Non-monotonic Lyapunov Functions for Stability of Nonlinear and Switched Systems: Theory and Computation

by

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B.S., Electrical Engineering; B.S., Mathematics (2006)
University of Maryland

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Abstract

Lyapunov's direct method, which is based on the existence of a scalar function of the state that decreases monotonically along trajectories, still serves as the primary tool for establishing stability of nonlinear systems. Since the main challenge in stability analysis based on Lyapunov theory is always to find a suitable Lyapunov function, weakening the requirements of the Lyapunov function is of great interest. In this thesis, we relax the monotonicity requirement of Lyapunov's theorem to enlarge the class of functions that can provide certificates of stability. Both the discrete time case and the continuous time case are covered. Throughout the thesis, special attention is given to techniques from convex optimization that allow for computationally tractable ways of searching for Lyapunov functions. Our theoretical contributions are therefore amenable to convex programming formulations.

In the discrete time case, we propose two new sufficient conditions for global asymptotic stability that allow the Lyapunov functions to increase locally, but guarantee an average decrease every few steps. Our first condition is nonconvex, but allows an intuitive interpretation. The second condition, which includes the first one as a special case, is convex and can be cast as a semidefinite program. We show that when non-monotonic Lyapunov functions exist, one can construct a more complicated function that decreases monotonically. We demonstrate the strength of our methodology over standard Lyapunov theory through examples from three different classes of dynamical systems. First, we consider polynomial dynamics where we utilize techniques from sum-of-squares programming. Second, analysis of piecewise affine systems is performed. Here, connections to the method of piecewise quadratic Lyapunov functions are made. Finally, we examine systems with arbitrary switching between a finite set of matrices. It will be shown that tighter bounds on the joint spectral radius can be obtained using our technique.

In continuous time, we present conditions invoking higher derivatives of Lyapunov functions that allow the Lyapunov function to increase but bound the rate at which the increase can happen. Here, we build on previous work by Butz that provides a nonconvex sufficient condition for asymptotic stability using the first three derivatives of Lyapunov functions. We give a convex condition for asymptotic stability that includes the condition by Butz as a special case. Once again, we draw the connection to standard Lyapunov functions. An example of a polynomial vector field is given to show the potential advantages of using higher order derivatives over standard Lyapunov theory. We also discuss a theorem by Yorke that imposes minor conditions on the first and second derivatives to reject existence of periodic orbits, limit cycles, or chaotic attractors. We give some simple convex conditions that imply the requirement by Yorke and we compare them with those given in another earlier work.

Before presenting our main contributions, we review some aspects of convex programming with more emphasis on semidefinite programming. We explain in detail how the method of sum of squares decomposition can be used to efficiently search for polynomial Lyapunov functions.

Thesis Supervisor: Pablo A. Parrilo
Title: Associate Professor

To my father

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Chapter 1

Introduction

1.1 Dynamical Systems and Stability

The world, as we know it, is comprised of entities in space that evolve through time. The idea of modeling the motion of a physical system with mathematical equations probably dates back to Sir Isaac Newton [1]. Today, mathematical analysis of dynamical systems places itself at the center of control theory and engineering, as well as, many sciences such as physics, chemistry, ecology, and economics. In this thesis, we study both discrete time dynamical systems

$$x_{k+1} = f(x_k), \tag{1.1}$$

and continuous time systems modeled as

$$\dot{x}(t) = f(x(t)). \tag{1.2}$$

The vector $x \in \mathbb{R}^n$, often referred to as the *state*, contains the information about the underlying system that is important to us. For example, if we are modeling an electrical circuit, components of $x(t)$ can represent the currents and voltages at different nodes in the circuit at a particular time instant t. On the other hand, if we are analyzing a discrete model of the population dynamics of rabbits in a particular forest, we might want to include in our state x_k information such as the number of male rabbits, the number of female rabbits, the number of wolves, and the amount of food available in the environment at day k. The mapping $f : \mathbb{R}^n \to \mathbb{R}^n$ expresses how the states change in time and it can be in general nonlinear, non-smooth, or even uncertain. In discrete time, f describes the evolution of the system by expressing the current state as a function of the previous state, whereas in continuous time the differential equation expresses the rate of change of the current state as a function of the current state. In either situation, we will be interested in long-term behavior of the states as time goes to infinity. Will the rabbits eventually go extinct? Will the voltages and currents in the circuit settle to a particular value in steady state? Stability theory deals with questions of this flavor. In order to make things more formal, we need to introduce the concept of an equilibrium point and present a rigorous notion of stability. We will do this for the continuous time case. The definitions are almost identical in discrete time once t is replaced with k.

A point $x = x^*$ in the state space is called and equilibrium point of (1.2) if it is a real root of the equation

$$f(x) = 0.$$

An equilibrium point has the property that if the state of the system starts at x^*, it will remain there for all future time. Loosely speaking, an equilibrium point is stable if nearby trajectories stay near it. Moreover, an equilibrium point is asymptotically stable if it attracts nearby trajectories. Without loss of generality, we study stability of the origin; i.e. we assume $x^* = 0$. If the equilibrium point is at any other point, one can simply shift the coordinates so that in the new coordinates the origin is the equilibrium point. The formal definitions of stability that we are going to be using are as follows.

Definition 1. *([20]) The equilibrium point $x = 0$ of (1.2) is*

- stable *(or sometimes called stable in the sense of Lyapunov) if for each $\varepsilon > 0$,* *there exists* $\delta = \delta(\varepsilon) > 0$ *such that*

$$||x(0)|| < \delta \;\Rightarrow\; ||x(t)|| < \varepsilon, \quad \forall t \geq 0.$$

- unstable *if not stable.*

- asymptotically stable *if it is stable and* δ *can be chosen such that*

$$||x(0)|| < \delta \;\Rightarrow\; \lim_{t\to\infty} x(t) = 0.$$

- globally asymptotically stable *if stable and*

$$\forall x(0) \in \mathbb{R}^n, \quad \lim_{t\to\infty} x(t) = 0.^1$$

Note that the question of global asymptotic stability only makes sense when the system has only one equilibrium point in the state space. The first three items in Definition 1 are *local* definitions; they describe the behavior of the system only near the equilibrium point. Asymptotic stability of an equilibrium can sometimes be determined by asymptotic stability of its linearization around the equilibrium. This technique is known as Lyapunov's *indirect* (or *first*) *method*. In this thesis, however, we will mostly be concerned with global asymptotic stability (GAS). In general, the question of determining whether the equilibrium of a nonlinear dynamics is GAS can be extremely hard. Even for special classes of systems several undecidability and NP-hardness results exist in the literature; see e.g. [9] and [6]. The main difficulty is that more often that not it is impossible to explicitly write a solution to the differential equation (1.2) or the difference equation (1.1). Nevertheless, in some cases, we are still able to make conclusions about stability of nonlinear systems, thanks to a brilliant idea by the famous Russian mathematician Aleksandr Mikhailovich Lyapunov. This

[1]Implicit in Definition 1 is the assumption that the differential equation (1.2) has well-defined solutions for all $t \geq 0$. Such global existence of solutions can be guranteed either by assuming that f is globally Lipschitz, or by assuming that f is locally Lipschitz together with the requirements of the Lyapunov theorem of Section 1.2 [20].

method is known as Lyapunov's *direct* (or *second*) *method* and was first published in 1892. Over the course of the past century, this theorem has found many new applications especially in control theory. Its many extensions and variants continue to be an active area of research. We devote the next section to a discussion of this theorem.

1.2 Lyapunov's Stability Theorem

We state below a variant of Lyapunov's direct method that establishes global asymptotic stability.

Theorem 1.2.1. [2]*([20]) Consider the dynamical system (1.2) and let $x = 0$ be its unique equilibrium point. If there exists a continuously differentiable function $V : \mathbb{R}^n \rightarrow \mathbb{R}$ such that*

$$\begin{aligned} V(0) &= 0 & (1.3) \\ V(x) &> 0 \quad \forall x \neq 0 & (1.4) \\ ||x|| \rightarrow \infty &\Rightarrow V(x) \rightarrow \infty & (1.5) \\ \dot{V}(x) &< 0 \quad \forall x \neq 0, & (1.6) \end{aligned}$$

then $x = 0$ is globally asymptotically stable.

Condition (1.6) is what we refer to as the *monotonicity requirement* of Lyapunov's theorem. In that condition, $\dot{V}(x)$ denotes the derivative of $V(x)$ along the trajectories of (1.2) and is given by

$$\dot{V}(x) = \langle \frac{\partial V(x)}{\partial x}, f(x) \rangle,$$

where $\langle ., . \rangle$ denotes the standard inner product in $\mathbb{R}^n$ and $\frac{\partial V(x)}{\partial x} \in \mathbb{R}^n$ is the gradient of $V(x)$. As far as the first two conditions are concerned, it is only needed to assume that $V(x)$ is lower bounded and achieves its global minimum at $x = 0$. There is no

[2]The original theorem by Lyapunov was formulated to imply local stability. This variant of the theorem is often known as the Barbashin-Krasovskii theorem [20].

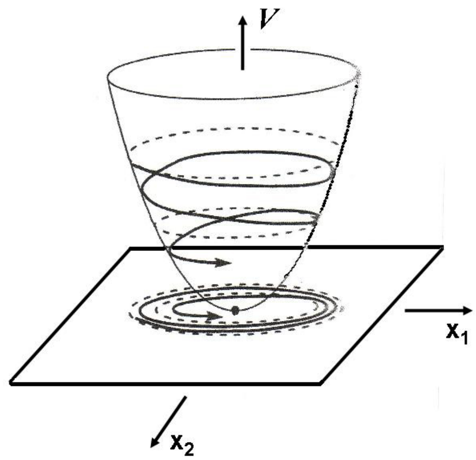


Figure 1-1: Geometric interpretation of Lyapunov's theorem.

conservatism, however, in requiring (1.3) and (1.4). A function satisfying condition (1.5) is called *radially unbounded*. We refer the reader to [20] for a formal proof of this theorem and for an example that shows condition (1.5) cannot be removed. Here, we give the geometric intuition of Lyapunov's theorem, which essentially carries all of the ideas behind the proof.

Figure 1-1[3] shows a hypothetical dynamical system in $\mathbb{R}^2$. The trajectory is moving in the (x_1, x_2) plane but we have no knowledge of where the trajectory is as a function of time. On the other hand, we have a scalar valued function $V(x)$, plotted on the z-axis, which has the guaranteed property that as the trajectory moves the value of this function along the trajectories strictly decreases. Since $V(x(t))$ is lower bounded by zero and is strictly decreasing, it must converge to a nonnegative limit as time goes to infinity. It takes a relatively straightforward argument appealing to continuity of $V(x)$ and $\dot{V}(x)$ to show that the limit of $V(x(t))$ cannot be strictly positive and indeed conditions (1.3)-(1.6) imply

$$V(x(t)) \to 0 \quad \text{as } t \to \infty.$$

Since $x = 0$ is the only point in space where $V(x)$ vanishes, we can conclude that $x(t)$ goes to the origin as time goes to infinity.

[3]Picture borrowed from [1].

It is also insightful to think about the geometry in the (x_1, x_2) plane. The level sets of $V(x)$ are plotted in Figure 1-1 with dashed lines. Since $V(x(t))$ decreases monotonically along trajectories, we can conclude that once a trajectory enters one of the level sets, say given by $V(x) = c$, it can never leave the set $\Omega_c := \{x \in \mathbb{R}^n \mid V(x) \leq c\}$. This property is known as *invariance of sub-level sets.* It is exactly a consequence of this invariance property that we can easily establish stability in the sense of Lyapunov as defined in Definition 1.

Once again we emphasize that the significance of Lyapunov's theorem is that it allows stability of the system to be verified without explicitly solving the differential equation. Lyapunov's theorem, in effect, turns the question of determining stability into a search for a so-called *Lyapunov function*, a positive definite function of the state that decreases monotonically along trajectories. There are two natural questions that immediately arise. First, do we even know that Lyapunov functions always exist? Second, if they do in fact exist, how would one go about finding one? In many situations, the answer to the first question is positive. The type of theorems that prove existence of Lyapunov functions for every stable system are called *converse theorems*. One of the well known converse theorems is a theorem due to Kurzweil that states if f in (1.2) is continuous and the origin is globally asymptotically stable, then there exists an infinitely differentiable Lyapunov function satisfying conditions of Theorem 1.2.1. We refer the reader to [20] and [2] for more details on converse theorems. Unfortunately, converse theorems are often proven by assuming knowledge of the solutions of (1.2) and are therefore useless in practice. By this we mean that they offer no systematic way of finding the Lyapunov function. Moreover, little is known about the connection of the dynamics f to the Lyapunov function V. Among the few results in this direction, the case of linear systems is well settled since a stable linear system always admits a quadratic Lyapunov function. It is also known that stable and smooth homogeneous[4] systems always have a homogeneous Lyapunov function [37].

As we are going to see in Chapter 2, recent advances in the area of *convex opti-*

[4] A homogeneous function f is a function that satisfies $f(\lambda x) = \lambda^d x$ for some constant d.

mization have enabled us to efficiently search for Lyapunov functions using computer software [43], [30], [26], and [32]. Since the main challenge in stability analysis of nonlinear dynamics is to find suitable Lyapunov functions, we are going to give a lot of emphasis to computational search techniques in this thesis. Therefore, we will make sure that our theoretical contributions to Lyapunov theory are amenable to convex optimization.

We end this section by stating Lyapunov's theorem in discrete time. The statement will be almost exactly the same, except that instead of requiring $\dot{V}(x) < 0$ we impose the condition that the value of the Lyapunov function should strictly decrease after each iteration of the map f.

Theorem 1.2.2. *Consider the dynamical system (1.1) and let $x = 0$ be its unique equilibrium point. If there exists a continuously differentiable function $V : \mathbb{R}^n \to \mathbb{R}$ such that*

$$V(0) = 0 \quad (1.7)$$

$$V(x) > 0 \quad \forall x \neq 0 \quad (1.8)$$

$$||x|| \to \infty \Rightarrow V(x) \to \infty \quad (1.9)$$

$$V(f(x)) < V(x) \quad \forall x \neq 0, \quad (1.10)$$

then $x = 0$ is globally asymptotically stable.

1.3 Outline and Contributions of the Thesis

Lyapunov's direct method appears ubiquitously in control theory. Applications are in no way limited to proving stability but also include synthesis via control Lyapunov functions, robustness analysis and dealing with uncertain systems, estimating basin of attraction of equilibrium points, proving instability or nonexistence of periodic orbits, performance analysis (e.g. rate of convergence analysis), proving convergence of combinatorial algorithms (e.g. consensus algorithms), finding bounds on joint spectral radius of matrices, and many more. Therefore, contributions to the core theory of Lyapunov's direct method are of great interest.

In this thesis, we weaken the requirements of Lyapunov's theorem by relaxing the condition that the Lyapunov function has to monotonically decrease along trajectories. This weaker condition allows for *simpler* functions to certify stability of the underlying dynamical system. This in effect makes the search process easier and from a computational point of view leads to saving decision variables. In order to relax monotonicity, two questions need to be answered. (i) Are we able to replace $\dot{V} < 0$ in continuous time and $V_{k+1} < V_k$ in discrete time with other conditions that allow Lyapunov functions to increase locally but yet guarantee their convergence to zero in the limit? (ii) Can the search for a Lyapunov function with the new conditions be cast as a convex program, so that already available computational techniques can be readily applied? The contribution of this thesis is to give an affirmative answer to both of these questions. Our answer will also illuminate the connection of non-monotonic Lyapunov functions to standard Lyapunov functions.

More specifically, the main contributions of this thesis are as follows:

- **In discrete time (Chapter 4):**

  - We propose two new sufficient conditions for global asymptotic stability that allow the Lyapunov functions to increase locally.

    * The first condition (Section 4.2.1) is nonconvex but allows for a intuitive interpretation. Instead of requiring the Lyapunov function to decrease at every step, this condition requires the Lyapunov function to decrease *on average* every m steps.

    * The second condition (Section 4.2.2) is convex and includes the first one as a special case. Here, we map the state space into multiple Lyapunov functions instead of one. The improvement in different steps is measured with different Lyapunov functions.

