Coadjoint Orbits and Induced Representations

by

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ABSTRACT

Coadjoint orbits of Lie groups play important roles in several areas of mathematics and physics. In particular, in representation theory, Kirillov showed in the 1960's that for nilpotent Lie groups there is a one-one correspondence between coadjoint orbits and irreducible unitary representations. Subsequently this result was extensively generalized by Kostant, Auslander-Kostant, Duflo, Vogan and others. In this thesis we give a formulation in the setting of coadjoint orbits, of the Mackey theory of induced representations.

Let G be a real Lie group, $N \subset G$ be a normal subgroup. Denote by $\mathfrak{g}$ and $\mathfrak{n}$ the Lie algebras of G and N respectively. Let $p \in \mathfrak{n}^*$, the dual space of $\mathfrak{n}$, and Y be the coadjoint orbit of N through p. There is a natural G action on $\mathfrak{n}^*$. Let

$$K = G_Y = \{g \in G;\ g \cdot q \in Y \text{ for all } q \in Y\}$$

be the stabilizer subgroup of Y, and $\mathfrak{k} = \mathrm{Lie}K$. Then restricting G action on $\mathfrak{n}^*$ to K gives us a K action on Y. Under the assumption that the stabilizer subgroup N_p is connected, we can always, by getting rid of so-called Mackey's obstruction, find a suitable K-homogeneous space W on which N acts trivially so that $Y \times W$ is a coadjoint orbit of K. We call $Y \times W$ the *little group data.* On the other hand, K acts on G on the right; this induces a Hamiltonian K action on T^*G which commutes with the left action of G. Therefore, K acts on $T^*G \times Y \times W$ in a Hamiltonian fashion. The reduced space of $T^*G \times Y \times W$ with respect to K at 0 is a well defined symplectic manifold, and we prove that this reduced space is a coadjoint orbit of G. We also show that any coadjoint orbit of G can be reconstructed this way. Quantization of this classical construction gives us the Mackey theory of induction. A special case of this theory, the semi-direct product case, was worked out by Sternberg and Rawnsley in the 1970's.

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Contents

Chapter 1

Preliminary Topics

1.1 Hamiltonian action and Marsden-Weinstein Reduction

We review some basic concepts of symplectic geometry which will be used throughout the whole paper. Let (M, ω) be a symplectic manifold, $C^\infty(M)$ be the set of the real valued smooth functions on M. For any function $f \in C^\infty(M)$, we denote by ξ_f the *Hamiltonian vector field of* f. Namely,

$$\iota(\xi_f)\omega = df, \tag{1.1}$$

the inner product of ξ_f and ω is df. An automorphism $\phi : M \longrightarrow M$ is said to be a *symplectomorphism* if it preserves the symplectic form,

$$\phi^*\omega = \omega. \tag{1.2}$$

Let G be a Lie group whose Lie algebra is $\mathfrak{g}$. Suppose G acts on M. We say that the G action on M is *symplectic* if each of the elements of G serves as a symplectomorphism of M. For any $\xi \in \mathfrak{g}$, we denote by $\xi^\sharp$ the infinitesimal generator of ξ for the $\mathfrak{g}$ action on M. It is defined as

$$\xi^\sharp(p) = \frac{d}{dt}|_{t=0} exp(-\xi \cdot p), \quad \text{for any } p \in M. \tag{1.3}$$

The infinitesimal version of the symplectic action is expressed as the Lie derivative of ω with respect to $\xi^\sharp$ for any $\xi \in \mathfrak{g}$ being zero:

$$\mathcal{L}_{\xi^\sharp}\omega = 0. \tag{1.4}$$

By the Weil formula

$$\mathcal{L}_{\xi^\sharp}\omega = d(\iota(\xi^\sharp)\omega) + \iota(\xi^\sharp)d\omega, \tag{1.5}$$

we see that, since ω is closed, (1.4) is equivalent to

$$d(\iota(\xi^\sharp)\omega) = 0. \tag{1.6}$$

Therefore the G action is symplectic means that all $\iota(\xi^\sharp)\omega$ are closed. Moreover, if they are exact, we have some function f^ξ for each ξ such that

$$\iota(\xi^\sharp)\omega = df^\xi. \tag{1.7}$$

It is clear that such a f^ξ is determined up to a constant. Suppose this gives rise to a map $\lambda :\ \mathfrak{g} \longrightarrow C(M)$ by $\lambda(\xi) = f^\xi$. We call f^ξ the *Hamiltonian function* for $\xi^\sharp$, or simply, ξ. A short calculation shows that the Poisson bracket $\{f^\xi, f^\eta\}$ for $\xi,\ \eta \in \mathfrak{g}$ is a Hamiltonian function $f^{[\xi,\eta]}$ for $[\xi,\eta]$. Namely

$$\iota([\xi,\eta]^\sharp)\omega = d\{f^\xi, f^\eta\}. \tag{1.8}$$

In particular, it follows that there is a skew-symmetric bilinear map $c_\lambda :\ \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathbf{R}$ so that

$$\{f^\xi, f^\eta\} = f^{[\xi,\eta]} + c_\lambda(\xi,\eta). \tag{1.9}$$

The Jacobi identity implies that the map c_λ satisfies the condition

$$c_\lambda([\xi,\eta],\zeta) + c_\lambda([\eta,\zeta],\xi) + c_\lambda([\zeta,\xi],\eta) = 0, \qquad \text{for all } \xi,\eta,\zeta \in \mathfrak{g}. \tag{1.10}$$

This condition is known as the *2-cocycle condition* for c_λ regarded as an element of $\Lambda^2(\mathfrak{g}^*)$. It is of our interest to see if we can choose λ so that c_λ is identically zero, in other words, λ is a Lie algebra homomorphism. In order to see whether it is possible, let us choose another linear map $\tilde{\lambda}: \mathfrak{g} \longrightarrow C(M)$ which also lifts the infinitesimal action of $\mathfrak{g}$ on M. Let the linear map $\rho: \mathfrak{g} \longrightarrow C(M)$ be $\rho(\xi) = \tilde{\lambda}(\xi) - \lambda(\xi)$. We now compute that

$$
\begin{aligned}
(1.11) \qquad \{\tilde{\lambda}(\xi), \tilde{\lambda}(\eta)\} &= \{\lambda(\xi), \lambda(\eta)\} = \lambda([\xi, \eta]) + c_\lambda(\xi, \eta) \\
&= \tilde{\lambda}([\xi, \eta]) + c_\lambda(\xi, \eta) - \rho([\xi, \eta]).
\end{aligned}
$$

Thus, $c_{\tilde{\lambda}}(\xi, \eta) = c_\lambda(\xi, \eta) - \rho([\xi, \eta])$. Therefore in order to be able to choose $\tilde{\lambda}$ so that $c_{\tilde{\lambda}} = 0$, there must exist a $\rho \in \mathfrak{g}^*$ such that $c_\lambda = -\delta\rho$ where $\delta\rho$ is the skew-symmetric bilinear map on $\mathfrak{g}$ satisfying $\delta\rho(\xi, \eta) = -\rho([\xi, \eta])$. This condition is known as the *2-coboundary condition.*

Definition 1.1.1. The symplectic g action on M is said to be *Hamiltonian* if there is a lifting of infinitesimal action which is a Lie algebra homomorphism. In this case, X is called a *Hamiltonian* G-space.

In the case that G action on M is Hamiltonian, we may define a *moment map* Φ : $M \longrightarrow \mathfrak{g}^*$ by

$$< \Phi(p), \xi >= f^\xi(p), \qquad \text{for any } p \in M \text{ and } \xi \in \mathfrak{g}.$$

There is a natural coadjoint action of G on $\mathfrak{g}^*$. With respect to this action and the G action on M, we have

Proposition 1.1.1. Φ is G-equivariant.

It is helpful at this moment to introduce the following terminology.

Definition 1.1.2. Let $\phi: X \longrightarrow Y$ be a smooth map between two differential manifolds. We say that $y \in Y$ is a *clean* value of ϕ if the set $\phi^{-1}(y) \subset X$ is a submanifold in X and, for each $x \in \phi^{-1}(y)$, we have $T_x\phi^{-1}(y) = \ker d\phi_x$.

Note that any regular value of ϕ is a clean value.

Now suppose $\beta \in \mathfrak{g}^*$ is a clean value of the moment map Φ. Proposition 1. 1. 1 implies that the stabilizer subgroup G_β acts on the submanifold $\Phi^{-1}(\beta)$. Suppose in addition that the space of G_β orbits on $\Phi^{-1}(\beta)$, $M_\beta = \Phi^{-1}(\beta)/G_\beta$, can be given a structure of a smooth manifold in such a way that the quotient map $\pi_\beta : \Phi^{-1}(\beta) \longrightarrow M_\beta$ is a smooth submersion. Then Marsden-Weinstein proved

Theorem 1.1.1. There is a symplectic form Ω on M_β so that

$$i^*\omega = \pi_\beta^*\Omega, \tag{1.12}$$

where $i : \Phi^{-1}(\beta) \longrightarrow M$ is the inclusion map.

This reduction is called the Marsden-Weinstein reduction, and the space M_β is called the Marsden-Weinstein reduced space. This will be the main tool we use to study the structure of coadjoint orbits.

Remark 1.1.1. Let G_p be the stabilizer subgroup for some $p \in \Phi^{-1}(\beta)$. Then the dimension of the Marsden-Weinstein reduced space is:

$$\dim M_\beta = \dim M - \dim G - \dim G_\beta + 2\dim G_p. \tag{1.13}$$

1.2 Coadjoint Orbits

We now review some fundamental properties of our main object, the coadjoint orbits of Lie groups. Let G be a Lie group and X be a coadjoint orbit of G through a point $p \in \mathfrak{g}^*$. Let G_p be the stabilizer subgroup of p. Then $X = G/G_p$. There is a canonical symplectic form on X described as follows.

Since X is a G-homogeneous space, any tangent vector at point $q \in X$ can be written as $\xi^\sharp(q)$ for some $\xi \in \mathfrak{g}$. We define Ω_X a two-form on X by

$$\Omega_X(q)(\xi^\sharp(q), \eta^\sharp(q)) = - < q, [\xi, \eta] > . \tag{2.14}$$

It is easy to see that (2.14) is well defined by checking that it only depends on $\xi^\sharp$ and $\eta^\sharp$ but not on the choice of ξ and η. Furthermore, Ω_X is also closed and non-degenerate.

Hence it defines a symplectic form on X. This is known as the Kirillov-Kostant symplectic form.

G acts on X naturally by the coadjoint action. The following result can be proved by the direct computation.

Proposition 1.2.1. G action on X is Hamiltonian with the moment map $\iota : X \longrightarrow \mathfrak{g}^*$ being the inclusion map.

Thus we see that any coadjoint orbit is a Hamiltonian G-homogeneous space. What is more significant is the converse assertion given by Kostant as well as Souriau.

Proposition 1.2.2. Suppose G action on X is Hamiltonian and transitive, then X is a covering space of some coadjoint orbit of G.

We give a proof by using the moment map. For this, we need the following lemmas for the general symplectic manifolds M. They are also useful later.

Lemma 1.2.1. The transpose map of $d\Phi_m : T_mM \longrightarrow \mathfrak{g}^*$ is the composition of the following two maps:

$$\mathfrak{g} \longrightarrow T_mM, \qquad \xi \longmapsto \xi^\sharp(m)$$

and

$$T_mM \longrightarrow T_m^*M, \qquad v \longmapsto \iota(v)\omega_m.$$

Lemma 1.2.2. The image of $d\Phi_m$ in $\mathfrak{g}^*$ is $\mathfrak{g}_m^0$, the annihilator of $\mathfrak{g}_m$ in $\mathfrak{g}^*$, where $\mathfrak{g}_m$ is the Lie algebra of the stabilizer subgroup G_m.

Lemma 1.2.3. The kernel of $d\Phi_m$ is the symplectic orthocomplement of the tangent space to the G orbit through m.

Lemma 1.2.4. The orbit of G through m is open if and only if $d\Phi_m$ is injective, i.e., Φ is an immersion at m.

Lemma 1.2.5. If G acts transitively on M, Φ is an immersion; and if M is connected, the converse is true.

Proof of Proposition 1.2.2: Since the G action on X is Hamiltonian, we have a moment map $\Phi : X \longrightarrow \mathfrak{g}^*$. Pick a point $m \in M$, let $p = \Phi(m) \in \mathfrak{g}^*$ and X be the coadjoint orbit of G through p. Since Φ is G equivariant, and G action on X is also transitive, we see that ImΦ is X. Locally, by **Lemma 1.2.5**, $d\Phi_m$ is injective. Hence Φ is locally diffeomophic. This implies that as manifolds $\Phi : M \longrightarrow X$ is a covering. We claim that this covering pulls back the symplectic form ω_X on X to the symplectic form $\omega = \omega_M$ on M,

$$\Phi^*\omega_X = \omega.$$

Indeed, for any $\xi \in \mathfrak{g}$, we denote by $\xi_1^\sharp$ the vector field on M corresponding to ξ, and $\xi_2^\sharp$ the vector field on X corresponding to ξ. For any $v, w \in T_mM$, $p = \Phi(m)$, we need to check

$$(\omega_X)_p(d\Phi_m(v), d\Phi_m(w)) = \omega_m(v, w). \tag{2.15}$$

Since G acts transitively on M, the map

$$\mathfrak{g} \longrightarrow T_mM, \quad \xi \longmapsto \xi_1^\sharp$$

is onto. There exist $\xi, \eta \in \mathfrak{g}$ such that $v = \xi_1^\sharp(m)$ and $w = \eta_1^\sharp(m)$. The G-equivariance of Φ implies

$$d\Phi_m(v) = \xi_2^\sharp(p) \quad \text{and} \quad d\Phi_m(w) = \eta_2^\sharp(p).$$

Hence,

$$\begin{aligned} \text{RHS of (2.15)} = \omega_m(\xi_1^\sharp(m), \eta_1^\sharp(m)) &= -\{f^\xi, f^\eta\}(m) \\ = -f^{[\xi,\eta]}(m) = -< [\xi,\eta], \Phi(m) > &= -< [\xi,\eta], p >= \text{LHS of (2.15)}. \end{aligned}$$

Thus we are done. $\square$

Corollary 1.2.1. Under the coditions of Proposition 1.2.2, if in addtion, the moment map Φ is one to one, M is symplectomorphic to a coadjoint orbit of G.

Corollary 1.2.2. Under the conditions of Proposition 1.2.2, if in addition, $\mathrm{H}^1(\mathfrak{g})$ and $\mathrm{H}^2(\mathfrak{g})$, the homology groups of Lie algebra $\mathfrak{g}$, are vanishing, M is symplectomorphic to a coadjoint orbit of G.

1.3 Kazhdan-Kostant-Sternberg Reduction

In this section, we introduce another type of symplectic reduction based on the following theorem given by Kazhdan-Kostant-Sternberg.

Let us use the notations developed in the previous sections. Namely, we have a moment map $\Phi : M \longrightarrow \mathfrak{g}^*$ for the Hamiltonian G action on M. Then we have (see [11])

Theorem 1.3.1.(Kazhdan-Kostant-Sternberg, 1978) If Φ intersects an orbit $X \subset \mathfrak{g}^*$ cleanly, then $\Phi^{-1}(X)$ is coisotropic and the leaf of the null foliation through m in $\Phi^{-1}(X)$ is the orbit of M under $G^0_{\Phi(m)}$, the connected component of the isotropic subgroup of $\Phi(m)$.

Theorem 1.3.1 associates a symplectic manifold to an orbit X by taking the quotient of $\Phi^{-1}(X)$ by its null foliation, $\Phi^{-1}(X)$/null foliation, provided this null foliation is fibrating. Let us denote by Z_X this quotient symplectic manifold and call this reduction the Kazhdan-Kostant-Sternberg reduction. On the other hand, we have another version of the Marsden-Weinstein reduced space M_X. Let us consider the product space $M \times X^-$ where X^- is the symplectic manifold X equipped with the negative symplectic form $-\omega_X$. The space $M \times X^-$ is equipped with the symplectic form $\omega - \omega_X$. It is easy to see that G acts on $M \times X^-$ by product in a Hamiltonian fashion with the moment map $\Psi : M \times X^- \longrightarrow \mathfrak{g}^*$ defined by $\Psi(m, q) = \Phi(m) - q$. A point $p \in X$ is a clean value of Φ if and only if $0 \in \mathfrak{g}^*$ is a clean value of Ψ, and in which case, the Marsden-Weinstein reduced space M_p is symplectomorphic to the reduced space $M_X = \Psi^{-1}(0)/G$. We call the latter space Guillemin's version of the Marsden-Weinstein reduction. Both of these two spaces will be used frequently later regarded as the same space.

We now suppose that the stabilizer subgroup G_p of some $p \in X$, hence of all points on X, are connected. The following theorem gives beautifully the relation between two types of reduced spaces we discussed above (see [10]).

Theorem 1.3.2. Under the hypotheses of **Theorem** 1.3.1, asssume that the null foliation is fibrating over a symplectic manifold, Z_X. Then Z_X is a Hamiltonian G space with the moment map Φ_{Z_X} given by $\Phi_{Z_X}[m] = \Phi(m)$ where $[m]$ is the equivalent class of $m \in \Phi^{-1}(X)$. We can identify Z_X/G with M_X. Furthermore, we have a symplectomorphism of Z_X with $M_X \times X$ and this is a G morphism when we regard M_X as a trivial G space.

Sketch of proof: Since $\Phi^{-1}(X)$ is invariant under the action of G, we know that each of the vector fields $\xi_M^\sharp$ is tagent to $\Phi^{-1}(X)$ and hence $\omega(\xi_M^\sharp, v) = 0$ for all $v \in$ null foliation. This is to say $df^\xi = 0$ on the null foliation where f^ξ is the lifting function of the infinitesimal action of ξ on M. Hence f^ξ is constant on the leaves of the null foliation. It defines a function F^ξ on Z_X. By **Theorem** 1.3.1, the G action preserves the null foliation and hence defines an action on Z_X preserving its symplectic structure. Moreover, by the relation between the symplectic form on Z_X and the restriction of ω to $\Phi^{-1}(X)$, we see that the action of G on Z_X is Hamiltonian with F^ξ being the lifting of the infinitesimal action of ξ on Z_X. This lifting gives rise to a moment map Φ_{Z_X} satisfying $\Phi_{Z_X}[m] = \Phi(m)$.

Let us denote by ρ the projection from $\Phi^{-1}(X)$ onto Z_X. Let $z = [m] \in Z_X$, and $p = \Phi_{Z_X}[m] = \Phi(m)$. By **Theorem** 1.3.1 $\rho(am) = \rho(m)$ for all $a \in G_p^0$. Since $\rho(am) = a\rho(m)$ we conclude that $G_p^0 \subset G_z$. On the other hand the moment map Φ_{Z_X} is a G morphism, we know that $G_z \subset G_p$. Since G_p is connected, we conclude that $G_z = G_p$. Thus the moment map gives a diffeomophism of the G orbit, $G \cdot z$ through

z and the orbit X. Let $F_p = \Phi_{Z_X}^{-1}(p) = \Phi^{-1}(p)/G_p = M_X$. We have the map

$$\phi : \; X \times F_p \longrightarrow Z_X, \qquad \phi(ap, z) = az,$$

which is well defined since $G_z = G_p$ for all $z \in F_p$. On the other hand we have the map

$$\Phi^{-1}(X) \longrightarrow \Psi^{-1}(0), \qquad m \longmapsto (m, \Phi(m)),$$

which induces a map $Z_X/G \longrightarrow \Psi^{-1}(0)/G = M_X$ and we denote by ψ the combination of this induced map and the projection $Z_X \longrightarrow Z_X/G$. So we have the map

$$\Phi_{Z_X} \times \psi : \; Z_X \longrightarrow X \times M_X$$

whose inverse is ϕ. This proves that ϕ is a diffeomophism. Furthermore, the G equivariance of Φ derives

$$< d\Phi_m(\eta_M^\sharp), \xi > = - < \Phi(m), [\eta, \xi] > . \tag{3.16}$$

By this formula and **Lemma 1.2.3** one may get that ϕ is a symplectomorphism. □

Chapter 2

Inductive Structure of Coadjoint Orbits

2.1 Semidirect Products

We start this chapter by looking at the structures of the coadjoint orbits of the groups of semidirect products given by Sternberg (see [7]).

