilde{Z}(-\lambda)$ in the right half-plane. The only feature that makes this separation problem slightly different from the more familiar one, is the possible occurrence of a double-order pole of $D_Z(-\lambda^2)$ at the origin. This can come about since a strictly minimum reactive PRDz function may have a first-order pole at $s = -\delta$ which is not removable by a reactance reduction (section 4.1).

The problem raised by possible poles as $s = -\delta$ can be solved by multiplying both sides of equation (152) by $-\lambda^2$ to obtain

$$-\lambda^2 D_Z(-\lambda^2)\big|_{\lambda = j\xi} = \text{Ev} \left[ G(\lambda) \right]_{\lambda = j\xi},$$

(153)
where we have defined
\[
G(\lambda) = -(\lambda(\lambda - \alpha) \tilde{Z}(\lambda).
\]

Having determined \(F(\lambda),\) or \(G(\lambda),\) we obtain \(\tilde{Z}(\lambda)\) from either
\[
\tilde{Z}(\lambda) = \frac{\lambda}{\lambda - \alpha} F(\lambda) \quad \text{or} \quad \tilde{Z}(\lambda) = \frac{G(\lambda)}{-\lambda(\lambda - \alpha)}.
\]

Due to the pole at \(\lambda = \alpha,\) however, the \(\tilde{Z}(\lambda)\) thus obtained is not strictly minimum reactive. The final step in either case is therefore to subtract off the pole at \(\lambda = \alpha.
\]

The extended Bode procedure for finding \(Z(\lambda)\) can be summarized as follows:
1. Substitute \(-\lambda^2\) for \(\xi^2\) in \(2D(j \xi)\) \[use \(2\xi^2 D(j \xi)\) instead if \(D(j \xi)\) has a double order pole at the origin\].
2. Make a partial fraction expansion of \(2D(-X^2)\) \[or \(2(-X^2) D(-X^2)\)].
3. Determine \(F(\lambda)\) \[or \(G(\lambda)\)] as the sum of the terms with poles in the left hand plane plus one half of the constant term of \(2D(-\lambda^2)\) \[or \(2(-\lambda^2) D(-\lambda^2)\)].
4. Determine the corresponding minimum reactive \(\tilde{Z}(\lambda)\) from \(F(\lambda)\) \[or \(G(\lambda)\)].
5. Determine the strictly minimum reactive \(\tilde{Z}_M(\lambda)\) from \(\tilde{Z}(\lambda)\) by subtracting off the pole at \(\lambda = \alpha.
\]

To illustrate this procedure we consider the following examples.

**EXAMPLE 2**

Consider the following \(D_Z\)-function
\[
D_Z(j \xi) = \frac{5 + \xi^2}{9 + \xi^2}
\]

and assume that \(\alpha = 1.\) Determine the corresponding strictly minimum reactive \(PRD_Z\) function, \(\tilde{Z}_M(\lambda).\)

**Step 1**
\[
D(-\lambda^2) = \frac{5 - \lambda^2}{9 - \lambda^2}
\]

Since \(D(-\lambda^2)\) has no double order pole at the origin the approach leading to \(F(\lambda)\) will be used.

**Step 2**
\[
2D(-\lambda^2) = \frac{-4}{3 - \lambda} + \frac{-4}{3 + \lambda} + 2
\]

**Step 3**
\[
F(\lambda) = \frac{\lambda + 5}{\lambda + 3}
\]

**Step 4**
\[
\tilde{Z}(\lambda) = \frac{\lambda}{\lambda - \alpha} F(\lambda) = \frac{\lambda(\lambda + 5)}{(\lambda - 1)(\lambda + 3)}
\]
EXAMPLE 3

Consider the following $D_Z$-function

$$D_Z(j \xi) = \frac{1 + \xi^2}{\xi^2(4 + \xi^2)}.$$  \hfill (158)

and assume that $\alpha = 1$. Determine the corresponding strictly minimum reactive PRD$_Z$ function, $\tilde{Z}_M(\lambda)$.

Step 1

$$D_Z(-\lambda^2) = \frac{1 - \lambda^2}{-\lambda^2(4 - \lambda^2)}.$$  

Since $D_Z(-\lambda^2)$ has a double-order pole at the origin the approach leading to $G(\lambda)$ will be used.

Step 2

$$2(-\lambda^2)D_Z(-\lambda^2) = \frac{-3}{2 - \lambda} + \frac{3}{2 + \lambda} + 2.$$  

Step 3

$$G(\lambda) = \frac{-3}{\lambda + 2} + 1 = \frac{\lambda + 1}{\lambda + 2}.$$  

Step 4

$$\tilde{Z}(\lambda) = \frac{G(\lambda)}{-\lambda(\lambda = \alpha)} = \frac{\lambda + 1}{-\lambda(\lambda - 1)(\lambda + 2)}.$$  

Step 5

$$\tilde{Z}_M(\lambda) = \tilde{Z}(\lambda) - \frac{k}{\lambda - 1}$$  

where $k = (\lambda - 1)\tilde{Z}(\lambda) \bigg|_{\lambda=1} = -\frac{1}{2}$.

ANSWER: $\tilde{Z}_M(\lambda) = \frac{1}{2} \frac{\lambda + 1}{\lambda(\lambda + 2)}$ (and $Z(s) = \tilde{Z}_M(\lambda) \bigg|_{\lambda=s+\delta}$) \hfill (159)
5.1.2 Extended Pantell Method

To reconstruct a minimum reactive PR function, $Z(s)$, from its even part, Pantell has suggested the following procedure. Express $Z(s)$ as the sum of the known even part and the unknown odd part. Determine the unknown numerator coefficients of the odd part by requiring a cancellation of all the right half plane poles of the sum.

This procedure is directly applicable for reconstructing $F(X)$ (or $G(W)$) from equations (152) (or (153)); the reason being that $F(X)$ (or $G(X)$) is a PR function in the $A$-plane as is easily seen from the equation just referred to.

To summarize the procedure, let us write

$$A X^2 + A_2 X^2(n-1) + \ldots + A_n = \frac{2n}{2(n-1)}$$

$$B + B X^n + B_2(n-1) + \ldots + B_n$$

Equation (160)

$F(X)$ can then be expressed as the sum of a function having (160) as its even part plus an odd function;

$$F(X) = \frac{A_2 X^n A_{2(n-1)} X^{2(n-1)} + \ldots + A_n}{B_2 X^n + B_{2(n-1)} X^{2(n-1)} + \ldots + B_n}$$

Equation (161)

The unknown numerator coefficients of the odd part are determined by requiring a cancellation of all the right half plane poles of the sum. The resulting $F(\lambda)$ is a PR function, and $\tilde{Z}(\lambda)$, which is related to $F(\lambda)$ by equation (151), will have a pole at $\lambda = \alpha$. Removing this pole leads to the desired strictly minimum reactive $\tilde{Z}_M(\lambda)$.

The procedure of reconstructing $G(\lambda)$, for the case where $D(-\lambda^2)$ has a double order pole at the origin, is entirely analogous.

As an example let us consider the same function as was used in example 2.

EXAMPLE 4

The following $D_Z$-function is given

$$D_Z(j\xi) = 5 + \frac{\xi^2}{9 + \xi^2}; \alpha = 1$$

Equation (162)

According to Eq. 161

$$F(\lambda) = \frac{5 - \lambda^2}{9 - \lambda^2} + \frac{C_0}{9 - \lambda^2}$$

where $C_0$ can be determined by setting the numerator of $F(\lambda)$ equal to zero at the right
half plane pole of $9 - \lambda^2$, i.e.,

$$
\left[ 5 - \lambda^2 + \lambda C_\alpha \right]_{\lambda=3} = 0
$$

or $C_\alpha = \frac{4}{3}$.

The PR function $F(\lambda)$ thus becomes

$$
F(\lambda) = \frac{5 - \lambda^2}{9 - \lambda^2} + \frac{4}{3} \frac{\lambda}{9 - \lambda^2} = \frac{\lambda + 5}{3 \lambda + 3}
$$

which is seen to check with the result of example 2.

Both of the foregoing methods suffer from the disadvantage of not being able to determine the strictly minimum reactive PRD$_Z$ function, $Z_M(\lambda)$, directly. Both require that one first determine the minimum reactive PRD$_Z$-function, $\tilde{Z}(\lambda)$, having a pole at $\lambda = \alpha$. It is possible to avoid this detour by modifying the Pantell approach in accordance with the following development.

The starting point is again the $D_Z$-function which can be expressed as follows

$$
D_Z(-\lambda^2) \bigg|_{\lambda=j\xi} = \left[ \text{Ev} \tilde{Z}_M(\lambda) - \frac{\alpha}{\lambda} \text{Odd} \tilde{Z}_M(\lambda) \right]_{\lambda=j\xi}.
$$

This relationship can be written in the form

$$
\left[ \text{Ev} \tilde{Z}_M(\lambda) + \text{Odd} \tilde{Z}_M(\lambda) \right]_{\lambda=j\xi} = \tilde{Z}_M(\lambda) \bigg|_{\lambda=j\xi} = \left[ D(-\lambda^2) + \frac{\lambda + \alpha}{\lambda} \text{Odd} \tilde{Z}_M(\lambda) \right]_{\lambda=j\xi}.
$$

Analytic continuation therefore suggests that

$$
\tilde{Z}_M(\lambda) = D_Z(-\lambda^2) + \frac{\lambda + \alpha}{\lambda} \text{Odd} \tilde{Z}_M(\lambda).
$$

By an argument similar to the Pantell scheme, we can then write

$$
\tilde{Z}_M(\lambda) = \frac{A_{2n+1} \lambda^{2n} + A_{2(n-1)} \lambda^{2(n-1)} + \cdots + A_o}{B_{2n+1} \lambda^{2n} + B_{2(n-1)} \lambda^{2(n-1)} + \cdots + B_o}
$$

$$
+ (\lambda + \alpha) \frac{C_{2(n-1)} \lambda^{2(n-1)} + \cdots + C_o}{B_{2n+1} \lambda^{2n} + \cdots + B_o}.
$$

where the first rational function in $\lambda^2$ on the right-hand side represents $D_Z(-\lambda^2)$.

Notice that $\tilde{Z}_M(\lambda)$ is assumed to be strictly minimum reactive such that the order of the unknown polynomial must be less than or equal to $2(n-1)$. The unknown coefficients, $C_i$, are again determined by requiring a cancellation of all right-hand plane poles of the denominator of $D_Z(-\lambda^2)$.

It should be clear that this approach will work whether $D_Z(-\lambda^2)$ has a double-
order pole at the origin or not.

We illustrate this procedure by again considering the difference functions considered in the two first examples of this section.

EXAMPLE 5

Consider

\[ D_Z(j\xi) = \frac{5 + \xi^2}{9 + \xi^2}; \quad \alpha = 1. \]

According to Eq. 166

\[ \tilde{Z}_M(\lambda) = \frac{5 - \lambda^2}{9 - \lambda^2} + (\lambda + 1) \frac{C_0}{9 - \lambda^2} \]  

(167)

where

\[ \left[ 5 - \lambda^2 + (\lambda + 1) C_0 \right]_{\lambda = 3} = 0 \]

or

\[ C_0 = 1. \]

We have thus found

\[ \tilde{Z}_M(\lambda) = \frac{5 - \lambda^2}{9 - \lambda^2} + \frac{\lambda + 1}{9 - \lambda^2} = \frac{\lambda + 2}{\lambda + 3} \]

which checks with Eq. 157.

EXAMPLE 6

Consider

\[ D_Z(j\xi) = \frac{1 + \xi^2}{\xi^2(4 + \xi^2)}; \quad \alpha = 1. \]

According to Eq. 166,

\[ \tilde{Z}_M(\lambda) = \frac{1 - \lambda^2}{-\lambda^2(4 - \lambda^2)} + (\lambda + 1) \frac{C_1\lambda^2 + C_0}{-\lambda^2(4 - \lambda^2)}. \]  

(168)

Cancellations must occur at \( \lambda = 0, 2 \), so we obtain the following two simultaneous equations for \( C_0 \) and \( C_1 \):

\[ C_0 = -1 \]
Solution of these equations yields

\[ C_0 = -1 \text{ and } C_1 = \frac{1}{2}. \]

We have thus found

\[
\mathcal{Z}_M(\lambda) = \frac{1 - \lambda^2}{-\lambda^2(4 - \lambda^2)} + (\lambda + 1) \frac{\frac{1}{2} \lambda^2 - 1}{-\lambda^2(4 - \lambda)} = \frac{1}{2} \frac{\lambda + 1}{\lambda(\lambda + 2)}
\]

which checks with the result of example 3.

In order to reconstruct a strictly minimum susceptive function \( Y(s) \) from its \( D_Y \)-function, one can proceed in a dual manner. Making the appropriate modifications in the preceding method leads to the following analogue of Eq. 166.

\[
\mathcal{Y}(\lambda) = \frac{A_{2n} \lambda^{2n} + A_{2(n-1)} \lambda^{2(n-1)} + \cdots + A_0}{B_{2n} \lambda^{2n} + B_{2(n-1)} \lambda^{2(n-1)} + \cdots + B_0}
+ (\lambda - \alpha) \frac{C_{2(n-1)} \lambda^{2(n-1)} + \cdots + C_0}{B_{2n} \lambda^{2n} + \cdots + B_0}
\]

where the first rational function on the right hand side corresponds to \( D_Y(-\lambda^2) \). The unknown coefficients \( C_i \) are again determined such as to cancel right half plane poles of \( D_Y(-\lambda^2) \).

5.2 EXTENDED DARLINGTON PROCEDURE

Darlington has shown how to realize any minimum reactive PR function as the driving point impedance of a lossless coupling network terminated in a single resistance. The Darlington configuration is shown in Fig. 32a.

Using the Darlington strategy, we will show how any strictly minimum reactive PRD\(_Z\) function can be realized as the driving point impedance of the analogous structure shown in Fig. 32b.

The following sequence of steps used by Darlington in the RLCT case;

\[ Z(s) \rightarrow \text{Re } Z(j\omega) \rightarrow |Z_{21}(j\omega)|^2 \rightarrow z_{22}, z_{21}, z_{11} \rightarrow \text{Figure 32a,} \]

suggests the following sequence of steps for the RL'C''T case;

\[ \mathcal{Z}(\lambda) \rightarrow D_Z(j\xi) \rightarrow |\mathcal{Z}_{21}(j\xi)|^2 \rightarrow \mathcal{z}_{22}, \mathcal{z}_{21}, \mathcal{z}_{11} \rightarrow \text{Fig. 32b.} \]

In the remainder of this section we shall show that these steps, when properly executed, lead to Darlington-like realizations for PRD\(_Z\) functions.
Fig. 32. Resistance-terminated reactance network relevant to the Darlington procedure for realizing (a) RLCT driving-point impedances; (b) RL'C'T driving-point impedances.

The first step requires no explanation.

The second step has already been given in section 3.1. The result is as follows

\[ |\tilde{Z}_{21}(j\xi)|^2 = \text{Re} \tilde{Z}(j\xi) - \frac{\alpha}{\xi} \text{Im} \tilde{Z}(j\xi). \] (172)

The third step of the procedure consists of deducing \( Z_{21}(X) \) from the \( |\tilde{Z}_{21}(j\xi)|^2 \) expression (Eq. 172). This can be done by employing the normal procedure for solving an equation of the form

\[ |\tilde{Z}_{21}(j\xi)|^2 = \frac{a_n\xi^{2n} + a_{n-1}\xi^{2(n-1)} + \cdots + a_0}{b_n\xi^{2n} + b_{n-1}\xi^{2(n-1)} + \cdots + b_0} \] (173)

leading to a transfer impedance of the form

\[ \tilde{Z}_{21}(\lambda) = \frac{m(\lambda)}{m_1(\lambda) + n_1(\lambda)} \] (174)

where \( m \) and \( n \) are even and odd polynomials in \( \lambda \), and \( m_1 + n_1 \) is a Hurwitz polynomial (from the assumption that \( \tilde{Z}(\lambda) \) is strictly minimum reactive).

The reader will recall that in order to make this separation, augmentation is usually necessary to make the zero-pattern of \( \tilde{Z}_{21}(\lambda)\tilde{Z}_{21}(-\lambda) \) possess zeros of even multiplicity only. (It should be noted that a zero of \( \tilde{Z}_{21}(\lambda)\tilde{Z}_{21}(-\lambda) \) at the origin must be of fourth order to be considered of even multiplicity (see Example 3).)

The extraction of \( \tilde{Z}_{21}(\lambda) \) and \( \tilde{Z}_{21}(-\lambda) \) from \( \tilde{Z}_{21}(\lambda)\tilde{Z}_{21}(-\lambda) \) is then accomplished by the following two steps:

1. Poles of \( \tilde{Z}_{21}(\lambda)\tilde{Z}_{21}(-\lambda) \) lying in the left-hand plane are assigned to \( Z_{21}(\lambda) \) and those in the right half-plane to \( \tilde{Z}_{21}(-\lambda) \).
2. Zeros of $\tilde{Z}_{21}(\lambda)\tilde{Z}_{21}(-\lambda)$ are divided evenly between $\tilde{Z}_{21}(\lambda)$ and $\tilde{Z}_{21}(-\lambda)$.]

The next to the last step in the 171 sequence is to rewrite Eq. 174 such as to
identify the two-port parameters $z_{21}$ and $z_{22}$. This is accomplished by the follow-
ing equation

$$\tilde{Z}_{21}(\lambda) = \frac{\frac{m}{(\lambda-\alpha)\frac{n_1}{\lambda}}}{1 + \frac{n_1}{\lambda} + \frac{m_1 + \alpha \frac{n_1}{\lambda}}{\lambda}}$$

which suggests the following relations:

$$\tilde{Z}_{21}(\lambda) = \frac{m}{(\lambda-\alpha)\frac{n_1}{\lambda}} \quad \text{and} \quad \tilde{Z}_{22} = \frac{m_1 + \alpha \frac{n_1}{\lambda}}{\lambda}$$

The final step is to show that these open-circuit impedance parameters satisfy
the conditions for $L'C''T$ realizability. These conditions were derived in Section II.
Of major importance in this connection is (i) the required PRD$_Z$ reactance character
of the driving-point impedance $\tilde{Z}_{22}(\lambda)$ and (ii) the requirement that the degree of the
numerator polynomial of $\tilde{Z}_{21}(\lambda)$ cannot exceed the degree of the numerator poly-
nomial of $\tilde{Z}_{22}(\lambda)$, i.e.

$$\deg m \leq \deg \left[ m_1 + \alpha \frac{n_1}{\lambda} \right].$$

This last requirement follows from observing that the open-circuit impedance
parameters must have either a pole or a zero at infinity and that $\tilde{Z}_{21}(\lambda)$ cannot have
a pole at infinity unless $\tilde{Z}_{22}(\lambda)$ also has a pole at this point, if the residue condition
is to be satisfied.