- We show that every time a non-monotonic Lyapunov function exists, we can construct a standard Lyapunov function from it. However, the standard Lyapunov function will have a more complicated structure.

- In Section 4.3, we show the advantages of our methodology over standard Lyapunov theory with examples from three different classes of dynamical systems.

  * We consider *polynomial systems*. Here, we use techniques from sum-of-squares programming, which is explained in detail in our introductory chapters.

  * We analyze *piecewise affine systems*. These are affine systems that undergo switching based on the location of the state. We explain the connection of quadratic non-monotonic Lyapunov functions to the well-known technique of piecewise quadratic Lyapunov functions.

  * We examine linear systems that undergo *arbitrary switching*. We explain how non-monotonic Lyapunov functions can be used to bound the *joint spectral radius* of a finite set of matrices. The bounds will be tighter than those obtained from standard Lyapunov theory.

- **In continuous time (Chapter 5):**

  - We relax the condition $\dot{V} < 0$ by imposing conditions on *higher order derivatives* of Lyapunov functions to bound the rate at which the Lyapunov function can increase. Here, we build on previous work by Butz [12]. In Section 5.2, we review the results by Butz which assert that using only $\dot{V}$ and $\ddot{V}$ is vacuous, but it is possible to infer asymptotic stability by using the first three derivatives. The formulation of the condition by Butz is *not* convex.

– We show in Section 5.3 that whenever the condition by Butz is satisfied, one can construct a standard Lyapunov function from it. We present a *convex* sufficient condition for global asymptotic stability using higher order derivatives of Lyapunov functions. This condition contains the standard Lyapunov's theorem and Butz's theorem as a special case.

– We show that unlike the result by Butz, the examination of only $\dot{V}$ and $\ddot{V}$ can be beneficial with the new convex condition. We give an example of a polynomial vector field that has no quadratic Lyapunov function but using our convex condition involving the higher derivatives, it suffices to search over quadratic functions to prove global asymptotic stability.

– In section 5.4, we review a result by Yorke [44] that imposes minor conditions on $\dot{V}$ and $\ddot{V}$ to infer that trajectories must either go to infinity or converge to the origin. This is particularly useful to reject existence of periodic orbits, limit cycles, or chaotic attractors. Once again, the conditions by Yorke are not convex. We give simple convex conditions that imply the condition by Yorke but can be more conservative in general. We compare them with conditions given by Chow and Dunninger [13].

Before presenting our main contributions, we review some aspects of convex programming and explain how it can be used to search for Lyapunov functions. This is done in Chapter 2 where we introduce semidefinite programming, sum of squares (SOS) programming, and the $\mathcal{S}$-procedure. We also present the analysis of linear systems in this chapter to give an example of how semidefinite programming can be used to search for quadratic Lyapunov functions. Chapter 3 is devoted to polynomial dynamics and polynomial Lyapunov functions. We explain how the method of sum of squares programming can be used to efficiently search for polynomial Lyapunov functions. Both the continuous time case and the discrete time case are covered. Our introductory chapters include some minor contributions as well. In particular, in Section 2.2.2 we make some observations on quadratic Lyapunov functions for linear systems and in Section 3.2 we investigate if sum of squares programming can potentially be conservative for finding polynomial Lyapunov functions. We give an example of a two dimensional stable vector field that admits a quadratic Lyapunov function, but the gap between nonnegativity and sum of squares avoids the SOS program to detect it. Finally, our conclusions and some future directions are presented in Chapter 6.

1.4 Mathematical Notation and Conventions

Our notation is mostly standard. We use superscripts V^1, V^2 to refer to different functions. $\dot{V}$ denotes the derivative of V with respect to time. Some of our Lyapunov functions will decrease monotonically and some will not. Whenever confusion may arise, we refer to a function satisfying Lyapunov's original theorem as a *standard* Lyapunov function. In discrete time, for simplicity, we denote $V(x_k)$ by V_k. Often, we refer to $V_{k+i} - V_k$ as the improvement in i steps, which can either be negative (a decrease in V) or positive (an increase in V). By f^i, we mean composition of f with itself i times.

As we mentioned before, $\langle .,. \rangle$ denotes the standard inner product in $\mathbb{R}^n$. By $A \succ 0$ ($A \succeq 0$), we mean that the symmetric matrix A is positive definite (positive semidefinite). A Hurwitz matrix is a matrix whose eigenvalues have strictly negative real part. By a Schur stable matrix, we mean a matrix with eigenvalues strictly inside the unit complex ball.

Chapter 2

Convex Programming and the Search for a Lyapunov Function

The goal of this chapter is to familiarize the reader with basics of some techniques from convex programming, which will be used in future chapters to search for Lyapunov functions. We start out by giving an overview of convex programming and then focus our attention on semidefinite programming. Stability analysis of linear systems using quadratic Lyapunov functions is done in this chapter since it fits well within the semidefinite programming framework. We present some observations on Lyapunov analysis of linear systems in Section 2.2.2. Finally, we build up the reader's background on the $\mathcal{S}$-procedure and sum of squares programming. Both of these concepts will come into play repeatedly in future chapters.

2.1 Why Convex Programming?

As we discussed in the previous chapter, Lyapunov theorems prove stability of dynamical systems if one succeeds in finding a Lyapunov function. In cases when one fails to find such function, no conclusion can be drawn regarding stability of the system since the theorems solely provide sufficient conditions. Although converse theorems guarantee the existence of a Lyapunov function for any stable system, they offer no information about how to construct one. When the underlying dynamics represents a physical system, it is natural to take the energy of the system as a candidate Lyapunov function. If the system loses energy over time and energy is never restored, then the states of the system must reach an equilibrium since the energy will eventually die out. However, in many situations where the models are not overly simplified, it can be difficult to write an expression for the energy of the system. More importantly, applicability of Lyapunov theorems goes beyond systems for which the concept of physical energy is available. In such cases, one would have to make an intelligent guess of a Lyapunov function and check the conditions of the theorem or maybe take a judicious trial-and-error approach.

In the past few decades, however, the story has changed. Recnet advances in the theory of *convex programming* have rejuvenated Lyapunov theory by providing systematic and efficient ways to search for Lyapunov functions. A convex program, is an optimization problem of the type

$$\begin{array}{ll} \min & g(x) \\ \textit{subject to} & x \in X, \end{array} \tag{2.1}$$

where $g:\ \mathbb{R}^n \to \mathbb{R}$ is a *convex function*, and the feasible set $X \subset \mathbb{R}^n$ is a *convex set*.

Surprisingly many problems in control engineering and operations research can be cast as a convex problem. We refer the reader to [11] for a thorough treatment of the theory and applications of convex programming. One of the simplest special cases of a convex program is a *linear program* (LP), in which the objective function g is linear, and the set X is defined by a set of linear equalities and inequalities and therefore has a polytopic structure. In 1984, Narendra Karmarkar proposed an algorithm for solving linear programs with a worst-case polynomial time guarantee [19]. The algorithm also worked reasonably fast for practical problems. Karmarkar's algorithm, known as an *interior point algorithm*, along with its polynomial complexity attribute was extended by Nesterov and Nemirovsky to a wide family of convex optimization problems in the late 1980s [25]. This numerical tractability is one of the main motivations for reformulating various problems as a convex program. Lyapunov theory is no exception.

The make this reformulation, one parameterizes a class of Lyapunov functions with restricted complexity (e.g., quadratics or polynomials), imposes the constraints of Lyapunov's theorem on the parameters, and then poses the search as a convex feasibility problem (i.e., a problem of the form (2.1) where there is no objective function to be minimized). Many examples of this methodology are discussed in the current and the next chapter to illustrate how this is exactly done. One valuable common feature among techniques based on convex programming is that if a Lyapunov function of a certain class exists, it will be found. If the problem is infeasible, the variables of the dual program[1] provide a certificate of nonexistence of a Lyapunov function of that class. We will make use of this fact several times in this thesis in situations where we claim nonexistence of Lyapunov functions of a certain class.

2.2 Semidefinite Programming

In almost every example of this thesis a semidefinite program (SDP) will be solved to find a Lyapunov function. Therefore, we devote this section to familiarize the reader with the basics of SDPs.

A semidefinite program is a convex program of the form

$$\begin{array}{ll} \min_x & c^T x \\ \text{subject to} & A_0 + \sum_{i=1}^m x_i A_i \succeq 0, \end{array} \tag{2.2}$$

where $x \in \mathbb{R}^m$ is the decision variable, and $c \in \mathbb{R}^m$ and the $m+1$ symmetric $n \times n$ matrices A_i are given data of the problem. The objective is to minimize a linear function of x subject to matrix positive semidefiniteness constraints. The constraint in (2.2) is called a Linear Matrix Inequality (LMI), and SDP problems are sometimes referred to as LMI problems. The feasible set of an SDP, which is the intersection of the cone of positive semidefinite matrices with an affine subspace, is a convex set. Notice that

[1] Every optimization problem comes with its *dual*, which is another optimization problem where essentially the role of decision variables and constraints have been reversed. Feasible and optimal solutions of each of the problems contain valuable information about the other. The reader is referred to [3] for a comprehensive treatment of duality theory.

linear programs can be interpreted as a special case of semidefinite programs where the matrices A_i are diagonal. Not only SDPs can be solved more or less as efficiently as LPs, but they also provide a much richer framework in terms of applications. A wide range of problems in controls and optimization such as matrix norm inequalities, Lyapunov inequalities, and quadratically constrained quadratic programs can be written as LMIs. Applications are in no way limited to controls but come from a variety of other fields including combinatorial optimization, relaxations of various NP-hard problems, pattern seperation by ellipsoids in statistics, and many more. A great introduction to the theory and applications of semidefinite programming is given in [43].

Depending on the particular problem, the most natural formulation of a semidefinite program may not be in the *standard form* of (2.2). Many semidefinite programming solvers such as SeDuMi [40], YALMIP [23], and SDPT3 [41] are capable of automatically reformulating the constraints in the standard form. What is important is that one should only write equality and matrix inequality constraints that appear *affinely* in the decision variables. For a discussion on several tricks of converting different SDPs to the standard form see again [43].

In the next subsection we explain how one can find a quadratic Lyapunov function for a stable linear system using semidefinite programming.

2.2.1 Linear Systems and Quadratic Lyapunov Functions

Consider the continuous time (CT) and discrete time (DT) linear dynamical systems

$$\dot{x}(t) = Ax(t) \tag{2.3}$$

$$x_{k+1} = Ax_k. \tag{2.4}$$

It is well known that (2.3) is globally asymptotically stable if and only if the matrix A is Hurwitz, and (2.4) is globally asymptotically stable if and only if A is Schur stable. Here, our goal is to prove stability using Lyapunov theory. We choose a quadratic

Lyapunov function candidate of the form

$$V(x) = x^T P x. \tag{2.5}$$

Notice that $V(x)$ has no linear or constant terms. As we discussed in Chapter 1, with no loss of generality we can take $V(0) = 0$, and therefore constant terms are not required. The linear terms are excluded because V is differentiable and achieves its minimum at $x = 0$. Therefore, the gradient of V should vanish at the origin, which would not be the case if V had linear terms. For this quadratic candidate Lyapunov function, after a little bit of algebra one can get

$$\begin{aligned} \dot{V}(x) &= x^T(A^T P + PA)x & (2.6) \\ V_{k+1}(x) - V_k(x) &= x^T(A^T PA - P)x. & (2.7) \end{aligned}$$

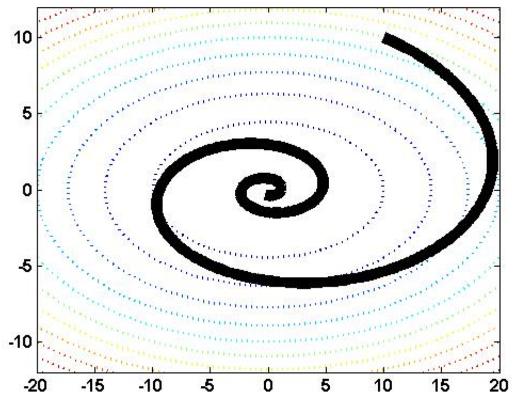
The CT Lyapunov theorem of Chapter 1 (Theorem 1.2.1) suggests that the linear system (2.3) is GAS if there exists a symmetric matrix P such that

$$\begin{aligned} P &\succ 0 \\ A^T P + PA &\prec 0. \end{aligned} \tag{2.8}$$

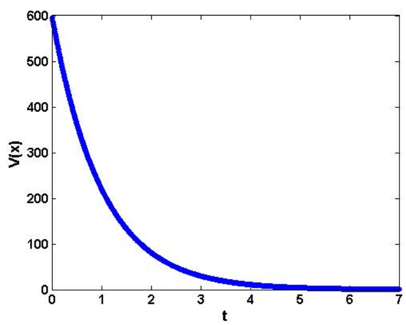
Similarly, Theorem 1.2.2 suggests that the DT linear system (2.4) is GAS if there exists a symmetric matrix P such that

$$\begin{aligned} P &\succ 0 \\ A^T PA - P &\prec 0. \end{aligned} \tag{2.9}$$

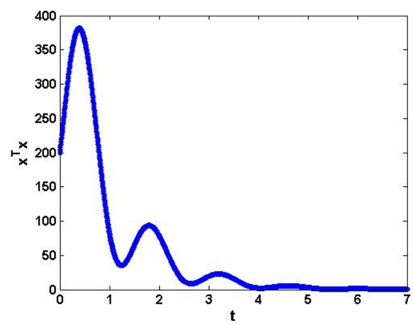
Notice that both (2.8) and (2.9) are semidefinite programs. There is no objective to be minimized (i.e., we have a feasibility problem), and the matrix inequalities are linear in the unknown parameter P. It turns out that (2.8) is feasible if and only if the matrix A is Hurwitz, and (2.9) is feasible if and only if A is Schur stable. In other words, stable linear systems always admit a quadratic Lyapunov function. We omit the proof of this classical result since it can be found in many textbooks; see e.g. [20].



(a) typical trajectory (solid), level sets of the Lyapunov function (dotted)



(b) value of the Lyapunov function on a typical trajectory



(c) square of the Euclidian norm on a typical trajectory

Figure 2-1: Quadratic Lyapunov function for the linear system of Example 2.2.1.

Instead, we give an example to illustrate the geometry.

Example 2.2.1. *Consider a continuous time linear system of the form (2.3) with*

$$A = \begin{bmatrix} -0.5 & 5 \\ -1 & -0.5 \end{bmatrix}.$$

The eigenvalues of A *are* $-\frac{1}{2} \pm j\sqrt{5}$. *Complex eigenvalues with negative real parts tell us that the system exhibits oscillations and all the trajectories converge to the origin. Indeed, the semidefinite program (2.8) can be solved to get a Lyapunov function*[2]

$$P = \begin{bmatrix} 1 & 0 \\ 0 & 5 \end{bmatrix}.$$

Of course, this is not the unique P satisfying (2.8) as we know Lyapunov functions are not necessarily unique. Since $x^T P x$ is a nonnegative quadratic function, it has ellipsoidal level curves. These level sets are plotted in Figure 2-1(a) along with a trajectory of the system starting at $x = (10, 10)^T$. The invariance property of sub-level sets of Lyapunov functions can be seen in this picture. Once the trajectory enters a level set, it never exits. Figure 2-1(b) shows the value of the Lyapunov function along the same trajectory starting at $x = (10, 10)^T$. Notice that the Lyapunov function is decreasing monotonically as expected.

Figure 2-1(c) shows that $V(x) = x^T I x$ is not a valid Lyapunov function as it does not decrease monotonically along the trajectory. Not surprisingly, $P = I$ does not satisfy (2.8) because $A + A^T$ is not negative definite. The level sets of $V(x) = x^T I x$ are circles. Indeed, it is not true that once the trajectory enters a circle of a given radius, it stays in it forever. As we know, linear transformations deform circles into ellipsoids. One can think of Lyapunov theory for linear systems as searching for a right linear coordinate transformation such that in the new coordinates the norm decreases monotonically.