Let $G = H \ltimes V$ be the semidirect product of a Lie group H and a vector space V on which H acts. We may write the elements of G as the matrices $g = \begin{pmatrix} a & v \\ 0 & 1 \end{pmatrix}$ where $a \in H$ and $v \in V$. The Lie algebra of G is the semidirect sum $\mathfrak{g} = \mathfrak{h} \oplus V$ where $\mathfrak{h}$ is the Lie algebra of H. The elements of $\mathfrak{g}$ can be written as the matrices $\xi = \begin{pmatrix} A & x \\ 0 & 0 \end{pmatrix}$ where $A \in \mathfrak{h}$ and $x \in V$. We may then compute the coadjoint action of G by the rule of matrix groups and end up with

$$\mathrm{Ad}_g^{*G}(\alpha, p) = (\mathrm{Ad}_a^{*H}\alpha + a^{-1}p \odot v, a^{-1}p), \tag{1.1}$$

for any $(\alpha, p) \in \mathfrak{g}^* = \mathfrak{h}^* \times V^*$, where H action on V^* is the one induced from the H action on V, and $p \odot v \in \mathfrak{h}^*$ is defined as the following:

$$< p \odot v, A > = < p, Av >, \qquad \text{for any } A \in \mathfrak{h}. \tag{1.2}$$

One can see from this formula that all G orbits are fibered over the H orbits in V^*. We may describe the G orbits fibering over the H orbit $\mathcal{O}$ through p in V^* as follows. Let H_p be the stabilizer subgroup of H at p, and let $\mathfrak{h}_p$ be its Lie algebra. Since $\mathfrak{h}_p$ is a subalgebra of $\mathfrak{h}$, restriction of any $\alpha \in \mathfrak{h}^*$ to $\mathfrak{h}_p$ will give an element in $\mathfrak{h}_p^*$. We have

Lemma 2.1.1. For any $\alpha, \beta \in \mathfrak{h}^*$,

$$\alpha - \beta = p \odot v \quad \text{for some } v \in V$$

if and only if

$$\alpha|_{\mathfrak{h}_p} = \beta|_{\mathfrak{h}_p}.$$

Sketch of Proof: Since $< p \odot v, \eta > = < \eta p, v >$ for any $\eta \in \mathfrak{h}$, but $\eta p = 0$ if $\eta \in \mathfrak{h}_p$, we see that $< p \odot v, \eta > = 0$ for all $\eta \in \mathfrak{h}_p$. This proves that $p \odot V \subset \mathfrak{h}_p^0$, the annihilator of $\mathfrak{h}_p$ in $\mathfrak{h}^*$. By counting the dimension, one may see that these two vector spaces are the same. $\square$

Lemma 2.1.1 gives us clearly the picture of all coadjoint orbits of G fibering over $\mathcal{O}$. Namely, the fiber over p is $\mathfrak{h}_p^0 \times W$, where W is the orbit of H_p through $\alpha|_{\mathfrak{h}_p}$. Another viewpoint of this picture is, when we identify $T_p\mathcal{O}$ with $\mathfrak{h}_p^0$, any coadjoint orbit of G is fibered over some cotangent bundle of $\mathcal{O}$ with the typical fiber above $(p, \xi) \in T^*\mathcal{O}$ being some coadjoint orbit of H_p. Note that V acts on V^* trivially. Thus the stabilizer subgroup of G at p is $G_p = H_p \ltimes V$ and the similar argument shows that G orbit through p is exactly H orbit through p and the coadjoint G_p orbit through $(\alpha, p) \in \mathfrak{g}_p^* = \mathfrak{h}_p^* \times V^*$ is $W \times \{p\}$ where W is the orbit of H_p through α. Therefore any coadjoint orbit of G can be described as a fibration over some cotangent bundle of G orbit $\mathcal{O}$ in V^* with the typical fiber above $(p, \xi) \in T^*\mathcal{O}$ being some coadjoint orbit of G_p. This gives us a complete inductive picture of coadjoint orbits of the semidirect product G via the data of G orbits in V^* and the coadjoint orbits of some "smaller" group G_p, which we call the "*little group*". Practically, to compute the coadjoint orbits of the high dimensional Lie groups is in general difficult. This inductive constructions offer us a very effective way to do it. This will be illustrated later.

2.2 Inductive Construction of Coadjoint Orbits

We now construct the coadjoint orbits of general Lie groups in terms of the data of some subgroups by using the symplectic induction method. Let G be a Lie group, $N \subset G$ be a connected normal subgroup. Denote by $\mathfrak{g}$ and $\mathfrak{n}$ their Lie algebras correspondingly. Let $Y \subset \mathfrak{n}^*$ be an arbitrary coadjoint orbit of N. There is a natural G action on $\mathfrak{n}$ given by conjugation. It induces a dual action of G on $\mathfrak{n}^*$. Let

$$K = G_Y = \{g \in G;\ g \cdot q \in Y \text{ for all } q \in Y\}$$

be the stabilizer subgroup of Y and $\mathfrak{k} = \text{Lie}K$.

Definition 2.2.1. The subgroup K above is called the *little group* of G related to the coadjoint orbit Y of the normal subgroup N.

The restriction of G action on $\mathfrak{n}^*$ to K gives rise to a K action on Y. Obviously this action is transitive. Let W be a K-homogeneous space on which N acts trivially. Then it's easy to see that K acts by product on $Y \times W$ transitively. Now we assume this action is Hamiltonian with the moment map

$$\Psi_1 : Y \times W \longrightarrow \mathfrak{k}^*.$$

Definition 2.2.2. The space $Y \times W$ is called *little group data* if the following conditions are satisfied:

1) W is a K-homogeneous space;

2) N acts on W trivially;

3) K acts on $Y \times W$ by product in a Hamiltonian fashion.

As a subgroup of G, K acts on G to the left converted from the right action, namely

$$a \cdot g = ga^{-1}$$

for all $a \in K, g \in G$.

This action induces a Hamiltonian action of K on T^*G. If we use the left trivialization of T^*G to get

$$T^*G = G \times \mathfrak{g}^*,$$

the K action on T^*G is

$$a \cdot (g, \alpha) = (ga^{-1}, Ad_a^* \alpha) \tag{2.3}$$

for all $a \in K$, $g \in G$ and $\alpha \in \mathfrak{g}^*$, where

$$< Ad_a^* \alpha, \xi > = < \alpha, Ad_{a^{-1}} \xi > \tag{2.4}$$

for all $\xi \in \mathfrak{g}$.

The moment map of this action is

$$\begin{aligned} \Psi_2 : T^*G &\longrightarrow \mathfrak{k}^*, \\ \Psi_2(g, \alpha) &= -\alpha|_{\mathfrak{k}}. \end{aligned} \tag{2.5}$$

Taking the sum symplectic form, we creat a new symplectic manifold

$$M = T^*G \times Y \times W.$$

By the product, K acts on M in a Hamiltonian fashion. The moment map is

$$\Psi : M \longrightarrow \mathfrak{k}^*,$$

where

$$\begin{aligned} \Psi(g, \alpha, p, w) &= \Psi_2(g, \alpha) + \Psi_1(p, w) \\ &= -\alpha \mid_{\mathfrak{k}} + \Psi_1(p, w). \end{aligned} \tag{2.6}$$

for all $g \in G, \alpha \in \mathfrak{g}^*, p \in Y$, and $w \in W$.

Obviously, the K action on M is free. By the Marsden-Weinstein reduction, we have a reduced space

$$X_1 = \Psi^{-1}(0)/K.$$

On the other hand, G acts on itself to the left, inducing a Hamiltonian G action on T^*G as follows:

$$g_1 \cdot (g, \alpha) = (g_1 g, \alpha) \tag{2.7}$$

for all $g_1, g \in G, \alpha \in \mathfrak{g}^*$.

The moment map of this action is

$$\begin{aligned} \Phi_1 : T^*G &\longrightarrow \mathfrak{g}^*, \\ \Phi_1(g, \alpha) &= Ad_g^* \alpha. \end{aligned}$$

Let G act on $Y \times W$ trivially. Then we have a Hamiltonian G action on M, commuting with the K action, therefore factorizing through the reduction to give rise to a Hamiltonian G action on X_1. The moment map of this action is

$$\begin{aligned} \Phi : X_1 &\longrightarrow \mathfrak{g}^*, \\ \Phi([g, \alpha, p, w]) &= Ad_g^* \alpha, \end{aligned} \tag{2.8}$$

where $[g, \alpha, p, w] \in X_1$ is the equivalent class of $(g, \alpha, p, w) \in \Psi^{-1}(0)$. We want to show that Φ gives us a covering map from X_1 to a coadjoint orbit of G.

We now start to look at the structure of the coadjoint orbits. First of all, let $x \in \mathfrak{g}^*$ such that $p = x|_{\mathfrak{n}} \in Y$. We would like to observe the orbit through x under the action of N_p, as a subgroup of G, on $\mathfrak{g}^*$.

Let

$$\mathfrak{k}^0 = \{\beta \in \mathfrak{g}^*,\ \beta|_{\mathfrak{k}} = 0\} \tag{2.9}$$

be the annihilator of $\mathfrak{k}$ in $\mathfrak{g}^*$, and N_p^0 be the connected component of N_p. We have

Lemma 2.2.1. $N_p^0 \cdot x = x + \mathfrak{k}^0$.

Proof: Let $\mathfrak{n}_p = \mathrm{Lie} N_p$. We first look at the behavior of the infinitesimal action. Define

$$\begin{aligned} j &: \mathfrak{n}_p \longrightarrow \mathfrak{g}^* \\ j(\xi) &= \xi \cdot x \end{aligned} \tag{2.10}$$

for any $\xi \in \mathfrak{n}_p$; the right hand side of (2.10) is the infinitesimal action of ξ on x. We may see that

$$\mathrm{Im} j \subseteq \mathfrak{k}^0. \tag{2.11}$$

Indeed, one can easily get

$$\mathfrak{k} = \mathfrak{g}_p + \mathfrak{n}. \tag{2.12}$$

Hence for any $\xi \in \mathfrak{n}_p, \eta = \eta_1 + \eta_2 \in \mathfrak{k}$, where $\eta_1 \in \mathfrak{g}_p, \eta_2 \in \mathfrak{n}$, and

$$< \xi \cdot x, \eta >=< \xi \cdot x, \eta_1 > + < \xi \cdot x, \eta_2 > .$$

But

$$\begin{aligned}
< \xi \cdot x, \eta_1 >_{\mathfrak{g}} &= - < x, [\xi, \eta_1] >_{\mathfrak{g}} \\
&= < x, [\eta_1, \xi] >_{\mathfrak{g}} \\
&= < p, [\eta_1, \xi] >_{\mathfrak{n}} \\
&= - < \eta_1 \cdot p, \xi >_{\mathfrak{n}} \\
&= 0,
\end{aligned}$$

and

$$\begin{aligned}
< \xi \cdot x, \eta_2 >_{\mathfrak{g}} &= - < x, [\xi, \eta_2] >_{\mathfrak{g}} \\
&= - < p, [\xi, \eta_2] >_{\mathfrak{n}} \\
&= - < \xi \cdot p, \eta_2 >_{\mathfrak{n}} \\
&= 0.
\end{aligned}$$

Thus

$$< \xi \cdot x, \eta >= 0$$

for any $\eta \in \mathfrak{k}$. Namely,

$$\xi \cdot x \in \mathfrak{k}^0$$

for any $\xi \in \mathfrak{n}_p$. Hence (2.11) holds. Furthermore, we have a short exact sequence

$$0 \longrightarrow \mathfrak{n}_x \xrightarrow{i} \mathfrak{n}_p \xrightarrow{j} \mathfrak{k}^0 \to 0 \tag{2.13}$$

where $\imath$ is the inclusion.

To prove this, it only remains to show that

$$\dim(\mathfrak{n}_p/\mathfrak{n}_x) = \dim\mathfrak{k}^0. \tag{2.14}$$

For this, let us suppose X to be the coadjoint orbit of G through x. As a subgroup of G, N acts on X in a Hamiltonian fashion. The moment map is

$$\begin{aligned} \pi : X &\longrightarrow \mathfrak{n}^* \\ \pi(x_1) &= x_1|_{\mathfrak{n}}. \end{aligned} \tag{2.15}$$

Let $\mathcal{O}$ be the G orbit in $\mathfrak{n}^*$ through p. Then π is a submersion from X onto $\mathcal{O}$. It's a basic result of symplectic geometry that

$$\mathrm{Im} d\pi_x = \mathfrak{n}_x^0,$$

where $\mathfrak{n}_x^0$ is the annihilator of $\mathfrak{n}_x$ in $\mathfrak{n}^*$.

Hence

$$\dim\mathcal{O} = \dim(\mathrm{Im} d\pi_x) = \dim\mathfrak{n}_x^0 = \dim N - \dim N_x. \tag{2.16}$$

It follows that

$$\dim N_x = \dim N - \dim\mathcal{O} = \dim N - \dim G/G_p. \tag{2.17}$$

But on the other hand,

$$Y \cong N/N_p \cong K/G_p.$$

So

$$\dim N - \dim N_p = \dim K - \dim G_p,$$

or

$$\dim N + \dim G_p = \dim N_p + \dim K. \tag{2.18}$$

By (2.17) and (2.18),

$$\begin{aligned} \dim N_x &= \dim N - (\dim G - \dim G_p) \\ &= \dim N + \dim G_p - \dim G \\ &= \dim N_p - (\dim G - \dim K) \\ &= \dim N_p - \dim\mathfrak{k}^0. \end{aligned}$$

Namely,

$$\dim(N_p/N_x) = \dim\mathfrak{k}^0.$$

It completes the proof of (2.13).

We now go to the group level in terms of exponential map. Note that

$$exp\xi \cdot x = x + \xi \cdot x + \frac{1}{2!}\xi^2 \cdot x + \cdots \tag{2.19}$$

for any $\xi \in \mathfrak{n}_p$.

One may observe that for any $x_1 \in \mathfrak{k}^0$,

$$\xi \cdot x_1 = 0. \tag{2.20}$$

Indeed, for any $\eta \in \mathfrak{g}$,

$$[\xi, \eta] \in \mathfrak{n} \subseteq \mathfrak{k}.$$

So

$$< x_1, [\xi, \eta] >= 0.$$

It implies

$$< \xi \cdot x_1, \eta >= 0$$

for all $\eta \in \mathfrak{g}$.

Hence (2.20) holds.

Therefore, (2.19) is simplified to

$$exp\xi \cdot x = x + \xi \cdot x \in x + \mathfrak{k}^0. \tag{2.21}$$

Since N_p^0 is connected, any $v \in N_p^0$ can be written as

$$v = exp\xi_1 \cdots exp\xi_l$$

for some $\xi_1, \cdots, \xi_l \in \mathfrak{n}_p$.

Then by (2.21) and (2.20), we have

$$v \cdot x = x + x_1 + \cdots + x_l \tag{2.22}$$

where $x_i = \xi_i \cdot x \in \mathfrak{k}^0$.

(2.22) gives us an injection from $N_p^0 \cdot x$ into $x + \mathfrak{k}^0$. We claim that it is also surjective since for any $y \in \mathfrak{k}^0$, we have by (2.13) an element $\xi \in \mathfrak{n}_p$ such that

$$\xi \cdot x = y.$$

Then

$$exp\xi \cdot x = x + y.$$

This completes the proof of the lemma. $\square$

Remark 2.2.1. By this lemma, we see that if N_p is connected, $N_p \cdot x = x + \mathfrak{k}^0$. This is an interesting situation. For example, if we can choose $\mathfrak{n}$ to be the nilradical of $\mathfrak{g}$ and N the connected and simply connected Lie group on $\mathfrak{n}$, all N_p are connected. In general, we need to work with the covering of G for this purpose. We will see in the next chapter that this condition is fairly important.

For our structure theorem we only need $N_p \cdot x \supseteq x + \mathfrak{k}^0$, which is generally true. Namely we need

Corollary 2.2.1. For any $x_1, x_2 \in \mathfrak{g}^*$, if $x_1|_{\mathfrak{k}} = x_2|_{\mathfrak{k}} \in Y$, then there is an element $v \in N_{x_2|_{\mathfrak{n}}}$ such that

$$v \cdot x_1 = x_2.$$

Proof: Since $x_2 - x_1 \in \mathfrak{k}^0$, by **Lemma 3.1**, there is an element $v \in N_p$ such that

$$v \cdot x_1 = x_1 + (x_2 - x_1) = x_2. \qquad \square$$

We also would like to list another corollary to **Lemma 2.2.1** here which we need to use in the next chapter.

Corollary 2.2.2. Under the assumption of **Lemma 2.2.1**, if N_p is connected, $G_x N_p$ is a closed subgroup of G.

Proof: First of all, since G_x stabilizes N_p, G_xN_p is a subgroup. To show that G_xN_p is closed, we only need to show that N_pG_x is closed. We claim

$$N_pG_x = \{a \in G | \; ax \in x + \mathfrak{k}^0\}.$$

Indeed, by **Lemma 2.2.1**, it is obvious that

$$N_pG_x \subseteq \{a \in G | \; ax \in x + \mathfrak{k}^0\}.$$

On the other hand, for any $a \in \{a \in G| \; ax \in x + \mathfrak{k}^0\}$, by **Lemma 2.2.1**, there exists an element $v \in N_p$ such that $ax = vx$. This implies $v^{-1}a \in G_x$. Namely, $a = vb$ for some $b \in G_x$. This concludes the proof. □

With **Corollary 2.2.1**, we are now ready to state that the reduced space X_1 given early in this section is a coadjoint orbit of G up to covering. Thanks to Kostant-Souriau's classification of the Hamiltonian G-homogeneous spaces (see **Proposition** 1. 2. 2) we only need to prove the following result.

Theorem 2.2.1. G acts on X_1 transitively.

Proof: The proof of the theorem is straitforward.

For any two elements

$$[g_i, \alpha_i, p_i, w_i] \in X_1, \; i = 1, 2,$$

there exists $a \in K$ such that

$$a \cdot w_1 = w_2. \tag{2.23}$$

But $a \cdot p_1 \in Y$, hence there exists $u \in N$ such that

$$ua \cdot p_1 = p_2. \tag{2.24}$$

Note that N acts on W trivially, we have by (2.23) and (2.24) that

$$a_1 \cdot (p_1, w_1) = (p_2, w_2), \tag{2.25}$$

where

$$a_1 = ua \in K.$$

Since the moment map

$$\Psi_1 : Y \times W \longrightarrow \mathfrak{k}^*$$

is K-equivariant, following (2.25), we have

$$\begin{aligned} \Psi_1(p_2, w_2) &= \Psi_1(a_1 \cdot (p_1, w_1)) \\ &= a_1 \cdot \Psi_1(p_1, w_1). \end{aligned} \tag{2.26}$$

Recall that the moment map for K action on $M = T^*G \times Y \times W$ is

$$\begin{aligned} \Psi : M &\longrightarrow \mathfrak{k}^*, \\ \Psi(g, \alpha, q, w) &= -\alpha|_{\mathfrak{k}} + \Psi_1(q, w) \end{aligned}$$

and

$$(g_i, \alpha_i, p_i, w_i) \in \Psi^{-1}(0),$$

it follows

$$\begin{aligned} (a_1 \cdot \alpha_1)|_{\mathfrak{k}} &= a_1 \cdot (\alpha_1|_{\mathfrak{k}}) = a_1 \cdot \Psi_1(p_1, w_1) \\ &= \Psi_1(p_2, w_2) = \alpha_2|_{\mathfrak{k}}. \end{aligned} \tag{2.27}$$

Hence, by **Corollary 2.2.1**, there exists $v \in N_{\alpha_2|_{\mathfrak{n}}}$ such that

$$va_1 \cdot \alpha_1 = \alpha_2. \tag{2.28}$$

Set

$$a = va_1 \in K.$$

We observe that the restriction of K action on $Y \times W$ to N is the product action of the coadjoint action of N on Y and the trivial one on W. It follows

$$\Psi_1(q, w)|_{\mathfrak{n}} = q.$$

In particular,

$$\Psi_1(p_2, w_2)|_{\mathfrak{n}} = p_2. \tag{2.29}$$

So

$$\alpha_2|_{\mathfrak{n}} = \Psi_1(p_2, w_2)|_{\mathfrak{n}} = p_2. \tag{2.30}$$

(2.30) tells us $v \in N_{p_2}$. Then by (2.25),

$$\begin{aligned} a \cdot (p_1, w_1) &= va_1 \cdot (p_1, w_1) \\ &= v \cdot (p_2, w_2) \\ &= (p_2, w_2). \end{aligned} \tag{2.31}$$

So far we get

$$a \cdot (\alpha_1, p_1, w_1) = (\alpha_2, p_2, w_2). \tag{2.32}$$

Let

$$g = g_2 a g_1^{-1} \in G.$$

Finally

$$\begin{aligned} g \cdot [g_1, \alpha_1, p_1, w_1] &= [gg_1, \alpha_1, p_1, w_1] \\ &= [g_2 a, \alpha_1, p_1, w_1] \\ &= [a^{-1} \cdot (g_2, a\alpha_1, ap_1, aw_1)] \\ &= [g_2, \alpha_2, p_2, w_2] \end{aligned}$$

Hence, G acts on X_1 transitively. □

Thus, we obtain a covering space of some coadjoint orbit of G from the little group data. In fact we may get the orbit itself rather than a covering space if we choose the little group data good enough.