Condition (i) is readily seen to be satisfied since $\tilde{Z}_{22}$ is the ratio of the "even"
over "odd" part of a Hurwitz polynomial factored according to Theorem 2.

The inequality of condition (ii) is also seen to be satisfied by observing that $Z_{21}(\lambda)$
(Eq. 174) cannot have any poles at infinity and be consistent with the difference func-
tion from which it was derived.

In order to complete the set of open-circuit impedances $\tilde{Z}_{11}$, $\tilde{Z}_{21}$, and $\tilde{Z}_{22}$, we
must find a $\tilde{Z}_{11}$ such that:

a. the set $\tilde{Z}_{11}$, $\tilde{Z}_{21}$, and $\tilde{Z}_{22}$ is realizable as an $L'C''T$ two-port.

b. the driving-point impedance resulting from the $L'C''T$ two-port terminated in
a one-ohm resistance is strictly minimum reactive.

It is readily shown by using standard arguments that $\tilde{Z}_{11}(\lambda)$ must be chosen to have
the same poles as $\tilde{Z}_{21}$ and $\tilde{Z}_{22}$, and such that the impedances $\tilde{Z}_{11}(\lambda)$, $\tilde{Z}_{21}(\lambda)$ and $\tilde{Z}_{22}(\lambda)$
are compact at all poles.

The network realization of \( \tilde{Z}(\lambda) \) is completed by realizing the \( L'C''T \) network by the modified Cauer procedure discussed in section 2.5 and then terminating this two-port in a one ohm resistance.

It is also easily verified that the network thus obtained actually realizes a driving point impedance with the desired \( D_Z \)-function. Since \( \tilde{Z}_{11}(\lambda) \) was chosen such that the driving point impedance of this realization is strictly minimum reactive, it follows from our discussion of difference function sufficiency in section 5.1, that the driving-point impedance thus obtained must be the given \( \tilde{Z}(\lambda) \).

We have, therefore, shown that all strictly minimum reactive \( PRD_Z \) functions can be realized as the driving-point impedance of a network of the type shown in Fig. 32b.

### 5.3 CASE B OF THE DARLINGTON PROCEDURE

The reader has no doubt discovered that the general \( RL'C''T \) realization procedure of the previous section was based on the so-called case A situation of the RLCT case. In order to make the procedure general, it is usually necessary to augment the pole-zero pattern of \( \tilde{Z}_{21}(\lambda)\tilde{Z}_{21}(-\lambda) \) in order to obtain a \( \tilde{Z}_{21}(\lambda) \) of the form of Eq. 174. Furthermore, it is permissible but generally undesirable (due to unnecessary complicated network realizations) to overaugment \( \tilde{Z}_{21}(\lambda)\tilde{Z}_{21}(-\lambda) \) in this procedure. There is one special case, however, where the case A procedure above leads to unnecessary augmentations. It happens whenever \( \tilde{Z}_{21}(\lambda)\tilde{Z}_{21}(-\lambda) \) has a pair of zeros of odd multiplicity located at \( \lambda = \pm \alpha \), and \( \tilde{Z}_{21}(\lambda)\tilde{Z}_{21}(-\lambda) \) has an even number of poles in the region \( -\alpha < \nu < \alpha \) along the real axis. This situation is better handled by a procedure analogous to the so-called case B development of RLCT networks.

To see how to proceed in this case, let us assume that \( \tilde{Z}_{21}^B(\lambda)\tilde{Z}_{21}^B(-\lambda) \) has been constructed by the scheme suggested in (171), and that the resulting pole-zero pattern has a pair of zeros of odd multiplicity located at \( \lambda = \pm \alpha \). In addition, we will assume that \( \tilde{Z}_{21}^B(\lambda)\tilde{Z}_{21}^B(-\lambda) \) has an even number of poles located between \( \lambda = -\alpha \) and \( \lambda = \alpha \) along the real axis. (The reason for this restriction will become clear when we later make use of Theorem 2).

The extraction of \( \tilde{Z}_{21}^B(\lambda) \) and \( \tilde{Z}_{21}^B(-\lambda) \) from \( \tilde{Z}_{21}^B(\lambda)\tilde{Z}_{21}^B(-\lambda) \) is the same as for case A except for the inclusion of a zero in \( \tilde{Z}_{21}^B(\lambda) \) at \( \lambda = -\alpha \) and a corresponding zero of \( \tilde{Z}_{21}^B(-\lambda) \) at \( \lambda = \alpha \). Let us therefore write

\[
\tilde{Z}_{21}^B(\lambda) = \frac{(\lambda + \alpha)m}{m_1 + n_1}, \tag{178}
\]

where \( m \) and \( n \) denote even and odd polynomials in \( \lambda \), respectively, and the denominator polynomial of \( \tilde{Z}_{21}^B(\lambda) \) is a Hurwitz polynomial with an even number of zeros to the right of \( \lambda = -\alpha \). Then, if we write
we can make the identifications

\[ \tilde{Z}_B(\lambda) = \frac{(\lambda+\alpha)m}{m_1-\alpha} \]  

\[ \tilde{Z}_{21}(\lambda) = \frac{(\lambda+\alpha)n_1}{1 + \tilde{Z}_{22}(\lambda)} \]  

\[ \tilde{Z}_{22}(\lambda) = \frac{(\lambda+\alpha)n_1}{m_1-\alpha} \]  

Having established in Theorem 3 that the "odd" over "even" parts of a polynomial like the denominator polynomial of \( \tilde{Z}_{21}(\lambda) \) must be a lossy reactance function, we know that \( \tilde{Z}_{22}(\lambda) \) is a PRDZ reactance function. Therefore, by the same arguments as in case A we can determine a Darlington realization of the form shown in Fig. 32b.

It should be noted that whereas a case A realization always exists, a case B realization requires a pair of zeros of \( \tilde{Z}_{21}(\lambda) \tilde{Z}_{21}(-\lambda) \) located at \( \lambda = \pm \alpha \). The requirement of an even number of poles of \( \tilde{Z}_{21}(\lambda) \tilde{Z}_{21}(-\lambda) \) between \( \lambda = -\alpha \) and \( \lambda = \alpha \) is no restriction, however, since this situation can always be produced by augmentation. If this is required, however, one might as well augment to a case A situation by making the zero pair located at \( \lambda = \pm \alpha \) of even multiplicity.

5.4 AN ALTERNATIVE METHOD FOR OBTAINING THE DARLINGTON RESULT

In the previous two sections we have several times referred to the polynomial factorizations of section 3.6. The real beauty of these factorizations lies in the resulting similarities that can thus be achieved between the RLCT and the RL'C'T cases. To illustrate this point let us establish the general Darlington procedure (case A) from a somewhat different point of view.

The strategy in this approach is analogous to the standard alternate way of introducing the Darlington realizations in the RLCT case.11

Expressing the driving point impedance of Fig. 32b in terms of the open-and short-circuit parameters of the two-port, we can write

\[ \tilde{Z}(\lambda) = \frac{\tilde{Z}_{11}\tilde{Z}_{22} - \tilde{Z}_{21}^2}{\tilde{Z}_{22} + 1} + \tilde{Z}_{11} = \frac{(\frac{1}{\tilde{Z}_{22}}) + 1}{\tilde{Z}_{22} + 1} \]  

where all quantities have been expressed in \( \lambda \)-coordinates.

To relate Eq. 181 to the given \( \tilde{Z}(\lambda) \) let us write the latter in factored form
\[
\tilde{Z}(\lambda) = \frac{P(\lambda)}{Q(\lambda)} = \frac{A_{2n+1}(\lambda+\alpha)^n(\lambda-\alpha)^n + A_{2n}(\lambda+\alpha)^n(\lambda-\alpha)^n + \cdots + A_1(\lambda+\alpha) + A_0}{B_{2n+1}(\lambda-\alpha)^n(\lambda+\alpha)^n + B_{2n}(\lambda-\alpha)^n(\lambda+\alpha)^n + \cdots + B_1(\lambda-\alpha) + B_0},
\]

(182)

where the 'even' and 'odd' parts of \(P(\lambda)\) and \(Q(\lambda)\) have been denoted \(M\) and \(N\), respectively. In section 3.6 we established that the ratio of certain 'even' over 'odd' parts of Hurwitz polynomials corresponds to physically realizable lossy reactance or susceptance functions, which suggests the following method for identifying (181) and (182).

Let

\[
\tilde{Z}(\lambda) = \frac{M_A + N_A}{M_B + N_B},
\]

(183)

whereupon a comparison with Eq. 181 shows that

\[
\tilde{z}_{11}(\lambda) = \frac{M_A}{N_B} = \frac{A_{2n}(\lambda+\alpha)^n(\lambda-\alpha)^n + \cdots + A_0}{(\lambda-\alpha)[B_{2n+1}(\lambda-\alpha)^n(\lambda+\alpha)^n + \cdots + B_1]}.
\]

(184)

\[
\tilde{z}_{22}(\lambda) = \frac{M_B}{N_B} = \frac{B_{2n}(\lambda-\alpha)^n(\lambda+\alpha)^n + \cdots + B_0}{(\lambda-\alpha)[B_{2n+1}(\lambda-\alpha)^n(\lambda+\alpha)^n + \cdots + B_1]}.
\]

(185)

\[
\tilde{y}_{22}(\lambda) = \frac{M_A}{N_A} = \frac{A_{2n}(\lambda+\alpha)^n(\lambda-\alpha)^n + \cdots + A_0}{(\lambda+\alpha)[A_{2n+1}(\lambda+\alpha)^n(\lambda-\alpha)^n + \cdots + A_1]}.
\]

(186)

The transfer impedance associated with these driving-point functions can be determined from the relation

\[
\tilde{z}_{11} \tilde{z}_{22} - \tilde{z}_{12}^2 = \frac{\tilde{z}_{11}}{\tilde{y}_{22}} = \frac{N_A}{N_B}.
\]

(187)

and is found to be

\[
\tilde{z}_{21}(\lambda) = \sqrt{\frac{M_AM_B - N_AN_B}{N_B}}.
\]

(188)

The quantity appearing under the radical in Eq. 188 is the numerator of the impedance derived difference function, as shown in section 3.6. In general, this polynomial is not a full square, so one must augment by supplying factors corresponding to zeros of

58
D_Z(λ) which are not evenly repeated. This corresponds to the augmentation discussed under case A of the preceding section. Let us therefore assume that the given impedance \( \tilde{Z}(λ) \) is already properly augmented such that the quantity under the radical of Eq. 188 is a full square. Then it is easily seen that \( \tilde{Z}_{22} \) and \( \tilde{Z}_{12} \) as given by Eqs. 185 and 188 are the same as in case A of the preceding section. It is also readily shown that \( \tilde{Z}_{11} \) as given by Eq. 184 is the same as the \( \tilde{Z}_{11} \) determined by requiring that all poles of \( \tilde{Z}_{22} \), \( \tilde{Z}_{21} \), and \( \tilde{Z}_{11} \) must have a compact set of residues.

Thus all strictly minimum reactive PRD_Z functions can be realized as the driving-point impedance of an L'C"T two-port, characterized by Eqs. 184-6, and terminated in a 1-ohm resistance.

5.5 ILLUSTRATIVE EXAMPLES

As our first example we will make a Darlington realization of the function that was used to illustrate the Brune procedure.

EXAMPLE 7

Consider the following driving-point impedance

\[
Z(s) = \frac{s^2 + 5s + 12}{s^2 + 6s + 9}, \quad \alpha = \frac{1}{2}(\alpha_L - \alpha_C) = 1
\]

\[
\delta = \frac{1}{2}(\alpha_L + \alpha_C) = 2
\]

The first step in the Darlington development is to determine

\[
D_Z(j\xi) = \text{Re}Z(j\xi) - \frac{\alpha}{\xi} \text{Im}Z(j\xi) = \frac{\xi^4 - 6\xi^2 + 17}{\xi^4 + 2\xi^2 + 1},
\]

(189)

where we have expressed everything in \( \lambda \)-coordinates such that

\[
\tilde{Z}(\lambda) = \frac{\lambda^2 + \lambda + 6}{\lambda^2 + 2\lambda + 1}
\]

\[
|\tilde{Z}_{21}(j\xi)|^2 \text{ equals } D_Z(j\xi) \text{ so we can write}
\]

\[
\tilde{Z}_{21}(\lambda)\tilde{Z}_{21}(-\lambda)|_{\lambda = j\xi} = \frac{\lambda^4 + 6\lambda^2 + 17}{\lambda^4 - 2\lambda^2 + 1}
\]

(192)

and by the identity theorem for analytic functions

\[
\tilde{Z}_{21}(\lambda)\tilde{Z}_{21}(-\lambda) = \frac{\lambda^4 + 6\lambda^2 + 17}{\lambda^4 - 2\lambda^2 + 1}
\]

\[
= (\lambda + .749 - j1.89)(\lambda + .749 + j1.89)(-\lambda + .749 - j1.89)(-\lambda + .749 + j1.89)
\]

(193)
Fig. 33. Pole-zero plot of
\[ \mathcal{Z}_{21}(\lambda) \mathcal{Z}_{21}(-\lambda) = \frac{\lambda^4 + 6\lambda^2 + 17}{\lambda^2 - 2\lambda^2 + 1}. \]

Fig. 35. Darlington realization of impedance (189).

Fig. 34. Extraction of \( \mathcal{Z}_{21}(\lambda) \) from \( \mathcal{Z}_{21}(\lambda) \mathcal{Z}_{21}(-\lambda) \) of Fig. 33, requires augmentation of the pole-zero plot as shown in (a); subsequent pole-zero separation then readily yields \( \mathcal{Z}_{21}(\lambda) \) as shown in (b).

Fig. 36. Network realizing the impedance function (195) in Darlington form.
The pole-zero plot of $Z_{21}(\lambda)Z_{21}(-\lambda)$ is shown in Fig. 33. Since there are no zeros located at $\lambda = \pm \alpha$, a case A approach is pertinent. Proper augmentation and separation of $Z_{21}(\lambda)Z_{21}(-\lambda)$ in this case is illustrated by Fig. 34.

From Fig. 34b we construct

$$
\mathcal{Z}_{21}^A(\lambda) = \frac{\lambda^4 + 6\lambda^2 + 17}{\lambda^4 + 3.5\lambda^3 + 8.12\lambda^2 + 9.74\lambda + 4.12}
$$

$$
= \frac{\lambda^4 + 6\lambda^2 + 17}{(\lambda - 1)(3.5\lambda^2 + 9.74)}
$$

which yields

$$
\mathcal{Z}_{22}(\lambda) = \frac{1}{3.5} \frac{\lambda^4 + 11.62\lambda^2 + 13.87}{(\lambda = 1)(\lambda^2 + 2.79)}
$$

and

$$
\mathcal{Z}_{21}(\lambda) = \frac{1}{3.5} \frac{\lambda^4 + 6\lambda^2 + 17}{(\lambda - 1)(\lambda^2 + 2.79)}
$$

If the open-circuit impedances are written as a partial-fraction expansion, and the residues at all poles are required to be compact, we obtain

$$
\mathcal{Z}_{22}(\lambda) = \frac{1}{3.5} \left[ (\lambda + 1) + \frac{7}{\lambda - 1} + \frac{2.84(\lambda + 1)}{\lambda^2 + 2.79} \right]
$$

$$
\mathcal{Z}_{21}(\lambda) = \frac{1}{3.5} \left[ (\lambda + 1) + \frac{6.34}{\lambda - 1} - \frac{2.13(\lambda + 1)}{\lambda^2 + 2.79} \right]
$$

$$
\mathcal{Z}_{11}(\lambda) = \frac{1}{3.5} \left[ (\lambda + 1) + \frac{5.74}{\lambda - 1} + \frac{1.59(\lambda + 1)}{\lambda^2 + 2.79} \right]
$$

The final realization is shown in Fig. 35.

As our second example let us realize a simple bilinear PRDZ function by the approach given in section 5.4.

**EXAMPLE 8**

Make a Darlington realization of the following function if it is found to be a PRDZ function

$$
\mathcal{Z}(\lambda) = \frac{\lambda + \frac{1}{\alpha}}{\lambda + 1}, \quad \alpha = 1 \text{ and } \delta = 2.
$$

(195)

In factored form $Z(\lambda)$ becomes

$$
\mathcal{Z}(\lambda) = \frac{M_1 + N_1}{M_2 + N_2} = \frac{A_1(\lambda + \alpha) + A_2}{B_1(\lambda - \alpha) + B_2} = \frac{(\lambda + 1) - \frac{1}{\alpha}}{(\lambda - 1) + \frac{1}{\alpha}}
$$

(196)

where the "even" and "odd" parts of the numerator and denominator polynomials are
\[
M_1 = -\frac{1}{2}, \quad M_2 = 2
\]

and

\[
N_1 = \lambda + 1, \quad N_2 = \lambda - 1.
\]

In order to test whether \(\tilde{Z}(\lambda)\) is a PRD\(_Z\) function, it is sufficient to establish the non-negativeness of the numerator polynomial of \(D_Z(j\xi)\). This follows since \(\tilde{Z}(\lambda)\) is clearly strictly minimum reactive in this case. We have

\[
\left[ M_1 M_2 - N_1 N_2 \right]_{\lambda = j\xi} = \left[ -1 - (\lambda^2 - 1) \right]_{\lambda = j\xi} = -\lambda^2 \quad \Rightarrow \quad \xi^2 \geq 0; \quad (198)
\]

\(\tilde{Z}(\lambda)\) is therefore a PRD\(_Z\) function.