Figure 2-1 also gives us a glimpse of what is to come in the later chapters of this thesis. Even though the square of the norm does not decrease monotonically along trajectories, it still goes to zero in a non-monotonic fashion. In fact, we will see in Chapter 5 that by changing the conditions of the original Lyapunov theorem and using derivatives of higher order, we can prove stability from Figure 2-1(c) (a non-monotonic Lyapunov function) instead of Figure 2-1(b) (a standard Lyapunov function).

[2] We sometimes abuse notation and write P is a Lyapunov function. By that of course we mean that the quadratic function $x^T P x$ is a Lyapunov function.

2.2.2 Some Observations on Linear Systems and Quadratic Lyapunov Functions

Much of what we discussed in the previous section may already be familiar to many readers with basic linear systems theory background. In this section, however, we list some of our observations on quadratic Lyapunov functions for linear systems that are not commonly found in textbooks. Our hope is that the reader will find them interesting.

Lemma 2.2.1. *If A is symmetric and Hurwitz (Schur stable), then $P = I$ satisfies (2.8) ((2.9)). (i.e., the square of the Euclidean norm (and therefore the Euclidian norm itself) is a Lyapunov function for the corresponding continuous time (discrete time) linear system.)*

Proof.

- In CT: with $P = I$, the decrease condition of (2.8) reduces to $A+A^T \prec 0$. since A is symmetric and Hurwitz, $A + A^T = 2A$ is also Hurwitz and therefore negative definite.

- In DT: with $P = I$ and by symmetry of A, the decrease condition of (2.9) reduces to $A^2 - I \prec 0$. Because A is Schur stable,

$$|\lambda_{max}(A)| < 1,$$

which imples

$$|\lambda_{max}(A^2)| = |\lambda_{max}^2(A)| < 1.$$

Therefore, $A^2 - I \prec 0$.

□

The lemma we just proved also makes intuitive sense. If A is symmetric, then it has real eigenvalues and eigenvectors. The trajectory starting from any point on the eigenvectors will stay on it and go directly towards the origin. At any other point in the space, the trajectory is pulled towards the eigenvectors and moves with an orientation towards origin. There are no oscillations to increase the norm at any point. The next lemma generalizes the same result to normal matrices, which do not necessarily have real eigenvalues.

Lemma 2.2.2. *If A is normal*[3]*, then $P = I$ satisfies (2.8).*

Proof. We need the following two facts before we start the proof:

- **Fact 1.** If A_1 and A_2 commute and are both Hurwitz, then there exists a common quadratic Lyapunov function for both of them. This means that $\exists P$ such that

$$A_1^T P + PA_1 \prec 0 \quad (2.10)$$

$$A_2^T P + PA_2 \prec 0 \quad (2.11)$$

  See [14] and references therein for a proof of this fact and other conditions for existence of a common Lyapunov function.

- **Fact 2.** If P is a common Lyapunov function for general matrices A_1 and A_2 (which may not necessarily commute), then P is also a Lyapunov function for any convex combination of A_1 and A_2. In other words, for any $\lambda \in [0, 1]$

$$(\lambda A_1^T + (1-\lambda)A_2^T)P + P(\lambda A_1 + (1-\lambda)A_2) \prec 0. \quad (2.12)$$

  This follows by multiplying (2.10) by λ, (2.11) by $(1 - \lambda)$, and adding them up.

Now we can proceed with the proof of Lemma 2.2.2. We know A^T has the same eigenvalues as A. Therefore, A^T is also Hurwitz. Since A and A^T commute, Fact 1 implies that there exists a common Lyapunov function P. By Fact 2 with $\lambda = \frac{1}{2}$, P is also a Lyapunov function for $A + A^T$. Therefore $A + A^T$ must be a Hurwitz matrix and hence negative definite. This implies that I is a Lyapunov function for $\dot{x} = Ax$. □

Lemma 2.2.3. *If A is a Schur stable matrix, then there exists m such that I is a Lyapunov function for the DT dynamical system $x_{k+1} = A^m x_k$*

[3] A normal matrix is a matrix that satisfies $A^T A = AA^T$

The point of this lemma is that even though I may not be a Lyapunov function for $x_{k+1} = Ax_k$, for every stable DT linear system, there exists a fixed m such that if you look at the trajectory every m iterations, the norm is monotonically decreasing.

Proof. (of Lemma 2.2.3) Recall the following characterization of the spectral radius of the matrix A:

$$\rho(A) = \lim_{k \to \infty} ||A^k||^{\frac{1}{k}}, \tag{2.13}$$

where the value of $\rho(A)$ is independent of the matrix norm used in (2.13). For this proof, we take the matrix norm $||.||$ to be the induced 2-norm. Since A is Schur stable, we must have

$$\rho(A) < 1.$$

We claim that there exists m such that

$$||A^m|| < 1.$$

Indeed, if this was not the case we would have $||A^k|| \geq 1 \ \forall k$, and hence $||A^k||^{\frac{1}{k}} \geq 1$ $\forall k$. By definition (2.13), this would contradict $\rho(A) < 1$.

The fact that $||A^m|| < 1$ means that the largest singular value of A^m is less than unity, and therefore

$$A^{m^T} A^m - I \prec 0.$$

By (2.9), the last inequality implies that I is a Lyapunov function for the dynamics $x_{k+1} = A^m x_k$. □

Lemma 2.2.4. *If A is Hurwitz, any P satisfying $A^T P + PA \prec 0$ will automatically satisfy $P \succ 0$.*

This lemma suggests that to find a Lyapunov function for a Hurwitz matrix, it suffices to impose only the second inequality in (2.8), i.e. the first inequality is redundant. It is possible to prove this lemma using only linear algebra. However, we give a much simpler proof based on a dynamical systems intuition. This is one instance where one notices the power of Lyapunov theory. In fact, one can reverse engineer

Lyapunov's theorem for linear systems to get many interesting theorems in linear algebra. We will mention some more instances of this in future chapters.

Proof. (of Lemma 2.2.4) Suppose there exists a matrix P that satisfies

$$A^T P + PA \prec 0, \tag{2.14}$$

but it is not positive definite. Therefore, there exists $\bar{x} \in \mathbb{R}^n$, $\bar{x} \neq 0$, such that $\bar{x}^T P \bar{x} \leq 0$. We evaluate the Lyapunov function $x^T P x$ along the trajectories of the system $\dot{x} = Ax$ starting from the initial condition $\bar{x}$. The value of the Lyapunov function is nonpositive to begin with and will strictly decrease because of (2.14). Therefore, the Lyapunov function can never be zero again, contradicting asymptotic stability of the dynamics. □

Lemma 2.2.5. *If A is Schur stable, any P satisfying $A^T PA - P \prec 0$ will automatically satisfy $P \succ 0$.*

Proof. This lemma is the discrete time analog of Lemma 2.2.4. The proofs are identical. □

2.3 $\mathcal{S}$-procedure

In many circumstances, one would like to impose nonnegativity of a quadratic form not on the whole space, but maybe only on specific regions of the space. A technique known as the *$\mathcal{S}$-procedure* enables us to do that. This technique will come in handy in particular in Section 4.3.2 when we analyze stability of switched linear systems.

What the $\mathcal{S}$-procedure allows us to do is to impose nonnegativity of a quadratic function whenever some other quadratic functions are nonnegative. Given

$$\sigma_i(x) = x^T Q_i x + L_i x + c_i \quad i = 0, \ldots, k, \tag{2.15}$$

suppose we are interested in the following condition

$$\sigma_0(x) \geq 0 \quad \forall x \text{ such that } \sigma_i(x) \geq 0 \; i = 1, \ldots, k. \tag{2.16}$$

If there exists nonnegative scalars $\tau_i, \; i = 1, \ldots, k$ such that

$$\sigma_0(x) \geq \sum_{i=1}^{k} \tau_i \sigma_i(x) \quad \forall x, \tag{2.17}$$

then (2.16) holds. This implication is obvious. What is less trivial is that the converse is also true when $k = 1$ provided there exists $\bar{x}$ such that $\sigma_1(\bar{x}) > 0$. For a proof of this fact and more details on $\mathcal{S}$-procedure the reader is referred to [10].

If we are searching for a quadratic function σ_0 that must be nonnegative on a region $\mathcal{R} \subset \mathbb{R}^n$, we can try to describe $\mathcal{R}$ as the set where some other quadratic functions σ_i are nonnegative and then imposes the constraint (2.17). Notice that once the functions σ_i are fixed, the inequality in (2.17) is linear in the decision variables σ_0 and τ_i. Therefore we can perform the search via a semidefinite program after converting (2.17) to an LMI.

2.4 Sum of Squares Programming

When the candidate Lyapunov function is polynomial or when the dynamical system is described by polynomial equations, conditions of Lyapunov's theorem reduce to checking nonnegativity of certain polynomials on the whole space. This problem is known to be NP-hard even for polynomials of degree 4 [28]. A tractable sufficient condition for global nonnegativity of a polynomial function is the existence of a sum of squares (SOS) decomposition. We postpone the stability analysis of polynomial systems until the next chapter. In this section we review basics of sum of squares programming, which was introduced in 2000 [27] and has found many applications since.

A multivariate polynomial $p(x_1, ..., x_n) := p(x)$ is a sum of squares, if there exist polynomials $q_1(x), ..., q_m(x)$ such that

$$p(x) = \sum_{i=1}^{m} q_i^2(x). \tag{2.18}$$

It is clear that $p(x)$ being SOS implies $p(x) \geq 0$. In 1888, Hilbert proved that the converse is true for a polynomial in n variables of degree $2d$ *only* in the following cases:

- Univariate polynomials ($n = 1$)
- Quadratic polynomials ($2d = 2$)
- Bivariate quartics ($n = 2,\ 2d = 4$)

In all other cases there are counter examples of nonnegative polynomials that are not sum of squares. Many such counter examples can be found in [35]. Unlike nonnegativity however, it was shown in [27] that the search for an SOS decomposition of a polynomial can be cast as an SDP, which we know how to solve efficiently in polynomial time. The result is summarized in the following theorem.

Theorem 2.4.1. *([27], [28]) A multivariate polynomial $p(x)$ in n variables and of degree $2d$ is a sum of squares if and only if there exists a positive semidefinite matrix Q (often called the Gram matrix) such that*

$$p(x) = z^T Q z, \tag{2.19}$$

where z is the vector of monomials of degree up to d

$$z = [1, x_1, x_2, \ldots, x_n, x_1 x_2, \ldots, x_n^d]. \tag{2.20}$$

Notice that given $p(x)$, the search for the matrix Q is a semidefinite program. By expanding the right hand side of (2.19) and matching coefficients of x, we get linear constraints on the entries of Q. We also have the constraint that Q must be positive semidefinite (PSD). Therefore, the feasible set is the intersection of an affine subspace with the cone of PSD matrices. As described in Section 2.2, this is exactly the structure of the feasible set of an SDP.

The size of the matrix Q depends on the size of the vector of monomials. When there is no sparsity to be exploited Q will be $\binom{n+d}{d} \times \binom{n+d}{d}$. If the polynomial $p(x)$ is homogeneous of degree $2d$ (i.e., only has terms of degree exactly $2d$), then it suffices to consider in (2.19) a vector z of monomials of degree exactly d [28]. This will reduce the size of Q to $\binom{n+d-1}{d} \times \binom{n+d-1}{d}$.

The conversion step of going from an SOS decomposition problem to an SDP problem is fully algorithmic and has been implemented in the SOSTOOLS [33] software package. We can input a polynomial $p(x)$ into SOSTOOLS and if the code is feasible, a Cholesky factorization of Q will give us an explicit SOS decomposition of $p(x)$. If the code is infeasible, we have a certificate that $p(x)$ is not a sum of squares (thought it might still be nonnegative). Moreover, using the same methodology, we can *search* for SOS polynomials or even optimize linear functionals over them.

Chapter 3

Polynomial Systems and Polynomial Lyapunov Functions

The evolution of many dynamical systems around us is most naturally modelled as polynomials. Examples include variety of chemical reactions, predator-pray models, and nonlinear electrical circuits. In contrary to linear systems, polynomial systems can exhibit significantly more complicated dynamics. Even in one dimension and with a polynomial of degree 2, it is possible to observe chaotic behavior; see e.g. the logistic map in [1]. Not surprisingly, proving stability of polynomial systems is a much more challenging task. In this chapter, we explain how the machinery of sum of squares programming that we introduced in the previous chapter can be utilized to efficiently search for polynomial Lyapunov functions for polynomial systems.

3.1 Continuous Time Case

Consider the dynamical system

$$\dot{x} = f(x), \tag{3.1}$$

where each of the elements of the vector valued mapping $f : \mathbb{R}^n \to \mathbb{R}^n$ is a multivariate polynomial. We say that f has degree d when the highest degree appearing in all of the n polynomials is d. For the polynomial dynamics in (3.1), it is natural to search for Lyapunov functions that are polynomial themselves, though it is not clear that polynomial dynamics always admit a global polynomial Lyapunov function. It is known, however, that exponentially stable nonlinear systems (not necessarily polynomial) have a polynomial Lyapunov function on bounded regions [31]. This fact is perhaps not surprising since we know any function can be approximated by polynomials arbitrarily well on compact sets. A more practical difficulty that arises for stability analysis of polynomial systems is that given the degree of the vector field f, no upper bounds on the degree of the Lyapunov function are known in general. There is yet a third obstacle, which we already touched on in Section 2.4. Namely, the conditions of Lyapunov's theorem (Theorem 1.2.1) for polynomial dynamics lead to checking nonnegativity of polynomials on the whole space; a problem known to be NP-hard. Recall from Chapter 1 that the Lyapunov function V must be continuous, radially unbounded, and must satisfy

$$\begin{aligned} V &> 0 \quad \forall x \neq 0 && (3.2)\\ \langle \frac{\partial V}{\partial x}, f \rangle &< 0 \quad \forall x \neq 0. && (3.3) \end{aligned}$$

When $V(x)$ is a polynomial, continuity is obviously satisfied. Moreover, we will require the degree of V to be even, which is a sufficient condition for radially unboundedness and a necessary one for positivity. As we discussed in section 2.4, we relax the positivity constraints to the more tractable condition

$$\begin{aligned} V &\quad SOS && (3.4)\\ -\langle \frac{\partial V}{\partial x}, f \rangle &\quad SOS. && (3.5) \end{aligned}$$

Note that the unknown polynomial V appears linearly in both (3.4) and (3.5). Therefore, for a V of fixed degree, we can perform the search by solving an SOS program. Usually, the approach is to start with a low degree candidate Lyapunov function, say degree 2, and increase the degree to the next even power every time the search is infeasible. For reasons that we discussed in previous chapters, we can always exclude constant and linear terms in the parametrization of V.

Another point that is worth mentioning is that SOS conditions of (3.4) and (3.5) imply nonnegativity, whereas the conditions (3.2) and (3.3) require *strict* positivity. For this reason, some authors [32] have proposed conditions of the type

$$V - \varepsilon \sum_{i=1}^{n} x_i^q \quad SOS, \tag{3.6}$$

and a similar expression for the derivative (3.5). Here, ε is a fixed small positive number and q is degree of V. Conditions of this kind can often be conservative in practice, and we claim that they are usually not needed. Sum of squares polynomials that vanish at some points in space lie on the boundary of the cone of SOS polynomials. When an interior point algorithm is used to solve a feasibility problem, it will aim for the *analytic center* [25] of the feasible set, which is away from the boundary. So, unless the original problem is only marginally feasible, conditions (3.4) and (3.5) will automatically imply strict positivity. One can (and should) always do some post-processing analysis and check that the polynomials obtained from SOSTOOLS are positive definite. This can be done, for instance, by checking the eigenvalues of the corresponding Grammian matrix as introduced in Section 2.4.

Sum of squares relaxation has shown to be a powerful technique for finding polynomial Lyapunov functions over the past few years. Several examples can be found in [30], [26], and [32]. Below, we give an example of our own to illustrate the procedure more concretely and develop a geometric intuition.

