Single-petaled K-types and Weyl Group Representations for Classical Groups

by

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B.S. in Mathematics, Seoul National University, 2003

Submitted to the Department of Mathematics
in partial fulfillment of the requirements for the degree of

Doctor of Philosophy

at the

MASSACHUSETTS INSTITUTE OF TECHNOLOGY

June 2008

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Abstract

In this thesis, we show that single-petaled K-types and quasi-single-petaled K-types for reductive Lie groups generalize petite K-types for split groups. First, we prove that a Weyl group algebra element represents the action of the long intertwining operator for each single-petaled K-type, and then we demonstrate that a Weyl group algebra element represents a part of the long intertwining operator for each quasi-single-petaled K-type. We classify irreducible Weyl group representations realized by quasi-single-petaled K-types for classical groups. This work proves that every irreducible Weyl group representation is realized by quasi-single-petaled K-types for $SL(n, \mathbb{C})$, $SL(n, \mathbb{R})$, $SU(m, n)$, $SO(m, n)$, and $Sp(n, \mathbb{R})$.

Thesis Supervisor: David A. Vogan, Jr.
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Acknowledgments

I feel happy to finally have a chance to thank all the people I owe during my graduate study at MIT. First and foremost, I want to thank my wonderful advisor, David Vogan, for his endless patience and encouragement. He has been always generous sharing his knowledge, time, and ideas. He gave me suggestions when I was stuck, pointed out when I was wrong, and helped me out when I had questions. I cannot imagine a better advisor, and I cannot thank him enough.

My graduate life at MIT was possible with help from many people. My registration advisor Michael Artin helped me settling down at MIT as a new student, and my cofactors, Boguk Kim and Jaehyuk Choi, helped me settling down when I moved to the United States to pursue graduate study, and I would like to thank them for that. I am sincerely grateful to Sigudur Helgason, Myung-Hwan Kim, Seok-Jin Kang, and Igor Pak for writing letters for me for the NSF fellowship, which helped me focus on my research for three years. One thing that I feel great about coming to MIT was that I could make many good friends, which kept me emotionally healthy even when I was stressed. I greatly appreciate the friendship and moral support from all my friends, especially Hyemin Chung and Jihye Seo. Staying in the office to work was much more enjoyable with my officemates and office neighbors. I want to thank Thomas Lam, Brian Osserman, Joungkeun Lim, Zhang Zhou, Fangyun Yang, Peter Lee, Max Lipyanskiy, Silvia Montarani, and Ting Xue especially. I want to thank my academic sisters Chuying Fang, Alessandra Pantano, and Wai Ling Yee for their advice and encouragement. I like to express my gratitude to my thesis committee; Bertram Kostant and JuLee Kim.

My mom and dad has always been concerned about me but believed in me at the same time, and I want to thank them for that. Last, certainly not the least, I want to thank Daihyun Lim, my husband, for comforting me when I am stressed and supporting me in real life for last four years.

This research was supported by an MIT Presidential Fellowship and a National Science Foundation Graduate Research Fellowship.

Contents

Chapter 1

Introduction

In unitary representation theory, the classification of irreducible unitary representations has been a popular area of research. One of the natural motivations to study unitary representations lies in harmonic analysis. If a reductive Lie group G acts on a space X, it is often possible to prove abstractly that there is a direct integral decomposition of $L^2(X)$ into irreducible unitary representations, even though finding such a decomposition explicitly is difficult. Unitary representations have been actively studied for the last several decades. For compact G, the Peter-Weyl theorem proves that all irreducible unitary representations are finite dimensional and they decompose $L^2(G)$ [17]. For noncompact G, the irreducible unitary representations are generally infinite dimensional, and the situation is much more complicated. For reductive G, Harish-Chandra proved that $L^2(G)$ could be decomposed using only the tempered unitary representations, which are completely classified [22]; but for other homogeneous spaces, like conjugacy classes in G, this is no longer true. We would therefore like to understand *all* irreducible unitary representations.

This thesis contributes to expanding the previously known list of correspondences between the irreducible representations of the Weyl group and the maximal compact subgroup for classical groups. We can use the expanded list for the classification of unitary representations. As a corollary, this work confirms the relation between unitarity in the real setting and unitarity in the p-adic setting for split groups. (cf. [5])

In Chapter 2, we explore the previous research in unitary representation theory. We can reduce the classification of irreducible unitary representations to the following situation thanks to the theorems of Harish-Chandra [16]:

- Every irreducible unitary representation is admissible, so it is enough to check unitarity for irreducible admissible representations.
- Irreducible admissible representations are classified as the Langlands quotients of principal series representations.

A general principal series representation depends on many parameters, but for spherical representations, we can simplify the parameters significantly. Here, a spherical representation is a representation with a copy of the trivial representation of the maximal compact subgroup K.

- For the spherical case, the Langlands quotient is the unique irreducible spherical quotient of the principal series representation $I(\nu)$ indexed by just one parameter. This parameter ν is a complex-valued linear functional on the real Cartan subspace $\mathfrak{a}$. The Langlands quotient is uniquely determined, so we may write it as $J(\nu)$ [16].
- The problem of deciding whether $J(\nu)$ is unitary or not can be reduced to the case when ν is a real-valued linear functional [16].

The remaining question is to find out which $J(\nu)$ is unitary for real ν. Harish-Chandra proved that an irreducible admissible representation is unitary when we have an invariant positive definite Hermitian form on it [16][25]. The invariant Hermitian form is represented by a linear operator from $I(\nu)$ to $I(\omega_0(\nu))$, which is called *the long intertwining operator* [19][20][21]. Here, ω_0 is the longest Weyl group element. Then, $J(\nu)$ is unitary if and only if the corresponding long intertwining operator

$$A(\nu) : I(\nu) \to I(\omega_0(\nu))$$

is positive semi-definite. We may pass the long intertwining operator $A(\nu)$ to the operator

$$A_\phi(\nu) : \mathrm{Hom}_K(\phi, I(\nu)) \to \mathrm{Hom}_K(\phi, I(\omega_0(\nu)))$$

for an irreducible representation of K, ϕ. Irreducible representations of K are also called **K-types**. The operator $A(\nu)$ is positive semi-definite if and only if $A_\phi(\nu)$ is positive semi-definite for all K-types ϕ. Using Frobenius reciprocity, we can regard $A_\phi(\nu)$ as a linear operator on the M-fixed vectors of ϕ, where M is subgroup of K that centralizes $\mathfrak{a}$. In this way, the long intertwining operator can be considered as a linear operator on the M-fixed vectors of each K-type. If this linear operator is not positive semi-definite, then the long intertwining operator fails to be positive semi-definite, and therefore, $J(\nu)$ is not unitary. We will call these calculations **non-unitarity tests using K-types**. On the other hand, the Weyl group W acts on V_ϕ^M, the M-fixed vectors of ϕ. We can write $A_\phi(\nu)$ as a Weyl group algebra element $A_{\omega_0}(\nu)$ acting on V_ϕ^M for some special K-types. The main technique is to decompose the long intertwining operator into the composition of simpler intertwining operators [10][26][16]. If the representation of W, τ, is realized on the M-fixed vectors of the special K-type ϕ, then $\tau(A_{\omega_0}(\nu))$ appears in the long intertwining operator on V_ϕ^M. Irreducible representations of W are also called **W-types**. If $\tau(A_{\omega_0}(\nu))$ is not positive semi-definite, then the long intertwining operator fails to be positive semi-definite, and $J(\nu)$ is not unitary. We will call these calculations **non-unitarity tests using W-types**.

We can relate non-unitarity tests using K-types with non-unitarity tests using W-types in this way, and we can pass the calculation to the setting of affine Hecke algebra [5][2]. To ensure unitarity, an infinite number of tests using K-types have to be conducted. However, we can check non-unitarity even with a finite number of tests. Therefore, we can use W-types corresponding to the special K-types for non-unitarity tests [2]. One family of these special K-types is petite K-types, and some of W-types $\leftrightarrow$ K-types correspondences have been studied [6][25]. This thesis contributes to generalizing petite K-types and expanding the known list of correspondences.

In particular, W-type tests ensure unitarity in the p-adic setting for split $G_{\mathbb{Q}_p}$ [4], so W-types $\leftrightarrow$ K-types correspondences reveal some information about the relation between non-unitarity in the real setting and non-unitarity in the p-adic setting.

In Chapter 3, we introduce single-petaled K-types and explain how they generalize petite K-types and how we can represent $A_\phi(\nu)$ using a Weyl group algebra element for single-petaled K-types ϕ. Let Δ be the restricted root system for $(\mathfrak{g}, \mathfrak{a})$, and let $\mathfrak{g}_\alpha$ be the root space corresponding to $\alpha \in \Delta$. If $\alpha \in \Delta$ and $2\alpha \notin \Delta$, then we call α reduced. Let Δ_1 be the set of reduced roots.

Definition 1.1 *A **quasi-spherical** K-type is a K-type with a nonzero M-fixed vector.*

In [24], Oda defined an interesting family of K-types, which is called single-petaled K-types, as the following.

Definition 1.2 *For each simple root α, fix X_α, θX_α, and H_α such that*

$$\begin{gathered} H_\alpha \textit{ is the coroot of } \alpha \\ [X_\alpha, \theta X_\alpha] = -H_\alpha \\ X_\alpha \in \mathfrak{g}_\alpha \end{gathered}$$

*Put $Z_\alpha = X_\alpha + \theta X_\alpha$. Then a quasi-spherical K-type (ϕ, V_ϕ) is **single-petaled** if and only if*

$$\phi(Z_\alpha)(\phi(Z_\alpha)^2 - 4)v = 0^1 \ , \ \forall v \in V_\phi^M, \ \forall \alpha \in \Delta_1.$$

This family of K-types can play an interesting role in unitary representation theory: it can be used to generalize petite K-types, for which we can construct a correspondence between W-types and K-types for non-unitarity tests [2][25]. We explain how

[1]In [24], the single-petaled K-types are defined in a different way, which is equivalent to this one.

single-petaled K-types generalize the petite K-types. The definition of petite K-types is given below.

Definition 1.3 *For each simple root α, we define a real rank one subalgebra $\mathfrak{g}^\alpha$ generated by $\mathfrak{g}_{n\alpha}$ for nonzero n. The real rank one subgroup G^α is the analytic subgroup of G corresponding to $\mathfrak{g}^\alpha$, and K^α is the maximal compact subgroup of G^α.*

Definition 1.4 *Suppose that G is a split real group, and (ϕ, V_ϕ) is a K-type. Since G is split, G^α is locally isomorphic to $SL(2,\mathbb{R})$, and K^α to $SO(2,\mathbb{R})$. Note that irreducible representations of $SO(2,\mathbb{R})$ are one-dimensional and they are indexed by $\mathbb{Z}$. Let*

$$\phi|_{K^\alpha} = \phi_1 \oplus \phi_2 \oplus \ldots \oplus \phi_k$$

*be the decomposition of ϕ into irreducible representations of K^α for $\alpha \in \phi_1$. We call ϕ **petite** if ϕ_i is a character 0, ± 1, ± 2 of K^α for all $\alpha \in \Delta_1$.*

For a general rank one group G^α (when G is not split), the group K^α may not be isomorphic to $SO(2,\mathbb{R})$. For example, K^α is isomorphic to $S(U(p-q+1)\times U(1))$ if G is $SU(p,q)$ and α is a short reduced root. Nevertheless, Kostant gave a complete list of the quasi-spherical K^α-types and calculated the long intertwining operator on each of them [23]. As a consequence of his calculation, we show that the long intertwining operator of G^α for K^α-type ϕ is given by a Weyl group algebra element if ϕ is either the trivial representation or a subrepresentation of $\mathfrak{p}^\alpha_{\mathbb{C}}$. Here, we use the Cartan decomposition

$$\mathfrak{g}^\alpha = \mathfrak{k}^\alpha \oplus \mathfrak{p}^\alpha$$

corresponding to a fixed Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. Note that $\mathfrak{p}^\alpha_{\mathbb{C}}$ is the complexification of $\mathfrak{p}^\alpha$.

We can write the long intertwining operator as a composition of simpler intertwining operators, which are the long intertwining operators of real rank one subgroups

[10][26][16]. Therefore, we can represent the long intertwining operator using a Weyl group algebra element for K-types satisfying the following condition. Since the long intertwining operator action that matters is only on nonzero M-fixed vectors, this condition, which is weaker than that of petite K-types, still works.

Condition 1.5 *Let*

$$\phi|_{MK^\alpha} = \phi_1 \oplus \phi_2 \oplus \ldots \oplus \phi_k$$

be the decomposition of ϕ into irreducible representations of MK^α for $\alpha \in \Delta_1$. If ϕ_i has a nonzero M-fixed vector, then we require that $\phi_i|_{K^\alpha}$ is either the trivial representation or the subrepresentation of $(\mathfrak{p} \cap \mathfrak{g}^\alpha)_\mathbb{C}$ for every reduced root α.

For real rank one groups, the single-petaled K-types are classified [24], and they are the trivial K-type and a irreducible subrepresentation of K on $\mathfrak{p}_\mathbb{C}$. The definition of the single-petaled K-type is given by the action of $Z_\alpha \in \mathfrak{k}^\alpha$, so a K-type ϕ is single-petaled if and only if every irreducible quasi-spherical direct summand of $\phi|_{MK^\alpha}$ is restricted to the single-petaled K^α-types for all $\alpha \in \Delta_1$. We obtain the following theorem.

Theorem 1.6 *Single-petaled K-types are the representations satisfying Condition 1.5.*

We classify single-petaled K-types for $SL(n, \mathbb{C})$ and $SU(m, n)$ in 3.4.1 and 3.5.1 using this theorem and the branching theorem 3.1.1.6 of Weyl [17]. We also calculate the corresponding Weyl group representations in 3.4.2 and 3.5.2.

In Chapter 4, we generalize this argument and relate K-types and W-types for a larger set of K-types, called quasi-single-petaled K-types, defined in [24].

Definition 1.7 *Let (ϕ, V_ϕ) be a quasi-spherical K-type and define $V^M_{\phi,single}$ to be*

$$\{v \in V^M_\phi \mid \phi(Z_\alpha)(\phi(Z_\alpha)^2 - 4)v = 0, \forall \alpha \in \Delta_1\}.$$

The K-type ϕ is called ***quasi-single-petaled*** *if $V^M_{\phi,single} \neq 0$. We call an M-fixed vector v* ***single-petaled*** *if $v \in V^M_{\phi,single}$.*

A Weyl group representation is defined on the single-petaled M-fixed vectors of a quasi-single-petaled K-type ϕ, and the action of the long intertwining operator stabilizes the single-petaled M-fixed vectors.

Definition 1.8 *A Weyl group representation ψ is called* ***single-petaled*** *(with respect to G) if it appears in $V^M_{\phi,single}$ for a quasi-single-petaled K-type (ϕ, V).*

Theorem 1.9 *The long intertwining operator on $V^M_{\phi,single} \subset V^M \simeq Hom_K(\phi, I(\nu))$ is represented by a Weyl group algebra element $A_{\omega_0}(\nu)$.*

Suppose ψ appears in the Weyl group representation on $V^M_{\phi,single}$. If $\psi(A_{\omega_0}(\nu))$ has a negative eigenvalue, the long intertwining operator on $V^M_{\phi,single}$ is not positive semi-definite. Therefore, the long intertwining operator on V^M_ϕ is not positive definite. We obtain the following theorem.

Theorem 1.10 *if ψ is a single-petaled W-type and $\psi(A_{\omega_0}(\nu))$ has a negative eigenvalue, then $J(\nu)$ is not unitary.*

We classify single-petaled W-types for classical groups case by case. The notation for W-types is from [11] and 2.4.4.

Theorem 1.11 *For the following groups, every W-type is single-petaled:*

$$SL(n,\mathbb{C}),\ SL(n,\mathbb{R}),\ SU(p,q),\ SO(p,q),\ Sp(n,\mathbb{R}).$$

Theorem 1.12 *For the following groups, we give the lists of single-petaled W-types:*

- $G = SL(n,\mathbb{H})$, $K = Sp(n)$, $W = S_n$:
  ψ^λ is single-petaled

$\iff \lambda = (a_1, a_2, \dots, a_n)^T$ satisfies $a_2 \leq 2$.

- $G = Sp(m,n)$, $K = Sp(m) \times Sp(n)$, $W = S_n \ltimes (\mathbb{Z}/2\mathbb{Z})^n$:
  $\psi^{(\lambda,\tau)}$ is single-petaled
  $\iff \tau = (1,1,\dots,1)$ and $\lambda^T = (k_1, k_2, \dots, k_{|\lambda|})$ such that $k_1 \leq 2$
  or $\tau = \emptyset$ and $\lambda^T = (k_1, k_2, \dots, k_{|\lambda|})$ such that $k_2 \leq 2$

- $G = SO(2n+1,\mathbb{C})$, $K = SO(2n+1)$, $W \simeq S_n \ltimes (\mathbb{Z}/2\mathbb{Z})^n$:
  $\psi^{(\lambda,\tau)}$ is single-petaled
  $\iff \lambda = (k) \vdash k$, $\tau = (r,1,1,\dots,1) \vdash n-k$

- $G = SO(2n,\mathbb{C})$, $K = SO(2n)$, $W \subset S_n \ltimes (\mathbb{Z}/2\mathbb{Z})^n$:
  $\psi^{\{\lambda,\tau\}}$ is single-petaled for $\lambda \neq \tau$
  $\iff \lambda = (k,1,1,\dots,1)$, $\tau = (r)$
  or $\lambda = (k,1)$, $\tau = (r,1)$
  $\psi^{\{\lambda,\lambda\}_I}$, $\psi^{\{\lambda,\lambda\}_{II}}$ is single-petaled when $n = 2m$
  $\iff \lambda = (m) \vdash m$
  or $\lambda = (m-1,1) \vdash m$

- $G = Sp(n,\mathbb{C})$, $K = Sp(n) \simeq Sp(n,\mathbb{C}) \cap U(2n)$, $W = S_n \ltimes (\mathbb{Z}/2\mathbb{Z})^n$:
  $\psi^{(\lambda,\tau)}$ is single-petaled
  $\iff \tau = \emptyset$, $\lambda = (k,1,1,\dots,1)$
  or $\tau = (1)$, $\lambda = (k_1,k_2,k_3)$ such that $0 \leq k_2, k_3 \leq 1$
  or $\tau = (1,1,\dots,1)$, $\lambda = (k_1,k_2)$ such that $0 \leq k_2 \leq 1$

- $G = SO^*(2n)$, $K = U(n)$, $W = S_n \ltimes (\mathbb{Z}/2\mathbb{Z})^n$:
  $\psi^{(\lambda,\tau)}$ is single-petaled
  $\iff \lambda = (k,1,1,\dots,1)$, $\tau = (r)$
  or $\lambda = (r)$, $\tau = (k,1,1,\dots,1)$.

We confirm each single-petaled W-type by constructing a single-petaled M-fixed vector space that realizes the W-type.

In Chapter 5, we propose some future research.

Chapter 2

Background

2.1 Irreducible Admissible Representations

2.1.1 Irreducible Admissible Representations

Let G be a reductive group with a fixed maximal compact group K. Note that K may be disconnected. By **K-types**, we mean irreducible representations of K. The irreducible representations of compact connected Lie groups are fully classified by highest weight representations [17], and we denote the K-type with highest weight Λ by ϕ_Λ.

Definition 2.1.1.1. *An irreducible representation π of G is **admissible** if*

$$\mathrm{Hom}_K(\phi, \pi)$$

is finite dimensional for every K-type ϕ.

If we restrict π to K, then it is expressed as a direct sum of K-types. Admissibility means that, for each K-type ϕ, only finite number of direct summands are equivalent to ϕ. The admissibility condition is not a harsh condition to add when we study irreducible unitary representations because of the following theorem of Harish-Chandra [16].

Theorem 2.1.1.2. *Every irreducible unitary representation of a reductive Lie group G is admissible.*

Classifying irreducible unitary representations is an important problem in unitary representation theory, and we only need to check the unitarity of irreducible admissible representations thanks to the theorem above.

2.1.2 Principal Series Representations and Harish-Chandra Subquotient Theorem

As a step toward classifying irreducible unitary representations, it is natural to ask the classification of admissible irreducible representations. The answer is given by Langlands: any irreducible admissible representation is an irreducible quotient of a "principal series representation" [16]. In this subsection, we will describe the principal series representations induced from the minimal parabolic subgroup. In general, a principal series representation is induced from a parabolic subgroup, not necessarily minimal. However, it is enough to study the principal series representations induced from the minimal parabolic subgroup to describe the spherical case, which is our interest. We will discuss the spherical case in 2.1.4.

Definition 2.1.2.1. *An irreducible representation π of G is called* ***spherical*** *if $\pi|_K$ has a copy of the trivial representation of K.*

Let $\mathfrak{g}$ be the Lie algebra of the reductive group G. Let us fix the Cartan involution θ and the corresponding Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. Note that $\mathfrak{k}$ is the Lie algebra of the maximal compact subgroup K. Let $\mathfrak{a}$ be a fixed maximal abelian subalgebra of $\mathfrak{p}$, and M the subgroup of K that centralizes $\mathfrak{a}$.

Definition 2.1.2.2. *A K-type is called* ***quasi-spherical*** *if it has a nonzero M-fixed vector.*

Quasi-spherical K-types are useful when we study spherical unitary representations. Using the abelian subalgebra $\mathfrak{a}$, we can obtain the restricted root space decomposition of $\mathfrak{g}$. The nonzero linear functional α on $\mathfrak{a}$ is a restricted root if

$$\mathfrak{g}_\alpha = \{X \in \mathfrak{g} \mid [a, X] = \alpha(a)X \text{ for all } a \in \mathfrak{a}\}$$

is nonzero. Let Δ be the set of restricted roots. If $\alpha \in \Delta$ and $2\alpha \notin \Delta$, then we call α reduced. Let Δ_1 be the set of reduced roots. The restricted root spaces $\mathfrak{g}_\alpha$ together with $\mathfrak{a}$ and $\mathfrak{m}$ span $\mathfrak{g}$, so we obtain the following restricted root decomposition of $\mathfrak{g}$ [17]:

$$\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{a} \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha.$$

We can choose simple restricted roots Π and corresponding positive restricted roots Δ^+. The minimal parabolic subalgebra $\mathfrak{q}$ corresponding to the set of simple roots Π is

$$\mathfrak{q} = \mathfrak{m} \oplus \mathfrak{a} \oplus \bigoplus_{\alpha \in \Delta^+} \mathfrak{g}_\alpha.$$

The nilpotent Lie algebra $\bigoplus_{\alpha \in \Delta^+} \mathfrak{g}_\alpha$ is also denoted by $\mathfrak{n}$. The minimal parabolic subgroup Q is the normalizer in G of the Lie algebra $\mathfrak{q}$. The minimal parabolic subgroup Q has the decomposition [17]

$$Q = MAN$$

where A and N are analytic subgroups of G corresponding to the Lie subalgebras $\mathfrak{a}$ and $\mathfrak{n}$ respectively. The subgroup N of Q is normal, and $Q/N \simeq M \times A$. We can construct a representation of $Q = MAN$ which is trivial on N from representations of M, A, and N:

δ is an irreducible unitary representation of M

ν is an element in dual of $\mathfrak{a}$

triv is the trivial representation of N.

From these representations, we can construct the representation of Q

$$\delta \otimes e^\nu \otimes \text{triv}$$

For ν in $\mathfrak{a}^*$, we can construct a corresponding character of A, e^ν such that

$$e^\nu(\exp(a)) = e^{\nu(a)}$$

for a in $\mathfrak{a}$. Because exp is a bijective map from $\mathfrak{a}$ to A, the equation above defines a character of A. Now, we demonstrate two ways to describe principal series representations [16][25].

- **Induced Picture:** The principal series representation $I_Q(\delta, \nu)$ is induced from the representation $\delta \otimes e^\nu \otimes \text{triv}$ of the minimal parabolic subgroup in the following way.

$$I_Q(\delta, \nu) = \text{Ind}_{Q=MAN}^G(\delta \otimes e^\nu \otimes \text{triv}).$$

Here, the representation space is the following Hilbert space:

$$H_Q(\delta, \nu) = \{F : G \to V_\delta \text{ measurable } \mid F(gman) = e^{-(\nu+\rho)log(a)}\delta(m)^{-1}F(g) \\ \text{for } m \in M, a \in A, n \in N, g \in G\}$$

with the norm

$$||F||^2 = \int_K |F(k)|^2 dk.$$

The representation of G is defined on $H_Q(\delta, \nu)$. Specifically, g sends a function $F \in H_Q(\delta, \nu)$ to another function $I_Q(\delta, \nu)(g)F \in H_Q(\delta, \nu)$, which is defined below:

$$[I_Q(\delta, \nu)(g)F](x) = F(g^{-1}x).$$

- **Compact Picture:** The compact picture is simply the restriction to K of the induced picture, and the restriction is one-to-one. The representation space is the following Hilbert space:

$$H_\delta^Q = \{F : K \to V_\delta \text{ measurable } \mid F(km) = \delta(m)^{-1}F(k) \text{ for } k \in K, m \in M\}$$

with the norm

$$||F||^2 = \int_K |F(k)|^2 dk.$$

Since G equals KAN, we can decompose $g \in G$ correspondingly as

$$g = \kappa(g) e^{H(g)} n.$$

The action of G is defined as

$$[I_Q(\delta, \nu)(g)F](k) = e^{-(\nu+\rho)H(g^{-1}k)} F(\kappa(g^{-1}k)).$$

Note that $I_Q(\delta, \nu)$ share the representation space H_δ^Q in the compact picture for all $\nu \in \mathfrak{a}^*$.

We notice that every principal series representation is admissible due to Frobenius reciprocity because the following space is finite dimensional for a fixed K-type ϕ [25]:

$$\mathrm{Hom}_K(\phi, \mathrm{Ind}_{Q=MAN}^G(\delta \otimes e^\nu \otimes 1)) = \mathrm{Hom}_K(\phi, \mathrm{Ind}_M^K(\delta)) = \mathrm{Hom}_M(\phi, \delta).$$

Definition 2.1.2.3. *A $\mathfrak{g}$-module (μ, V_μ) is a $(\mathfrak{g}, K)$**-module** when the following holds.*

- *(μ, V_μ) is also a representation of K.*
- *The differential of the representation of K is the same as the representation of $\mathfrak{g}$ restricted to $\mathfrak{k}$.*
- *If $X \in \mathfrak{g}$ and $k \in K$, then $\mu(Ad(k)X) = \mu(k)\mu(X)\mu(k^{-1})$.*
- *Every element v in V_μ lies in a finite dimensional K-invariant subspace.*

We say a vector v in V_μ is K**-finite** when v lies in a finite dimensional K-invariant subspace. Similarly, in a representation (π, V_π) of G, we can define $v \in V_\pi$ to be K-finite when K translates of v span a finite dimensional space. When we study admissible representations, it is often helpful to make use of K-finite vectors of them.

Proposition 2.1.2.4. *For an irreducible admissible representation (π, V_π) of G, its K-finite part is a $(\mathfrak{g}, K)$-module $(\pi_{(K)}, V_{\pi,(K)})$.*

Moreover, for a unitary irreducible representation of G, its K-finite part is a unitary $(\mathfrak{g}, K)$-module [16].

Definition 2.1.2.5. *Two representations of G π^1 and π^2 are called* ***infinitesimally equivalent*** *if the $(\mathfrak{g}, K)$-modules $\pi^1_{(K)}$ and $\pi^2_{(K)}$ are isomorphic.*

Two irreducible unitary representations of G are unitarily equivalent if and only if they are infinitesimally equivalent [16]. Therefore, classifying the spherical unitary dual is equivalent to classifying spherical unitary $(\mathfrak{g}, K)$-modules. In order to study unitarity of $(\mathfrak{g}, K)$-modules, we use the long intertwining operator, which is introduced in the next subsection. For a principal series representation, the K-finite part is actually a Harish-Chandra module [16], which is defined below.

Definition 2.1.2.6. *A $(\mathfrak{g}, K)$-module (μ, V_μ) is* ***admissible*** *if*

$$Hom_K(\phi, \mu)$$

is finite dimensional for every K-type ϕ. A $(\mathfrak{g}, K)$-module is ***finitely generated*** *if there exist finitely many K-finite vectors $\{v_j \mid j \in J\}$ such that*

$$\Sigma_{j \in J} U(\mathfrak{g}^{\mathbb{C}}) v_j = V_\mu.$$

A finitely generated admissible $(\mathfrak{g}, K)$-module is called a ***Harish-Chandra module****.*

We call the vector space spanned by K-finite vectors of $H_Q(\delta, \nu)$,

$$\{v \in H_Q(\delta, \nu) \mid I_Q(\delta, \nu)(K)(v) \text{ is finite dimensional}\},$$

the **Harish-Chandra module corresponding to $I_Q(\delta, \nu)$.** The principal series representation is not generally irreducible, but it partly classifies irreducible admissible representations according to the following theorem of Harish-Chandra [16].

Theorem 2.1.2.7. ***(Harish-Chandra Subquotient Theorem)*** *Every irreducible admissible representation of G is infinitesimally equivalent to a composition factor of a principal series representation induced from the minimal parabolic subgroup.*

In other words, an irreducible admissible representation of G is infinitesimally equivalent to

$$\mathcal{M}_1/\mathcal{M}_2$$

for some subrepresentations $\mathcal{M}_1 \supset \mathcal{M}_2$ of a principal series representation $I_Q(\delta, \nu)$.

2.1.3 The Long Intertwining Operator and the Langlands Quotient

The principal series representation is not an irreducible representation generally, and the Langlands quotient $J_Q(\delta, \nu)$ is an irreducible quotient of the principal series representation [16]. To define the Langlands quotient, we need to define the long intertwining operator first. The minimal parabolic subgroup Q has the decomposition MAN where

$$Lie(N) = \bigoplus_{\alpha \in \Delta^+} \mathfrak{g}_\alpha$$

as in 2.1.2. We choose another set of simple roots $\widetilde{\Pi}$ such that every root is uniquely expressed as a sum of elements of $\widetilde{\Pi}$ and the coefficient is all non-negative or all non-positive. Let $\widetilde{\Delta^+}$ be the set of positive roots corresponding to $\widetilde{\Pi}$. Note that $\widetilde{\Pi}$ is expressed as $w(\Pi)$ for some Weyl group element w [13]. Then $\widetilde{Q} = MA\widetilde{N}$ is another minimal parabolic subgroup where

$$Lie(\widetilde{N}) = \bigoplus_{\alpha \in \widetilde{\Delta^+}} \mathfrak{g}_\alpha.$$

We have two principal series representations $I_Q(\delta, \nu)$ and $I_{\widetilde{Q}}(\delta, \nu)$. Then an intertwining operator

$$A(\widetilde{Q} \colon Q \colon \delta \colon \nu) : H_Q(\delta, \nu) \to H_{\widetilde{Q}}(\delta, \nu)$$

is defined such that

$$A(\widetilde{Q} \colon Q \colon \delta \colon \nu)(F)(x) = \int_{\overline{N} \cap \widetilde{N}} F(x\overline{n}) d\overline{n}$$

where

$$Lie(\overline{N}) = \bigoplus_{\alpha \in -\Delta^+} \mathfrak{g}_\alpha.$$

If $Re(\nu)$ is strictly dominant, $A(\widetilde{Q}\colon Q\colon \delta\colon \nu)$ converges [16][25]. This operator has an analytic continuation to $A(\widetilde{Q}\colon Q\colon \delta\colon \nu)$ for weakly dominant $Re(\nu)$ [16]. Therefore, we obtain the following theorem.

Theorem 2.1.3.1. *For weakly dominant $Re(\nu)$, $A(\widetilde{Q}\colon Q\colon \delta\colon \nu)$ is well-defined, and we can define the kernel and the image of this operator in a natural way.*

For $\widetilde{Q}$ given by $\omega.Q$, we will denote $A(\widetilde{Q}\colon Q\colon \delta\colon \nu)$ as $A(\omega\colon Q\colon \delta\colon \nu)$. We can factorize the intertwining operator $A(\widetilde{Q}\colon Q\colon \delta\colon \nu)$ into a product of simpler intertwining operators [25][10][26][16].

Theorem 2.1.3.2. *Let $Q_0 = MAN_0$, $Q_1 = MAN_1$, and $Q_2 = MAN_2$ be the minimal parabolic subgroups satisfying*

$$Lie(N_0) \cap Lie(N_2) \subset Lie(N_1) \cap Lie(N_2).$$

Then,

$$A(Q_2\colon Q_0\colon \delta\colon \nu) = A(Q_2\colon Q_1\colon \delta\colon \nu) \circ A(Q_1\colon Q_0\colon \delta\colon \nu)$$

holds. Suppose that

$$\omega.Q_0 = Q_2,\ \omega_1.Q_0 = Q_1,\ \text{and}\ \omega_2.Q_1 = Q_2$$

and that $\omega = \omega_2\omega_1$ satisfies $l(\omega) = l(\omega_1) + l(\omega_2)$. Then,

$$A(\omega\colon Q_0\colon \delta\colon \nu) = A(\omega_2\colon \omega_1.Q_0\colon \delta\colon \nu)A(\omega_1\colon Q_0\colon \delta\colon \nu).$$

Definition 2.1.3.3. *Let ν be an element of $\mathfrak{a}^*$ such that $Re(\nu)$ is weakly dominant. Let $A(\overline{Q}\colon Q\colon \delta\colon \nu)$ be the **long intertwining operator** between $I_Q(\delta, \nu)$ and $I_{\overline{Q}}(\delta, \nu)$. Note that $\overline{Q}$ is given by $\omega_0.Q$ for the longest Weyl group element ω_0, and $A(\overline{Q}\colon Q\colon \delta\colon \nu)$ is also denoted by $A(\omega_0\colon Q\colon \delta\colon \nu)$.*

Theorem 2.1.3.4. *(Langlands and Miličić [7]) Let ν be an element in $\mathfrak{a}_{\mathbb{C}}^*$ such that $Re(\nu)$ is weakly dominant. Let $Q = MAN$ be the minimal parabolic subgroup and δ be a representation of M. The image of the operator $A(\overline{Q}\colon Q\colon \delta\colon \nu)$ is an irreducible submodule of $I_{\overline{Q}}(\delta, \nu)$, and the image is isomorphic to*

$$J_Q(\delta, \nu) = \frac{I_Q(\delta, \nu)}{Ker(A(\overline{Q}\colon Q\colon \delta\colon \nu))}.$$

Now, we are ready to define the long intertwining operator and the Langlands quotient [16][7].