Furthermore, with

\[
M_1 M_2 - N_1 N_2 = -\lambda^2 \quad (199)
\]

this is seen to be a case A situation. Augmentation is therefore necessary in order to make the numerator of \(\tilde{Z}_{21}(\lambda), \sqrt{M_1 M_2 - N_1 N_2}\), of the right form. The proper augmentation factor of \(Z(\lambda)\) is readily seen to be \(\lambda/\lambda\), such that

\[
\tilde{Z}(\lambda) = \frac{\lambda}{\lambda} \frac{(\lambda + \frac{1}{2})}{(\lambda + 1)} \frac{(\lambda^2 - 1) + \frac{1}{2}(\lambda + 1) + \frac{1}{2}}{(\lambda^2 - 1) + (\lambda - 1) + 2} \quad (200)
\]

The "even" and "odd" parts of the augmented numerator and denominator polynomials are therefore

\[
M'_1 = (\lambda^2 - 1) + \frac{1}{2} \quad M'_2 = (\lambda^2 - 1) + 2
\]

and

\[
N'_1 = \frac{1}{2}(\lambda + 1) \quad N'_2 = \lambda - 1,
\]

such that

\[
M'_1 M'_2 - N'_1 N'_2 = \lambda^4.
\]

Using relations (184) - (186) we then obtain

\[
\tilde{Z}_{22} = \frac{M'_2}{N'_2} = \frac{(\lambda^2 - 1) + 2}{\lambda - 1} = (\lambda + 1) + \frac{2}{\lambda - 1}
\]

\[
\tilde{Z}_{21} = \sqrt{\frac{M'_1 M'_2 - N'_1 N'_2}{N'_2}} = \frac{\lambda^2}{\lambda - 1} = (\lambda + 1) + \frac{1}{\lambda - 1}
\]

\[
\tilde{Z}_{11} = \frac{M'_1}{N'_2} = \frac{(\lambda^2 - 1) + \frac{1}{2}}{\lambda - 1} = (\lambda + 1) + \frac{1}{2} \frac{1}{\lambda - 1}, \text{ and}
\]

the final realization of \(\tilde{Z}(\lambda)\) becomes Fig. 36.
VI. TRANSFORMERLESS SYNTHESIS PROCEDURES

The fact that all RL'C"T driving point impedances are PRD functions, and that all PRD functions are realizable as the driving-point impedance of RL'C"T networks, has now definitely been established. The ideal transformer has been useful in showing the latter part, however, it is not indispensable in the realization problem, as we shall now show.

To achieve this objective, we will modify a procedure suggested by Pantell leading to a Bott and Duffin realization. Finally, the elegant although not completely general Miyata procedure, will be extended to the RL'C" case.

6.1 BOTT AND DUFFIN TYPE REALIZATIONS

Instead of arriving at a Bott and Duffin realization via the well known Richard's transformation, Pantell has given a somewhat more systematic procedure based upon even-part decompositions. We will now show how to modify this procedure in order to arrive at an analogous transformerless realization for PRD functions.

Our strategy is to decompose a known PRD function into the sum of two PRD functions, each with either poles and zeros that can be removed by the procedure outlined in section 4.1. Whether this decomposition is performed on an impedance basis or an admittance basis depends on the nature of the given Z(λ), which we will assume has been reduced as far as possible by the procedure of section 4.1. Z(λ) is therefore both minimum reactive and minimum resistive; its impedance derived difference function is therefore zero somewhere along the jξ-axis, say at λ = jξ_o. It follows that

\[ Z(jξ_o) = k_o(α + jξ_o) \]  

where \( k_o \) can be either positive or negative. If \( k_o < 0 \), an impedance decomposition is employed; while if \( k_o > 0 \), an admittance decomposition is used. We will consider these two cases separately.

Case 1: \( k_o < 0 \)

To obtain a suitable decomposition of \( \tilde{Z}(λ) \), we shall decompose its difference function \( D_Z \) into components \( D_{Z_1} \) and \( D_{Z_2} \), and then determine the PRD functions \( \tilde{Z}_1 \) and \( \tilde{Z}_2 \) that are associated with \( D_{Z_1} \) and \( D_{Z_2} \). The specific decomposition of \( D_Z \), that we select, is the following

\[ D_{Z_1} = \frac{a^2}{a^2 - (λ^2 - α^2)} D_Z \]

(203)
and

\[
D_{Z_2} = \frac{-(\lambda^2 - \alpha^2)}{a^2 - (\lambda^2 - \alpha^2)} D_Z
\]  

(204)

where "\(a^2\)" is a real constant. (Note that both \(D_{Z_1}\) and \(D_{Z_2}\) are non-negative along the \(j\xi\)-axis for any real "\(a\)."

The associated impedances \(\tilde{Z}_1\) and \(\tilde{Z}_2\) can easily be constructed by the modified Pantell scheme given in section 5.1. The results are:

\[
\tilde{Z}_1(\lambda) = a^2 \frac{\tilde{Z}(\lambda) - \frac{\tilde{Z}(r)}{r + \alpha}(\lambda + \alpha)}{a^2 - (\lambda^2 - \alpha^2)}
\]  

(205)

and

\[
\tilde{Z}_2(\lambda) = \frac{-(\lambda^2 - \alpha^2)\tilde{Z}(\lambda)}{a^2 - (\lambda^2 - \alpha^2)} + \frac{a^2\tilde{Z}(r)}{r + \alpha}(\lambda + \alpha)
\]  

(206)

where we have defined

\[
r = \sqrt{\alpha^2 + a^2}.
\]  

(207)

Note that \(\tilde{Z}_1\) and \(\tilde{Z}_2\) have denominator polynomials of degree \(n+1\) (after cancellation of the common pole and zero at \(\lambda = r\)) and, therefore, are more complicated than the original impedance \(\tilde{Z}\). By an appropriate choice of "\(a\)" however, we can put \(\tilde{Z}_1\) and \(\tilde{Z}_2\) in such a form to make degree reducing reactance removals possible.

A proper choice of "\(a\)" is to require that the numerator of \(Z_2\) has a zero at \(\lambda = j\xi_o\), that is,

\[
\left(\xi_o^2 + \alpha^2\right)k_o(\alpha + j\xi_o) + \frac{a^2\tilde{Z}(r)}{r + \alpha}(\alpha + j\xi_o) = 0,
\]  

(208)

which we can write

\[
(r - \alpha)Z(r) + k_o\left(\xi_o^2 + \alpha^2\right) = 0.
\]  

(209)

[Note: "\(a\)" can also sometimes be chosen so as to make \(\tilde{Z}_1(j\xi_o) = 0\). This leads to a slightly different Bott and Duffin realization. The existence of such an "\(a\)" however, cannot be guaranteed, except in the special case of uniform loss.]

That Eq. 209 has a real positive solution for \(r > \alpha\) is seen from the continuity of its left-hand side together with the following two observations:

(i) for \(r = \alpha\), the left-hand side is negative

(ii) as \(r \to +\infty\), the left-hand side becomes positive.

Taking the solution of this equation as our value for "\(a\)" we can write (205) and (206) as follows
\[
\tilde{Y}_1(\lambda) = \tilde{Z}_1^{-1}(\lambda) = \frac{1}{-k_0(\xi_0^2+\omega^2)} (\lambda-\omega) + \frac{1}{-k_0(\xi_0^2+\omega^2)} \gamma_y(\lambda) \tag{210}
\]

and

\[
\tilde{Y}_2(\lambda) = \tilde{Z}_2^{-1}(\lambda) = \frac{a^2}{-k_0(\xi_0^2+\omega^2)} \frac{1}{(\lambda+\omega)} + \frac{1}{-k_0(\xi_0^2+\omega^2)} \frac{\lambda-\omega}{\lambda+\omega} \frac{1}{\gamma_y(\lambda)} \tag{211}
\]

where we have defined

\[
\gamma_y(\lambda) = -\frac{(\lambda-\omega)\tilde{Z}(\lambda) + k_0(\xi_0^2+\omega^2)}{Z(\lambda)+\frac{(\xi_0^2+\omega^2)k_0(\lambda+\omega)}{a^2}}. \tag{212}
\]

From \(\tilde{Y}_1\) and \(\tilde{Y}_2\) we can clearly make the susceptance removals shown in Fig. 37 and obtain the remainder admittances

\[
\tilde{Y}_3(\lambda) = \frac{1}{-k_0(\xi_0^2+\omega^2)} \gamma_y(\lambda) \tag{213}
\]

and

\[
\tilde{Y}_4(\lambda) = \frac{1}{-k_0(\xi_0^2+\omega^2)} \frac{\lambda-\omega}{\lambda+\omega} \frac{1}{\gamma_y(\lambda)}, \tag{214}
\]

The function \(\gamma_y\), that characterizes the remainder functions \(\tilde{Y}_3\) and \(\tilde{Y}_4\), has the following detailed properties:

a. \(\gamma_y(\lambda)\) is a PRD\(_Y\) function. (This follows directly from Eq. 210 since \(\tilde{Z}_1(\lambda)\) is known to be a PRD\(_Z\) function.)

b. \(\frac{\lambda+\omega}{\lambda-\omega} \gamma_y(\lambda)\) is a PRD\(_Z\) function. (This follows directly from Eq. 211 and the PRD\(_Y\) property of \(\tilde{Y}_2\). It can also be shown as a special case of the following general property of PRD functions: If \(\tilde{Z}(\lambda)\) is a PRD\(_Z\) function, then \(\frac{\lambda+\omega}{\lambda-\omega} \tilde{Z}(\lambda)\) is a PRD\(_Y\) function; and if \(\tilde{Y}(\lambda)\) is a PRD\(_Y\) function then \(\frac{\lambda+\omega}{\lambda-\omega} \tilde{Y}(\lambda)\) is a PRD\(_Z\) function. The proof is straightforward.)

c. \(\gamma_y(\pm j\xi_0) = 0\). (This is readily seen from Eqs. 202 and 212.)

d. The numerator and denominator polynomials of \(\gamma_y\) have a common factor \((\lambda - \sqrt{\alpha^2 + \omega^2})\). (This is most easily seen if we consider the function

\[
(\lambda+\omega)\gamma_y(\lambda) = \frac{-a^2}{\alpha^2 Z(\lambda) + (\xi_0^2+\omega^2)k_0(\lambda+\omega)} \frac{(\lambda^2-\omega^2)\tilde{Z}(\lambda) + (\xi_0^2+\omega^2)k_0(\lambda+\omega)}{(\lambda^2+\omega^2)k_0(\lambda+\omega)}. \tag{215}
\]

From (208) we see that both numerator and denominator of \((\lambda+\omega)\gamma_y(\lambda)\) vanish at \(\lambda = \sqrt{a^2 + \omega^2}\), which shows that \(\gamma_y\) has a pole and a zero superimposed at \(\lambda = \sqrt{a^2 + \omega^2}\).

e. The order of \(\gamma_y(\lambda)\) is \(\frac{a}{\omega}\) (after cancellation of the common pole and zero at \(\lambda = \sqrt{a^2 + \omega^2}\)). (This follows directly from Eq. 212 and property d.)
From properties a, b and c it is seen that both \( Z_3 = \tilde{Y}_3^{-1} \) and \( \tilde{Y}_4 \) have a pair of removable \( j \xi \)-axis poles located at \( \lambda = \pm j \xi_0 \). We can therefore represent the original \( \tilde{Z}(\lambda) \) as shown in Fig. 38 with a pair of PRD \(_Z\) remainder functions, \( \tilde{Z}_5 \) and \( \tilde{Z}_6 \).

A pertinent question, at this point, would be to ask for the order of complexity of \( \tilde{Z}_5 \) and \( \tilde{Z}_6 \). It follows from Eqs. 213 and 214, property e and Fig. 38 that

\[
\text{Order } \tilde{Z}_5 = \frac{n-2}{n-2}
\]

(216)

and

\[
\text{Order } \tilde{Z}_6 = \frac{n-1}{n-1}
\]

(217)

The reason for this difference in order is the factor \( \frac{\lambda-\alpha}{\lambda+\alpha} \) in Eq. 214. Let us take a closer look at \( \tilde{Z}_6 \) to see if its order can be reduced further.

Since \( \tilde{Y}_4 \) is seen to have a pole at \( \lambda = -\alpha \) and a zero at \( \lambda = \alpha \), it follows from Fig. 38 that \( \tilde{Y}_6 = \tilde{Z}_6^{-1} \) must also have this property. \( \tilde{Z}_6 \) must therefore be of the form

\[
\tilde{Z}_6(\lambda) = \frac{\lambda + \alpha}{\lambda - \alpha} \tilde{Y}(\lambda)
\]

(218)

\( \tilde{Z}_6 \) is therefore not minimum reactive. It has a removable pole at \( \lambda = \alpha \). When this removal is made, the development of \( \tilde{Z}(\lambda) \) takes the form shown in Fig. 39.

The remainder function \( \tilde{Z}_8 \) is guaranteed to be a PRD \(_Z\) function of order \( \frac{n-2}{n-2} \).

We have thus been successful in demonstrating how to carry out one cycle of the Bott and Duffin procedure for the case where \( k_0 < 0 \) (in Eq. 202).

Before we consider the other possible case, however, it is pertinent to make a few comments concerning the number of elements required. It is seen that Fig. 39 contains one more element than is to be expected based on the ordinary Bott and Duffin realization. If there is anything objectionable about a Bott and Duffin realization, it is the large number of elements that it requires. Let us therefore see if the capacitor in series with \( \tilde{Z}_8 \) must always enter the realization.

The following possibility has no doubt occurred to the reader. Suppose that instead of removing the pole of \( \tilde{Z}_6 \) at \( \lambda = \alpha \) we remove the pole of \( \tilde{Y}_6 = \tilde{Z}_6^{-1} \) at \( \lambda = -\alpha \) by a shunt inductor. This inductor could then be combined with the inductor already shunting \( \tilde{Z}_6 \) in Fig. 38. The success of this scheme is obviously dependent upon the residue of \( \tilde{Y}_6 \) in its pole at \( \lambda = -\alpha \), which can have either sign. However, the residue of \( \tilde{Y}_6 \) at \( \lambda = -\alpha \) is no less than \( \frac{1}{k_0(\xi_0^2 + \alpha^2)} \), since \( \tilde{Y}_2 \) has a non-negative residue at \( \lambda = -\alpha \); this removal will therefore save one (or possibly two) element provided the remainder admittance is non-negative at \( \lambda = \alpha \) and thus PRD \(_Y\). The realization in this case, will be as shown in Fig. 38 and the order of both remainder functions will equal \( \frac{n-2}{n-2} \).
Fig. 37. Lossy susceptance removals relevant to the Bott and Duffin procedure.

Fig. 38. Partially completed Bott and Duffin cycle for the RL'C'' case.
Fig. 39. Typical Bott and Duffin cycle for the RL'C'' case. Notice the extra circuit element in series with $Z_8$ (compared with its RLC counterpart).

Fig. 40. An alternative Bott and Duffin cycle (the "dual" of Fig. 38).
Case 2: \( k > 0 \)

In this case

\[
\tilde{Z}(j \xi_0) = k_o(\alpha + j \xi_0) \quad \text{with} \quad k_o > 0
\]

such that

\[
\tilde{Y}(j \xi_0) = \frac{1}{\tilde{Z}(j \xi_0)} = -k'_o(\alpha - j \xi_0) \quad \text{with} \quad k'_o = -\frac{1}{k_o(\alpha^2 + \xi_0^2)} < 0.
\]

To obtain a suitable decomposition in this case we shall proceed on an admittance basis. Thus employing a dual approach leads to the following decomposition

\[
\tilde{Y}_1(\lambda) = a^2 \frac{\tilde{Y}(\lambda) - \tilde{Y}(r) \frac{\lambda - \alpha}{r + \alpha}}{a^2 - (\lambda^2 - \alpha^2)}
\]

and

\[
\tilde{Y}_2(\lambda) = \frac{-(\lambda^2 - \alpha^2)\tilde{Y}(\lambda) + \frac{a^2}{r + \alpha}\tilde{Y}(r) (\lambda - \alpha)}{a^2 - (\lambda^2 - \alpha^2)}.
\]

The sum of \( \tilde{Y}_1 \) and \( \tilde{Y}_2 \) is clearly equal to \( \tilde{Y} \), and it is easily shown that both \( \tilde{Y}_1 \) and \( \tilde{Y}_2 \) are PRD_y functions for any real value of the constant "a".

A proper choice of "a" in this case is obtained by setting \( \tilde{Y}_2 \) equal to zero at \( \lambda = j \xi_0 \), that is,

\[
(\xi_0^2 + \alpha^2)k'_o(-\alpha + j \xi_0) + \frac{a^2}{r + \alpha}\tilde{Y}(r) (-\alpha + j \xi_0) = 0
\]

or

\[
(r - \alpha) \tilde{Y}(r) + k'_o(\xi_0^2 + \alpha^2) = 0,
\]

where \( a^2 = r^2 - \alpha^2 \).

The same argument as was used in connection with equation 209 shows that a solution exists also in this case. Using this value for "a" we can write (221) and (222) as follows

\[
\tilde{Z}_1(\lambda) = \frac{\lambda + \alpha}{-k_o(\xi_0^2 + \alpha^2)} + \frac{1}{-k'_o(\xi_0^2 + \alpha^2)} \gamma_z(\lambda)
\]

and

\[
\tilde{Z}_2(\lambda) = \frac{a^2}{-k_o(\xi_0^2 + \alpha^2)} \lambda - \alpha + \frac{1}{-k'_o(\xi_0^2 + \alpha^2)} \lambda - \alpha \gamma_z(\lambda),
\]

where we have defined
\[
\gamma_z(\lambda) = -\frac{(\lambda+\alpha) \tilde{Y}(\lambda) + k'_0 (\xi_o^2 + \alpha^2)}{\tilde{Y}(\lambda) + \frac{(\xi_o^2 + \alpha^2)k'_0 (\lambda-\alpha)}{\alpha^2}}. \tag{226}
\]

The properties of \(\gamma_z\) are analogous to those of \(\gamma_y\) and are listed below.

a. \(\gamma_z\) is a PRD\(_Z\) function.

b. \(\frac{\lambda-\alpha}{\lambda+\alpha} \gamma_z\) is a PRD\(_Y\) function.

c. \(\gamma_z(\pm j\xi_o) = 0\).

d. The numerator and denominator polynomials of \(\gamma_z\) have a common factor \((\lambda - \sqrt{\alpha^2 - \alpha^2})\).

e. The order of \(\gamma_z = \frac{n}{n}\) The decomposition \(\tilde{Y} = \tilde{Y}_1 + \tilde{Y}_2\) in this case, thus leads to the network realization shown in Fig. 40. As one would expect this realization is simply the "dual" of Fig. 38.

The remainder admittances \(\tilde{Y}_3\) and \(\tilde{Y}_4\) are PRD\(_Y\) functions of order \(\frac{n-2}{n-2}\).