Example 3.1.1. *Consider the dynamical system*

$$\begin{aligned} \dot{x_1} &= -0.15x_1^7 + 200x_1^6x_2 - 10.5x_1^5x_2^2 - 807x_1^4x_2^3 + 14x_1^3x_2^4 + 600x_1^2x_2^5 - 3.5x_1x_2^6 + 9x_2^7 \\ \dot{x_2} &= -9x_1^7 - 3.5x_1^6x_2 - 600x_1^5x_2^2 + 14x_1^4x_2^3 + 807x_1^3x_2^4 - 10.5x_1^2x_2^5 - 200x_1x_2^6 - 0.15x_2^7 \end{aligned} \tag{3.7}$$

We would like to establish global asymptotic stability of the origin by searching for a polynomial Lyapunov function V. Since the vector field is homogeneous, we can restrict our search to homogeneous Lyapunov functions [37]. The corresponding SOS

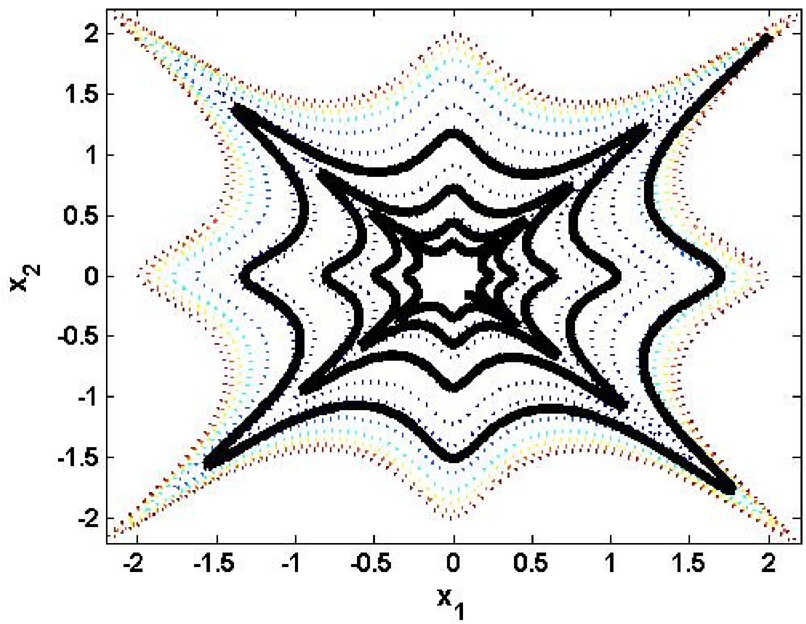


Figure 3-1: A typical trajectory of Example 3.1.1 (solid), level sets of a degree 8 Lyapunov function (dotted).

program is infeasible for a V *of degree* 2, 4, *and* 6. *The infeasibility of the underlying semidefinite program gives us a certificate that in fact no polynomial of degree less than or equal to* 6 *satisfies (3.4) and (3.5). We finally increase the degree to* 8 *and SOSTOOLS and SDP solver SeDuMi find the following Lyapunov function*

$$\begin{aligned} V &= 0.02x_1^8 + 0.015x_1^7x_2 + 1.743x_1^6x_2^2 - 0.106x_1^5x_2^3 - 3.517x_1^4x_2^4 \\ &\quad +0.106x_1^3x_2^5 + 1.743x_1^2x_2^6 - 0.015x_1x_2^7 + 0.02x_2^8. \end{aligned} \tag{3.8}$$

Figure 3-1 shows a trajectory of (3.7) starting from the point $(2,2)^T$. *Level sets of the Lyapunov function (3.8) are also plotted with dotted lines. Note that the trajectory stretches out in* 8 *different directions as it moves towards the origin. We know that level sets of a Lyapunov function should have the property that once the trajectory enters them, it never exits. For this reason, we expect the level sets to also have* 8 *preferred directions. At an intuitive level, this explains why we were unable to find a Lyapunov function of lower degree. This suggests that for this particular example,* *we should not blame our failure of finding lower degree Lyapunov functions on the conservatism of SOS relaxation (i.e. the gap between SOS and nonnegativity).*

3.2 How Conservative is the SOS Relaxation for Finding Lyapunov Functions?

An interesting and natural question in the study of polynomial Lyapunov functions is to investigate how significant the gap between SOS and nonnegativity can be in terms of existence of Lyapunov functions. Is it true that whenever a polynomial Lyapunov function of a certain degree exists, one can always take it to be a sum of squares? Is it true that when a positive polynomial is a valid Lyapunov function, there exists a nearby SOS polynomial of maybe slightly higher degree that is also a valid Lyapunov function? Or is it the case that some family of dynamical systems naturally admit polynomial Lyapunov functions that are not SOS?

Because SOS relaxation is the main tool that we currently have for finding polynomial Lyapunov functions, it is important to know the answer to these questions. To the best knowledge of the author, no one has yet carefully investigated this topic. In this section, we will make a small first step in this direction by giving an explicit example where a valid polynomial Lyapunov function is not detected through SOS programming.

Recall that every time we search for a Lyapunov function through the methodology we described, we use the SOS relaxation twice. First for nonnegativity of V, and second for nonpositivity of $\dot{V}$. Each of these relaxations may in general be conservative. Many of the well known counter examples of nonnegative functions that are not SOS have local minimas [35]. On the other hand, we know that in continuous time, Lyapunov functions cannot have local minimas. The reason is that if a trajectory starts exactly at the local minima of V, irrespective of the direction in which f moves the trajectory, the Lyapunov function will locally increase[1]. As a consequence, many

[1] In future chapters of this thesis, we will introduce non-monotonic Lyapunov functions. These Lyapunov functions are allowed to increase locally and therefore can, in theory, have local minimas.

of the well known nonnegative polynomials that are not SOS cannot be a Lyapunov function. However, the following example shows that an SOS relaxation on $\dot{V}$ can be conservative.

Example 3.2.1. *Consider the dynamical system*

$$\begin{aligned}\dot{x_1} &= -x_1^3x_2^2 + 2x_1^3x_2 - x_1^3 + 4x_1^2x_2^2 - 8x_1^2x_2 + 4x_1^2 - x_1x_2^4 + 4x_1x_2^3 - 4x_1 + 10x_2^2 \\ \dot{x_2} &= -9x_1^2x_2 + 10x_1^2 + 2x_1x_2^3 - 8x_1x_2^2 - 4x_1 - x_2^3 + 4x_2^2 - 4x_2.\end{aligned} \tag{3.9}$$

One can verify that the origin is the only equilibrium point for this system, and therefore it makes sense to investigate global asymptotic stability. If we search for a quadratic Lyapunov function for (3.9) using SOS programming, we will not find one. Therefore, there exists no quadratic Lyapunov function whose decrease rate satisfies (3.5). Nevertheless, we claim that

$$V = \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2 \tag{3.10}$$

is a valid Lyapunov function. Indeed, one can check that

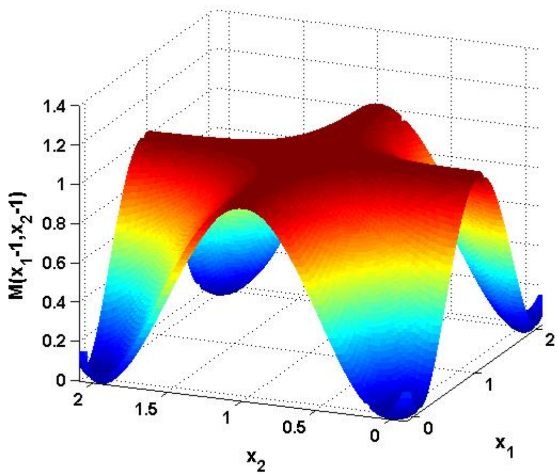
$$\dot{V} = x_1\dot{x_1} + x_2\dot{x_2} = -M(x_1 - 1, x_2 - 1), \tag{3.11}$$

where $M(x_1, x_2)$ is the famous Motzkin polynomial

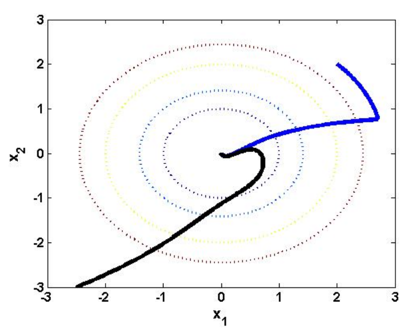
$$M(x_1, x_2) = x_1^4x_2^2 + x_1^2x_2^4 - 3x_1^2x_2^2 + 1. \tag{3.12}$$

This polynomial is known to be nonnegative but not SOS [35]. $\dot{V}$ is strictly negative everywhere in the space, except for the origin and three other points $(0,2)^T$, $(2,0)^T$, and $(2,2)^T$, where $\dot{V}$ is zero. However, at each of these three points we have $\dot{x} \neq 0$. Once the trajectory reaches any of these three points, it will be kicked out to a region where $\dot{V}$ is strictly negative. Therefore, by LaSalle's invariance principle [20], the quadratic Lyapunov function in (3.10) proves GAS of the origin of (3.9).

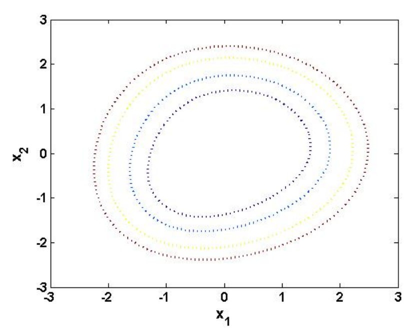
The fact that $\dot{V}$ is zero at three points other than the origin is not the reason



(a) Shifted Motzkin polynomial is nonnegative but not SOS.



(b) Typical trajectories of (3.9) (solid), level sets of V (dotted).



(c) Level sets of a quartic Lyapunov function found through SOS programming.

Figure 3-2: Example 3.2.1. The quadratic polynomial $\frac{1}{2}x_1^2 + \frac{1}{2}x_2^2$ is a Lyapunov function but it is not detected through SOS programming.

why SOS programming is failing. After all, when we impose the condition that $-\dot{V}$ *should be SOS, we allow for the possibility of a non-strict inequality. The reason why our SOS program does not recognize (3.10) as a Lyapunov function is that the shifted Motzkin polynomial in (3.11) is nonnegative but it is not a sum of squares. This sextic polynomial is plotted in Figure 3-2(a). Trajectories of (3.9) starting at* $(2,2)^T$ *and* $(-2.5,-3)^T$ *along with level sets of* V *are shown in Figure 3-2(b).*

If we increase the degree of the candidate Lyapunov function from 2 *to* 4*, SOS-TOOLS succeeds in finding a quartic Lyapunov function*

$$\begin{aligned} W &= 0.08x_1^4 - 0.04x_1^3 + 0.13x_1^2x_2^2 + 0.03x_1^2x_2 + 0.13x_1^2 \\ &\quad +0.04x_1x_2^2 - 0.15x_1x_2 + 0.07x_2^4 - 0.01x_2^3 + 0.12x_2^2. \end{aligned} \tag{3.13}$$

The level sets of this function are close to circles and are plotted in Figure 3-2(c).

We should mention that the example we just described was contrived to make our point. For many practical problems, SOS programming has provably shown to be a powerful technique [30], [26], [32]. There are some recent results [5], however, that show for a fixed degree, as the dimension goes up the gap between nonnegativity and SOS broadens. The extent to which this can impact existence of Lyapunov functions in higher dimensions is yet to be investigated.

3.3 Discrete Time Case

In this section we consider a dynamical system of the type

$$x_{k+1} = f(x_k), \tag{3.14}$$

where f is again a multivariate polynomial. All of the methodology developed in the continuous time case carries over in a straightforward manner to the discrete time case. We will once again replace the conditions of Lyapunov's theorem

$$\begin{aligned} V(x) &> 0 \quad \forall x \neq 0 & (3.15)\\ V(f(x)) - V(x) &< 0 \quad \forall x \neq 0. & (3.16)\end{aligned}$$

with the more tractable conditions

$$\begin{aligned} V & \quad SOS & (3.17)\\ -(V(f(x)) - V(x)) & \quad SOS. & (3.18)\end{aligned}$$

Since f is fixed, condition (3.18) is still linear in the coefficients of the decision variable V. Therefore, once the degree of the candidate Lyapunov function is fixed, we can search for a V that satisfies (3.17) and (3.18) via a sum of squares program.

Examples of discrete time polynomial systems that do not admit a quadratic Lyapunov function but have a higher order Lyapunov function seem to be missing from the literature. For completeness, we give one such example.

Example 3.3.1. *Consider the dynamical system*

$$x_{k+1} = f(x_k),$$

with

$$f = \begin{pmatrix} \frac{1}{2}x_1 - x_1^2 - x_2 \\ \frac{1}{2}x_1^2 + \frac{1}{2}x_2 - (\frac{1}{2}x_1 - x_1^2 - x_2)^2. \end{pmatrix} \tag{3.19}$$

No quadratic Lyapunov function is found using SOS relaxation. However, an SOS quartic Lyapunov function exists:

$$\begin{aligned} V \;=\; & 1.976x_1^4 - 0.012x_1^3x_2 - 0.336x_1^3 + 0.001x_1^2x_2^2 + 4.011x_1^2x_2 \\ & +0.680x_1^2 - 0.012x_1x_2^2 - 0.360x_1x_2 + 0.0002x_2^3 + 2.033x_2^2. \end{aligned} \tag{3.20}$$

A level set of this Lyapunov function along with two trajectories is shown in Figure 3-3.

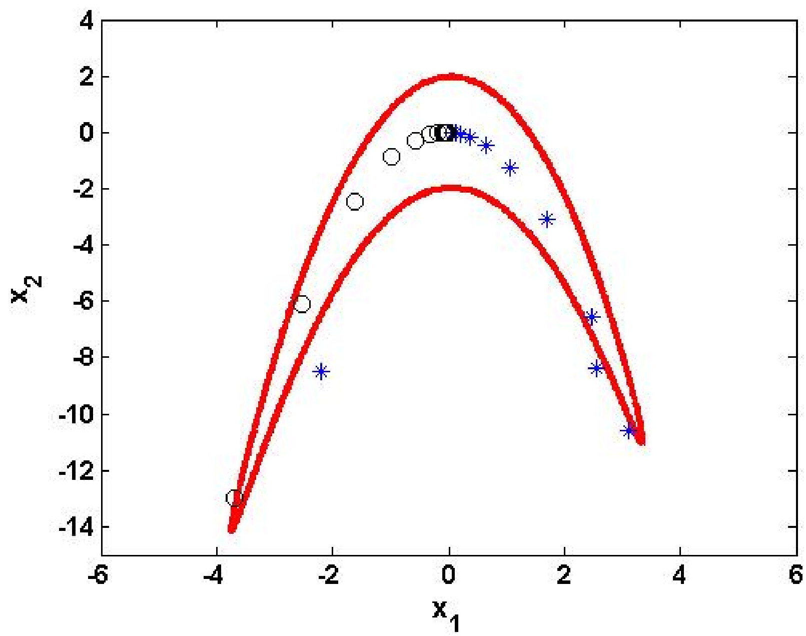


Figure 3-3: Trajectories of Example 3.3.1 and a level set of a quartic Lyapunov function.

Chapter 4

Non-Monotonic Lyapunov Functions in Discrete Time

4.1 Motivation

Despite all the recent positive progress in Lyapunov theory due to availability of techniques based on convex programming, it is not too difficult to find stable systems where most of the techniques fail to find a Lyapunov function. Even if one is found, in many situations, the structure of the Lyapunov function can be very complicated. This setback encourages one to think whether the conditions of Lyapunov's theorem are overly conservative.

The rest of this thesis will address the following natural question: if it is enough to show $V \to 0$ as $k \to \infty$, why should we require V to decrease monotonically? It is perhaps not immediate to see whether relaxing monotonicity would help simplify the structure of Lyapunov functions. Figure 4-1 explains why we would conceptually expect this to happen. In the top part of the figure, a hypothetical trajectory is plotted along with a level curve of a candidate Lyapunov function. The problem is that a simple dynamics f (e.g., polynomial of low degree) can produce such trajectory. However, a Lyapunov function V with such level curve must be very complicated (e.g., polynomial of high degree). On the other hand, much simpler functions (maybe even a quadratic) can decrease in a non-monotonic fashion as plotted in the bottom right.