Definition 2.1.3.5. *$J_Q(\delta, \nu)$ is the **Langlands quotient** of $I_Q(\delta, \nu)$. Note that the Langlands quotient is defined if $Re(\nu)$ is weakly dominant.*

In this subsection, we described the Langlands quotient of the principal series representation induced from the minimal parabolic subgroup. Generally, the principal series representation and its Langlands quotient are defined for non-minimal parabolic subgroups as well [16]. In this thesis, we will only use the principal series representation induced from the minimal parabolic subgroup and its Langlands quotient, so we will not explain the general case.

2.1.4 The Spherical Case

Suppose that a principal series representation induced from the minimal parabolic subgroup $I_Q(\delta, \nu)$ is spherical, which means that $\mathrm{Hom}_K(\mathrm{triv}_K, I_Q(\delta, \nu))$ is nonzero. By Frobenius reciprocity, $\mathrm{Hom}_M(\mathrm{triv}_K, \delta)$ is nonzero, which means δ is the trivial representation of M. Therefore, we assume that δ is the trivial representation of M. Notice that $\mathrm{Hom}_M(\mathrm{triv}_K, \mathrm{triv}_M)$ is one-dimensional.

Theorem 2.1.4.1. *[18][16] Assume that $Re(\nu)$ is weakly dominant. The lowest K-type of $I_Q(\delta, \nu)$ is not annihilated by the long intertwining operator. In other words,*

$$\dim(\mathrm{Hom}_K(\mathrm{triv}_K, I_Q(\delta, \nu))) = \dim(\mathrm{Hom}_K(\mathrm{triv}_K, J_Q(\delta, \nu))).$$

For the spherical principal series representation, the lowest K-type is the trivial K-type. The definition of the lowest K-type is given below [16].

Definition 2.1.4.2. *Fix a Cartan subalgebra* $\mathfrak{b}$ *of* $\mathfrak{k}$ *and a positive system for* $\Delta(\mathfrak{b}^{\mathbb{C}}, \mathfrak{k}^{\mathbb{C}})$. *Among* K*-types* $\phi_{\Lambda'}$ *occurring in* π, *the minimal* K*-types of* π *are* ϕ_Λ *for which* $|\Lambda' + 2\delta_K|^2$ *is minimized by* $\Lambda' = \Lambda$.

From 2.1.2.7, an irreducible admissible representation π of G is infinitesimally equivalent to $\mathcal{M}_1/\mathcal{M}_2$ for some subrepresentations $\mathcal{M}_1$ and $\mathcal{M}_2$ of $I_Q(\delta, \nu)$. Also, because

$$I_Q(\mathrm{triv}_M, \nu) \simeq I_Q(\mathrm{triv}_M, \omega\nu)$$

for a Weyl group element ω [16], so it is enough to think the case where $Re(\nu)$ is weakly dominant. If π is spherical, then δ is the trivial representation of M. Since the dimension of $\mathrm{Hom}_M(\mathrm{triv}_K, \mathrm{triv}_M)$ is 1, the dimension of $\mathrm{Hom}_K(\mathrm{triv}, \mathcal{M}_1/\mathcal{M}_2)$ is 1. Therefore, $\dim(\mathrm{Hom}_K(\mathrm{triv}, \mathcal{M}_1))$ is 1, and $\dim(\mathrm{Hom}_K(\mathrm{triv}, \mathcal{M}_2))$ is 0. Only one composition factor of $I_Q(\delta, \nu)$ satisfies this condition, and the Langlands quotient satisfies this condition. Therefore, we obtain the following proposition.

Proposition 2.1.4.3. *The spherical irreducible admissible representation of* G *is expressed as the Langlands quotient of* $I(\nu) := I_Q(\mathrm{triv}, \nu)$,

$$J(\nu) := J_Q(\mathrm{triv}, \nu).$$

Here, $Re(\nu)$ *is weakly dominant.*

Definition 2.1.4.4. *The set of equivalence classes of irreducible unitary representations of* G *is called the* ***unitary dual****. Also, the set of equivalence classes of irreducible spherical unitary representations is called the* ***spherical unitary dual****.*

Therefore, classifying the spherical unitary dual of G is reduced to deciding whether $J(\nu)$ is infinitesimally unitary, which means that there is a positive-definite invariant Hermitian form on it. We discuss this Hermitian form more specifically in the next section.

2.2 Unitary Representations and the Long Intertwining Operator

2.2.1 Unitary Representations and the Invariant Hermitian Form

Suppose $\nu \in \mathfrak{a}^*_{\mathbb{C}}$ is a complex linear functional. The problem of deciding whether $J(\nu)$ is unitary can be reduced to the case when ν is a real linear functional on $\mathfrak{a}$ [16]. Therefore, we focus on only real ν. Note that for weakly dominant real ν, the long intertwining operator is guaranteed to converge.

Theorem 2.2.1.1. *[16][25] A representation of G is unitary if and only if we have a positive definite invariant Hermitian form on it.*

On a $(\mathfrak{g}_{\mathbb{C}}, K)$-module (π, V_π), an invariant Hermitian form is the sesquilinear form satisfying the following conditions [27]:

- $< v, w >= \overline{< w, v >}$
- $< \pi(X)(v), w >=< v, \pi(-\overline{X})(w) >$ for $X \in \mathfrak{g}^{\mathbb{C}}$
- $< \pi(k)(v), \pi(k)(w) >=< v, w >$ for $k \in K$.

Note that $<, >$ is **positive definite** if and only if $< v, v >> 0$ and $<, >$ is **positive semi-definite** if and only if $< v, v >\geq 0$ for all nonzero v. We would like to check if there exists a positive semi-definite invariant Hermitian on $I(\nu)$, and to do this, we need to know under what condition the invariant Hermitian form exists.

Proposition 2.2.1.2. *If there exists an invariant Hermitian form on $I(\nu)$, then the following conditions hold: the longest Weyl group element ω_0 satisfies*

$$\omega_0\delta \simeq \delta,\ \omega_0\nu \simeq -\overline{\nu},\ \omega_0.Q = \omega_0 Q \omega_0^{-1} = \overline{Q}$$

This condition is called ***the formal symmetry condition*** *[25].*

The formal symmetry condition is necessary because $I_Q(\delta, \nu)$ and $I_{\bar{Q}}(\delta, -\overline{\nu})$ should be equivalent to define an invariant Hermitian form on $I_Q(\delta, \nu)$. Suppose that the formal symmetry condition holds for Q, δ, and ν for the rest of the thesis. An invariant Hermitian form on $I_Q(\mathrm{triv}, \nu) = I(\nu)$ is represented by an intertwining operator between $I_Q(\nu)$ and $I_{\omega_0.Q}(\omega_0\delta, \omega_0\nu)$ [19][20][21]. By normalizing this intertwining operator as in [25], we obtain an intertwining operator for weakly dominant real ν.

$$A(\nu) : I(\nu) \to I(\omega_0^{-1}(\nu)).$$

We can obtain $A(\nu)$ by normalizing the long intertwining operator [25]

$$A(\omega_0\colon Q\colon \mathrm{triv}\colon \nu) : I_Q(\nu) \to I_{\omega_0.P}(\nu).$$

Likewise, by normalizing $A(\omega\colon Q\colon \mathrm{triv}\colon \nu)$, we can define $A(\omega, \nu) : I(\nu) \to I(\omega^{-1}(\nu))$. Note that $A(\omega_0, \nu)$ is $A(\nu)$. We factorize $A(\omega, \nu)$.

Theorem 2.2.1.3. *Suppose that $\omega = \omega_1\omega_2$, and $l(\omega) = l(\omega_1) + l(\omega_2)$. Then,*

$$A(\omega, \nu) = A(\omega_1\omega_2, \nu) = A(\omega_1, \omega_2^{-1}\nu) \circ A(\omega_2, \nu).$$

Note that $A(\omega_0\colon Q\colon \mathrm{triv}\colon \nu)$ and $A(\nu)$ are the same linear operator, so we will refer to both of the intertwining operators as the long intertwining operator. We will analyze this operator using K-types in the following subsection.

2.2.2 Unitarity Tests Using K-types

We may canonically pass the long intertwining operator

$$A(\nu) : I(\nu) \to I(\omega_0^{-1}\nu) = I(\omega_0\nu)$$

to the operator

$$A_\phi(\nu) : \mathrm{Hom}_K(\phi, I(\nu)) \to \mathrm{Hom}_K(\phi, I(\omega_0\nu))$$

for K-type ϕ [25]. Since $I(\nu)$ is an admissible representation, $\mathrm{Hom}_K(\phi, I(\nu))$ is finite dimensional. For the intertwining operator

$$A(\omega,\nu) : I(\nu) \to I(\omega^{-1}(\nu)),$$

we will write the produced operator as

$$A_\phi(\omega,\nu) : \mathrm{Hom}_K(\phi, I(\nu)) \to \mathrm{Hom}_K(\phi, I(\omega^{-1}(\nu))).$$

The operator $A(\nu)$ is positive semi-definite if and only if $A_\phi(\nu)$ is positive semi-definite for every K-type ϕ [25]. Using Frobenius reciprocity, we can regard $A_\phi(\nu)$ as a linear operator on M-fixed vectors of ϕ, V_ϕ^M because

$$\mathrm{Hom}_K(\phi, I(\nu)) \simeq V_\phi^M \simeq \mathrm{Hom}_K(\phi, I(\omega^{-1}(\nu))).$$

In this way, the long intertwining operator can be considered as a linear operator on M-fixed vectors of each K-type. Therefore, $J(\nu)$ is unitary if and only if $A_\phi(\nu)$ is positive semi-definite for every K-type ϕ. If $A_\phi(\nu)$ is not positive semi-definite for some K-type ϕ, then the long intertwining operator $A(\nu)$ fails to be positive semi-definite, and therefore, $J(\nu)$ is not unitary [25]. We will call this calculation **non-unitarity test using K-type ϕ**. Note that if the formal symmetry condition holds, $A_\phi(\omega_0,\nu)$ on V_ϕ^M is a Hermitan operator. We choose a basis of V_ϕ^M to represent this hermitian operator as a matrix A. Regardless of the basis that we choose, the set of eigenvalues of A is fixed, and $A_\phi(\omega_0,\nu)$ is positive definite if and only if the eigenvalues of A are all positive. Therefore, to check that the invariant Hermitian form on V_ϕ^M is positive definite, it is enough to check that every eigenvalue of A is nonnegative: we will calculate an example for $SL(3,\mathbb{R})$ in 2.3.3 and 2.4.3.

2.3 Calculating the Long Intertwining Operators

2.3.1 The Long Intertwining Operators for Real Rank One Groups

Throughout this subsection, let G be a Lie group with real rank one. For real rank one groups such as $SU(n,1)$, $SO(n,1)$, $Sp(n,1)$, and $SL(2,\mathbb{R})$, every K-type has one dimensional M-fixed vectors. For example, for $G = SO(n,1)$, the representation of K on $\mathfrak{p}$ has the M-fixed vector space $\mathfrak{a}$, which is one dimensional for real rank one groups. Therefore, $A_\phi(\nu)$ is a scalar operator. Note that $A_\phi(\nu)$ is positive semi-definite if and only if the scalar is non-negative. The scalar is calculated by Kostant [23], which we are going to explain in this subsection. For real rank one groups, the K-types with nonzero M-fixed vectors are basically parametrized by two non-negative integers [23]. We will write the K-type corresponding to (i,j) as $\phi_{i,j}$. For this paper, we will mainly focus on the K-types corresponding to $(0,1)$ and $(0,0)$, and we will not discuss how to generally parametrize the K-types with nonzero M-fixed vectors. Those who are interested should refer to [23].

$$\phi_{(0,0)} \Leftrightarrow \text{triv } K\text{-type}$$
$$\phi_{(0,1)} \Leftrightarrow \text{irreducible subrepresentation of } K \text{ on } \mathfrak{p}_{\mathbb{C}}$$

The representation of K on $\mathfrak{p}_{\mathbb{C}}$ may not be irreducible, but every irreducible subrepresentation corresponds to $(0,1)$ [23][1].

Definition 2.3.1.1. *We call an irreducible subrepresentation of K on $\mathfrak{p}_{\mathbb{C}}$ as a* ***$\mathfrak{p}$-representation****.*

Theorem 2.3.1.2. *(Kostant) Let a be an element in $\mathfrak{a}$ such that $\alpha(a) = 1$. Note that a is a basis of $\mathfrak{a}$. Suppose $\nu(a) = \frac{<\nu,\alpha^\vee>}{2} = h$. The long intertwining operator $A(\nu)$ acts on the M-fixed vector of $\phi_{i,j}$ with the following scalar after normalization.*

$$\frac{(m-h)(m-(h+2))(m-(h+4))...(m-(h+2(i+j)-2))}{(m+h)(m+(h+2))(m+(h+4))...(m+(h+2(i+j)-2))} \times \frac{(m-(h-s+1))(m-(h-s+3))...(m-(h-s+2i-1))}{(m+(h-s+1))(m+(h-s+3))...(m+(h-s+2i-1))}$$

Here, $s = \dim \mathfrak{g}_{2\alpha}$, and $m = \frac{\dim \mathfrak{g}_\alpha}{2} + s$.

Note that

- This scalar is 1 for $\phi_{0,0}$.
- This scalar is $\frac{m-h}{m+h} = \frac{m-<\nu,\alpha^\vee>}{m+<\nu,\alpha^\vee>}$ for $\phi_{0,1}$. Here, α is the simple root of G.

2.3.2 Decomposition of the Long Intertwining Operator

For groups of real rank higher than 1, a K-type has the M-fixed vector space whose dimension is larger than one generally. If the M-fixed vector space has dimension higher than 1, then the action of the long intertwining operator is not guaranteed to be scalar anymore. However, we can still calculate the long intertwining operators by decomposing it into simpler intertwining operators – the long intertwining operator of real rank one subgroups [10][26][16]. In this subsection, we are going to explain the decomposition and the calculation of the long intertwining operator. We need to define a real rank one subgroup corresponding to a reduced root.

Definition 2.3.2.1. *Fix a reduced restricted root α. The **real rank one subalgebra corresponding to** α is denoted by $\mathfrak{g}^\alpha$. It is a Lie subalgebra of $\mathfrak{g}$ generated by $\mathfrak{g}_{n\alpha}$, where n runs over nonzero integers. **The real rank one subgroup corresponding to** α is the analytic subgroup of G with Lie algebra $\mathfrak{g}^\alpha$, and it is denoted by G^α. The maximal compact subgroup of G^α is K^α. We define the Cartan decomposition*

$$\mathfrak{g}^\alpha = \mathfrak{k}^\alpha \oplus \mathfrak{p}^\alpha$$

such that $\mathfrak{k}^\alpha = \mathfrak{g}^\alpha \cap \mathfrak{k}$ and $\mathfrak{p}^\alpha = \mathfrak{g}^\alpha \cap \mathfrak{p}$.

The subgroup G^α has real rank one, so the long intertwining operator is a scalar operator for each irreducible K^α representation. Here are the steps to decompose the long intertwining operator [25].

- Calculate a reduced decomposition of ω_0:

$$\omega_0 = s_{\alpha_l} s_{\alpha_{l-1}} s_{\alpha_{l-2}} \dots s_{\alpha_1}$$

using Kostant's cascade construction. Here, l is the length of ω_0.

- Decompose the long intertwining operator $A(\omega_0, \nu)$ correspondingly:

$$A(\omega_0, \nu) = A(s_{\alpha_l} s_{\alpha_{l-1}} s_{\alpha_{l-2}} \dots s_{\alpha_1}, \nu)$$
$$= A(s_{\alpha_1}, s_{\alpha_{l-1}} s_{\alpha_{l-2}} \dots s_{\alpha_1} \nu) A(s_{\alpha_{l-1}}, s_{\alpha_{l-2}} \dots s_{\alpha_1} \nu) \dots A(s_{\alpha_1}, \nu).$$

- Decompose $A_\phi(\omega_0, \nu)$, the long intertwining operator for K-type ϕ, correspondingly:

$$A_\phi(\omega_0, \nu) = A_\phi(s_{\alpha_l} s_{\alpha_{l-1}} s_{\alpha_{l-2}} \dots s_{\alpha_1}, \nu)$$
$$= A_\phi(s_{\alpha_1}, s_{\alpha_{l-1}} s_{\alpha_{l-2}} \dots s_{\alpha_1} \nu) A_\phi(s_{\alpha_{l-1}}, s_{\alpha_{l-2}} \dots s_{\alpha_1} \nu) \dots A_\phi(s_{\alpha_1}, \nu).$$

- We can interpret the intertwining operators $A_\phi(s_\beta, \mu)$ as a linear operator on V_ϕ^M.

- Decompose ϕ into irreducible representations of MK^β: $\phi = \bigoplus_i \phi_i$. Then,

$$(V_\phi|_{MK^\beta})^M = \bigoplus_i V_{\phi_i}^M.$$

- $A(s_\beta, \mu)$ satisfying $< \mu, \beta^\vee >> 0$ can be interpreted as a long intertwining operator for real rank one subgroup MG^β.

- $A(s_\beta, \mu)$ acts on the M-fixed vectors of each direct summand by scalar. Calculate the scalar using Kostant's theorem 2.3.1.2.

2.3.3 Example of $GL(3, \mathbb{C})$

In this subsection, we calculate an example of 2.3.2, $GL(3, \mathbb{C})$: we will calculate the long intertwining operator for the K-type on $\mathfrak{p}_\mathbb{C} \cap sl(3, \mathbb{C})$. The followings are the notations that we will use in this subsection.

- $G = GL(3, \mathbb{C})$

- $K = U(3)$

- $\mathfrak{p} = \{X \mid X^* = X\}$
- $\mathfrak{a} = \{\begin{pmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{pmatrix} \mid d_1, d_2, d_3 \in \mathbb{R}\} = \text{span}_{\mathbb{R}} < E_{1,1}, E_{2,2}, E_{3,3} >$
- $M = U(1)^3$
- reduced roots: $\Delta = \{\alpha_i - \alpha_j | i \neq j\}$ where $\alpha_i(\Sigma_{j=1}^3 d_j E_{j,j}) = d_i$
- simple roots: $\Pi = \{\alpha_1 - \alpha_2, \alpha_2 - \alpha_3\}$
- $G^{\alpha_1 - \alpha_2} = \{\begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix} \mid A \in SL(2, \mathbb{C})\}$
- $K^{\alpha_1 - \alpha_2} = \{\begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix} \mid A \in SU(2)\}$
- $MG^{\alpha_1 - \alpha_2} = \{\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \mid A \in GL(2, \mathbb{C}), |\det(A)| = 1, B \in U(1)\}$
- $MK^{\alpha_1 - \alpha_2} = \{\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \mid A \in U(2), B \in U(1)\}$
- $G^{\alpha_2 - \alpha_3} = \{\begin{pmatrix} 1 & 0 \\ 0 & A \end{pmatrix} \mid A \in SL(2, \mathbb{C})\}$
- $K^{\alpha_2 - \alpha_3} = \{\begin{pmatrix} 1 & 0 \\ 0 & A \end{pmatrix} \mid A \in SU(2)\}$
- $MG^{\alpha_1 - \alpha_2} = \{\begin{pmatrix} B & 0 \\ 0 & A \end{pmatrix} \mid A \in GL(2, \mathbb{C}), |\det(A)| = 1, B \in U(1)\}$
- $MK^{\alpha_1 - \alpha_2} = \{\begin{pmatrix} B & 0 \\ 0 & A \end{pmatrix} \mid A \in U(2), B \in U(1)\}$
- $\omega_0 = s_{\alpha_1 - \alpha_3} = s_{\alpha_2 - \alpha_3} s_{\alpha_1 - \alpha_2} s_{\alpha_2 - \alpha_3}$

- $m = \dim g_\alpha + 2\dim g_{2\alpha}$=2
- $V_\phi = \{X \in \mathfrak{p}_\mathbb{C} \mid tr(X) = 0\}$
- $V_\phi^M = \{X \in \mathfrak{a}_\mathbb{C} \mid tr(X) = 0\} = \text{span} < v_1 = E_{1,1} - E_{2,2}, v_2 = E_{2,2} - E_{3,3} >$

Let us put

$$\nu = x\alpha_1 + y\alpha_2 + z\alpha_3.$$

Since we are assuming ν is real and the formal symmetry condition holds, x, y, and z is real and $y = 0$ and $x = -z$. We represent the following long intertwining operator.

$$A(\omega_0, \nu) = A(s_{\alpha_2-\alpha_3}s_{\alpha_1-\alpha_2}s_{\alpha_2-\alpha_3}, \nu) =$$
$$A(s_{\alpha_2-\alpha_3}, s_{\alpha_1-\alpha_2}s_{\alpha_2-\alpha_3}\nu)A(s_{\alpha_1-\alpha_2}, s_{\alpha_2-\alpha_3}\nu)A(s_{\alpha_2-\alpha_3}, \nu)$$

We restrict ϕ to $MK^{\alpha_1-\alpha_2}$. Then, it is expressed as a direct sum of irreducible representations of $MK^{\alpha_1-\alpha_2}$. Only two direct summands have nonzero M-fixed vectors. One copy is the trivial representation and the other copy is the representation on $V_\phi \cap \mathfrak{g}^{\alpha_1-\alpha_2}$. Each direct summand has a one-dimensional M-fixed vector space, and $A_\phi(s_{\alpha_1-\alpha_2}, \nu)$ acts by scalar on M-fixed vectors in each direct summand. The scalar is 1 for the M-fixed vectors of the trivial representations, and it is

$$\frac{2- < \nu, (\alpha_1 - \alpha_2)^\vee >}{2+ < \nu, (\alpha_1 - \alpha_2)^\vee >}$$

for the M-fixed vectors in $V_\phi \cap \mathfrak{g}^{\alpha_1-\alpha_2}$. The M-fixed vector space in $V_\phi \cap \mathfrak{g}^{\alpha_1-\alpha_2}$ is

$$\text{span} < E_{1,1} - E_{2,2} >,$$

and the trivial representation of $MK^{\alpha_1-\alpha_2}$ has the representation space

$$\text{span} < E_{1,1} + E_{2,2} - 2E_{3,3} >,$$

which are M-fixed. In the same way, when we restrict ϕ to $MK^{\alpha_2-\alpha_3}$, it is expressed as a direct sum of representations of $MK^{\alpha_2-\alpha_3}$. Only two direct summands have nonzero M-fixed vectors. One copy is the trivial representations, and the other copy is the representation on $V_\phi \cap \mathfrak{g}^{\alpha_2-\alpha_3}$. Each direct summand has a 1-dimensional M-fixed vector space, and $A_\phi(s_{\alpha_2-\alpha_3},\nu)$ acts by scalar on M-fixed vectors in each direct summand. The scalar is 1 for M-fixed vectors of trivial representations, and it is

$$\frac{2-<\nu,(\alpha_2-\alpha_3)^\vee>}{2+<\nu,(\alpha_2-\alpha_3)^\vee>}$$

for the M-fixed vectors in $V_\phi \cap \mathfrak{g}^{\alpha_2-\alpha_3}$. The M-fixed vector space in $V_\phi \cap \mathfrak{g}^{\alpha_2-\alpha_3}$ is

$$\text{span} < E_{2,2} - E_{3,3} >,$$

and the trivial representation of $MK^{\alpha_2-\alpha_3}$ has the representation space

$$\text{span} < E_{2,2} + E_{3,3} - E_{1,1} >,$$

which are M-fixed. Therefore, the action of the intertwining operators on the M-fixed vectors are calculated as follows. For matrix representation, we choose $\{v_1, v_2\}$ for the basis of V_ϕ^M.

- The intertwining operator $A_\phi(s_{\alpha_2-\alpha_3},\nu)$ acts on V_ϕ^M by

$$A_3 = \begin{pmatrix} 1 & 0 \\ \frac{1}{2} & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \frac{2-y+z}{2+y-z} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\frac{1}{2} & 1 \end{pmatrix}$$

  since $<\nu,(\alpha_2-\alpha_3)^\vee>=<x\alpha_1+y\alpha_2+z\alpha_3, \alpha_2-\alpha_3>= y-z$.

- The intertwining operator $A_\phi(s_{\alpha_1-\alpha_2}, s_{\alpha_2-\alpha_3}\nu)$ acts on V_ϕ^M by

$$A_2 = \begin{pmatrix} 1 & \frac{1}{2} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \frac{2-x+z}{2+x-z} \end{pmatrix} \begin{pmatrix} 1 & -\frac{1}{2} \\ 0 & 1 \end{pmatrix}$$

  since $<s_{\alpha_2-\alpha_3}(\nu),(\alpha_1-\alpha_2)^\vee>=<x\alpha_1+z\alpha_2+y\alpha_3, \alpha_1-\alpha_2>= x-z$.

- The intertwining operator $A_\phi(s_{\alpha_2-\alpha_3}, s_{\alpha_1-\alpha_2}s_{\alpha_2-\alpha_3}\nu)$ acts on V_ϕ^M by

$$A_1 = \begin{pmatrix} 1 & 0 \\ \frac{1}{2} & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \frac{2-x+y}{2+x-y} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\frac{1}{2} & 1 \end{pmatrix}$$

since $< s_{\alpha_1-\alpha_2}s_{\alpha_2-\alpha_3}(\nu), (\alpha_2-\alpha_3)^\vee >=< z\alpha_1 + x\alpha_2 + y\alpha_3, \alpha_2 - \alpha_3 >= x - y$.

Note that this operator is not generally Hermitian, but it is a similar linear operator to the Hermitian operator that defines the invariant Hermitian form. The long intertwining operator $A(\omega_0, \nu)$ acts on V_ϕ^M by the matrix $A_1A_2A_3$. The Hermitian form, which is represented by the long intertwining operator, is not positive semi-definite when one of the eigenvalues of $A_1A_2A_3$ is negative. However, it is very hard to calculate the eigenvalues of a product of many matrices. Therefore, we like to use Weyl group algebra, which is explained in the next section.

2.4 Weyl Group Representations

2.4.1 The Long Intertwining Operators for Real Rank One Groups

The long intertwining operator acts on the M-fixed vectors of each K-type as we explained in the previous section. On the other hand, Weyl group elements act on the M-fixed vectors of each K-type as well. Weyl group elements are represented by $N_K(\mathfrak{a})/Z_K(\mathfrak{a})$, and the Weyl group representatives in $N_K(\mathfrak{a})$ stabilizes the M-fixed vectors. Therefore, we can define a Weyl group representation on the M-fixed vectors of each K-type. **Let us denote the Weyl group representation on V_ϕ^M for K-type (ϕ, V_ϕ) by ψ_ϕ.** In this subsection, we explain the real rank one case, and in the next two subsections, we explain how we can extend this idea to higher real rank groups.

Suppose that G is a real rank one group, and α is the unique simple root. Then s_α acts on the M-fixed vectors of each K-type, by scalar $+1$ or -1. For example, for the trivial K-type, s_α acts by 1, and for a $\mathfrak{p}$-representation of K, s_α acts by -1. Actually, these two K-types are K-types of our interest, because we can represent the action of the long intertwining operator using Weyl group elements. As we explained in 2.3.2, the long intertwining operator $A(\nu)$ acts by scalar 1 for M-fixed vectors of the trivial K-type, the K-type with Kostant's parameters $i = 0$, $j = 0$. The long intertwining operator acts by scalar

$$\frac{m - < \nu, \alpha^\vee >}{m + < \nu, \alpha^\vee >}$$

on the M-fixed vectors of $\mathfrak{p}$-representation of K, the K-type with Kostant's parameters $i = 0$, $j = 1$. Here, m depends on the multiplicity of α and 2α. Specifically,

$$m = \dim \mathfrak{g}_\alpha + 2 \dim \mathfrak{g}_{2\alpha}.$$

Therefore, we can write the action of the long intertwining operator as

$$A_{s_\alpha}(\nu) := \frac{m}{m + < \nu, \alpha^\vee >} + \frac{< \nu, \alpha^\vee >}{m + < \nu, \alpha^\vee >} s_\alpha$$

for the trivial K-type and a $\mathfrak{p}$-representation of K. Note that $A_{s_\alpha}(\nu)$ is in the Weyl group algebra $\mathbb{C}[w]$, and we extend the representation of Weyl group to the representation of Weyl group algebra canonically. The Weyl group representations ψ_{triv} and $\psi_{\mathfrak{p}}$ act on $A_{s_\alpha}(\nu)$ by

$$\psi_{triv}\left(\frac{m}{m + < \nu, \alpha^\vee >} + \frac{< \nu, \alpha^\vee >}{m + < \nu, \alpha^\vee >} s_\alpha\right) = 1$$

$$\psi_{\mathfrak{p}}\left(\frac{m}{m + < \nu, \alpha^\vee >} + \frac{< \nu, \alpha^\vee >}{m + < \nu, \alpha^\vee >} s_\alpha\right) = \frac{m - < \nu, \alpha^\vee >}{m + < \nu, \alpha^\vee >}$$

respectively. For other K-types corresponding to Kostant's parameter (i, j) with $2i + j > 1$, the scalar by which the long intertwining operator acts is more complicated, and it is not possible to express the long intertwining operator action simply using a Weyl group algebra element.

2.4.2 The Long Intertwining Operators for Groups with Higher Real Rank

For groups with real rank greater than 1, the long intertwining operator is not guaranteed to be scalar. However, roughly speaking, we can decompose the long intertwining operator into a composite of the long intertwining operators of real rank one subgroups [16] as in 2.2.1.3 and 2.3.2:

$$A(\omega_0,\nu) = A(s_{\alpha_l}s_{\alpha_{l-1}}s_{\alpha_{l-2}}\dots s_{\alpha_1},\nu)$$
$$= A(s_{\alpha_1}, s_{\alpha_{l-1}}s_{\alpha_{l-2}}\dots s_{\alpha_1}\nu)A(s_{\alpha_{l-1}}, s_{\alpha_{l-2}}\dots s_{\alpha_1}\nu)\dots A(s_{\alpha_1},\nu).$$

Correspondingly,

$$A_\phi(\omega_0,\nu) = A_\phi(s_{\alpha_l}s_{\alpha_{l-1}}s_{\alpha_{l-2}}\dots s_{\alpha_1},\nu)$$
$$= A_\phi(s_{\alpha_1}, s_{\alpha_{l-1}}s_{\alpha_{l-2}}\dots s_{\alpha_1}\nu)A_\phi(s_{\alpha_{l-1}}, s_{\alpha_{l-2}}\dots s_{\alpha_1}\nu)\dots A_\phi(s_{\alpha_1},\nu).$$

Suppose that we can describe the action of $A_\phi(s_\beta,\mu)$ using a Weyl group algebra element for every $\mu \in \nu^*$ and $\beta \in \Pi$ satisfying

$$< \mu, \beta^\vee > \geq 0.$$

Then, we can describe $A_\phi(\omega_0,\nu)$ using a Weyl group algebra element. A K-type ϕ is restricted to direct sum of irreducible representations of MK^β, and suppose that the $A(s_\beta,\mu)$ acts by

$$1 \text{ or } \tfrac{m-<\mu,\beta^\vee>}{m+<\mu,\beta^\vee>}$$

on M-fixed vectors of each direct summand. Here, $m = \dim \mathfrak{g}_\beta + 2\dim \mathfrak{g}_{2\beta}$. Note that s_β acts by 1 or -1 on the M-fixed vectors of MK^β-types. Suppose the following condition holds for the K-type ϕ.

Condition 2.4.2.1. *The Weyl group element s_β acts by 1 on an M-fixed vector v if and only if $A_\phi(s_\beta,\mu)$ acts by 1 on v. Also, s_β acts by -1 on an M-fixed vector v if and only if $A_\phi(s_\beta,\mu)$ acts by $\frac{m-<\mu,\beta^\vee>}{m+<\mu,\beta^\vee>}$ on v.*

Proposition 2.4.2.2. *If 2.4.2.1 holds for K-type ϕ, we can represent the action of $A_\phi(s_\beta, \mu)$ using a Weyl group algebra element*

$$A_{s_\beta}(\mu) := \frac{m}{m+ < \mu, \beta^\vee >} + \frac{< \mu, \beta^\vee >}{m+ < \mu, \beta^\vee >} s_\beta.$$

If we can repeat this calculation for $A_\phi(s_\beta, \mu)$ that appears in the composite of $A_\phi(\omega_0, \nu)$, then we can write the long intertwining operator as a product of Weyl group algebra elements. Specifically, we replace each factor in the decomposition of the long intertwining operator by the Weyl group algebra element of the form $A_{s_\beta}(\mu)$. Now,

$$A_\phi(\omega_0, \nu) = A_\phi(s_{\alpha_l} s_{\alpha_{l-1}} s_{\alpha_{l-2}} \dots s_{\alpha_1}, \nu)$$
$$= A_\phi(s_{\alpha_1}, s_{\alpha_{l-1}} s_{\alpha_{l-2}} \dots s_{\alpha_1}\nu) A_\phi(s_{\alpha_{l-1}}, s_{\alpha_{l-2}} \dots s_{\alpha_1}\nu) \dots A_\phi(s_{\alpha_1}, \nu)$$

becomes

$$A_{\omega_0}(\nu) := A_{s_{\alpha_1}}(s_{\alpha_{l-1}} s_{\alpha_{l-2}} \dots s_{\alpha_1}\nu) A_{s_{\alpha_{l-1}}}(s_{\alpha_{l-2}} \dots s_{\alpha_1}\nu) \dots A_{s_{\alpha_1}}(\nu).$$

Note that $A_{\omega_0}(\nu)$ does not depend on ϕ. Since $A_{\omega_0}(\nu)$ is the multiplication of many Weyl group algebra elements, it is also a Weyl group algebra element. Using this Weyl group algebra element and an irreducible representation of W, we obtain a unitarity test in the p-adic setting. **Irreducible representations of W are also called W-types.** Using a K-type ϕ satisfying 2.4.2.1,

$$A_\phi(\omega_0, \nu) = A_\phi(s_{\alpha_l} s_{\alpha_{l-1}} s_{\alpha_{l-2}} \dots s_{\alpha_1}, \nu)$$
$$= A_\phi(s_{\alpha_1}, s_{\alpha_{l-1}} s_{\alpha_{l-2}} \dots s_{\alpha_1}\nu) A_\phi(s_{\alpha_{l-1}}, s_{\alpha_{l-2}} \dots s_{\alpha_1}\nu) \dots A_\phi(s_{\alpha_1}, \nu)$$

is represented by

$$\psi(A_{\omega_0}(\nu)) = \psi(A_{s_{\alpha_l} s_{\alpha_{l-1}} s_{\alpha_{l-2}} \dots s_{\alpha_1}}(\nu))$$
$$= \psi(A_{s_{\alpha_1}}(s_{\alpha_{l-1}} s_{\alpha_{l-2}} \dots s_{\alpha_1}\nu)) \psi(A_{s_{\alpha_{l-1}}}(s_{\alpha_{l-2}} \dots s_{\alpha_1}\nu)) \dots \psi(A_{s_{\alpha_1}}(\nu))$$

where ψ is the Weyl group representation on V_ϕ^M. We note that $A_\phi(\omega_0, \nu)$ is positive definite if and only if every eigenvalue of $\psi(A_{\omega_0}(\nu))$ is nonnegative.