It should be noted that the pole of \(\frac{\lambda+\alpha}{\lambda-\alpha} \gamma_z(\lambda)\) at \(\lambda = \alpha\), has been removed and combined with the series capacitor of \(\tilde{Y}_2\). No extra element (as in Fig. 39) is, therefore, necessary in this case.

We have thus demonstrated that any PRD function is realizable as the driving-point function of a transformerless RL'C' network.

6.2 ILLUSTRATIVE EXAMPLE

To illustrate the discussion of the preceding section, we will make a transformerless realization of the driving point impedance used in connection with both the Brune and the Darlington procedures (Examples 1 and 7).

**EXAMPLE 9**

Consider the following PRD\(_Z\) function

\[
Z(s) = \frac{s^2 + 5s + 12}{s^2 + 6s + 9}; \quad \alpha = 1 \text{ and } \delta = 2, \tag{227}
\]

which can be expressed as

\[
\tilde{Z}(\lambda) = \frac{\lambda^2 + \lambda + 6}{\lambda^2 + 2\lambda + 1} = \frac{(\lambda^2 - 1) + (\lambda + 1) + 6}{(\lambda^2 - 1) + 2(\lambda - 1) + 4}.
\]

If we define the "even" and "odd" parts of \(\tilde{Z}(\lambda)\) as
\[ M_1 = (\lambda^2 - 1) + 6 \quad M_2 = (\lambda^2 - 1) + 4 \]

and
\[ N_1 = (\lambda + 1) \quad N_2 = 2(\lambda - 1) \, , \]

then
\[ D_Z(j\xi) = \frac{M_1 M_2 - N_1 N_2}{M_2 + N_2^2} = \frac{\xi^4 - 6\xi^2 + 17}{\xi^4 + 2\xi^2 + 1} . \]

On page 44 it was found that
\[ \text{Min } D_Z(+j\xi) = D_Z(+j\xi_0) = D_Z(+j\sqrt{5}) = \frac{1}{3} . \]

Subtracting off a resistance equal to \( \text{Min } D_Z(+j\xi) \), we obtain a minimum resistive PRD \( Z \) remainder function

\[ Z'(\lambda) = Z(\lambda) - \frac{1}{3} = \frac{2\lambda^2 + \lambda + 17}{\lambda^2 + 2\lambda + 1} . \]  

(228)

To determine whether this is case I (\( k_o < 0 \)) or case II (\( k_o > 0 \)), we evaluate

\[ Z'(+j\xi_0) = k_o(\alpha + j\xi_0) = -\frac{1}{6} (1 + j\sqrt{5}) . \]

(229)

Since
\[ k_o = -\frac{1}{6} < 0 , \]

we proceed on an impedance basis.

According to (209) we must choose "\( a \)" such that

\[ \frac{a^2}{1 + \sqrt{a^2 + 1}} Z'(\sqrt{a^2 + 1}) = \frac{1}{6} (5 + 1) = 1 , \]

which factors as follows

\[ (a^2 + 6)(a^2 + 1) = (6 + a^2) \sqrt{a^2 + 1} . \]  

(230)

Cancelling the common factor of \( a^2 + 6 \), one obtains

\[ a^2(a^2 - 3) = 0 , \]

which yields the positive real solution

\[ a^2 = 3 . \]

[Note that Eq. 209 always admits of a pair of complex conjugate roots for

\[ a = \pm j\sqrt{\xi_0^2 + a^2} , \]
which implies that it can be reduced from an \( n \)th-order equation to an \((n-2)\)th-order equation without any search for roots.]  

Equation 212 for \( \gamma_Y(\lambda) \) thus becomes

\[
\gamma_Y(\lambda) = 2 \frac{(\lambda-2)(\lambda^2+5)}{(\lambda-2)(\lambda^2+3\lambda+8)} = \frac{1}{\lambda^2+\frac{2}{3} \lambda+\frac{1}{2}},
\]

which is readily seen to have all of the properties from a to e of section 6.1.

Inserting \( \gamma_Y \) into (210) and (211), we obtain

\[
\tilde{Z}_1(\lambda) = \frac{1}{(\lambda-1) + \frac{1}{\lambda+\frac{2}{3} \lambda+\frac{1}{2}}}
\]

and

\[
\tilde{Z}_2(\lambda) = \frac{1}{3} \frac{1}{\lambda+1} \frac{1}{(\lambda-1) + \frac{1}{\lambda+\frac{2}{3} \lambda+\frac{1}{2}}} = \frac{3}{\lambda+1} + \frac{1}{2} \lambda+\frac{1}{2}.
\]

The "all pass" component that shows up in the denominator of \( \tilde{Z}_2(\lambda) \) can be written as

\[
y(\lambda) = \frac{3}{2} \frac{\lambda-1}{\lambda+1} = \frac{1}{3} \left[ \frac{1}{\lambda+1} + \frac{1}{3} \right] = \frac{1}{3} \frac{1}{\lambda+1}.
\]

and the final realization takes the form shown in Fig. 41.

Notice the "extra" lossy capacitor in the remainder function of \( \tilde{Z}_2 \). As was pointed out on page 66 it might be possible to save one (or two) component if we instead extract a shunt inductor from \( y(\lambda) \). We can write

\[
y(\lambda) = \frac{-3}{\lambda+1} + \frac{3}{2},
\]

and the shunt inductors are seen to cancel (that is, a saving of two elements). The final realization is shown in Fig. 42.

6.3 MODIFIED MIYATA REALIZATION

Miyata has shown how to make a transformerless driving point realization of any minimum reactive PR function \( Z(s) = \frac{P(s)}{m_2(s) + n_2(s)} \) whose even part can be decomposed in the following manner
Fig. 41. Network produced by the Bott and Duffin procedure for realizing impedance (227).

Fig. 42. An alternative Bott and Duffin realization for the impedance (227), where the extra circuit element of Fig. 39 has been eliminated.
\[ \text{Ev } Z(s) = \sum_{i=0}^{k} a_i \frac{(-s^2)^i}{m_2(s) - n_2(s)}; \quad a_i \geq 0. \] (232)

The realization takes the form shown in Fig. 43, where the component impedances are related to the even-part decomposition as follows

\[ \text{Ev } Z_1(s) = a_i \frac{(-s^2)^i}{m_2(s) - n_2(s)}; \quad a_i > 0. \] (233)

We shall show how certain strictly minimum reactive PRD \( Z \) functions \( \tilde{Z}(\lambda) = \frac{P(\lambda)}{m_2(\lambda) + n_2(\lambda)} \) can be realized by the Miyata-like configuration shown in Fig. 44. The realization procedure is based upon the following decomposition of the difference function

\[ D_Z(\lambda) = \sum_{i=0}^{k} A_i \frac{(-\lambda^2 + \alpha^2)^i}{m_2(\lambda) - n_2(\lambda)}; \quad A_i \geq 0. \] (234)

The explicit relationship between the \( A_i \)'s and the \( a_j \)'s of

\[ D_Z(\lambda) = \sum_{j=0}^{k} a_j \frac{(-\lambda^2)^j}{m_2(\lambda) - n_2(\lambda)} \]

is as follows;

\[ A_i = \sum_{j=i}^{k} (-1)^{(j+i)} \left( \begin{array}{c} j \\ i \end{array} \right) \alpha^{2(j-i)} a_j \]

The condition for success is that all of the \( A_i \) be non-negative.

The relationships between the component impedances \( \tilde{Z}_1 \) and the terms of the difference function decomposition are as follows:

\[ D_{Z_1} = A_i \frac{(-\lambda^2 + \alpha^2)^i}{m_2(\lambda) - n_2(\lambda)}; \quad A_i > 0, \] (235)
where \( D_{Z_i} \) is used to denote the impedance derived difference function of the component impedance \( \tilde{Z}_i \).

\( D_{Z_i} = \frac{Z_{21}(\lambda)Z_{21}(-\lambda)}{m_2(\lambda) - n_2(\lambda)} \); \( A_i > 0 \).

(236)

To establish the validity of the procedure one must show that the driving-point impedance corresponding to a \( D_{Z_i} \)-function having its zeros evenly distributed between \( \lambda = \alpha \) and \( \lambda = -\alpha \) or at infinity, can be realized by means of an \( L'C'' \) ladder structure terminated in a resistance.

Using the results of the Darlington procedure, we make the following observations with respect to the \( i^{th} \) component of the \( D_{Z_i} \)-function;

\( D_{Z_i} = \frac{Z_{21}(\lambda)Z_{21}(-\lambda)}{m_2(\lambda) - n_2(\lambda)} \);

(236)

\( D_{Z_i} \) leads to a case A situation (section 5.2) if "i" is even and a case B situation (section 5.3) if "i" is odd. The nature of the open-circuit transfer impedance, \( Z_{21}^{i}(\lambda) \), is therefore

\[
Z_{21}^{i}(\lambda) = \frac{\frac{i}{2}(\lambda+\alpha)^2(\lambda-\alpha)^2}{(\lambda-\alpha)m(\lambda)} \text{ for case A}
\]

(237)

and

\[
Z_{21}^{i}(\lambda) = \frac{\frac{i-1}{2}(\lambda+\alpha)^2(\lambda-\alpha)^2}{m(\lambda)} \text{ for case B,}
\]

where \( m(\lambda) \) is an even polynomial in \( \lambda \). Should the case B components fail to have an even number of poles in the interval \(-\alpha < \nu < 0\), then augmentation is necessary (Theorem 3). The transfer impedances \( Z_{21}^{i}(\lambda) \), will therefore have their transmission zeros evenly distributed between \( \lambda = \pm \alpha \) or at infinity.

With this point in mind it is readily seen that a Miyata realization can be obtained by steps analogous to the ordinary RLC case. The steps are as follows:

1. Decompose the impedance derived difference function according to Eq. 234.
2. Extract from each component \( D_{Z_1} \) function the corresponding \( Z_{21}^i \) and identify \( Z_{21}^i \) and \( Z_{22}^i \).

3. Realize each component network according to Fig. 45 by a ladder development of \( Z_{22}^i \).

4. Impedance scale each of the component impedances, \( Z_{21}^i \), by the factor \( \frac{A_i}{K_i} \) where \( \sqrt{K_i} \) is the multiplier of the transfer impedance, \( Z_{21}^i \), obtained in step 3.

5. Make a series connection of each component impedance as shown in Fig. 44.

**EXAMPLE 10**

To illustrate the previous discussion we make a Miyata realization of the PRD function

\[
\mathcal{Z}(\lambda) = \frac{\lambda + 3}{\lambda + 7} = \frac{(\lambda + 1) + 2}{(\lambda + 1) + 8}, \quad \alpha = 1 \text{ and } \delta = 2.
\]

A step-by-step procedure will be employed.

**Step 1:** Decomposition of the impedance derived difference function

\[
D_{Z}(\lambda) = \frac{(-\lambda + 1)(\lambda + 1)}{-\lambda^2 + 49} + \frac{16}{-\lambda^2 + 49}.
\]

We make the following identifications

\[
D_{Z_0}(\lambda) = \frac{Z_{21}^0(\lambda)}{Z_{21}^0(-\lambda)} = \frac{16}{-\lambda^2 + 49}
\]

and

\[
D_{Z_1}(\lambda) = \frac{Z_{21}^1(\lambda)}{Z_{21}^1(-\lambda)} = \frac{(-\lambda + 1)(\lambda + 1)}{-\lambda^2 + 49}.
\]

**Step 2:** Extractions of transfer impedances, \( Z_{21}^i \), and identifications of \( Z_{21}^i \) and \( Z_{22}^i \) \((i = 0, 1)\).

\[
i) \quad Z_{21}^0(\lambda) = \frac{4}{(\lambda + 1) + 8} = \frac{4}{1 + \frac{8}{\lambda + 1}}
\]

such that

\[
Z_{21}^0(\lambda) = \frac{4}{\lambda - 1} \quad \text{and} \quad Z_{22}^0(\lambda) = \frac{8}{\lambda - 1}
\]

\[
ii) \quad Z_{21}^1(\lambda) = \frac{(\lambda + 1)}{(\lambda + 1) + 6} = \frac{\lambda + 1}{1 + \frac{6}{\lambda + 1}}
\]

such that

\[
76
\]
\[ Z_{21}^0(\pm \lambda) Z_{21}^0(-\lambda) = \frac{A_0}{m_2^2(\lambda) - n_2^2(\lambda)} \Rightarrow Z_{21}^0(\lambda) \]

\[ Z_{21}^1(\lambda) Z_{21}^1(-\lambda) = \frac{A_1(-\lambda+\alpha)(\lambda+\alpha)}{m_2^2(\lambda) - n_2^2(\lambda)} \Rightarrow Z_{21}^1(\lambda) \]

\[ Z_{21}^n(\lambda) Z_{21}^n(-\lambda) = \frac{A_n((-\lambda+\alpha)(\lambda+\alpha))^n}{m_2^2(\lambda) - n_2^2(\lambda)} \Rightarrow Z_{21}^n(\lambda) \]

Fig. 45. Networks produced by each component impedance of the Miyata decomposition.
\[
\frac{1}{Z_{21}(\lambda)} = \frac{1}{Z_{22}(\lambda)} = \frac{\lambda + 1}{6}
\]

**Step 3:** Component realizations with one ohm terminations.

\[
\tilde{Z}_0 = \frac{8}{\lambda - 1}
\]

\[
\tilde{Z}_0' = \frac{1}{Z_0}(\lambda) = \frac{8}{\lambda - 7}
\]

\[
\tilde{Z}_1 = \frac{\lambda - 1}{6}
\]

\[
\tilde{Z}_1' = \frac{\lambda + 1}{\lambda + 7}
\]

**Step 4:** Scaling and final realization. Since \(\tilde{Z}_{21}^0(\lambda) = \frac{1}{2} \tilde{Z}_{21}^0(\lambda)\), we must impedance-scale \(N_0\) by \(\frac{1}{4}\). Since \(\tilde{Z}_{21}^1(\lambda) = \tilde{Z}_{21}^1(\lambda)\), no scaling is necessary for \(N_1\). The final realization is shown in Fig. 46.

**Fig. 46.** Network produced by the Miyata procedure for realizing the impedance \(\tilde{Z}(\lambda) = \frac{\lambda + 3}{\lambda + 7}\) with \(\alpha = 1\) and \(\delta = 2\).
VII. TRANSFER FUNCTION SYNTHESIS

We shall now show how our knowledge of RL'C"T driving-point functions can be utilized to synthesize voltage transfer ratios of both singly loaded and doubly loaded L'C"T two-ports. Only voltage transfer ratios will be considered, since other types of transfer ratios can be handled by straightforward modifications.

The first section will be concerned with the synthesis of lowpass filters with single resistive terminations.

The second section will be concerned with doubly resistive terminated L'C"T two-ports. A solution to this problem has been achieved by Darlington from a different point of view. We believe, however, that our knowledge of the PRD properties of RL'C"T driving-point functions can be used to obtain additional insight.

7.1 LOWPASS FILTERS WITH A SINGLE RESISTANCE TERMINATION

It is well known that any voltage transfer function of the form

\[ \frac{E_2}{E_1} = \frac{A}{a_n s^n + \cdots + a_1 s + a_0} \]  

(with a Hurwitz denominator) can be realized by the structure of Fig. 47a provided the constant multiplier \( A \) is properly chosen. It is also true that any voltage transfer ratio of the form

\[ \frac{E_2}{E_1} = \frac{A}{a_n \lambda^n + \cdots + a_1 \lambda + a_0} \]  

(with a Hurwitz denominator) can be realized by the structure of Fig. 47b, provided the constant multiplier \( A \) is properly chosen. The procedure for doing this should be more or less evident from the discussion of section 6.3 and Theorem 2. Accordingly, we merely summarize the steps as follows:

1. Adjust the multiplier to the value \( A_0 \), so that \( T(\alpha) = 1 \).

2. Write \( T(\lambda) = \frac{A}{M(\lambda) + N(\lambda)} \) (where \( M \) and \( N \) are "even" and "odd," respectively, as defined in Theorem 2) in the following form:

\[ T(\lambda) = \frac{A_0}{1 + \frac{M(\lambda)}{N(\lambda)}} = \frac{\tilde{Z}_{21}(\lambda)}{1 + \tilde{Z}_{11}(\lambda)}. \]

3. Make the identifications

\[ \tilde{Z}_{11}(\lambda) = \frac{M(\lambda)}{N(\lambda)} \text{ and } \tilde{Z}_{21}(\lambda) = \frac{A_0}{N(\lambda)}. \]
4. Develop $\tilde{Z}_{11}$ in an L'C' ladder having series inductors and shunt capacitors; the required $\tilde{Z}_{21}$ then automatically results.

The Butterworth and Tschebyscheff lowpass functions possess the functional form of Eq. 239, provided $\delta$ is smaller than the distance from the $j\omega$-axis to the rightmost pole in the $s$-plane. All such functions can therefore be realized by this procedure. As an illustration we realize a third-order Butterworth function.

**EXAMPLE 11**

Realize the following transfer ratio

$$\frac{E_2}{E_1} = T(s) = \frac{A}{s^3 + 2s^2 + 2s + 1} = \frac{A}{(s + 1)(s^2 + s + 1)}$$

as shown in Fig. 47b for the dissipation ratios $\alpha_L = \frac{1}{2}$ and $\alpha_C = 0$.

![Fig. 47. Lowpass filter with a single resistance termination: (a) employing ideal reactance elements; (b) employing reactance elements with semiuniform loss.](image)

From the given values of $\alpha_L$ and $\alpha_C$ we find that $\alpha = \delta = \frac{1}{4}$. Equation 240 can therefore be written, in $\lambda$-coordinates, as follows:

$$T(\lambda) = \frac{A}{\lambda^3 + \frac{5}{4} \lambda^2 + \frac{19}{16} \lambda + \frac{39}{64}}$$

**Step 1.**

$$T(\alpha) = \frac{A}{1} = 1 \rightarrow A = 1.$$

**Step 2.**

$$T(\lambda) = \frac{1}{(\lambda^2 - \frac{1}{16})(\lambda - \frac{1}{4}) + \frac{3}{2}(\lambda^2 - \frac{1}{16}) + \frac{5}{4}(\lambda - \frac{1}{4}) + 1}$$
\[ \begin{align*} \frac{1}{(\lambda^2 - \frac{1}{16})(\lambda - \frac{1}{4}) + \frac{5}{4}(\lambda - \frac{1}{4})} \cdot \frac{2}{2(\lambda^2 - \frac{1}{16}) + 1} & = \frac{(\lambda^2 - \frac{1}{16})(\lambda - \frac{1}{4})}{(\lambda^2 - \frac{1}{16})(\lambda - \frac{1}{4}) + \frac{5}{4}(\lambda - \frac{1}{4})} \cdot \frac{2}{2(\lambda^2 - \frac{1}{16}) + 1} \\
\end{align*} \]

Step 3.

\[ Z_{11} = \frac{\frac{2}{3}(\lambda^2 - \frac{1}{16}) + 1}{(\lambda^2 - \frac{1}{16})(\lambda - \frac{1}{4}) + \frac{5}{4}(\lambda - \frac{1}{4})} \cdot \frac{2}{2(\lambda^2 - \frac{1}{16}) + 1} \]

Step 4.

\[ Z_{11} = \frac{\frac{2}{3}(\lambda^2 - \frac{1}{16}) + 1}{(\lambda^2 - \frac{1}{16})(\lambda - \frac{1}{4}) + \frac{5}{4}(\lambda - \frac{1}{4})} \cdot \frac{2}{2(\lambda^2 - \frac{1}{16}) + 1} \]

The resulting circuit is shown in Fig. 48.