Lemma 2.2.2. The moment map Φ is injective provided Ψ_1 is injective.

Proof: Suppose $[g_1, x_1, p_1, w_1], [g_2, x_2, p_2, w_2] \in X_1$ and

$$\Phi[g_1, x_1, p_1, w_1] = \Phi[g_2, x_2, p_2, w_2].$$

By (2.8), it means

$$\mathrm{Ad}^*_{g_1} x_1 = \mathrm{Ad}^*_{g_2} x_2.$$

But $x_1|_{\mathfrak{n}} = p_1 \in Y$ and $x_2|_{\mathfrak{n}} = p_2 \in Y$, we have $g_1 p_1 = g_2 p_2$. It follows that $a = g_2^{-1} g_1 \in K$. Thus restriction of $\mathrm{Ad}^*_{g_1} x_1 = \mathrm{Ad}^*_{g_2} x_2$ to $\mathfrak{k}$ gives

$$a\Psi_1(p_1, w_1) = \Psi_1(p_2, w_2).$$

Therefore,

$$ap_1 = p_2, \quad \text{and} \quad aw_1 = w_2,$$

since for Ψ_1 is K-equivariant and one-one. It then follows

$$\begin{aligned} [g_1, x_1, p_1, w_1] &= [g_2 a, a^{-1} x_2, p_1, w_1] \\ &= [a^{-1}(g_2, x_2, p_2, w_2)] \\ &= [g_2, x_2, p_2, w_2]. \end{aligned}$$

Thus we are done. □

It says that if we choose our little group data good enough in the sense that $Y \times W$ is a coadjoint orbit of K other than only a covering space to some coadjoint orbit, the constructed space X_1 is also a coadjoint orbit of G rather than merely a covering space.

2.3 Symplectic bundles

Let $P \to M$ be a principal G bundle for some Lie group G, where G acts on P to the right, and $\rho : T^*M \to M$ be the natural projection. We may pull the bundle $P \to M$ back to T^*M to get a principal G bundle $\rho^\sharp P \to T^*M$. Let us denote by $\tilde{P}$ this pull back bundle. For any Hamiltonian G space Q we may form a fiber bundle $\tilde{P} \times_G Q = (\tilde{P} \times Q)/G$ associated to $\tilde{P} \to T^*M$. Sternberg showed in [21] that this associated bundle is a symplectic manifold with a modified symplectic form depending on the choice of the connection on the principal bundle $\tilde{P} \to T^*M$. On the other hand, Let us consider the Hamiltonian G space $T^*P \times Q$ where G action on T^*P is induced from the left action of G on P converted from the right action by $g \cdot p = p \cdot g^{-1}$. Let Ψ_1 and Ψ_2 be the moment maps for the G actions on T^*P and Q respectively, $\Psi_1 + \Psi_2$ is then a moment map for the product G action on $T^*P \times Q$. This action is free and $0 \in \mathfrak{g}^*$ is a regular value of $\Psi_1 + \Psi_2$. By the Marsden-Weinstein reduction we get a reduced symplectic space $(T^*P \times Q)_0$. Weinstein showed (see [23])

Theorem 2.3.1. The reduced space $(T^*P \times Q)_0$ is symplectomorphic to the symplectic bundle $\tilde{P} \times_G Q \to T^*M$.

Sketch of Proof: Let $p \in P$ lie over $m \in M$. By the infinitesimal action of $\mathfrak{g}$ on P and the natural projection we have an exact sequence:

$$0 \longrightarrow \mathfrak{g} \longrightarrow T_pP \longrightarrow T_mM \longrightarrow 0. \tag{3.33}$$

Its dual is

$$0 \longleftarrow \mathfrak{g}^* \xleftarrow{\Psi_1} T_p^*P \longleftarrow T_m^*M \longleftarrow 0. \tag{3.34}$$

We choose a connection Θ on $P \to M$. By definition it is a linear map from TP to $\mathfrak{g}$, whose dual splits the exact sequence (3.34). In particular, we get a linear map $T_p^*P \to T_m^*M$ and consequently a map $T^*P \to T^*M$ by putting together these maps for all $p \in P$. It is not difficult to check that this map is constant on G -orbits, so it induces a map $(T^*P \times Q)_0 \to T^*M$. On the other hand we notice that $\tilde{P}$ is the pullback of T^*M to P as well as being the pullback of P to T^*M. Thus the maps $T_p^*P \to T_m^*M$ define a map $T^*P \to \tilde{P}$. Taking the product with Q we get a G-equivariant map $T^*P\times \xrightarrow{\lambda} \tilde{P} \times Q$.

The next thing to notice is that the restriction of λ to $(\Psi_1 + \Psi_2)^{-1}(0)$ is a diffeomorphism. Indeed, if we fix $(p, q) \in P \times Q$, the map restricted to $T_p^*P \times Q$ takes (ξ, q) to (η, q) where η is the horizontal part of ξ with respect to the connection Θ. Requiring that $\Psi_1(\xi) = \Psi_2(q)$ means fixing the vertical component $\Psi_1(\xi)$ of ξ. Hence, ξ is uniquely determined by η and q. It follows that λ induces a diffeomorphism λ_0 from $(T^*P \times Q)_0 = (\Psi_1 + \Psi_2)^{-1}(0)/G$ to $(\tilde{P} \times Q)/G = \tilde{P} \times_G Q$. Furthermore one may see that this diffeomorphism is a symplectomorphism with respect to the canonical symplectic form on the Marsden-Weinstein reduced space $(T^*P \times Q)_0$ and the modified symplectic form on the symplectic bundle $\tilde{P} \times_G Q$ related to the connection Θ. $\square$

Now let us look at the inductive construction of the coadjoint orbits. We have a principal K bundle $G \to G/K$ and a Hamiltonian K space $Y \times W$. In the way discussed above we may construct a symplectic bundle $\tilde{G} \times_K (Y \times W) \to T^*(G/M)$. Hence, by Theorem 2.3.1 our reduced space X_1 constructed in the last section can be realized as this symplectic bundle. In terms of this picture we may give an intuitive description of the transitive G action on X_1. Namely, G acts transitively on G/K; K acts transitively on $Y \times W$ leaving G/K fixed; finally, N_p acts transitively on the cotangent space $T^*_\bullet(G/K)$, leaving everything else fixed. Combination of all those effects, we get the transitive G action on X_1.

Remark 2.3.1. We would like to mention here that the symplectic bundle picture shows clearly that our inductive construction is a generalization of Guillemin-Sternberg's result introduced in the begining of this chapter.

Chapter 3

Converse Problem

3.1 Little Group Data

In this chapter we consider the converse problem. For any given coadjoint orbit X of G, can we find the suitable little group data so that X can be reconstructed from such data in the way we introduced in the last chapter?

Throughout this chapter, we assume that N_p for all $p \in \mathfrak{n}^*$ are connected if not otherwise mentioned. We may find the little group data naturally as follows.

Take a point $x_0 \in X$, let $p_0 = x_0|_{\mathfrak{n}} \in \mathfrak{n}^*$, and Y the coadjoint orbit of N through p. As before we denote by K the stabilizer subgroup G_Y. We assume the stabilizer subgroup N_p is connected through out this chapter. Recall (2.15) in **Chapter 2**

$$\begin{aligned} \pi : X &\longrightarrow \mathfrak{n}^*, \\ \pi(x) &= x|_{\mathfrak{n}} \end{aligned}$$

is the moment map for the N action on X.

We put the negative Kirillov-Kostant symplectic form on Y, denoted by Y^-, and by product N acts on $X \times Y^-$ in a Hamiltonian fashion. The corresponding moment map turns out to be

$$\pi_1 : X \times Y^- \longrightarrow \mathfrak{n}^*,$$

$$\pi_1(x,p) = x|_{\mathfrak{n}} - p. \tag{1.1}$$

The Marsden-Weinstein reduction gives rise to a symplectic manifold

$$W = \pi_1^{-1}(0)/N.$$

Remark 3.1.1. Note that $\pi^{-1}(p_0) = K_{p_0}/G_{x_0}$. To see $\pi^{-1}(p_0)/N_{p_0}$ is a well defined manifold, we only need to see that $G_{x_0}N_{p_0}$ is a closed subgroup. This is true due to Corollary 2.2.2. So $W \cong \pi^{-1}(p_0)/N_{p_0}$ is a well defined manifold.

There is a natural K action on $X \times Y^-$ by product, which induces a K action on W since N is a normal subgroup of both G and K. Let us have a close look at this induced action. First we have an easy result.

Lemma 3.1.1. The K action on W is transitive.

Proof: For any $(x_i, p_i) \in \pi_1^{-1}(0)$, $i = 1, 2$, we can find $a \in G$ such that

$$a \cdot x_1 = x_2. \tag{1.2}$$

Restrict (1.2) to $\mathfrak{n}$, it gives by (1.1)

$$a \cdot p_1 = p_2. \tag{1.3}$$

So

$$a \cdot (x_1, p_1) = (x_2, p_2),$$

and

$$a \cdot [x_1, p_1] = [x_2, p_2], \tag{1.4}$$

where $[x, p]$ is the equivalent class in W of $(x, p) \in \pi_1^{-1}(0)$.

Note that (1.3) also implies that $a \in K$, which completes the proof. □

Note: In the case that the K action on Y is Hamiltonian, so will be the K action on W. Consequently, the K action on $Y \times W$ is Hamiltonian. But the last assertion is always true. We will prove this next.

So far we already have the materials Y and W. N acts on W, of course, trivially. To see that such materials really meet our requirement, we need the following result.

Lemma 3.1.2. The K action on $Y \times W$ by product is Hamiltonian.

Proof: An equivalent version to see the reduced space W is to look at the moment map

$$\pi : X \longrightarrow \mathfrak{n}^*.$$

For $p_0 \in Y \subset \mathfrak{n}^*$, the Marsden-Weinstein reduction at p_0 is

$$\pi^{-1}(p_0)/N_{p_0}.$$

Our $W = \pi_1^{-1}(0)/N$ is just Guillemin's version of this reduction, Namely

$$W \cong \pi^{-1}(p_0)/N_{p_0}. \tag{1.5}$$

On the other hand, let us consider the Kazhdan-Kostant-Sternberg reduction

$$Z_Y = \pi^{-1}(Y)/\text{Null foliation}.$$

We take a look at the K action on X. Since for any $x \in \pi^{-1}(Y)$, and $a \in K$,

$$(a \cdot x)|_{\mathfrak{n}} = a \cdot (x|_{\mathfrak{n}}) \in Y.$$

Namely, $a \cdot x \in \pi^{-1}(Y)$.

So $\pi^{-1}(Y)$ is an invariant subspace of the K action. On the other hand, for any $p \in Y$, if $v \in N_p^0$, it is easy to see that $ava^{-1} \in N_{a \cdot p}^0$. Therefore, by **Theorem 1.3.1**, there is an induced K action on Z_Y. In fact, this action is also Hamiltonian. To see this, for any $\xi \in \mathfrak{k}$, let

$$f^\xi : X \longrightarrow \mathbf{R}$$

be the lift of the infinitesimal action of $\mathfrak{k}$ on X. So we have

$$f^\xi : \pi^{-1}(Y) \longrightarrow \mathbf{R}.$$

We observe that f^ξ is constant on the null foliation.

Indeed,

$$\iota(\xi^\sharp)\omega_X = df^\xi, \tag{1.6}$$

where ω_X is the canonical symplectic form on X. For $\eta \in$ Null foliation,

$$\iota(\eta)\omega_X|_{\pi^{-1}(Y)} = 0.$$

Since

$$\pi^{-1}(Y) = K \cdot x_0, \tag{1.7}$$

$\xi^\sharp$ is tangent to $\pi^{-1}(Y)$. Hence,

$$\omega_X(\xi^\sharp, \eta) = 0. \tag{1.8}$$

So

$$df^\xi(\eta) = 0. \tag{1.9}$$

This implies that $\xi^\sharp$ is constant on the null foliation.

Therefore, $\xi^\sharp$ induces a function

$$\tilde{f}^\xi : Z_Y \longrightarrow \mathbf{R},$$

which gives us the moment map for K action on Z_Y.

Meanwhile since N_{p_0} is connected, **Theorem 1.3.2** says that $Z_Y \cong Y \times W$. It is not difficult to check from the proof of **Theorem 1.3.2** that the action of K on Z_Y corresponds to the product action of K on $Y \times W$ under the above symplectomorphism. Thus, the K action on $Y \times W$ is Hamiltonian.

Therefore, we conclude

Proposition 3.1.1. The space $Y \times W$ we choose above is a little group datum of K.

3.2 The Complete Construction of Coadjoint Orbits

We are now ready to reconstruct X from the little group datum $Y \times W$ we choose in the last section. by some fundamental computation, one can write out explicitly the formula of the moment map for this K action on $Y \times W$ without difficulty. Note that we can write $u \cdot p$, $u \in N$, for the elements on Y, and $[x, p_0]$, $x \in \pi^{-1}(p_0)$, for elements on $W \cong \pi_1^{-1}(0)/N$. Then the moment map is

$$\Psi_1 : Y \times W \longrightarrow \mathfrak{k}^*$$

$$\Psi_1(u \cdot p_0, [x, p_0]) = (u \cdot x)|_{\mathfrak{k}}. \tag{2.10}$$

It can be seen directly from (2.10) that Ψ_1 is K equivariant. Namely, for any $a \in K$, let $u' \in N$ such that

$$au = u'a$$

since for N is a normal subgroup of K; and

$$a \cdot p_0 = u_a \cdot p_0$$

for some $u_a \in N$. Let $w = [x, p_0]$, then

$$\begin{aligned}
\Psi_1(a \cdot (u \cdot p_0, w)) &= \Psi_1(au \cdot p_0, a \cdot w) \\
&= \Psi_1(u'a \cdot p_0, [a \cdot x, a \cdot p_0]) \\
&= \Psi_1(u'u_a \cdot p_0, [a \cdot x, u_a \cdot p_0]) \\
&= \Psi_1(u'u_a \cdot p_0, [u_a^{-1} a \cdot x, p_0]) \\
&= u'u_a \cdot (u_a^{-1} a \cdot x) \\
&= u'a \cdot x \\
&= au \cdot x \\
&= a \cdot \Psi_1(u \cdot p_0, w).
\end{aligned}$$

It follows (2.10) that Ψ_1 sends $Y \times W$ to the coadjoint orbit of K through $x_0|_{\mathfrak{k}}$.

Lemma 3.2.1. The moment map Ψ_1 is injective.

Proof: Suppose

$$\Psi_1(u_1 \cdot p_0, [x_1, p_0]) = \Psi_1(u_2 \cdot p_0, [x_2, p_0]).$$

Hence $u_1 x_1|_{\mathfrak{k}} = u_2 x_2|_{\mathfrak{k}}$. Restricting this to $\mathfrak{n}$ gives us $u_1 p_0 = u_2 p_0$. Hence $u = u_1^{-1} u_2 \in N_{p_0}$. Therefore,

$$\begin{aligned}(u_2 \cdot p_0, [x_2, p_0]) &= (u_1 \cdot p_0, [x_2, u^{-1} p_0]) \\ &= (u_1 \cdot p_0, [x_1, p_0]).\end{aligned}$$

This completes the proof. □

We may now construct a reduced space X_1 from the chosen little group data $Y \times W$. We have

Theorem 3.2.1. $\Phi(X_1) = X$. Furthermore, Φ is a symplectomorphism.

Proof: To show $\Phi(X_1) = X$, it suffices to show that

$$\Phi(X_1) \cap X \neq \emptyset.$$

In fact, if we take

$$\lambda = [e, x_0, p_0, w_0],$$

where e is the identity element of G and $w_0 = [x_0, p_0] \in W$, by (4.45) λ is a well defined element in X_1. But

$$\Phi(\lambda) = Ad_e^* x_0 = x_0 \in X.$$

So X_1 covers X. By Lemma 3.2.1 and Lemma 2.2.2, Φ is a symplectomorphism. Thus we are done. □

Hence, the constructed space X_1 exactly recovers the coadjoint orbit X under the symplectomorphism Φ. We summarize our result as follows.

Theorem 3.2.2. For Lie group G and a normal subgroup N, the inductive construction from any little group data $Y \times W$ gives rise to a covering space of some coadjoint orbit of G. Furthermore, it gives rise to a coadjoint orbit if the little group data is chosen good enough in the sense that it is symplectomorphic to a coadjoint orbit of the corresponding little group. Conversely, if the stabilizer subgroups N_p for all $p \in \mathfrak{n}^*$ are connected, all coadjoint orbits of G can be constructed inductively from the suitable little group data.

By means of Theorem 2.3.1 we conclude that

Proposition 3.2.1. Let G be any Lie group with a normal subgroup N such that the stabilizer subgroups N_p for all $p \in \mathfrak{n}^*$ are connected. Then any coadjoint orbit of G can be realized as a symplectic fiber bundle over $T^*(G/K)$ with the typical fiber being some coadjoint orbit of K where K is some certain little group.

Remark 3.2.1. We emphasize that in the converse problem we assumed N_{p_0} therefore all N_p for $p \in Y$ are connected. The coadjoint orbits of G can be recovered by our symplectic inductive costruction only up to the covering without this assumption. In the cases of nilpotent and solvable groups, as well as semi-direct products, we may always be able to choose N to be the connected and simply connected nilpotent groups. In this case N_p are connected. In general we may not be able to choose such normal subgroups. It is also noticed that the orbit method works out the representations of the nilpotent and solvable groups (see [12] and [2]) but only for the representation of covering of groups in general.

3.3 Examples

To illustrate our results, we will look at some examples of constructing coadjoint orbits of Lie groups on the little group data.

Example 6.1. $G_m = GL(m, \mathbf{R}) \ltimes \mathbf{R}^m$, the semidirect product of $GL(m, \mathbf{R})$ and $\mathbf{R}^m$. Take $N = V_m = \mathbf{R}^m$, and denote $H_m = GL(m, \mathbf{R})$; as we remarked, all coadjoint orbits of G_m can be built up as fiber bundles over the cotangent bundles of G orbits in V_m^* with the typical fibers being coadjoint orbits of the stabilizer subgroups of H_m at the corresponding points in V_m^*. Let us do it by induction.

First of all, let us look at

$$G_1 = H_1 \ltimes V_1.$$

In this very simple case,

$$G = G_1 = \left\{ \begin{pmatrix} a & x \\ 0 & 1 \end{pmatrix} ;\ a \in \mathbf{R} - \{0\},\ x \in \mathbf{R} \right\} \tag{3.11}$$

and

$$\mathfrak{g} = \mathrm{Lie}G = \left\{ \begin{pmatrix} \lambda & x \\ 0 & 0 \end{pmatrix} ;\ \lambda, x \in \mathbf{R} \right\}. \tag{3.12}$$

As the vector spaces,

$$\mathfrak{g}^* \cong \mathbf{R}^2.$$

Take

$$e_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},\ e_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

a basis of $\mathfrak{g}$, and $\{e_1^*, e_2^*\}$ the dual basis of $\mathfrak{g}^*$. Let $\alpha = (x_0, y_0) \in \mathfrak{g}^*$, $\mathcal{O}_\alpha$ be the coadjoint orbit through α. We have

1. $y_0 \neq 0$, $\mathcal{O}_\alpha$ is the union of the upper-hemiplane $y > 0$ and the lower-hemiplane $y < 0$;

2. $y_0 = 0$, $\mathcal{O}_\alpha = \{\alpha\}$;

Now we turn to study the general case

$$G_m = H_m \ltimes V_m,\ m \geq 2$$

Denote now $G = G_m$, $H = H_m$, $V = V_m$. There are totally two G orbits in V^*, namely,

$$\mathcal{O}_0 = \{0\}, \text{ and } \mathcal{O} = V^* - \{0\}.$$

Over $\mathcal{O}_0$, the orbits of G correspond to the orbits of the stabilizer subgroup $H_0 = H$. Let's build up coadjoint orbits of G over $T^*\mathcal{O}$. Take, for example, the point

$$p = (1, 0, \cdots, 0) \in \mathcal{O}.$$

By the straitforward calculation, we get

$$H_p \cong G_{n-1}. \tag{3.13}$$

Hence every coadjoint orbit of G_m through some $x \in \mathfrak{g}^*$ with $q = x|_{V_m} \neq 0$ can be realized as a fiber bundle over $T^*\mathcal{O}$ with the typical fiber being some coadjoint orbit of G_{n-1}. While we already know the case of G_1, we can then build up the orbits of all G_m by induction.