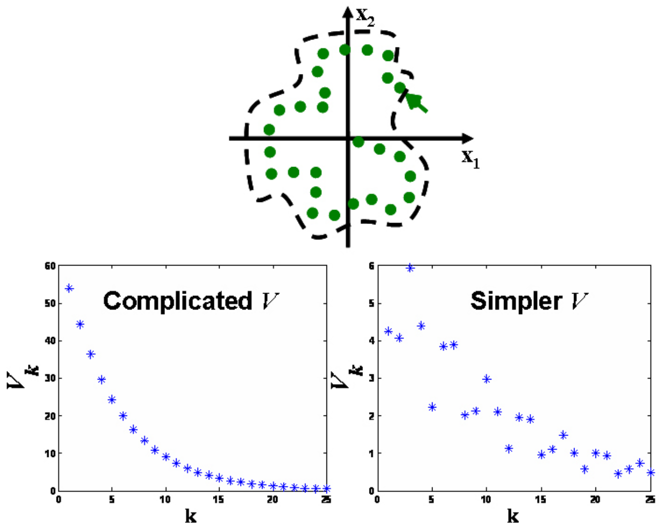


Figure 4-1: Motivation for relaxing monotonicity. Level curves of a standard Lyapunov function can be complicated. Simpler functions can decrease on average every few steps.

Later in the chapter, we will verify this intuition with specific examples.

Throughout this chapter, we will be concerned with a discrete time dynamical system

$$x_{k+1} = f(x_k). \tag{4.1}$$

It is assumed that the origin is the unique equilibrium point of (4.1) and our goal is to prove global asymptotic stability (GAS). We will relax Lyapunov's requirement $V_{k+1} < V_k$ by presenting new conditions that allow Lyapunov functions to increase locally but yet guarantee that in the limit they converges to zero. We will pay special attention to writing conditions that can be checked by a convex program.

The organization of this chapter is as follows. In Section 4.2 we present our theorems and give some interpretations. In Section 4.3.1, we apply our results to polynomial systems by using SOS programming. Section 4.3.2 analyzes stability of piecewise affine systems. In Section 4.3.3, we use non-monotonic Lyapunov functions to find upper bounds on the joint spectral radius of a finite set of matrices. Throughout Section 4.3, we draw comparisons with techniques based on standard Lyapunov theory.

4.2 Non-monotonic Lyapunov Functions

In this section we state our main results which are comprised of two sufficient conditions for global asymptotic stability. Both theorems impose conditions on higher order differences of Lyapunov functions. For clarity, we state our theorems with formulations that only use up to a two-step difference. The generalized versions are presented as corollaries.

4.2.1 The Non-Convex Formulation

Our first theorem has a non-convex formulation and it will turn out to be a special case of our second theorem. On the other hand, it allows for an intuitive interpretation of relaxing the monotonicity requirement $V_{k+1} < V_k$. For this reason, we present it as a motivation.

Theorem 4.2.1. *Consider the dynamical system (4.1). If there exists a scalar $\tau \geq 0$, and a continuous radially unbounded function $V : \mathbb{R}^n \to \mathbb{R}$, such that*

$$\begin{aligned} V(x) > 0 \quad \forall x \neq 0 \\ V(0) = 0 \\ \tau(V_{k+2} - V_k) + (V_{k+1} - V_k) < 0 \end{aligned} \tag{4.2}$$

then the origin is a GAS equilibrium of (4.1).

Note that we have a product of decision variables V and τ in (4.2). Therefore, this condition cannot be checked via an SDP. We shall overcome this problem in the next subsection. But for now, our approach will be to fix τ through a binary search, and then search for V.

Before we provide a proof of the theorem, we shall give an interpretation of condition (4.2). When $\tau = 0$, we recover Lyapunov's theorem. For $\tau > 0$, condition (4.2) requires a weighted average of the improvement in one step and the improvement in two steps to be negative. Meaning that V has to decrease on average every two steps. This allows the Lyapunov function to increase in one step (i.e. $V_{k+1} > V_k$), as long as the improvement in two steps is negative enough. Similarly, at some other points in space, we may have $V_{k+2} > V_k$ when there is enough decrease in the first step. The special case of $\tau = 1$ has a nice interpretation. In this case (4.2) reduces to

$$V_k > \frac{1}{2}(V_{k+1} + V_{k+2}),$$

i.e., at every point in time, the value of the Lyapunov function should be more than the average of the value at the next two future steps. It should intuitively be clear that condition (4.2) should imply $V_k \to 0$ as $k \to \infty$. The formal proof is as follows.

Proof. (of Theorem 4.2.1) Consider the sequence $\{V_k\}$. For any given V_k, (4.2) and the fact that $\tau \geq 0$ imply that either V_{k+1} or V_{k+2} should be strictly less than V_k. Therefore, there exists a subsequence of $\{V_k\}$ that is monotonically decreasing. Since the subsequence is lower bounded by zero, it must converge to some $c \geq 0$. It can be shown (for e.g. by contradiction) that because of continuity of $V(x)$, c must be zero. This part of the proof is similar to the proof of standard Lyapunov theory (see e.g. [20]). Now that we have established a converging subsequence, for any $\varepsilon > 0$, we can find $\bar{k}$ such that $V_{\bar{k}} < \min\{\frac{\varepsilon}{1+\tau}, \frac{\tau\varepsilon}{1+\tau}\}$. Because of positivity of V and condition (4.2), we have $V_k < \varepsilon \; \forall k > \bar{k}$. Therefore, $V_k \to 0$, which implies $x \to 0$. □

We shall provide an alternative proof in Section 4.2.2 for the more general theorem. Note that by construction, Theorem 4.2.1 should work better than requiring $V_{k+1} < V_k$ ($\tau = 0$) and $V_{k+2} < V_k$ (τ large). The following example illustrates that the improvement can be significant.

Example 4.2.1. *(piecewise linear system in one dimension) Consider the piecewise linear dynamical system:*

$$x_{k+1} = f(x_k)$$

with

$$f = \begin{cases} A_1 x & |x| \in R_1 = [9, \infty) \\ A_2 x & |x| \in R_2 = [7, 9) \\ A_3 x & |x| \in R_3 = [6, 7) \\ A_4 x & |x| \in R_4 = [0, 6) \end{cases}$$

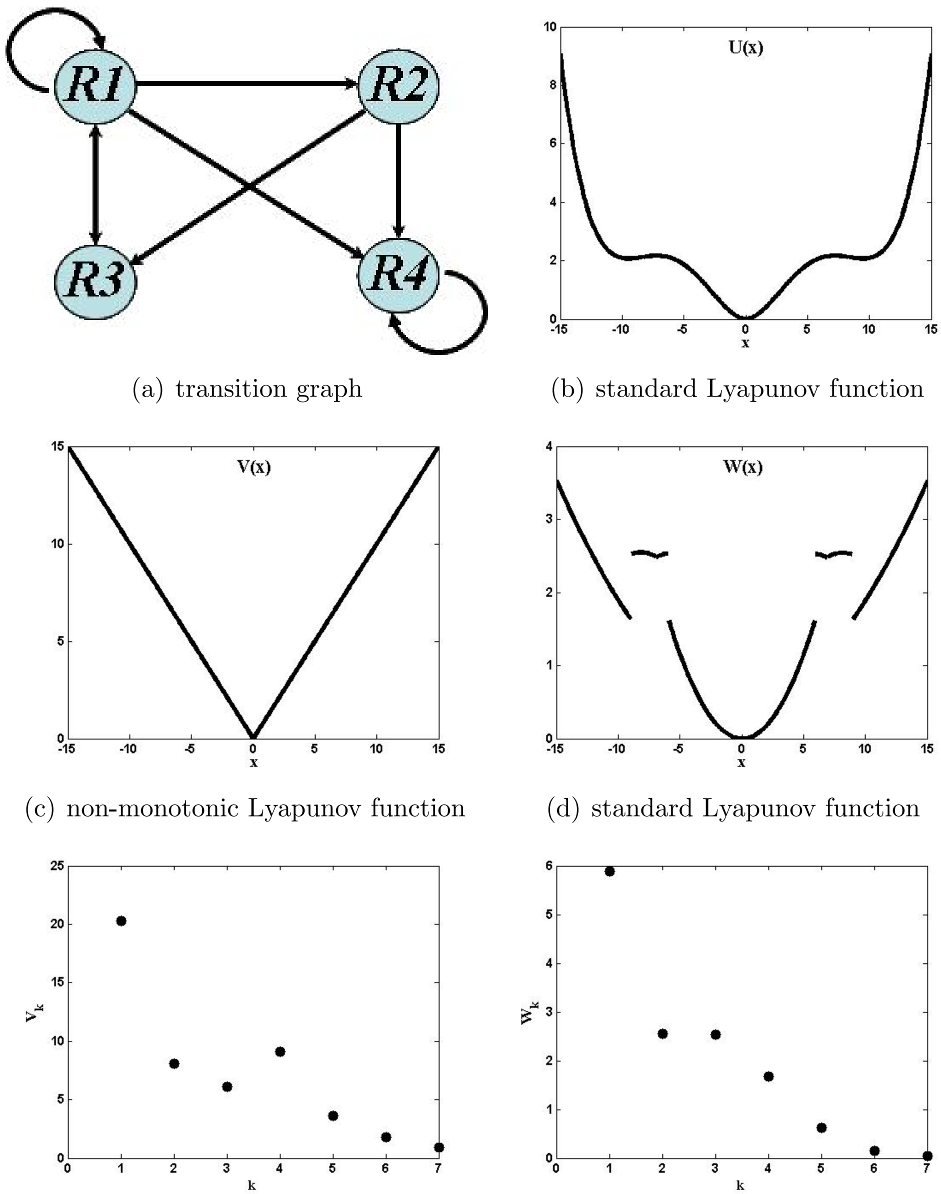
where $A_1 = \frac{2}{5}$, $A_2 = \frac{3}{4}$, $A_3 = \frac{3}{2}$, *and* $A_4 = \frac{1}{2}$.

We would like to establish global asymptotic stability using Lyapunov theory. Since f *is odd, it suffices to find a Lyapunov function for half of the space (e.g.,* $x \geq 0$*) and use its mirror image on the other half space. Figure 4-2(a) illustrates the possible switchings among the four regions. Note that* $A_3 > 1$ *and* $A_3 A_2 > 1$*. We claim that no quadratic Lyapunov function exists. Moreover, no quadratic function can satisfy* $V_{k+2} < V_k$*. These facts can easily be seen by noting that any positive definite quadratic function will increase if the trajectory moves away from the origin. Therefore, transitions* A_3 *and* $A_3 A_2$ *respectively reject the existence of a quadratic Lyapunov function that would decrease monotonically in one or two steps.*

In order to satisfy the monotonic decrease of Lyapunov's theorem, we should search for functions that are more complicated than quadratics. Figure 4-2(b) and 4-2(d) illustrate two such functions. The first function, U*, is a polynomial of degree* 4 *(on the nonnegative half-space) found through SOS programming. The second function,* W*, is a piecewise quadratic with four pieces that is obtained by solving an SDP. Figure 4-2(f) shows the value of* W *on a trajectory that starts in* R_1*, visits* R_2*,* R_3*,* R_1*,* R_4*, and stays in* R_4 *before it converges to the origin. The corresponding plot for* U *is omitted to save space.*

Next, we apply Theorem 4.2.1 to prove stability. As shown in Figure 4-2(c), we can simply take V *to be a linear function with slope of 1 on the positive half-space. This* V *along with any* $\tau \in (1.25, 2)$ *satisfies (4.2). Figure 4-2(e) shows the value of* V *on the same trajectory described before. Even though from* $k = 2$ *to* $k = 4$ V *is increasing, at any point in time condition (4.2) is satisfied.*

This example clearly demonstrates that relaxing monotonicity can simplify the structure of Lyapunov functions. From a computational point of view, the search



(a) transition graph
(b) standard Lyapunov function
(c) non-monotonic Lyapunov function
(d) standard Lyapunov function
(e) $\tau(V_{k+2} - V_k) + (V_{k+1} - V_k) < 0$
(f) $W_{k+1} < W_k$

Figure 4-2: Comparison between non-monotonic and standard Lyapunov functions for Example 4.2.1. The non-monotonic Lyapunov function has a simpler structure and therefore fewer decision variables.

for the non-monotonic Lyapunov function only had 2 *decision variables: the slope of the line in* V *and the value of* τ. *On the other hand, each of the four quadratic pieces of* W *have three free parameters. If we take advantage of the fact that the piece containing the origin should have no constant or linear terms, we end up with a total of* 10 *decision variables. As we shall see in Section 4.3.2, both methods will have the same number of constraints. The quartic polynomial* U *has no constant or linear terms and therefore has* 3 *decision parameters. However, as the dimension of the space goes up, the difference between the number of free parameters of a quadratic and a quartic grows quadratically in the dimension. We will make many more similar comparisons in Section 4.3 for different types of dynamical systems.*

We end this section by stating the general version of Theorem 4.2.1, which requires the Lyapunov function to decrease on average every m steps.

Corollary 4.2.1. *Consider the dynamical system (4.1). If there exists* $m-1$ *non-negative scalars* τ_i, *and a continuous radially unbounded function* $V: \mathbb{R}^n \to \mathbb{R}$, *such that*

$$\begin{aligned} V(x) &> 0 \quad \forall x \neq 0 \\ V(0) &= 0 \\ \tau_{m-1}(V_{k+m} - V_k) + \cdots + (V_{k+1} - V_k) &< 0 \end{aligned} \tag{4.3}$$

then the origin is a GAS equilibrium of (4.1).

Proof. The proof is a straightforward generalization of the proof of Theorem 4.2.1. □

4.2.2 The Convex Formulation

In this section we present our main theorem, which will be used throughout Section 4.3.

Theorem 4.2.2. *Consider the dynamical system (4.1). If there exists two continuous* *functions* $V^1, V^2 : \mathbb{R}^n \to \mathbb{R}$ *such that*

$$\begin{aligned} V^2 \text{ and } V^1 + V^2 &\text{ are radially unbounded} \\ V^2(x) &> 0 \;\forall x \neq 0 \\ V^1(x) + V^2(x) &> 0 \;\forall x \neq 0 \\ V^1(0) + 2V^2(0) &= 0 \\ (V^2_{k+2} - V^2_k) + (V^1_{k+1} - V^1_k) &< 0 \end{aligned} \tag{4.4}$$

then the origin is a GAS equilibrium of (4.1).

The inequality (4.4) is linear in the decision variables V^1 and V^2. This will allow us to check condition (4.4) via a semidefinite program. Note that Theorem 4.2.1 is a special case of Theorem 4.2.2, when $V^1 = V$ and $V^2 = \tau V$. Unlike Theorem 4.2.1, Theorem 4.2.2 maps the state into two Lyapunov functions instead of one. In this fashion, the improvement in one and two steps are measured using two different metrics. The theorem states that as long as the sum of the two improvements is negative at any point in time, stability is guaranteed and both V^1 and V^2 will converge to zero. Figure 4-3 illustrates the trajectory of a hypothetical dynamical system at three consecutive instances of time. Here, V^1 and V^2 ar taken to be quadratics and therefore have ellipsoidal level sets. Since the decrease in the horizontal ellipsoid in two steps is larger than the increase of the vertical ellipsoid in the first step, inequality (4.4) is satisfied.

The following proof will use the conditions of Theorem 4.2.2 to explicitly construct a standard Lyapunov function.

Proof. (of Theorem 4.2.2) We start by rewriting (4.4) in the form

$$V^2_{k+2} + V^1_{k+1} < V^2_k + V^1_k.$$

Adding V^2_{k+1} to both sides and rearranging terms we get

$$V^1_{k+1} + V^2_{k+1} + V^2_{k+2} < V^1_k + V^2_k + V^2_{k+1}.$$

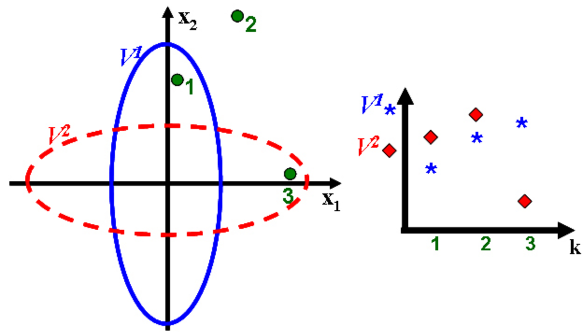


Figure 4-3: Interpretation of Theorem 4.2.2. On the left, three consecutive instances of the trajectory are plotted along with level sets of V^1 and V^2. V^1 measures the improvement in one step, and V^2 measures the improvement in two steps. The plot on the right shows that inequality (4.4) is satisfied.