Definition 2.4.2.3. *The unitarity test using W-type ψ is to check if every eigenvalue of $\psi(A_{\omega_0}(\nu))$ is nonnegative.*

We construct K-types satisfying 2.4.2.1 in Chapter 3.

Proposition 2.4.2.4. *If a K-type ϕ satisfies 2.4.2.1, then $A_\phi(\nu)$ can be represented by $\psi(A_{\omega_0}(\nu))$ for the Weyl group representation ψ on V_ϕ^M.*

2.4.3 Example of $GL(3,\mathbb{C})$

In this subsection, we calculate an example 2.4.2 for $G = SL(3,\mathbb{C})$. The notation is the same as 2.3.3. We will represent the long intertwining operator $A_\phi(\nu)$ using a Weyl group algebra element. Since we are assuming

$$\nu = xE_{1,1} + yE_{2,2} + zE_{3,3}$$

is real, x, y, and z are real. Also, $y = 0$ and $x = -z$ since we assume that the formal symmetry condition holds. This is possible because every $v \in V_\phi^M$ satisfies 2.4.2.1, which we will check later in this subsection. Note that the following holds.

$$s_\beta \text{ acts by } 1 \text{ on } v \Leftrightarrow A_\phi(s_\beta,\mu) \text{ acts by } 1 \text{ on } v$$
$$s_\beta \text{ acts by } -1 \text{ on } v \Leftrightarrow A_\phi(s_\beta,\mu) \text{ acts by } \tfrac{m-<\mu,\beta^\vee>}{m+<\mu,\beta^\vee>} \text{ on } v$$

We restrict ϕ to $MK^{\alpha_1-\alpha_2}$. Then it is expressed as a direct sum of two irreducible representations of $MK^{\alpha_1-\alpha_2}$. One copy is the representation on $V_\phi \cap \mathfrak{g}^{\alpha_1-\alpha_2}$, and the other copy is the trivial representation. Each direct summand has the one-dimensional M-fixed vector space, and $A(s_{\alpha_1-\alpha_2},\nu)$ acts by scalar on the M-fixed vectors in each direct summand. The scalar is 1 for the M-fixed vectors of the trivial $MK^{\alpha_1-\alpha_2}$-type, and it is $\frac{2-<\nu,(\alpha_1-\alpha_2)^\vee>}{2+<\nu,(\alpha_1-\alpha_2)^\vee>}$ for the M-fixed vectors of the $MK^{\alpha_1-\alpha_2}$-type on $V_\phi \cap \mathfrak{g}^{\alpha_1-\alpha_2}$. On the other hand, $s_{\alpha_1-\alpha_2}$ acts by $+1$ on the M-fixed vectors of the trivial $MK^{\alpha_1-\alpha_2}$-type, and $s_{\alpha_1-\alpha_2}$ acts by -1 on the M-fixed vectors of the $MK^{\alpha_1-\alpha_2}$-type on $V_\phi \cap \mathfrak{g}^{\alpha_1-\alpha_2}$. Therefore, $A(s_{\alpha_1-\alpha_2},\nu)$ acts on M-fixed vectors of ϕ by

$$\frac{2}{2+<\nu,(\alpha_1-\alpha_2)^\vee>} + \frac{<\nu,(\alpha_1-\alpha_2)^\vee>}{2+<\nu,(\alpha_1-\alpha_2)^\vee>} s_{\alpha_1-\alpha_2}.$$

Likewise, if we restrict ϕ to $MK^{\alpha_2-\alpha_3}$, then it is expressed as a direct sum of two irreducible representations of $MK^{\alpha_2-\alpha_3}$. One copy is the $MK^{\alpha_2-\alpha_3}$-type on $V_\phi \cap \mathfrak{g}^{\alpha_2-\alpha_3}$ and the other copy is the trivial $MK^{\alpha_2-\alpha_3}$-type. Each direct summand has the one-dimensional M-fixed vector space, and $A(s_{\alpha_2-\alpha_3}, \nu)$ acts by scalar on the M-fixed vectors in each direct summand. The scalar is 1 for the M-fixed vectors of the trivial $MK^{\alpha_2-\alpha_3}$-type, and it is $\frac{2-<\nu,(\alpha_2-\alpha_3)^\vee>}{2+<\nu,(\alpha_2-\alpha_3)^\vee>}$ for the M-fixed vectors of $MK^{\alpha_2-\alpha_3}$-type on $V_\phi \cap \mathfrak{g}^{\alpha_2-\alpha_3}$. On the other hand, $s_{\alpha_2-\alpha_3}$ acts by $+1$ on the M-fixed vectors of the trivial $MK^{\alpha_2-\alpha_3}$-type, and $s_{\alpha_2-\alpha_3}$ acts by -1 on the M-fixed vectors of $MK^{\alpha_2-\alpha_3}$ representation on $V_\phi \cap \mathfrak{g}^{\alpha_2-\alpha_3}$. Therefore, $A(s_{\alpha_2-\alpha_3}, \nu)$ acts on the M-fixed vectors by

$$\frac{2}{2+<\nu,(\alpha_2-\alpha_3)^\vee>} + \frac{<\nu,(\alpha_2-\alpha_3)^\vee>}{2+<\nu,(\alpha_2-\alpha_3)^\vee>} s_{\alpha_2-\alpha_3}.$$

We note that 2.4.2.1 holds for ϕ, so we can represent $A_\phi(\nu)$ by a Weyl group algebra element.

We apply the computations above to the long intertwining operator $A(\nu)$:

$$A(\omega_0,\nu) = A(s_{\alpha_2-\alpha_3}s_{\alpha_1-\alpha_2}s_{\alpha_2-\alpha_3},\nu) =$$
$$A(s_{\alpha_2-\alpha_3}, s_{\alpha_1-\alpha_2}s_{\alpha_2-\alpha_3}\nu)A(s_{\alpha_1-\alpha_2}, s_{\alpha_2-\alpha_3}\nu)A(s_{\alpha_2-\alpha_3},\nu).$$

We proceed part by part.

- $A(s_{\alpha_2-\alpha_3},\nu)$ acts on V_ϕ^M by

  $$\frac{2}{2+<\nu,(\alpha_2-\alpha_3)^\vee>} + \frac{<\nu,(\alpha_2-\alpha_3)^\vee>}{2+<\nu,(\alpha_2-\alpha_3)^\vee>} s_{\alpha_2-\alpha_3} = \frac{2}{2+y-z} + \frac{y-z}{2+y-z} s_{\alpha_2-\alpha_3}$$

  since $<\nu,(\alpha_2-\alpha_3)^\vee> = y-z$

- $A(s_{\alpha_1-\alpha_2}, s_{\alpha_2-\alpha_3}\nu)$ acts on V_ϕ^M by

  $$\frac{2}{2+x-z} + \frac{x-z}{2+x-z} s_{\alpha_1-\alpha_2}$$

  since $< s_{\alpha_2-\alpha_3}(\nu), (\alpha_1-\alpha_2)^\vee > = < x\alpha_1 + z\alpha_2 + y\alpha_3, (\alpha_1-\alpha_2)^\vee > = x-z$

- $A(s_{\alpha_2-\alpha_3}, s_{\alpha_1-\alpha_2}s_{\alpha_2-\alpha_3}\nu)$ acts on V_ϕ^M by

$$\frac{2}{2+x-y}+\frac{x-y}{2+x-y}s_{\alpha_2-\alpha_3}$$

since $< s_{\alpha_1-\alpha_2}s_{\alpha_2-\alpha_3}\nu, (\alpha_2-\alpha_3)^\vee >=< z\alpha_1+x\alpha_2+y\alpha_3, (\alpha_2-\alpha_3)^\vee >= x-y$

Therefore, the long intertwining operator $A(\omega_0, \nu)$ acts on V_ϕ^M by the Weyl group algebra element

$$A := A_{\omega_0}(\nu) = (\tfrac{2}{2+x-y}+\tfrac{x-y}{2+x-y}s_{\alpha_2-\alpha_3})(\tfrac{2}{2+x-z}+\tfrac{x-z}{2+x-z}s_{\alpha_1-\alpha_2})(\tfrac{2}{2+y-z}+\tfrac{y-z}{2+y-z}s_{\alpha_2-\alpha_3})$$

Note that $A_\phi(\nu)$ is positive semi-definite if and only if every eigenvalue of $\psi_\phi(A)$ is positive. Since ψ_ϕ is a standard representation on $\mathfrak{a}$, $s_{\alpha_1-\alpha_2}$ is represented by $E_{1,2}-E_{2,1}+E_{3,3}$ and $s_{\alpha_2-\alpha_3}$ is represented by $E_{1,1}+E_{2,3}-E_{3,2}$. Then, $\psi_\phi(A)$ is represented by $B_1B_2B_3$ where

$$B_1 = \frac{2}{2+x-y}\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}+\frac{x-y}{2+x-y}\begin{pmatrix}1&0&0\\0&0&1\\0&-1&0\end{pmatrix}$$

$$B_2 = \frac{2}{2+x-z}\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}+\frac{x-z}{2+x-z}\begin{pmatrix}0&1&0\\-1&0&0\\0&0&1\end{pmatrix}$$

$$B_3 = \frac{2}{2+y-z}\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}+\frac{y-z}{2+y-z}\begin{pmatrix}1&0&0\\0&0&1\\0&-1&0\end{pmatrix}.$$

Note that every eigenvalue of $B_1B_2B_3$ is positive if and only if every eigenvalue of $\psi_\phi(A)$ is positive.

One of the benefits of using a Weyl group algebra element is that we do not need to calculate "the change of basis" matrices, which keep track of M-fixed vectors in each irreducible direct summand when restricted to K^α with varying simple root α.

Another benefit is that we can make correspondence between the unitarity tests using K-types and the unitarity tests using Weyl group representations.

2.4.4 Irreducible Representation of W

In this subsection, we introduce the parametrization of W-types for classical groups. The details and proofs are provided in [11].

The Weyl groups of classical groups have the following forms:

- S_n: $SL(n,\mathbb{C})$, $SL(n,\mathbb{R})$, $SL(n,\mathbb{H})$
  generated by $\{s_{\alpha_i-\alpha_j} \mid 1 \le i \ne j \le n\}$

- $S_n \times (\mathbb{Z}/2\mathbb{Z})^n$: $SU(m,n)$, $SO(m,n)$, $Sp(m,n)$, $Sp(n,\mathbb{R})$, $Sp(n,\mathbb{C})$, $SO(2n+1,\mathbb{C})$
  generated by $\{s_{\alpha_i-\alpha_j} \mid 1 \le i \ne j \le n\} \cup \{s_{\alpha_i} \mid 1 \le i \le n\}$

- R_n: Index 2 subgroup of $S_n \times (\mathbb{Z}/2\mathbb{Z})^n$: $SO(2n,\mathbb{C})$
  generated by $\{s_{\alpha_i\pm\alpha_j} \mid 1 \le i \ne j \le n\}$.

We illustrate irreducible representations of S_n, $S_n \times (\mathbb{Z}/2\mathbb{Z})^n$, and R_n.

(1) irreducible representations of S_n:

The irreducible representations of S_n are parametrized by partitions of n, which are Young diagrams of size n. Let $\lambda = (a_1, a_2, \ldots, a_n)$ be a partition of n. Then the S_n-type parametrized by λ is a subrepresentation of

$$\mathrm{Ind}_{S_{a_1}\times S_{a_2}\times \ldots S_{a_n}}^{S_n}(\mathrm{triv}\otimes\mathrm{triv}\otimes\ldots\mathrm{triv}),$$

but it is not a subrepresentation of

$$\mathrm{Ind}_{S_{c_1}\times S_{c_2}\times \ldots S_{c_n}}^{S_n}(\mathrm{triv}\otimes\mathrm{triv}\otimes\ldots\mathrm{triv})$$

for

$$(c_1, c_2, c_3, \ldots, c_n) \ngeq (a_1, a_2, a_3, \ldots, a_n).$$

We denote the S_n representation parametrized by λ by ψ^λ. Then, the S_n representation $\psi^\lambda \otimes \text{sign}$ is parametrized by ψ^{λ^T}. **Here, λ^T is a Young diagram such that ith column has a_i boxes.** For each Young tableau A of shape λ, we define a vector

$$u_A = e_{x_1} \otimes \ e_{x_2} \otimes \ldots \otimes e_{x_n}$$

such x_i is j if and and only if i is in the jth row of A.

$$Row(A) = \{s \in S_n | s \text{ preserves the rows of } A\}$$
$$Col(A) = \{s \in S_n | s \text{ preserves the columns of } A\}$$

Associated to a tableau A are the *row symmetrizer*

$$R(A) = \Sigma_{r \in Row(A)} r$$

and the *column skew-symmetrizer*

$$C(A) = \Sigma_{c \in Col(A)} sgn(c)c.$$

We define $T(A)$ as follows:

$$T(A) = C(A)R(A).$$

Then, $\text{span}_{A \in \text{Tab}\lambda} < T(A)u_A >$ generated the S_n-type ψ^λ.

Proposition 2.4.4.1. *Let $\lambda^T = (c_1, c_2, \ldots, c_n) \vdash n$. The S_n-type ψ^λ is a subrepresentation of*

$$\text{Ind}_{S_{c_1} \times S_{c_2} \times \ldots \times S_{c_n}}^{S_n} (\text{sign} \otimes \text{sign} \otimes \ldots \otimes \text{sign})$$

but it is not a subrepresentation of

$$\text{Ind}_{S_{d_1} \times S_{d_2} \times \ldots \times S_{d_n}}^{S_n} (\text{sign} \otimes \text{sign} \otimes \ldots \otimes \text{sign})$$

for $(d_1, d_2, d_3, \ldots, d_n) \not\leq (c_1, c_2, c_3, \ldots, c_n)$.

(2) irreducible representations of $S_n \ltimes (\mathbb{Z}/2\mathbb{Z})^n$:

The irreducible representations of $S_n \ltimes (\mathbb{Z}/2\mathbb{Z})^n$ is parametrized by two Young diagrams such that the number of boxes in each diagram sums up to be n. In other words, the $S_n \ltimes (\mathbb{Z}/2\mathbb{Z})^n$-types are parametrized by (λ, τ) such that $\lambda \vdash k$ and $\tau \vdash n-k$ for some $0 \leq k \leq n$, and **we denote this representation by $\psi^{(\lambda,\tau)}$.** Note that $\psi^{(\lambda,\tau)}$ is the following representation:

$$\mathrm{Ind}_{(S_k \ltimes (\mathbb{Z}/2\mathbb{Z})^k) \times (S_{n-k} \ltimes (\mathbb{Z}/2\mathbb{Z})^{n-k})}^{S_n \ltimes (\mathbb{Z}/2\mathbb{Z})^n} (\psi^\lambda \ltimes \mathrm{triv}) \otimes (\psi^\tau \ltimes \mathrm{sign}).$$

(3) irreducible representations of R_n:

The irreducible representations of R_n are parametrized by unordered pairs of two Young diagrams such that the number of boxes in each diagram sums up to be n. Specifically, the R_n-types fall into one of the following two categories.

(A) It is parametrized by $\{\lambda, \tau\}$ such that $\lambda \vdash k$ and $\lambda \neq \tau \vdash n-k$, and **we denote this representation by $\psi^{\{\lambda,\tau\}}$.** Note that $\psi^{\{\lambda,\tau\}}$ is just the restricted representation of $S_n \ltimes (\mathbb{Z}/2\mathbb{Z})^n$-type $\psi^{(\lambda,\tau)}$ to R_n.

(B) It is parametrized by $\{\lambda, \lambda\}_I$ or $\{\lambda, \lambda\}_{II}$ for $\lambda \vdash \frac{n}{2}$ if n is an even number. Note that $\psi^{\{\lambda,\lambda\}_I}$ and $\psi^{\{\lambda,\lambda\}_{II}}$ are two inequivalent representations of R_n, and their sum is the restriction of the $S_n \ltimes (\mathbb{Z}/2\mathbb{Z})^n$-type $\psi^{(\lambda,\lambda)}$ to R_n.

2.4.5 Petite K-type and Relevant W-types for Split Groups

Split rank one subgroups have various forms. They might be isomorphic to $SL(2,\mathbb{R})$, $SL(2,\mathbb{C})$, or $SU(n,1)$ [17]. However, for a special groups called **split groups**, the split rank one subgroups are always isomorphic to $SL(2,\mathbb{R})$.

Definition 2.4.5.1. *The Lie group G is **split** when $\mathfrak{m} = Z_{\mathfrak{k}}(\mathfrak{a})$ equals 0. In other words, for the split group G, M is a finite group.*

For split groups, the unitarity test using W-type can be considered in the setting of affine Hecke algebra [5][2]. Moreover, unitarity tests using W-types ensure unitarity in the p-adic setting.

Theorem 2.4.5.2. *(Barbasch-Moy [4]) Let G be a split p-adic group of classical type. Then, the Langlands quotient $J(\nu)_{\mathbb{Q}_p}$ is unitary if and only if every eigenvalue* *of $\psi(A_{\omega_0}(\nu))$ is positive for every W-type ψ.*

We can represent the long intertwining operator using a Weyl group algebra element for K-types satisfying 2.4.2.1 as shown in 2.4.1 and 2.4.2. For split groups, petite K-types are such K-types [25][2][6].

Definition 2.4.5.3. *Since G is split, G^α is locally isomorphic to $SL(2,\mathbb{R})$, and K^α to $SO(2,\mathbb{R})$. Note that irreducible representations of $SO(2,\mathbb{R})$ are one-dimensional and they are indexed by $\mathbb{Z}$. Let $\phi|_{K^\alpha} = \phi_1 \oplus \phi_2 \oplus \ldots \oplus \phi_k$ be the decomposition of a K-type ϕ into irreducible representations of K^α for $\alpha \in \Pi$. ϕ is called* ***petite*** *if ϕ_i is a character 0, ± 1, ± 2 of K^α for all $\alpha \in \Pi$.*

For a petite K-type ϕ, we can represent $A_\phi(\nu)$ using $A_{\omega_0}(\nu)$ and ψ_ϕ. Note that every eigenvalue of $\psi_\phi(A_{\omega_0}(\nu))$ is positive if and only if $A_\phi(\nu)$ is positive semi-definite. If G is a split group over $\mathbb{Q}$ or $\mathbb{Q}_p$, we can define $J(\nu)_\mathbb{Q}$ or $J(\nu)_{\mathbb{Q}_P}$ similarly as we defined $J(\nu) = J(\nu)_\mathbb{R}$ for minimal parabolic subgroup Q, ν satisfying the formal symmetry condition in 2.2.1. By 2.4.5.2, we know that the $J(\nu)_{\mathbb{Q}_P}$ is unitary if $\psi(A_{\omega_0}(\nu))$ is positive definite for every irreducible representation of Weyl group ψ [4]. Actually, it is enough to check some W-types, called the **relevant W-types** [2]. If there exists a K-type corresponding to each relevant W-type, the following relation between non-unitarity in the real setting and in the p-adic setting holds:

$$\begin{gathered}\text{non-unitarity in the } p\text{-adic setting}\\ \Longrightarrow \psi(A_{\omega_0}(\nu)) \text{ is not positive semi-definite for some relevant } \psi \in \hat{W}\\ \Longrightarrow A_\phi(\nu) \text{ is not positive semi-definite for some } \phi \in \hat{K}\\ \Longrightarrow \text{non-unitarity in the real setting.}\end{gathered}$$

Note that petite K-types match all the relevant W-types for classical split groups.

Chapter 3

Single-Petaled K-types

3.1 Generalizing Petite K-types

3.1.1 Generalizing Petite K-types

One benefit of using petite K-types is that we can make use of a Weyl group algebra element to express the long intertwining operator. We repeat the definition given in 2.4.5.3.

Definition 3.1.1.1. *Suppose that G is a split real group and (ϕ, V_ϕ) is a K-type. Since G is split, G^α is locally isomorphic to $SL(2, \mathbb{R})$, and K^α to $SO(2, \mathbb{R})$. Note that irreducible representations of $SO(2, \mathbb{R})$ are one-dimensional and they are indexed by $\mathbb{Z}$. Let*

$$\phi|_{K^\alpha} = \phi_1 \oplus \phi_2 \oplus \ldots \oplus \phi_k$$

*be the decomposition of ϕ into irreducible representations of K^α for $\alpha \in \Delta_1$. The K-type ϕ is called **petite** if ϕ_i is a character 0, ± 1, ± 2 of K^α for all $\alpha \in \Delta_1$ for all i.*

We can generalize the petite K-types. Since $A_\phi(\nu)$ acts on M-fixed vectors, it can be represented by a Weyl group algebra element if ϕ satisfies the following:

Proposition 3.1.1.2. *Suppose*

$$\phi|_{MK^\alpha} = \phi'_1 \oplus \phi'_2 \oplus \ldots \oplus \phi'_{k'}$$

and ϕ'_i *has a nonzero* M*-fixed vector only if* $\phi'_i|_{K^\alpha}$ *is a* K^α *representation* (0), (± 2) *or* $(2) \oplus (-2)$. *Then* ϕ *is a single-petaled* K*-type of the split group* G.

This adjustment seems to be small, but there are many more single-petaled K-types than petite K-types. For non-split groups as well, we can think of $A_\phi(\nu)$ as a linear operator on M-fixed vectors of a K-type ϕ. For non-split groups, K^α is not guaranteed to be locally isomorphic to $SO(2, \mathbb{R})$. For example, $K^\alpha \simeq S(U(m-n+1) \times U(1))$ for $G = SU(m,n)$ $(m > n)$ and short restricted root α. Kostant parametrized quasi-spherical K-types of real rank one groups, and some "small" K-types play the roles of character (0) and (± 2) of $SO(2, \mathbb{R})$ for split groups. They satisfy 2.4.2.1, and from 2.3.1.2, we notice that they are the K-types with a parameter of either $(0,0)$ or $(0,1)$. They are

$$(0,0) \leftrightarrow \text{the trivial representation}$$
$$(0,1) \leftrightarrow \text{an irreducible subrepresentation of } \mathfrak{p}_\mathbb{C}\text{-representation.}$$

The representation of K on $\mathfrak{p}_\mathbb{C}$ is sometimes irreducible, or it is a sum of two irreducible subrepresentations of K. We call the irreducible subrepresentations of $\mathfrak{p}_\mathbb{C}$-representation as $\mathfrak{p}$-representations. These K-types are discussed in 2.3.1.

We can interpret 2.4.2.1 and 2.4.2.2 as follows:

Condition 3.1.1.3. *Suppose*

$$\phi|_{MK^\alpha} = \phi''_1 \oplus \phi''_2 \oplus \ldots \oplus \phi''_{k''}$$

be the decomposition of ϕ *into irreducible representations of* MK^α. *If* ϕ''_i *has a nonzero* M*-fixed vector, then we require that* $\phi''_i|_{K^\alpha}$ *be either the trivial representation or a subrepresentation of* $(\mathfrak{p} \cap \mathfrak{g}^\alpha)_\mathbb{C}$ *for every* $\alpha \in \Delta_1$.

Proposition 3.1.1.4. *Suppose that the K-type ϕ satisfies the 3.1.1.3. Then we can represent $A_\phi(\nu)$ using a Weyl group algebra element.*

We illustrate an example for $G = GL(2, \mathbb{C})$: we calculate K-types ϕ such that $A_\phi(\nu)$ can be represented by a Weyl group algebra element.

- $K^\alpha \simeq SU(2)$
- $MK^\alpha \simeq U(2)$
- $M = U(1) \times U(1)$

Throughout this thesis, we use the equivalence classes in the character of $U(1)^n$ to denote the highest weight of a $SU(n)$-type which is a character of $S(U(1)^n)$. The character of $U(1)^n$ is denoted by $(a_1, a_2, \ldots, a_n)$ where a_i are integers. We denote the restriction of this character to $S(U(1)^n)$ as

$$(a_1, a_2, \ldots, a_n)$$

as well. Because the two characters of $U(1)^n$

$$(a_1, a_2, \ldots, a_n) \text{ and } (a_1 + k, a_2 + k, \ldots, a_n + k)$$

have the same restriction to $S(U(1)^n)$, we can write a character of $S(U(1)^n)$ as an equivalent class

$$[(a_1, a_2, \ldots, a_n) \in \mathbb{Z}^n]/[(a_1, a_2, \ldots, a_n) \sim (a_1 + k, a_2 + k, \ldots, a_n + k)].$$

For the highest weight of $SU(n)$, we use a dominant integral weight which is

$$(a_1, a_2, \ldots, a_n) \text{ where } a_i \geq a_{i+1} \text{ and } a_i \in \mathbb{Z}.$$

The highest weight representation of $SU(n)$ with highest weight $(a_1, a_2, \ldots, a_n)$ can be extended to the representation of $U(n)$ with highest weight $(a_1+l, a_2+l, \ldots, a_n+l)$ with integer l. We use the following proposition.

Proposition 3.1.1.5. *The restriction of a quasi-spherical $U(2)$-type is a quasi-spherical $SU(2)$-type, and a quasi-spherical $SU(2)$-type can be extended to a quasi-spherical $U(2)$-type.*

Proof: We prove the first part in 3.4.1.3, so we omit the proof. We prove the second part. We denote an $SU(2)$-type ϕ is denoted by the highest weight

$$\lambda = (\lambda_1, \lambda_2)$$

for $\lambda_i \in \mathbb{Z}$, $\lambda_1 \geq \lambda_2$. The $U(2)$-type with highest weight (λ_1, λ_2) restricts to the $SU(2)$-type with highest weight (λ_1, λ_2), which is equivalent to the $SU(2)$-type with highest weight $(\lambda_1 + k, \lambda_2 + k)$ for $k \in Z$. Therefore, the $SU(2)$-type with highest weight

$$(\lambda_1, \lambda_2)$$

can be extended to the $U(2)$-type with highest weight

$$(\lambda_1 + k, \lambda_2 + k)$$

for integer k. From 3.1.1.5, the quasi-spherical $SU(2)$-type can be extended to a quasi-spherical $U(2)$-type. If ϕ has a nonzero $S(U(1) \times U(1))$-fixed vector, then the $U(2)$-type with highest weight

$$(\lambda_1 + k, \lambda_2 + k)$$

is quasi-spherical for some $k \in \mathbb{Z}$, which means that

$$\lambda_1 + k + \lambda_2 + k = 0 \text{ for some } k.$$

Therefore $\lambda_1 + \lambda_2$ is even. We may view the $SU(2)$-type with highest weight (λ_1, λ_2) as the $SU(2)$-type with highest weight

$$(\lambda_1 - \frac{\lambda_1 + \lambda_2}{2}, \lambda_2 - \frac{\lambda_1 + \lambda_2}{2})$$

if $\lambda_1 + \lambda_2$ is even. Then, this representation has a $U(1) \times U(1)$-fixed vector. This is because the quasi-spherical $U(2)$-types have the highest weights of the form

$$(a, -a)$$

for $a \in \mathbb{N} \cup \{0\}$ due to the following theorem of Weyl. □

Theorem 3.1.1.6. *(Weyl [17]) If we restrict $U(n)$ representation ϕ with highest weight $(a_1, a_2, \ldots, a_n)$ to the representation of $U(n-1) \times U(1)$, the representation $\phi_{(b_1, b_2, \ldots, b_{n-1})(b_n)}$ with highest weight $(b_1, b_2, \ldots, b_{n-1})(b_n)$ has a nonzero multiplicity if and only if*

$$a_1 \geq b_1 \geq a_2 \geq b_2 \geq \ldots \geq b_{n-1} \geq a_n$$

$$\Sigma_{i=1}^{i=n} a_i = \Sigma_{i=1}^{i=n} b_i.$$

If these conditions hold, $\phi_{(b_1, b_2, \ldots, b_{n-1})(b_n)}$ has multiplicity 1 in ϕ.

The trivial $SU(2)$-type has the highest weight $(0, 0)$. On the other hand, the $SU(2)$-type on $\mathfrak{p}_{\mathbb{C}}$ has the highest weight $(1, -1)$. These two quasi-spherical $SU(2)$-types are extended to quasi-spherical $U(2)$-types with highest weights $(0, 0)$ and $(1, -1)$ respectively. Therefore, the $U(2)$-type ϕ satisfying 3.1.1.3 has the highest weight $(0, 0)$ or $(1, -1)$.

Proposition 2.4.2.2 suggests that we should study that K-types satisfying 3.1.1.3, which turn out to be **single-petaled K-types**. In [24], Oda defined single-petaled K-types.

Definition 3.1.1.7. *For each reduced root α, fix X_α ,θX_α, and H_α such that*

$$H_\alpha \text{ is the coroot of } \alpha$$
$$[X_\alpha, \theta X_\alpha] = -H_\alpha$$
$$X_\alpha \in \mathfrak{g}_\alpha.$$

*Put $Z_\alpha = X_\alpha + \theta X_\alpha$. Then a quasi-spherical K-type (ϕ, V_ϕ) is **single-petaled** if and only if*

$$\phi(Z_\alpha)(\phi(Z_\alpha)^2 - 4)v = 0^1, \ \forall v \in V_\phi^M, \ \forall \alpha \in \Delta.$$

Note that the definition of single-petaled K-type is independent of the choice of Z_α [24]. Suppose that an MK^α-type $\widetilde{\phi}$ is quasi-spherical, and the K^α-type in $\widetilde{\phi}|_{K^\alpha}$ is

[1] In [24], the single-petaled K-types are defined in a different way, which is equivalent to this one.

parameterized by (i, j) in [23]. Then, Z_α has eigenvalues $\pm 2k$ such that $|k| \leq 2i + j$ on $V_{\tilde{\phi}}^M$. Since

$$\phi(Z_\alpha)(\phi(Z_\alpha)^2 - 4)$$

is the same as

$$\phi(Z_\alpha)(\phi(Z_\alpha) - 2)(\phi(Z_\alpha) + 2),$$

this operator annihilates a vector if the vector is expressed as a sum of eigenvectors corresponding to eigenvalues ± 2 or 0 under the action of Z_α.

Proposition 3.1.1.8. *The single-petaled K-types are the K-types that satisfy 3.1.1.3.*

Using this proposition and 3.1.1.2, we get the following theorem.

Theorem 3.1.1.9. *We can represent the action of the long intertwining operator on M-fixed vectors of a single-petaled K-type using a Weyl group algebra element.*

3.2 Correspondence Between K-types and W-types

3.2.1 Correspondence Between K-types and W-types

We can use K-types to test unitarity of a representation of G, and it is often useful to make a relation between tests using K-types and tests using W-types. We can derive this relation naturally for single-petaled K-types, and this is one benefit of classifying single-petaled K-types. As discussed in Chapter 2, we have a natural Weyl group representation on M-fixed vectors of each K-type: the Weyl group is realized as $N_K(\mathfrak{a})/M$, and it acts on M-fixed vectors of a K-type, hence defining a Weyl group representation. For single-petaled K-type ϕ, the long intertwining operator on $\mathrm{Hom}_K(\phi, I(\nu))$ is represented by a Weyl group algebra element because of 3.1.1.4. From 2.4.2, this Weyl group algebra element is $A_{\omega_0}(\nu)$.

$A_{\omega_0}(\nu)$.

Definition 3.2.1.1. *For single-petaled K-type ϕ, $A_{\omega_0}(\nu)$ is the Weyl group algebra element that represents the action of the long intertwining operator on* $\mathrm{Hom}_K(\phi, I(\nu))$.

We can apply the argument similar to 2.4.5 for single-petaled K-types. **If the Weyl group representation on V_ϕ^M is ψ_ϕ, then the long intertwining operator on $\mathrm{Hom}_K(\phi, I(\nu))$ is $\psi_\phi(A_{\omega_0}(\nu))$.** In other words, the long intertwining operator on $\mathrm{Hom}_K(\phi, I(\nu))$ is positive semi-definite if and only if every eigenvalue of $\psi_\phi(A_{\omega_0}(\nu))$ is positive. Therefore, if $\psi_\phi(A_{\omega_0}(\nu))$ has a negative eigenvalue, then the long intertwining operator on $\mathrm{Hom}_K(\phi, I(\nu))$ is not positive semi-definite, hence $J(\nu)$ is not a unitary representation. In this way, we relate unitarity tests using K-types and unitarity tests using W-types as in 2.4.5. Using this relation, we relate non-unitarity in the p-adic setting and unitarity in the real setting as in 2.4.5 for single-petaled K-types of split groups. From 2.4.5.2, for split p-adic groups of classical type, it is known (cf. [4]) that $J(\nu)$ is unitary if and only if

$$\psi(A_{\omega_0}(\nu))$$

is positive semi-definite for every W-type ψ. Note that we defined the unitarity test using W-type ψ checking if $\psi(A_{\omega_0}(\nu))$ is positive semi-definite in 2.4.2.3. If $J(\nu)_{\mathbb{Q}_p}$ is not unitary, then $\psi(A_{\omega_0}(\nu))$ is not positive semi-definite for some W-type ψ. If ψ is a Weyl group representation corresponding to a single-petaled K-type ϕ, then $\psi(A_{\omega_0}(\nu))$ is a linear operator on V_ϕ^M which is similar to $A_\phi(\nu)$, the long intertwining operator on V_ϕ^M. Therefore, $J(\nu)$ is not a unitary representation in the real group setting either.