For instance, in the case of $n = 2$, we may get that G_2 has

1. One open orbits;
2. one-parameter family orbits of codimension 2;
3. two-parameter family orbits of codimension 4;
4. one-parameter family orbits of codimension 6.

Example 6.2. Conformal Heisenberg Lie groups.

Let H_n be the $2n+1$ dimensional Heisenberg group, $G = \mathbf{R}^+ \ltimes H_n$, the semidirect product of $\mathbf{R}^+$ and H_n, where the $\mathbf{R}^+$ action on H_n is

$$\lambda \cdot (v, t) = (\lambda v, \lambda^2 t) \tag{3.14}$$

for all $\lambda \in \mathbf{R}^+$ and $(v, t) \in H_n$.

Take $N = H_n$.

$H_n = \mathbf{R}^{2n} \times \mathbf{R}$, the product on H_n is

$$(u, s) \cdot (v, t) = (u + v, t + s + \frac{1}{2}\Omega(u, v)) \tag{3.15}$$

where Ω is the standard symplectic form on $\mathbf{R}^{2n}$.

The Lie algebra of H_n is $\mathfrak{n} = \mathbf{R}^{2n+1} = \mathbf{R}^{2n} \times \mathbf{R}$, and the Lie bracket on $\mathfrak{n}$ is

$$[(\xi, t),\ (\eta, s)] = (0, \Omega(\xi, \eta)).$$

Take the standard basis $\{e_i\}$ of $\mathfrak{n}$, where

$$e_i = (\underbrace{0, \cdots, 0}_{i}, 1, 0, \cdots, 0),\ i = 1, \cdots, 2n+1.$$

Let $\{e_i^*\}$ be the dual basis of $\mathfrak{n}^* = \mathbf{R}^{2n+1}$. It is well known that for any point $p = (c_1, \cdots, c_{2n+1})$ in $\mathfrak{n}$, the coadjoint orbit of N through p, say Y_p, is

1. the hyperplane $x_{2n+1} = c_{2n+1}$ if $c_{2n+1} \neq 0$, or
2. just the single point p if $c_{2n+1} = 0$.

In the first case, we have

$$K = G_Y = 1 \times H_n \cong H_n.$$

Therefore, K acts on any Y_p in the Hamiltonian fashion. By the **Note** to **Lemma 4.1**, we only need to pick the little group data W to be orbits of K on which N acts trivially. Of course, they are those fixed points of coadjoint action of N on the hyperplane $x_{2n+1} = 0$.

If we write

$$\mathfrak{g} = \mathbf{R} \times \mathbf{R}^{2n+1} \text{ with } \mathfrak{n} = \mathbf{R}^{2n+1} \subset \mathfrak{g},$$

let $f_0 = (\underbrace{1, 0, \cdots, 0}_{2n+2})$ and $f_i = (\underbrace{0, \cdots, 0}_{i}, 1, 0, \cdots, 0) = e_i$, $i = 1, \cdots, 2n+1$. Equip $\mathfrak{g}^*$ with the dual basis $\{f_0^*, \cdots, f_{2n+1}^*\}$. Then for those $\alpha = (c_0, \cdots, c_{2n+1})$ with $c_{2n+1} \neq 0$ the coadjoint orbit of G through α is the fiber bundle over $T^*(\mathbf{R}^+)$ with the typical fiber $\mathbf{R}^{2n}$.

In the second case, if $p \neq 0$, once again,

$$K = G_{Y_p} = G_p = 1 \times H_n \cong H_n.$$

The construction of coadjoint orbits of G is similar to the first case.

If $p = 0$, $K = G$. Our theorems tell us that all coadjoint orbits of G through $(c_0, 0, \cdots, 0)$ are acted by H_n trivially.

Example 6.3. Consider

$$G = \left\{ \begin{pmatrix} a & d & e \\ 0 & b & f \\ 0 & 0 & c \end{pmatrix} \in GL(3, \mathbf{R});\ a, b, c > 0 \right\},$$

the group of the upper triangular matrices. Then

$$\mathfrak{g} = \left\{ \begin{pmatrix} x & u & v \\ 0 & y & w \\ 0 & 0 & z \end{pmatrix} \in gl(3, \mathbf{R}) \right\}$$

Take

$$N = \left\{ \begin{pmatrix} 1 & d & e \\ 0 & 1 & f \\ 0 & 0 & 1 \end{pmatrix} \in GL(3, \mathbf{R}) \right\}.$$

A basis of $\mathfrak{g}$ is chosen as

$$\begin{aligned} e_1 &= \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, e_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, e_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \\ e_4 &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, e_5 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, e_6 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \end{aligned}$$

Then e_1, e_2, e_3 form a basis of $\mathfrak{n}$.

Let e_1^*, e_2^*, e_3^* be the dual basis in $\mathfrak{n}^*$, (x^1, x^2, x^3) be the corresponding coordinate system. For $p = (\lambda_1, \lambda_2, \lambda_3) \in \mathfrak{n}^*$, it is not difficult to see that the orbit of N through p , say Y_p, is

1. hyperplane $x^3 = \lambda_3$ if $\lambda_3 \neq 0$;
2. single point p if $\lambda_3 = 0$.

In the first case,

$$K = G_{Y_p} = \left\{ \begin{pmatrix} a & d & e \\ 0 & b & f \\ 0 & 0 & a \end{pmatrix} \in GL(3, \mathbf{R});\ a, b > 0 \right\}.$$

$\mathfrak{k} = \mathrm{Lie}K$, and $e_1, \cdots, e_5$ form a basis of $\mathfrak{k}$.

Let $e_1^*, \cdots, e_5^*$ be the dual basis in $\mathfrak{k}^*$. It's easy to see that Y_p is not an orbit of K.

By the moment map (2.10), we may find the corresponding little group data $Y_p \times W_p$ such that

$$\Psi_1(Y_p \times W_p) = K\text{-orbit through } \beta = x|_{\mathfrak{k}}$$

where $x \in \mathfrak{g}^*$ satisfying $x\,|_{\mathfrak{k}} = p$.

Let $B_\beta = K$-orbit through β. By examining the proof of **Theorem 4.3**, we see that in the construction of X_1, $Y_p \times W_p$ can be replaced by B_β, while, of course, Ψ_1 being replaced by the inclusion map.

Practically, for example, if we take $p = (0, \lambda_2, 0) \in Y$, where $\lambda_2 \neq 0$, let $\beta = (0, \lambda_2, 0, 0, 0) \in \mathfrak{k}^*$, then it can be calculated that

$$B_\beta = \{\lambda_2(f, 1, -d, df, -df) \in \mathfrak{k}^*,\ f, d \in \mathbf{R}\}. \tag{3.16}$$

We conclude that the coadjoint orbit of G through x such that $x|_{\mathfrak{k}} = \beta$ is the fiber bundle over the 2-dimensional space $T^*(G/K)$ with the typical fiber B_β given by (3.16).

Similarly, we may construct other orbits of G in this way.

Chapter 4

The Mackey Obstruction

4.1 Extension of Representations and the Mackey Obstruction

One may ask that why we choose our little group data in the way given in Chapter 2 and Chapter 3. We will see that realizing the orbit Y of N as a coadjoint orbit of the little group K is the classical version of extending an irreducible unitary representation of N to a representation of K. There is in general an obstruction to this extension. In this chapter we will review such obstructions and discuss it at the classical level. This will explain the choice of the little group data.

Let G be a real Lie group, $N \subset G$ be a normal subgroup, and $N^\wedge$ be the set of the equivalent classes of the irreducible unitary representations of N. Since N is normal in G, there is a natural action of G on $N^\wedge$. Pick a representation $\rho \in N^\wedge$, we assume that G fixes ρ. One would like to ask that if ρ can extend to an irreducible unitary representation of G in the sense that there is such a representation of G with the restriction of it to N being ρ? The answer in general is no. There is an obstruction (called the Mackey obstruction) to extend ρ to a unitary representation of G. In general, ρ can only extend to a projective representation ρ^σ of G. Namely

$$\rho^\sigma(ab) = \sigma(a,b)\rho(a)\rho(b) \tag{1.1}$$

for any $a, b \in G$, where the 2-cocycle σ satisfies

$$|\sigma(a,b)| = 1, \tag{1.2}$$

$$\sigma(e,a) = \sigma(a,e) = 1, \quad \text{and} \tag{1.3}$$

$$\sigma(ab,c)\sigma(a,b) = \sigma(a,bc)\sigma(b,c), \tag{1.4}$$

where e is the identity of G. ρ^σ is a representation only if $\sigma \equiv 1$. But it extends to a representation of some central extension G^σ of G with respect to the circle group T. Namely, we define

$$G^\sigma = \{(t,a) \in T \times G\},$$

where the multiplication of this group is defined as

$$(t,a)\cdot(s,b) = (ts/\sigma(a,b), ab). \tag{1.5}$$

Then G^σ is a group with the identity $(1,e)$ and $(\sigma(a,a^{-1})/t, a^{-1})$ is the inverse of (t,a). We have the exact sequence of groups

$$1 \longrightarrow T \xrightarrow{i} G^\sigma \xrightarrow{j} G \rightarrow e, \tag{1.6}$$

where

$$i(t) = (t,e), \;\; j(t,a) = a.$$

We define

$$\rho_0^\sigma : G^\sigma \longrightarrow \mathrm{Aut}(\mathcal{H}),$$

where $\mathcal{H}$ is the representation space of ρ;

$$\rho_0^\sigma(t,a) = t\rho^\sigma(a). \tag{1.7}$$

It is easy to check that ρ_0^σ is a unitary representation of G^σ. The 2-cocycle σ is the obstruction to extend ρ to a representation of G. We may define the Mackey obstruction by the following procedure.

First, let us recall the definition of the Baer product.

Let

1) $1 \longrightarrow Q \longrightarrow E_1 \xrightarrow{\phi_1} F \longrightarrow 1$

2) $1 \longrightarrow Z \longrightarrow E_2 \xrightarrow{\phi_2} F \longrightarrow 1$

be two exact sequences of groups where Z is in the center of Q. Let $R \subset E_1 \times E_2$ consist of all pairs $(e_1, e_2) \in E_1 \times E_2$ such that

$$\phi_1(e_1) = \phi_2(e_2).$$

Let S be the subgroup of R of the form (z, z^{-1}), $z \in Z$. Define the Baer product

$$E_1 \otimes E_2 = R/S.$$

Note that if we define

$$\phi(e_1, e_2) = \phi_1(e_1) = \phi_2(e_2), \ (e_1, e_2) \in R,$$

then ϕ defines a homomorphism of $E_1 \otimes E_2$ onto F with the kernel $(Q \times Z)/S$. Also $(Q \times Z)/S$ is isomorphic to Q, so $E_1 \otimes E_2$ satisfies the exact sequence

3) $1 \longrightarrow Q \longrightarrow E_1 \otimes E_2 \longrightarrow F \longrightarrow 1.$

We call (3) the Baer product of (1) and (2).

For $\rho \in N^\wedge$, we define a unitary representation $\tilde{\rho}$ of $T \times N$ as follows:

$$\tilde{\rho}(tv) = t\rho(v), \text{ for any } t \in T, \ v \in N.$$

We have

Theorem 2.1. (Mackey) Given the group extension

1) $1 \longrightarrow T \times N \longrightarrow T \times G \longrightarrow G/N \longrightarrow 1,$

there exists a unique group extension

2) $1 \longrightarrow T \longrightarrow F \longrightarrow G/N \longrightarrow 1$

satisfying the following conditions:

A) T is central in F;

B) if $G^\natural = F \otimes (T \times G)$ is the Baer product, then there exists a unitary representation $\rho^\natural$ of $G^\natural$ such that $\rho^\natural\mid_{T\times N} = \tilde{\rho}$.

Definition 2.1. The group extension (2) above is called the Mackey obstruction of G at ρ or the obstruction to extend ρ from N to G.
In the case that the exact sequence (2) is trivial, or splitting, **Theorem** 2.1 says that ρ can extend to a representation of G.

Remark 2.1. Auslander-Kostant and Brezin showed in [2] and [4] how to compute the Mackey obstruction from a particular 2-cocycle on Lie algebras in certain cases. Roughly, let $\mathfrak{g}$ and $\mathfrak{n}$ be the Lie algebras of G and N respectively. There is a subspace $\mathfrak{a}$ of $\mathfrak{g}$ such that

$$\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{n} \text{ as vector spaces, and } \beta[\mathfrak{a}, \mathfrak{n}] = 0.$$

where β is some element in $\mathfrak{g}^*$, the dual space of $\mathfrak{g}$. Then let $\tau: \ \mathfrak{g} \to \mathfrak{g}/\mathfrak{n}$ and let $\theta: \ \mathfrak{g}/\mathfrak{n} \to \mathfrak{a}$ be the unique linear mapping such that $\tau \circ \theta$ is the identity mapping of $\mathfrak{g}/\mathfrak{n}$. Define a bilinear form σ on $\mathfrak{g}/\mathfrak{n}$ by

$$\sigma(\bar{\xi}, \bar{\eta}) = -\beta([\theta\bar{\xi}, \theta\bar{\eta}]), \ \bar{\xi}, \bar{\eta} \in \mathfrak{g}/\mathfrak{n}.$$

σ gives rise to a 2-cocycle on $\mathfrak{g}$. It was also used to construct the Mackey obstruction in the articles mentioned above. We call it the Mackey 2-cocycle.

4.2 Classical version of the Mackey obstruction

We now translate the Mackey obstruction into the classical language. Let G and N be as above where N is connected. Let $Y \subset \mathfrak{n}^*$ be a coadjoint orbit of N such that the natural action of G on $\mathfrak{n}^*$ fixes it.We assume that the stabilizer subgroup N_p of coadjoint action of N at some point p, hence all points, on Y is connected. The Mackey obstruction can be interpreted as the obstruction to realize Y as a coadjoint orbit of G.

Proposition 4.1. The G action on Y is symplectic.

Proof: For any automorphism $\phi : \mathfrak{n} \longrightarrow \mathfrak{n}$, ϕ^t preserves the Poisson structure on $\mathfrak{n}^*$ so it carries symplectic leaves into symplectic leaves. Moreover, if Y_1 is a symplectic leaf and $Y_2 = \phi^t(Y_1)$ then ϕ^t is a symplectomorphism of Y_1 onto Y_2.

Let Ω_Y be the canonical symplectic form on Y. **Proposition 4.1** says that $\iota(\xi^\sharp)\Omega_Y$ is closed. In addition, if N is simply connected it is easy to see that Y is simply connected and hence $\iota(\xi^\sharp)\Omega_Y$ is exact. Indeed, $N \rightarrow Y$ is a fibration with fiber N_p. By the long exact sequence in homotopy

$$\cdots \longrightarrow \pi_1(N) \longrightarrow \pi_1(Y) \longrightarrow \pi_0(N_p) \longrightarrow \pi_0(N),$$

we see that if $\pi_1(N) = \pi_0(N) = 0$, then $\pi_1(Y) = \pi_0(N_p)$. In other words, if N is simply connected, then N_p is connected if and only if Y is simply connected. We will however show that $\iota(\xi^\sharp)\Omega_Y$ is exact even when N is not simply connected and in fact we will construct a canonical function $\phi^\xi : Y \rightarrow \mathbf{R}$ such that $\iota(\xi^\sharp)\Omega_Y = d\phi^\xi$. Pick a point $p_0 \in Y$. We can write

$$Y = \{v \cdot p_0;\ v \in N\}$$

Let x_0 be an element in $\mathfrak{g}^*$ such that $x_0|_{\mathfrak{n}} = p_0$. Define the map

$$\phi^\xi : Y \longrightarrow \mathbf{R}$$

by

$$\phi^\xi(v \cdot p_0) = < v \cdot x_0, \xi > . \tag{2.8}$$

By **Lemma 2.2.1** we see that it is well defined. We want to show that

$$\iota(\xi^\sharp)\Omega_Y = d\phi^\xi. \tag{2.9}$$

Note that Y is an N-homogeneous space. Any vector at $q \in Y$ can be written as $\delta^\sharp(q)$ for some $\delta \in \mathfrak{n}$.

We need to show that

$$\iota(\xi^\sharp)\Omega_Y(\delta^\sharp) = d\phi^\xi(\delta^\sharp). \tag{2.10}$$

Let $q = vp_0 \in Y$, at q,

$$\begin{aligned}
\text{RHS of (2.10)} &= \delta^\sharp(\phi^\xi)(q) \\
&= \frac{d}{dt}|_{t=0}\phi^\xi(\exp - t\delta \cdot q) \\
&= \frac{d}{dt}|_{t=0} < \exp - t\delta \cdot vx_0, \xi >_\mathfrak{g} \\
&= - < \delta \cdot vx_0, \xi >_\mathfrak{g} \\
&= < vx_0, [\delta, \xi]_\mathfrak{g} >_\mathfrak{g} \\
&= < x_0, v^{-1}[\delta, \xi] >_\mathfrak{g} \\
&= < p_0, v^{-1}[\delta, \xi] >_\mathfrak{n} \\
&= < q, [\delta, \xi] >_\mathfrak{n} \\
&= - < q, [\xi, \delta] > . \qquad (2.11)
\end{aligned}$$

On the other hand, there exists an $\delta_\xi \in \mathfrak{n}$ such that $\delta_\xi^\sharp(q) = \xi^\sharp(q)$. Thus

$$\begin{aligned}
\text{LHS of (2.10)} &= \Omega_Y(\delta_\xi^\sharp(q), \delta^\sharp(q)) \\
&= - < q, [\delta_\xi, \delta] > . \qquad (2.12)
\end{aligned}$$

But $\delta_\xi^\sharp(q) = \xi^\sharp(q)$ means $\xi \cdot q = \delta_\xi \cdot q$ on $\mathfrak{n}$, so

$$< \xi \cdot q, \delta > = < \delta_\xi \cdot q, \delta > .$$

Namely,

$$< q, [\xi, \delta] > = < q, [\delta_\xi, \delta] > .$$

Hence (2.11) = (2.12), which says (2.10) holds.

Therefore, the G-action on Y has an infinitesimal lift $\phi : \ \xi \longmapsto \phi^\xi$. In general, ϕ is not a Lie algebra homomorphism. There is a 2-cocycle

$$\sigma : \ \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathbf{R}$$

satisfying

$$\sigma(\xi, \eta) = -\sigma(\eta, \xi), \qquad (2.13)$$

and

$$\sigma([\xi,\eta],\zeta)+\sigma([\eta,\zeta],\xi)+\sigma([\zeta,\xi],\eta)=0 \tag{2.14}$$

such that

$$\{\phi^{\xi},\phi^{\eta}\}=\sigma(\xi,\eta)+\phi^{[\xi,\eta]}. \tag{2.15}$$

Classically, this 2-cocycle is the Mackey 2-cocycle. It prevents Y being a coadjoint orbit of G. Y is a coadjoint orbit of some central extension of G with respect to the circle group T. At the algebra level, define

$$\mathfrak{g}^{\sigma}=\{(s,\xi),\ s\in\mathbf{R},\ \xi\in\mathfrak{g}\},$$

where

$$[(s,\xi),(t,\eta)]=(\sigma(\xi,\eta),[\xi,\eta]_{\mathfrak{g}}). \tag{2.16}$$

It's easy to check that (2.16) does define a Lie bracket for $\mathfrak{g}^{\sigma}$.