![Fig. 48. A third-order Butterworth realization of a lowpass filter with a single resistance termination.](image)

7.2 DOUBLY RESISTIVE TERMINATED L'C''T TWO-PORTS

The problem of realizing doubly resistive terminated two-ports (Fig. 49a) was first solved by Darlington. Darlington has also pointed out how to realize a voltage ratio for a given source-to-load resistance ratio, r, when the coupling network is an L'C''T two-port (Fig. 49b).

Our objective here is to arrive at Darlington's result from a different point of view. By taking advantage of the established PRD properties of driving-point functions of RL'C''T networks, we believe additional insight into this problem is to be gained.

The design of Fig. 49b places certain restrictions on the nature of T(\lambda). The poles of T(\lambda) represent frequencies at which a voltage might be observed across the load.
resistor after a voltage impulse excitation has been applied. They are, therefore, restricted to the left-half of the \( \lambda \)-plane, except for the possibility of simple poles along the real axis in the interval \( 0 \leq \lambda < \alpha \). The only restriction on the numerator polynomial of \( T(\lambda) \) is that its degree does not exceed the degree of the denominator polynomial.

For this discussion, let us therefore assume that we are to realize a voltage ratio of the form

\[
\frac{E_2}{E_1} = T(\lambda) = \frac{a_m \lambda^m + a_{m-1} \lambda^{m-1} + \ldots + a_0}{b_n \lambda^n + b_{n-1} \lambda^{n-1} + \ldots + b_0},
\]

(241)

where \( m \leq n \) and the denominator is Hurwitz, except possibly for some simple poles along the real axis in the interval \( 0 \leq \nu < \alpha \).

Our strategy is again one of power separation along the \( j\xi \)-axis. Thus we obtain an equation relating \( D_Z \) and \( T(\lambda) \), from which we shall show that a \( \text{PRD}_Z \) impedance, \( \tilde{Z}(\lambda) \), can always be extracted. \( \tilde{Z}(\lambda) \) is then realized by the Darlington procedure of section 5.2.

The power delivered to the load resistance, for excitation frequencies on the \( j\xi \)-axis, must equal \( |I_1|^2 D_Z(j\xi) \), that is,

\[
|I_1|^2 D_Z(j\xi) = \frac{|V_2|^2}{I_1} = |T(j\xi)|^2,
\]

(242)

which we can write as follows:

\[
\frac{\left[ j\xi - \alpha \right] \tilde{Z}(j\xi) + \left[ -j\xi - \alpha \right] \tilde{Z}(-j\xi)}{2|Z(j\xi) + r|^2} = |T(j\xi)|^2.
\]

(243)

In order to solve this equation for \( \tilde{Z}(\lambda) \), we introduce the quantity

\[
\rho(j\xi) = \frac{\tilde{Z}(j\xi) - r j\xi + \alpha}{\tilde{Z}(j\xi) + r}
\]

(244)

and write Eq. 243 in the form

\[
|\rho(j\xi)|^2 + 4r \frac{\xi^2}{\xi^2 + \alpha^2} |T(j\xi)|^2 = 1.
\]

(245)

[For \( \alpha = 0 \), \( \rho(j\xi) \) is just the familiar reflection coefficient that arises in connection with terminated lossless two-ports and in transmission line theory.]

Equation 245 represents the desired relationship between \( T(\lambda) \) and \( \tilde{Z}(\lambda) \) and can be solved for \( \tilde{Z}(\lambda) \) by the following sequence of steps:

\[
|\rho(j\xi)|^2 \rightarrow \rho(\lambda) \rho(-\lambda) \rightarrow \rho(\lambda) \rightarrow \tilde{Z}(\lambda).
\]

(246)
To carry out the second step we need to know the detailed properties of \( \rho \), which we now enumerate.

1. \( \rho(0) = 1 \).
2. \( |\rho(j\xi)|^2 \) is a quotient of even polynomials in \( \xi \).
3. \( 1 \geq |\rho(j\xi)|^2 \geq 0 \).
4. Poles and zeros of \( \rho(\lambda)\rho(-\lambda) \) (the analytic continuation of \( \rho(j\xi)\rho(-j\xi) \)) must display quadrantal symmetry.
5. \( \rho(\lambda)\rho(-\lambda) \) has a pair of poles located at \( \lambda = \pm\alpha \), provided \( T(\lambda) \) does not have zeros at either \( \lambda = \alpha \) or \( \lambda = -\alpha \).
6. \( \rho(\lambda)\rho(-\lambda) \) is analytic on the j\( \xi \)-axis.
7. The relationship between \( \rho(\lambda) \) and \( \tilde{Z}(\lambda) \) is as follows:

\[
\rho(\lambda) = \frac{\tilde{Z}(\lambda) - r \frac{\lambda + \alpha}{\lambda - \alpha}}{\tilde{Z}(\lambda) + r} \tag{247}
\]

or equivalently

\[
\tilde{Z}(\lambda) = r \frac{\lambda + \alpha + \rho(\lambda)}{1 - \rho(\lambda)} \tag{248}
\]

8. \( \rho(\lambda) \) must have a simple pole at \( \lambda = \alpha \) if \( \tilde{Z}(\lambda) \) is regular at this point and vice versa.

We are now in a position to extract an appropriate \( \rho(\lambda) \) from \( \rho(\lambda)\rho(-\lambda) \). In this connection we shall not discuss the case in which \( T(\lambda) \) has a transmission zero at
\[ \lambda = \alpha, \text{ since the procedure then is identical to the ordinary RLC case.} \]

Assume, therefore, that \( T(\lambda) \) is non-zero at \( \lambda = \alpha \), so that \( \rho(\lambda) \rho(-\lambda) \) has a pole-pair at \( \lambda = \pm \alpha \) (property 5). \( \tilde{Z}(\lambda) \) must be regular at \( \lambda = \alpha \), so that one must assign the pole at \( \lambda = \alpha \) to \( \rho(\lambda) \). The remaining pole-zero pattern of \( \rho(\lambda) \rho(-\lambda) \) is then distributed between \( \rho(\lambda) \) and \( \rho(-\lambda) \) as in the ordinary case, that is, all left-hand plane poles and any half of the zeros to \( \rho(\lambda) \). [Zeros in the interval \((0, \alpha)\), if any, must occur in even numbers, however.]

In assigning the sign to the various choices of \( \rho \), the reader undoubtedly recalls that either sign can be used in the ordinary case. In this case, however, the sign of \( \rho(\lambda) \) must be chosen such that the residue of its pole at \( \lambda = \alpha \) is negative, that is,

\[
0 > \left[ \frac{(\lambda - \alpha) \rho(\lambda)}{\lambda} \right] \bigg|_{\lambda = \alpha}.
\] (249)

This is necessary in order for \( \tilde{Z}(\lambda) \) to be a PRD function and be regular at \( \lambda = \alpha \), as can be seen from

\[
Z(\alpha) > 0 \rightarrow \frac{2\alpha + (\lambda - \alpha) \rho(\lambda)}{- (\lambda - \alpha) \rho(\lambda)} \bigg|_{\lambda = \alpha} > 0.
\] (250)

This inequality can only be satisfied if

\[
0 > \left[ \frac{(\lambda - \alpha) \rho(\lambda)}{\lambda} \right] \bigg|_{\lambda = \alpha} > -2\alpha.
\]

It will be shown in Appendix B that the lower bound of this inequality is automatically satisfied because of the nature of \( \rho(\lambda) \). We shall also show in Appendix B that the resulting \( \tilde{Z}(\lambda) \) is guaranteed to be PRD, provided \( \rho(\lambda) \) has the following properties:

(i) A pole at \( \lambda = \alpha \) with a negative residue,
(ii) No other right half-plane poles, and
(iii) \( \rho(0) = 1 \).

\( \tilde{Z}(\lambda) \) can therefore be realized by the Darlington driving-point realization of section 5.2. The final realization of \( T(\lambda) \) will be of the form shown in Fig. 49b.

The reader might ask what happens if \( \rho(\lambda) \) is extracted from \( \rho(\lambda) \rho(-\lambda) \) as in the RLCT case. The answer is simple. Going through the steps above will lead to a network realization of the type shown in Fig. 49b that will be consistent with Eq. 245. There will be a transmission zero, however, at \( \lambda = \alpha \), since \( \tilde{Z}(\lambda) \) has a pole at this point (\( \rho(\lambda) \) being regular). Therefore, instead of arriving at a network realization of \( T(\lambda) \), the network realizes

\[
T'(\lambda) = \frac{\lambda - \alpha}{\lambda + \alpha} T(\lambda).
\] (251)

The magnitude of \( T' \) and \( T \) along the \( j \xi \)-axis is, of course, the same, so both transfer ratios will satisfy Eq. 245.

The following example has been included to illustrate this procedure. The choice
of a lowpass transfer function as an example was made to have an opportunity to
discuss briefly the elimination of ideal transformers in lowpass filter design with
two resistive terminations for coupling networks composed of dissipative reac-
tance elements.

EXAMPLE 12

Realize the following third-order Butterworth function

\[ T(s) = \frac{A}{s^3 + 2s^2 + 2s + 1} \]  

as the voltage transfer ratio of a network of the form shown in Fig. 49b under the
following conditions:

(a) \( \alpha_L = \frac{1}{2} \) and \( \alpha_C = 0 \) such that \( \alpha = 5 = \frac{1}{4} \)
(b) \( r = 2 \).

The transfer ratio can be written in \( \lambda \)-coordinates as follows:

\[ T(\lambda) = \frac{A}{\lambda^3 + \frac{5}{4}\lambda^2 + \frac{19}{16}\lambda + \frac{39}{64}}, \]  

and Eq. 245 becomes

\[ |\rho(j\theta)|^2 = 1 - 4rA^2 \left| \frac{\xi^2}{\xi^2 + \alpha^2} \right|^2 |T(j\xi)|^2 \]

\[ = 1 - 4rA^2 \left( \frac{\xi^2}{4} - \frac{3}{4}\xi^6 - \frac{21}{128}\xi^4 + \frac{373}{1024}\xi^2 + \frac{1521}{65536} \right). \]

To maximize the transfer ratio \( T \), we select its constant multiplier so that \( 4rA^2 = \frac{1}{4} \).

\( \rho(\lambda)\rho(-\lambda) \) then becomes

\[ \rho(\lambda)\rho(-\lambda) = \frac{(\lambda + j\cdot.788)^2 (\lambda - j\cdot.788)^2 (\lambda + .495) (\lambda - .495) (\lambda + .496) (\lambda - .496)}{(\lambda + \frac{1}{4})(\lambda - \frac{1}{4})(\lambda + \frac{3}{4})(\lambda - \frac{3}{4})(\lambda^2 + \frac{1}{2}\lambda + \frac{13}{16})(\lambda^2 - \frac{1}{2}\lambda + \frac{13}{16})} \]

whose pole-zero pattern is shown in Fig. 50.

According to the previous discussion there are four acceptable ways of selecting
an appropriate \( \rho(\lambda) \) in this case. Figure 51 indicates two of these, provided the sign
of \( \rho(\text{sgn } \rho) \) is so chosen as to give a negative residue at \( \lambda = \alpha \).

Let us arbitrarily select Fig. 51b and write

\[ \rho(\lambda) = \frac{-\frac{\lambda^4}{4} - .991\lambda^3 + .867\lambda^2 - .606\lambda + .153}{(\lambda - \frac{1}{4})(\lambda^3 + \frac{5}{4}\lambda^2 + \frac{19}{16}\lambda + \frac{39}{64})}. \]

The corresponding \( \tilde{Z}(\lambda) \) then becomes
\[ \tilde{Z}(\lambda) = \frac{4.982\lambda^2 + 1.268\lambda + 3.042}{2\lambda^3 + 0.009\lambda^2 + 1.741\lambda - 0.302}. \]

This impedance is known to be a \( PRD_Z \) function and therefore realizable by the Darlington procedure.

\[ \text{Fig. 50. Pole-zero plot of } \rho(\lambda) \rho(-\lambda) \text{ in expression (255).} \]

\[ \text{Fig. 51. Appropriate extraction of } \rho(\lambda) \text{ from Fig. 50: (a) with } \text{Sgn } \rho = 1, \text{ (b) with } \text{Sgn } \rho = -1. \]

It can be shown, however, by the same reasoning as in the regular case \( (\alpha = \delta = 0) \) that this impedance has some very special properties. The reason for this is the low-pass character of the corresponding transfer ratio from which \( \tilde{Z}(\lambda) \) was derived. Because of this property, \( \tilde{Z}(\lambda) \) can be developed in a ladder structure composed of inductive series arms and capacitive shunt arms, and terminated in a resistance.
For this purpose, let us factorize \( \tilde{Z}(\lambda) \) as suggested in connection with the ladder developments of section 2.4, that is,

\[
\tilde{Z}(\lambda) = \frac{4.982 \left( \lambda^2 - \frac{1}{16} \right) + 1.268 \left( \lambda + \frac{1}{4} \right) + 3.036}{2 \left( \lambda - \frac{1}{4} \right) \left( \lambda^2 - \frac{1}{16} \right) + 0.509 \left( \lambda^2 - \frac{1}{16} \right) + 1.866 \left( \lambda - \frac{1}{4} \right) + 0.166}.
\]

(258)

From the factored form of \( \tilde{Z}(\lambda) \), it is a simple matter of repeated division and inversion to obtain the final realization shown in Fig. 52.

![Fig. 52. Final realization of a doubly resistive terminated L'C'' two-port for the voltage transfer ratio (252).](image)

The reader may recall that by properly adjusting the constant multiplier of a lowpass transfer function in the ordinary case, it is possible to realize almost all such functions by a network of the form shown in Fig. 53. No corresponding transformerless procedure has been found for realizing a given lowpass voltage transfer ratio for the case in which the coupling network is an L'C'' two-port.

![Fig. 53. Typical transformerless realization of a doubly loaded lossless coupling network.](image)

7.3 RL'C''T versus RLCT NETWORKS

The preceding section concludes our discussion of RL'C''T networks. Let us therefore summarize our results.

An expression for the difference between the power entering an RL'C''T network and the power dissipated in the lossy reactance elements has been introduced and called the difference function. The difference function plays a role in the RL'C''T case analogous to that played by the real-part function in the normal RLCT case. The positive and real (PR) character of the difference function over an appropriate portion of the s-plane has been shown to represent a necessary and sufficient condition for a rational function to be the driving-point immittance of an RL'C''T network. Most of the familiar RLCT synthesis procedures are extended to the RL'C''T case by employing the difference function in much the same manner as one employs the real part in the normal RLCT case.

The analogies between the two cases are rather striking. To repeat these analogies and give a concise summary of our results, we include Table 1.
<table>
<thead>
<tr>
<th>LC NETWORKS</th>
<th>RC NETWORKS</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Properties of Driving Point Impedance</strong></td>
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</tr>
<tr>
<td><em>Note:</em> The frequency variable is ( \omega = \omega_0 f ) in this case.</td>
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</tr>
<tr>
<td><strong>Graphical Method</strong></td>
<td><strong>Graphical Method</strong></td>
</tr>
<tr>
<td><strong>Solution Procedure</strong></td>
<td><strong>Solution Procedure</strong></td>
</tr>
<tr>
<td><strong>Direction of Flow</strong></td>
<td><strong>Direction of Flow</strong></td>
</tr>
<tr>
<td><strong>Solution Procedure</strong></td>
<td><strong>Solution Procedure</strong></td>
</tr>
</tbody>
</table>

### Table 1. Summary of results.

The table above summarizes the results for LC and RC networks. The columns detail the properties of driving point impedance, graphical methods, solution procedures, and directions of flow. The rows correspond to specific conditions and equations relevant to each network type.

**LC Networks**
- Properties of Driving Point Impedance
  - Equivalent circuit
  - Graphical method
  - Solution procedure

**RC Networks**
- Properties of Driving Point Impedance
  - Equivalent circuit
  - Graphical method
  - Solution procedure

### Notes
- All elements are positive.
- The power delivered to the load must meet the difference between the power dissipated by the generator and the power dissipated to the load.
- The power delivered to the load can be expressed as follows:

\[
\text{Power delivered} = P_{\text{load}} = V_1^2 Z_0
\]

where \( V_1 \) and \( Z_0 \) are the voltage and impedance, respectively, at the source.
Table 1. (continued).

<table>
<thead>
<tr>
<th>RLC NETWORKS</th>
<th>RLC NETWORKS</th>
</tr>
</thead>
<tbody>
<tr>
<td>Prep. of Driving-Point Impedance</td>
<td>Nondispersive Case</td>
</tr>
<tr>
<td><img src="image1" alt="Diagram" /></td>
<td><img src="image2" alt="Diagram" /></td>
</tr>
</tbody>
</table>

**Fundamental Equations:**

If $Re\left[Z(j\omega)\right] = 0$, then $Z(j\omega) = k_2 j\omega$, where $k_2$ is a positive or negative constant.

Assume $k_2 < 0$ and choose the constant "a" such that it satisfies:

$$Z(a) = \omega^2 k_2 = 0$$

A solution to this equation always exists, and we can write:

$$Z_1(a) = \frac{1}{\omega^2} Z_1(a) + Z_1(a)$$

$$Z_2(a) = \frac{1}{\omega^2} Z_2(a) + Z_2(a)$$

**Miyoota Procedure**

Consider a PR function, $Z(b)$, having its even part zeros at the origin and/or at infinity, i.e., an impedance whose even part is of the form:

$$Z(b) = \sum_{p=0}^{\infty} a_p b^{2p}$$

Any impedance of this form can be realized as a lossless ladder terminated in a resistance. Using this fact, one can get a transformless realization of any PR function that can be

$$Z(b) = \sum_{p=0}^{\infty} \left[\frac{a_p}{b^{2p}}\right] Z(b)$$

where $a_p > 0$

by realizing each component impedance from its even part, and connecting the $Z_p$'s in series as shown in the figure.
Table 1. (concluded).