Therefore, it is interesting to figure out which Weyl group representations correspond to single-petaled K-types. For example, for $GL(n, \mathbb{C})$, the maximal compact subgroup is $U(n)$. The $U(n)$-types can be parameterized by their highest weights. We will show in the next section that single-petaled K-types are the ones with the highest weight

$$(a_1 - 1, a_2 - 1, \ldots, a_n - 1)$$

such that every a_i is nonnegative and $\Sigma_{i=1}^n a_i = n$. Suppose ϕ is the K-type with highest weight $(a_1 - 1, a_2 - 1, \ldots, a_n - 1)$ such that $\Sigma_{i=1}^n a_i$. Note that we may interpret

$(a_1, a_2, \ldots, a_n)$ as a partition of n, or a Young diagram of size n. By $(a_1, a_2, \ldots, a_n)^T$, we mean the partition of n represented by a Young diagram whose ith column has a_i boxes. Let

$$(b_1, b_2, \ldots, b_n) = (a_1, a_2, \ldots, a_n)^T.$$

On the other hand, $GL(n, \mathbb{C})$ has the Weyl group S_n, and irreducible representations of S_n are parametrized by Young diagrams of size n as in 2.4.4. The Weyl group representation on M-fixed vectors of ϕ is parametrized by $(b_1, b_2, \ldots, b_n)$. We will prove this in 3.4.2. Therefore, all the W-types are realized by single-petaled K-types.

In the next section, we introduce a strategy to classify single-petaled K-types.

3.3 Strategy to Classify Single-petaled K-types

3.3.1 Strategy to Classify Single-petaled K-types

We introduce a strategy to classify single-petaled K-types using 3.1.1.2. Specifically, we follow the next steps.

1. Calculate the real rank one subgroup G^α and its maximal compact subgroup K^α corresponding to each $\alpha \in \Delta_1$.

2. Classify the irreducible representation $\widetilde{\phi}$ of MK^α such that

   (a) $\widetilde{\phi}^M$ is nonzero.

   (b) $\widetilde{\phi}|_{K^\alpha}$ is either the trivial representation or a $\mathfrak{p}^\alpha$-representation.

   We denote $\widetilde{\phi}$ by triv or $\widetilde{\mathfrak{p}^\alpha}$ respectively.

3. Classify quasi-spherical MK^α-types: calculate MK^α-types with nonzero M-fixed vectors.

4. Classify quasi-spherical K-types.

5. Calculate the restriction of ϕ to the representations of MK^α using a branching formula.

6. Classify quasi-spherical K-types ϕ such that quasi-spherical MK^{α}-types appearing in $\phi|_{MK^{\alpha}}$ are triv or $\widetilde{\mathfrak{p}^{\alpha}}$. These are the single-petaled K-types.

3.4 Single-petaled K-types of $SL(n, \mathbb{C})$ and Corresponding Weyl Group Representations

3.4.1 Single-petaled K-types of $SL(n, \mathbb{C})$

A $U(n)$-type is irreducible when restricted to $SU(n)$ as well. The representations of $SU(n)$ are parametrized by their highest weights in the manner introduced in 2.3.3.

Proposition 3.4.1.1. *A quasi-spherical $U(n)$-type restricts to a quasi-spherical $SU(n)$-type, and a quasi-spherical $SU(n)$-type can be extended to a quasi-spherical $U(n)$-type.*

Proof: When a $U(n)$-type $\widetilde{\phi}$ is given, it is irreducible as a representation of $SU(n)$:

$$\phi = \widetilde{\phi}|_{SU(n)}.$$

Then, $V_{\widetilde{\phi}}^{U(1)^n} \neq 0$ implies $V_{\phi}^{S(U(1)^n)} \neq 0$, and a quasi-spherical $U(n)$-type restricts to a quasi-spherical $SU(n)$-type.

An element u of $U(n)$ can be expressed as $s \times d$ such that $s \in SU(n)$ and d is a scalar matrix. When given a quasi-spherical $SU(n)$-type ϕ, we can extend ϕ to a $U(n)$ representation $\widetilde{\phi}$ by defining

$$\widetilde{\phi}(u) = \phi(s).$$

To prove that this is a well-defined representation of $U(n)$, we suppose that

$$u = s \times d = s' \times d'$$

where $s, s' \in SU(n)$ and d, d' are scalar matrices. Then

$$s^{-1}s' = dd'^{-1}$$

is an element of $SU(n)$ because $s, s' \in SU(n)$, and it is a scalar matrix because d and d' are scalar matrices. Therefore,

$$s^{-1}s' = dd'^{-1} = e^{\frac{2k\pi i}{n}} I$$

for $k \in \mathbb{Z}$, and this is an element of M, so

$$\phi(s^{-1}s') = \phi(dd'^{-1})$$

sends the nonzero $S(U(1)^n)$-fixed vector v to v. Also, it is a scalar operator because of Schur's lemma [17], so it is the identity linear operator on V_ϕ. Therefore, $\phi(s)$ equals $\phi(s')$, and $\widetilde{\phi}$ is well-defined. Note that $\widetilde{\phi}$ is a quasi-spherical $U(n)$-type. Suppose that $v \in V_\phi^{S(U(1)^n)}$ and $m \in U(1)^n$. We can write $m = m'd'$ such that $m' \in S(U(1)^n)$ and d' is a scalar matrix. Because

$$\widetilde{\phi}(m)(v) = \phi(m')(v) = v,$$

a nonzero vector v is in $V_{\widetilde{\phi}}^{U(1)^n}$. Therefore, $\widetilde{\phi}$ is a quasi-spherical $U(n)$-type. □

A nonzero $\widetilde{M}$-fixed vector v is the trivial representation with highest weight $(0)^n$ of $\widetilde{M} \simeq U(1)^n$. Therefore, the $U(n)$-type with highest weight $(a_1, a_2, \ldots, a_n)$ has a nonzero $U(1)^n$-fixed vector if and only if it has a subrepresentation with highest weight $(0)^n$ when branched to $U(1)^n$.

Proposition 3.4.1.2. *The $U(n)$-type has a nonzero $U(1)^n$-fixed vector if and only if it has the highest weight $(a_1, a_2, \ldots, a_n)$ such that $\Sigma_{i=1}^n a_i = 0$.*

Proof: $(\Rightarrow)$ We prove by induction on n. If the $U(n-1) \times U(1)$-type with highest weight

$$(b_1, b_2, \ldots, b_{n-1})(b_n)$$

appears in the $U(n)$-type with highest weight

$$(a_1, a_2, \ldots, a_n),$$

then

$$\Sigma_{i=1}^n a_i = \Sigma_{i=1}^n b_i$$

from 3.1.1.6. If the $U(n-1) \times U(1)$-type with highest weight $(b_1, b_2, \ldots, b_{n-1})(b_n)$ has a $U(1)^n$-fixed vector, then $b_n = 0$ and $\Sigma_{i=1}^{n-1} b_i = 0$ using inductive hypothesis. Therefore, the $U(n)$-type has a nonzero $U(1)^n$-fixed vector only if it has the highest weight $(a_1, a_2, \ldots, a_n)$ such that $\Sigma_{i=1}^n a_i = \Sigma_{i=1}^n b_i = 0$.

($\Leftarrow$) We prove by induction on n. This is true for $n = 1$. Since $\Sigma_{i=1}^n a_i = 0$, there exists k such that $a_k \geq 0$ and $a_{k+1} \leq 0$. Note that

$$a_k \geq a_k + a_{k+1} \geq a_{k+1}.$$

We define $b_1, b_2, b_3, \ldots, b_{n-1}$ such that

$$b_k = a_k + a_{k+1},$$
$$b_i = a_i \text{ for } i < k,$$
$$b_i = a_{i+1} \text{ for } i > k.$$

Then,

$$a_1 \geq b_1 \geq a_2 \geq b_2 \geq \ldots \geq b_{n-1} \geq a_n \text{ and } \Sigma_{i=1}^{n-1} b_i = 0$$

holds. From 3.1.1.6, the $U(n)$-type with highest weight

$$(a_1, a_2, \ldots, a_n)$$

contains the representation of $U(n-1) \times U(1)$ with highest weight

$$(b_1, b_2, \ldots, b_{n-1})(0).$$

From inductive hypothesis, we know that $U(n-1)$-type with highest weight

$$(b_1, b_2, b_3, \ldots, b_{n-1})$$

has a copy with highest weight $(0)^{n-1}$ when branched to $U(1)^{n-1}$. Therefore, the $U(n)$-type with highest weight $(a_1, a_2, \ldots, a_n)$ has a nonzero $U(1)^n$-fixed vector. $\square$

Proposition 3.4.1.3. *Restriction of a $U(n)$-type to $SU(n)$ defines a bijection from quasi-spherical $U(n)$-types in $GL(n, \mathbb{C})$ to quasi-spherical $SU(n)$-types in $SL(n, \mathbb{C})$. This bijection carries single-petaled representations of $GL(n, \mathbb{C})$ to single-petaled representations of $SL(n, \mathbb{C})$.*

Sketch of Proof: We use an argument similar to 3.1.1.5. From 3.4.1.1, a quasi-spherical $SU(n)$-type ϕ can be extended to a quasi-spherical $U(n)$-type $\widetilde{\phi}$ from 3.4.1.1. Therefore, ϕ has the highest weight $(a_1, a_2, \ldots, a_n)$ such that $\Sigma_{i=1}^n a_i$ is a multiple of n: if $\Sigma_{i=1}^n a_i$ is a multiple of n, then ϕ can be extended to $\widetilde{\phi}$ with highest weight

$$(a_1 - \frac{a_1 + a_2 + \ldots + a_n}{n}, a_2 - \frac{a_1 + a_2 + \ldots + a_n}{n}, \ldots, a_n - \frac{a_1 + a_2 + \ldots + a_n}{n}).$$

If $\widetilde{\phi}$ is a single-petaled $\widetilde{K}$-type, then the $\widetilde{M}$-fixed vector in $\widetilde{\phi}$ satisfies $Z_\alpha(Z_\alpha^2 - 4)v = 0$. When $\widetilde{\phi}$ is restricted to the $SU(n)$ representation ϕ, we have

$$V_{\widetilde{\phi}}^{U(1)^n} = V_{\widetilde{\phi}}^{S(U(1)^n)} = V_{\phi}^{S(U(1)^n)},$$

and therefore ϕ is a single-petaled $SU(n)$-type.

Conversely, if ϕ is a single-petaled $SU(n)$-type, then we can extend ϕ to be a quasi-spherical representation of $U(n)$, $\widetilde{\phi}$ as in 3.4.1.1. Then, we have

$$V_{\phi}^{S(U(1)^n)} = V_{\widetilde{\phi}}^{S(U(1)^n)} = V_{\widetilde{\phi}}^{U(1)^n}.$$

These vectors are annihilated by $Z_\alpha(Z_\alpha^2 - 4)$, and $gl(n, \mathbb{C})$ and $sl(n, \mathbb{C})$ share root systems and Z_α, so we conclude that $\widetilde{\phi}$ is a single-petaled $U(n)$-type. Therefore, we may classify single-petaled $U(n)$-types for $GL(n, \mathbb{C})$ instead, and they are the single-petaled $SU(n)$-types for $SL(n, \mathbb{C})$ when restricted to $SU(n)$. $\square$

Therefore, to classify single-petaled $SU(n)$-types in $SL(n, \mathbb{C})$, it is enough to classify single-petaled $U(n)$-types in $GL(n, \mathbb{C})$ and restrict them to $SU(n)$.

Now, we begin to classify single-petaled $U(n)$-types for $GL(n, \mathbb{C})$.

- $G = GL(n, \mathbb{C})$
- $K = U(n)$
- $\mathfrak{a} \simeq \mathbb{R}^n$: $\{d_1 E_{1,1} + d_2 E_{2,2} + \ldots + d_n E_{n,n} \mid d_i \in \mathbb{R}\}$
- restricted root system Δ:
  $\{\alpha_i - \alpha_j | n \geq i \neq j \geq 1\}$ where $\alpha_i(d_1 E_{1,1} + d_2 E_{2,2} + \ldots + d_n E_{n,n}) = d_i$
- simple restricted roots: $\Pi = \{\alpha_i - \alpha_{i+1} | i = 1, 2, \ldots, n-1\}$
- Weyl group $W = S_n$
- $\mathfrak{g}_{\alpha_i - \alpha_j} = \text{span}_{\mathbb{R}} < E_{i,j}, iE_{i,j} >$
- $Z_{\alpha_i - \alpha_j} = E_{i,j} - E_{j,i}$
- $\mathfrak{g}^{\alpha_i - \alpha_j} \simeq sl(2, \mathbb{C})$ with nonzero entries in $(i,i), (i,j), (j,i), (j,j)$
- $G^{\alpha_i - \alpha_j} \simeq SL(2, \mathbb{C})$
- $\mathfrak{k}^{\alpha_i - \alpha_j} \simeq su(2)$ with nonzero entries in $(i,i), (i,j), (j,i), (j,j)$
- $K^{\alpha_i - \alpha_j} \simeq SU(2, \mathbb{C})$
- $MK^{\alpha_i - \alpha_j} \simeq U(2) \times U(1)^{n-2}$

For $n = 2$, K^α is isomorphic to $SU(2)$ for a simple root α. The representations of $U(2)$ with a nonzero M-fixed vector are the ones with the highest weight $(m, -m)$ with non-negative integer m. For $m = 0$, $(0, 0)$ is the trivial representation. For $m = 1$, $(1, -1)$ is the $\mathfrak{p}$-representation, and it is also $\widetilde{\mathfrak{p}^\alpha}$-representation. Therefore, the $U(2)$-types with highest weight

$$(0, 0) \text{ and } (1, -1)$$

are single-petaled for $GL(2,\mathbb{C})$. A quasi-spherical representation of $MK^\alpha \simeq U(2) \times U(1)^{n-2}$ has a nonzero M-fixed vector annihilated by $Z_\alpha(Z_\alpha^2-4)$ if and only if it has the highest weight

$$(0,0)(0)^{n-2} \text{ or } (1,-1)(0)^{n-2}$$

when restricted to K^α. From 3.3.1, to calculate single-petaled K-types for $GL(n,\mathbb{C})$, we need to calculate the representation of $U(n)$ satisfying the following condition: if the subrepresentation with highest weight

$$(m,-m)\times(0)^{n-2}$$

appears in the restriction to $U(2)\times U(1)^{n-2}$, then $m=0$ or 1.

Theorem 3.4.1.4. *A single-petaled K-type of $GL(n,\mathbb{C})$ has the highest weights of the following form:*

$$(a_1,a_2,\ldots,a_n)-(1,1,\ldots,1) \textit{ where } a_i \geq a_{i+1} \geq 0 \textit{ and } \Sigma a_i = n$$
$$\textit{or } (1,1,\ldots,1)-(a_1,a_2,\ldots,a_n) \textit{ where } a_i \geq a_{i+1} \geq 0 \textit{ and } \Sigma a_i = n.$$

Proof: First, we show that the K-type with highest weight

$$(a_1,a_2,\ldots,a_n)-(1,1,\ldots,1)$$

is a single-petaled K-type. The restriction of $\phi_{(a_1,a_2,\ldots,a_n)-(1,1,\ldots,1)}$ to $U(2)\times U(1)^{n-2}$ has a nonzero M-fixed vector if and only if it has the highest weight

$$(1,-1)\times(0)^{n-2} \text{ or } (0,0)\times(0)^{n-2}.$$

We prove this by mathematical induction.

(1) For $n=2$, the statement is true.

(2) For $n=k\geq 3$, the $U(n)$-type with highest weight

$$(a_1,a_2,\ldots,a_n)-(1,1,\ldots,1)$$

contains $U(n-1) \times U(1)$-type with highest weight

$$(b_1, b_2, \ldots, b_{n-1})(b_n).$$

The $U(n-1) \times U(1)$-type with highest weight $(b_1, b_2, \ldots, b_{n-1})(b_n)$ has a nonzero M-fixed vector only if b_n is 0. If b_n is 0, then $\Sigma_{i=1}^{n-1} b_i = 0$ and $b_i \geq -1$ for $n-1 \geq i \geq 1$ from the proof of 3.4.1.2. Therefore,

$$(b_1 + 1, b_2 + 1, \ldots, b_{n-1} + 1)$$

is a partition of $n-1$, and

$$(b_1, b_2, \ldots, b_{n-1})$$

is the highest weight of a single-petaled $U(n-1)$-type in $GL(n-1, \mathbb{C})$ by inductive hypothesis. From (1), (2), and mathematical induction, the irreducible subrepresentation of $U(2) \times U(1)^{n-2}$ that appears in $\phi_{(a_1,a_2,\ldots,a_n)-(1,1,\ldots,1)}$ has a nonzero M-fixed vector if and only if it the highest weight $(1,-1) \times (0)^{n-2}$ or $(0,0) \times (0)^{n-2}$. We can prove that $\phi_{(1,1,\ldots,1)-(a_1,a_2,\ldots,a_n)}$ is a single-petaled K-type similarly.

Second, we show that every single-petaled K-type has the highest weight expressed in the following forms:

$$(a_1, a_2, \ldots, a_n) - (1, 1, \ldots, 1) \text{ where } a_i \geq a_{i+1} \geq 0 \text{ and } \Sigma a_i = n$$
$$\text{or } (1, 1, \ldots, 1) - (a_1, a_2, \ldots, a_n) \text{ where } a_i \geq a_{i+1} \geq 0 \text{ and } \Sigma a_i = n.$$

If the K-type with highest weight $(b_1, b_2, \ldots b_n)$ with a nonzero M-fixed vector is not expressed in one of the above forms, then the following holds:

$$\Sigma_{i=1}^{n} b_i = 0,\ b_1 \geq 2 \text{ and } b_n \leq -2.$$

Then the K-type with highest weight $(b_1, b_2, \ldots b_n)$ contains the representation of $U(2) \times U(1)^{n-2}$ with highest weight

$$(\min(|b_1|, |b_n|), -\min(|b_1|, |b_n|))(0)^{n-2}.$$

If this statement holds, then $\phi_{(b_1,b_2,\ldots,b_n)}$ is not a single-petaled K-type. We prove this statement by induction on n.

(3) For $n = 2$, the statement is true.

(4) For $n = k \geq 3$, since $\Sigma_{i=1}^{n} b_i = 0$, there exists k $(1 \leq k \leq n-1)$ such that

$$b_k \geq 0 \text{ and } b_{k+1} \leq 0.$$

Note that

$$b_k \geq b_k + b_{k+1} \geq b_{k+1}.$$

We define $c_1, c_2, c_3, \ldots, c_{n-1}$ such that

$$c_k = b_k + b_{k+1},$$
$$c_i = b_i \text{ for } i < k,$$
$$c_i = b_{i+1} \text{ for } i > k.$$

- If $1 < k < n-1$, then $|b_1| = |c_1|$ and $|b_n| = |c_{n-1}|$.
- If $k = 1$, then $\min(|b_1|, |b_n|) = |b_n| = |c_{n-1}| = \min(|c_1|, |c_{n-1}|)$.
- If $k = n-1$, then $\min(|b_1|, |b_n|) = |b_1| = |c_1| = \min(|c_1|, |c_{n-1}|)$.

Then, the irreducible subrepresentation of $U(n-1) \times U(1)$ that appears in $\phi_{(b_1,b_2,\ldots b_n)}$ has the highest weight $(c_1, c_2, \ldots, c_{n-1})(0)$ satisfying

$$\min(|b_1|, |b_n|) = \min(|c_1|, |c_{n-1}|).$$

Therefore, we prove that the K-type with highest weight $(b_1, b_2, \ldots b_n)$ contains the representation of $U(2) \times U(1)^{n-2}$ with highest weight

$$(\min(|b_1|, |b_n|), -\min(|b_1|, |b_n|))(0)^{n-2}.$$

from (3), (4), and mathematical induction. $\square$

3.4.2 Corresponding W-types for $SL(n, \mathbb{C})$

In this subsection, we calculate the corresponding W-type to each single-petaled $U(n)$-type of $G = GL(n, \mathbb{C})$. Using the argument in 3.4.1, we can show that the W-type corresponding to single-petaled $U(n)$-type with highest weight $(a_1 - 1, a_2 - 1, \ldots, a_n - 1)$ is the same as the W-type corresponding to the single-petaled $SU(n)$-type with highest weight $(a_1 - 1, a_2 - 1, \ldots, a_n - 1)$.

Theorem 3.4.2.1. *(Gutkin [12]) Suppose a single-petaled $U(n)$-type ϕ has the highest weight*

$$(a_1 - 1, a_2 - 1, \ldots, a_n - 1)$$

such that $a_1 \geq a_2 \geq \ldots a_n \geq 0$ and $\Sigma_{i=1}^n a_i = n$. The Weyl group representation on $V_\phi^{U(1)^n}$ is parametrized by

$$(a_1, a_2, \ldots, a_n)^T.$$

By the partition $(a_1, a_2, \ldots, a_n) \vdash n$, we mean the partition $(a_1, a_2, \ldots, a_k) \vdash n$ if $a_k \geq 1$ and $a_{k+1} = 0$. This is proved in [12], so we omit the proof.

3.5 Single-petaled K-types of $SU(m, n)$ and Corresponding Weyl Group Representations

3.5.1 Single-petaled K-types of $SU(m, n)$

Without loss of generality, we will assume $m \geq n$. Let $M(G)$ be the subgroup of the maximal compact subgroup of G that centralizes the maximal noncompact abelian subalgebra of $\mathfrak{g}$. Specifically, for $G = SU(m, n)$ and for $G = U(m, n)$, $M(G)$ is the following:

$$M(U(m,n)) = \{x \times (y_1 \times \ldots \times y_n) \times (z_n \times \ldots \times z_1) \mid x \in U(m-n), y_i = z_i \in U(1)\}$$
$$M(SU(m,n)) = M(U(m,n)) \cap SU(m,n).$$

Proposition 3.5.1.1. *The single-petaled $S(U(m) \times U(n))$-types in $SU(m, n)$ can be extended to the single-petaled $U(m) \times U(n)$-types in $U(m, n)$, and the single-petaled*

$U(m) \times U(n)$-types in $U(m,n)$ restrict to single-petaled $S(U(m) \times U(n))$-types in $SU(m,n)$.

The proof is similar to that of 3.4.1.3, so we just sketch the proof without details. To show 3.5.1.1, we need the following lemmas.

Lemma 3.5.1.2. *A quasi-spherical $S(U(m) \times U(n))$-type can be extended to a quasi-spherical $U(m) \times U(n)$-type, and a quasi-spherical $U(m) \times U(n)$-type restrict to a quasi-spherical $S(U(m) \times U(n))$-type.*

The proof is similar to that of 3.4.1.1, so we omit the proof.

Lemma 3.5.1.3. *A $U(m) \times U(n)$-type $\widetilde{\phi}$ is irreducible when restricted to $S(U(m) \times U(n))$.*

Proof: An element u in $U(m) \times U(n)$ is expressed as $s \times d$ where d is a scalar matrix and s is an element of $S(U(m) \times U(n))$, and the $\widetilde{\phi}(d)$ is a scalar operator on $V_{\widetilde{\phi}}$ due to Schur's lemma [17]. Therefore, $V_{\widetilde{\phi}} = \text{span}_{k \in U(m) \times U(n)} < \phi(k)v >$ and $\text{span}_{k \in S(U(m) \times U(n))} < \phi(k)v >$ have the same dimension for a nonzero vector $v \in V_{\widetilde{\phi}}$. □

Lemma 3.5.1.4. *The $U(m) \times U(n)$-type with highest weight*

$$(c_1, c_2, \ldots, c_m)(d_1, d_2, \ldots, d_n)$$

has an $M(U(m) \times U(n))$-fixed vector if and only if it contains an irreducible representation of $U(m-n) \times U(1)^n \times U(1)^n$ with highest weight

$$(0, 0, \ldots, 0)(k_1)(k_2)\ldots(k_n)(-k_n)\ldots(-k_2)(-k_1).$$

Therefore, a quasi-spherical $U(m) \times U(n)$-type satisfies

$$\Sigma_{i=1}^{m} c_i + \Sigma_{i=1}^{n} d_i = 0.$$

This lemma is true because

$$\Sigma_{i=1}^{m} c_i = \Sigma_{i=1}^{n} k_i \text{ and } \Sigma_{i=1}^{n} d_i = \Sigma_{i=1}^{n} - k_i$$

from 3.1.1.6 as in the proof of 3.4.1.2.

Lemma 3.5.1.5. *Suppose that $\widetilde{\phi}$ is a quasi-spherical $U(m) \times U(n)$-type and*

$$\widetilde{\phi}|_{SU(m)\times SU(n)} = \phi.$$

Then,

$$V_{\widetilde{\phi}}^{M(U(m,n))} = V_{\phi}^{M(SU(m,n))}$$

and they are annihilated by $Z_\alpha(Z_\alpha^2 - 4)$ for $\alpha \in \Delta_1$.

Proof: Suppose that a quasi-spherical $U(m) \times U(n)$-type $\widetilde{\phi}$ has the highest weight

$$(c_1, c_2, \ldots, c_m)(d_1, d_2, \ldots, d_n).$$

From 3.5.1.3, we may assume

$$\widetilde{\phi}|_{S(U(m)\times U(n))} = \phi$$

for an $S(U(m) \times U(n))$-type ϕ. We note that $V_{\widetilde{\phi}}^{M(U(m,n))}$ is the sum of the representations with highest weight

$$(0, 0, \ldots, 0)(k_1)(k_2)\ldots(k_n)(-k_n)\ldots(-k_2)(-k_1)$$

when restricted to $U(m-n) \times U(1)^n \times U(1)^n$. Likewise, the $V_{\widetilde{\phi}}^{M(SU(m,n))}$ is the sum of representations with highest weight

$$(k, k, \ldots, k)(k_1)(k_2)\ldots(k_n)(-k_n)\ldots(-k_2)(-k_1)$$

when restricted to $U(m-n) \times U(1)^n \times U(1)^n$. From 3.5.1.4, if $U(m-n) \times U(1)^n \times U(1)^n$-type with highest weight

$$(k, k, \ldots, k)(k_1)(k_2)\ldots(k_n)(-k_n)\ldots(-k_2)(-k_1)$$

appears in $\widetilde{\phi}$, then k is 0. We conclude that the $M(U(m,n))$-fixed vectors in $\widetilde{\phi}$ are the same as the $M(SU(m,n))$-fixed vector in ϕ.

$$V_{\widetilde{\phi}}^{M(U(m,n))} = V_{\widetilde{\phi}}^{M(SU(m,n))} = V_{\phi}^{M(SU(m,n))}.$$

Because $U(m,n)$ and $SU(m,n)$ share the restricted-root system, we may assume that they also share Z_α for simple restricted root α. Therefore, $V_{\widetilde{\phi}}^{M(U(m,n))} = V_{\phi}^{M(SU(m,n))}$ is annihilated by $Z_\alpha(Z_\alpha^2 - 4)$. □

A $SU(m) \times SU(n)$-type with highest weight

$$(a_1, a_2, \ldots, a_m)(b_1, b_2, \ldots, b_n)$$

is equivalent to the representation with highest weight

$$(a_1 + k, a_2 + k, \ldots, a_m + k)(b_1 + s, b_2 + s, \ldots, b_n + s)$$

for $k, s \in \mathbb{Z}$. This representation can be extended to the irreducible representation of $U(m) \times U(n)$ with the highest weight

$$(a_1 + k, a_2 + k, \ldots, a_m + k)(b_1 + s, b_2 + s, \ldots, b_n + s)$$

for $k, s \in \mathbb{Z}$ from the proofs of 3.4.1.3 and 3.4.1.1. Likewise, $S(U(m) \times U(n))$-type with the highest weight

$$(a_1, a_2, \ldots, a_m)(b_1, b_2, \ldots, b_n)$$

is equivalent to the representation with highest weight

$$(a_1 + k, a_2 + k, \ldots, a_m + k)(b_1 + k, b_2 + k, \ldots, b_n + k).$$

Note that a quasi-spherical $S(U(m) \times U(n))$-type with highest weight

$$(a_1, a_2, \ldots, a_m)(b_1, b_2, \ldots, b_n)$$

is extended to a $U(m) \times U(n)$-type with highest weight

$$(a_1 + k, a_2 + k, \ldots, a_m + k)(b_1 + k, b_2 + k, \ldots, b_n + k),$$

and it is quasi-spherical for unique k because

$$(m + n)k + \Sigma_{i=1}^{m} a_i + \Sigma_{i=1}^{n} b_i = 0.$$

Proposition 3.5.1.1 is immediate from 3.5.1.2 and 3.5.1.5. From 3.5.1.1, we may classify single-petaled $U(m) \times U(n)$-types of $U(m, n)$ instead of single-petaled $S(U(m) \times U(n))$-types of $SU(m, n)$, and they are single-petaled $S(U(m) \times U(n))$-types when restricted to $S(U(m) \times U(n))$. The next corollary to 3.5.1.5 is useful in 3.5.2.

Corollary 3.5.1.6. *Suppose that $\widetilde{\phi}$ is the single-petaled $U(n)$-type with highest weight*

$$(c_1, c_2, \ldots, c_m)(d_1, d_2, \ldots, d_n).$$

Then, $\phi = \widetilde{\phi}|_{SU(n)}$ is the single-petaled $SU(n)$-type with highest weight

$$(c_1, c_2, \ldots, c_m)(d_1, d_2, \ldots, d_n).$$

The W-type on $V_{\widetilde{\phi}}^{M(U(m,n))}$ is the same as the W-type on $V_{\phi}^{M(SU(m,n))}$.

A $U(m) \times U(n)$-type is denoted by a tensor product of a $U(m)$-type ϕ_1 and a $U(n)$-type ϕ_2:

$$\phi_1 \otimes \phi_2.$$

We clarify the notations that we will use for the rest of this subsection.

- $G = U(m, n)$

- $\mathfrak{g} = u(m,n) = \{\begin{pmatrix} a & b \\ b^* & c \end{pmatrix} \mid a, c$ are skew Hermitian $m \times m$ and $n \times n$ matrices respectively$\}$

- $\mathfrak{p} = \{\begin{pmatrix} 0 & x \\ x^* & 0 \end{pmatrix} \mid x$ is an $m \times n$ complex matrix$\}$

- $\mathfrak{a} = \{\begin{pmatrix} 0 & x \\ x^* & 0 \end{pmatrix} \mid x$ is an $m \times n$ real matrix, $x_{i,j}$ is zero if $i + j \neq m + 1\} \subset \mathfrak{p}$

- $\alpha_i \in \mathfrak{a}^*$: $\alpha_i(\begin{pmatrix} 0 & x \\ x^* & 0 \end{pmatrix}) = x_{m+1-i,i}$ for $1 \leq i \leq n$

- restricted-roots:
  $\{\pm\alpha_i \pm \alpha_j | i \neq j, 1 \leq i, j \leq n\} \cup \{\pm 2\alpha_i | 1 \leq i \leq n\} \cup \{\pm\alpha_i | 1 \leq i \leq n\}$ if $m > n$
  $\{\pm\alpha_i \pm \alpha_j | i \neq j, 1 \leq i, j \leq n\} \cup \{\pm 2\alpha_i | 1 \leq i \leq n\}$ if $m > n$ if $m = n$.