If we define $W(x) = V^1(x) + V^2(x) + V^2(f(x))$, the last inequality implies that $W_{k+1} < W_k$. It is easy to check from the assumptions of the theorem that W will be continuous, radially unbounded, and will satisfy $W(x) > 0 \ \forall x \neq 0$, and $W(0) = 0$. Therefore, W is a *standard* Lyapunov function for (4.1). $\square$

The explicit construction of a standard Lyapunov function in this proof suggests that non-monotonic Lyapunov functions are equivalent to standard Lyapunov functions of a very specific structure. The function $W(x)$ is parameterized not only with the value of the current state x, but also with the future value of the state $f(x)$. We will demonstrate in Section 4.3 that parameterizing W in this fashion and searching for V^1 and V^2 can often be advantageous over a direct search for a standard Lyapunov function of similar complexity. The reason is that depending on f itself, $W(x)$ will have a more complicated structure than $V^1(x)$ and $V^2(x)$. For example, if f is a polynomial of degree d and V^1 and V^2 are polynomials of degree q, then W will be of higher degree dq. As a second example, suppose f is piecewise linear with $\mathcal{R}$ pieces. If two smooth quadratic functions V^1 and V^2 satisfy the conditions of Theorem 4.2.2, then there will be a standard Lyapunov function W which is piecewise quadratic with $\mathcal{R}$ pieces. From a computational point of view, this additional complexity directly translates into more decision variables. These facts will become more clear in Section 4.3, where we compare standard Lyapunov techniques to our methodology for specific examples.

Next, we generalize Theorem 4.2.2 to m-step differences.

Corollary 4.2.2. *Consider the dynamical system (4.1). If there exist continuous functions* $V^1, \ldots, V^m : \mathbb{R}^n \to \mathbb{R}$ *such that*

$$\begin{aligned}
\sum_{i=j}^{m} V^i \text{ radially unbounded for } j = 1, \cdots, m \\
\sum_{i=j}^{m} V^i(x) > 0 \ \forall \ x \neq 0 \text{ for } j = 1, \cdots, m \\
\sum_{i=1}^{m} iV^i(0) = 0 \\
(V^m_{k+m} - V^m_k) + \cdots + (V^1_{k+1} - V^1_k) < 0
\end{aligned} \tag{4.5}$$

then the origin is a GAS equilibrium of (4.1).

Proof. Similar to the proof of Theorem 4.2.2, it can be shown that $\sum_{j=1}^{m} \sum_{i=j}^{m} V^i(f^{j-1})$ is a standard Lyapunov function. □

4.3 Applications and Examples

In this section, we apply our results to polynomial systems, piecewise affine systems, and linear systems with arbitrary switching. In all of the examples, our approach will be as follows. We fix a certain class of Lyapunov functions (e.g., quadratics) and we show that no function within that class satisfies $V_{k+1} < V_k$ or $V_{k+2} < V_k$. Then, we find functions V^1 and V^2 of the same class that prove stability based on Theorem 4.2.2. In most cases, we will write out the LMIs explicitly to provide guidelines for the users. Throughout, the reader should keep in mind that Corollary 4.2.2 with $m > 2$ can lead to better results than Theorem 4.2.2 at the expense of computing higher order differences.

4.3.1 Polynomial Systems

In Section 2.4, we introduced sum of squares (SOS) programming and in Chapter 3 we explained how it can be used to efficiently search for polynomial Lyapunov functions. Luckily, we can readily apply the same methodology to find non-monotonic polynomial Lyapunov functions. More specifically, we will search for V^1 and V^2 that satisfy

$$\begin{aligned} V^2(x) &\quad \text{SOS} \\ V^1(x) + V^2(x) &\quad \text{SOS} \\ -\{V^2(f(f(x))) - V^2(x) + V^1(f(x)) - V^1(x)\} &\quad \text{SOS.} \end{aligned} \tag{4.6}$$

Example 4.3.1. *Consider the discrete time polynomial dynamics in dimension two:*

$$f = \begin{pmatrix} \frac{3}{10}x_1 \\ -x_1 + \frac{1}{2}x_2 + \frac{7}{18}x_1^2 \end{pmatrix}.$$

One can check that no quadratic SOS function V can satisfy

$$-\{V(f(x)) - V(x)\} \quad \mathit{SOS}.$$

As we mentioned in Chapter 2, there is no gap between SOS and nonnegativity in dimension two and degree up to four. Therefore, we can be certain that in fact no quadratic Lyapunov function exists for this system. We can also check that no quadratic SOS function will satisfy

$$-\{V(f(f(x))) - V(x)\} \quad \mathit{SOS}.$$

On the other hand, from SOSTOOLS and the SDP solver SeDuMi [40] we get that condition (4.6) is satisfied with

$$V^1 = 0.063x_1^2 - 0.123x_1x_2 - 1.027x_2^2$$

$$V^2 = 0.731x_1^2 + 0.095x_1x_2 + 1.756x_2^2.$$

Stability follows from Theorem 4.2.2. It is easy to check that $W(x) = V^1(x)+V^2(x)+V^2(f(x))$ *will be a standard Lyapunov function of degree four. Alternatively, we could have directly searched for a standard Lyapunov function of degree four. However, a polynomial of degree* d *in* n *variables has* $\binom{n+d}{d}$ *coefficients. Therefore, as the dimension goes up, one quartic will have significantly more decision parameters than two quadratics.*

4.3.2 Piecewise Affine Systems

Piecewise affine (PWA) systems are systems of the form

$$x_{k+1} = A_i x_k + a_i, \qquad \text{for } x_k \in R_i \tag{4.7}$$

where R_i's are polyhedral partitions of the state space. There has been much recent interest in systems of this type because, among other reasons, they provide a practical framework for modeling and approximation of hybrid and nonlinear systems. Checking stability of PWA systems is in general undecidable [6]. It is well-known that Schur stability of the A_i's is not necessary, nor is it sufficient, for the overall system (4.7) to be stable [14]. In [34], the method of piecewise quadratic (PWQ) Lyapunov functions was introduced to analyze stability of continuous time PWA systems. Discrete time analogs of this technique have also been studied (see e.g. [15], [36]). A detailed comparison of different stability techniques for discrete time PWA systems is presented in [4]. In this section, we compare the strength of non-monotonic Lyapunov functions to some of the other techniques through an example. It will be shown that, in some cases, instead of a standard piecewise quadratic Lyapunov function, smooth non-monotonic Lyapunov functions can prove stability.

Example 4.3.2. *(Discretized flower dynamics) Consider the the following PWA system*

$$x_{k+1} = \begin{cases} A_1 x_k, & x_k^T H x_k > 0 \\ A_2 x_k, & x_k^T H x_k \leq 0 \end{cases}$$

where $A_1 = \lambda e^{2A_1^{CT}}$, *and* $A_2 = \frac{1}{\lambda} e^{2A_2^{CT}}$.

The matrices A_1^{CT}, A_2^{CT}, *and* H *are as in [34] (with a minor correction)*

$$A_1^{CT} = \begin{bmatrix} -0.1 & 5 \\ -1 & -0.1 \end{bmatrix}, \quad A_2^{CT} = \begin{bmatrix} -0.1 & 1 \\ -5 & -0.1 \end{bmatrix},$$

$$H = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},$$

and $\lambda \geq 1$ *will be a scaling parameter in this problem. We will compare different techniques based on the range of* λ *for which they can prove stability.*

If we search for a smooth[1] *quadratic Lyapunov function satisfying* $V_{k+1} < V_k$, *the problem will be infeasible even for* $\lambda = 1$. *As a second attempt, we search for a smooth quadratic function that satisfies* $V_{k+2} < V_k$. *Stability is proven for* $\lambda \in [1, 1.114]$. *Our next purpose is to show that by combining the improvement in one step and the improvement in two steps using quadratic non-monotonic Lyapunov functions, better results will be obtained. By taking* V^i *to be* $x^T P_i x$, *the conditions of Theorem 4.2.2 reduce to the following set of LMIs*

$$\begin{aligned} P_2 &\succ 0 \\ P_1 + P_2 &\succ 0 \\ (A_i^T A_j^T P_2 A_j A_i - P_2) + (A_i^T P_1 A_i - P_1) &\prec 0 \\ \text{when } x \in R_i \text{ and } A_i x \in R_j, \quad &\forall i, j \in \{1, 2\}. \end{aligned} \tag{4.8}$$

In order to impose the last inequality in (4.8) only on regions of space where $A_j A_i$ *is a possible future transition, we use the* $\mathcal{S}$*-procedure technique [10]. The LMIs in (4.8) will prove stability for* $\lambda \in [1, 1.221)$, *which is a strictly larger range than what*

[1]Reference [4] refers to this as a common quadratic Lyapunov function. This is not to be confused with common quadratic in the context of arbitrary switching. We avoid using this terminology to emphasize that the $\mathcal{S}$-procedure relaxation is used on the regions.

was obtained before.

Next, we shall comment on the connection of this approach to piecewise quadratic Lyapunov functions, which we denote by $x^T Q_i x$.

A search for a PWQ Lyapunov function can be posed by the following set of LMIs [4]

$$
\begin{aligned}
Q_1 &\succ 0 \\
Q_2 &\succ 0 \\
(A_i^T Q_j A_i - Q_i) &\prec 0
\end{aligned} \tag{4.9}
$$

$$\text{when } x \in R_i \text{ and } A_i x \in R_j, \quad \forall i, j \in \{1, 2\}.$$

If we ignore the positivity conditions, (4.8) and (4.9) show that the two methods have the same number of constraints. It is relatively straightforward to check that whenever P_1 *and* P_2 *satisfy (4.8),*

$$Q_i = P_1 + P_2 + A_i^T P_2 A_i \quad i = 1, 2 \tag{4.10}$$

will satisfy the LMIs in (4.9). This is in agreement with the standard Lyapunov function that we constructed in the proof of Theorem 4.2.2. On the other hand, existence of PWQ Lyapunov functions does not in general imply feasibility of the LMIs in (4.8). However, for the example discussed above, piecewise quadratic Lyapunov functions also prove stability for $\lambda \in [1, 1.221)$.

We should point out that the method of smooth non-monotonic Lyapunov functions is searching only for two functions P_1 and P_2 independent of the number of regions. On the other hand, PWQ Lyapunov functions have to find as many quadratic functions as the number of regions. This in turn results in more decision variables and more positivity constraints.

To obtain a method that works at least as well as (and most likely strictly better than) standard PWQ Lyapunov functions, one can take V^1, V^2, or both in Theorem 4.2.2 to be piecewise quadratic.

4.3.3 Approximation of the Joint Spectral Radius

In this section, we consider a dynamical system of the type

$$x_{k+1} = A_{\sigma(k)} x_k \tag{4.11}$$

where σ is a mapping from the integers to a finite set of indices $\{1, ..., m\}$. The question of interest is to determine whether the discrete inclusion (4.11) is *absolutely asymptotically stable* (AAS), i.e., asymptotically stable for all switching sequences.

It turns out [39] that (4.11) is AAS if and only if the *joint spectral radius* (JSR) of the matrices $A_1, ..., A_m$ is strictly less than one. The joint spectral radius represents the maximum growth rate obtained by taking arbitrary products of the matrices A_i. It is formally defined as [38]:

$$\rho(A_1, \cdots, A_m) := \lim_{k\to\infty} \max_{\sigma\in\{1,\cdots,m\}^k} \|A_{\sigma_k} \cdots A_{\sigma_2} A_{\sigma_1}\|^{\frac{1}{k}} \tag{4.12}$$

where the value of ρ is independent of the norm used in (4.12). For a given set of matrices, testing whether $\rho \leq 1$ is undecidable [8]. Moreover, computation and even approximation of the JSR is difficult [42]. Here, we will be interested in providing bounds on the JSR. Clearly, the spectral radius of any finite product of matrices gives a lower bound on ρ. Computing upper bounds is a much more challenging task. We explain our technique for a pair of matrices A_1, A_2. The generalization to a finite set of matrices is straightforward.

Because of the scaling property of the JSR, for any $\lambda \in (0, \infty)$, if we can prove AAS of (4.11) for the scaled pair of matrices λA_1 and λA_2, then $\frac{1}{\lambda}$ is an upper bound on $\rho(A_1, A_2)$. References [7] and [29] have respectively used common quadratic and common SOS polynomial Lyapunov functions to prove upper bounds on ρ. Here, we will use common non-monotonic Lyapunov functions for this purpose. For the special case where V^1 and V^2 are quadratics (i.e. $V^i = x^T P_i x$), Theorem 4.2.2 suggests that the following LMIs have to be satisfied to get an upper bound of $\frac{1}{\lambda}$ on $\rho(A_1, A_2)$.

$$\begin{aligned} P_2 &\succ 0 \\ P_1 + P_2 &\succ 0 \\ (\lambda^4 A_i^T A_j^T P_2 A_j A_i - P_2) + (\lambda^2 A_i^T P_1 A_i - P_1) &\prec 0 \\ \forall i, j \in \{1, 2\}. \end{aligned} \tag{4.13}$$

When P_2 is set to zero, the method of common quadratics is recovered. Similarly, when P_1 is set to zero, the LMIs will find a common quadratic that satisfies $V_{k+2} < V_k$. It is easy to see that the existence of a common quadratic in one step implies the existence of a common quadratic in two steps, but the converse is not true. Therefore, setting $P_1 = 0$ will produce upper bounds that are at least as tight as those obtained from setting $P_2 = 0$. Below, we show with two examples that when we use both P_1 and P_2 in (4.13) to combine the improvement in one and two steps, we can provide strictly tighter bounds on the JSR.

Example 4.3.3. *([29], Example 2) We consider the problem of finding an upper bound for the JSR of the following pair of matrices:*

$$A_1 = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix}$$

It is not difficult to show that $\rho(A_1, A_2) = 1$. Using common quadratic standard Lyapunov functions, one would obtain an upper bound of $\sqrt{2} \approx 1.41$. A common quadratic standard Lyapunov function for A_1A_1, A_2A_1, A_1A_2, and A_2A_2 would produce an upper bound of $\sqrt[4]{2} \approx 1.19$. On the other hand, common quadratic non-monotonic Lyapunov functions can achieve an upper bound of $1 + \varepsilon$ for any $\varepsilon > 0$. Given ε, the LMIs (4.13) will be feasible with

$$P_1 = \begin{bmatrix} -\alpha & 0 \\ 0 & -\alpha \end{bmatrix}, \quad P_2 = \begin{bmatrix} \beta & 0 \\ 0 & \beta \end{bmatrix}$$

with any $\quad \beta > 0, \quad 1 - \frac{4\varepsilon}{1+\varepsilon} < \frac{\alpha}{\beta} < 1.$

We should mention that in [29], it is shown that a common SOS quartic Lyapunov function also achieves an upper bound of $1+\varepsilon, \quad \forall \varepsilon > 0$.

Example 4.3.4. *([29], Example 4) We consider the following three randomly generated* 4×4 *matrices:*

$$A_1 = \begin{bmatrix} 0 & 1 & 7 & 4 \\ 1 & 6 & -2 & -3 \\ -1 & -1 & -2 & -6 \\ 3 & 0 & 9 & 1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -3 & 3 & 0 & -2 \\ -2 & 1 & 4 & 9 \\ 4 & -3 & 1 & 1 \\ 1 & -5 & -1 & -2 \end{bmatrix}$$

$$A_3 = \begin{bmatrix} 1 & 4 & 5 & 10 \\ 0 & 5 & 1 & -4 \\ 0 & -1 & 4 & 6 \\ -1 & 5 & 0 & 1 \end{bmatrix}$$

A lower bound on the JSR is $\rho(A_1A_3)^{\frac{1}{2}} \approx 8.91$ *[29]. Method of common quadratic satisfying* $V_{k+1} < V_k$*, common quadratic satisfying* $V_{k+2} < V_k$*, and common non-monotonic quadratic satisfying Theorem 4.2.2 respectively produce upper bounds equal to* $9.77, 9.19,$ *and* 8.98. *A common SOS quartic satisfying* $V_{k+1} < V_k$ *produces an upper bound of* 8.92 *[29]. This bound is tighter than what we obtained from quadratic non-monotonic functions. However, the latter technique will have* 20 *decision parameters for this example in contrast with* 35 *needed to find a homogeneous quartic function.*

Even though, throughout Section 4.3 we have used quadratic non-monotonic Lyapunov functions, the reader should keep in mind that better results can be obtained by taking V^1 and V^2 of Theorem 4.2.2 to be SOS polynomials.