We have an exact sequence of Lie algebras

$$0\longrightarrow\mathbf{R}\xrightarrow{i}\mathfrak{g}^{\sigma}\xrightarrow{j}\mathfrak{g}\longrightarrow 0,$$

where $i(s)=(s,0)$, and $j(s,\xi)=\xi$. $\mathfrak{g}^{\sigma}$ is a central extension of $\mathfrak{g}$ with respect to $\mathbf{R}$. Let G^{σ} be the corresponding central extension of G with Lie algebra $\mathfrak{g}^{\sigma}$, we have

$$1\longrightarrow T\xrightarrow{I}G^{\sigma}\xrightarrow{J}G\longrightarrow e.$$

G^{σ} acts on Y by the recipe

$$\tilde{a}\cdot q=J(\tilde{a})\cdot q,\ \text{ for any } \tilde{a}\in G^{\sigma},\ q\in Y.$$

The corresponding infinitesimal action is

$$(s,\xi)\cdot q=\xi\cdot q.$$

Define

$$\psi^{(s,\xi)}:\ Y\longrightarrow\mathbf{R}$$

$$\psi^{(s,\xi)}(vp_0)=<vx_0,\xi>+s. \tag{2.17}$$

Since

$$(s,\xi)^\sharp = \xi^\sharp, \text{ and } \psi^{(s,\xi)} = \phi^\xi + s,$$

we have

$$\iota((s,\xi)^\sharp)\Omega_Y = d\psi^{(s,\xi)}. \tag{2.18}$$

So $\psi^{(s,\xi)}$ is a lift of the (s,ξ) action. Furthermore,

$$\begin{aligned} \{\psi^{(s,\xi)}, \psi^{(t,\eta)}\} &= \{\phi^\xi + s, \phi^\eta + t\} = \{\phi^\xi, \phi^\eta\} \\ &= \sigma(\xi,\eta) + \phi^{[\xi,\eta]}. \end{aligned} \tag{2.19}$$

But

$$\begin{aligned} \psi^{[(s,\xi),(t,\eta)]} &= \psi^{(\sigma(\xi,\eta),[\xi,\eta])} \\ &= \sigma(\xi,\eta) + \phi^{[\xi,\eta]}. \end{aligned} \tag{2.20}$$

Hence,

$$\{\psi^{(s,\xi)}, \psi^{(t,\eta)}\} = \psi^{[(s,\xi),(t,\eta)]}. \tag{2.21}$$

This implies that $\psi : (s,\xi) \longmapsto \psi^{(s,\xi)}$ is a Lie algebra homomorphism.

Therefore, ψ defines a moment map Ψ for G^σ action on Y; in other words, the G^σ action on Y is Hamiltonian. Moreover, $\Psi(vp_0) \in (\mathfrak{g}^\sigma)^*$, and

$$< \Psi(vp_0), (s,\xi) >= \psi^{(s,\xi)}(vp_0) =< vx_0, \xi > +s, \tag{2.22}$$

so

$$\Psi(vp_0) = vx_0 + 1^*, \quad \text{where } 1^* \in \mathbf{R}^*, 1^*(s) = s.$$

If $v_1 x_0 = v_2 x_0$, then $v_1 p_0 = v_2 p_0$. Thus Ψ is 1-1. By **Proposition** 1.2.2, we have proved

Proposition 4.2.2. Y is a coadjoint orbit of G^σ.

4.3 Auslander-Kostant's treatment

To get a coadjoint orbit of G, we need to get rid of the 2-cocycle σ. For this purpose, we consider another G-homogeneous symplectic space (W, Ω_W) satisfying:

1) N acts on W trivially;

2) G acts on W symplectically with the infinitesimal lifting which has the obstruction $-\sigma$.

We form the product space $Y \times W$ with the product symplectic form. The first condition implies G acts on $Y \times W$ transitively. The second condition implies that W is a coadjoint orbit of the group extension $G^{-\sigma}$. When Y and W product together, the obstructions cancel out. Therefore we have

Proposition 4.3.1. $Y \times W$ is a covering space of some coadjoint orbit of G.

Proof: Let us denote by ϕ_1^ξ the lift of the infinitesimal ξ action on W. Set

$$\psi_1 : \mathfrak{g} \longrightarrow \mathrm{Poisson}(Y \times W),$$

$\psi_1(\xi) = \psi_1^\xi$, where

$$\psi_1^\xi(p, w) = \phi^\xi(p) + \phi_1^\xi(w). \tag{3.23}$$

Then $\xi \longmapsto \psi_1^\xi$ is a lift of the infinitesimal ξ action on $Y \times W$. Moreover,

$$\begin{aligned} \{\psi_1^\xi, \psi_1^\eta\}(p, w) &= \{\phi^\xi, \phi^\eta\}(p) + \{\phi_1^\xi, \phi_1^\eta\}(w) \\ &= \sigma(\xi, \eta) + \phi^{[\xi,\eta]} - \sigma(\xi, \eta) + \phi_1^{[\xi,\eta]} \\ &= \psi_1^{[\xi,\eta]}. \end{aligned}$$

Hence, ψ_1 is a Lie algebra homomorphism. Therefore, it gives rise to a moment map $\Psi_1 : \; Y \times W \longrightarrow \mathfrak{g}^*$. By Proposition 1.2.2, $Y \times W$ is a covering of some coadjoint orbit of G.

If we can choose W good enough so that the moment map Ψ_1 is 1-1, we then get a coadjoint orbit of G from Y. As matter of fact, such a "good" W always exists.

Let us discuss this in a little bit more generality. We consider now the situation of Chapter 3; namely, we no longer assume G fixes Y and consider the stabilizer subgroup $K = G_Y$. The problem now is to get a coadjoint orbit of K from Y. We take a coadjoint orbit X of G sitting over Y in the sense that $X|_{\mathfrak{n}} \supseteq Y$. Let

$$\pi : X \longrightarrow \mathfrak{n}^*$$

be the projection. It is actually the moment map for the N action on X. Denote by Y^- the space Y equipped with the negative symplectic form $-\Omega_Y$, let

$$\pi_1 : X \times Y^- \longrightarrow \mathfrak{n}^*$$

be the map $\pi_1(x,p) = \pi(x) - p$. It is the moment map for product action of N on $X \times Y^-$ with 0 as a regular value. By the Marsden-Weinstein reduction, we get a reduced space $W = \pi_1^{-1}(0)/N$. Since $N \subset K$ is normal, we can define a K action on W as follows:

$$a \cdot [x,p] = [ax, ap], \ \ a \in K, \ [x,p] \in W.$$

K actions on X and Y^- are symplectic. It follows that the K action on $X \times Y^-$ is symplectic. The latter action induces a symplectic K action on W.

We have the infinitesimal lifting $\phi : \mathfrak{k} \longrightarrow \text{Poisson}(Y)$. Let us now define a map

$$\phi_1 : \mathfrak{k} \longrightarrow \text{Poisson}(W),$$

$\phi_1(\xi) = \phi_1^\xi$, where

$$\phi_1^\xi[x,p] = < \xi, x >_{\mathfrak{g}} - \phi^\xi(p) \tag{3.24}$$

for any $[x,p] \in W$.

To see that ϕ_1 is well defined, we only need to check that

$$\phi_1^\xi[x,p_0] = \phi_1^\xi[x_1,p_0] \tag{3.25}$$

for $x, x_1 \in X, x|_{\mathfrak{n}} = p_0, x_1|_{\mathfrak{n}} = p_0$ and $[x,p_0] = [x_1,p_0] \in W$.

But $[x,p_0] = [x_1,p_0]$ implies that there exists $u \in N_{p_0}$ such that $x_1 = ux$. Hence,

$$\begin{aligned}
\phi_1^\xi[x_1,p_0] &= < \xi, x_1 >_{\mathfrak{g}} - \phi^\xi(p_0) \\
&= < \xi, x_1|_{\mathfrak{k}} >_{\mathfrak{k}} - \phi^\xi(p_0) \\
&= < \xi, (ux)|_{\mathfrak{k}} >_{\mathfrak{k}} - \phi^\xi(p_0) \\
&= < \xi, u(x|_{\mathfrak{k}}) >_{\mathfrak{k}} - \phi^\xi(p_0).
\end{aligned}$$

By **Lemma** 2.2.1, since N_{p_0} is connected, N_{p_0} fixes $x|_{\mathfrak{k}}$. Thus, we conclude

$$\phi_1^{\xi}[x_1, p_0] = <\xi, x|_{\mathfrak{k}}>_{\mathfrak{k}} - \phi^{\xi}(p_0) = \phi_1^{\xi}[x, p_0].$$

Proposition 4.3.1. ϕ_1^{ξ} is the lifting of the infinitesimal ξ action on W with the obstruction $-\sigma$.

Proof: Let

$$\phi_2^{\xi} : X \longrightarrow \mathbf{R}$$

$$\phi_2^{\xi}(x) = <\xi, x>_{\mathfrak{g}} . \tag{3.26}$$

It is well known that ϕ_2^{ξ} is the lift of the infinitesimal K action on X, which is Hamiltonian. There is no obstruction for ϕ_2^{ξ}.

First, we show that

$$\iota(\xi_W^{\sharp})\Omega_W = d\phi_1^{\xi}. \tag{3.27}$$

For any $\Xi \in \mathfrak{X}(W)$, a vector field on W, there exists a vector field $\Xi_1 \in \mathfrak{X}(\pi_1^{-1}(0)) \subset \mathfrak{X}(X \times Y^-)$ such that $j_{1*}(\Xi_1) = \Xi$, where $j_1 : \pi_1^{-1}(0) \longrightarrow W$ is the projection. We denote by i_1 the inclusion $\pi_1^{-1}(0) \longrightarrow X \times Y^-$. Now for any $(x, p) \in \pi_1^{-1}(0)$,

$$\Xi_1(x, p) = \Xi_1'(x) \oplus \Xi_1''(p),$$

where $\Xi_1'(x) \in T_xX$ and $\Xi_1''(p) \in T_pY$. So

$$\begin{aligned}
d\phi_1^{\xi}(\Xi)[x,p] &= d\phi_1^{\xi}(j_{1*}(\Xi_1))[x,p] \\
&= j_{1*}d\phi_1^{\xi}(\Xi_1)(x,p) \\
&= d(j_1^*\phi_1^{\xi})(\Xi_1)(x,p) \\
&= \Xi_1(j_1^*\phi_1^{\xi})(x,p) \\
&= \Xi_1'(\phi_2^{\xi})(x) - \Xi_1''(\phi^{\xi})(p) \\
&= d\phi_2^{\xi}(\Xi_1')(x) - d\phi^{\xi}(\Xi_1'')(p). \qquad (3.28)
\end{aligned}$$

On the other hand,

$$\iota(\xi_W^{\sharp})\Omega_W(\Xi)[x,p] = \Omega_W(\xi_W^{\sharp}, \Xi)[x,p]$$

$$\begin{aligned} &= \Omega_W(j_{1*}\xi^\sharp_{X\times Y^-}, \Xi_1)[x,p] \\ &= i^*\Omega_{X\times Y^-}(\xi^\sharp_{X\times Y^-}, \Xi_1)(x,p) \\ &= \Omega_X(\xi^\sharp_X, \Xi'_1)(x) - \Omega_Y(\xi^\sharp_Y, \Xi''_1)(p). \end{aligned} \tag{3.29}$$

Comparing (3.28) and (3.29) we see that (3.27) holds.

To compute the obstruction,

$$\begin{aligned} \{\phi_1^\xi, \phi_1^\eta\}[x,p] &= \{\phi_2^\xi, \phi_2^\eta\}(x) - \{\phi^\xi, \phi^\eta\}(p) \\ &= \phi_2^{[\xi,\eta]}(x) - \phi^{[\xi,\eta]}(p) - \sigma(\xi,\eta) \\ &= \phi_1^{[\xi,\eta]}[x,p] - \sigma(\xi,\eta). \end{aligned} \tag{3.30}$$

Therefore, the obstruction for ϕ_1 is $-\sigma$.

Corollary 4.3.1. W is a coadjoint orbit of the central extension $K^{-\sigma}$ of K with respect to T.

Our W now satisfies two conditions we set before. Also we have

Lemma 4.3.2. K acts on W transitively.

Proof: For any two points $[x_1, p_0], [x_2, p_0] \in W$, we may find $g \in G$ such that $g \cdot x_1 = x_2$. It follows that $g \cdot p_0 = p_0$ so that $g \in K$. But then $a \cdot [x_1, p_0] = [x_2, p_0]$.

We now have the material we need to get a coadjoint orbit of K.

Proposition 4.3.2. $Y \times W$ is a coadjoint orbit of K.

Proof: It only remains to show that the moment map

$$\Psi_1 : Y \times W \longrightarrow \mathfrak{k}^*$$

$$\Psi_1(vp_0, [x, p_0]) = vx_0 + x|_{\mathfrak{k}} - x_0, \tag{3.31}$$

where x_0 is a point in $\mathfrak{k}^*$ sitting over $p_0 \in Y$, is 1-1.

We claim that this moment map can be rewritten as

$$\Psi_1(vp_0, [x, p_0]) = vx|_{\mathfrak{k}}. \tag{3.32}$$

To see this, we need to show

$$vx_0 + x|_{\mathfrak{k}} - x_0 = vx|_{\mathfrak{k}}, \tag{3.33}$$

in other words,

$$v(x|_{\mathfrak{k}} - x_0) = x|_{\mathfrak{k}} - x_0, \quad \text{for any } v \in N. \tag{3.34}$$

Note that

$$(x|_{\mathfrak{k}} - x_0)|_{\mathfrak{n}} = p_0 - p_0 = 0.$$

In fact, we claim that for any $w \in \mathfrak{k}^*$, $w|_{\mathfrak{n}} = 0$, then $v \cdot w = w$ on $\mathfrak{k}$. Since N is connected, we only need to consider it at the algebra level. For any $\delta \in \mathfrak{n}$, $\eta \in \mathfrak{k}$,

$$< \delta \cdot w, \eta >_{\mathfrak{k}} = - < w, [\delta, \eta] > = 0$$

since for $[\delta, \eta] \in \mathfrak{n}$.

Hence, $\delta \cdot w = 0$.

We now show this moment map is 1-1. Suppose $v_1 x_1|_{\mathfrak{k}} = v_2 x_2|_{\mathfrak{k}}$, then $v_1 p_0 = v_2 p_0$ since for x_1 and x_2 all sit over p_0. It follows that $v_2^{-1} v_1 \in N_{p_0}$. Since $v_2^{-1} v_1 x_1|_{\mathfrak{k}} = x_2|_{\mathfrak{k}}$,by **Lemma 2.2.1**, there exists an element $v \in N_{p_0}$ such that $v v_2^{-1} v_1 x_1 = x_2$. Let $u = v v_2^{-1} v_1 \in N_{p_0}$, we conclude that

$$(v_1 p_0, [x_1, p_0]) = (v_2 p_0, [u x_1, u p_0]) = (v_2 p_0, [x_2, p_0]).$$

So Ψ_1 is 1-1. By **Proposition 1.2.2**, $Y \times W$ is a coadjoint orbit of K.

Remark 4.3.1. Our method here reflects at the classical level Auslander-Kostant's treatment to the Mackey obstruction in their paper [2] on representation theory of solvable Lie groups.

Chapter 5

Geometric Quantization and Induced Representations

5.1 Prequantization

From now on, we would like to apply our inductive structure of coadjoint orbits to the group representation theory to get the corresponding quantum picture by using the geometric quantization method. For this we review some basic facts about the geometric quantization. Let us start with the prequantization. We copy down some important facts from [14].

Let (M,ω) be a symplectic manifolds, $L \xrightarrow{\pi} M$ be a line bundle over M, and S be the set of the smooth sections of L. Let ∇ be a connection on L, namely a linear map

$$\nabla : \ \mathfrak{X}(M) \longrightarrow \text{End}\, S, \quad \text{where} \quad v \longmapsto \nabla_v,$$

such that for any $f \in C^\infty(M)$ one has

$$\nabla_{fv} = f\nabla_v \tag{1.1}$$

and for any $s \in S$

$$\nabla_v fs = (vf)s + f\nabla_v s. \tag{1.2}$$

We observe that if $U \subseteq M$ is an open set and $s \in S(U)$ is a nowhere vanishing section on U then one could associate to s a 1-form $\alpha(s) \in \Omega^1(U)$ so that

$$(1.3) \qquad \nabla_v s = 2\pi i < \alpha, v > s, \qquad \text{for all } v \in \mathfrak{X}(M).$$

Let us denote by $S^\times(U)$ the set of the nowhere vanishing smooth sections over U and $L^\times$ the open subset of L given by $L^\times = \cup L_m^\times$ over all $m \in M$ where $L_m^\times$ is $L_m - 0$. Let $\tilde{\pi} = \pi|_{L^\times}$. Then $L^\times$ is a $\mathbf{C}^\times$ bundle over M. By multiplication, $\mathbf{C}^\times$ acts as a group of diffeomorphisms of $L^\times$ where the orbits are just the fibers $L_m^\times$. The 1-form $\frac{1}{2\pi i}\frac{dz}{z}$ on $\mathbf{C}^\times$ is invariant under the multiplication by $\mathbf{C}^\times$. Denote by $\tau : \mathbf{C}^\times \to L_m^\times$ the map given by the action. Then there exists a unique 1-form β_m on $L_m^\times$ such that $\tau^*(\beta_m) = \frac{1}{2\pi i}\frac{dz}{z}$.

Definition 5.1.1. A connection 1-form on $L^\times$ is a 1-form $\alpha \in \Omega^1(L^\times)$ such that

1) α is invariant under $\mathbf{C}^\times$;

2) For all $m \in M$ one has $\alpha|_{L_m^\times} = \beta_m$.

We have

Proposition 5.1.1. Let M be a manifold and let L be a line bundle over M. If ∇ is a connection in L, there exists a unique connection form $\alpha \in \Omega^1(L^\times)$ such that for all open $U \subseteq M$ and all $s \in S^\times(U)$ one has

$$(1.4) \qquad \alpha(s) = s^*(\alpha).$$

Conversely if α is a connection 1-form on $L^\times$ then there is a unique connection ∇ on L such that (1.4) is satisfied for all $s \in S^\times(U)$ and all open $U \subseteq M$.

Therefore, equivalent to the line bundle with connection (L, ∇), we may say the line bundle with the connection 1-form (L, α). For such a line bundle we have

Proposition 5.1.2. Let (L, α) be a line bundle with connection over M. Then there exists a unique 2-form $\Omega \in \Omega^2(M)$ such that

$$(1.5) \qquad d\alpha = \tilde{\pi}^*\Omega.$$

Moreover, if $U \subseteq M$ is open and $s \in S^{\times}(U)$ is arbitrary then

$$d\alpha(s) = \Omega|_U. \tag{1.6}$$

Definition 5.1.2. The 2-form Ω is called the *curvature* of (L, α) and is written

$$\Omega = \mathrm{curv}(L, \alpha).$$

Lemma 5.1.1. If $\mathrm{curv}(L, \alpha) = 0$ and if M is simply connected, then the parallel translation is independent of path.

We now consider the metric on L.

Definition 5.1.3. A Hermitian structure on L is a function H on the set of all $(x, y) \in L \times L$ where $\pi(x) = \pi(y)$ such that

1) H induces a 1-dimensional Hilbert space structure on L_m for all $m \in M$ and

2) one has $|H|^2 \in C^{\times}(L^{\times})$ where $|H|^2$ is the positive-valued function on $L^{\times}$ defined by $|H|^2(x) = H(x, x)$.

We simply write (x, y) for $H(x, y)$. If α is a connection form on $L^{\times}$, H is called α-invariant if

$$v(s, t) = (\nabla_v s, t) + (s, \nabla_v t) \tag{1.7}$$

for all $s, t \in S$ and all $v \in \mathfrak{X}(M)$. We have

Lemma 5.1.2. (L, α) has an invariant Hermitian structure if and only if $2\pi i(\alpha - \overline{\alpha})$ is exact.

We say that two line bundles with connections are isomorphic if there is a vector bundle isomorphism between them which preserves the connection forms. In this sense we may talk about the equivalent classes of line bundles with connections, $\iota = [(L, \alpha)]$. Obviously, the curvature is fixed for all members in the same equivalent class. We denote by $\mathfrak{L}_c(M, \Omega)$ the set of equivalent classes of the line bundles with connections whose curvatures are Ω.

Now we look at the symplectic form ω on M. Note that $[\omega] \in \mathrm{H}^2(X, \mathbf{R})$. There is a natural inclusion $\mathrm{H}^2(X, \mathbf{Z}) \to \mathrm{H}^2(X, \mathbf{R})$.

Definition 5.1.4. ω is called *integral* if $[\omega]$ is in the image of $\mathrm{H}^2(X, \mathbf{Z})$.

We have

Proposition 5.1.3. $\mathfrak{L}_c(M, \omega)$ is not empty if and only if ω is integral.

In the case that ω is integral and $\iota = [(L, \alpha)] \in \mathfrak{L}_c(M, \omega)$, let R be the set of the smooth functions on M, we have a well defined map

$$\begin{aligned} \delta: \quad R &\longrightarrow \mathrm{End}\, S, \\ \delta(f)s &= (\nabla_{\xi_f} + 2\pi i f)s, \end{aligned} \tag{1.8}$$

where ξ_f is the Hamiltonian vector field of f.