- positive roots: $\{\alpha_i \pm \alpha_j | n \geq j > i \geq 1\} \cup \{2\alpha_i | n \geq i \geq 1\} \cup \{\alpha_i | n \geq i \geq 1\}$ if $m > n$
  $\{\alpha_i \pm \alpha_j | n \geq j > i \geq 1\} \cup \{2\alpha_i | n \geq i \geq 1\}$ if $m = n$

- simple roots: $\{\alpha_i - \alpha_{i+1} | 1 \leq i \leq n - 1\} \cup \{\alpha_n\}$ if $m > n$
  $\{\alpha_i - \alpha_{i+1} | 1 \leq i \leq n - 1\} \cup \{2\alpha_n\}$ if $m = n$

We describe the restricted-root spaces as in [17]. We first need to define the following 2×2 complex matrices:

$$J(z) = \begin{pmatrix} 0 & z \\ -\overline{z}^* & 0 \end{pmatrix}, \; I_+(z) = \begin{pmatrix} z & 0 \\ 0 & \overline{z} \end{pmatrix}, \; I_-(z) = \begin{pmatrix} z & 0 \\ 0 & -\overline{z} \end{pmatrix}.$$

The restricted-root spaces for $\pm\alpha_i \pm \alpha_j$ with $i < j$ are 2-dimensional and are nonzero only in the 16 entries corresponding to row and column indices $m - j + 1$, $m - j + 1$, $m + i$, and $m + j$, where they are

$$\mathfrak{g}_{\alpha_i - \alpha_j} = \{\begin{pmatrix} J(z) & -I_+(z) \\ -I_+(\bar{z}) & -J(\bar{z}) \end{pmatrix}\}, \; \mathfrak{g}_{-\alpha_i + \alpha_j} = \{\begin{pmatrix} J(z) & I_+(z) \\ I_+(\bar{z}) & -J(\bar{z}) \end{pmatrix}\},$$

$$\mathfrak{g}_{\alpha_i+\alpha_j} = \{\begin{pmatrix} J(z) & -I_-(z) \\ -I_-(\bar{z}) & J(\bar{z}) \end{pmatrix}\}, \mathfrak{g}_{-\alpha_i-\alpha_j} = \{\begin{pmatrix} J(z) & I_-(z) \\ I_-(\bar{z}) & J(\bar{z}) \end{pmatrix}\}.$$

The restricted-root spaces for $\pm 2\alpha_i$ have dimension 1 and are nonzero only in the 4 entries corresponding to row and column indices $m-i+1$ and $m+i$, where they are

$$\mathfrak{g}_{2\alpha_i} = i\mathbb{R}\begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix} \text{ and } \mathfrak{g}_{-2\alpha_i} = i\mathbb{R}\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}.$$

The restricted-root spaces for $\pm\alpha_i$ have dimension $2(m-n)$ and are nonzero only in the entries corresponding to row and column indices 1 to $m-n$, $m-i+1$, and $m+i$, where they are

$$\mathfrak{g}_{\alpha_i} = \{\begin{pmatrix} 0 & v & -v \\ -v^* & 0 & 0 \\ -v^* & 0 & 0 \end{pmatrix}\} \text{ and } \mathfrak{g}_{-\alpha_i} = \{\begin{pmatrix} 0 & v & v \\ -v^* & 0 & 0 \\ v^* & 0 & 0 \end{pmatrix}\}.$$

We can choose $Z_{\alpha_i-\alpha_j}$ for $i < j$ such that they are nonzero only in the 16 entries corresponding to row and column indices $m-i+1$, $m-j+1$, $m+i$ and $m+j$, where they are

$$Z_{\alpha_i-\alpha_j} = \begin{pmatrix} J(1) & 0 \\ 0 & -J(1) \end{pmatrix}.$$

Likewise, we can choose Z_{α_i} for $\pm\alpha_i$ such that they are nonzero only in the entries corresponding to row and column indices 1 to $m-n$, $m-i+1$, and $m+i$. We may choose

$$Z_{\alpha_i} = \begin{pmatrix} 0 & v & 0 \\ -v* & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \text{ where } v = e_1 \in \mathbb{C}^{m-n}.$$

For $m = n$, $2\alpha_i$ is a reduced root, and we can choose $Z_{2\alpha_i}$ such that they are nonzero only in the 16 entries corresponding to row and column indices $m-i+1$ and $m+i$, where they are

$$Z_{2\alpha_i} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}.$$

The Weyl group representatives in $N_K(\mathfrak{a})$ are defined as follows. Let A be a matrix such that the entries corresponding to row and column indices $m-j+1$, $m-j+1$, $m+i$ and $m+j$ are

$$\begin{pmatrix} J(1) & 0 \\ 0 & -J(1) \end{pmatrix},$$

and all other entries are zero. Let B be a matrix such that the entries corresponding to row and column indices $m-j+1$, $m-j+1$, $m+i$ and $m+j$ are

$$\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix},$$

and all other entries are zero. Then the Weyl group representatives are given as follows:

$$W_{\alpha_i-\alpha_j} = I - B + A \text{ and } W_{\alpha_i} = I - 2E_{m+i,m+i}.$$

A simple restricted root for $U(m,n)$ is either $\alpha_i - \alpha_{i+1}$ or α_n. The real rank one subalgebra $\mathfrak{g}^{\alpha_n}$ is isomorphic to $su(m-n+1,1)$. We can calculate that MK^{α_n} is the following:

$$\{x \times z \times (y_1 \times y_2 \times \ldots \times y_{n-1}) \times (y_{n-1} \times \ldots \times y_2 \times y_1) \mid z, y_i \in U(1), x \in U(m-n+1)\}.$$

The MK^{α}-type with a nonzero M-fixed vector restricts to $U(m-n+1) \times U(1)$-type with highest weight

$$(a, 0, 0, \ldots, 0, 0, -b)(b-a)$$

where a, b are nonnegative integers. For real rank one group $SU(m-n+1,1)$, the quasi-spherical $S(U(m-n+1) \times U(1))$-types corresponding to Kostant's parameter $(0,0)$ and $(0,1)$ are extended to the $U(m-n+1) \times U(1)$-types with a nonzero $M(U(m-n+1,1))$-fixed vector according to 3.5.1.1. They have the following highest weight:

$$(0,0,0,\ldots,0)(0),\ (1,0,0,\ldots,0)(-1),\ \text{and}\ (0,0,\ldots,0,-1)(1).$$

Therefore we should look for a $U(m) \times U(n)$-type such that its restriction to $U(m-n+1) \times U(1) \times U(1)^{n-1} \times U(1)^{n-1}$ has the irreducible subrepresentation with highest weight

$$(a, 0, 0, \ldots, 0, 0, -b) \times (b-a) \times (k_1)(k_2) \ldots (k_{n-1}) \times (k_{n-1}) \ldots (k_2)(k_1)$$

only if (a, b) is either $(0, 0)$, $(1, 0)$ or $(0, 1)$.

On the other hand, the real rank one subalgebra $\mathfrak{g}^{\alpha_i - \alpha_{i+1}}$ is isomorphic to a subalgebra of $su(2,2)$, and $MK^{\alpha_i - \alpha_{i+1}}$ is isomorphic to a subgroup of $U(m-n) \times U(2) \times U(1)^{n-2} \times U(1)^{n-2} \times U(2)$. Specifically, $MK^{\alpha_i - \alpha_{i+1}}$ is the following:

$$\{x_1 \times x_2 \times (y_1 \times \ldots \times y_{n-2}) \times (y_{n-2} \times \ldots \times y_1) \times x_3 \mid x_1 \in U(m-n), x_2 = I_-(1)J(1)x_3 I_-(1)J(1)\}.$$

The real rank one subgroup $G^{\alpha_i - \alpha_j}$, which is isomorphic to a subgroup of $SU(2,2)$, has the maximal compact group

$$\{A \times B \in S(U(2) \times U(2)) \mid A = I_-(1)J(1)BI_-(1)J(1)\}.$$

Therefore, we conclude that $K^{\alpha_i - \alpha_{i+1}} \simeq SU(2)$. The $SU(2)$-type corresponding to the Kostant's parameter $(0, 0)$ and $(0, 1)$ have the irreducible subrepresentation $(0, 0)$ and $(1, -1)$ respectively. Therefore, we should look for a representation of $U(m) \times U(n)$ ϕ satisfying the following condition:

$$\begin{gathered}[\phi|_{U(m-n) \times U(2) \times U(2) \times U(1)^{n-2} \times U(1)^{n-2}} \text{ has a subrepresentation with highest weight} \\ (0, 0, 0, \ldots, 0) \times (\phi_1) \times (\phi_2) \times (k_1)(k_2) \ldots (k_{n-2}) \times (-k_{n-2}) \ldots (-k_2)(-k_1)] \\ \Longrightarrow [\phi_1 \otimes \phi_2 \text{ has a subrepresentation with highest weight } (k, -k) \\ \text{only if } k = 0 \text{ or } 1.]\end{gathered}$$

To classify single-petaled $U(m) \times U(n)$-types in $U(m, n)$, we use 3.1.1.6 and 3.5.1.4.

Proposition 3.5.1.7. *For $U(m, n)$ with $m > n$, a single-petaled K-type has the highest weight*

$$(a_1, a_2, \ldots, a_m)(b_1, b_2, \ldots, b_n)$$

such that

$$[a_m \geq 0,\ b_n \geq -1,\ \Sigma_{i=1}^n b_i + \Sigma_{i=1}^m a_i = 0] \ \mathit{or}\ [a_1 \leq 0,\ b_1 \leq 1,\ \Sigma_{i=1}^n b_i + \Sigma_{i=1}^m a_i = 0].$$

The proof is similar to the proof of 3.4.1.4, so we omit the proof.

For $m = n$, the simple restricted roots are $\{\alpha_i - \alpha_{i+1} | n \geq i \geq 1\} \cup \{2\alpha_n\}$. We can still make use of 3.5.1.5. A $U(n) \times U(n)$-type ϕ that is quasi-spherical in $U(n,n)$ contains a subrepresentation of $U(1)^n \times U(1)^n$ with highest weight

$$(a_1)(a_2)\ldots(a_n)(-a_n)\ldots(-a_2)(-a_1).$$

Note that $Z_{2\alpha_i}(Z_{2\alpha_i}^2 - 4)$ annihilates the M-fixed vectors in the $U(1)^n \times U(1)^n$-type with highest weight

$$(a_1)(a_2)\ldots(a_n)(-a_n)\ldots(-a_2)(-a_1)$$

only if $|a_i| \leq 1$. Using a similar argument as before, we get the single-petaled K-types of $U(n,n)$.

Proposition 3.5.1.8. *A single-petaled K-type is the representation with highest weight*

$$(a_1, a_2, \ldots, a_n)(b_1, b_2, \ldots, b_n)$$

such that

$$[a_n \geq 0,\ b_n \geq -1,\ \Sigma b_i + \Sigma a_i = 0] \ \mathit{or}\ [a_1 \leq 0,\ b_1 \leq 1,\ \Sigma b_i + \Sigma a_i = 0].$$

3.5.2 Corresponding W-types for $SU(m,n)$

In this subsection, we calculate the corresponding W-type to each single-petaled $U(m) \times U(n)$-type ϕ of $G = U(m,n)$. Because of 3.5.1.6, this calculation gives the corresponding W-type to each single-petaled $S(U(m) \times U(n))$-type $\phi_{S(U(m)\times U(n))}$ in $SU(m,n)$. We make use 3.4.2.1: in $GL(n,\mathbb{C})$, the $U(n)$-type with highest weight

$$(a_1 - 1, a_2 - 1, \ldots, a_n - 1)$$

such that

$$\Sigma_{i=1}^{n} a_i = n \text{ and } a_i \in \mathbb{N} \cup \{0\}$$

realizes the Weyl group representation $\psi^{(a_1,a_2,\ldots,a_n)^T}$ on its $U(1)^n$-fixed vectors. Note that G has the Weyl group $S_n \ltimes (\mathbb{Z}/2\mathbb{Z})^n$, and its irreducible representation is parametrized by two Young diagrams (λ, τ), where $\lambda \vdash q$ and $\tau \vdash n-q$. Suppose that

$$\lambda^T = (a_1, a_2, \ldots, a_q) \vdash q \text{ and } \tau^T = (b_1, b_2, \ldots, b_{n-q}) \vdash n-q.$$

The representation of S_q, ψ^λ is realized by $U(q)$-type with highest weight

$$(a_1, a_2, \ldots, a_q) - (1, 1, \ldots, 1)$$

on its $U(1)^q$-fixed vectors, and the representation of S_{n-q}, ψ^τ is realized by $U(n-q)$-type with highest weight

$$(b_1, b_2, \ldots, b_{n-q})^T$$

on its $(1, 1, \ldots, 1)$ weight vectors. Here, we choose the same representatives for S_n as in 3.4.2.

Let us illustrate two examples. Throughout these two examples, we assume that

$$\Sigma_{i=1}^{n} a_i = n \text{ and } a_i \geq 0.$$

The reflections with respect to long simple roots generate a subgroup $W_1 \simeq S_n$ of Weyl group. The W_1-type on the M-fixed vectors of K-type ϕ' with highest weight

$$(0, 0, \ldots, 0)(a_1 - 1, a_2 - 1, \ldots, a_n - 1)$$

is ψ^λ where $\lambda = (a_1, a_2, \ldots, a_n)^T$ from 3.4.2.1. The reflection with respect to the short root generate a subgroup $W_2 \simeq (\mathbb{Z}/2\mathbb{Z})^n$ of Weyl group, and its representation on $V_{\phi'}^M$ is $(\text{triv})^n$. Therefore, the Weyl group representation on $V_{\phi'}^M$ is parametrized by

$$(\lambda, \emptyset).$$

For another example, the representation of W_1 on the M-fixed vectors of K-type ϕ'' with highest weight

$$(a_1, a_2, \ldots, a_n, 0, 0, \ldots, 0)(-1, -1, \ldots, -1)$$

is $(a_1, a_2, \ldots, a_n) = \lambda^T$. The representation of W_2 on $V_{\phi''}^M$ is $(\text{sign})^n$. Therefore, the Weyl group representation on $V_{\phi''}^M$ is parametrized by

$$(\emptyset, \lambda).$$

Using there two examples, we calculate correspondences between K-types and W-types.

Theorem 3.5.2.1. *Let ϕ be the single-petaled $U(m) \times U(n)$-type with highest weight*

$$(b_1, b_2, \ldots, b_{n-q}, 0, 0, \ldots, 0)(a_1 - 1, a_2 - 1, \ldots, a_q - 1, -1, -1, \ldots, -1)$$

such that

$$a_1 \geq a_2 \geq \ldots a_q \geq 0,\ b_1 \geq b_2 \geq \ldots b_{n-q} \geq 0,\ \text{and } \Sigma a_i + \Sigma b_i = n.$$

The Weyl group representation on $M(U(m,n))$-fixed vector of ϕ is

$$\psi^{\lambda,\tau}$$

where

$$\lambda = (a_1, a_2, \ldots, a_q)^T \vdash q$$

$$\tau = (b_1, b_2, \ldots, b_{n-q})^T \vdash n - q.$$

Sketch of Proof: The irreducible representation $\psi^{\lambda,\tau}$ of Weyl group is parametrized by two Young diagrams (λ, τ), where $\lambda \vdash q$ and $\tau \vdash n - q$. Suppose that

$$\lambda^T = (a_1, a_2, \ldots, a_q) \vdash q \text{ and } \tau^T = (b_1, b_2, \ldots, b_{n-q}) \vdash n - q.$$

The $S_q \ltimes (\mathbb{Z}/2\mathbb{Z})^q$-type

$$\psi^{(\lambda,\emptyset)}$$

is realized by $U(q) \times U(q)$-type with highest weight

$$(0,0,\ldots,0)(a_1-1,a_2-1,\ldots,a_q-1)$$

on the vectors fixed by $\{(x_1,x_2,\ldots,x_q)\times(x_q,x_{q-1},\ldots,x_1) \mid x_i \in U(1)\}$.
The $S_{n-q} \ltimes (\mathbb{Z}/2\mathbb{Z})^{n-q}$-type

$$\psi^{(\emptyset,\tau)}$$

is realized by $U(n-q)$-type with highest weight

$$(b_1,b_2,\ldots,b_{n-q})(-1,-1,\ldots,-1)$$

on its vectors fixed by $\{(x_1,x_2,\ldots,x_{n-q})(x_{n-q},x_{n-q-1},\ldots,x_1) \mid x_i \in U(1)\}$.
The $U(m)\times U(n)$-type ϕ with highest weight

$$(b_1,b_2\ldots,b_{n-q},0,0,\ldots,0)(a_1-1,a_2-1,\ldots,a_q-1,-1,-1,\ldots,-1)$$

contains the $U(m-n)\times U(q)\times U(n-q)\times U(n-q)\times U(q)$-type ϕ_{res} with highest weight

$$(0,0,\ldots,0)(0,0,\ldots,0)(b_1,b_2,\ldots,b_{n-q})(-1,-1,\ldots,-1)(a_1-1,a_2-1,\ldots,a_q-1).$$

Note that ϕ is a single-petaled $U(m)\times U(n)$-type of $U(m,n)$, so V_ϕ^M is annihilated by $Z_\alpha(Z_\alpha^2-4)$ for reduced root α. The representation of $(S_q\times(\mathbb{Z}/2\mathbb{Z})^q)\times(S_{n-q}\times(\mathbb{Z}/2\mathbb{Z})^{n-q})$, which is a subgroup of W, on $V_{\phi_{res}}^M$ is

$$(\lambda\otimes(\text{triv})^q)\times(\tau\otimes(\text{sign})^{n-q}).$$

Therefore, the $S_n \ltimes (\mathbb{Z}/2\mathbb{Z})^n$-type

$$\psi^{(\lambda,\tau)} = \text{Ind}_{(S_q\times(\mathbb{Z}/2\mathbb{Z})^q)\times(S_{n-q}\times(\mathbb{Z}/2\mathbb{Z})^{n-q})}^{S_n\ltimes(\mathbb{Z}/2\mathbb{Z})^n}(\lambda\otimes(\text{triv})^q)\times(\tau\otimes(\text{sign})^{n-q})$$

is a subrepresentation of the representation of W on V_ϕ^M. In fact, we can prove that the representation space of $\psi^{(\lambda,\tau)}$ has the same dimension with V_ϕ^M, so $\psi^{(\lambda,\tau)}$ is the representation of W on V_ϕ^M. We omit the detailed proof. $\square$

Chapter 4

Quasi-single-petaled K-types

We classified single-petaled K-types for $G = SL(n, \mathbb{C})$ and $G = SU(m, n)$ in 3.4.1 and 3.5.1. For these groups, single-petaled K-types realize all W-types, but for some classical groups, we do not have enough single-petaled K-types. Therefore, we would like to generalize the single-petaled K-types to the quasi-single-petaled K-types, which we will introduce in this chapter. Even for a K-type ϕ such that V_ϕ^M is not annihilated by $\phi(Z_\alpha)(\phi(Z_\alpha)^2 - 4)$ for all $\alpha \in \Pi$, there often exists a nonzero M-fixed vector annihilated by all of them. On these M-fixed vectors, it is still possible to represent the long intertwining operator using a Weyl group algebra element, and quasi-single-petaled K-types are the ones with such M-fixed vectors. In this chapter, we define quasi-single-petaled K-types, associate W-types to the quasi-single-petaled K-types, and classify which W-types can be realized in this way for each classical group.

4.1 Quasi-single-petaled K-types

4.1.1 Quasi-single-petaled K-types

Definition 4.1.1.1. *For a K-type ϕ, $v \in V_\phi^M$ is* ***single-petaled*** *if*

$$\phi(Z_\alpha)(\phi(Z_\alpha)^2 - 4)v = 0 \textit{ for all } \forall\alpha \in \Delta_1.$$

The vector space spanned by single-petaled M-fixed vectors of ϕ is denoted by $V_{\phi,single}^M$.

The action of the long intertwining operator on $\mathrm{Hom}_K(\phi, I(\nu)) \simeq V_\phi^M$ is not fully represented by a Weyl group algebra element if ϕ is not a single-petaled K-type. However, the action of the long intertwining operator on $V_{\phi,single}^M$ can be represented by a Weyl group algebra element still. This is because $V_{\phi,single}^M$ is stabilized by the action of the Weyl group and the long intertwining operator.

Lemma 4.1.1.2. *The action of the Weyl group stabilizes* $V_{\phi,single}^M$.

Proof: We prove that the Weyl group representatives in $N_K(\mathfrak{a})$ stabilize V_ϕ^M. Let us denote the Weyl group representative for s_β by r_β.
Then, r_β acts on X_α, θX_α, H_α, and

$$\begin{aligned} Ad(r_\beta)\mathrm{span}_{\mathbb{R}} < H_\alpha >= \mathrm{span}_{\mathbb{R}} < H_{s_\beta(\alpha)} > \\ Ad(r_\beta)\mathfrak{g}^\alpha = \mathfrak{g}_{s_\beta(\alpha)} \\ Ad(r_\beta)\theta\mathfrak{g}^\alpha = \theta(\mathfrak{g}^{s_\beta(\alpha)}) \\ Ad(r_\beta)(Z_\alpha) \in \mathfrak{k} \cap \mathfrak{g}^\alpha = \mathfrak{k}^\alpha. \end{aligned}$$

Suppose that

$$A = \{Z_\alpha \mid \alpha \in \Delta_1\} \text{ and } B = \{Ad(r_\beta)(Z_\alpha) \mid \alpha \in \Delta_1\}.$$

Since the definitions of single-petaled K-type and single-petaled M-fixed vectors do not depend on the choice of Z_α, the following holds:

$$\begin{aligned} &v \text{ is single-petaled} \\ &\Leftrightarrow \phi(X)(\phi(X)^2 - 4)v = 0 \text{ for all } X \in A \\ &\Leftrightarrow \phi(X)(\phi(X)^2 - 4)v = 0 \text{ for all } X \in B. \end{aligned}$$

Suppose v is single-petaled. We show that $s_\beta.v = Ad(r_\beta)v$ is single-petaled as follows:

$$\begin{aligned} &v \in V_\phi^M \text{ is single-petaled} \\ &\Leftrightarrow \phi(Z_\alpha)(\phi(Z_\alpha)^2 - 4)v = 0 \ \forall \alpha \in \Delta_1 \\ &\Leftrightarrow \phi(r_\beta)\phi(Z_\alpha)(\phi(Z_\alpha)^2 - 4)v = 0 \ \forall \alpha \in \Delta_1 \\ &\Leftrightarrow \phi(Ad(r_\beta)Z_\alpha)(\phi(Ad(r_\beta)Z_\alpha)^2 - 4)\phi(r_\beta)v = 0 \ \forall \alpha \in \Delta_1 \\ &\Leftrightarrow \phi(Ad(r_\beta)Z_\alpha)(\phi(Ad(r_\beta)Z_\alpha)^2 - 4)\phi(r_\beta)v = 0 \ \forall \alpha \in \Delta_1 \end{aligned}$$

$$\Leftrightarrow \phi(X)(\phi(X)^2 - 4)\phi(r_\beta)v = 0 \text{ for all } X \in B$$
$$\Leftrightarrow \phi(X)(\phi(X)^2 - 4)\phi(r_\beta)v = 0 \text{ for all } X \in A$$
$$\Leftrightarrow \phi(r_\beta)v \in V_\phi^M \text{ is single-petaled.}$$

We conclude that the Weyl group stabilizes $V^M_{\phi,single}$. □

Lemma 4.1.1.3. *The action of the long intertwining operator stabilizes* $V^M_{\phi,single}$.

Proof: To prove that the long intertwining operator stabilizes $V^M_{\phi,single}$, we use the decomposition of the long intertwining operator. If $v \in V^M_{\phi,single}$, then

$$v = v_0 + v_2,$$

with v_0 an eigenvector of the eigenvalue 0, and v_2 a sum of eigenvectors of eigenvalues ± 2 under action of Z_α. Then

$$s_\alpha v = v_0 - v_2,$$

and it is a single-petaled M-fixed vector. Because

$$A(s_\alpha, \nu)v = v_0 + kv_2$$

for some scalar k, it is a linear combination of $v_0 + v_2$ and $v_0 - v_2$. Therefore, $A(s_\alpha, \nu)v$ is a single-petaled M-fixed vector as well. Since the long intertwining operator is expressed as a composition of $A(s_\alpha, \nu)$ with varying α and ν, we conclude that the long intertwining operator stabilizes $V^M_{\phi,single}$ □

Definition 4.1.1.4. *A K-type is* ***quasi-single-petaled*** *if it has a nonzero single-petaled M-fixed vector.*

Definition 4.1.1.5. *We denote the a Weyl group representation on* $V^M_{\phi,single}$ *by* $\psi_{\phi,single}$.

4.2 Correspondence Between W-types and K-types

4.2.1 Correspondence Between W-types and K-types

As discussed in 3.2.1, it is often useful to make a relation between unitarity tests using K-types and unitarity tests using W-types. For a single-petaled K-type ϕ, we derived this relation naturally – we matched the W-type that is realized on the M-fixed vectors of ϕ. For quasi-single-petaled K-type ϕ, the long intertwining operator on $V^M_{\phi,single}$, which is a subspace of $V^M_\phi \simeq \mathrm{Hom}_K(\phi, I(\nu))$, is represented by the Weyl group algebra element $A_{\omega_0}(\nu)$ defined in 2.4.2.

Lemma 4.2.1.1. *For a K-type ϕ, the long intertwining operator on $V^M_{\phi,single}$ is*

$$\psi_{\phi,single}(A_{\omega_0}(\nu)).$$

Therefore, if the long intertwining operator on $V^M_{\phi,single}$ is positive semi-definite, then every eigenvalue of $\psi_{\phi,single}(A_{\omega_0}(\nu))$ is positive.

Proposition 4.2.1.2. *If there exists a K-type ϕ such that an eigenvalue of $\psi_{\phi,single}(A_{\omega_0}(\nu))$ is negative, then the long intertwining operator on*

$$V^M_{\phi,single} \subset V^M_\phi \simeq \mathrm{Hom}_K(\phi, I(\nu))$$

is not positive semi-definite, so $J(\nu)$ is not a unitary representation.

Proposition 4.2.1.3. *Suppose the following:*

(1) G is split.

(2) For each W-type τ, there exists a K-type ϕ such that τ is a subrepresentation of $\psi_{\phi,single}$.

Then, if $J(\nu)_p$ is not unitary, $J(\nu)$ is not unitary either.

Sketch of Proof: From the theorem of Barbasch-Moy in 2.4.5.2, it is known for split p-adic groups that $J(\nu)_p$ is unitary if and only if $\psi(A_{\omega_0}(\nu))$ is positive semi-definite for every W-type ψ. Suppose that $J(\nu)_p$ is not positive semi-definite. Then, $\psi(A_{\omega_0}(\nu))$ is not positive semi-definite for some W-type ψ. From (2), ψ is contained in $\psi_{\phi,single}$ for some quasi-single-petaled K-type ϕ. Then, $\psi(A_{\omega_0}(\nu))$ is a block in the long intertwining operator on

$$V^M_{\phi,single} \subset V^M_\phi \simeq \mathrm{Hom}_K(\phi, I(\nu)),$$

so the long intertwining operator is not positive definite on $\mathrm{Hom}_K(\phi, I(\nu))$. Therefore, $J(\nu)$ is not a unitary representation in the real group setting either. □

Therefore, it is meaningful to figure out which W-types are realized on the single-petaled M-fixed vectors of quasi-single-petaled K-types. In 3.4.2 and 3.5.2, we proved that the single-petaled K-types realize all W-types for $SL(n,\mathbb{C})$ and $SU(m,n)$. We can realize all W-types using quasi-single-petaled K-types also for $SL(n,\mathbb{R})$, $SO(m,n)$, $Sp(n,\mathbb{R})$, which we show in 4.4.

4.3 Strategy to Classify Single-petaled W-types

4.3.1 Strategy to Classify Single-petaled W-types

In this subsection, we introduce a strategy to classify W-types realized on the single-petaled M-fixed vectors of quasi-single-petaled K-types.

Definition 4.3.1.1. *A W-type ψ is **single-petaled** for G if a K-type ϕ exists such that $\psi_{\phi,single}$ on $V^M_{\phi,single}$ has ψ as a subrepresentation.*

To classify single-petaled W-types, we use the following strategy: for each single-petaled W-type, we construct a K-type that realizes the W-type on its single-petaled M-fixed vectors, or we construct a vector space spanned by single-petaled M-fixed vectors which realizes the W-type. To construct such vector spaces, we use a tensor product or a wedge product of simple representations, such as standard representations. To show a W-type ψ is not single-petaled, we make use of the following lemma.

Proposition 4.3.1.2. *Suppose that G_1 is a Levi subgroup of G, K_1 is the maximal compact subgroup of G_1, and W_1 is the Weyl group of G_1. Note that W_1 is the subgroup of W. If a W_1-type τ is not single-petaled for G_1 and a W-type ψ has τ as a subrepresentation when restricted to W_1, then ψ is not single-petaled.*

Proof: It is enough to show that if ψ is a single-petaled W-type in G, then every irreducible subrepresentation of ψ when restricted to W_1 is single-petaled in G_1. Suppose τ is a W_1-type in the restricted representation of ψ to W_1. If ψ is single-petaled in G, then there is a K-type ϕ that realizes ψ on $V^M_{\phi,single}$. Suppose that

$$\phi|_{K_1} = \phi_1 \oplus \phi_2 \oplus \ldots \oplus \phi_s.$$

Then, $V^M_{\phi,single}$ is a subspace of

$$\bigoplus_{i=1}^{s} V^{M\cap K_1}_{\phi_i,single}.$$

Because $\psi_{\phi,single}$ has ψ as a subrepresentation, it has τ as a subrepresentation when restricted to W_1. As a result, the W_1 representation on

$$V^{M\cap K_1}_{\phi_1,single} \oplus V^{M\cap K_1}_{\phi_2,single} \oplus \ldots \oplus V^{M\cap K_1}_{\phi_s,single}$$

has τ as a subrepresentation. Therefore, the W-type τ appears in $V^{M\cap K_1}_{\phi_i,single}$ for some i, and τ is a single-petaled W_1-type in G_1. □

4.4 Classification of Single-petaled W-types

Using 3.3.1 and 4.3.1, we classify single-petaled W-types for classical groups in this section.

4.4.1 $SL(n, \mathbb{C})$

In 3.4.2.1, we showed that, for each W-type of $SL(n, \mathbb{C})$, there exists a single-petaled K-type ϕ that realizes the W-type on $V^M_\phi = V^M_{\phi,single}$.

Proposition 4.4.1.1. *Every irreducible Weyl group representation is single-petaled in $SL(n,\mathbb{C})$.*

4.4.2 $SL(n,\mathbb{R})$

Proposition 4.4.2.1. *Every irreducible Weyl group representation is single-petaled in $SL(n,\mathbb{R})$.*

Proof: Suppose a single-petaled $U(n)$-type ϕ of $GL(n,\mathbb{C})$ has the highest weight

$$(a_1, a_2, \ldots, a_n) - (1, 1, \ldots, 1)$$

such that

$$\Sigma_{i=1}^n a_i = n \text{ and } a_i \geq 0.$$

Suppose that

$$\phi|_{SO(n,\mathbb{R})} = \phi_1 \oplus \phi_2 \oplus \ldots \oplus \phi_k$$

is the decomposition of ϕ into a direct sum of irreducible representations of $SO(n,\mathbb{R})$. Then,

$$V_\phi^{U(1)^n} \subset V_\phi^{S(O(1)^n)} = \bigoplus_{i=1}^k V_{\phi_i}^{S(O(1)^n)}.$$

Because $GL(n,\mathbb{C})$ and $SL(n,\mathbb{R})$ share the restricted root system, we can assume that they share Z_α for every reduced root α and the Weyl group representatives as well. In fact, we can fix a subgroup isomorphic to $SO(n,\mathbb{R})$ in $U(n)$ and choose Z_α and Weyl group representative such that they are in $so(n,\mathbb{R})$ and $SO(n,\mathbb{R})$ respectively. We may thus assume that Z_α and Weyl group representatives act on V_{ϕ_i}. If $v \in V_\phi^{U(1)^n} \subset \oplus_{i=1}^k V_{\phi_i}^{S(O(1)^n)}$, then we can express v as

$$v = v_1 + v_2 + \ldots + v_k$$

such that $v_i \in V_{\phi_i}^{S(O(1)^n)}$. Then,

$$Z_\alpha(Z_\alpha^2 - 4)v_i = 0 \ \forall \alpha \in \Delta, \ \forall i$$

because

$$Z_\alpha(Z_\alpha^2 - 4)v = 0 \ \forall \alpha \in \Delta.$$

This means that a nonzero v_i is single-petaled. Therefore, we can conclude that

$$V_\phi^{U(1)^n} = V_{\phi,single}^{U(1)^n} \subset V_{\phi_1,single}^{S(O(1)^n)} \oplus V_{\phi_2,single}^{S(O(1)^n)} \oplus \ldots \oplus V_{\phi_k,single}^{S(O(1)^n)}.$$

As we showed in 3.4.2.1, the Weyl group representation ψ_ϕ on $V_\phi^{U(1)^n}$ is ψ^λ where

$$\lambda = (a_1, a_2, \ldots, a_n)^T.$$

The Weyl group acts on $V_{\phi_i,single}^{S(O(1)^n)}$. Therefore, it acts on $\bigoplus_{i=1}^k V_{\phi_i,single}^{S(O(1)^n)}$. Because ψ_ϕ is an irreducible subrepresentation of $\bigoplus_{i=1}^k V_{\phi_i,single}^{S(O(1)^n)}$, it appears in $V_{\phi_i,single}^{S(O(1)^n)}$ for some i. Therefore, ψ^λ is realized by a quasi-single-petaled $SO(n,\mathbb{R})$-type ϕ_i, and it is a single-petaled Weyl group representation of $SL(n,\mathbb{R})$. Because every Weyl group representation is single-petaled in $GL(n,\mathbb{C})$, we prove the proposition. $\square$

Now, we specify which representation of $SO(n,\mathbb{R})$ realizes the W-type ψ^λ where

$$\lambda = (a_1, a_2, \ldots, a_n)^T.$$

We make use of the fact that ψ^λ is the W-type on the $U(1)^n$-fixed vectors of the $U(n)$-type with highest weight

$$(a_1, a_2, \ldots, a_n) - (1, 1, \ldots, 1)$$

where $a_i \geq a_{i+1}$ and $a_i \geq 0$ from 3.4.2. The $U(1)^n$-fixed vector space of the $U(n)$-type with highest weight

$$(a_1, a_2, \ldots, a_n) - (1, 1, \ldots, 1)$$

correspond to the $(1, 1, \ldots, 1)$ weight vector space of the $U(n)$-type ϕ with highest weight

$$(a_1, a_2, \ldots, a_n).$$

Note that

$$\phi^{(a_1-1,a_2-1,\ldots,a_n-1)} \otimes \det = \phi^{(a_1,a_2,\ldots,a_n)}.$$

The Weyl group acts on both vector space, and we choose the same Weyl group representatives for S_n as in 3.4.2. Instead of $O(1)^n$-fixed vectors, we think of $O(1)^n$-signed vectors in V_ϕ. **We say $v \in V_\phi$ is $O(1)^n$-signed if**

$$\phi(x)v = \det(x)v$$

for $x \in O(1)^n \subset O(n) \subset U(n)$. An $O(1)^n$-signed vector is a sum of weight vectors and each weight vector that appears in the sum corresponds to $(b_1, b_2, \ldots, b_n)$ such that each b_i is odd. Because the only weight satisfying this condition in ϕ is $(1, 1, \ldots, 1)$, the $(1, 1, \ldots, 1)$ weight vectors of ϕ are exactly the $O(1)^n$-signed vectors. Note that, for an $O(n)$-type ϕ', $\phi' \otimes \det$ has a nonzero $O(1)^n$-fixed vector if and only if ϕ' has a nonzero $O(1)^n$-signed vector.

Lemma 4.4.2.2. *Suppose $U(n)$-type ϕ has the highest weight*

$$(a_1, a_2, \ldots, a_n)$$

such that $\Sigma_{i=1}^n a_i = n$ and $a_i \geq 0$. If

$$\phi|_{O(n)} = \phi_1 \oplus \phi_2 \oplus \ldots \oplus \phi_r$$

is the decomposition of ϕ into a direct sum of irreducible representations of $O(n)$. Then, ϕ_i has a nonzero $O(1)^n$-signed vector only for one i.

Proof: The W-type ψ^λ is realized on the $O(1)^n$-signed vectors of ϕ where

$$\lambda = (a_1, a_2, \ldots, a_n)^T.$$

We can define a Weyl group representation ψ_i (which may be 0) on the $O(1)^n$-signed vectors of ϕ_i. Then,

$$\psi^\lambda = \psi_1 \oplus \psi_2 \oplus \ldots \oplus \psi_r$$

and since ψ^λ is irreducible, ψ_i is 0 except only for one i. $\square$

Definition 4.4.2.3. *$\mathscr{H}_k$ is the representation of $O(k)$ on the harmonic polynomial of degree k.*

The $\mathscr{H}_{a_i}$ is irreducible because it is an irreducible representation of $SO(a_i)$ for $a_i > 2$, and we can easily check that it is an irreducible representation of $O(a_i)$ for $a_i = 1$ or 2.