Chapter 5

Non-monotonic Lyapunov Functions in Continuous Time

Our motivation for relaxing monotonicity of Lyapunov functions is the same in continuous time. Replacing the strict decreasing condition on V with less restrictive conditions that still imply convergence of V to zero in the limit enlarges the class of functions that can prove stability. By doing so, simpler functions may satisfy the new condition and this in turn may make the search process easier and cut down on the number of decision variables. Once again, our focus will be on coming up with conditions that can be checked by a convex program. This will enable us to utilize the techniques form semidefinite and sum of squares programming to search for Lyapunov functions in an efficient and computationally tractable manner.

5.1 Literature Review

In order to allow $\dot{V}$ to be occasionally positive, we will impose conditions on the higher derivatives of V to bound the rate at which V can increase. Although not with the motivation of relaxing monotonicity, there has been earlier work invoking conditions on higher order derivatives of Lyapunov functions. In [12], Butz gives a sufficient condition for global asymptotic stability (GAS) using the first three derivatives. However, the formulation of his condition is nonconvex. Heinen and Vidyasagar show in [17] that Butz's condition, when imposed only on complements of compact sets implies boundedness of the trajectories. Gunderson proves a comparison lemma for higher order derivatives in [16], which can be used to generalize the results by Butz to even higher order derivatives. More recently, Jiang shows that GAS is still achieved if $\dot{V} < 0$ is weakened to $\dot{V} \leq 0$, but some additional conditions on higher order derivatives are satisfied [18]. Another set of papers study conditions on only the first and second derivative of Lyapunov functions [44], [13], and [21]. Yorke shows in [44] that with very minor assumptions on $\dot{V}$ and $\ddot{V}$, it can be concluded that the trajectories either go to the origin or to infinity. Kudaev has some independent similar results, but his conditions seem to be harder to satisfy [21], [44].

We devote the main part of this chapter to build on the result by Butz. This is done in Sections 5.2 and 5.3. Our approach will be similar in spirit to the methodology of the previous chapter for the discrete time case. We will illustrate the connection of a non-monotonic Lyapunov function satisfying some differential inclusion with higher order derivatives, to a standard Lyapunov function satisfying $\dot{V} < 0$. This observation will enable us to change Butz's condition to a *convex* condition that is even stronger. In fact, the condition by Butz will only be a special case of our convex condition. We give some examples to show the potential advantages of this methodology over standard Lyapunov theory and we make some conjectures. In Section 5.4, we state the results of Yorke that are particularly useful to reject existence of periodic orbits or limit cycles. We give simple convex conditions that would imply the required conditions of Yorke. However, these convex conditions may in general be more conservative. We compare them with conditions given by Chow and Dunninger [13] in some examples.

Throughout this chapter, we will be concerned with the dynamical system

$$\dot{x} = f(x), \tag{5.1}$$

where $f : \mathbb{R}^n \to \mathbb{R}^n$ is continuously differentiable and has a unique equilibrium at the origin. By the first three derivatives of the Lyapunov function $V : \mathbb{R}^n \to \mathbb{R}$, we mean

$$\begin{aligned} \dot{V}(x) &= \langle \frac{\partial V(x)}{\partial x}, f(x) \rangle & (5.2) \\ \ddot{V}(x) &= \langle \frac{\partial \dot{V}(x)}{\partial x}, f(x) \rangle & (5.3) \\ \dddot{V}(x) &= \langle \frac{\partial \ddot{V}(x)}{\partial x}, f(x) \rangle. & (5.4) \end{aligned}$$

We will make some remarks about the special case where f is linear ($\dot{x} = Ax$) and V is quadratic ($V(x) = x^T P x$). In this case the first three derivatives will become

$$\begin{aligned} \dot{V}(x) &= x^T(PA + A^T P)x & (5.5) \\ \ddot{V}(x) &= x^T(PA^2 + 2A^T PA + A^{T^2} P)x & (5.6) \\ \dddot{V}(x) &= x^T(PA^3 + 3A^T PA^2 + 3A^{T^2} PA + A^{T^3} P)x. & (5.7) \end{aligned}$$

5.2 A Discussion on the Results by Butz

In [12], Butz investigated the use of higher order derivatives of Lyapunov functions for proving global asymptotic stability. We begin by reviewing his contributions that are comprised of two theorems (Theorems 5.2.1 and 5.2.2) and one example (Example 5.2.2 below).

One would naturally start with conditions using only the first and second derivatives. The following theorem shows that a large class of conditions using $\dot{V}$ and $\ddot{V}$ alone, which seem to imply stability, are vacuous in the sense that no dynamical system will satisfy them, unless it already satisfies $\dot{V} < 0$.

Theorem 5.2.1. *(Butz, [12]) Consider the dynamical system (5.1) and a twice differentiable Lyapunov function V, with its first two derivatives given in (5.2) and (5.3). If it were true that*

$$\min[\dot{V}(x), \tau \ddot{V}(x)] < 0 \tag{5.8}$$

for all $x \neq 0$ with some $\tau \geq 0$, then a proof of global asymptotic stability would be routine. However, it is not possible that (5.8) holds unless $\dot{V}(x) < 0$ for all $x \neq 0$. In *particular, a condition of the type*

$$\tau \ddot{V}(x) + \dot{V}(x) < 0 \quad \text{for all } x \neq 0, \text{ with some } \tau \geq 0 \tag{5.9}$$

is vacuous.

Butz does not specify what he exactly means by V being a Lyapunov function. We show by a simple example that he must have had in mind that V is lower bounded or else Theorem 5.2.1 is not true.

Example 5.2.1. *Consider the dynamical system*

$$\begin{aligned} \dot{x_1} &= x_2 \\ \dot{x_2} &= -x_2 - x_1^2. \end{aligned} \tag{5.10}$$

Letting $V(x) = x_1$*, we will have* $\dot{V}(x) = x_2$*, which is not negative definite. However, taking* $\tau = 1$ *in (5.9) we get*

$$\dot{V} + \ddot{V} = -x_1^2,$$

which is negative definite.

In [12], Butz gives a simple geometric proof of Theorem 5.2.1. Here, we give an alternative argument, which at an intuitive level shows why conditions of the type (5.8) or (5.9) are not helpful. Note that both of these conditions do not allow for $\dot{V}$ and $\ddot{V}$ to be simultaneously positive. We claim that if $V(x(t))$ is decreasing at some point as a function of time, it can never start to increase unless both $\dot{V}(x(t))$ and $\ddot{V}(x(t))$ are positive for a period of time. The reason is that when V starts to increase, we will obviously have $\dot{V} > 0$. Moreover, since V was decreasing before and it is now increasing, $\dot{V}$ must have changed sign from negative to positive. This implies that $\ddot{V}$ must be positive when the transition occurs (and at least for a short while after the transition by continuity). Therefore, conditions (5.8) or (5.9) do not allow for a non-monotonic behavior of $V(x(t))$ as a function of time.

Next, we state the main result of [12], which shows that using the first three derivatives of Lyapunov functions we can get a non-vacuous sufficient condition for global asymptotic stability, which contains the standard Lyapunov theorem (Theorem 1.2.1 of Chapter 1) as a special case.

Theorem 5.2.2. *(Butz, [12]) Consider the continuous time dynamical system (5.1). If there exists scalars $\tau_1 \geq 0$ and $\tau_2 \geq 0$, and a three times differentiable Lyapunov function V with its first three derivatives given as in (5.2)-(5.4), such that*

$$\tau_2 \dddot{V}(x) + \tau_1 \ddot{V}(x) + \dot{V}(x) < 0 \tag{5.11}$$

for all $x \neq 0$, then for any $x(0)$, $V(x(t)) \to 0$ as $t \to \infty$, and the origin of (5.1) is globally asymptotically stable.

Note than when $\tau_1 = \tau_2 = 0$, we recover the original Lyapunov's theorem. Once again, for Theorem 5.2.2 to be correct even for the special case where $\tau_1 = \tau_2 = 0$, we need the Lyapunov function V to be lower bounded. Without loss of generality, we assume that $V(0) = 0$ and $V(x) > 0$ for all $x \neq 0$.

Butz proves this theorem in [12] by comparison lemma [20] type arguments after appealing to some results from ordinary differential equations. Below, we give a proof for the special case of linear systems and quadratic Lyapunov functions using only basic linear algebra.

Corollary 5.2.1. *Consider the linear dynamical system $\dot{x} = Ax$. If there exist scalars $\tau_1 \geq 0$ and $\tau_2 \geq 0$, and a quadratic function $V(x) = x^T P x$ with $P \succ 0$, such that*

$$\tau_2 \dddot{V}(x) + \tau_1 \ddot{V}(x) + \dot{V}(x) < 0,$$

then A is Hurwitz.

Proof. We begin by replacing (5.5)-(5.7) in (5.11)

$$\tau_2(PA^3 + 3A^T PA^2 + 3{A^T}^2 PA + {A^T}^3 P) + \tau_1(PA^2 + 2A^T PA + {A^T}^2 P) + (PA + A^T P) \prec 0. \tag{5.12}$$

Let λ be an eigenvalue of A with eigenvector v. So, we have

$$Av = \lambda v.$$

If we multiply (5.12) by v from the right hand side and by v^*[1] from the left, after factoring and regrouping some terms we get

$$\begin{aligned}(\lambda^3 + 3\lambda^*\lambda^2 + 3\lambda^{*2}\lambda + \lambda^{*3})(\tau_2 v^* P v) + (\lambda^2 + 2\lambda^*\lambda + \lambda^{*2})(\tau_1 v^* P v) + (\lambda + \lambda^*)(v^* P v) &< 0 \\ \Rightarrow [\tau_2(\lambda + \lambda^*)^3 + \tau_1(\lambda + \lambda^*)^2 + (\lambda + \lambda^*)](v^* P v) &< 0.\end{aligned} \tag{5.13}$$

Since $P \succ 0$, we must have

$$\tau_2(\lambda + \lambda^*)^3 + \tau_1(\lambda + \lambda^*)^2 + (\lambda + \lambda^*) < 0.$$

By nonnegativity of τ_1 and τ_2, we get

$$\lambda + \lambda^* = 2\Re(\lambda) < 0,$$

and therefore A is Hurwitz. □

Example 5.2.2. *(Butz, [12]) Consider the linear system $\dot{x} = Ax$ with*

$$A = \begin{bmatrix} -4 & -5 \\ 1 & 0 \end{bmatrix}.$$

The eigenvalues of A are $-2 \pm j$. We know from our discussion in Section 2.2.1 that there exists a positive definite quadratic function that decreases monotonically along trajectories of this linear system. However, instead of searching for that quadratic

[1] Recall that $*$ denotes the conjugate transpose.

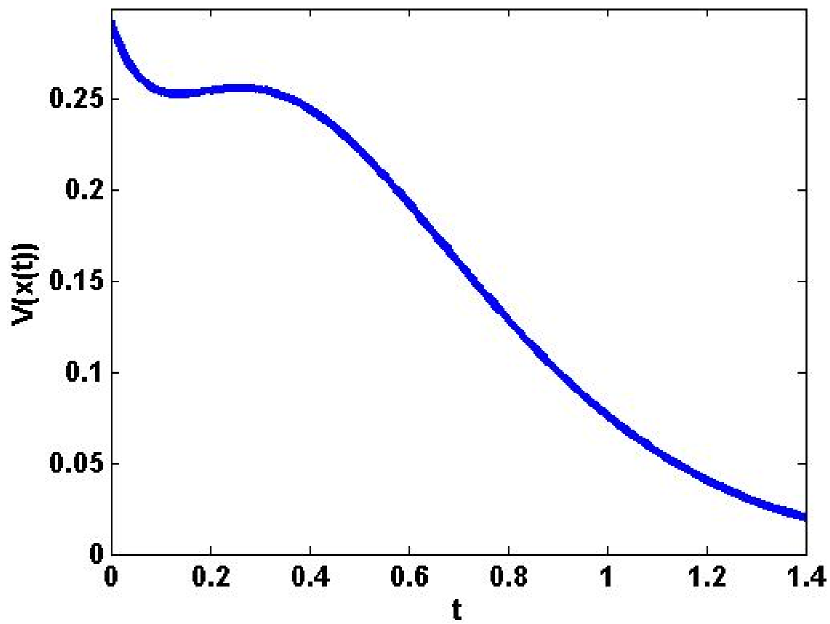


Figure 5-1: $V(x) = \frac{1}{2}x^T Px$ does not decrease monotonically along a trajectory of the linear system in Example 5.2.2. However, stability can still be proven by Theorem 5.2.2.

function, let us pick

$$V(x) = \frac{1}{2}x^T Px, \quad \textit{with } P = \begin{bmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{bmatrix}.$$

Figure 5-1 shows the value of this positive definite Lyapunov function along a typical trajectory of the given linear system. Notice that the Lyapunov function increases over a period of time and then decreases again. Indeed, we can calculate the first derivative

$$\dot{V}(x) = \frac{1}{2}x^T Qx, \quad \textit{with } Q = \begin{bmatrix} -7 & -6 \\ -6 & -5 \end{bmatrix},$$

and observe that the matrix Q is not negative definite. This explains why in Figure 5-1 the Lyapunov function can occasionally increase. The second and third derivatives *of V are given by*

$$\begin{aligned} \ddot{V}(x) &= \frac{1}{2}x^T R x, \quad R = QA + A^T Q \\ \dddot{V}(x) &= \frac{1}{2}x^T S x, \quad S = RA + A^T R. \end{aligned} \tag{5.14}$$

Some calculation gives

$$R = \begin{bmatrix} 44 & 54 \\ 54 & 60 \end{bmatrix} \quad \textit{and} \quad S = \begin{bmatrix} -244 & -376 \\ -376 & -540 \end{bmatrix}.$$

For this particular example we can set $\tau_1 = 0$ and not use the second derivative. Condition (5.11) then reduces to

$$\tau_2 S + Q \prec 0. \tag{5.15}$$

Since this is an LMI in one variable τ_2, by solving a generalized eigenvalue problem [10] and computing $eig(Q, -S)$, we can conclude that (5.15) is feasible for

$$0.0021 < \tau_2 < 0.0486. \tag{5.16}$$

Global asymptotic stability follows from Theorem 5.2.2.

This example demonstrated that even though our quadratic Lyapunov function did not satisfy $\dot{V}(x) < 0$, we were able to find τ_1 and τ_2 that satisfied condition (5.11). We will next conjecture that this fact was not special to this particular example. In other words, for an asymptotically stable second order linear system, one can choose *any* positive definite quadratic function $V(x)$, and there will always exist nonnegative scalars τ_1 and τ_2 satisfying condition (5.11). For simplicity, we take $V(x) = x^T x$ (i.e. $P = I$). If the conjecture is true with this Lyapunov function candidate, by changing coordinates it must be true for any positive definite quadratic function. The exact statement of the conjecture written in an LMI form is as follows.