<table>
<thead>
<tr>
<th>RLCT NETWORKS</th>
<th>Nonlative Case</th>
<th>Disparative Case</th>
</tr>
</thead>
<tbody>
<tr>
<td>Voltage-Transfer Function</td>
<td>Any voltage transfer function of the form</td>
<td>Any voltage transfer function of the form</td>
</tr>
<tr>
<td>$E_2$</td>
<td>$E_2 = T(0) = \frac{A}{a_n^2 + \ldots + a_1^2 + s}$</td>
<td></td>
</tr>
<tr>
<td>Loop-by Filter with a Single Resistance Termination</td>
<td>can be realized as shown in the figure, provided the constant multiplier $A$ is properly chosen. One can accomplish this by carrying out the following steps:</td>
<td></td>
</tr>
<tr>
<td></td>
<td>1) adjust the multiplier $A$ to the value $A_0$ so that $T(0) = 1$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>2) write $T(s) = \frac{A_0}{a_n^2 + \ldots + a_1^2 + s}$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>read off $s_0$ and $A_0$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>3) develop $E_2$ in a ladder structure. The required $s_0$ automatically results.</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Because the Butterworth and Chebyshev loop-by functions possess this functional form, all such functions can be realized by the foregoing procedure.</td>
<td></td>
</tr>
</tbody>
</table>

<table>
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<th>Disparative Case</th>
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<th>Two-Port</th>
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<td>Power in: $</td>
<td>E_1</td>
</tr>
<tr>
<td>Power out: $\frac{</td>
<td>E_1</td>
</tr>
<tr>
<td>Therefore: $Z(\phi) = Z(\phi) =</td>
<td>Z(\phi)</td>
</tr>
</tbody>
</table>

To solve this equation for $Z(\phi)$ we introduce the so-called reflection coefficient $\rho(\phi)$, defined as follows: $\rho(\phi) = \frac{Z(\phi) - \rho(\phi)}{Z(\phi) + \rho(\phi)}$, and obtain: $|\rho(\phi)|^2 = 1 - |\rho(\phi)|^2 < 1$. |

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To solve for $Z(\phi)$ we introduce the reflection coefficient $\rho(\phi)$, defined as follows: $\rho(\phi) = \frac{Z(\phi) - \rho(\phi)}{Z(\phi) + \rho(\phi)}$, and obtain: $|\rho(\phi)|^2 = 1 - 4|\rho(\phi)|^2 < 1$. |

Following the same scheme as in the nonlative case, we get the following conditions for selecting an appropriate $\rho(\phi)$ from $\rho(\phi)$: $\rho(\phi)$ must be negative on both sides of the $J$-plane. 

This will guarantee that $Z(\phi)$ obtained from $\rho(\phi)$ will be proto $Z$. |
VIII. NETWORKS COMPOSED OF MORE GENERAL CIRCUIT ELEMENTS

In view of the foregoing discussion one might expect that the difference-function approach would yield some rewarding results also in the case of networks composed of any three linear circuit elements ($Z_1Z_2Z_3T$ networks). This is indeed the case, although necessary and sufficient conditions for a function to be the driving-point impedance of a $Z_1Z_2Z_3T$ network have only been established for certain classes of circuit elements.

Specifically, the following results have been obtained:
1. Another pairing of a specific network class with a function class.
2. Descriptions of more general classes of networks for which the non-negativeness of the difference function over appropriate regions of the s-plane completely specifies the driving-point functions in the network class.
3. A procedure that, in most cases of interest, will lead to a set of necessary conditions (Generalized Positive Real (GPR) conditions or ABC conditions) on the driving-point functions of any $Z_1Z_2Z_3T$ network.
4. An example to show that the GPR conditions are not always sufficient to completely describe driving-point functions in the network class.

These results are obtained by studying several representative network classes; the various classes being defined in Fig. 54. We shall refer to these as $RL'C''''T$ networks, $Z_{RL}Z_{RC}Z_{RT}$ networks, $RLC_LT$ networks, and $LCC''''T$ networks, respectively.

8.1 $RL'C''''T$ NETWORKS

We begin by considering $RL'C''''T$ networks. This class of networks is not introduced just as another specific example but as a means of introducing the larger class of $Z_{RL}Z_{RC}Z_{RT}$ networks below. The development in this section is therefore made somewhat more comprehensive than is necessary just to arrive at a set of necessary and sufficient conditions for driving-point functions of $RL'C''''T$ networks. As a matter of fact, the procedure outlined in this section will establish a strong set of necessary conditions for driving-point impedances of almost all networks composed of any three linear circuit elements.

The five considerations discussed below can be considered an algorithm which will be applied to each of the following classes of networks. The resulting differences for the various cases are thus readily compared step by step.

Consideration 1: Root-Locus Plots

Employing root-locus techniques (as discussed in section 2.1) we shall determine where the driving-point impedances of the three possible two-element-kind networks, composed of the three circuit elements of Fig. 54a, may have their poles and zeros.
Fig. 54. Standard parts defining: (a) RL'C'T networks; (b) ZRLZRCRT networks; (c) RLC'T networks; (d) LCC'T networks.

These contours are obtained as the loci (Cij), which constitute solutions of the equations

\[ 1 + k \frac{z_i}{z_j} = 0 \text{ for } k > 0, \quad (259) \]

and are shown in Fig. 55.

Fig. 55. Root-locus plots for the RL'C'T case.

Notice that the end points of the open-ended contours have been labeled by a pole or a zero, depending upon whether the driving-point impedance composed of the two elements in question can have a pole or a zero at these points. The end points of these
contours can therefore be considered as the root-locus plots of the three elements taken one at the time. (The root locus of the elements taken three at the time will, as usual, be determined later (consideration 3) via the difference function.)

**Consideration 2: Difference Functions**

The experiment shown in Fig. 11 can be generalized to any three linear circuit elements as shown in Fig. 56. By utilizing conservation of complex power, it can be readily shown that this experiment leads to two equations:

\[
\text{Re } Z(s) - \frac{1}{|I|^2} \text{Re} \left[ \sum_{z_1 \text{ and } z_2 \text{ in } N_1} V_b l_b^* \right] = \frac{1}{|I|^2} \left[ \sum_{i = 1}^{n} \text{Re } z_3 |I_1| \right]^2 \tag{260}
\]

\[
\text{Im } Z(s) - \frac{1}{|I|^2} \text{Im} \left[ \sum_{z_1 \text{ and } z_2 \text{ in } N_1} V_b l_b^* \right] = \frac{1}{|I|^2} \left[ \sum_{i = 1}^{n} \text{Im } z_3 |I_1| \right]^2. \tag{261}
\]

Fig. 56. Representation of a general three-element-kind network in a form suitable for complex power considerations.

In order to obtain a necessary condition on \( Z(s) \) consistent with these equations, we observe that the vectors representing \( z_1(s) \) and \( z_2(s) \) for points on \( C_{12} \) are \( 180^\circ \) out of phase. It follows that

\[
\frac{\text{Re } z_1(s)}{\text{Im } z_1(s)} \bigg|_{s \in C_{12}} = \frac{\text{Re } z_2(s)}{\text{Im } z_2(s)} \bigg|_{s \in C_{12}} = \frac{\text{Re} \left[ \sum_{z_1 \text{ and } z_2 \text{ in } N_1} V_b l_b^* \right]}{\text{Im} \left[ \sum_{z_1 \text{ and } z_2 \text{ in } N_1} V_b l_b^* \right]} \bigg|_{s \in C_{12}}. \tag{262}
\]
Substituting Eqs. 261 and 262 in Eq. 260, we obtain

\[
\sum_{i=1}^{n} \left| \frac{I_i}{I} \right|^2 \mathcal{Z}_i = \left. \begin{array}{c}
\Re Z(s) - f(s) \Im Z(s) \\
\Re \mathcal{Z}_3(s) - f(s) \Im \mathcal{Z}_3(s)
\end{array} \right|, \quad s \in C_{12}
\]

where

\[
f(s) = \left. \frac{\Re z_1(s)}{\Im z_1(s)} \right| = \left. \frac{\Re z_2(s)}{\Im z_2(s)} \right| = \frac{1}{a+b} \left[ \frac{\Re z_1}{\Im z_1} + b \frac{\Re z_2}{\Im z_2} \right]. \quad (264)
\]

\[a \text{ and } b \text{ are constants}
\]

Now, if we define the impedance derived difference function, or just \(E_Z\)-function, of \(RL'C'T\) networks as follows:

[Note that we have rather arbitrarily chosen

\[
f(s) = \frac{1}{a+b} \left[ \frac{\Re z_1}{\Im z_1} + b \frac{\Re z_2}{\Im z_2} \right] = \frac{\Re z_1}{\Im z_1}
\]

\[s=1; \ b=0
\]

in this case. It can be shown that any other choice for \(a\) and \(b\) will lead to the same ABC conditions in consideration 3.]

\[
E_Z(s) = \frac{\Re Z(s) - \frac{\Re z_1(s)}{\Im z_1(s)} \Im Z(s)}{\Re z_3(s) - \frac{\Re z_1(s)}{\Im z_1(s)} \Im z_3(s)} = \Re Z(s) - \frac{\sigma + 5}{\omega} \Im Z(s), \quad (265)
\]

then it is seen that

\[
E_Z(s) \gtrless 0, \quad s \in C'_{12}
\]

where \(C'_{12}\) is \(C_{12}\) excluding the real axis.

In order to determine the regions of the s-plane (if any), where \(E_Z(s)\) is strictly positive, let us express the driving-point impedance \(Z(s)\) in terms of its component impedances as follows:

\[
Z(s) = \frac{\sum \Re z_1 |I_1|^2}{|I|^2} z_1 + \frac{\sum \Re z_2 |I_2|^2}{|I|^2} z_2 + \frac{\sum \Re z_3 |I_3|^2}{|I|^2} z_3. \quad (267)
\]

If this expression is inserted into the \(E_Z\)-function, and \(E_{Z_1}\) is used to denote the \(E_Z\)-function of \(z_1\), then we can write
\[
E_Z(s) = \frac{\sum_{\text{all } z_1} |I_{z_1}|^2}{|I|^2} E_{z_1}(s) + \frac{\sum_{\text{all } z_2} |I_{z_2}|^2}{|I|^2} E_{z_2}(s) + \frac{\sum_{\text{all } z_3} |I_{z_3}|^2}{|I|^2} E_{z_3}(s)
\]

\[
= \frac{\sum_{\text{all } z_2} |I_{z_2}|^2}{|I|^2} E_{z_2}(s) + \frac{\sum_{\text{all } z_3} |I_{z_3}|^2}{|I|^2} E_{z_3}(s),
\]

(268)

since \(E_{z_1}(s) = 0\) and \(E_{z_3}(s) = 1\).

We see that \(E_Z(s)\) must be positive in the same regions of the \(s\)-plane as \(E_{z_2}(s)\).

Let us therefore determine the contour where \(E_{z_2}(s)\) changes sign, that is \(E_{z_2}(s) = 0\).

We can write

\[
E_{z_2}(s) = \Re z_2 - \frac{\Re z_1}{\Im z_1} \Im z_2 = \frac{\Re z_2}{\Re z_1} \frac{\Im z_1}{\Im z_2} - \frac{\Re z_1}{\Im z_1} \frac{\Im z_1}{\Im z_2} = e_{21}
\]

(269)

where we have defined

\[
e_{ij} = \Re z_i \Im z_j - \Re z_j \Im z_i.
\]

(270)

In this case \(z_3 = 1\), which implies that \(e_{31} = \Im z_1\). [Notice that the ordering of the subscripts is significant, that is,

\[
e_{21} = -e_{12}.
\]

The \(e_{ij}\)-functions are introduced, since they enable us to obtain the various regions where \(E_Z(s)\) is non-negative by the following straightforward procedure.

First we observe that

\[
e_{ij} = 0
\]

(271)

describes the root locus

\[
1 + k \frac{z_i}{z_j} = 0 \quad \text{for all real } k.
\]

(272)

It follows immediately that the locus described by \(e_{12} = 0\), say, is a closed contour and thus divides the \(s\)-plane into well-defined regions. Since \(e_{12}\) is a continuous function of \(\sigma\) and \(\omega\), it must be either strictly positive or strictly negative in each of these regions. A similar statement holds for \(e_{13}\) and \(e_{23}\).

Then we observe that we can express the impedance derived difference function (Eq. 265) in terms of the \(e_{ij}\)-functions in the following manner:
The problem of obtaining the region where the difference function is positive, is therefore readily solved by means of sign-maps of the various $e_{ij}$-functions (i.e., $s$-planes labeled $\text{Sgn } e_{ij}$).

The appropriate sign-maps in connection with the impedance derived difference function for $RL'C'T$ networks are shown in Fig. 57.

Since $E_{Z}(s)$ is not defined along the real axis, we should, as in the $RL'C'T$ case, determine whether (Eq. 260) imposes any additional restrictions on $Z(s)$ along the real axis (Eq. 261 is automatically satisfied along the real axis, of course, for circuit elements described by linear differential equations with real coefficients).
For the three circuit elements in the RL'C'T case, we see that

\[ z_1(\sigma) > 0 \quad \text{for } \sigma > -5 \]
\[ z_2(\sigma) > 0 \quad \text{for } \sigma > -2 \]
\[ z_3(\sigma) = 1 \]

It follows that

\[ Z(\sigma) > 0 \quad \text{for } \sigma > -2. \]

At this point one should investigate the possibility of one or two more difference functions resulting from considering the experiments shown in Fig. 58. The relations correpodnings to Eqs. 260-265 for these experiments follow by appropriate interchanges of the subscripts associated with the three circuit elements.

Fig. 58. Representation of a general three-element-kind network in a form suitable for complex power considerations with (a) a \( Z_1Z_3T \) coupling network; (b) a \( Z_2Z_3T \) coupling network.

For the particular case that we are considering, however, the resulting boundary conditions are undefined on the contours \( C_{13} \) and \( C_{23} \), both of which are located along the real axis. No additional conditions, therefore, result from the experiments shown in Fig. 58 in the RL'C'T case.
We have thus established that driving-point impedances of RL'C'T networks satisfy the following conditions

1. $E_z(s) > 0$ for $s \in C_{12}^1$
2. $E_z(s) > 0$ for $s \in R$
3. $Z(\sigma) > 0$ for $\sigma > -2$.

where $C_{12}^1$ and $R$ have been defined previously. [Notice that it is also true that if we had used any other choice for "a" and "b" in defining $f(s)$ (Eq. 264), then the impedance derived difference function would be different off the contour $C_{12}^1$; however, the conditions above would have been of the same form.]

A rational function that satisfies these conditions will hereafter be called a PREZ function.

By pursuing a dual line of reasoning one can derive analogous properties for RL'C'T admittances. The properties in question are most easily stated in terms of the following quantity, which we shall refer to as the admittance derived difference function of RL'C'T networks, or just the $E_y$-function,

$$E_y(s) = \frac{\text{Re } Y(s) - \frac{\text{Im } Y(s)}{\text{Re } Y(s)}}{\text{Re } y_3(s) - \frac{\text{Im } y_3(s)}{\text{Re } y_3(s)}} = \text{Re } Y(s) + \frac{\sigma + 5}{\omega} \text{Im } Y(s).$$

The conditions are as follows:

1. $E_y(s) \geq 0$ for $s \in C_{12}^1$
2. $E_y(s) > 0$ for $s \in R$
3. $Y(\sigma) > 0$ for $\sigma > -2$.

A rational function that satisfies these conditions will hereafter be called a PREY function.

It can readily be shown that the PREZ conditions on $Z$ imply that $Y = Z^{-1}$ satisfies the PREY conditions and vice versa.

Consideration 3: ABC conditions

Conditions analogous to the so-called ABC conditions of PRDZ functions are readily obtained from the PRE conditions. These are stated below on an impedance basis.

Any PREZ function has the following properties:
A. It is analytic in $R$ (Fig. 57c).
B. Its impedance derived difference function, $E_z(s)$, is non-negative on the contour $C_{12}^1$.  

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C. Its poles on the various boundaries of \( R \) must satisfy the following residue conditions:

(i) Poles on \( C'_{12} \) must be simple and occur in complex conjugate pairs.

If \( Z(s) \) has a pole at \( s = s_v \) with residue \( k_v = |k_v| e^{i\phi} \), then

\[
\phi = -\tan^{-1} \left( \frac{\cos \Psi + \frac{\text{Re } z_1(s_v)}{\text{Im } z_1(s_v)} \sin \Psi}{\sin \Psi - \frac{\text{Re } z_1(s_v)}{\text{Im } z_1(s_v)} \cos \Psi} \right);
\]

the ambiguity in sign is resolved by having the \( k_v \)-vector pointing into the region \( R \).

The angle \( \Psi \) is defined in Fig. 59a. [Note: \( \phi \) is written in a general form that applies to poles along the smooth portion of the boundaries of any region where the difference function,

\[
\text{Re } Z(s) - \frac{\text{Re } z_1(s)}{\text{Im } z_1(s)} \text{ Im } Z(s),
\]

is non-negative. This result is readily obtained by the method used in connection with RL'C'T networks.]

(ii) Poles on the contour \( CA \) of Fig. 59b must be simple and have a positive real residue. At point A, however, there might be a double-order pole, provided a Laurent expansion of \( Z(s) \) about this point can be written

\[
Z(s) = \frac{k_{-2}}{(s+4)^2} + \frac{k_{-1}}{s+4} + \sum_{i=0}^{\infty} k_i (s+4)^i,
\]

with

\[
k_{-1} > k_{-2} > 0.
\]

(iii) Poles on \( CB \) of Fig. 59b must be simple and have a negative real residue. At point B, however, there might be a double-order pole, provided a Laurent expansion about this point can be written

\[
Z(s) = \frac{k_{-2}}{(s+8)^2} + \frac{k_{-1}}{s+8} + \sum_{i=0}^{\infty} k_i (s+8)^i,
\]

with

\[
k_{-1} < \frac{k_{-2}}{3} < 0.
\]

These conditions will hereafter be referred to as the ABCZ conditions of \( \text{PRE}_Z \) functions.