Proposition 5.1.4. The map δ is a representation of R on S.

Definition 5.1.5. The representation δ is called *prequantization.*

Now we focus on the special case of coadjoint orbit. Take X to be a coadjoint orbit of some Lie group G. X has a canonical symplectic form $\omega = \omega_X$. One would ask that when this ω is integral. One beautiful answer given by Kostant is the following:

Theorem 5.1.1. Let $G_p^\sharp$ be the set of the characters χ of the stabilizer subgroup G_p of some point $p \in X$ satisfying $d\chi = 2\pi\sqrt{-1}p|_{\mathfrak{g}_p}$. Then ω is integral if and only if $G_p^\sharp$ is not empty.

Practically, this theorem offers us a very effective criterion to tell whether an orbit is integral. One could see this point later.

5.2 Polarization and Quantization

The prequantization gives us a representation of R, which might induces a representation of the group if we start with a coadjoint orbit. The problem is this representation space is "too big". We need to "cut" half of it. This idea originated from physics. From group representation point of view, this representation space is too big to give the irreducible representation. For this surgery we need to introduce polarizations.

Definition 5.2.1. A *polarization* of a symplectic manifold (M, ω) is a map F which assigns to each point $m \in M$ a subspace $F_m \subset (T_m)^{\mathbf{C}}$ of the complexified tangent space at m, satisfying:

1) F is involutary and smooth;

2) F is maximally isotropic with respect to the extension of ω to $(TM)^{\mathbf{C}}$;

3) For each $m \in M$, $D_m^{\mathbf{C}} = F_m \cap \overline{F}_m$ has constant dimension k.

Two extreme but very important examples of the polarizations are the following.

Example 5.2.1. $k = n = \dim M$. In this case $F = \overline{F}$ and the polarization is called real polarization. We only need to consider TM other than $(TM)^{\mathbf{C}}$. A real polarization is then a Lagrangian foliation on M. One example is the cotagent bundle TM. We may choose fiber foliation as a real polarization. This is known as the *vertical polarization.*

Example 5.2.2. Another extreme case is $k = 0$. Let us look at a Kähler manifold (M, ω, J). It has a natural polarization defined by

$$F_m = \{v_m \in (TM)^{\mathbf{C}} \mid J_m v = \sqrt{-1} v_m\}, \quad m \in M. \tag{2.9}$$

In the local complex coordinates $\{z^a\}$ F_m is the linear span of the set $\{\frac{\partial}{\partial \bar{z}^a}\}$ of anti-holomorphic coordinate vectors at m. In this case $F_m \cap \overline{F}_m = \{0\}$. A polarization with this property is called a *Kähler polarization.*

Given a polarization F, we have a real distribution E such that $F + \overline{F} = E^{\mathbf{C}}$. Remember another distribution D we difined above, we have

$$E_m = \{v \in T_m M | \omega(v, u) = 0, \text{ for all } u \in D_m\} \tag{2.10}$$

and

$$D_m = \{u \in T_m M | \omega(u, v) = 0, \text{ for all } v \in E_m\}. \tag{2.11}$$

The involutivity of F implies that D is an involutive distribution so that D defines a foliation of M. We denote by M/D the space of all leaves and by $\pi_D : M \to M/D$ the canonical projection.

Definition 5.2.2. A polarization F is said to be *geometrically admissible* if
1) E is involutive;
2) the leaves of D are simply connected;
3) The spaces M/D and M/E of leaves of D and E, respectively, are quotient manifolds of M and the canonical projection $\pi_{ED} : M/D \to M/E$ is a submersion.

Let F be a geometrically admissible polarization. We denote by $\mathfrak{B}^F$ the frame bundle of F in the sense that the fiber $\mathfrak{B}^F_m$ over each point $m \in M$ is the set of ordered basis of the complex vector space F_m. This is a principal $GL(n, \mathbf{C})$ bundle. There is a unique double covering $\tau : ML(n, \mathbf{C}) \to GL(n, \mathbf{C})$ where the covering group $ML(n, \mathbf{C})$ is called the *complex metalinear group.* Therefore, there exists a principal $ML(n, \mathbf{C})$ bundle $\tilde{\mathfrak{B}}^F_m$ which double covers $\mathfrak{B}^F_m$, so that the diagram

$$\begin{array}{ccc} \tilde{\mathfrak{B}}^F_m \times ML(n,\mathbf{C}) & \longrightarrow & \tilde{\mathfrak{B}}^F_m \\ \downarrow & & \downarrow \\ \mathfrak{B}^F_m \times GL(n,\mathbf{C}) & \longrightarrow & \mathfrak{B}^F_m \end{array}$$

commutes, where the vertical maps are given by the double coverings and the horizontal maps by the group actions. An element of $\tilde{\mathfrak{B}}^F_m$ is call a *metalinear frame* of F_m. Thus, we have a pricipal $ML(n,\mathbf{C})$ bundle $\tilde{\mathfrak{B}}^F = \cup \tilde{\mathfrak{B}}^F_m$. A section $\sigma : U \to \tilde{\mathfrak{B}}^F$ over an open set $U \subseteq M$ is said to be *Hamiltonian* if there exist $\phi_1, \cdots, \phi_n \in C^\infty(U)$ such that $\mu\sigma(m) = (\xi_{\phi_1}(m), \cdots, \xi_{\phi_n}(m))$ for all $m \in U$, where $\mu : \tilde{\mathfrak{B}}^F_m \to \mathfrak{B}^F_m$ is the covering map.

Let $\chi : ML(n,\mathbf{C}) \to \mathbf{C}$ be the unique holomorphic function such that

1) $\chi(g)^2 = \det \tau(g)$,

2) $\chi(1) = 1$

for all $g \in ML(n,\mathbf{C})$. Let $ML(n,\mathbf{C})$ act on $\mathbf{C}$ by multiplying $\chi(a)$, and associate it to the principal $ML(n,\mathbf{C})$ bundle $\tilde{\mathfrak{B}}^F \to M$, we get a complex line bundle L^F, whose fiber L^F_m is isomorphic to the space of the complex-valued functions v on $\tilde{\mathfrak{B}}^F_m$ such that

$$v(\tilde{b}g) = \chi(g^{-1})v(\tilde{b})$$

for all metalinear frames $\tilde{b}$ and all $g \in ML(n,\mathbf{C})$. L^F is called the *complex line bundle of half-F-forms*. A C^∞ section $\nu : M \to L^F$ is called a *half-F-form* and the space of all such sections is denoted by $\Gamma(L^F)$. A half-F-form τ is said to be *Hamiltonian* at $m \in M$ if there exists a Hamiltonian section σ of metalinear bundle $\tilde{\mathfrak{B}}^F$ on a neighbourhood U of m such that $\nu_p(\sigma(p)) = 1$ for all $p \in U$. There is a unique connection ∇ on L^F such that $(\nabla_\xi \nu)(m) = 0$ for all $\xi \in F$ if ν is Hamiltonian at m.

In terms of the symplectic form ω we may identify E_m with the space of all complex-valued linear functions on T_mM/D, which is denoted by $(T_mM/D)^*_{\mathbf{C}}$.

Let (M,ω,L) be a prequantization, F be a geometrically admissiable polarization on M, and L^F be the bundle of half-F forms over M.

Definition 5.2.3. A polarization F is said to be *positive* if

$$-\omega(\xi,\bar{\xi}) \geq 0 \text{ for all } \xi \in F. \tag{2.12}$$

We now suppose in addition that F is positive. We consider the line bundle $L \otimes L^F$. There is a unique connection on $L \otimes L^F$ such that

$$\nabla_\xi(s \otimes \nu) = (\nabla_\xi s) \otimes \nu + s \otimes \nabla_\xi \nu \tag{2.13}$$

for all $s \in \Gamma(L)$ and all $\nu \in \Gamma(L^F)$.

Write

$$\Gamma^F = \psi \in \Gamma(L \otimes L^F);\ \ \nabla_\xi \psi = 0 \text{ for all } \xi \in F.$$

Each pair of the wave functions $\psi_1 = s_1 \otimes \nu_1, \psi_2 = s_2 \otimes \nu_2$ in Γ^F defines a density on M/D. Explicitly, let $m \in M$ and let $\tilde{b} \in \tilde{\mathfrak{B}}^F_m$ be a metaframe at m such that $\mu(\tilde{b}) = (\xi_1, \cdots, \xi_n) \in \mathfrak{B}^F_m$ aand $\xi_1, \cdots, \xi_k$ is a basis for D_m. Choose $\zeta_1, \cdots, \zeta_n \in T_m M^{\mathbf{C}}$ so that $\{\xi_1, \cdots, \xi_n, \zeta_1, \cdots, \zeta_n\}$ is a symplectic basis. Then $c = \{\pi_{D*}\xi_{k+1}, \cdots, \pi_{D*}\xi_n, \pi_{D*}\zeta_1, \cdots, \pi_{D*}\zeta_n\}$ will be a basis for $(T_x M/D)^{\mathbf{C}}$ at $x = \pi_D(m)$. We define

$$\begin{aligned} &(\psi_1, \psi_2)(x, c) \\ = \ &< s_1, s_2 > (m) |\omega^{n-k}(\xi_{k+1}, \bar{\xi}_{k+1}, \cdots, \xi_n, \bar{\xi}_n)|^{\frac{1}{2}} \nu_1(\tilde{b}) \nu_2(\tilde{b}). \end{aligned} \tag{2.14}$$

It can be checked that (ψ_1, ψ_2) is a well defined density on M/D. We consider

$$\mathcal{H}^F_0 = \{\psi \in \Gamma^F | \int_{M/D} (\psi, \psi) < \infty\}.$$

Then we can define an inner product on $\mathcal{H}^F_0$ as the following:

$$< \psi_1, \psi_2 >= \int_{M/D} (\psi_1, \psi_2) \tag{2.15}$$

$\mathcal{H}^F_0$ is a pre-Hilbert space under this inner product. Denote by $\mathcal{H}^F$ the Hilbert space obtained by completion of $\mathcal{H}^F_0$. This is the quantized space. We call it *quantization.*

Remark 5.2.1. The quantizable functions in $C^\infty(M)$ are those whose Hamiltonian vector fields preserve the polarization F under Lie derivative. When there is a Hamiltonian group action on M preserving the polarization, all lifting of the infinitesimal action are quantizable. In this case, the quantized space $\mathcal{H}^F$ carries a representation of the Lie algebra.

An equivalent definition of the geometric quantization can be obtained by using the half densities instead of the half forms.

Definition 5.2.4. Let V be an n-dimensional vector space over $\mathbf{R}$. Let $\alpha > 0$. Then $|\Lambda^n|^\alpha V$ is the vector space of all $\mathbf{C}$-valued functions ν defined on the space $\mathfrak{B}(V)$ of the ordered bases of V such that

$$\nu(v_1, \cdots, v_n) = |\det(a_{ij})|^\alpha \nu(w_1, \cdots, w_n) \tag{2.16}$$

whenever $(v_1, \cdots, v_n)$, $(w_1, \cdots, w_n) \in \mathfrak{B}$ and satisfy

$$w_i = \sum_{j=1}^{n} a_{ji} v_j, \qquad i = 1, \cdots, n. \tag{2.17}$$

For the cotangent bundle $T^*M \to M$ of any n-dimensional manifold M, we may then define the bundle of α-densities $|\Lambda^n|^\alpha T^*M$ over M by asking the fiber over $m \in M$ to be the space $|\Lambda^n|^\alpha T^*_m M$. When $\alpha = 1$, we say simply densities. It is easy to observe that for any two sections ν_1, $\nu_2 \in \Gamma|\Lambda^n|^{\frac{1}{2}} T^*M$, the product $\nu_1\nu_2$ is a section of the bundle of densities. Therefore we can use the half densities instead of the half forms to define the integration. More explicitly, let F, D and E be as before, under the assumption that all D-leaves are connected and simply connected, any polarized section of $L \to M$ corresponds to a section of the reduced bundle $L/D \to M/D$. Then as we have done by using the half forms, we may define a pre-Hilbert space structure on the space of all compactly supported sections of the line bundle $|\Lambda^n|^{\frac{1}{2}} T^*(M/D) \otimes L/D$. This space naturally corresponds bijectively to the space of all sections of $|\Lambda^n|^{\frac{1}{2}} (TM/D)^* \otimes L$ covariant constant along the leaves of D and with compact support modulo D. Therefore, we may turn the latter into a pre-Hilbert space whose completion $\mathcal{H}^F$ is the quantized space. Readers are refered to [3] for the details.

5.3 Induced Representations

In this section we give an application of our inductive construction of coadjoint orbits on group representations in terms of the geometric quantization method. Let G be a connected Lie group, $N \subset G$ be a normal subgroup. We will consider the cases of G being nilpotent or solvable Lie groups. In these cases, N can be chosen to be some connected and simply connected nilpotent subgroups so that all N_p are connected for all $p \in \mathfrak{n}^*$ and $exp : \mathfrak{n} \to N$ are diffeomorphic. Recall that, using our notations before, a coadjoint orbit X of G is symplectomorphic to the reduced space $X_1 = \Psi^{-1}(0)/K$. We denote by $B = \Psi_1(Y \times W)$ the image of $Y \times W$ under the moment map $\Psi_1 : Y \times W \to \mathfrak{k}^*$, it is a coadjoint orbit of K. In the case that the quantizations of B and X give rise to the unitary representations of K and G respectively, we want to describe the relation between these two representations. Let us start with the prequantization.

Let $x_0 \in X$ sit over $b_0 \in B$ and $b_0 \in B$ sit over $p_0 \in Y$. By Theorem 5.1.1, to see whether the prequantizations exist for X and B, we only need to see whether $G^\sharp_{x_0}$ and $K^\sharp_{b_0}$ are not empty. In fact, we have the following result.

Proposition 5.3.1. $G^\sharp_{x_0}$ is not empty if and only if $K^\sharp_{b_0}$ is not empty.

Proof: First of all, we observe that $G_{x_0} \subset K_{b_0}$. Indeed, for any $g \in G_{x_0}$, $gx_0 = x_0$ implies $gp_0 = p_0$. Hence, $g \in K$. Then by $x_0|_{\mathfrak{k}} = b_0$ we get that $gb_0 = b_0$. Therefore, if $K^\sharp_{b_0} \neq \emptyset$, we may pick $\chi_1 \in K^\sharp_{b_0}$, and let $\chi = \chi_1|_{G_{x_0}}$. It is straightforward to see that χ is a character of G_{x_0} and

$$d\chi = 2\pi\sqrt{-1}\chi|_{\mathfrak{g}_{x_0}}.$$

It means that $\chi \in G^\sharp_{x_0}$. Conversely, if $\chi \in G^\sharp_{x_0}$, we want to construct a character $\chi_1 \in K^\sharp_{b_0}$ from χ. For this, we notice that according our assumption, N is simply connected and nilpotent so that the exponential map $exp : \mathfrak{n} \to N$ is a diffeomorphism. It follows that $N^\sharp_{p_0}$ is nonempty and contains a unique character. We denote by χ_2 this character. Note that $N_{x_0} \subset N_{p_0}$ is again simply connected and nilpotent, χ and χ_2 agree on $\mathfrak{n}_{x_0}$ by the defining properties, these two characters also agree on

N_{x_0}. For any $a \in K_{b_0}^{\sharp}$, we have $ab_0 = b_0$. This implies that $ax_0|_{\mathfrak{k}} = x_0|_{\mathfrak{k}}$. Thus, by Corollary 2.2.1, there exists $v \in N_{p_0}$ such that $ax_0 = vx_0$. It follows that $a = vg_a$ for some $g_a \in G_{x_0}$. We define

$$\begin{aligned} \chi_1 : K_{b_0} &\longrightarrow \mathbf{C} \\ \chi_1(a) &= \chi_2(v)\chi(g_a). \end{aligned} \tag{3.18}$$

(3.18) is well defined. Indeed, if $a = vg_a = v'g'_a$, then $v'^{-1}v = g'_a g_a^{-1}$. This implies $v'^{-1}v = g'_a g_a^{-1} \in N_{p_0} \cap G_{x_0} = N_{x_0}$. Therefore,

$$\chi(g'_a g_a^{-1}) = \chi_2(v'^{-1}v).$$

It follows that

$$\chi(g'_a)\chi(g_a^{-1}) = \chi_2(v'^{-1})\chi_2(v).$$

This is equivalent to

$$\chi_2(v)\chi(g_a) = \chi_2(v')\chi(g'_a).$$

Hence, $\chi_1(a)$ does not depend on the choice of v and g_a.

To see χ_1 is a character, we need to show that

$$\chi_1(aa') = \chi_1(a)\chi_1(a'), \qquad \text{for all } a, a' \in K_{b_0}. \tag{3.19}$$

Suppose $a = vg_a$, $a' = v'g_{a'}$, then

$$\begin{aligned} \chi_1(aa') &= \chi_1(vg_a v' g_a^{-1} g_a g_{a'}) \\ &= \chi_2(v)\chi_2(g_a v' g_a^{-1})\chi(g_a)\chi(g_{a'}). \end{aligned}$$

So it remains to show that

$$\chi_2(g_a v' g_a^{-1}) = \chi_2(v'). \tag{3.20}$$

Since $v' = exp\zeta$ for some $\zeta \in \mathfrak{n}_{p_0}$, and

$$g_a v' g_a = exp(g_a \zeta),$$

we have

$$\begin{aligned}
\chi_2(g_a v' g_a) &= \chi_2(exp(g_a\zeta)) \\
&= exp(< 2\pi\sqrt{-1}p_0, g_a\zeta >) \\
&= exp(2\pi\sqrt{-1} < x_0, g_a\zeta >) \\
&= exp(2\pi\sqrt{-1} < g_a^{-1}x_0, \zeta >) \\
&= exp(2\pi\sqrt{-1} < x_0, \zeta >) \\
&= exp(2\pi\sqrt{-1} < p_0, \zeta >) \\
&= \chi_2(v').
\end{aligned}$$

To see that $\chi_1 \in K^\sharp_{b_0}$, we need to show

$$d\chi_1 = 2\pi\sqrt{-1}b_0|_{\mathfrak{k}_{b_0}}. \tag{3.21}$$

Indeed, for any $\xi \in \mathfrak{k}_{b_0}$, we may write

$$\xi = \zeta + \eta, \qquad \zeta \in \mathfrak{n}_{p_0},\ \eta \in \mathfrak{g}_{x_0} = \mathfrak{k}_{x_0} \subset \mathfrak{k}_{b_0}.$$

Hence,

$$\begin{aligned}
< d\chi_1, \xi > &= < d\chi_2, \zeta > + < d\chi, \eta > \\
&= 2\pi\sqrt{-1}(< p_0, \zeta > + < x_0, \eta >) \\
&= 2\pi\sqrt{-1}(< b_0, \zeta > + < b_0, \eta >) \\
&= 2\pi\sqrt{-1}(< b_0, \zeta + \eta >) \\
&= 2\pi\sqrt{-1}(< b_0, \xi >). \qquad \square
\end{aligned}$$

We suppose now X is integral so that B is also integral. Let F^0 be a K-invariant polarization on B. Since G acts on B trivially, F^0 is also G-invariant. Take the vertical polarization F^G on T^*G, which is $G \times K$-invariant, we get a $G \times K$-invariant polarization $\tilde{F}$ on the product space $M = T^*G \times B$. The reduction of $\tilde{F} \cap (T\Psi^{-1}(0))^{\mathbf{c}}$ gives rise to a G-invariant polarization F on X. Let $\mathcal{H}_0$ and $\mathcal{H}^F$ be the corresponding quantized spaces of (B, F^0) and (X, F) respectively. Suppose $\mathcal{H}_0$ carries a unitary representation of K and $\mathcal{H}^F$ carries a unitary representation of G. Our main result of this section is the following theorem.

Theorem 5.3.1. $\mathcal{H}^F$ is equivalent to the induced representation of $\mathcal{H}_0$ from K to G.