Lemma 4.4.2.4. *When the $U(a_i)$-type ϕ with highest weight $(a_i, 0, 0, \ldots, 0)$ is restricted to $O(a_i)$, only one irreducible subrepresentation has a nonzero $O(1)^{a_i}$-signed vector, and it is $\mathscr{H}_{a_i}$.*

Proof: We can think of ϕ as representation of $U(a_i)$ on degree a_i polynomials on a_i variables $x_1, x_2, \ldots, x_{a_i}$. The $O(1)^n$-signed vector space is 1-dimensional and it is spanned by $x_1 x_2 \ldots x_{a_i}$. When $U(a_i)$-type with highest weight $(a_i, 0, 0, \ldots, 0)$ is restricted to $O(a_i)$, only one irreducible subrepresentation has a nonzero $O(1)^{a_i}$-signed vector due to 4.4.2.2. The $\mathscr{H}_{a_i}$ appears in ϕ, and it contains a nonzero $O(1)^{a_i}$-signed vector $x_1 x_2 \ldots, x_{a_i}$. $\square$

Suppose S is a subgroup of $U(n)$ isomorphic to $\Pi_{i=1}^n U(a_i)$. Because

$$\phi_{(a_1, a_2, \ldots, a_n)}|_S$$

has the subrepresentation with highest weight

$$\Pi_{i=1}^n (a_i, 0, 0, \ldots, 0),$$

we notice that $\phi_{(a_1, a_2, \ldots, a_n)}|_{\Pi_{i=1}^n O(a_i)}$ has the subrepresentation

$$\Pi_{i=1}^k \mathscr{H}_{a_i}.$$

Also, the representation of $\Pi_{i=1}^n U(a_i)$ with highest weight $\Pi_{i=1}^n(a_i, 0, 0, \ldots, 0)$ has one dimensional $O(1)^n$-signed vectors because the representation of $U(a_i)$ with highest weight $(a_i, 0, 0, \ldots, 0)$ has one-dimensional $O(1)^{a_i}$-signed vectors. If a $O(n)$-type contains the representation of $O(a_1) \times O(a_2) \times \ldots O(a_n)$

$$\Pi_{i=1}^n \mathscr{H}_{a_i}$$

then it has a nonzero $O(1)^n$-signed vector. Therefore, we look for a subrepresentation of $O(n)$ in the $U(n)$-type with highest weight $(a_1, a_2, \ldots, a_n)$, which has a subrepresentation of $\Pi_{i=1}^n O(a_i)$ with highest weight $\Pi_{i=1}^n \mathscr{H}_{a_i}$. Note that this representation has subrepresentation of $\Pi_{i=1}^n SO(a_i)$ with highest weight

$$\Pi_{i=1}^n (a_i, 0, 0, \ldots, 0).$$

We state the following lemma without proof.

Lemma 4.4.2.5. *If a $\Pi_{i=1}^n O(a_i)$-type appears in $\Pi_{i=1}^n U(a_i)$-type with highest weight*

$$\Pi_{i=1}^n (a_i, 0, 0, \ldots, 0),$$

and it has a subrepresentation of $\Pi_{i=1}^n SO(a_i)$ with highest weight

$$\Pi_{i=1}^n (a_i, 0, 0, \ldots, 0),$$

then it is

$$\Pi_{i=1}^n \mathscr{H}_{a_i}.$$

Therefore, we look for a subrepresentation of $O(n)$ in the $U(n)$-type with highest weight $(a_1, a_2, \ldots, a_n)$, which has a subrepresentation of $\Pi_{i=1}^n SO(a_i)$ with highest weight $\Pi_{i=1}^n (a_i, 0, 0, \ldots, 0)$.

Lemma 4.4.2.6. *Fix a partition*

$$n = a_1 + a_2 + \ldots + a_n$$

of n such that $a_i \geq a_{i+1}$. Define k to be the number of nonzero parts of the partition (the largest integer such that $a_k \geq 1$) and s to be the number of parts of size at least 2 (the largest integer such that $a_s \geq 2$). Suppose σ is an irreducible representation of $SO(n, \mathbb{R})$ with highest weight

$$(a_1 - a_n, a_2 - a_{n-1}, a_3 - a_{n-2}, \ldots).$$

Then,

(1) It is the subrepresentation of the $U(n)$-type with highest weight $(a_1, a_2, \ldots, a_n)$.

(2) $\sigma|_{\Pi_{i=1}^{n} SO(a_i,\mathbb{R}) \simeq \Pi_{i=1}^{s} SO(a_i,\mathbb{R})}$ contains the irreducible subrepresentation with highest weight $\Pi_{i=1}^{s}(a_i, 0, 0, \ldots, 0)$.

Proof: We prove this statement by mathematical induction on n.

(i) $n = 2$:

(A) The $U(2)$-type with highest weight $(2, 0)$ restricts to the sum of the three $SO(2, \mathbb{R})$-types: (2), (-2) and (0). The $SO(2, \mathbb{R})$-type (2) restricts to the $SO(2, \mathbb{R})$-type (2).

(B) The representation of $U(2)$ with highest weight $(1, 1)$ is the determinant representation, and it restricts to the trivial $SO(2, \mathbb{R})$-type (0).

(ii) $n > 2$:

Suppose that the statement is true for n smaller than m. We prove the statement for $n = m$.

(C-1) $k > \frac{m}{2}$, m is even:

Then, $a_k = 1$ and $a_{m-k+1} = 1$ because $a_1 \geq a_2 \geq \ldots \geq a_n$ and $\Sigma_{i=1}^{m} a_i = m$. Suppose the $U(m)$-type ϕ has the highest weight $(a_1, a_2, \ldots, a_m)$. Then, $\phi|_{U(m-2)\times U(2)}$ has a subrepresentation ϕ_{res} with highest weight

$$(a_1, a_2, \ldots, a_{m-k}, a_{m-k+2}, \ldots, a_{k-1}, a_{k+1}, \ldots, a_m) \times (1, 1).$$

By inductive hypothesis, ϕ_{res} has the subrepresentation of $\Pi_{i=1}^{s} SO(a_i, \mathbb{R})$ with highest weight $\Pi_{i=1}^{s}(a_i, 0, 0, \ldots, 0)$, and it appears in the irreducible subrepresentation ϕ'_{res} of

$\phi_{res}|SO(m-2,\mathbb{R}) \times SO(2,\mathbb{R})$ with highest weight

$$(a_1 - a_m, a_2 - a_{m-1}, \ldots, a_{m-k} - a_{k+1}, \widehat{a_{m-k+1} - a_k}, a_{m-k+2} - a_{k-1}, \ldots) \times \text{triv}.$$

Note that ϕ'_{res} appears in the $SO(m)$-type ϕ' with highest weight

$$(a_1 - a_m, a_2 - a_{m-1}, \ldots, a_{m-k} - a_{k+1}, a_{m-k+1} - a_k, a_{m-k+2} - a_{k-1}, \ldots).$$

In the similar way, we can show that ϕ' appears in ϕ. Therefore, the proposition holds for $n = m$.

(C-2) $k > \frac{m}{2}$, m is odd:

Then, $a_{\frac{m+1}{2}}$ is 1 because $a_1 \geq a_2 \geq \ldots \geq a_n$ and $\Sigma_{i=1}^{m} a_i = m$. Suppose the $U(m)$-type ϕ has the highest weight $(a_1, a_2, \ldots, a_m)$. Then, $\phi|_{U(m-1)\times U(1)}$ has a subrepresentation ϕ_{res} with highest weight

$$(a_1, a_2, \ldots, a_{\frac{m-1}{2}}, a_{\frac{m+3}{2}}, \ldots, a_{m-1}, a_m) \times (1).$$

By inductive hypothesis, ϕ_{res} has the subrepresentation ϕ'_{res} of $SO(m-1) \times SO(1)$ with highest weight

$$(a_1 - a_m, a_2 - a_{m-1}, \ldots, a_{\frac{m-1}{2}} - a_{\frac{m+3}{2}}) \times \text{triv},$$

and ϕ'_{res} has the subrepresentation of $\Pi_{i=1}^{s} SO(a_i, \mathbb{R})$ with highest weight $\Pi_{i=1}^{s}(a_i, 0, 0, \ldots, 0) \times$ triv. Note that ϕ'_{res} is a subrepresentation of $SO(m)$-type ϕ' with highest weight

$$(a_1 - a_m, a_2 - a_{m-1}, \ldots, a_{\frac{m-1}{2}} - a_{\frac{m+3}{2}}).$$

In the similar way, we can show that this $SO(m)$-type appears in ϕ. Therefore, the proposition holds for $n = m$.

(D) $k \leq \frac{m}{2}$:

Suppose the $U(m)$-type ϕ has the highest weight $(a_1, a_2, \ldots, a_m)$, and $\phi|_{U(m-a_s)\times U(a_s)}$

contains a subrepresentation ϕ_{res} with highest weight

$$(a_1, a_2, \ldots, \widehat{a_s}, \ldots, a_k, 0, 0, 0, \ldots, 0) \times (a_s, 0, 0, \ldots).$$

By inductive hypothesis, ϕ_{res} has a subrepresentation ϕ'_{res} of $SO(m-a_s, \mathbb{R}) \times SO(a_s)$ with the highest weight

$$(a_1, a_2, \ldots, \hat{a}_s, \ldots, a_k, 0, 0, \ldots, 0) \times (a_s, 0, \ldots),$$

and ϕ'_{res} has the subrepresentation with highest weight

$$\Pi_{i=1}^{s}(a_i, 0, 0, \ldots, 0)$$

when restricted to $\Pi_{i=1}^{n} SO(a_i) = \Pi_{i=1}^{s} SO(a_i)$. Note that ϕ'_{res} is a subrepresentation of the $SO(m)$-type ϕ' with highest weight

$$(a_1, a_2, \ldots, a_k, 0, 0, \ldots, 0) = (a_1, a_2, \ldots, a_{[\frac{m}{2}]}).$$

In the similar way, we can show that this $SO(m)$-type appears in ϕ. Therefore, the proposition holds for $n = m$.
By (i), (ii) and mathematical induction, the proposition is proved. □

Corollary 4.4.2.7. *The representation of* $O(n, \mathbb{R})$

$$\mathrm{Ind}_{SO(n,\mathbb{R})}^{O(n,\mathbb{R})} \phi_{(a_1-a_n, a_2-a_{n-1}, a_3-a_{n-2}, \ldots)}$$

has the subrepresentation $\Pi_{i=1}^{s} \mathscr{H}_{a_i}$ *of* $\Pi_{i=1}^{s} O(a_i)$.

Proposition 4.4.2.8. *In* $SL(n, \mathbb{R})$*, the* W*-type* $\psi^{(a_1, a_2, \ldots, a_n)}$ *is realized on the quasi-single-petaled* $SO(n)$*-type with highest weight*

$$(a_1 - a_n, a_2 - a_{n-1}, a_3 - a_{n-2}, \ldots).$$

Proof: The representation of $O(n,\mathbb{R})$

$$\phi'' = \operatorname{Ind}_{SO(n,\mathbb{R})}^{O(n,\mathbb{R})} \phi_{(a_1-a_n, a_2-a_{n-1}, a_3-a_{n-2}, \ldots)}$$

realizes the W-type ψ^λ on the $O(1)^n$-signed vectors from 4.4.2.7. Therefore, the W-type ψ^λ is realized on the $O(1)^n$-fixed vectors of

$$\phi'' \otimes \det$$

when we choose the same Weyl group representatives $k_{\alpha_p-\alpha_q}$ for $s_{\alpha_p-\alpha_q}$ to be

$$k_{\alpha_p-\alpha_q} = I - E_{p,p} - E_{q,q} + E_{p,q} - E_{q,p}.$$

The $O(n)$-type ϕ'' is restricted to $SO(n,\mathbb{R})$ such that

$$\phi''|_{SO(n,\mathbb{R})} = \phi''_1 \oplus \phi''_2.$$

Here, ϕ''_1 and ϕ''_2 are the $SO(n,\mathbb{R})$-types with highest weight

$$(b_1, b_2, \ldots, b_{[\frac{n}{2}]}) = (a_1 - a_n, a_2 - a_{n-1}, a_3 - a_{n-2}, \ldots)$$

and

$$(-b_{[\frac{n}{2}]}, -b_{[\frac{n}{2}]-1}, \ldots, -b_1)$$

respectively. Thus, we have

$$\phi'' \otimes \det|_{SO(n,\mathbb{R})} = \phi''_1 \oplus \phi''_2$$

and

$$V_{\phi'',single}^{O(1)^n} \subset V_{\phi''_1,single}^{S(O(1)^n)} \oplus V_{\phi''_2,single}^{S(O(1)^n)}.$$

Because we have the Weyl group representations on $V_{\phi''_1}^{S(O(1)^n)}$ and $V_{\phi''_2}^{S(O(1)^n)}$ respectively, we conclude that ψ^λ is the subrepresentation of the Weyl group representation on $V_{\phi_1''}^{S(O(1)^n)} \oplus V_{\phi_2''}^{S(O(1)^n)}$. Actually, the Weyl group representations on $V_{\phi_1''}^{S(O(1)^n)}$ and $V_{\phi_2''}^{S(O(1)^n)}$ are the same, so we obtain the proposition. □

4.4.3 $SU(m,n)$

By 3.5.2.1, every W-type is realized on the $M(U(m,n))$-fixed vectors of a single-petaled $U(m)\times U(n)$-type $\widetilde{\phi}$ of $U(m,n)$. By 3.5.1.1, $\widetilde{\phi}|_{S(U(m)\times U(n))}$ is a single-petaled $S(U(m)\times U(n))$-type of $SU(m,n)$, and it realizes the same W-type. We obtain the following proposition.

Proposition 4.4.3.1. *Every W-type is single-petaled for $SU(m,n)$.*

4.4.4 $SO(m,n)$

For $G = SO(m,n)$, the maximal compact subgroup K is $S(O(m)\times O(n))$, and $M = M(SO(m,n))$ is

$$\{x \times (x_1 \times x_2 \times \ldots x_n) \times (x_n \times x_{n-2} \times \ldots \times x_1) \mid x \in SO(m-n), x_i \in O(1)\}$$
$$\subset SO(m-n) \times O(1)^n \times O(1)^n.$$

Let $\widetilde{G}$ be $U(m,n)$, $\widetilde{K}$ be $U(m)\times U(n)$, and $\widetilde{M} = M(U(m,n))$ be

$$\{x \times (e^{i\theta_1} \times e^{i\theta_2} \times \ldots \times e^{i\theta_n}) \times (e^{i\theta_n} \times e^{i\theta_{n-1}} \times \ldots \times e^{i\theta_1}) \mid x \in U(m-n)\}$$
$$\subset U(m-n) \times U(1)^n \times U(1)^n.$$

We make use of 3.5.2.1.

Proposition 4.4.4.1. *Every W-type is single-petaled for $SO(m,n)$.*

Proof: For each W-type ψ, we can find a single-petaled $U(m)\times U(n)$-type $\widetilde{\phi}$ in $U(m,n)$ such that the Weyl group representation on $V_{\tilde{\phi}}^{M}$ is ψ from 3.4.2.1. Then, $V_{\tilde{\phi}}^{\widetilde{M}} = V_{\tilde{\phi},single}^{\widetilde{M}}$ is the subspace of $V_{\tilde{\phi}}^{M}$. Suppose

$$\phi|_K = \phi_1 \oplus \phi_2 \oplus \ldots \oplus \phi_s$$

is the decomposition of ϕ into a sum of irreducible representations of K. Then,

$$V^M_{\widetilde{\phi}} = V^M_{\phi_1} \oplus V^M_{\phi_2} \oplus \ldots \oplus V^M_{\phi_s}$$

holds.

Note that $U(m,n)$ and $SO(m,n)$ share the restricted root system, and we can assume that they share Z_α for all reduced root α and the Weyl group representatives as well. In fact, we can fix a subgroup isomorphic to $SO(m,n)$ in $U(m,n)$ and choose Z_α and the Weyl group representatives of $U(m,n)$ such that they are in $so(m,n)$ and $SO(m,n)$ respectively. If $v \in V^M_{\widetilde{\phi}}$, then

$$v = v_1 + v_2 + \ldots v_s$$

such that $v_i \in V^M_{\phi_i}$. Then,

$$Z_\alpha(Z_\alpha^2 - 4)v_i = 0, \ \forall\ i, \ \forall \alpha \in \Delta_1$$

because $Z_\alpha(Z_\alpha^2 - 4)v = 0$ and Z_α acts on each ϕ_i. Therefore,

$$V^{\widetilde{M}}_{\widetilde{\phi},single} \subset V^M_{\phi_1,single} \oplus V^M_{\phi_2,single} \oplus \ldots \oplus V^M_{\phi_s,single}.$$

We note that the W-type ψ on $V^{\widetilde{M}}_{\widetilde{\phi}}$ appears as a subrepresentation of $\bigoplus_{i=1}^s V^M_{\phi_i,single}$. The Weyl group representation is defined on $V^M_{\phi_i,single}$ for each i, so ψ appears in $V^M_{\phi_i,single}$ for some i. Therefore, every W-type is single-petaled for $SO(m,n)$. $\square$

The $S(O(m) \times O(n))$-type that realizes each Weyl group representation on its single-petaled $M(SO(m,n))$-fixed vectors is given by the followings proposition. The method of the proof is similar to 4.4.2.8, so we omit the proof.

Proposition 4.4.4.2. *Suppose that*
(1) $\lambda = (a_1, a_2, \ldots, a_q)^T \vdash q$
(2) $\tau = (b_1, b_2, \ldots, b_{n-q})^T \vdash n - q$.

The W-type $\psi^{(\lambda,\tau)}$ is realized on the single-petaled M-fixed vectors of an irreducible subrepresentation of

$$\mathrm{Ind}_{SO(m)\times SO(n)}^{S(O(m)\times O(n))}(\phi')$$

where the $SO(m)\times SO(n)$-type ϕ' has the highest weight Λ which is sorted

$$(b'_1, b'_2, \ldots, b'_{[\frac{n-q}{2}]}, 0, 0, \ldots, 0)(a'_1, a'_2, \ldots, a'_{[\frac{q}{2}]}, 0, 0, \ldots, 0).$$

where

$$(a_1 - a_q, a_2 - a_{q-1}, \ldots) = (a'_1, a'_2, \ldots, a'_{[\frac{q}{2}]})$$
$$\textit{and } (b_1 - b_{n-q}, b_2 - b_{n-q-1}, \ldots) = (b'_1, b'_2, \ldots, b'_{[\frac{n-q}{2}]}).$$

4.4.5 $Sp(n, \mathbb{R})$

We start with clarifying notations that we will use in this subsection.

- $G = Sp(n, \mathbb{R}) = \{g \in GL(2n, \mathbb{R}) \mid g^t J_{n,n} g = J_{n,n}\}$
  where $J_{n,n}$ is a $2n$-by-$2n$ matrix $\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$
- $\mathfrak{g} = sp(n, \mathbb{R}) = \{X \in gl(2n, \mathbb{R}) \mid X^t J_{n,n} + J_{n,n} X = 0\}$
- $K = Sp(n) \cap Sp(n, \mathbb{R}) \simeq U(2n) \cap Sp(n, \mathbb{C}) \cap Sp(n, \mathbb{R}) \simeq U(2n) \cap Sp(n, \mathbb{R})$

We define a Lie algebra isomorphism isomorphism $f : \mathfrak{k} \simeq sp(n, \mathbb{R}) \to u(n)$ by

$$f(E_{k,n+s} - E_{n+s,k} + E_{s,n+k} - E_{n+k,s}) = iE_{k,s} + iE_{s,k}$$
$$\text{and } f(E_{k,s} + E_{n+k,n+s} - E_{s,k} - E_{n+s,n+k}) = E_{k,s} - E_{s,k}.$$

Since $sp(n, \mathbb{R}) \cap u(2n)$ is spanned by

$$(E_{k,n+s} - E_{n+s,k} + E_{s,n+k} - E_{n+k,s}) \text{ and } (E_{k,s} + E_{n+k,n+s} - E_{s,k} - E_{n+s,n+k}),$$

and $u(n)$ is spanned by

$$(iE_{k,s} + iE_{s,k}) \text{ and } (E_{k,s} - E_{s,k}),$$

f is surjective, and it is a Lie algebra isomorphism. We can define a Lie group isomorphism between $Sp(n,\mathbb{R}) = exp(u(2n) \cap sp(n,\mathbb{R}))$ and $U(n) = exp(u(n))$ correspondingly. We refer the corresponding isomorphism as f as well.

- $\mathfrak{a} = \text{span}_{\mathbb{R}} < E_{k,k} - E_{n+k,n+k} >$
- $M = Z_K(\mathfrak{a}) = \{ \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix} \mid A = \pm E_{1,1} \pm E_{2,2} \ldots \pm E_{n,n} \}$

An element of K which centralizes $\mathfrak{a}$ is a diagonal matrix, and the diagonal matrix in K has the form of $\begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix}$ where $A = \pm E_{1,1} \pm E_{2,2} \ldots \pm E_{n,n}$. Note that $f(M) = \pm E_{1,1} \pm E_{2,2} \ldots \pm E_{n,n}$ in $U(n)$.

- $\alpha_j(\Sigma_{k=1}^n h_k(E_{k,k} - E_{n+k,n+k})) = h_j$ for $1 \leq j \leq n$
- Restricted roots: $\Delta = \{\pm\alpha_j \pm \alpha_k \mid 1 \leq k \neq j \leq n\} \cup \{\pm 2\alpha_k \mid 1 \leq k \leq n\}$
- $\Pi = \{\alpha_k - \alpha_{k+1} \mid 1 \leq k \leq n-1\} \cup \{2\alpha_n\}$
- $\mathfrak{g}_{\alpha_j - \alpha_k} = \text{span}_{\mathbb{R}} < E_{j,k} - E_{k+n,j+n} >$
- $\mathfrak{g}_{\alpha_j + \alpha_k} = \text{span}_{\mathbb{R}} < E_{j,k+n} + E_{k,j+n} >$
- $\mathfrak{g}_{-\alpha_j - \alpha_k} = \text{span}_{\mathbb{R}} < E_{j+n,k} + E_{k+n,j} >$
- $\mathfrak{g}_{2\alpha_k} = \text{span}_{\mathbb{R}} < E_{k,k+n} >$
- $\mathfrak{g}_{-2\alpha_k} = \text{span}_{\mathbb{R}} < E_{k+n,k} >$
- The Weyl group representative for $s_{2\alpha_k}$ is

  $$I - E_{k,k} - E_{n+k,n+k} + E_{k,n+k} - E_{n+k,k} \in Sp(n,\mathbb{R}),$$

  which maps to

  $$I - E_{k,k} + iE_{k,k} \in U(n) \text{ by } f$$

- The Weyl group representative for $s_{\alpha_j-\alpha_k}$ is

$$I-(E_{j,j}+E_{k,k}+E_{n+j,n+j}+E_{n+k,n+k})+(E_{j,k}-E_{k,j}+E_{n+j,n+k}-E_{n+k,n+j}),$$

which maps to

$$I-E_{j,j}-E_{k,k}+E_{j,k}-E_{k,j}\in U(n) \text{ by } f$$

- The Weyl group representative for $s_{\alpha_j+\alpha_k}$ is

$$I-(E_{j,j}+E_{k,k}+E_{n+j,n+j}+E_{n+k,n+k})+(-E_{j,n+k}-E_{k,n+j}+E_{n+j,k}+E_{n+k,j}),$$

which maps to

$$I-E_{j,j}-E_{k,k}-iE_{j,k}-iE_{k,j}\in U(n) \text{ by } f$$

- $Z_{\alpha_j-\alpha_k}$ maps to $E_{j,k}-E_{k,j}\in u(n)$ by f
- $Z_{\alpha_j+\alpha_k}$ maps to $iE_{j,k}+iE_{k,j}\in u(n)$ by f
- $Z_{2\alpha_k}$ maps to $iE_{k,k}\in u(n)$ by f

The real rank one subgroup $G^{\alpha_k-\alpha_{k+1}}$ has the maximal compact subgroup isomorphic to $SO(2)$, and $G^{2\alpha_k}$ has the maximal compact subgroup isomorphic to $U(1)$. The trivial representation and the $\mathfrak{p}$-representation have the following highest weights respectively:

$K^{\alpha_k-\alpha_{k+1}}\simeq SO(2)$: the trivial representation (0) and $\mathfrak{p}$-representation (2) or (-2)

$K^{\alpha_k}\simeq U(1)$: the trivial representation (0) and $\mathfrak{p}$-representation (2) or (-2)

Proposition 4.4.5.1. *Every W-type is single-petaled for $Sp(n,\mathbb{R})$.*

To prove this proposition, we need a few lemmas. The Weyl group of $Sp(n,\mathbb{R})$ is $S_n\ltimes(\mathbb{Z}/2\mathbb{Z})^n$, and the irreducible representations are represented by two Young diagrams (λ,τ) such that $\lambda\vdash q$ and $\tau\vdash(n-q)$ for some q as in 2.4.4. Suppose that

$$\lambda = (a_1, a_2, \ldots, a_q)^T \text{ and } \tau = (b_1, b_2, \ldots, b_{n-q})^T,$$

which means that $\Sigma_{k=1}^{q} a_q = q$ and $\Sigma_{k=1}^{n-q} b_k = n - q$.

Lemma 4.4.5.2. *For $\lambda = (a_1, a_2, \ldots, a_q)^T$, the $S_q \ltimes (\mathbb{Z}/2\mathbb{Z})^q$-type*

$$\psi^{\lambda,\emptyset} = \psi^{\lambda} \ltimes (\text{triv})^q$$

is realized by the $U(q)$-type with highest weight

$$(1 - a_q, 1 - a_{q-1}, \ldots, 1 - a_1)$$

on its $U(1)^q$-fixed vectors.

Proof: Since $(\pm 1)^q$ is a subgroup of $U(1)^q$, a vector is $(\pm 1)^q$-fixed if it is $U(1)^q$-fixed. Let $W_1(q)$ be the subgroup of the Weyl group such that $W_1(q) \simeq S_q$ is generated by the reflections with respect to the short roots. Then, the representation of $W_1(q)$ on the $U(1)^q$-fixed vectors is ψ^{λ} from 3.4.2. On the other hand, let $W_2(q)$ be the subgroup of the Weyl group such that $W_2(q) \simeq (\mathbb{Z}/2\mathbb{Z})^q$ is generated by the reflections with respect to the long roots. Then, the representation of W_2 is $(\text{triv})^q$ since the $U(1)^q$-fixed vector has weight $(0)^q$. Therefore, the Weyl group representation on the $U(1)^q$-fixed vector is $\phi^{\lambda,\emptyset}$. $\square$

Lemma 4.4.5.3. *For $\tau = (b_1, b_2, \ldots, b_{n-q})^T$, the $S_{n-q} \ltimes (\mathbb{Z}/2\mathbb{Z})^{n-q}$-type*

$$\psi^{\tau} \ltimes (\text{sign})^{n-q}$$

is realized on the single-petaled $(\pm 1)^{n-q}$-fixed vectors of $U(n-q)$ representation ϕ with highest weight

$$(b_1 + 1, b_2 + 1, \ldots, b_{n-q} + 1, 1, 1, \ldots, 1).$$

Proof: Suppose ϕ_0 is the $U(k)$-type with the highest weight

$$(k + 1, 1, 1, \ldots, 1).$$

Then, ϕ_0 has one-dimensional $(\pm 1)^k$-fixed vectors spanned by v, and v is the vector in the restricted representation of $U(1)^k$,

$$(2)^k.$$

Let $W_2(k)$ be the subgroup of the Weyl group such that $W_2(k) \simeq (\mathbb{Z}/2\mathbb{Z})^k$ is generated by the reflections with respect to the long roots. Then, the representation of $W_2(k)$ is $(\text{sign})^k$ since the $U(1)^k$-fixed vector has weight $(2)^k$. From this fact, we can show that the representation of $W_2(n-q)$ on the $(\pm 1)^{n-q}$-fixed vectors of ϕ is the sign representation.

On the other hand,

$$\begin{gathered}(2,2,\ldots,2)\text{-weight vectors of } U(n-q)\text{-type with highest weight}\\ (b_1+1, b_2+1, \ldots, b_{n-q}+1)\end{gathered}$$

have one-to-one correspondence to

$$\begin{gathered}(0,0,\ldots,0)\text{-weight vectors of } U(n-q)\text{-type with highest weight}\\ (b_1-1, b_2-1, \ldots, b_{n-q}-1),\end{gathered}$$

which realizes the S_{n-q}-type ψ^τ by 3.4.2.1 when we choose the Weyl group representatives in the manner of 3.5.1.

The restriction of ϕ to $U(2) \times U(1)^{n-q-2}$ has a nonzero $(\pm 1)^{n-q}$-fixed vector only if the highest weight is $(3,1)(2)^{n-q-2}$ or $(2,2)(2)^{n-q-2}$. From this fact, we can easily check that v is single-petaled. □

Lemma 4.4.5.4. *The W-type $\psi^{(\lambda,\tau)}$ is realized on part of the single-petaled M-fixed vectors of the $U(n)$-type ϕ with highest weight*

$$(b_1+1, b_2+1, \ldots, b_{n-q}+1, 1-a_q, 1-a_{q-1}, \ldots, 1-a_1)$$

where

$$\tau = (a_1, a_2, \ldots, a_q)^T \vdash q$$
$$\lambda = (b_1, b_2, \ldots, b_{n-q})^T \vdash n - q.$$

Proof: Fix a subgroup A isomorphic to $U(q) \times U(n-q)$. Then $\phi|_A$ has subrepresentation ϕ_A with highest weight

$$(1 - a_q, 1 - a_{q-1}, \ldots, 1 - a_1)(b_1 + 1, b_2 + 1, \ldots, b_{n-q} + 1).$$

The $(\pm 1)^n$-fixed vectors in ϕ_A are the weight vector corresponding to the weight

$$(0)^q (2)^{n-q},$$

and these vectors realized the representation of $(S_q \ltimes (\mathbb{Z}/2\mathbb{Z})^q) \times (S_{n-q} \ltimes (\mathbb{Z}/2\mathbb{Z})^{n-q})$,

$$(\psi^\lambda \ltimes (\text{triv})^q) \times (\psi^\tau \ltimes (\text{sign})^{n-q})$$

from 4.4.5.2 and 4.4.5.3. Therefore, $\psi^{\lambda,\tau}$ is realized on some $(\pm 1)^n$-fixed vectors of ϕ: we can prove this in the similar way to 4.4.7.4, and we skip the detailed proof here.

Now we show that the $(\pm 1)^n$-fixed vectors of ϕ which realize $\psi^{\lambda,\tau}$ are single-petaled. It is enough to show that the $(\pm 1)^n$-fixed vectors in ϕ_A are single-petaled. First, since the $(\pm 1)^n$-fixed vectors in ϕ_A are the weight vectors corresponding to the weight $(0)^q(2)^{n-q}$, they are annihilated by $Z_\alpha(Z_\alpha^2 - 4)$ for long roots α.
Second, we show that $(\pm 1)^n$-fixed vectors in ϕ_A are annihilated by $Z_\alpha(Z_\alpha^2 - 4)$ for short root α. We note that

$$\phi = \phi^{(b_1, b_2, \ldots, b_{n-q}, -a_q, -a_{q-1}, \ldots, -a_1)} \otimes \phi^{(1,1,\ldots,1)}.$$

Therefore, the $(\pm 1)^n$-fixed vectors in ϕ_A is annihilated by $Z_\alpha(Z_\alpha^2 - 4)$ for short root α if and only if the $(\pm 1)^n$-signed vectors in the $U(q) \times U(n-q)$-type ϕ'_A with highest weight

$$(-a_q, -a_{q-1}, \ldots, -a_1) \otimes (b_1, b_2, \ldots, b_{n-q})$$

are annihilated by $Z_\alpha(Z_\alpha^2-4)$. The $(\pm1)^n$-signed vectors in ϕ'_A are the weight vectors corresponding to the weight

$$(-1,-1,\ldots,-1)\otimes(1,1,\ldots,1),$$

and we can easily check that these vectors are annihilated by $Z_\alpha(Z_\alpha^2-4)$ for long roots α. $\square$

Proposition 4.4.5.1 is immediate from this lemma.

4.4.6 $Sp(n,\mathbb{C})$

We start with clarifying the notations that we will use in this subsection.

- $G = Sp(n,\mathbb{C}) = \{g \in GL(2n,\mathbb{C}) \mid g^t J_{n,n} g = J_{n,n}\}$ where $J_{n,n}$ is a $2n$-by-$2n$ matrix $\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$
- $\mathfrak{g} = sp(n,\mathbb{C}) = \{X \in gl(2n,\mathbb{C}) \mid X^t J_{n,n} + J_{n,n}X = 0\}$
- $K = U(2n) \cap Sp(n,\mathbb{C})$

We define a Lie algebra isomorphism isomorphism

$$f : \mathfrak{k} = u(2n) \cap sp(n,\mathbb{C}) \to sp(n)$$

by

$$\begin{aligned} f(E_{k,n+s}-E_{n+s,k}+E_{s,n+k}-E_{n+k,s}) &= jE_{k,s}+jE_{s,k}, \\ f(iE_{k,n+s}+iE_{n+s,k}+iE_{s,n+k}+iE_{n+k,s}) &= kE_{k,s}+kE_{s,k}, \\ f(E_{k,s}+E_{n+k,n+s}-E_{s,k}-E_{n+s,n+k}) &= E_{k,s}-E_{s,k}, \text{ and} \\ f(iE_{k,s}-iE_{n+k,n+s}+iE_{s,k}-iE_{n+s,n+k}) &= iE_{k,s}+iE_{s,k}, \end{aligned}$$

Note that $sp(n,\mathbb{C}) \cap u(2n)$ is spanned by

$$E_{k,n+s}-E_{n+s,k}+E_{s,n+k}-E_{n+k,s},\ iE_{k,n+s}+iE_{n+s,k}+iE_{s,n+k}+iE_{n+k,s},$$
$$E_{k,s}+E_{n+k,n+s}-E_{s,k}-E_{n+s,n+k}, \text{ and } iE_{k,s}-iE_{n+k,n+s}+iE_{s,k}-iE_{n+s,n+k},$$

and $sp(n)$ is spanned by

$$iE_{k,s} + iE_{s,k},\ jE_{k,s} + jE_{s,k},\ kE_{k,s} + kE_{s,k}, \text{ and } E_{k,s} - E_{s,k}.$$

Therefore, f is surjective, and it is a Lie algebra isomorphism. We can define a Lie group isomorphism between $U(2n) \cap Sp(n, \mathbb{C}) = exp(u(2n) \cap sp(n, \mathbb{C}))$ and $Sp(n) = exp(sp(n))$ correspondingly. We refer the corresponding isomorphism as f as well.