For \( \text{PRE}_Y \) functions, a dual set of conditions of the same form can easily be established. These conditions (the \( \text{ABC}_Y \) conditions for \( \text{PRE}_Y \) functions) will not be included.
THEOREM 10. Let $Z(s)$ be a rational function with real coefficients. Then, if $Z(s)$ satisfies the $ABC_Z$ conditions for $PRE_Z$ functions, it must be a $PRE_Z$ function.

Proof:

The proof in this case follows the same line of reasoning as was used in the proof of Theorem 1; some of the details are therefore left out.

In order to show that $E_Z(s) > 0$ for $s \in R$, we introduce the auxiliary function

$$A(s) = \frac{Z(s)}{z_1(s)}$$

and make the observation

$$\text{Im} A(s) = \text{Im} \frac{Z(s)}{z_1(s)} = \frac{\text{Re} z_1 \text{Im} Z(s) - \text{Im} z_1 \text{Re} Z(s)}{\text{Re}^2 z_1 + \text{Im}^2 z_1} = -\frac{\text{Im} z_1}{\text{Re}^2 z_1 + \text{Im}^2 z_1} E_Z(s),$$

where

$$\text{Im} z_1 = \omega$$
$$\text{Re} z_1 = \sigma + 5.$$ 

From (278) we can construct Fig. 60, from which it follows that

$$\text{Im} A(s) < 0 \quad \text{for} \quad s \in R_U$$
$$\text{Im} A(s) > 0 \quad \text{for} \quad s \in R_L$$

or equivalently

$$E_Z(s) > 0 \quad \text{for} \quad s \in R = R_U \cup R_L.$$ 

It is also true that the $ABC_Z$ conditions imply

$$Z(\sigma) > 0 \quad \text{for} \quad \sigma > -2.$$ 

This can be shown by the same line of reasoning that was used in the $RL'C''T$ case.

Therefore it is evident that the $ABC_Z$ conditions of $PRE_Z$ functions $\leftrightarrow$ $PRE_Z$ conditions.

A dual set of arguments can be used to establish the equivalence between the $ABC_Y$ conditions of $PRE_Y$ functions and the $PRE_Y$ conditions.
Fig. 59. Relevant to residue evaluations of RL'C''T networks on (a) contour C'\textsubscript{12}; (b) contours C\textsubscript{A} and C\textsubscript{B}.

Fig. 60. Behavior of the auxiliary function A on the modified boundary of (a) the upper half-plane; (b) the lower half-plane.
Consideration 5: Realization Procedure

It can be readily shown that the Brune procedure can be extended to handle any PRE function. This extension is similar to the one in connection with the RL'C'T case (Section IV). This time, therefore, it should be sufficient just to illustrate the two Brune cycles (cases 1 and 2). This is done in Fig. 61.

The design shown as case I (Fig. 61a) is pertinent to the case in which the direction of the $Z_1(s)$ vector is the same as that of $Z_2(s)$. $k_1$ is therefore negative, and the admittance function $Y_2(s)$ must be a PRE function with a pair of conjugate poles at $s = \pm s_v$. The negative element $k_1 z_1$ can be absorbed by the use of an ideal transformer in the usual manner.

The design shown as case 2 (Fig. 61b) is pertinent to the case in which the direction of the $Z_1(s)$ vector is the same as that of $Z_2(s)$. $k_1$ is therefore also in this case negative, and the admittance function $Y_2(s)$ must be a PRE function with a pair of conjugate poles at $s = \pm s_v$. The negative element $k_1 z_2$ can also, in this case, be absorbed by the use of an ideal transformer.

To summarize, we have outlined a five-step procedure for obtaining a set of necessary and sufficient conditions, the so-called PRE conditions, for a function to be the driving-point impedance of an RL'C'T network.
8.2 $Z_{RL}Z_{RC}RT$ NETWORKS

RLCT, RL'C'^T, and RL'C''T networks are all special cases of networks composed of the three circuit elements of Fig. 54b ($Z_{RL}Z_{RC}RT$ networks). In all of these cases the five steps outlined above were sufficient to obtain a complete description of the driving-point functions belonging to the network class. Let us now determine whether this is the case for all $Z_{RL}Z_{RC}RT$ networks.

In view of the foregoing discussion, it should be clear that the first four steps of section 8.1 can be applied to all $Z_{RL}Z_{RC}RT$ networks to obtain a set of conditions of the form

1. $F_Z(s) \geq 0$ for $s \in C_{12}$

[Note that we are here assuming that $C_{12}$ (i.e., the root locus $1 + k\frac{z_{RL}}{z_{RC}} = 0$ for real $k$) is not entirely confined to the real axis; this will be the case when the poles and zeros of $\frac{z_{RL}}{z_{RC}}$ alternate along the real axis. In other words, we exclude cases in which $\frac{z_{RL}}{z_{RC}}$ is an RL impedance. This makes sense, since in this case, we are effectively working with the building blocks $\frac{z_{RL}}{z_{RC}}$ (RL impedance), $\frac{z_{RC}}{z_{RC}} = 1$, and $\frac{1}{z_{RC}}$ (RL impedance), which we do not know how to handle if the building blocks are independent (i.e., neither of the RL impedances is realizable by means of the other two elements).]

2. $F_Z(s) > 0$ for $s \in R$

3a. $Z(s) > 0$ along certain intervals along the real axis

3b. $Z(s) < 0$ along certain other intervals along the real axis.

Here $F_Z(s)$ denotes the impedance derived difference function of some $Z_{RL}Z_{RC}RT$ network, that is

$$F_Z(s) = \text{Re} Z(s) - f(s) \text{Im} Z(s),$$ where

$$f(s) = \frac{1}{a+b} \left[ \frac{\text{Re} z_1}{\text{Im} z_1} + b \frac{\text{Re} z_2}{\text{Im} z_2} \right];$$ $a$ and $b$ are real constants.

$C_{12}$, $R$, and the intervals along the real axis in connection with conditions 3a and 3b, are determined, in each case, from the nature of the circuit elements as outlined in the previous section.

In trying to establish a realization procedure, however, some difficulties are encountered in the general $Z_{RL}Z_{RC}RT$ case when the degree of the numerator polynomial of the driving-point impedance

$$z_{sh}(s) = k_1z_1 + k_2z^2; k_1 \text{ and } k_2 \text{ are positive real constants},$$

exceeds two. The reason for this can be explained in connection with the Brune realizations shown in Fig. 61. Figure 61a shows that $z_{sh}(s)$ represents the shunt...
impedance of a typical Brune cycle. The purpose of $z_{sh}(s)$ is to remove a pole-pair $Y_2(s)$ on the contour $C_{12}$. The preceding steps of the cycle are guaranteed to set up a pair of conjugate poles of $Y_2(s)$ at $s = \pm s_v$, say, but not necessarily any additional poles at other points along $C_{12}$. Now, assume that the degree of the numerator polynomial of $z_{sh}(s)$ exceeds two. Then it follows that $y_{sh} = z_{sh}^{-1}$ has poles along $C_{12}$ in addition to just a single pair of conjugate poles at $s = \pm s_v$. Removing the shunt admittance $y_{sh}$ from $Y_2(s)$ will therefore, in general, leave a remainder function that is not in the function class. This is due to one or more admittance poles along $C_{12}$ with residues pointing in the wrong direction. It might happen, of course, that these last poles are located on $C_{12}$ in a region where there is no restriction on the residues, in which case the Brune cycle can be completed. This cannot be guaranteed, however.

Notice that all $Z_{RL}Z_{RC}$RT networks for which $z_{RL}$ and $z_{RC}$ contain only one energy storage element each, satisfy condition (282). These can, therefore, be treated, in general.

8.3 RLC\textsubscript{LT} NETWORKS

For RLC\textsubscript{LT} networks the conditions that result from the first two steps will have two new features. The more important one of these is two distinct difference functions. As a result of this property, we shall not be able to establish a general realization procedure in the final step. In consideration 5 we shall therefore discuss the reason why the ordinary realization procedures do not adapt themselves (as in the RL'C'T case) to classes of networks described by more than one difference function.

Consideration 1: Root-Locus Plots

We have seen the significance of obtaining two kinds of contours in the previous cases:

(i) The contours for poles and zeros of driving-point impedances composed of positive multiples of $z_i$ and $z_j$; these contours ($C_{ij}$) are described by

$$1 + k \frac{z_i}{z_j} = 0 \quad \text{for real positive } k. \quad (283)$$

(ii) The closed contours given by the points of the $s$-plane, where $z_i$ and $z_j$ are either "in phase" or $180^\circ$ "out of phase"; these contours are described by

$$h_{ij} = 0, \quad (284)$$

where

$$h_{ij} = \text{Re } z_i \text{ Im } z_j - \text{Re } z_j \text{ Im } z_i.$$

For the RLC\textsubscript{LT} case the various contours are shown in Fig. 62.
Consideration 2: Difference Functions

In this case both $C_{12}$ and $C_{13}$ include points off the real axis, so we obtain two distinct impedance derived difference functions. Let us define these as $H_{Z_{12}}$ and $H_{Z_{13}}$ pertaining to contours $C_{12}$ and $C_{13}$, respectively. We can then write

$$H_{Z_{12}}(s) = \frac{\text{Re} Z(s) - \frac{\text{Re} z_2}{\text{Im} z_2} \text{Im} Z(s)}{\text{Re} z_3(s) - \frac{\text{Re} z_2}{\text{Im} z_2} \text{Im} z_3(s)} = \text{Re} Z(s) - \frac{\sigma}{\omega} \text{Im} Z(s) \quad (285)$$

and
Now, it is readily shown, by the procedure outlined in connection with the RL'C''T case, that

\[ H^1_{Z}(s) = \frac{\text{Im} z_3}{\text{Re} z_3} \frac{\text{Re} Z(s) - \text{Im} Z(s)}{\text{Im} z_1(s) - \text{Im} z_1(s)} = \frac{\text{Im} Z(s)}{\text{Im} z_1(s)}. \]  

(286)

where the contours \( C_{ij} \) and the regions \( R_{ij} \) are shown in Fig. 63.

This procedure amounts to inserting the sign of \( h_{ij} \) into the various regions of Fig. 62d-f and considering the expressions \( H^1_Z = k_1 + k_2 + k_3 \) and \( H^2_Z = k_1 + k_2 + k_3 \) for real positive \( k \)'s.

One might ask whether or not these conditions are redundant. In particular, is it necessary to specify

\[ H^1_{Z}(s) \geq 0 \quad \text{for } s \in C_{12}' \cap R_{13}', \]

and

\[ H^2_{Z}(s) \geq 0 \quad \text{for } s \in C_{13}' \cap R_{12}? \]

Examples have been found that show that these conditions are not redundant.

One might also ask whether the following conditions are redundant:

\[ H^1_{Z}(s) > 0 \quad \text{for } s \in R_{12} \cap R_{13}, \]

and

\[ H^2_{Z}(s) > 0 \quad \text{for } s \in R_{12} \cap R_{13}. \]

Although no example has been found in this case, it appears that these conditions are not redundant.

In addition to these conditions, another independent condition on \( H^1_{Z} \) has been found. This condition comes about by the following special nature of the circuit elements -- a common pole of \( z_1 \) and \( z_2 \) at infinity. The additional condition follows directly from Theorem 1 of Huang and Lee, and can be stated as follows.

Because of the common \( j\omega \)-axis pole of \( z_1 \) and \( z_2 \) at infinity, all reactive elements are effectively open-circuited at this point and

\[ \text{Max Re } Z(j\omega) = \text{Max } H^1_{Z}(j\omega) = H^1_{Z}(j\infty) \geq H^1_{Z}(j\omega). \]  

(287)
Fig. 63. Contour $C'_{ij}$ defining the shaded region $R'_{ij}$ within which the impedance derived difference function $H_{ij}^Z$ must be positive for (a) $i = 1$ and $j = 2$; (b) $i = 1$ and $j = 3$.

The following necessary conditions for a rational function to be the driving-point impedance of an RLC T network have thus been established:

1a. $H_Z^{12}(s) \geq 0$ for $s \in C'_{12}$
1b. $H_Z^{13}(s) \geq 0$ for $s \in C'_{13}$
2a. $H_Z^{12}(s) > 0$ for $s \in R_{12}$
2b. $H_Z^{13}(s) > 0$ for $s \in R_{13}$
3. $H_Z^{12}(j\omega) \geq H_Z^{12}(j\omega)$.

A rational function that satisfies these conditions will be called a PRH$_Z$ function. There are two new features to be observed in this case:

a. Two distinct difference functions.
b. Overlapping poles in the circuit elements leading to condition 3.
Considerations 3 and 4.

Although these steps could be carried out, in this case, very little is to be gained, so we proceed to the final step.

Consideration 5. Realization procedure

We have not been able to establish a general realization procedure for PRH functions. The reason follows.

All known realization procedures depend, in one form or another, on the degree-reducing operation of two-element-kind pole removals along the boundaries C_{ij}. This reduction has the nice feature that it does not affect the difference function along the contour from which the poles have been removed; the remainder function, therefore, will satisfy the boundary condition along this contour. With two distinct difference functions, however, the problem arises at the other contour, since there is no guarantee that if a pole-pair is removed from a PRH function along C_{12}, say, that the difference function \( H_Y^{13}(s) \) of the remainder function remains non-negative.

It is therefore not known whether the PRH conditions are sufficient to guarantee that a rational function with the PRH properties represents the driving-point impedance of an RLCL network.

8.4 LCC'T NETWORKS

We shall now show that the positive real character of the difference functions does not always represent a sufficient condition for a function to be a driving-point function of the function class.

For this purpose it is convenient to employ the particular set of circuit elements shown in Fig. 54d, since it represents a combination of two classes of networks with which we are already familiar (namely RLCT and RLC'T excluding resistances). This enables us to state the results of the first 4 steps more or less directly. Only impedance derived conditions will be given.

Consideration 1: Root-Locus Plots

The root loci of importance, in this case, are shown in Fig. 64.

Consideration 2: Difference Functions

From Fig. 64 we see that we have another double-contour case (i.e., two contours off the real axis) which implies two distinct impedance derived difference functions. These we call \( K_{Z12} \) and \( K_{Z13} \), pertaining to contours \( C_{12} \) and \( C_{13} \) respectively.

There is a certain amount of freedom in defining \( K_{Z12} \) and \( K_{Z13} \) owing to "a" and "b" in...
\[ f(s) = \frac{1}{a+b} \left[ a \frac{\text{Re} z_1}{\text{Im} z_1} + b \frac{\text{Re} z_2}{\text{Im} z_2} \right] \] (Eq. 264).

Since the resulting ABCZ conditions, however, can be shown to be independent of "a" and "b", let us choose a convenient combination, namely

\[ a = b = 1. \]

[Notice that this is also the choice made in the RL'C'T case, and the choice that will lead to the well-known PR conditions of RLCT networks.]

Then

\[
K_Z^{12}(s) = \frac{\text{Re} Z(s) - \frac{1}{2} \left( \frac{\text{Re} z_1}{\text{Im} z_1} + \frac{\text{Re} z_2}{\text{Im} z_2} \right) \text{Im} Z(s)}{\text{Re} z_3(s) - \frac{1}{2} \left( \frac{\text{Re} z_1}{\text{Im} z_1} + \frac{\text{Re} z_2}{\text{Im} z_2} \right) \text{Im} z_3(s)} = \frac{(\sigma + \alpha)^2 + \omega^2}{\sigma + \alpha} \text{Re} Z(s). \tag{288}
\]

\[ K_Z^{13}(s) = \frac{\text{Re} Z(s) - \frac{1}{2} \left( \frac{\text{Re} z_1}{\text{Im} z_1} + \frac{\text{Re} z_3}{\text{Im} z_3} \right) \text{Im} Z(s)}{\text{Re} z_2(s) - \frac{1}{2} \left( \frac{\text{Re} z_1}{\text{Im} z_1} + \frac{\text{Re} z_3}{\text{Im} z_3} \right) \text{Im} z_2(s)} = \frac{\sigma^2 + \omega^2}{\sigma - \frac{\alpha}{2}} \left[ \text{Re} Z(s) + \frac{1}{2} \frac{\alpha}{\omega} \text{Im} Z(s) \right]. \tag{289}
\]

The following necessary conditions for a function to be the driving-point impedance of an LCC"T network are then readily established.

Fig. 64. Root-locus plot of \( 1 + k \frac{z_i}{z_j} = 0 \) for the LCC"T case, with (a) \( i = 1 \) and \( j = 2 \); (b) \( i = 1 \) and \( j = 3 \); (c) \( i = 2 \) and \( j = 3 \).
1a. $K_{12}(s) > 0$ for $s \in C_{12}$
1b. $K_{13}(s) > 0$ for $s \in C_{13}$
2a. $K_{12}(s) > 0$ for $\text{Re } s > 0$
2b. $K_{13}(s) > 0$ for $\text{Re } s < -\frac{\alpha}{2}$ and $\text{Im } s \neq 0$.
3. $Z(\sigma) < 0$ for $\sigma < -\alpha$.

A rational function that satisfies these conditions will be called a PRK$_Z$ function.

**Consideration 3: ABC$_Z$ conditions**

The ABC$_Z$ conditions for a PRK$_Z$ function follow directly from the conditions established in step 2 and are given below.

A. It must be analytic in region $R$ of Fig. 65.

B. $K_{12}(s) > 0$ for $s \in C_{12}$

C. $K_{13}(s) > 0$ for $s \in C_{13}$.

C. Poles of $Z(s)$ on $C_{12}$ must satisfy the residue condition of PR functions; poles on $C_{13}$ must satisfy the residue conditions of PRD$_Z$ functions.

![Fig. 65. Illustrating the region $R$ in which LCC$''$T driving-point impedances must be analytic.](image)

**Consideration 4: ABC$_Z$ Conditions $\leftrightarrow$ PRK$_Z$ Conditions**

By the same approach as in the RLCT and the RL/C$''$T cases, it is readily shown that the ABC$_Z$ conditions imply the PRK$_Z$ conditions; accordingly, we leave out the details.
Consideration 5: Realization Procedure

Again we have a double-contour case, so, for the same reasons as in the RLC\textsubscript{L}T case, we have not been able to establish a realization procedure for PRK\textsubscript{Z} functions. This is not surprising in this case, however, since the ABC\textsubscript{Z} conditions are not sufficient for a function to be the driving-point impedance of an LCC''T network. To show this, we have constructed a rational function that satisfies the ABC\textsubscript{Z} conditions and has a partial pole-zero pattern as shown in Fig. 66. This will establish that the ABC\textsubscript{Z} conditions do not completely describe driving-point impedances of LCC''T networks, since a driving-point impedance with a partial pole-zero pattern as shown in Fig. 66 can be shown not to belong to any LCC''T network for the following reason.