For the proof of this theorem, we need to introduce the ***bundle of little group orbits.*** Let G, N, X, Y, K, W, and B be as before. We construct a fiber bundle over the space G/K as follows. For any $[g] = gK \in G/K$, let us take $p_0 \in Y$, then $g \cdot Y$ is a coadjoint orbit of N through the point $p = gp_0$. We denote by $Y_{[g]}$ this orbit. For any $b \in B$, we define $gb \in \mathfrak{g}^*_{Y_{[g]}}$ by

$$< gb, \eta > = < b, g^{-1}\eta >, \qquad \text{for any } \eta \in \mathfrak{k}. \tag{3.22}$$

It is well defined since for any $d \in K^{[g]} = G_{Y_{[g]}}$, $g^{-1}dg \in K$, so $g^{-1}\eta \in \mathfrak{k}$. Let $\mathcal{B}_{[g]} = gB$. It is easy to see that $\mathcal{B}_{[g]}$ is a coadjoint orbit of $K^{[g]}$. Actually it is a copy of B. $\mathcal{B}_{[e]} = B$. We can construct a fiber bundle $\mathcal{B} \xrightarrow{\rho_1} G/K$ with the fiber over $[g]$ being $\mathcal{B}_{[g]}$. This is associated to the principal K-bundle $G \to G/K$ with the typical fiber B. G acts on $\mathcal{B}_{[g]}$ as transformations, carrying F^0 to $F^{[g]}$, a $K^{[g]}$-invariant polarization on $\mathcal{B}_{[g]}$. If B is integral, each of all $\mathcal{B}_{[g]}$ is integral. The quantized space $\mathcal{H}_{[g]}$ carries a unitary representation of $K^{[g]}$. We can therefore construct a Hilbert bundle $\mathcal{H} \to G/K$ such that the fiber over $[g]$ is $\mathcal{H}_{[g]}$. G acts on this bundle unitarily. Therefore, taking a volume form on G/K which will be specified next, the completion of the space of the square integrable smooth sections of $\mathcal{H}$, $\Gamma_0\mathcal{H}$, carries a unitary representation of G.

Define the map $\rho_2 : X \to \mathcal{B}$ as follows. For any $x = [g, \beta, q, w] \in \Psi^{-1}(0)/K = X$, let

$$\rho_2(x) = g(q, w) \in \mathcal{B}_{[g]}. \tag{3.23}$$

It is easy to check that ρ_2 is well defined. We define another map

$$\begin{aligned} \rho_3 : X &\longrightarrow G/K \\ \rho_3[g, \beta, q, w] &= [g]. \end{aligned} \tag{3.24}$$

This map is also well defined, and

$$\rho_1 \circ \rho_2 = \rho_3.$$

Let us now look at the relation between F and F^0. Recall the reconstruction of X from the little group data. $x \in X$ corresponds to $[e, x, b] \in X_1$ where $b \in B$ and $x|_{\mathfrak{k}} = b$. Let $\sigma : \mathfrak{g}^* \to \mathfrak{k}^*$ be the natural projection. $\sigma(x) = b$. Note that $F \subset TX^{\mathbf{C}} \subset (T\mathfrak{g}^*)^{\mathbf{C}}$ and $F^0 \subset TB^{\mathbf{C}} \subset (T\mathfrak{k}^*)^{\mathbf{C}}$, F_x and F_b^0 can be regarded as subspaces of $(T_x\mathfrak{g}^*)^{\mathbf{C}}$ and $(T_b\mathfrak{k}^*)^{\mathbf{C}}$ respectively. We have the following result.

Lemma 5.3.1. $F_x = \sigma_*^{-1}(F_b^0)$.

Proof: By the definition,

$$F_x = ((F^G_{(e,x)} \oplus F_b^0) \cap T_{(e,x,b)}\Psi^{-1}(0))/K.$$

Let us write F_x for both $\Phi_{*x}(F_x) \subset T_xX$ and $F_x \subset T_{[e,x,b]}X_1$. We have

$$\begin{aligned} &((F^G_{(e,x)} \oplus F_b^0) \cap T_{(e,x,b)}\Psi^{-1}(0)) = ((F^G_{(e,x)} \oplus F_b^0) \cap \Psi^{-1}_{*(e,x,b)}(0) \\ = \; & \{(z_1, z_2),\ z_1 \in F^G_{(e,x)} \subset (T_x\mathfrak{g}^*)^{\mathbf{C}},\ z_2 \in F_b^0,\ \sigma_*(z_1) = z_2\}. \end{aligned}$$

Let $j : \Psi^{-1}(0) \to X_1$ be the natural projection. Then we have a map

$$\begin{aligned} \sigma \circ \Phi \circ j : \Psi^{-1}(0) &\longrightarrow \mathfrak{k}^* \\ (g, \beta, b_1) &\longmapsto (Ad_g^*\beta)|_{\mathfrak{k}}. \end{aligned}$$

This implies that $F_x \subset \sigma_*^{-1}(F_b^0)$. Since the both sides have the dimension $\dim(G/K) + \frac{1}{2}\dim B$, they are evntually the same. $\square$

Remark 5.3.1. In general, if we think of a point $x' = [g, x, b] \in X$, $\rho_2(x') = g \cdot b \in \mathcal{B}_{[g]}$, then $\rho_{2*}^{-1}(F^{[g]}) = F_{x'}$.

We are now ready to prove **Theorem 5.3.1**. First of all, let us look at the quantized space of $M = T^*G \times B$. Since the line bndle L^G over T^*G is a trivial bundle and the space of the polarized sections of this bundle with respect to the polarization F^G can be identified with the space of all smooth functions on T^*G which are constant on the fibers of T^*G, and the latter is identified with $C^\infty(G)$. Therefore, by definition and the basic properties of Hilbert space, if we choose dg to be a left-invariant measure on G, the quantized space $\tilde{\mathcal{H}}$ for M is

$$\tilde{\mathcal{H}} = \{\psi : G \longrightarrow \mathcal{H}_0 \mid \int_G < \psi(g), \psi(g) >_{\mathcal{H}_0} dg < \infty\}.$$

When one quantizes the action of G on M, one obtains a unitary representation of G on $\tilde{\mathcal{H}}$ given by

$$(g \cdot \psi)(g_1) = \psi(g^{-1} g_1). \tag{3.25}$$

Let $\tilde{\mathcal{V}}$ be the space of all smooth maps from G to $\mathcal{H}_0$. By pushing down the K-invariant polarized sections of the line bundle $\tilde{L}$ over M to X, one can get the reduced line bundle L over X and the polarized sections of L. In this way, one can obtain an one to one correspondence between the space of the polarized sections of $\tilde{L}$ which are polarized with respect to $\tilde{F}$ and the space of the polarized sections of L with respect to F (see, for example, [8]). On the other hand, the space $M/\tilde{D}$ is reduced to X/D and the space of all half densities of X/D is identified with the space of all half densities ν of $M/\tilde{D}$ satisfying (see [22])

$$a \cdot \nu = \det(Ad_K(a))^{-1/2}\nu, \qquad \text{for all } a \in K. \tag{3.26}$$

Therefore, we have a natural map

$$\{\psi \in \tilde{\mathcal{V}} \mid a \cdot \psi = \det(Ad_K(a))^{-1/2}\psi, \quad \text{for all } a \in K\} \longrightarrow \mathcal{H}^F. \tag{3.27}$$

By the definition of the half densities, one can easily calculate that

$$(a \cdot \psi)(g) = \det(Ad_G(a))^{-1/2} a(\psi(gh)). \tag{3.28}$$

The factor $\det(Ad_G(a))^{-1/2}$ is due to the fact that the left invariant measure dg on G is not invariant under the action of K, but transforms with $\det(Ad_G(a))$. Hence, we have the map

$$\{\psi \in \tilde{\mathcal{V}} \mid \psi(ga^{-1}) = \gamma(a)a(\psi(g)), \quad \text{for all } a \in K,\ g \in G\} \longrightarrow \mathcal{H}^F, \tag{3.29}$$

where the function γ on K is defined by

$$\gamma(a) = \sqrt{\frac{\det(Ad_K(a))}{\det(Ad_G(a))}}. \tag{3.30}$$

Denote by $\mathcal{V}$ the space of the left hand side of (3.29). We need to define a measure on $\mathcal{V}$ so that (3.29) is unitary. For this, we consider for $\psi, \phi \in \mathcal{V}$ the volume form $< \psi(g), \phi(g) > dg$ on G. Take a basis $\eta_1, \cdots, \eta_k$ of $\mathfrak{k}$, we have the infinitesimal generators $\eta_1^\sharp, \cdots, \eta_k^\sharp$ on G. We then contract the form $< \psi(g), \phi(g) > dg$ with $\eta_1^\sharp, \cdots, \eta_k^\sharp$ to obtain a form $\alpha =< \psi(g), \phi(g) > dg(\eta_1^\sharp, \cdots, \eta_k^\sharp)$ on G. Due to the defining property of $\mathcal{V}$, α is K-invariant, and thus is the pull-back of a volume form on G/K. Integration of this volume form gives the inner product of ψ and ϕ. This can be described in the following way.

We may define a strictly positive function ρ on G satisfying

$$\rho(ga^{-1}) = \gamma(a)^2 \rho(g), \qquad \text{for all } a \in K,\ g \in G. \tag{3.31}$$

We denote by $d\mu$ the volume form on G/K obtained from the volume form $\rho(g)dg$ on G in the procedure described above. We can then define

$$< \psi, \phi >= \int_{G/K} \frac{< \psi(g), \phi(g) >}{\rho(g)} d\mu. \tag{3.32}$$

From this we see immediately that the left hand side of (3.29) is nothing but $\mathrm{Ind}_K^G \mathcal{H}_0$. Thus, we have set up a G-equivariant map

$$\mathrm{Ind}_K^G \mathcal{H}_0 \longrightarrow \mathcal{H}^F. \tag{3.33}$$

We now want to show that this map is bijective. To do this, we will find the inverse map of (3.33). From the discuss above we see that the function γ is introduced in by using the half densities. So what we need to do is to find a map from the space of all polarized sections of L to the space of all square integrable smooth sections of the bundle over G/K whose fiber above $[g]$ is the space of all polarized sections of the line bundle $L^{[g]} \to \mathcal{B}_{[g]}$, translated from the line bundle $L^0 \to B$, with respect to the polarization $F^{[g]}$ so that it will give us the inverse map of (3.33) by combining the indentity of the half densities (3.26). To see this, we need to look at the structures of the leaves of the polarizations.

Lemma 5.3.2. F is a geometrically admissible polarization on X if F^0 is a geometrically admissible polarization on B.

Proof: We only need to examine the situation at $x \in X$. Let us denote by $\hat{P}$ the leaf of an involutive distribution P. By the definition, we have

$$F_x = ((F^G_{(e,x)} \oplus F^0_b) \cap (T_{(e,x,b)}\Psi^{-1}(0))^{\mathbf{C}}/K.$$

Since F^0 is K invariant, so is $\overline{F}^0$; and $\overline{F}^G = F^G$, it follows that

$$\overline{F} = (F^G \oplus \bar{F}^0) \cap (T\Psi^{-1}(0))^{\mathbf{C}}/K.$$

Hence,

$$F + \overline{F} = (F^G \oplus (F^0 + \overline{F}^0)) \cap (T\Psi^{-1}(0))^{\mathbf{C}}/K,$$

and

$$E = (F + \overline{F}) \cap TX = (F^G \oplus E^0) \cap T\Psi^{-1}(0)/K.$$

If E^0 is involutive, since $E_G = F^G \cap TX$ is involutive too, we conclude that $E^G \oplus E^0$ is involutive. The reduced space of the integral submanifold of $E^G \oplus E^0$ gives us an integral submanifold of E. Thus E is involutive. On the other hand, according to Weinstein [23], $X = \Psi^{-1}(0)/K$ is a fiber bundle over $T^*(G/K)$ with the typical fiber B. Looking carefully at his proof one sees that the E-leaf

$$\hat{E}_x = (\hat{E}^G_{(e,x)} \oplus \hat{E}^0_b) \cap T\Psi^{-1}(0)/K$$

corresponds symplectically to the fiber bundle over $T^*(G/K)$ with the typical fiber $\hat{E}^0_b$. This concludes that all E-leaves have the same dimension.

Similarly,

$$D = (D^G \oplus D^0) \cap T\Psi^{-1}(0)/K.$$

By using the fiber bundle argument as above, if D^0-leaves are simply connected, since $T^*_e(G/K)$ is simply connected too, D-leaves are also simply connected.

The last condition of the geometrical admissibility, namely the canonical map $X/D \rightarrow X/E$ is a submersion, is obviously true since our symplectic manifolds are coadjoint orbits and we may describe X/D, X/E by the qotient spaces of the subalgebras of $\mathfrak{g}$. We can see this in the next section when we discuss the induced representations of solvable Lie groups. $\square$

Now let us suppose that the polarization F^0 is chosen to be geometrically admissible, and so is F. It follows that D-leaves are simply connected. Since D is isotropic and $\mathrm{Curv}(L,\alpha) = \Omega_X$, we see that D-leaves are absolutely parrallel submanifolds for the line bundle $(L,\alpha) \rightarrow X$, giving rise to a quotient bundle $L/D \rightarrow X/D$ by "sticking" the fibers of L over D-leaf together in the following way.

For any $y = \hat{D}_x$, let

$$L'_y = \{s : \ D_x \longrightarrow L \mid \nabla_v s = 0, \ v \in D\}.$$

Since $\hat{D}_x$ is absolutely parallel, s is uniquely determined by the equations $\nabla_v s = 0$ for $v \in D$ when $s(x)$ is given, not depending on the curves on $\hat{D}_x$. So L'_y is an 1-dimensional space which is isomorphic to $L_{x'}$ for any $x' \in \hat{D}_x$. This defines the bundle $L/D \rightarrow X/D$ with the fiber L'_y above $y = \hat{D}_x$.

Let s be any polarized section of L. Then s gives rise to a section s/D of L/D. Consider the map

$$\rho_2 : \ X \longrightarrow \mathcal{B}.$$

For any $b' \in \mathcal{B}_{[g]} \subset \mathcal{B}$ and any $x' \in {\rho_2}^{-1}(b')$, by **Lemma 5.3.1** we have that

$$\begin{aligned} T_x({\rho_2}^{-1}(b')) &\subseteq \ker(\rho_{2*})^{-1}_{x'} \subseteq (\rho_{2*})^{-1}_{x'}(F^{[g]}) \cap TX \\ &= F_{x'} \cap TX = D_{x'}. \end{aligned}$$

Hence, ${\rho_2}^{-1}(b') \subseteq D$-leaf through x'. So π_D factorizes ρ_2. Namely, there exists a map $\rho_4 : \ \mathcal{B} \rightarrow X/D$ such that

$$\rho_4 \circ \rho_2 = \pi_D.$$

Let $L^{[g]}$ be the line bundle over $\mathcal{B}_{[g]}$. Then the pull back of $L/D|_{\rho_4(\mathcal{B}_{[g]})}$ is just $L^{[g]}$.

And the restriction $s/D|_{\rho_4(\mathcal{B}_{[g]})}$ corresponds to

$$s^{[g]} : \mathcal{B}_{[g]} \longrightarrow L^{[g]},$$

polarized with respect to $F^{[g]}$ since ρ_2 preserves the polarizations. This is the map we need to give an inverse map of (3.33). This concludes the proof of **Theorem 5.3.1**.

5.4 Representations of Nilpotent and Solvable Lie Groups

In the theories of nilpotent Lie groups and solvable Lie groups, the project of connecting the irreducible unitary representations with the coadjoint orbits of the Lie groups has been worked out perfectly. We will construct all irreducible unitary representations from the representations of "little groups" in such cases.

5.4.1 Nilpotent Case

We start with the Kirillov theory. The correspondence between the irreducible unitary representations and the coadjoint orbits of a nilpotent group is well described by the following theorem given by Kirillov (see [12]).

Theorem 5.4.1. Let G be a connected and simply connected nilpotent Lie group. Then the quantization of any coadjoint orbit of G, which is, up to unitary equivalence, independent of the choice of the polarizations, gives rise to an irreducible unitary representation of G. Conversely, any irreducible unitary representation of G is equivalent to such a quantization. Two irreducible unitary representations are not equivalent to each other if and only if they correspond to the different coadjoint orbits.

Kirillov's theorem sets up an one to one correspondence between the irreducible unitary representations and the coadjoint orbits of any connected and simply connected nilpotent Lie group. Let G be a connected and simply connected nilpotent Lie group,

$N \subset G$ be a connected and simply connected normal subgroup. For any coadjoint orbit X of G, let Y be a coadjoint orbit of N over which X sits, K be the little group related to Y and B be the little group data. The quantizations of X and B with respect to the polarizations chosen in the discussed fashion give rise to the unitary representations $\mathcal{H}$ and $\mathcal{H}_0$ of G and K respectively. By Theorem 5.3.1, $\mathcal{H} = \mathrm{Ind}_K^G \mathcal{H}_0$. Combine this and Kirillov's theorem, we have

Theorem 5.4.2. Let G and N as above. Then for any little group K related to some coadjoint orbit of N and any irreducible unitary representation $\mathcal{H}_0$ of K, $\mathrm{Ind}_K^G \mathcal{H}_0$ is an irreducible unitary representation of G. Conversely, given any irreducible unitary representation $\mathcal{H}$ of G, we may find a little group K and an irreducible unitary representation $\mathcal{H}_0$ of K such that $\mathcal{H}$ is equivalent to $\mathrm{Ind}_K^G \mathcal{H}_0$.

Proof: By Kirillov's theorem, we may transfer the problem to quantization. Since in the nilpotent case, according to Kirillov, any coadjoint orbit is integral, Theorem 5.3.1 completes the proof. $\square$

5.4.2 Exponential Solvable Case

We now consider a more general case. Let $\mathfrak{g}$ be a real solvable Lie algebra. Then there exists a sequence of the ideals

$$\mathfrak{g} = \mathfrak{g}_0 \supset \mathfrak{g}_1 \supset \cdots \supset \mathfrak{g}_m = 0, \quad m \leq n = \dim \mathfrak{g},$$

such that codimension of $\mathfrak{g}_{i+1}$ in $\mathfrak{g}_i$ is 1 or 2. If for some i, the codimension of $\mathfrak{g}_{i+1}$ in $\mathfrak{g}_i$ is 1, we may pick ξ_0 a nonzero element of $\mathfrak{g}_i$ not contained in $\mathfrak{g}_{i+1}$, so that for all $\xi \in \mathfrak{g}$,

$$ad_\xi(\xi_0) = \gamma(\xi)\xi_0 \pmod{\mathfrak{g}_{i+1}}, \tag{4.34}$$

where $\gamma(\xi)$ is a linear form vanishing on $[\mathfrak{g}, \mathfrak{g}]$.

If the codimension of $\mathfrak{g}_{i+1}$ in $\mathfrak{g}_i$ is 2, there exist the elements ξ_1 , ξ_2, linearly independent mod $\mathfrak{g}_{i+1}$, of $\mathfrak{g}_i$ so that for all $\xi \in \mathfrak{g}$,

$$ad_\xi(\xi_1) = \gamma_1(\xi)\xi_1 - \gamma_2(\xi)\xi_2 \quad (\mathrm{mod}\,\mathfrak{g}_{i+1}) \quad \text{and} \tag{4.35}$$

$$(4.36) \qquad ad_\xi(\xi_2) = \gamma_2(\xi)\xi_1 + \gamma_1(\xi)\xi_2 \pmod{\mathfrak{g}_{i+1}},$$

where γ_1, γ_2 are linear forms on $\mathfrak{g}$ vanishing on $[\mathfrak{g}, \mathfrak{g}]$. Then the set $\{\gamma(\xi),\ \gamma_1(\xi) + \sqrt{-1}\gamma_2(\xi)\}$ where γ, γ_1, γ_2 are obtained in the indicated fashion for all $\mathfrak{g}_i$ is called the system of roots of $\mathfrak{g}$.

Definition 5.4.1. A real solvable Lie algebra $\mathfrak{g}$ is *exponential* if its complex roots are of the form $(1+\sqrt{-1}\lambda)\delta(\xi)$ where λ is a nonzero real number and $\delta(\xi)$ is a real linear form on $\mathfrak{g}$. A connected and simply connected Lie group G is called *exponential* if its Lie algebra $\mathfrak{g}$ is exponential.

Obviously, all nilpotent Lie algebras are exponential.

Proposition 5.4.1. If G is an exponential solvable Lie group, $\mathfrak{g} = \mathrm{Lie}G$, then the exponential map $exp: \mathfrak{g} \to G$ is an analytic isomorphism.

An easy fact is that any subalgebra or factoralgebra of an exponential Lie algebra is again of the same kind. We will call an exponential solvable Lie group a *type E* group. In the remainder of this subsection, our groups will be the type E groups except otherwise mentioned. We want to describe Mackey's picture for all type E groups. First, we quote the following theorem which generalizes Kirillov's result.