- $\widetilde{M} = Sp(1)^n \subset K$

- $\mathfrak{a} = \text{span}_{\mathbb{R}} < E_{p,p} - E_{n+p,n+p} >$

- $M = Z_K(\mathfrak{a})$
  $= \{ \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \mid A = e^{i\theta_1}E_{1,1} + e^{i\theta_2}E_{2,2} + \ldots + e^{i\theta_n}E_{n,n}, B = A^{-1}\} \subset \widetilde{M}$

An element of M, which is an element of K which centralizes $\mathfrak{a}$, is a diagonal matrix, and the diagonal matrix in K has the form of $\begin{pmatrix} A & 0 \\ 0 & \bar{A} \end{pmatrix}$ where $A = e^{i\theta_1}E_{1,1} + e^{i\theta_2}E_{2,2} + \ldots + e^{i\theta_n}E_{n,n}$. Note that $f(M) = e^{i\theta_1}E_{1,1} + e^{i\theta_2}E_{2,2} + \ldots + e^{i\theta_n}E_{n,n}$ in $Sp(n)$.

- $\alpha_q(\Sigma_{p=1}^n h_p(E_{p,p} - E_{n+p,n+p})) = h_q$ for $1 \le q \le n$

- $\Delta = \{\pm\alpha_p \pm \alpha_q \mid 1 \le p \ne q \le n\} \cup \{\pm 2\alpha_p \mid 1 \le p \le n\}$

- $\Pi = \{\alpha_p - \alpha_{p+1} \mid 1 \le p \le n-1\} \cup \{2\alpha_n\}$

- $\mathfrak{g}_{\alpha_p - \alpha_q} = \text{span}_{\mathbb{R}} < E_{p,q} - E_{q+n,p+n}, i(E_{p,q} - E_{q+n,p+n}) >$

- $\mathfrak{g}_{\alpha_p + \alpha_q} = \text{span}_{\mathbb{R}} < E_{p,q+n} + E_{q,p+n}, i(E_{p,q+n} + E_{q,p+n}) >$

- $\mathfrak{g}_{-\alpha_p - \alpha_q} = \text{span}_{\mathbb{R}} < E_{p+n,q} + E_{q+n,p}, i(E_{p+n,q} + E_{q+n,p}) >$

- $\mathfrak{g}_{2\alpha_p} = \text{span}_{\mathbb{R}} < E_{p,p+n}, iE_{p,p+n} >$

- $\mathfrak{g}_{-2\alpha_p} = \text{span}_{\mathbb{R}} < E_{p+n,p}, iE_{p+n,p} >$

- $W \simeq S_n \ltimes (\mathbb{Z}/2\mathbb{Z})^n$

- The Weyl group representative for $s_{2\alpha_p}$ is

$$I - E_{p,p} - E_{n+p,n+p} + E_{p,n+p} - E_{n+p,p} \in Sp(n, \mathbb{C}).$$

This maps to

$$I - E_{p,p} + jE_{p,p} \in Sp(n) \text{ by } f$$

- The Weyl group representative for $s_{\alpha_p - \alpha_q}$ is

$$I - E_{p,p} - E_{q,q} - E_{n+p,n+p} - E_{n+q,n+q} + E_{p,q} - E_{q,p} + E_{n+p,n+q} - E_{n+q,n+p} \in Sp(n, \mathbb{C}).$$

This is

$$I - E_{p,p} - E_{q,q} + E_{p,q} - E_{q,p} \in Sp(n)$$

- $Z_{\alpha_p - \alpha_q}$ maps to $E_{p,q} - E_{q,p} \in sp(n)$ by f
- $Z_{\alpha_p + \alpha_q}$ maps to $jE_{p,q} + jE_{q,p} \in sp(n)$ by f
- $Z_{2\alpha_p}$ maps to $jE_{p,p} \in sp(n)$ by f

Lemma 4.4.6.1. *The Weyl group representation parameterized by* $(\emptyset, (2))$ *is not single-petaled.*

Proof: Suppose there exists an $Sp(2)$-type that realizes $\psi^{(\emptyset,(2))}$ on its single-petaled M-fixed vectors. Here, M is isomorphic to

$$\{A_1 \times A_2 \times B_1 \times B_2 \mid A_p = e^{i\theta_p}, B_i = A_i^{-1}\}$$

in $K \simeq Sp(2, \mathbb{C}) \cap U(4)$. Since $\psi^{(\emptyset,(2))}$ is a one-dimensional Weyl group representation, it is enough to show that every M-fixed vector that realizes $\psi^{(\emptyset,(2))}$ is not single-petaled. An irreducible representation of $Sp(2) \simeq U(4) \cap Sp(2, \mathbb{C})$ is realized by a highest weight representation. A highest weight vector can be constructed using wedge products of a direct sum of standard representations. Let

$$V^x = \text{span}_{\mathbb{C}} < e_1^x, e_2^x, e_3^x, e_4^x >$$

be the standard representation of $K \simeq U(4) \cap Sp(2, \mathbb{C})$. Then,

$$\bigoplus_{x=1}^{k} V^x,$$

a direct sum of standard representations, is also a representation of K. Also, we can define a representation of K on

$$\bigwedge_{j=1}^{l} (\bigoplus_{x=1}^{k} V^x).$$

For example, a highest weight vector of $\phi_{(1,1)}$ can be expressed as

$$e_1^1 \wedge e_2^1 \in V^1 \wedge V^1.$$

For another example, a highest weight vector of $\phi_{(2,0)}$ can be expressed as

$$e_1^1 \wedge e_1^2 \in (V^1 \oplus V^2) \wedge (V^1 \oplus V^2).$$

The highest weight vector of $Sp(2)$-type with highest weight (a, b) can be expressed as

$$(\bigwedge_{x=1}^{a} e_1^x) \wedge (\bigwedge_{x=1}^{b} e_2^x) \in \bigwedge_{z=1}^{a+b} (\bigoplus_{x=1}^{a} V^x).$$

A highest weight representation is generated by a highest weight vector, and we can conclude that an M-fixed vector in this representation is expressed as a sum of wedge products of this form.

The M-fixed vectors are generated by wedge products of the following forms:

$$e_1^x \wedge e_3^x,\ e_2^x \wedge e_4^x,\ e_1^x \wedge e_3^y \text{ or } e_2^x \wedge e_4^y.$$

If an M-fixed vector v realizes $\psi^{(\emptyset,(2))}$, then

$$\omega_{2\alpha_1} v = -v \text{ and } \omega_{2\alpha_2} v = -v.$$

Therefore, v is generated by wedge products of the following forms:

$$p_1 = (\bigwedge_{x=1}^{k}(e_1^{a_x} \wedge e_3^{b_x})) - (-1)^k(\bigwedge_{x=1}^{k}(e_3^{a_x} \wedge e_1^{b_x})) \in \bigwedge_{z=1}^{2k}(\bigoplus_{y=1}^{l} V^y)$$
$$q_1 = (\bigwedge_{x=1}^{k}(e_2^{a_x} \wedge e_4^{b_x})) - (-1)^k(\bigwedge_{x=1}^{k}(e_4^{a_x} \wedge e_2^{b_x})) \in \bigwedge_{z=1}^{2k}(\bigoplus_{y=1}^{l} V^y)$$
$$\text{where } 1 \leq a_x, b_x \leq l$$

such that $a_x \neq a_y$, $b_x \neq b_y$ if $x \neq y$, and

$$A := \{a_x \mid 1 \leq x \leq k\} \neq B := \{b_y \mid 1 \leq x \leq k\}.$$

Note that p_1 is the scalar multiple of

$$p_2 = \bigwedge_{x=1}^{k} e_1^{a_x} \wedge \bigwedge_{x=1}^{k} e_3^{b_x} - (-1)^k \bigwedge_{x=1}^{k} e_3^{a_x} \wedge \bigwedge_{x=1}^{k} e_1^{b_x}$$

and q_1 is the scalar multiple of

$$q_2 = \bigwedge_{x=1}^{k} e_2^{a_x} \wedge \bigwedge_{x=1}^{k} e_4^{b_x} - (-1)^k \bigwedge_{x=1}^{k} e_4^{a_x} \wedge \bigwedge_{x=1}^{k} e_2^{b_x}.$$

Note that p_1 and q_1 depends only on $\{A, B\}$ (a set consisting two sets). Since $A \neq B$, we may choose $A', C \subset A$ and $B', C \subset B$ such that

$$A' \cap C = \emptyset,\ B' \cap C = \emptyset, \text{ and } A' \cap B' = \emptyset.$$

Put

$$A' = \{a'_1, a'_2, \ldots, a'_{k-s}\},\ B' = \{b'_1, b'_2, \ldots, b'_{k-s}\}, \text{ and } C = \{c_1, c_2, \ldots, c_s\}.$$

Note that p_2 is the scalar multiple of

$$p_3 = \bigwedge_{x=1}^{s} e_1^{c_x} \wedge \bigwedge_{x=1}^{s} e_3^{c_x} \wedge \bigwedge_{x=1}^{k-s} e_1^{a'_x} \wedge \bigwedge_{x=1}^{k-s} e_3^{b'_x} - (-1)^k \bigwedge_{x=1}^{s} e_3^{c_x} \wedge \bigwedge_{x=1}^{s} e_1^{c_x} \wedge \bigwedge_{x=1}^{k-s} e_3^{a'_x} \wedge \bigwedge_{x=1}^{k-s} e_1^{b'_x}.$$

Likewise, q_2 is the scalar multiple of

$$q_3 = \bigwedge_{x=1}^{s} e_2^{c_x} \wedge \bigwedge_{x=1}^{s} e_4^{c_x} \wedge \bigwedge_{x=1}^{k-s} e_2^{a'_x} \wedge \bigwedge_{x=1}^{k-s} e_4^{b'_x} - (-1)^k \bigwedge_{x=1}^{s} e_4^{c_x} \wedge \bigwedge_{x=1}^{s} e_2^{c_x} \wedge \bigwedge_{x=1}^{k-s} e_4^{a'_x} \wedge \bigwedge_{x=1}^{k-s} e_2^{b'_x}.$$

Note that p_1 and q_1 depends only on $\{A', B'\}$ and C. If an M-fixed vector v realizes $\psi^{(\emptyset,(2))}$, then

$$\omega_{\alpha_1-\alpha_2} v = v.$$

Therefore, v is spanned by wedge products of the form of $p_3 \wedge q_3$, which is the scalar multiple of the following:

$$\begin{aligned} w = &\textstyle\bigwedge_{x=1}^{s} e_1^{c_x} \wedge \bigwedge_{x=1}^{s} e_3^{c_x} \wedge \bigwedge_{x=1}^{s} e_2^{c_x} \wedge \bigwedge_{x=1}^{s} e_4^{c_x} \\ &\wedge(\textstyle\bigwedge_{x=1}^{k-s} e_1^{a'_x} \wedge \bigwedge_{x=1}^{k-s} e_3^{b'_x} - (-1)^{k-s} \bigwedge_{x=1}^{k-s} e_3^{a'_x} \wedge \bigwedge_{x=1}^{k-s} e_1^{b'_x}) \\ &\wedge(\textstyle\bigwedge_{x=1}^{k-s} e_2^{a'_x} \wedge \bigwedge_{x=1}^{k-s} e_4^{b'_x} - (-1)^{k-s} \bigwedge_{x=1}^{k-s} e_4^{a'_x} \wedge \bigwedge_{x=1}^{k-s} e_2^{b'_x}). \end{aligned}$$

where

$$\begin{gathered} 1 \le a'_x, b'_x, c_x \le l, \\ a'_x \ne a'_y,\ b'_x \ne b'_y \text{ if } x \ne y \\ \text{and } a'_x \ne b'_y,\ a'_x \ne c_y,\ c_x \ne b'_y. \end{gathered}$$

We notice that v is not single-petaled because v is not annihilated by $Z_\alpha(Z_\alpha^2 - 4)$ for the short root $\alpha = \alpha_1 - \alpha_2$. To show this, we find the eigenvectors in V^x

$$v_1^x,\ w_1^x,\ v_{-1}^x, \text{ and } w_{-1}^x$$

under action of Z_α corresponding to the eigenvalues 1, 1, -1, and -1 such that

$$e_1^x = v_1^x + v_{-1}^x,\ e_2^x = v_1^x - v_{-1}^x,\ e_3^x = w_1^x + w_{-1}^x, \text{ and } e_4^x = w_1^x - w_{-1}^x.$$

Rewriting w, v is generated by wedge products of the following form:

$$\begin{aligned} &\textstyle\bigwedge_{x=1}^{s}(v_1^{c_x} + v_{-1}^{c_x}) \wedge \bigwedge_{x=1}^{s}(w_1^{c_x} + w_{-1}^{c_x}) \wedge \bigwedge_{x=1}^{s}(v_1^{c_x} - v_{-1}^{c_x}) \wedge \bigwedge_{x=1}^{s}(w_1^{c_x} - w_{-1}^{c_x}) \\ &\wedge(\textstyle\bigwedge_{x=1}^{k-s}(v_1^{a'_x} + v_{-1}^{a'_x}) \wedge \bigwedge_{x=1}^{k-s}(w_1^{b'_x} + w_{-1}^{b'_x}) - (-1)^{k-s} \bigwedge_{x=1}^{k-s}(w_1^{a'_x} + w_{-1}^{a'_x}) \wedge \bigwedge_{x=1}^{k-s}(v_1^{b'_x} + v_{-1}^{b'_x})) \\ &\wedge(\textstyle\bigwedge_{x=1}^{k-s}(v_1^{a'_x} - v_{-1}^{a'_x}) \wedge \bigwedge_{x=1}^{k-s}(w_1^{b'_x} - w_{-1}^{b'_x}) - (-1)^{k-s} \bigwedge_{x=1}^{k-s}(w_1^{a'_x} - w_{-1}^{a'_x}) \wedge \bigwedge_{x=1}^{k-s}(v_1^{b'_x} - v_{-1}^{b'_x})) \end{aligned}$$

which is the scalar multiple of

$$\begin{aligned} v' = &\textstyle\bigwedge_{x=1}^{s} v_1^{c_x} \wedge \bigwedge_{x=1}^{s} w_1^{c_x} \wedge \bigwedge_{x=1}^{s} v_{-1}^{c_x} \wedge \bigwedge_{x=1}^{s} w_{-1}^{c_x} \\ &\wedge(\textstyle\bigwedge_{x=1}^{k-s}(v_1^{a'_x} + v_{-1}^{a'_x}) \wedge \bigwedge_{x=1}^{k-s}(w_1^{b'_x} + w_{-1}^{b'_x}) - (-1)^{k-s} \bigwedge_{x=1}^{k-s}(w_1^{a'_x} + w_{-1}^{a'_x}) \wedge \bigwedge_{x=1}^{k-s}(v_1^{b'_x} + v_{-1}^{b'_x})) \\ &\wedge(\textstyle\bigwedge_{x=1}^{k-s}(v_1^{a'_x} - v_{-1}^{a'_x}) \wedge \bigwedge_{x=1}^{k-s}(w_1^{b'_x} - w_{-1}^{b'_x}) - (-1)^{k-s} \bigwedge_{x=1}^{k-s}(w_1^{a'_x} - w_{-1}^{a'_x}) \wedge \bigwedge_{x=1}^{k-s}(v_1^{b'_x} - v_{-1}^{b'_x})). \end{aligned}$$

Note that v' only depends on $\{A', B'\}$ (a set consisting two disjoint sets) and C up to scalar. Let

$$v'' = (\bigwedge_{x=1}^{k-s}(v_1^{a'_x}+v_{-1}^{a'_x})\wedge\bigwedge_{x=1}^{k-s}(w_1^{b'_x}+w_{-1}^{b'_x})-(-1)^{k-s}\bigwedge_{x=1}^{k-s}(w_1^{a'_x}+w_{-1}^{a'_x})\wedge\bigwedge_{x=1}^{k-s}(v_1^{b'_x}+v_{-1}^{b'_x}))$$
$$\wedge(\bigwedge_{x=1}^{k-s}(v_1^{a'_x}-v_{-1}^{a'_x})\wedge\bigwedge_{x=1}^{k-s}(w_1^{b'_x}-w_{-1}^{b'_x})-(-1)^{k-s}\bigwedge_{x=1}^{k-s}(w_1^{a'_x}-w_{-1}^{a'_x})\wedge\bigwedge_{x=1}^{k-s}(v_1^{b'_x}-v_{-1}^{b'_x})).$$

It is enough to show that v'' is not annihilated by $Z_\alpha(Z_\alpha^2-4)$. Note that v'' only depends on $\{A', B'\}$ (a set consisting two disjoint sets) up to scalar.

(1) Suppose $k-s=1$. Then, v'' is not annihilated by $Z_\alpha(Z_\alpha^2-4)$ because a term in v'',

$$(v_1^{a'_1}\wedge w_1^{b'_1}+w_1^{a'_1}\wedge v_1^{b'_1})\wedge(v_1^{a'_1}\wedge w_1^{b'_1}+w_1^{a'_1}\wedge v_1^{b'_1}) = 2v_1^{a'_1}\wedge w_1^{b'_1}\wedge w_1^{a'_1}\wedge v_1^{b'_1},$$

is nonzero. Since $k-s=1$, this term cannot be canceled in the nonzero linear combination of the same form.

(2) Suppose $k-s\geq 2$. We can show that v'' is not annihilated by $Z_\alpha(Z_\alpha^2-4)$. In

$$v'' = (\bigwedge_{x=1}^{k-s}(v_1^{a'_x}+v_{-1}^{a'_x})\wedge\bigwedge_{x=1}^{k-s}(w_1^{b'_x}+w_{-1}^{b'_x})-(-1)^{k-s}\bigwedge_{x=1}^{k-s}(w_1^{a'_x}+w_{-1}^{a'_x})\wedge\bigwedge_{x=1}^{k-s}(v_1^{b'_x}+v_{-1}^{b'_x}))$$
$$\wedge(\bigwedge_{x=1}^{k-s}(v_1^{a'_x}-v_{-1}^{a'_x})\wedge\bigwedge_{x=1}^{k-s}(w_1^{b'_x}-w_{-1}^{b'_x})-(-1)^{k-s}\bigwedge_{x=1}^{k-s}(w_1^{a'_x}-w_{-1}^{a'_x})\wedge\bigwedge_{x=1}^{k-s}(v_1^{b'_x}-v_{-1}^{b'_x})),$$

we have the following term

$$(\bigwedge_{x=1}^{k-s}(v_1^{a'_x}+v_{-1}^{a'_x})\wedge\bigwedge_{x=1}^{k-s}(w_1^{b'_x}+w_{-1}^{b'_x}))\wedge(\bigwedge_{x=1}^{k-s}(w_1^{a'_x}-w_{-1}^{a'_x})\wedge\bigwedge_{x=1}^{k-s}(v_1^{b'_x}-v_{-1}^{b'_x}))$$
$$+(\bigwedge_{x=1}^{k-s}(v_1^{a'_x}-v_{-1}^{a'_x})\wedge\bigwedge_{x=1}^{k-s}(w_1^{b'_x}-w_{-1}^{b'_x}))\wedge(\bigwedge_{x=1}^{k-s}(w_1^{a'_x}+w_{-1}^{a'_x})\wedge\bigwedge_{x=1}^{k-s}(v_1^{b'_x}+v_{-1}^{b'_x})),$$

and therefore we have the nonzero term which is the scalar multiple of

$$((\Sigma_{x=1}^{k-s}(\bigwedge_{y=1}^{k-s} v_{1-2\delta_{x,y}}^{a'_x}))\wedge(\bigwedge_{x=1}^{k-s} w_1^{b'_x}))\wedge((\Sigma_{x=1}^{k-s}(\bigwedge_{y=1}^{k-s} w_{1-2\delta_{x,y}}^{a'_x}))\wedge(\bigwedge_{x=1}^{k-s} v_1^{b'_x})).$$

This vector is not annihilated by $Z_\alpha(Z_\alpha^2-4)$ because $2(k-s-2)+2(k-s)\geq 4$. This term cannot be canceled in the nonzero linear combination of the same form. From (1) and (2), we conclude that $\psi^{(\emptyset,(2))}$ is not single-petaled. □

We state the following lemmas without proof. We can prove them in the similar way to 4.4.6.1.

Lemma 4.4.6.2. *The W-type parametrized by $((1,1,1,1),(1))$ is not single-petaled.*

Lemma 4.4.6.3. *The W-type parametrized by $((1,1,1),(1,1))$ is not single-petaled.*

Lemma 4.4.6.4. *The W-type parametrized by $((2,2),\emptyset)$ is not single-petaled.*

Therefore, from 4.3.1.2, the candidates for single-petaled W-types are $\psi^{(\lambda,\tau)}$ satisfying the following:

$$\begin{gathered}[\tau=\emptyset,\ \lambda=(k,1,1,\dots,1)] \\ \text{or } [\tau=(1),\ \lambda=(k_1,k_2,k_3) \text{ such that } 0\le k_2,k_3\le 1] \\ \text{or } [\tau=(1,1,\dots,1),\ \lambda=(k_1,k_2) \text{ such that } 0\le k_2\le 1].\end{gathered}$$

Lemma 4.4.6.5. *The W-type $\psi^{(\lambda,\tau)}$ is single-petaled if*

$$\tau=(1,1,\dots,1)\vdash n \text{ and } \lambda=\emptyset.$$

Sketch of Proof: Suppose that

$$V^x=\mathrm{span}_{\mathbb{C}} < e_1^x, e_2^x,\dots,e_{2n}^x >$$

is the standard representation of $K\simeq U(2n)\cap Sp(n,\mathbb{C})$. Then,

$$v=\bigwedge_{j=1}^{n}(e_j^0\wedge e_{n+j}^j+e_{n+j}^0\wedge e_j^j)\in\bigwedge_{y=1}^{2n}\bigoplus_{x=0}^{n}V^x$$

is M-fixed, and

$$\mathrm{span}_{A\in\mathrm{Tab}(\tau)} < c(A)\circ R(A)\circ v >$$

realizes $S_n\ltimes(\mathbb{Z}/2\mathbb{Z})^n$-type $\psi^{(\emptyset,(1,1,\dots,1))}$ when the Young tableau

$$\begin{array}{|c|}\hline 1\\ \hline 2\\ \hline \cdots\\ \hline n\\ \hline\end{array}$$

is A. Note that v is annihilated by $Z_\alpha(Z_\alpha^2-4)$ for every root α. This is true because

$$(e_1^0\wedge e_{n+1}^1+e_{n+1}^0\wedge e_1^1)\wedge(e_2^0\wedge e_{n+2}^2+e_{n+2}^0\wedge e_2^2)-(e_2^0\wedge e_{n+2}^1+e_{n+2}^0\wedge e_2^1)\wedge(e_1^0\wedge {e_{n+1}}^2+e_{n+1}^0\wedge e_1^2)$$

in $\bigwedge^4(V^0\oplus V^1\oplus V^2)$ is annihilated by $Z_\alpha(Z_\alpha^2-4)$ for every root α. We can confirm this by case-by-case calculation. Therefore, we conclude that there exists a quasi-single-petaled K-type which realizes $\psi^{(\emptyset,(1,1,\ldots,1))}$ on its single-petaled M-fixed vectors. □

Lemma 4.4.6.6. *The Weyl group representation $\psi^{(\lambda,\tau)}$ is single-petaled if*

$$\tau=(1,1,\ldots,1)\vdash k \text{ and } \lambda=(n-k-1,1)\vdash n-k$$
$$\text{or } \tau=(1,1,\ldots,1)\vdash k \text{ and } \lambda=(n-k)\vdash n-k.$$

Sketch of Proof: We construct single-petaled M-fixed vectors which realize $\psi^{(\lambda,\tau)}$. First, we define $\mathrm{Tab}(\lambda,\tau)$: we enter the numbers from 1 to n in each blocks of λ or τ exactly one time for each number, and those elements consists $\mathrm{Tab}(\lambda,\tau)$. By $A_{i,j}$, we mean the number in the ith row and jth column of A. Likewise, by $B_{i,j}$, we mean the number in the ith row and jth column of B. We fix an element

$$(A_0,B_0)\in \mathrm{Tab}(\lambda,\tau).$$

Suppose that

$$V^x=\mathrm{span}_{\mathbb{C}}<e_1^x,e_2^x,\ldots,e_{2n}^x>$$

is the standard representation of $K\simeq U(2n)\cap Sp(n,\mathbb{C})$. Then,

$$\mathrm{span}_{(A_0,B)\in\mathrm{Tab}(\lambda,\tau)}<R(B)\circ C(B)\circ v_B>$$

realizes $\psi^{\emptyset,\tau}$ from 4.4.6.5 for

$$v_B:=\bigwedge_{i=1}^{k}(e_{B_i}^0\wedge e_{n+B_i}^i+e_{n+B_i}^0\wedge e_i^i)\in\bigwedge_{i=1}^{2k}(\bigoplus_{j=0}^{k}V^j).$$

On the other hand,

$$\mathrm{span}_{(A,B_0)\in \mathrm{Tab}(\lambda,\tau)} < R(A)\circ C(A)\circ v_A >$$

realizes $\psi^{\lambda,\emptyset}$ from 4.4.6.6 for

$$v_A := e^0_{A_{1,1}} \wedge e^0_{A_{1,1}+n} \in V^0 \wedge V^0 \text{ if } \lambda = (n-k-1,1)$$
$$\text{and } v_A := 1 \text{ if } \lambda = (n-k).$$

We notice that

$$U := \mathrm{span}_{(A,B)\in \mathrm{Tab}(\lambda,\tau)} < (R(B)\circ C(B)\circ v_B) \wedge (R(A)\circ C(A)\circ v_A) >$$

realizes the Weyl group representation $\psi^{(\lambda,\tau)}$. We can prove this in the similar way to 4.4.7.4, so we omit the detailed proof. Moreover, since $v_A \wedge v_B$ is annihilated by $Z_\alpha(Z_\alpha^2 - 4)$ for every root α, every vector in U is single-petaled. We conclude that $\psi^{(\lambda,\tau)}$ is a single-petaled W-type. □

Lemma 4.4.6.7. *The Weyl group representation* $\psi^{(\lambda,\tau)}$ *is single-petaled if*

$$\tau = (1) \vdash 1,\ \lambda = (k_1,k_2,k_3) \vdash n-1$$

such that $0 \le k_2, k_3 \le 1$.

Sketch of Proof: We construct single-petaled M-fixed vectors which realize $\psi^{(\lambda,(1))}$. Let

$$V^x = \mathrm{span}_{\mathbb{C}} < e^x_1, e^x_2, \ldots, e^x_{2n} >$$

be the standard representation of $Sp(n) \simeq U(2n) \cap Sp(n,\mathbb{C})$. We fix an element

$$(A_0, B_0) \in \mathrm{Tab}(\lambda,\tau).$$

Using the numbers that comprise A_0 as coordinates, we fix a subgroup of G isomorphic to $Sp(n-1,\mathbb{C})$, and we call the corresponding Weyl group $W_{A_0} \simeq S_{n-1} \ltimes (\mathbb{Z}/2\mathbb{Z})^{n-1}$. Likewise, we define W_{B_0}.For $(A,B_0) \in \mathrm{Tab}(\lambda,\tau)$, we define v_A as follows:

(1) if $k_2 = k_3 = 0$, $v_A = 1$

(2) if $k_2 = 1$, $k_3 = 0$, $v_A = e^0_{A_{1,1}} \wedge e^0_{A_{1,1+n}} \in V^0 \wedge V^0$

(3) if $k_2 = k_3 = 1$,

$v_A = e^0_{A_{1,1}} \wedge e^0_{A_{1,1+n}} \wedge e^0_{A_{2,1}} \wedge e^1_{A_{2,1+n}} - e^0_{A_{1,1}} \wedge e^0_{A_{1,1+n}} \wedge e^0_{A_{2,1+n}} \wedge e^1_{A_{2,1}} \in \bigwedge^4(V^0 \oplus V^1)$.

Then

$$\mathrm{span}_{(A,B_0)\in \mathrm{Tab}(\lambda,\tau)} < C(A) \circ R(A) \circ v_A >$$

realizes the W_{A_0}-type $\psi^{\lambda,\emptyset}$.

Likewise, let

$$v_B = e^0_{B_{1,1}} \wedge e^1_{B_{1,1+n}} + e^0_{B_{1,1+n}} \wedge e^1_{B_{1,1}} \in (V^0 \oplus V^1) \wedge (V^0 \oplus V^1).$$

Then

$$\mathrm{span}_{(A_0,B)\in \mathrm{Tab}(\lambda,\tau)\in \mathrm{Tab}(\lambda,\tau)} < v_B >$$

realizes W_B-type $\psi^{\emptyset,(1)}$.

Then,

$$U := \mathrm{span}_{(A,B)\in \mathrm{Tab}(\lambda,\tau)} < (R(A) \circ C(A) \circ v_A) \wedge (R(B) \circ C(B) \circ v_B) >$$

realizes the Weyl group representation $\psi^{(\lambda,\tau)}$. We can prove this in a similar way to 4.4.7.4, so we omit the detailed proof. Moreover, we can easily check that $v_A \wedge v_B$ is annihilated by $Z_\alpha(Z_\alpha^2 - 4)$ for every root α, and therefore every vector in U is single-petaled. We conclude that $\psi^{(\lambda,\tau)}$ is a single-petaled W-type. □

Using lemmas in this subsection, we obtain the following proposition.

Proposition 4.4.6.8. *The W-type $\psi^{(\lambda,\tau)}$ is single-petaled if and only if (λ, τ) has one of the following forms:*

$$[\tau = \emptyset,\ \lambda = (k, 1, 1, \dots, 1)]$$
$$or\ [\tau = (1),\ \lambda = (k_1, k_2, k_3)\ such\ that\ 0 \le k_2, k_3 \le 1]$$
$$or\ [\tau = (1, 1, \dots, 1),\ \lambda = (k_1, k_2)\ such\ that\ 0 \le k_2 \le 1].$$

4.4.7 $SL(n, \mathbb{H})$

We start with clarifying notations that we will use in this subsection.

- $G = SL(n, \mathbb{H})$
- $K = Sp(n) \simeq Sp(n, \mathbb{C}) \cap U(2n)$

The $Sp(n)$-types are denoted by their highest weight

$$(a_1, a_2, \ldots, a_n) \text{ such that } a_1 \geq a_2 \geq \ldots \geq a_n \geq 0.$$

- $\mathfrak{a} = \{\Sigma_{i=1}^{n} a_i E_{i,i} \mid a_i \in \mathbb{R}, \Sigma_{i=1}^{n} a_i = 0\}$
- $M \simeq Sp(1)^n$, $\mathfrak{m} = sp(1)^n$
- simple restricted-roots: $\Pi = \{\alpha_p - \alpha_{p+1} | p = 1, 2, \ldots, n-1\}$
  where $\alpha_p(d_1 E_{1,1} + d_2 E_{2,2} + \ldots + d_n E_{n,n}) = d_p$
- $\Delta = \Delta_1 = \{\alpha_p - \alpha_q | n \geq p \neq q \geq 1\}$
- Weyl group $W = S_n$
- $\mathfrak{g}_{\alpha_p - \alpha_q} = \text{span}_{\mathbb{R}} < E_{p,q}, iE_{p,q}, jE_{p,q}, kE_{p,q} >$
- $Z_{\alpha_p - \alpha_q} = E_{p,q} - E_{q,p}$
- The Weyl group representative for $s_{\alpha_p - \alpha_q} \in W$

$$k_{s_{\alpha_p - \alpha_q}} = I - E_{p,p} - E_{q,q} + E_{p,q} - E_{q,p}$$

- $\mathfrak{g}^{\alpha_p - \alpha_q} \simeq sl(2, \mathbb{H})$ with all the nonzero entries in $(p,p), (p,q), (q,p), (q,q)$
- $G^{\alpha_p - \alpha_q} \simeq SL(2, \mathbb{H})$
- $\mathfrak{k}^{\alpha_p - \alpha_q} \simeq sp(2)$ with all the nonzero entries in $(p,p), (p,q), (q,p), (q,q)$
- $K^{\alpha_p - \alpha_q} \simeq Sp(2)$
- $\mathfrak{m} \cap \mathfrak{k}^{\alpha_p - \alpha_q} \simeq \{x \times y | x, y \in sp(1)\}$ with all the nonzero entries in $(p,p), (q,q)$

- $MK^{\alpha_p-\alpha_q} \simeq Sp(2) \times Sp(1)^{n-2}$

We can calculate the branching from $Sp(n)$ to $Sp(n-1)\times Sp(1)$ using the following theorem of Lepowsky.

Theorem 4.4.7.1. *(Lepowsky [17]) The $Sp(n)$-type with highest weight*

$$\lambda = (a_1, a_2, \ldots, a_n)$$

restricts to the sum of $Sp(n-1) \times Sp(1)$-types with highest weights

$$\mu = (c_1, c_2, \ldots, c_{n-1})(c_0)$$

with the multiplicities $m_\lambda(\mu)$ in the following. The multiplicity is 0 unless the integers

$$\begin{gathered} A_1 = a_1 - max(a_2, c_1) \\ A_2 = min(a_2, c_1) - max(a_3, c_2) \\ \ldots \\ A_{n-1} = min(a_{n-1}, c_{n-2}) - max(a_n, c_{n-1}) \\ A_n = min(a_n, c_{n-1}) \end{gathered}$$

are all equal to or greater than 0 and also c_0 has the same parity as $\Sigma_{p=1}^{n} a_p - \Sigma_{p=1}^{n-1} c_p$. In this case, the multiplicity is

$$m_\lambda(\mu) = P(A_1e_1 + \ldots + A_ne_n - c_0e_n) - P(A_1e_1 + \ldots + A_ne_n + (c_0+2)e_n).$$

Here, P is the Kostant partition function defined relative to the set $\{e_p \pm e_n \mid 1 \le p \le n-1\}$ where $e_p(iE_{q,q}) = \delta_{p,q}$.

Using this theorem, we obtain the following lemma.

Lemma 4.4.7.2. *Suppose ϕ_0 is an $Sp(n)$-type with highest weight*

$$(n-1, 1, 1, \ldots, 1).$$

The Weyl group representation on $V^{Sp(1)^n}_{\phi_0, single}$ is the sign representation.