![Fig. 66. Pole-zero configuration that cannot belong to an LCC''T driving-point impedance.](image)

The conjugate zero pair located at ±s\textsubscript{v} (on C\textsubscript{13}) implies that the K\textsubscript{Z}\textsuperscript{13}-function goes to zero at these points. It follows that there can be no current through the pure capacitance elements, z\textsubscript{2}, at these frequencies. We can therefore remove all of the z\textsubscript{2}'s without changing the terminal behavior of the network at these excitation frequencies. The remaining structure must have a path between the terminals composed of elements z\textsubscript{1} and z\textsubscript{3} only (and possibly ideal transformers).

Consider the same network excited with a DC current (excitation frequency s = 0). Again all of the pure capacitance elements can be removed without affecting the terminal behavior of the network. The remaining structure is the same as in the previous case, so it follows that the DC current can flow through the same path as the current with the excitation frequency s\textsubscript{v}. There can, therefore, be no pole at the origin. So, if we can construct a rational function with the ABC\textsubscript{Z} properties and a partial pole-zero pattern as shown in Fig. 66, we have found an additional condition from topological considerations alone. Such a function exists and can be constructed as follows.

Let Z(s) be the driving-point impedance of an LCC''T network with a pair of conjugate zeros on C\textsubscript{13} and regular at s = -c. The simplest such function (topologically) is shown in Fig. 67a and its pole-zero pattern in Fig. 67b. [The exact
location of the poles of $Z(s)$ is of little concern; we know that $Z(s)$ must satisfy the ABC conditions.]  

$Z(s)$ is clearly a PRK function and must therefore satisfy the ABC conditions.

Let us multiply $Z(s)$ by a function that has a pole at the origin and does not affect the argument of $Z(s)$ along the line $s = -\frac{a}{2}$. One such function is

$$Z_1(s) = \frac{s + \frac{1}{2}}{s(s+1)} Z(s),$$

if we define

$$Z_1(s) = \frac{(s + \frac{1}{2})^2}{s(s+1)} Z(s),$$

then it is clear that $Z_1(s)$ satisfies the ABC conditions, except for the condition $K_{Z_1}^{12}(j\omega) > 0$. But we can easily make $K_{Z_1}^{12}(j\omega) > 0$ without affecting the $K_{Z_1}^{13}$-function along $C_{13}$ and without destroying the partial pole-zero pattern shown in Fig. 66. This is accomplished by adding to $Z_1(s)$ a series combination of elements $z_1$ and $z_2$ with a pair of zeros located at $s = \pm s_v$ and a large enough multiplier to make $K_{Z_1}^{12}(j\omega) > 0$.

The result is shown in Fig. 68.

It is just a matter of simple algebra to show that if we choose $k = \frac{1}{8}$, then the $K_s^{12}$-function of $Z_2(s)$ will be non-negative (actually positive) along the $j\omega$-axis, and

$$Z_2(s) = \frac{1}{8} \frac{s^2 + s + 2}{s(s+1)} \frac{s^4 + s^3 + 11s^2 + 9s + 2}{s^3 + s^2 + 3s + 1}$$

with a pole-zero pattern as shown in Fig. 69.
Fig. 68. Relevant to the construction of a PRK\(_Z\) function, \(Z_2\), from a non-PRK\(_Z\) function, \(Z_1\).

\[
k \frac{s^2 + s + 2}{s + 1}
\]

Fig. 69. Pole-zero plot of a PRK\(_Z\) function that possesses the partial pole-zero pattern of Fig. 66. The PRK\(_Z\) property is therefore not sufficient to define driving-point impedances of LCC"T networks.

\[Z_2(s)\] satisfies the ABC\(_Z\) conditions and is therefore a PRK\(_Z\) function. \(Z_2\) does not, however, represent the driving-point impedance of any LCC"T network, since it has a partial pole-zero pattern as shown in Fig. 66. The ABC\(_Z\) conditions are therefore not sufficient to guarantee that a function is the driving-point impedance of an LCC"T network. A sufficient set of conditions has not been found, in this case, although we can tighten up the ABC\(_Z\) conditions somewhat by ruling out partial pole-zero patterns as shown in Fig. 66.

Notice that none of the poles and zeros along the contours \(C_{12}\) and \(C_{13}\) can be removed from \(Z_2(s)\) and leave a simpler PRK\(_Z\) remainder function. Any of the ordinary
realization procedures (extended as shown in connection with RL'C"T networks) would therefore fail to simplify the impedance function $Z_2(s)$.

The reader might possibly expect a section on general three-element-kind $(Z_1Z_2Z_3T)$ networks at this point. Since a general theory for $Z_1Z_2RT$ networks can be used to handle the $Z_1Z_2Z_3T$ case, by a simple impedance multiplication, we have nothing essential to add beyond what has already been said, namely, that the procedure outlined in section 8.1 will lead to a strong set of necessary conditions on driving-point functions for almost all $Z_1Z_2Z_3T$ networks, and that only for certain single-contour cases can these conditions also be shown to be sufficient.
IX. CONCLUDING REMARKS

9.1 CONCLUSIONS

The primary objective of this report has been to extend our knowledge of RLCT networks to networks composed of three more complicated linear circuit elements. This objective has been achieved.

We cannot claim total success, however, since the function class completely describing driving-point functions of any linear three-element-kind network has not been found. Nevertheless, we do claim that the approach established by the introduction of the so-called difference function brings us closer to this ultimate goal. The thesis research has also established a procedure for obtaining positive real (PR) requirements on the difference functions associated with all linear three-element-kind networks.

Unfortunately, the PR requirements on the difference functions are not always sufficient to describe driving-point functions in the network class; this has been established by an example. Realization procedures have, therefore, only been established for certain cases with well-defined restrictions on the three building blocks. A significant restriction is the lack of realization procedures, in all cases, for building blocks describing double- or triple-contour situations.

9.2 SUGGESTIONS FOR FUTURE RESEARCH

The restrictions above suggest several questions worthy of further attention. These follow.

1. Can the difference-function point of view be modified to yield additional conditions on the driving-point functions of three-element-kind networks?

2. Are there modifications of existing realization procedures, or perhaps more powerful realization procedures, which can be used to realize the driving-point functions of

   (i) double-contour cases?
   (ii) triple-contour cases?
   (iii) all $Z_1Z_2Z_3^T$ networks?

3. Are any additional conditions necessary to specify driving-point functions of single-contour cases when Eq. 282 is not satisfied?

The first two problems suggested above are clearly related, and the answer to either one is likely to be a major step in closing the discussion connected with the general problem of linear three-element-kind synthesis. The third problem is of a somewhat lesser magnitude, but is still important to the general theory.
APPENDIX A

A Factorization Theorem

Let $P(\lambda)$ be a Hurwitz polynomial; then if and only if it has an even number of zeros along the real axis in the interval $(-\alpha, 0)$ can it be factorized as follows

$$P(\lambda) = \lambda^{2n+1} + a_{2n} \lambda^{2n} + \cdots + a_0$$

(A.1)

where all the coefficients, $A_i$, are non-negative and no intermediate coefficient missing.

Proof: Our strategy is to show that the theorem is true for $n = 1$ and $n = 2$, and then by induction for any $n$ ($n$ refers to the order of the polynomial).

For a first-degree polynomial ($n = 1$) the statement is clearly true since the polynomial must be of the form

$$P_1(\lambda) = \lambda + a_0,$$  \hspace{1cm} (A.2)

where $a_0 \geq \alpha$.

For a second-degree polynomial let us write

$$P_2(\lambda) = (\lambda + \lambda_1)(\lambda + \lambda_2)$$

(A.3)

$$(\lambda^2 - \alpha^2) + (\lambda_1 + \lambda_2)(\lambda + \alpha) + (\lambda_1 - \alpha)(\lambda_2 - \alpha),$$

such that

$$A_1 = \lambda_1 + \lambda_2$$

and

$$A_0 = (\lambda_1 - \alpha)(\lambda_2 - \alpha).$$

In the case of a pair of complex conjugate roots occurring in the left half-plane, both coefficients are seen to be positive, i.e.,

$$A_1 = 2\text{Re} \lambda_1 = 2\text{Re} \lambda_2 > 0$$

(A.4)

$$A_0 = |\lambda_1 - \alpha|^2 = |\lambda_2 - \alpha|^2 > 0.$$  

If both roots are real they must both occur on the same side of $\lambda = -\alpha$. The coefficients in this case are easily seen to be non-negative.

Assume the statement to be true for any $2n$. We will then show that it must be true for $(2n+1)$ and $(2n+2)$. What we propose to show is the following. If a portion of a given zero pattern has been represented in factored form with non-negative coefficients only, then adding a simple zero on the real axis in the interval $(-\infty, -\alpha]$, or
a pair of real zeros located on the same side of \( \lambda = -\alpha \), or a pair of complex conjugate zeros, leaves the augmented polynomial with only non-negative coefficients. In other words, if \( P_{2n} \) is of the right form, i.e.,

\[
P_{2n}(\lambda) = A_{2n}(\lambda+\alpha)^{2n}(\lambda-\alpha)^{2n} + \cdots + A_1(\lambda+\alpha) + A_0
\]

where \( A_1 \geq 0 \), then

\[
P_{2n+1} = P_{2n} \cdot P_1 \quad \text{and} \quad P_{2n+2} = P_{2n} \cdot P_2
\]

have only non-negative coefficients.

It is clear that the expression for \( P_{2n+1} \) is of the right form except for terms of the form

\[
A_{2i}^i(\lambda+\alpha)^{i+1}(\lambda-\alpha)^{i-1} \quad \text{for} \quad i = 1, \ldots, n.
\]

These terms, however, can be expressed as follows

\[
A_{2i}^i[(\lambda+\alpha)^{i-1}(\lambda-\alpha)^{i-1}] (\lambda+\alpha)^2 = A_{2i}^i[(\lambda+\alpha)^{i-1}(\lambda-\alpha)^{i-1}] \cdot [(\lambda^2-\alpha^2) + 2\alpha(\lambda+\alpha)]
\]

\[
= A_{2i}^i(\lambda+\alpha)^i(\lambda-\alpha)^i + 2\alpha A_{2i}^i(\lambda+\alpha)^i(\lambda-\alpha)^{i-1}
\]

which are seen to have only non-negative coefficients. \( P_{2n+1}(\lambda) \) is therefore of proper form.

Next, we consider

\[
P_{2n+2}(\lambda) = P_{2n}(\lambda) [A_2^2(\lambda^2-\alpha^2) + A_1^i(\lambda+\alpha) + A_0^i]
\]

\[
= A_2^i(\lambda+\alpha)(\lambda-\alpha)P_{2n}(\lambda) + A_1^i(\lambda+\alpha)P_{2n}(\lambda) + A_0^iP_{2n}(\lambda).
\]

Each term on the right-hand side is readily seen to be of the proper form.

Finally, it should be clear that if \( P(\lambda) \) has an odd number of zeros on the real axis in the interval \((-\infty, -\alpha)\) then the factored polynomial must have a negative \( A_0 \).

Q.E.D.
APPENDIX B

Supplement to Section 7.2

First we establish that the lower bound of the inequality

\[ 0 > \left( \frac{\lambda - \alpha}{\lambda + \alpha} \right) \rho(\lambda) > -2\alpha \]  

for \( \lambda = \alpha \) (B.1)

is automatically satisfied due to the nature of \( \rho(\lambda) \). To this end, we introduce the function

\[ \rho'(\lambda) = -\frac{\lambda - \alpha}{\lambda + \alpha} \rho(\lambda) \]  

(B.2)

which is seen to have the following properties:

i. analytic in the right half-plane and on the j\( \xi \)-axis

ii. \( \left| \rho'(j\xi) \right| = \left| \frac{j\xi - \alpha}{j\xi + \alpha} \right| \left| \rho(j\xi) \right| \leq 1 \). (From property 3 of \( \rho \))

iii. \( \rho'(\alpha) > 0 \). (From Eqs. 249 and B.2)

From the maximum modulus theorem and the first two of these properties, it follows that

\[ \left| \rho'(\lambda) \right| < 1 \text{ for } \text{Re} \lambda > 0 \]  

(B.3)

Therefore, if we write

\[ (\lambda - \alpha)\rho(\lambda) = -(\lambda + \alpha)\rho'(\lambda) \]  

(B.4)

and let \( \lambda = \alpha \), we find that

\[ \left( \frac{\lambda - \alpha}{\lambda + \alpha} \right) \rho(\lambda) = -2\alpha \rho'(\alpha) > -2\alpha \]  

for \( \lambda = \alpha \) (B.5)

which shows that the lower bound of Eq. B.1 is automatically satisfied.

\( \rho'(\lambda) \) is also useful in establishing that a \( \rho(\lambda) \) with the following properties is sufficient to guarantee the PRD \( Z \) character of \( \tilde{Z}(\lambda) \):

i. a pole at \( \lambda = \alpha \) with a negative residue

ii. no other right half-plane poles

iii. \( \rho(0) = 1 \).

To show this, it is convenient to normalize \( \tilde{Z}(\lambda) \) with respect to the source resistance \( r \) in Fig. 49b.

Let therefore

\[ \tilde{Z}_N(\lambda) = \frac{\tilde{Z}(\lambda)}{r} = \frac{\lambda + \alpha}{\lambda - \alpha} + \rho(\lambda) = \frac{1 + \frac{\lambda - \alpha}{\lambda + \alpha} \rho(\lambda)}{1 - \rho(\lambda)} \]  

(B.6)

such that

\[ \tilde{Y}_N(\lambda) = \tilde{Z}_N^{-1}(\lambda) = \frac{\frac{\lambda - \alpha}{\lambda + \alpha} + \rho'(\lambda)}{1 - \rho'(\lambda)} = \frac{1 + \frac{\lambda - \alpha}{\lambda + \alpha} \rho(\lambda)}{1 - \rho'(\lambda)} = \frac{1 + \rho'(\lambda) - \alpha}{1 + \alpha} \]  

(B.7)
To establish the PRD₂ character of \( \tilde{Z}(\lambda) \) we show that the properties of \( \rho'(\lambda) \) imply that \( \tilde{Y}_N(\lambda) \) must be a PRD₁ function.

To this end, it is convenient to define

\[
\begin{align*}
    s(\lambda) &= \frac{1 + \rho'(\lambda)}{1 - \rho'(\lambda)}, \quad (B.8) \\
    \tilde{Y}_N(\lambda) &= \frac{s(\lambda) - \alpha}{1 + \frac{\alpha}{\lambda}} = \frac{\lambda s(\lambda) - \alpha}{\lambda + \alpha} \quad (B.9)
\end{align*}
\]

The important property of \( s(\lambda) \) is its PR character, which follows from the properties of \( \rho'(\lambda) \) and the fractional bilinear transformation relating \( \rho' \) and \( s \). This property guarantees that \( \tilde{Y}_N(\lambda) \) is analytic in the right half of the \( \lambda \)-plane. Therefore, what remains to be established in order for \( \tilde{Y}_N(\lambda) \) to be a PRD₁ function is that the admittance derived difference function of \( \tilde{Y}_N(\lambda) \) is non-negative along the \( j \xi \)-axis, \( \tilde{Y}_N(\alpha) > 0 \), and that \( j \xi \)-axis poles satisfy the residue condition of section 3.3.

The first two conditions are easily verified. We see that

\[
D_Y(j \xi) = \text{Re } Y_N(j \xi) + \frac{\alpha}{\xi} \text{Im } Y_N(j \xi) = \text{Re } s(j \xi) > 0
\]

since \( s(\lambda) \) is P.R. (This result could also have been seen directly from Eq. 242.)

Furthermore, we see that

\[
\tilde{Y}_N(\alpha) = \frac{s(\alpha)}{2} - \frac{1}{2} = \frac{1 + \rho'(\alpha)}{2} - \frac{1}{2} > 0,
\]

since \( 0 < \rho'(\alpha) < 1 \).

Concerning the residue condition, it is observed that \( \tilde{Y}_N(\lambda) \) can certainly have \( j \xi \)-axis poles due to \( j \xi \)-axis poles of \( s(\lambda) \) at frequencies where \( \rho'(j \xi) = 1 \). \( j \xi \)-axis poles of \( s(\lambda) \) must, of course, occur in complex conjugate pairs and have positive real residues.

Assume, therefore, that \( \tilde{Y}_N(\lambda) \) has a pole at \( \lambda = j \xi_o \). The residue of \( \tilde{Y}_N(\lambda) \) at \( \lambda = j \xi_o \) is given by

\[
\text{Res } \tilde{Y}_N(j \xi_o) = \lambda \frac{s(\lambda) - \alpha}{\lambda + \frac{\alpha}{\lambda}} \bigg|_{\lambda = j \xi_o} = \frac{j \xi_o}{j \xi_o + \alpha} = \frac{k}{1 - j \alpha}, \quad (B.11)
\]

where \( k \) is the residue of the corresponding pole of \( s(\lambda) \); \( k \) is therefore positive and the residue conditions for PRD₁ functions are seen to be satisfied.

We have thus shown that if \( \rho(\lambda) \) is extracted from \( \rho(\lambda)\rho(-\lambda) \) according to the conditions mentioned above, then a PRD₂ function, \( \tilde{Z}(\lambda) \), is assured.
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9. The author is indebted to Professor Harry B. Lee for making these observations, thereby initiating the thesis research.


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Realizability Conditions for New Classes of Three-Element-Kind Networks

The assumption of ideal elements places a practical limitation on the theory of Modern Network Synthesis. To extend the theory to cope with more realistic elements, several new classes of three-element-kind networks have been studied. Realizability conditions have been derived for driving-point impedances of these networks. This is achieved through physical arguments leading to a property of driving-point functions which we call the difference function. It is shown that the positiveness of the difference function over an appropriate portion of the s-plane represents a necessary and sufficient condition for a rational function to be realizable as the driving-point impedance of a given network class.

The most notable contribution is perhaps the establishment of a general theory for networks composed of resistances and reactances with semiuniform loss (i.e., all inductors have one Q; all capacitors have another Q). This theory is almost as comprehensive as that now available for RLCT networks.
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