Theorem 5.4.3. (Bernat) For each coadjoint orbit X of G, there exists a suitable polarization F so that the quantized space $\mathcal{H}^F$, which is independent of the choice of such polarizations, gives an irreducible unitary representation of G. On the other hand, any irreducible unitary representation of G can be obtained in this way up to equivalence.

Note that such "suitable polarizations" could be chosen real. The meaning of "suitable" was obtained by Pukanszky, which is well known as the Pukanszky condition. One version of this condition is described as follows.

Theorem 5.4.4. (Pukanszky) The representation $\mathcal{H}^F$ is irreducible if and only if each of all leaves of F is closed in $\mathfrak{g}^*$.

In the nilpotent case, any polarization automatically satisfies the Pukanszky condition. It is not the case for the type E groups. For example, let

$$G = \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \in GL(2, \mathbf{R}),\ a > 0 \right\}.$$

Then the Lie algebra of G is

$$\mathfrak{g} = \left\{ \begin{pmatrix} x & y \\ 0 & 0 \end{pmatrix} \in gl(2, \mathbf{R}) \right\}.$$

Let

$$e_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Then $[e_1, e_2] = e_2$.

Let e_1^*, e_2^* be the dual basis of $\mathfrak{g}^* \cong \mathbf{R}^2$, then the upper hemiplane is a coadjoint orbit. The only polarization satisfying the Pukanszky condition is the one whose leaves are parallel to e_1^*-axis. All other polarizations will give representations which are the sum of two irreducible representations.

We are now ready to quantize the classical structure to get the induced representation picture of the type E groups.

Let G be a type E group, N be a connected and simply connected nilpotent normal subgroup of G. Let X a coadjoint orbit of G, and Y, W, B, K, $\mathcal{B}_{[g]}$, x, p etc. be defined as before. Then K is again a type E group. Choose a K-invariant real polarization F^0 on B satisfying the Pukanszky condition. By reduction, we have a

G-invariant polarization F on X. First of all, we have

Proposition 5.4.2. F satisfies the Pukanszky condition.

Proof: Since F^0 satisfies the Pukanszky condition, F^0-leaves are closed in $\mathfrak{k}^*$. By translation, all $F^{[g]}$-leaves in $\mathfrak{k}^*_{gp}$ are closed. It follows Lemma 5.3.1 that F-leaves are closed in $\mathfrak{g}^*$. Thus F satisfies the Pukanszky condition. $\square$

Therefore we have the representation $\mathcal{H}_0$ of K and $\mathcal{H}^F$ of G given by quantization. Theorem 5.3.1 says that $\mathcal{H}^F = \mathrm{Ind}_K^G \mathcal{H}_0$. Combining this with Theorem 5.4.3 we summarize the representation picture as follows.

Theorem 5.4.5. Let G be a type E group and N be a connected and simply connected nilpotent normal subgroup. Then for any little group K related to some coadjoint orbit of N and any irreducible unitary representation $\mathcal{H}_0$ of K, $\mathrm{Ind}_K^G \mathcal{H}_0$ is an irreducible unitary representation of G. Conversely, given any irreducible unitary representation $\mathcal{H}$ of G, we may find a little group K and an irreducible unitary representation $\mathcal{H}_0$ of K such that $\mathcal{H}$ is equivalent to $\mathrm{Ind}_K^G \mathcal{H}_0$.

5.4.3 General Solvable Case

We continue to describe the induced representation picture of general solvable Lie groups based on Auslander-Kostant's theory. Things are more complicated in this case since we need to deal with the complex polarizations and require more conditions to get the representations from the quantization. First of all, let us review Auslander-Kostant's theory. For this, we would like to introduce the algebraic version of the quantization.

For moment let G be a general Lie group and $\mathfrak{g}$ be its Lie algebra. We define an alternating bilinear form on $\mathfrak{g}$ for any $\beta \in \mathfrak{g}^*$ as

$$J_\beta(\xi, \eta) = - < \beta, \ [\xi, \eta] >, \qquad \text{for all } \xi, \eta \in \mathfrak{g}. \tag{4.37}$$

Then the stabilizer subalgebra $\mathfrak{g}_\beta$ of the coadjoint G action on $\mathfrak{g}^*$ is exactly the radical of J_β. Let us regard β as a complex valued linear functional on $\mathfrak{g}^{\mathbf{C}} = \mathfrak{g} + \sqrt{-1}\mathfrak{g}$.

Definition 5.4.2. A polarization at β is a complex subalgebra $\mathfrak{h} \subset \mathfrak{g}^{\mathbf{C}}$ such that

1) $\mathfrak{g}_\beta \subset \mathfrak{h}$ and $\mathfrak{h}$ is stable under AdG_β;

2) $\dim_{\mathbf{C}}(\mathfrak{g}^{\mathbf{C}}/\mathfrak{h}) = \frac{1}{2}\dim_{\mathbf{R}}(\mathfrak{g}/\mathfrak{g}_\beta)$;

3) $\beta|_{[\mathfrak{h},\mathfrak{h}]} = 0$;

4) $\mathfrak{h} + \bar{\mathfrak{h}}$ is a Lie subalgebra of $\mathfrak{g}^{\mathbf{C}}$.

The definition says $\mathfrak{h}$ is a maximal isotropic subalgebra of $\mathfrak{g}^{\mathbf{C}}$ containing $(\mathfrak{g}_\beta)^{\mathbf{C}}$.

Let

$$\mathfrak{d} = \mathfrak{h} \cap \mathfrak{g} \quad \text{and} \quad \mathfrak{e} = (\mathfrak{h} + \bar{\mathfrak{h}}) \cap \mathfrak{g}.$$

Then

$$\mathfrak{h} \cap \bar{\mathfrak{h}} = \mathfrak{d}^{\mathbf{C}} \quad \text{and} \quad \mathfrak{h} + \bar{\mathfrak{h}} = \mathfrak{e}^{\mathbf{C}}.$$

Definition 5.4.3. A polarization $\mathfrak{h}$ is said to be *positive* if $-\sqrt{-1}J_\beta(z, \bar{z}) \geq 0$ for all $z \in \mathfrak{h}$.

Let $\mathfrak{n} \subset \mathfrak{g}$ be the nilradical of $\mathfrak{g}$ and $p = \beta|_{\mathfrak{n}} \in \mathfrak{n}^*$

Definition 5.4.4. A polarization $\mathfrak{h}$ is said to be *admissible* if it is positive and $\mathfrak{h} \cap \mathfrak{n}^{\mathbf{C}}$ is a polarization at p.

Let us denote by E the connected subgroup whose Lie algebra is $\mathfrak{e}$. We say that the polarization $\mathfrak{h}$ satisfies the Pukanszky condition if the orbit $E \cdot \beta$ is closed in $\mathfrak{g}^*$.

Auslander-Kostant's theory works for the type I group. In the solvable case we have a beautiful geometric interpretation of the type I condition.

Theorem 5.4.6. (Auslander-Kostant) Let G be a connected and simply connected solvable Lie group. Then G is of type I if and only if the following conditions are satisfied.

1) Any $\beta \in \mathfrak{g}^*$ is integrable. Equivalently, $G_\beta^\sharp \neq \emptyset$, and

2) any orbit $G \cdot \beta \subset \mathfrak{g}^*$ is the intersection of a closed and an open set.

When β is integrable, namely $G_\beta^\sharp \neq \emptyset$, let $\chi_\beta \in G_\beta^\sharp$. $\chi_\beta : G_\beta \to \mathbf{T}$ is a character and

$$d\chi_\beta = 2\pi\sqrt{-1}\beta|_{\mathfrak{g}_\beta}.$$

We are now ready to state Auslander-Kostant's celebrated theorem of representations of solvable Lie groups.

Theorem 5.4.7. Let G be a connected and simply connected solvable Lie group, $\mathfrak{g}$ be the Lie algebra of G. For any $\beta \in \mathfrak{g}^*$, whether or not G is of type I, there exists an admissible polarization at β. Moreover, any admissible polarization $\mathfrak{g}$ satisfies the Pukanszky condition so that if β is integrable, a unitary representation of G, $\mathrm{ind}_G(\chi_\beta, \mathfrak{h})$, is defined, and is independent of the choice of the polarizations $\mathfrak{h}$. Furthermore, if G is of type I, then $\mathrm{ind}_G(\chi_\beta, \mathfrak{h})$ is irreducible and every irreducible unitary representation is equivalent to a representation of this form. Finally, if G is of type I, then $\mathrm{ind}_G(\chi_\beta, \mathfrak{h})$ and $\mathrm{ind}_G(\chi_{\beta_1}, \mathfrak{h}_1)$ are equivalent if and only if $G\cdot\beta = G\cdot\beta_1$ and χ_β corresponds to χ_{β_1} under the action of an element $g \in G$ such that $g\beta = \beta_1$.

We are going to make use of this theorem to derive the relations between the quantized representations of G and K. For this purpose, let us look at the relations of the geometric and algebraic versions of polarizations.

Our underline symplectic manifolds now are coadjoint orbits. Let X be a coadjoint orbit of a Lie group G. We know that X can be equipped with the canonical symplectic form we stated in **Chapter 1**, namely,

$$\omega_x(\xi^\sharp, \eta^\sharp) = - < x, \ [\xi, \eta] > . \tag{4.38}$$

Let $\mathfrak{h}$ be an (algebraic) polarization at x. We may construct a (geometric) polarization on X. Notice that $X \cong G/G_x$, hence, $T_xX \cong \mathfrak{g}/\mathfrak{g}_x$. Let

$$F_x = \mathfrak{h}/(\mathfrak{g}_x)^{\mathbf{C}} \subset \mathfrak{g}^{\mathbf{C}}/(\mathfrak{g}_x)^{\mathbf{C}} \cong (T_xX)^{\mathbf{C}}.$$

Since $\mathfrak{h}$ is AdG_x invariant, for any $x_1 = gx$, let

$$F_{x_1} = g \cdot F_x.$$

Then the distribution F on X with $(F)_{x_1} = F_{x_1}$ is a polarization on X. Furthermore, such a correspondence is one to one, and $\mathfrak{h}$ is positive if and only if F is. In the case that x is integrable, X is integral, the representation given by the geometric quantization is exactly $\text{ind}_G(\chi_\beta, \mathfrak{h})$ we mentioned in Auslander-Kostant's theory.

From now on, let us restrict ourselves to the solvable case. Let G be a connected and simply connected solvable Lie group of type I, $\mathfrak{g}$ be the Lie algebra of G, and $\mathfrak{n} \subset \mathfrak{g}$ be the nilradical of $\mathfrak{g}$. Let N be the connected subgroup of G whose Lie algebra is $\mathfrak{n}$. Let X be a coadjoint orbit of G sitting over a coadjoint Y of N, and K be the little group related to Y. $\mathfrak{k} = \text{Lie}K$. Then $\mathfrak{n} \subset \mathfrak{k}$ is also an ideal of $\mathfrak{k}$. In fact, we have

Proposition 5.4.3. $\mathfrak{n}$ is the nilradical of $\mathfrak{k}$.

Proof: It is well known that

$$\mathfrak{n} = \{\xi \in \mathfrak{g} | \quad ad_{\mathfrak{g}}\xi \text{ is nilpotent}\}.$$

Let $\mathfrak{n}_1$ be the nilradical of $\mathfrak{k}$. Then

$$\mathfrak{n}_1 = \{\eta \in \mathfrak{k} | \quad ad_{\mathfrak{k}}\eta \text{ is nilpotent}\}.$$

By definition, $\mathfrak{n}_1 \supseteq \mathfrak{n} \cap \mathfrak{k} = \mathfrak{n}$. On the other hand, for any $\eta \in \mathfrak{n}_1 \subset \mathfrak{g}$, since $\mathfrak{n}_1 \subset \mathfrak{k}$ and $\mathfrak{n} \subset \mathfrak{k}$,

$$(ad_{\mathfrak{g}}\eta)|_{\mathfrak{n}} = (ad_{\mathfrak{k}}\eta)|_{\mathfrak{n}} : \ \mathfrak{n} \longrightarrow \mathfrak{n}$$

is nilpotent.

But

$$[\mathfrak{g},\mathfrak{g}] \subseteq \mathfrak{n},$$

thus,

$$ad_{\mathfrak{g}}\eta : \ \mathfrak{g} \longrightarrow \mathfrak{n} \subset \mathfrak{g}.$$

There exists an integer j such that

$$(ad_{\mathfrak{g}}\eta)^j(ad_{\mathfrak{g}}\eta \cdot \xi) = (ad_{\mathfrak{g}}\eta)^j|_{\mathfrak{n}}(ad_{\mathfrak{g}}\eta \cdot \xi) = 0$$

for all $\xi \in \mathfrak{g}$.

So

$$ad_{\mathfrak{g}}\eta : \ \mathfrak{g} \to \mathfrak{g}$$

is nilpotent. It follows that $\eta \in \mathfrak{n}$. This shows $\mathfrak{n} \supseteq \mathfrak{n}_1$. Hence

$$\mathfrak{n} = \mathfrak{n}_1 \qquad \square$$

Therefore, we know that $\mathfrak{n}$ is the nilradical of $\mathfrak{g}$ and $\mathfrak{k}$ simultaneously. Using our notations before, $x \in X$ and $b \in B$ such that $x|_{\mathfrak{k}} = b$. From a K-invariant polarization F^0 on B we get a G-invariant polarization F on X. Under the correspondence described above, we assume that F and F^0 correspond to the polarizations $\mathfrak{h} \subset \mathfrak{g}^*$ and $\mathfrak{h}_0 \subset \mathfrak{k}^*$ respectively. Let $\sigma : \ \mathfrak{g}^* \to \mathfrak{k}^*$ be the natural projection. **Lemma 5.3.1** says that $F_x = \sigma_*^{-1} F_b^0$.

Note that $X = G \cdot x$. Hence, $T_x X = \mathfrak{g} \cdot x$, and by the bridge between the geometric and the algebraic version of polarizations we just contructed, we have $F_x = \mathfrak{h} \cdot x$. Similarly, $F_b^0 = \mathfrak{h}_0 \cdot b$.

We now observe that actually

$$F_x = \mathfrak{h}^0 = \{\beta \in \mathfrak{g}^* | \ \beta|_{\mathfrak{h}} = 0\}. \tag{4.39}$$

Indeed, since for any $z_1 x \in F_x$, $z_1 \in \mathfrak{h}$, and any $z_2 \in \mathfrak{h}$,

$$< z_1 x, z_2 >= - < x, \ [z_1, z_2] >= 0.$$

Hence,

$$F_x \subseteq \mathfrak{h}^0.$$

By counting dimension, we conclude that $F_x = \mathfrak{h}^0$ in $\mathfrak{g}^*$.

Similarly, we have $F^0{}_b = \mathfrak{h}_0{}^0$ in $\mathfrak{k}^*$.

By Lemma 5.3.1, we know that

$$\sigma_*(\mathfrak{h}^0) = \mathfrak{h}_0{}^0. \tag{4.40}$$

It follows that $\mathfrak{h} \supset \mathfrak{h}_0$ as the subalgebras of $\mathfrak{g}^{\mathbf{C}}$. This implies that

$$\mathfrak{h} \cap \mathfrak{k}^{\mathbf{C}} \supset \mathfrak{h}_0 \cap \mathfrak{k}^{\mathbf{C}} = \mathfrak{h}_0.$$

On the other hand, notice that $\mathfrak{h} \cap \mathfrak{k}^{\mathbf{C}}$ is an isotropic subalgebra of $\mathfrak{k}^{\mathbf{C}}$, which forces

$$\mathfrak{h} \cap \mathfrak{k}^{\mathbf{C}} \subset \mathfrak{h}_0.$$

Therefore, we have

$$\mathfrak{h}_0 = \mathfrak{h} \cap \mathfrak{k}^{\mathbf{C}}. \tag{4.41}$$

Thus,

$$\mathfrak{h} \cap \mathfrak{n}^{\mathbf{C}} = \mathfrak{h} \cap \mathfrak{k}^{\mathbf{C}} \cap \mathfrak{n}^{\mathbf{C}} = \mathfrak{h}_0 \cap \mathfrak{n}^{\mathbf{C}}. \tag{4.42}$$

This implies that $\mathfrak{h} \cap \mathfrak{n}^{\mathbf{C}}$ is a polarization of $p = x|_{\mathfrak{n}} = b|_{\mathfrak{n}}$ if and only if $\mathfrak{h}_0 \cap \mathfrak{n}^{\mathbf{C}}$ is a polarization of p.

On the other hand, one can easily see from the definition and the Kirillov-Kostant symplectic form on the coadjoint orbits that F (or F^0) is positive if and only if $\mathfrak{h}$ (or $\mathfrak{h}_0$) is positive. The following proposition solves the problem of positivity.

Proposition 5.4.4. If F^0 is positive, so is F.

Proof: Let us denote by $(\ ,\ )_M$ the symplectic form on M. Note that

$$(\ ,\ )_{T^*G\times B} = (\ ,\ )_{T^*G} + (\ ,\ )_B.$$

Let

$$i:\ \Psi^{-1}(0) \longrightarrow T^*G \times B$$

be the inclusion map and

$$j:\ \Psi^{-1}(0) \longrightarrow X$$

be the canonical projection. Then the Marsden-Weistein reduction theorem tells us that

$$j^*(\ ,\ )_X = i^*(\ ,\ )_{T^*G\times B}.$$

So for $[(Z_1, Z_2)] \in F$ where $Z_1 \in (F^G)^{\mathbf{C}}$, $Z_2 \in F^0$, since $\overline{[(Z_1, Z_2)]} = [(\overline{Z}_1, \overline{Z}_2)]$, we have

$$-\sqrt{-1}([(Z_1, Z_2)], \overline{[(Z_1,\ Z_2)]})_X = -\sqrt{-1}(Z_1, \overline{Z}_1)_{T^*G} - \sqrt{-1}(Z_2, \overline{Z}_2)_B.$$

Note that F^G is chosen to be the vertical polarization, and $(\ ,\ )_{T^*G}$ is the canonical symplectic form on the cotangent bundle, so

$$(Z_1,\ \overline{Z}_1)_{T^*G} = 0.$$

But

$$-\sqrt{-1}(Z_2, \overline{Z}_2)_B \geq 0.$$

So

$$-\sqrt{-1}([(Z_1, Z_2)], \overline{[(Z_1,\ Z_2)]})_X \geq 0.$$

Namely, F is positive. □

Combine the argument above, we conclude that

Proposition 5.4.5. If $\mathfrak{h}_0$ is admissible, so is $\mathfrak{h}$.

Remark 5.4.1. In the proof of Proposition 5.3.2, we mentioned that we would discuss the last condition of the geometrical admissiblity in this section. Namely, we need

$$\pi_{DE}:\ X/D \longrightarrow X/E$$

to be a submersion. Notice that $D_x = \mathfrak{d}/\mathfrak{g}_x$ and $E_x = \mathfrak{e}/\mathfrak{g}_x$, so

$$(\pi_{DE})_*:\ T_{\hat{D}_x}(X/D) \longrightarrow T_{\hat{E}_x}(X/E)$$

is the projection

$$\mathfrak{g}/\mathfrak{d} \longrightarrow \mathfrak{g}/\mathfrak{e},$$

which is surjective.

We can now summarize the induced representation picture of the solvable Lie groups as follows. Let G be a connected and simply connected solvable Lie group of type I, $X \subset \mathfrak{g}^*$ be a coadjoint orbit of G. By Theorem 5.4.6, X is integral. Take a point $x \in X$, x is then integrable. Let $\mathfrak{n}$ be the nilradical of $\mathfrak{g}$ and N be the connected subgroup whose Lie algebra is $\mathfrak{n}$. Let p, Y, K, B, b, etc. be defined as before. We choose a polarization F^0 on B such that F^0 is geometrically admissible and the corresponding (algebraic) polarization $\mathfrak{h}_0$ is admissible. By our canonical way we get a polarization F on X. Then F is geometrically admissible and the corresponding (algebraic) polarization $\mathfrak{h}$ is admissible, too. According to Auslander-Kostant, the quantization of B gives a unitary representation $\mathcal{H}_0$ of K, which is independent of the choice of polarizations; the quantization of X gives an irreducible unitary representation $\mathcal{H}$ of G, which is also independent of the choice of polarizations. Furthermore, any irreducible unitary representation of G can be given by the quantization of some coadjoint orbit X. Therefore, we have

Theorem 5.4.8. $\mathcal{H} = \mathrm{Ind}_K^G \mathcal{H}_0$. Furthermore, any irreducible unitary representation of G can be induced from a unitary representation of some certain little group of G.

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