Proof: From 4.4.7.1, ϕ_0 is restricted to the representation of $Sp(n-1)\times Sp(1)$ with highest weight,

$$(c_1, c_2, \ldots, c_{n-1})(0)$$

if and only if

$$(c_1, c_2, \ldots, c_{n-1}) = (n-1, 1, 1, \ldots, 1, 0) \text{ or } (c_1, c_2, \ldots, c_{n-1}) = (n-2, 1, 1, \ldots, 1, 1).$$

Each subrepresentation of $Sp(n-1)\times Sp(1)$ has multiplicity one. The $Sp(n-1)$-type with highest weight $(c_1, c_2, \ldots, c_{n-1}) = (n-1, 1, 1, \ldots, 1, 0)$ does not have $Sp(1)^{n-1}$-fixed vector because it is not branched to $Sp(1)^{n-1}$-type with highest weight $(0)^{n-1}$. Therefore, the irreducible subrepresentation of $Sp(n-1)\times Sp(1)$ has a nonzero M-fixed vector if it has the highest weight

$$(n-2, 1, 1, \ldots, 1, 1)(0).$$

Using this fact repeatedly, we conclude that an irreducible subrepresentation of $Sp(2)\times Sp(1)^{n-2}$ has a nonzero $Sp(1)^n$-fixed vector if and only if it has the highest weight $(1,1)(0)^{n-2}$. The $Sp(2)\times Sp(1)^{n-2}$-type with highest weight $(1,1)(0)^{n-2}$ has multiplicity one in ϕ_0, so we have one-dimensional $Sp(1)^n$-fixed vectors in ϕ_0, and it is single-petaled.

For $n=2$, the $Sp(2)$-type with highest weight $(1,1)$ has one dimensional $Sp(1)^2$-fixed vectors and $k_{\alpha_1-\alpha_2}$ acts by (-1) on the M-fixed vectors. Therefore, we can conclude that $k_{\alpha_i-\alpha_j}$ acts by -1 on $V_{\phi_0,single}^{Sp(1)^n} = V_{\phi_0}^{Sp(1)^n}$ and the Weyl group representation on $V_{\phi_0,single}^{Sp(1)^n}$ is the sign representation. □

Lemma 4.4.7.3. *The S_6-type $\psi^{(2,2,2)}$ is not single-petaled.*

The proof is similar to that of 4.4.6.5 and 4.4.6.6, so we omit the proof.

From 4.3.1.2, the candidates for single-petaled W-types of $SL(n,\mathbb{H})$ are ψ^λ where

$$\lambda^T = (a_1, a_2, \ldots, a_n) \vdash n$$

satisfies $a_2 \leq 2$. We prove that ψ^λ is single-petaled.

Lemma 4.4.7.4. *Let $\lambda^T = (a_1, a_2, \ldots, a_n)$ be a partition of n such that $a_1 \geq 2$ and $a_2 \leq 2$. Let s be the number of entries in λ^T greater than 1. In other words, $a_{s+1} \geq 2$ and $a_{s+2} \leq 1$. The W-type ψ^λ is realized on the single-petaled M-fixed vectors of the $Sp(n)$-type ϕ with highest weight*

$$h_\lambda = (a_1 - 1, 1, 1, \ldots, 1, 0, 0, \ldots, 0)$$

with $a_1 + 2s$ nonzero entries.

Conversely, we define p such that $p(\phi^{h_\lambda}) = \lambda$.

Remark: If $a_1 = a_2 = \ldots = a_n = 1$, then $\lambda = (n)$ and ψ^λ is the trivial representation of the Weyl group. The trivial K-type realizes ψ^λ, so ψ^λ is a single-petaled W-type of $SL(n, \mathbb{H})$.

Proof: First, we show that ϕ is a single-petaled $Sp(n)$-type. The $Sp(2)$-type with a nonzero $Sp(1) \times SP(1)$-fixed vector has the highest weight

$$(m, m).$$

We note that the $Sp(2) \times Sp(1)^{n-2}$-type with highest weight $(m, m)(b_1)(b_2)\ldots(b_{n-2})$ with $m > 1$ does not appear in ϕ, and therefore, if the $Sp(2) \times Sp(1)^{n-2}$-type with highest weight $(m, m)(0)^{n-2}$ appears in ϕ, then $m = 0$ or $m = 1$. We conclude that ϕ is single-petaled.

Second, we parametrize V_ϕ^M using Young tableau of shape λ. We write the highest weight of $Sp(n)$-type ϕ as

$$h_\lambda = (b_1 = a_1 - 1, b_2 = 1, b_3 = 1, \ldots, b_{a_1+2s} = 1, 0, 0, \ldots, 0).$$

We will show that the part of V_ϕ^M realizes the W-type parametrized by

$$\lambda = (b_1 + 1 = a_1, b_2 + 1 = 2, \ldots, b_s + 1 = 2, d_1 = 1, d_2 = 1, \ldots, d_r = 1)^T \vdash n$$

where $r = n - (s+1) - \Sigma_{i=1}^{s+1} b_i = n - a_1 - 2s$. We parametrize vectors in V_ϕ^M using subgroups of $Sp(n)$ isomorphic to

$$\Pi_{i=1}^{s+1} Sp(b_i + 1) \times \Pi_{j=1}^{r} Sp(d_j).$$

Let S_A be a subgroup isomorphic to

$$\Pi_{i=1}^{s+1} Sp(b_i + 1) \times \Pi_{j=1}^{r} Sp(d_j)$$

corresponding to $A \in \mathrm{Tab}(\lambda)$ in the following way: the numbers in the ith column in A denote the coordinates that comprise $Sp(b_i + 1)$ for $1 \leq i \leq s+1$ and the number in the $(s+1+j)$th column in A denotes the coordinate that comprises $Sp(d_j)$ for $1 \leq j \leq r$. One irreducible subrepresentation of S_A in ϕ has the highest weight

$$\Pi_{i=1}^{s+1} (b_i, 1, 1, \ldots, 1) \times (0)^r,$$

and **it has one-dimensional M-fixed vector space from 4.4.7.2, which is spanned by a vector that we denote by v_A.** The M-fixed vector v_A is a single-petaled vector because ϕ is a single-petaled K-type.
Third, we show that the Weyl group representation on $\mathrm{span}_{A \in \mathrm{Tab}(\lambda)} < v_A >$ is the S_n-type ψ^λ.

- The action of S_n on $\mathrm{Tab}(\lambda)$ is naturally defined such that $s_{i,j} := (i, j) \in S_n$ exchanges i and j in an element of $\mathrm{Tab}(\lambda)$.

- Correspondingly, we define the action of S_n on $\mathrm{span}_{A \in \mathrm{Tab}(\lambda)} < u_A >$ such that $s.(u_A) = u_{s.A}$.

- We define an operator $T(A)$ on A and on u_A correspondingly.

Note that u_A and $T(A)$ are defined in 2.4.4. We define the free vector space V_λ over $\mathbb{C}$ with basis $\mathrm{Tab}(\lambda)$.

- The map

$$f : V_\lambda \to \mathrm{span}_{A \in Tab(\lambda)} < u_A >$$

such that $f(A) = u_A$ is an S_n homomorphism.

- We can define an S_n-homomorphism

$$g : V_\lambda \to \text{span}_{A\in\text{Tab}(\lambda)} < v_A > .$$

Since $\text{span}_{s\in S_n} < s.A >= V_\lambda$, we can define g such that it is an S_n-homomorphism.

- The restricted map of f

$$f : \text{span}_{A\in\text{Tab}(\lambda)} < R(A)A >\to \text{span}_{A\in\text{Tab}(\lambda)} < R(A)u_A >$$

  is an S_n-isomorphism.

- The restricted map of f

$$f : \text{span}_{A\in\text{Tab}(\lambda)} < T(A)A >\to \text{span}_{A\in\text{Tab}(\lambda)} < T(A)u_A >$$

  is an S_n-isomorphism.

- The restricted map of g

$$g : \text{span}_{A\in\text{Tab}(\lambda)} < T(A)A >\to \text{span}_{A\in\text{Tab}(\lambda)} < T(A)v_A >$$

  is surjective and nonzero.(*)

- $g \circ f^{-1} : \text{span}_{A\in\text{Tab}(\lambda)} < T(A)u_A >\to \text{span}_{A\in\text{Tab}(\lambda)} < T(A)v_A >$ is a well-defined surjective S_n-homomorphism.

- $\text{span}_{A\in Tab(\lambda)} < T(A)u_A >$ is an S_n-type ψ^λ. [11]

- $g \circ f^{-1}$ is injective, thus an S_n-isomorphism.

Therefore, we conclude that the Weyl group representation on $\text{span}_{A\in\text{Tab}(\lambda)} < T(A)v_A >$ is ψ^λ. Since

$$\text{span}_{A\in\text{Tab}(\lambda)} < T(A)v_A >\subset \text{span}_{A\in\text{Tab}(\lambda)} < v_A >\subset V_\phi^M$$

holds, ϕ is quasi-single-petaled K-type that realizes ψ^λ. We prove $(*)$ in the following separate lemma. □

The notations in the two following lemmas are the same as in the proof of 4.4.7.4.

Lemma 4.4.7.5. *For $A \in Tab(\lambda)$, $T(A)v_A$ is nonzero.*

Proof: We prove using mathematical induction on $n = |\lambda|$.

(1) If $|\lambda| = 1$, then $T(A)v_A$ is nonzero.

(2) Suppose $|\lambda| > 1$. An irreducible representation of $Sp(n) \simeq U(2n) \cap Sp(n, \mathbb{C})$ is realized by a highest weight representation. A highest weight vector can be expressed using a wedge product of a direct sum of standard representations as in the proof of 4.4.6.1. Specifically, let

$$V^x = \mathrm{span}_{\mathbb{C}} < e_1^x, e_2^x, \ldots, e_{2n}^x >$$

be the standard representation of $K = Sp(n) \simeq U(2n) \cap Sp(n, \mathbb{C})$. Then

$$\bigoplus_{x=1}^{k} V^x,$$

a direct sum of standard representations, is also a representation of K. Also, we can define a representation of K on

$$\bigwedge_{j=1}^{l} \bigoplus_{x=1}^{k} V^x.$$

We can realize the $Sp(n)$-type with highest weight $(b_1, b_2, \ldots, b_n)$ as a representation spanned by the highest weight vector

$$\bigwedge_{j=1}^{n} (\bigwedge_{i=1}^{b_j} e_j^i) \in \bigwedge^{\Sigma_{j=1}^{n} b_j} (\bigoplus_{x=1}^{b_1} V^x).$$

We can express the highest weight vector of ϕ as follows:

$$\textstyle\bigwedge_{x=1}^{b_1} e_1^x \wedge \bigwedge_{y=2}^{a_1+2s} e_y^1.$$

Without loss of generality, we choose $A \in \mathrm{Tab}(\lambda)$ such that

$$A^T_{i,j} \geq A^T_{k,l} \Leftrightarrow (i,j) \geq k,l \text{ in the dictionary order.}$$

(A) Suppose $a_1 > 2$. The M-fixed vector v_A is a scalar multiple of

$$w_A \wedge \bigwedge_{y=1}^{s}(e^1_{a_1+2y-1} \wedge e^1_{a_1+2y-1+n} - e^1_{a_1+2y} \wedge e^1_{a_1+2y+n})$$

where w_A is an $Sp(1)^{a_1}$-fixed vector of $Sp(a_1)$-type with highest weight

$$(a_1, 1, 1, \ldots, 1)$$

and the highest weight vector

$$\bigwedge_{x=1}^{a_1} e^x_1 \wedge \bigwedge_{y=2}^{a_1} e^1_y.$$

We note that w_A has the nonzero term which is a scalar multiple of

$$u_A \wedge (e^{a_1}_{a_1} \wedge e^1_{n+a_1} + e^{a_1}_{n+a_1} \wedge e^1_{a_1})$$

where u_A is an $Sp(1)^{a_1-1}$-fixed vector of $Sp(a_1-1)$-type with highest weight $(a_1 - 2, 1, 1, \ldots, 1)$ and the highest weight vector

$$\bigwedge_{x=1}^{a_1-1} e^x_1 \wedge \bigwedge_{y=2}^{a_1-1} e^1_y.$$

Then, $T(A)v_A$ has the nonzero term which is a scalar multiple of

$$(T(B)(u_A \wedge \bigwedge_{y=1}^{s}(e^1_{a_1+2y-1} \wedge e^1_{a_1+2y-1+n} - e^1_{a_1+2y} \wedge e^1_{a_1+2y+n}))) \wedge (e^{a_1}_1 \wedge e^1_{n+1} + e^{a_1}_{n+1} \wedge e^1_1).$$

Here, B is the tableau of shape τ such that $\tau^T = (a_1 - 1, a_2, \ldots, a_n)$ and $B_{i,j} = A_{i,j}$. From induction hypothesis

$$T(B)(u_A \wedge \bigwedge_{y=1}^{s}(e^1_{a_1+2y-1} \wedge e^1_{a_1+2y-1+n} - e^1_{a_1+2y} \wedge e^1_{a_1+2y+n})) \neq 0,$$

and therefore $T(A)v_A \neq 0$.

(B) Suppose $a_1 = 2$. The M-fixed vector v_A is the scalar multiple of

$$\bigwedge_{y=1}^{s+1}(e^1_{2y-1} \wedge e^1_{2y-1+n} - e^1_{2y} \wedge e^1_{2y+n})$$

Then, $T(A)v_A$ has a nonzero term which is a scalar multiple of

$$\bigwedge_{y=1}^{s+1}(e^1_{2y-1} \wedge e^1_{2y-1+n}),$$

and therefore $T(A)v_A \neq 0$.

From (1), (2), and mathematical induction, we prove the lemma. □

From lemmas in this subsection, we obtain the following proposition.

Proposition 4.4.7.6. *Every irreducible Weyl group representation ψ^λ is single-petaled for $SL(n, \mathbb{H})$ if $\lambda = (a_1, a_2, \ldots, a_n)^T$ satisfies $a_2 \leq 2$.*

4.4.8 $Sp(m, n)$

We start with clarifying notations that we will use in this subsection. Without loss of generality, we assume $m \geq n$.

- $G = Sp(m, n)$
- $K = Sp(m) \times Sp(n)$
- $\mathfrak{g} = sp(m, n) = \{ \begin{pmatrix} a & b \\ b^* & c \end{pmatrix} \mid a, c$ are $m \times m$ and $n \times n$ matrices over quaternions respectively satisfying $a + a^* = 0$, $c + c^* = 0\}$
- $\mathfrak{p} = \{ \begin{pmatrix} 0 & x \\ x^* & 0 \end{pmatrix} \mid x$ is an $m \times n$ matrix over quaternions.$\}$
- $\mathfrak{a} = \{ \begin{pmatrix} 0 & x \\ x^* & 0 \end{pmatrix} \mid x$ is an $m \times n$ matrix over real numbers, $x_{i,j}$ is zero unless $i + j = m + 1\} \subset \mathfrak{p}$.
- $\alpha_i \in \mathfrak{a}^*$: $\alpha_i(\begin{pmatrix} 0 & x \\ x^* & 0 \end{pmatrix}) = x_{m+1-i,i}$
- restricted roots:

  (1) $m > n$: $\{\pm 2\alpha_i \mid 1 \leq i \leq n\} \cup \{\pm \alpha_i \mid 1 \leq i \leq n\}$

$\cup\{\pm\alpha_i \pm \alpha_j \mid, 1 \leq i \neq j \leq n\}$

(2) $m = n$: $\{\pm\alpha_i \pm \alpha_j | i \neq j, 1 \leq i, j \leq n\} \cup \{\pm 2\alpha_i \mid 1 \leq i \leq n\}$

- positive roots:

  (1) $m > n$: $\{\alpha_i \pm \alpha_j \mid n \geq j > i \geq 1\} \cup \{2\alpha_i \mid n \geq i \geq 1\} \cup \{\alpha_i \mid n \geq i \geq 1\}$.

  (2) $m = n$: $\{\alpha_i \pm \alpha_j \mid n \geq j > i \geq 1\} \cup \{2\alpha_i \mid n \geq i \geq 1\}$

- simple roots:

  (1) $m > n$: $\{\alpha_i - \alpha_{i+1} \mid n > i \geq 1\} \cup \{\alpha_n\}$

  (2) $m = n$: $\{\alpha_i - \alpha_{i+1} \mid n > i \geq 1\} \cup \{2\alpha_n\}$

We choose the Weyl group representatives as in 3.5.1. The Weyl group representatives in $N_K(\mathfrak{a})$ are defined as follows. Let A be a matrix which is nonzero only in the entries corresponding to row and column indices $m-i+1$, $m-j+1$, $m+i$ and $m+j$, where they are

$$\begin{pmatrix} J(1) & 0 \\ 0 & -J(1) \end{pmatrix}.$$

Let B be a matrix which is nonzero only in the entries corresponding to row and column indices $m-i+1$, $m-j+1$, $m+i$ and $m+j$, where they are

$$\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}.$$

Then the Weyl group representatives are given as follows:

$$W_{\alpha_i - \alpha_j} = I - B + A \text{ and } W_{\alpha_i} = W_{\alpha_{2i}} = I - 2E_{m+i,m+i}.$$

In this subsection, we state the lemmas that we need to classify single-petaled W-types in $Sp(m,n)$ and skip the proofs. The method of proofs is similar to that of 4.4.6.1.

Lemma 4.4.8.1. *The W-type $\psi^{((2,2,2),\emptyset)}$ is not single-petaled.*

Lemma 4.4.8.2. *The W-type $\psi^{((1,1,1),(1))}$ is not single-petaled.*

Lemma 4.4.8.3. *The W-type $\psi^{(\emptyset,(2))}$ is not single-petaled.*

From 4.3.1.2, the W-types without a subrepresentation $\psi^{((1,1,1),(1))}$, $\psi^{((2,2,2),\emptyset)}$ or $\psi^{(\emptyset,(2))}$ are candidates for single-petaled W-types. These W-types are $\psi^{(\lambda,\tau)}$ for

$$[\lambda^T = (k_1, k_2, \ldots, k_{|\lambda|}) \text{ such that } k_2 \le 2,\ \tau = \emptyset]$$
$$\text{or } [\lambda^T = (k_1, k_2, \ldots, k_{|\lambda|}) \text{ such that } k_1 \le 2,\ \tau = (1, 1, \ldots, 1)].$$

Proposition 4.4.8.4. *The W-type $\psi^{(\lambda,\tau)}$ of $Sp(m, n)$ is single-petaled if and only if (A) or (B) holds.*
(A) $\tau = (1, 1, \ldots, 1)$ and $\lambda^T = (k_1, k_2, \ldots, k_{|\lambda|})$ such that $k_1 \le 2$
(B) $\tau = \emptyset$ and $\lambda^T = (k_1, k_2, \ldots, k_{|\lambda|})$ such that $k_2 \le 2$

To prove this proposition, we need a few lemmas. An irreducible representation of $Sp(n) \simeq U(2n) \cap Sp(n, \mathbb{C})$ is realized by a highest weight representation. A highest weight vector can be expressed using a wedge product of a direct sum of standard representations as in the proof of 4.4.6.1. Specifically, let

$$V^x = \mathrm{span}_{\mathbb{C}} < e_1^x, e_2^x, \ldots, e_m^x, f_1^x, f_2^x, \ldots, f_m^x >$$

be the standard representation of $Sp(m) \simeq U(2m) \cap Sp(m, \mathbb{C})$. Then,

$$\bigoplus_{x=1}^{k} V^x,$$

a direct sum of standard representations, is also a representation of $Sp(m)$. Also, we can define a representation of $Sp(m)$ on

$$\bigwedge_{j=1}^{l} \bigoplus_{x=1}^{k} V^x.$$

The $Sp(n)$-type with highest weight $(b_1, b_2, \ldots, b_m)$ as a representation spanned by the highest weight vector

$$\bigwedge_{j=1}^{m} (\bigwedge_{i=1}^{b_j} e_j^i) \in \bigwedge^{\Sigma_{j=1}^{m} b_j} (\bigoplus_{x=1}^{b_1} V^x).$$

We can regard V^x as a representation of $Sp(m) \times Sp(n)$ by defining that $Sp(n)$ acts trivially on V^x.

Likewise, let

$$W^x = \text{span}_{\mathbb{C}} < E_1^x, E_2^x, \ldots, E_n^x, F_1^x, F_2^x, \ldots, F_n^x >$$

be the standard representation of $Sp(n) \simeq U(2n) \cap Sp(n, \mathbb{C})$. Then

$$\bigoplus_{x=1}^{k} W^x,$$

a direct sum of standard representations, is also a representation of $Sp(n)$. Also, we can define a representation of $Sp(n)$ on

$$\bigwedge_{j=1}^{l} \bigoplus_{x=1}^{k} W^x.$$

The $Sp(n)$-type with highest weight $(b_1, b_2, \ldots, b_n)$ as a representation spanned by the highest weight vector

$$\bigwedge_{j=1}^{n} (\bigwedge_{i=1}^{b_j} e_j^i) \in \bigwedge^{\Sigma_{j=1}^{n} b_j} (\bigoplus_{x=1}^{b_1} V^x).$$

We can regard W^x as a representation of $Sp(m) \times Sp(n)$ by defining that $Sp(m)$ acts trivially on W^x.

Lemma 4.4.8.5. *The W-type $\psi^{(\lambda,\tau)}$ is single-petaled where*

$$\lambda^T = (k_1, k_2, \ldots, k_n) \text{ such that } k_2 \leq 2 \text{ and } \tau = \emptyset.$$

Sketch of Proof: From 4.4.7.4, S_n-type ψ^λ is realized on the single-petaled $Sp(1)^n$-fixed vectors of a $Sp(n)$-type ϕ. Suppose ϕ has the highest weight $(a_1, a_2, \ldots, a_n)$. The $S_n \ltimes (\mathbb{Z}/2\mathbb{Z})^n$-type

$$\psi^\lambda \ltimes (\text{triv})^n$$

is realized by the K-type with highest weight

$$(a_1, a_2, \ldots, a_n, 0, 0, \ldots, 0)(0, 0, \ldots, 0)$$

on its M-fixed vectors. Therefore, $\psi^{\lambda} \otimes (\text{triv})^n$ is a single-petaled W-type. □

Lemma 4.4.8.6. *The W-type $\psi^{(\lambda,\tau)}$ is single-petaled where*

$$\lambda = \emptyset \text{ and } \tau = (1, 1, \ldots, 1).$$

Sketch of Proof: The K-type ϕ with highest weight

$$(n, 0, 0, \ldots, 0)(1, 1, \ldots, 1)$$

realizes $\psi^{(\lambda,\tau)}$ on its single-petaled M-fixed vector. We can construct ϕ using the highest weight vector

$$(\bigwedge_{k=1}^{n} e_1^k) \wedge (\bigwedge_{k=1}^{n} E_k^1) \in \bigwedge^{2n}((\bigoplus_{k=1}^{n} V^1) \oplus W^1).$$

Let

$$v := \bigwedge_{k=1}^{n} (e_k^k \wedge F_k - f_k^k \wedge E_k) \in V_\phi^M.$$

Then, v is annihilated by $Z_\alpha(Z_\alpha^2 - 4)$ for all reduced root α. We can define the action of Weyl group on V_ϕ^M, so we can define $C(A)v$ where the Young tableau is

$$\begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline \cdots \\ \hline n \\ \hline \end{array}$$

is A. Note that $C(A)v$ is nonzero. In a similar way to 4.4.7.4, we can show that the Weyl group stabilizes $\text{span}_{\mathbb{C}} < C(A) \circ R(A) \circ fv >$, and the Weyl group representation on this vector space is the sign representation. Therefore, $\psi^{\emptyset,(1,1,\ldots,1)}$ is single-petaled.
□

Lemma 4.4.8.7. *The W-type $\psi^{(\lambda,\tau)}$ is single-petaled where*

$$\lambda^T = (k_1, k_2, \ldots, k_{|\lambda|}) \text{ such that } k_1 \leq 2,\ \tau = (1, 1, \ldots, 1).$$

We skip the proof. The method of proof is similar to that of 4.4.7.4, and we use 4.4.8.5 and 4.4.8.6 instead of 4.4.7.2.

Proposition 4.4.8.4 is immediate from 4.4.8.5, 4.4.8.6, and 4.4.8.7.

4.4.9 $SO(2n+1, \mathbb{C})$

In this subsection, we summarize the steps to classify single-petaled W-types of $SO(2n+1, \mathbb{C})$. We use the following lemmas.

Lemma 4.4.9.1. *The W-type $\psi^{((1,1),\emptyset)}$ is not single-petaled.*

Lemma 4.4.9.2. *The W-type $\psi^{((\emptyset),(2,2))}$ is not single-petaled.*

We skip the proof because the method of proof is similar to 4.4.6.1. From 4.3.1.2, the candidates for single-petaled W-types are parametrized by the following pairs of Young diagrams:

$$((k), (r, 1, 1, \ldots, 1)).$$

We can confirm that $\psi^{((k),(r,1,1,\ldots,1))}$ is single-petaled in the similar way to 4.4.6.5, 4.4.6.6, and 4.4.7.4.

4.4.10 $SO(2n, \mathbb{C})$

In this subsection, we summarize the steps to classify single-petaled W-types of $SO(2n, \mathbb{C})$. We use the following lemmas.

Lemma 4.4.10.1. *The W-type $\psi^{\{(2,2),\emptyset\}}$ is not single-petaled.*

Lemma 4.4.10.2. *The W-type $\psi^{\{(1,1,1),(1,1)\}}$ is not single-petaled.*

Lemma 4.4.10.3. *The W-type $\psi^{\{(2,2),(2,2)\}_I}$ is not single-petaled.*

Lemma 4.4.10.4. *The W-type $\psi^{\{(2,2),(2,2)\}_{II}}$ is not single-petaled.*

Lemma 4.4.10.5. *The W-type $\psi^{\{(1,1,1),(1,1,1)\}_{I}}$ is not single-petaled.*

Lemma 4.4.10.6. *The W-type $\psi^{\{(1,1,1),(1,1,1)\}_{II}}$ is not single-petaled.*

We skip the proof because the method of proofs is similar to 4.4.6.1. From 4.3.1.2, the candidates for single-petaled W-types are parametrized by the following pairs of Young diagrams:

$$\{(k,1,1,\ldots,1),(r)\},\ \{(k,1),(r,1)\},$$
$$\{(k,1),(k,1)\}_I,\ \{(k,1),(k,1)\}_{II}\ \{(k),(k)\}_I,\ \{(k),(k)\}_{II}.$$

We can confirm that these W-types are single-petaled in the similar way to 4.4.6.5, 4.4.6.6, and 4.4.7.4.

4.4.11 $SO^*(2n)$

In this subsection, we summarize the steps to classify single-petaled W-types of $SO^*(2n)$-types. We use the following lemmas.

Lemma 4.4.11.1. *The W-type $\psi^{((1,1),(1,1))}$ is not single-petaled.*

Lemma 4.4.11.2. *The W-type $\psi^{((2,2),\emptyset)}$ is not single-petaled.*

Lemma 4.4.11.3. *The W-type $\psi^{(\emptyset,(2,2))}$ is not single-petaled.*

We and skip the proof because the method of proofs is similar to 4.4.6.1. From 4.3.1.2, the candidates for single-petaled W-types are parametrized by the following pairs of Young diagrams:

$$((k,1,1,\ldots,1),(r)) \text{ or } ((r),(k,1,1,\ldots,1)).$$

We can confirm that these W-types are single-petaled in the similar way to 4.4.6.5, 4.4.6.6, and 4.4.7.4.

Chapter 5

Future Research

In this chapter, we discuss how single-petaled K-types and single-petaled W-types can be used in the future research.

One of the research projects is to complete the list of

$$\text{quasi-single-petaled } K\text{-types} \Leftrightarrow \text{single-petaled } W\text{-types}$$

for exceptional groups of type E_6, E_7, E_8, F_4, and G_2. Unitary representations are fully classified only for G_2 among exceptional cases [15], so the classification of single-petaled Weyl group representations and quasi-single-petaled K-types will provide a strong non-unitarity test for split real groups. Barbasch and Ciubotaru discovered a great deal about spherical unitary representations for p-adic groups of type E and F [3][9][8]. If we figure out the relation between W-types and K-types for these cases, we will be able to conclude some non-unitarity for split real groups of type E and F. Barbasch and Pantano identified the set of pairs

$$(\text{petite } K\text{-types} \Leftrightarrow \text{relevant } W\text{-types})$$

for some exceptional groups [6], and we expect the set of pairs

$$(\text{single-petaled } W\text{-types} \Leftrightarrow \text{quasi-single-petaled } K\text{-types})$$

to be larger. Therefore, identifying the correspondence between quasi-single-petaled K-types and single-petaled W-types will provide a stronger non-unitarity test.

The non-spherical unitary dual is still a mysterious field in unitary representation theory. One of the available techniques is to use two pairs of irreducible representations of maximal compact subgroups in the bottom layers with matching signatures, invented by Vogan [18][14]. Using this bottom layer argument, we can transfer information about signatures from spherical representations of smaller groups to non-spherical representations of larger groups. To use this technique, it will be helpful to have more K-types for which we can actually calculate signatures. It will be relatively easier to calculate (some part of) the signatures for quasi-single-petaled K-types, so we can make use of these K-types to determine non-unitarity in the non-spherical case.

Appendix A

Notations

- G: reductive Lie group in 2.1.1
- $\mathfrak{g}$: $Lie(G)$ in 2.1.1
- K: maximal compact subgroup in 2.1.1
- $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$: the Cartan decomposition of $\mathfrak{g}$ in 2.1.1
- $\mathfrak{a}$: the maximal abelian subalgebra of $\mathfrak{p}$ in 2.1.1
- (π, V_π): an irreducible representation of G in 2.1.1
- ϕ: a K-type in 2.1.1
- K-type with highest weight Λ : ϕ_Λ in 2.1.1
- W: Weyl group generated by restricted root reflections in 2.1.1
- Δ: restricted roots in 2.1.2
- Δ_1: reduced roots in 2.1.2
- Π: simple roots in 2.1.2
- Δ^+: positive roots in 2.1.2
- δ: an irreducible unitary representation of M in 2.1.2

- ν: an element in dual of $\mathfrak{a}$ in 2.1.2
- M: subgroup of K centralizes $\mathfrak{a}$ in 2.1.2
- A: analytic subgroup of G corresponding to $\mathfrak{a}$ in 2.1.2
- Q: the minimal parabolic subgroup in 2.1.2
- ω_0: the longest Weyl group element in 2.1.2
- $V_{\pi,(K)}$: K-finite vectors in V_π in 2.1.2.4
- $\pi_{(K)}$: $(\mathfrak{g}, K)$-module defined on $V_{\pi,(K)}$ in 2.1.2.4
- $I_Q(\delta, \nu)$: the principal series representation in 2.1.2
- $J_Q(\delta, \nu)$: the Langlands quotient of $I_Q(\delta, \nu)$ in 2.1.3.4
- $I(\nu)$: principal series representation in 2.1.4.3
- $J(\nu)$: Langlands quotient of $I(\nu)$ in 2.1.4.3
- $A(\nu)$: the long intertwining operator between $I(\nu)$ and $I(\omega_0\nu)$ in 2.2.1
- $A(\omega, \nu)$: intertwining operator between $I(\nu)$ and $I(\omega^{-1}, \nu)$ in 2.2.1
- $A_\phi(\omega, \nu)$: intertwining operator between $\mathrm{Hom}_K(V_\phi, I(\nu))$ and $\mathrm{Hom}_K(V_\phi, I(\omega^{-1}(\nu)))$ in 2.2.2
- $A_\phi(\nu) : \mathrm{Hom}_K(\phi, I(\nu)) \to \mathrm{Hom}_K(\phi, I(\omega_0\nu))$ in 2.2.2
- K-type tests (non-unitarity tests using K-types): checking if

  $$Hom_K(\phi, I(\nu)) \to Hom_K(\phi, I(\omega_0\nu))$$

  is positive definite in 2.2.2
- $\mathfrak{p}$-representation: a irreducible subrepresentation of K in $\mathfrak{p}_{\mathbb{C}}$ in 2.3.1.1

- $\mathfrak{g}^\alpha$: real rank one subalgebra of $\mathfrak{g}$ generated by $\mathfrak{g}_{n\alpha}$ where n runs over nonzero integers in 2.3.2.1
- $\mathfrak{p}^\alpha$: $\mathfrak{g}^\alpha \cap \mathfrak{p}$ in 2.3.2.1
- $\mathfrak{k}^\alpha$: $\mathfrak{g}^\alpha \cap \mathfrak{k}$ in 2.3.2.1
- G^α: analytic subgroup corresponding to $\mathfrak{g}^\alpha$ in 2.3.2.1
- K^α: maximal subgroup of G^α in 2.3.2.1
- ψ_ϕ: Weyl group representation on V_ϕ^M for K-type (ϕ, V_ϕ) in 2.4.1
- $A_{s_\alpha}(\nu) = \frac{m}{m+<\nu,\alpha^\vee>} + \frac{<\nu,\alpha^\vee>}{m+<\nu,\alpha^\vee>} s_\alpha$ in 2.4.1
- ψ: a W-type in 2.4.2
- W-type tests (non-unitarity tests using W-types): checking if every eigenvalues of $\psi(A_{s_{\omega_0}}(\nu))$ is nonnegative in 2.4.2
- ψ^λ: S_n-type parameterized by λ in 2.4.4
- $\psi^{(\lambda,\tau)}$: $S_n \ltimes (\mathbb{Z}/\mathbb{Z})^n$-type parameterized by (λ, τ) in 2.4.4
- R_n: the index 2 subgroup of $S_n \ltimes (\mathbb{Z}/\mathbb{Z})^n$ in 2.4.4
- $\psi^{\{\lambda,\tau\}}$: R_n-type parameterized by $\{\lambda, \tau\}$ in 2.4.4
- $\psi^{\{\lambda\}_I}$, $\psi^{\{\lambda\}_{II}}$: R_n-types parameterized by $\lambda \vdash n$ in 2.4.4
- $V^M_{\phi,single}$: single-petaled M-fixed vectors in ϕ in 4.1.1.1
- $\psi_{\phi,single}$: the Weyl group representation on $V^M_{\phi,single}$ in 4.1.1.5

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