Conjecture 5.2.1. *Suppose A is a 2×2 Hurwitz matrix. Then there always exist scalars $\tau_1 \geq 0$ and $\tau_2 \geq 0$ such that*

$$\tau_2(A^3 + 3A^TA^2 + 3{A^T}^2A + {A^T}^3) + \tau_1(A^2 + 2A^TA + {A^T}^2) + (A + A^T) \prec 0.$$

We have evidence to believe that this conjecture is true. If we succeed in proving this conjecture it would imply that instead of searching for the three free parameters of a 2×2 positive definite symmetric matrix P such that $V(x) = x^TPx$ decreases monotonically along trajectories, one can always fix $P = I$ and search for the two unknowns τ_1 and τ_2 in (5.11). The natural extension of this conjecture would be that when $x \in \mathbb{R}^n$, instead of searching for $\frac{1}{2}n(n+1)$ entries of P as in standard Lyapunov theory, it is enough to fix $P = I$ and search for n unknown coefficients multiplying derivatives of order up to $n + 1$. Although from a practical point of view this is not so significant[2], the result is of theoretical interest. We know that the stability of a continuous time linear system is completely determined by the real part of its n eigenvalues. In that sense, it makes intuitive sense that one should be able to reduce the free parameters of a quadratic Lyapunov function to n numbers.

5.3 A Convex Sufficient Condition for Stability Using Higher Order Derivatives of Lyapunov Functions

The main difficulty in the practical application of Theorem 5.2.2 of Butz is that there is a product of decision variables τ_i and V in condition (5.11), which makes the feasible set nonconvex. Therefore, we cannot search for τ_i and V simultaneously through a convex program. So far, our approach has been to either fix τ_1 and τ_2 through an a

[2]In practice, there is no advantage in finding a Lyapunov function for a linear system over solving for the eigenvalues of A directly. When a Lyapunov equation is solved, A is often transformed into its real Schur form, which already gives the eigenvalues. However, when the purpose is not solely to test stability but for instance robustness to perturbation, Lyapunov theory can be quite advantageous [20].

priori nested binary search and then only search for V, or fix V and search for τ_1 and τ_2 (as was the case, for example, in Conjecture 5.2.1). In this section, we are going to overcome this problem by changing condition (5.11) to a convex condition that still implies global asymptotic stability and is even stronger than condition (5.11). The key point that will enable us to do this is a simple observation that makes a connection between a non-monotonic Lyapunov function satisfying (5.11) and a standard Lyapunov function. We state this observation in the following theorem.

Theorem 5.3.1. *Consider the continuous time dynamical system (5.1). If there exists τ_1, τ_2, and $V(x)$ satisfying conditions of Theorem 5.2.2, then*

$$W(x) = \tau_2 \ddot{V}(x) + \tau_1 \dot{V}(x) + V(x) \tag{5.17}$$

is a standard Lyapunov function for (5.1).

Proof. We need to show that $W(x) > 0$ for all $x \neq 0$, $W(0) = 0$, and $\dot{W}(x) < 0$ for all $x \neq 0$. Since $f(0) = 0$, it follows from (5.2) and (5.3) that $\ddot{V}(0) = 0$ and $\dot{V}(0) = 0$. This together with the assumption that $V(0) = 0$ implies that $W(0) = 0$. Negative definiteness of $\dot{W}(x)$ follows from condition (5.11). It remains to show that $W(x) > 0$ for all $x \neq 0$. Assume by contradiction that there exists a point $\bar{x} \in \mathbb{R}^n$ such that $W(\bar{x}) \leq 0$. We evaluate the Lyapunov function $W(x)$ along the trajectories of system (5.1) starting from the initial condition $\bar{x}$. The value of the Lyapunov function is nonpositive to begin with and will strictly decrease because $\dot{W}(x) < 0$. Therefore, the value of the Lyapunov function can never become zero. On the other hand, since conditions of Theorem 5.2.2 are satisfied, trajectories of (5.1) must all go to the origin, where we have $W(0) = 0$. This gives us a contradiction. □

Example 5.3.1. *We verify Theorem 5.3.1 by revisiting Example 5.2.2. Recall from (5.16) that for $\tau_2 \in [0.0021, 0.0486]$ we had*

$$\tau_2 S + Q \prec 0. \tag{5.18}$$

Theorem 5.3.1 implies that for any τ_2 in that range, we must have

$$\tau_2 R + P \succ 0. \tag{5.19}$$

Indeed, by solving the generalized eigenvalue problem eig$(-P, R)$ we see that (5.19) holds for

$$-0.0139 < \tau_2 < 0.1951,$$

a range that strictly contains $[0.0021, 0.0486]$. *In fact, for any $\tau_2 \in [0.0021, 0.0486]$, $W(x) = x^T (\tau_2 R + P) x$ is a standard Lyapunov function for the linear system of Example 5.2.2.*

Theorem 5.3.1 suggests that non-monotonic Lyapunov functions satisfying (5.11) can be interpreted as standard Lyapunov functions of a specific structure given in (5.17). Therefore, we can convexify this parametrization by looking for a Lyapunov function of the form

$$W(x) = \ddot{V}^3(x) + \dot{V}^2(x) + V^1(x). \tag{5.20}$$

In other words, we will search for functions V^1, V^2, and V^3, but with no sign conditions on them individually. Instead, just like standard Lyapunov theory, we require

$$\begin{aligned} W(0) &= 0 && (5.21) \\ W(x) &= \ddot{V}^3(x) + \dot{V}^2(x) + V^1(x) > 0 \quad \text{for all } x \neq 0 && (5.22) \\ \dot{W}(x) &= \dddot{V}^3(x) + \ddot{V}^2(x) + \dot{V}^1(x) < 0 \quad \text{for all } x \neq 0. && (5.23) \end{aligned}$$

Note now that the inequalities are linear in the unknowns V^1, V^2, and V^3. Therefore, we can perform the search by a convex program using the methodologies of Chapter 2. Furthermore, Butz's condition is only a special case of this parametrization. Theorem 5.3.1 shows that whenever conditions of Theorem 5.2.2 of Butz are satisfied, a standard Lyapunov function $W(x)$ of the form (5.20) will exist. This specific parametrization enters the dynamics f multiplied by derivatives of functions V^i into the structure of the standard Lyapunov function W. Parameterizing W in this fashion and searching for $V^i s$ can be advantageous over a direct search for a W of similar complexity. The reason is that depending on f itself, $W(x)$ will have a more complicated structure than $V^i(x)$. For example when f is a polynomial system, the degree of $W(x)$ will be higher than the degree of each of the $V^i s$. From a computational point of view, this would lead to saving decision variables.

We will show next that when one uses different functions V^1 and V^2, examination of the first and second derivatives alone is *not* vacuous. By this we mean that it is possible to have $\ddot{V}^2 + \dot{V}^1 < 0$ without $\dot{V}^1$ being negative definite. This is in contrast to Theorem 5.2.1 by Butz.

Example 5.3.2. *Consider a continuous time linear system $\dot{x} = Ax$ with*

$$A = \begin{bmatrix} -0.5 & 5 \\ -1 & -0.5 \end{bmatrix}.$$

We have already analyzed this system in Chapter 2. Here, we let

$$V^1(x) = x^T P_1 x, \quad \textit{with } P_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

and

$$V^2(x) = x^T P_2 x, \quad \textit{with } P_2 = \begin{bmatrix} -1.74 & 0.11 \\ 0.11 & -9.35 \end{bmatrix}.$$

With some calculation we get

$$\dot{V}^1(x) = x^T Q_1 x, \quad \textit{with } Q = \begin{bmatrix} -1 & 4 \\ 4 & -1 \end{bmatrix},$$

which is not negative definite. On the other hand, one can check that

$$\ddot{V}^2(x) + \dot{V}^1(x) = x^T M x, \quad \textit{with } M = \begin{bmatrix} 2.52 & 0.53 \\ 0.53 & 10.46 \end{bmatrix}$$

is positive definite, and its derivative

$$\ddot{V}^2(x) + \dot{V}^1(x) = x^T Z x, \quad \textit{with } Z = \begin{bmatrix} -3.58 & 0.61 \\ 0.61 & -6.17 \end{bmatrix}$$

is negative definite.

So far, our examples have dealt with linear systems, which is not an interesting case from a practical point of view. After all, we know linear systems always admit a standard quadratic Lyapunov function. We now give an example of a polynomial system with no quadratic Lyapunov function, and we show that the parametrization in (5.21) can be beneficial.

Example 5.3.3. *Consider the following polynomial dynamics*

$$\begin{aligned} \dot{x_1} &= -0.8x_1^3 - 1.5x_1x_2^2 - 0.4x_1x_2 - 0.4x_1x_3^2 - 1.1x_1 \\ \dot{x_2} &= x_1^4 + x_3^6 + x_1^2x_3^4 \\ \dot{x_3} &= -0.2x_1^2x_3 - 0.7x_2^2x_3 - 0.3x_2x_3 - 0.5x_3^3 - 0.5x_3. \end{aligned} \tag{5.24}$$

If we look for a standard quadratic Lyapunov function using SOSTOOLS, the search will be infeasible. Instead, we search for V^1 and V^2 that satisfy $\dot{V}^2 + V^1 > 0$ and $\ddot{V}^2 + \dot{V}^1 < 0$. In principle, we can start with a linear parametrization for V^1 and V^2 since there is no positivity constraint on V^1 or V^2 directly. For this example, a linear parametrization will be infeasible. However, if we search for a linear function V^2 and a quadratic function V^1, SOSTOOLS and the SDP solver SeDuMi will find

$$\begin{aligned} V^1(x) &= 0.47x_1^2 + 0.89x_2^2 + 0.91x_3^2 \\ V^2(x) &= 0.36x_2. \end{aligned}$$

Therefore, the origin of (5.24) is globally asymptotically stable. We can in fact construct a sextic standard Lyapunov function from V^1 and V^2 given by

$$W(x) = \dot{V}^2(x) + V^1(x) = 0.36x_1^4 + 0.36x_1^2x_3^4 + 0.47x_1^2 + 0.89x_2^2 + 0.36x_3^6 + 0.91x_3^2.$$

Computing higher order derivatives of Lyapunov functions in continuous time is not as expensive as computing higher order compositions of the vector field in discrete time. If the vector field f is a polynomial of degree d, the degree of the $(m+1)^{th}$ derivative of V is in general $d-1$ higher than the degree of the m^{th} derivative. So, as higher order derivatives are computed, the degree goes up linearly. This is in contrast to the exponential growth in the degree of V composed with higher order compositions of f as we saw in the previous chapter. We remind the reader that [16] derives a generalized comparison lemma for higher order derivatives, which can be used to investigate conditions on derivatives of degree higher than three.

Finally, as we mentioned before, Heinen and Vidyasagar showed in [17] that condition (5.11) when imposed only on complements of bounded sets can imply Lagrange stability (boundedness of trajectories). It is possible to generalize and convexify their result in a similar fashion to what was done in this section.

5.4 Lyapunov Functions Using $\ddot{V}$

In this section we discuss an interesting theorem by Yorke [44] that imposes conditions on $\dot{V}$ and $\ddot{V}$. The implication of the main theorem is not global asymptotic stability but rather the conclusion that the trajectories either converge to the origin or go to infinity. This result can be particularly useful to show nonexistence of periodic orbits, limit cycles, or chaotic attractors. The main theorem in [44] is slightly more general than what we state below. However, this special case is more relevant for our purposes.

Theorem 5.4.1. *(Yorke, [44]) Consider the dynamical system (5.1) in dimension $n \neq 2$ and a twice differentiable Lyapunov function V with its first two derivatives given as in (5.2) and (5.3). If*

$$\textit{either} \quad \dot{V}(x) \neq 0 \quad \textit{or} \quad \ddot{V}(x) \neq 0 \quad \forall x \neq 0, \tag{5.25}$$

then starting from any initial condition $x(0) \in \mathbb{R}^n$, either $x(t) \to 0$ or $|x(t)| \to \infty$ as

$t \to \infty$.

We refer the reader to [44] for a proof of this theorem, which involves algebraic topology and manifold theory. In his proof, Yorke requires $\mathbb{R}^n - \{0\}$ to be simply connected, which is not the case for $n = 2$. In fact, [44] gives a simple counterexample of a second order linear system with periodic solutions that satisfies (5.25). Notice that the theorem imposes no sign conditions on either V, $\dot{V}$, or $\ddot{V}$. As the paper [44] reads: "it is rather surprising that such results are true because it seems at first that almost nothing is assumed".

Condition (5.25) is equivalent to requiring

$$\dot{V}(x)^2 + \ddot{V}(x)^2 > 0 \quad \forall x \neq 0. \tag{5.26}$$

Unfortunately, neither condition (5.25) nor condition (5.26) are convex. We can replace these conditions, however, by more restrictive conditions that imply (5.25) and are convex. For example, any of the following three conditions imply (5.25).

$$\begin{aligned} \dot{V}(x) &< 0 \quad \forall x \neq 0 && (5.27) \\ \ddot{V}(x) &< 0 \quad \forall x \neq 0 && (5.28) \\ \dot{V}(x) + \ddot{V}(x) &< 0 \quad \forall x \neq 0. && (5.29) \end{aligned}$$

Of course, the same implication holds if the sign of the inequality is reversed in any of the conditions (5.27)-(5.29). Note that all of these conditions are linear in the decision variable V.

Example 5.4.1. *Consider the dynamical system*

$$\begin{aligned} \dot{x_1} &= x_2 \\ \dot{x_2} &= x_3 \\ \dot{x_3} &= -x_1^2 - x_2^2 - x_3. \end{aligned} \tag{5.30}$$

We choose to search for a V that satisfies condition (5.29). We can do this using SOSTOOLS, but in this case it is easy enough to see that $V(x) = x_2$ satisfies (5.29).

Indeed, we have

$$\dot{V}(x) + \ddot{V}(x) = x_3 - x_1^2 - x_2^2 - x_3 = -x_1^2 - x_2^2,$$

which is negative definite. This implies that the condition of Theorem 5.4.1 is satisfied and therefore trajectories of (5.30) either go to zero or to infinity. In particular, solutions of (5.30) can never go on periodic orbits.

Chow and Dunninger argue in [13] for the use of condition (5.28) as a relaxation for Yorke's condition. They show that with this stronger assumption, the implication holds for $n = 2$ as well. They also claim that condition (5.28) is only slightly stronger than (5.25) and it is met in most applications. The author does not necessarily agree with this claim. We will illustrate the connection between the three conditions (5.27)-(5.29), and then compare them with an example.

First of all, note that the condition $\dot{V} < 0$, in any dimension, implies nonexistence of periodic orbits. This is easy to see by contradiction. If there were a periodic orbit, we could start from any point $\bar{x}$ on it and evaluate the value of the Lyapunov function V. When the trajectory returns to the starting point $\bar{x}$, the value $V(\bar{x})$ should be revisited. This is in contradiction to the fact that $V(x(t))$ is strictly decreasing as a function of time.

We claim that whenever conditions (5.28) and (5.29) are satisfied, condition (5.27) will also be satisfied with a possibly more complicated function. The converse is not necessarily true. Suppose there exists a V that satisfies $\ddot{V} < 0$. Then, $W = \dot{V}$ will satisfy $\dot{W} < 0$. Similarly, if there exists a V that satisfies $\dot{V} + \ddot{V} < 0$, then $W = V + \dot{V}$ will satisfy $\dot{W} < 0$. However, working with conditions (5.28) and (5.29) can still be beneficial from a computational point of view since W will have higher degree than V.

The next example is borrowed from [20]. We show that we are able to demonstrate nonexistence of periodic orbits with condition (5.27), but not with conditions (5.28) or (5.29). This is in contrary to the claim made in [13].

Example 5.4.2. *Consider the polynomial dynamical system*

$$\begin{aligned} \dot{x_1} &= x_2 \\ \dot{x_2} &= x_1 + bx_2 - x_1^2 x_2 - x_1^3. \end{aligned} \tag{5.31}$$

It is shown in [20] by an application of the Bendixson Criterion that for any $b < 0$, *(5.31) has no periodic orbits. Indeed for any* $b < 0$, *SOSTOOLS finds a quartic polynomial* V *that satisfies* $\dot{V} < 0$. *On the other hand, no polynomial* V *of degree even up to seven is found to satisfy either* $\ddot{V} < 0$ *or* $\dot{V} + \ddot{V} < 0$.

Chapter 6

Conclusions and Future Work

In this thesis we addressed the following natural question: why should we require a Lyapunov function to decrease *monotonically* along trajectories of a dynamical system if we only need to guarantee that it converges to zero in the limit? We relaxed this monotonicity assumption both in discrete time and continuous time by writing new conditions that are amenable to convex programming formulations.

In discrete time, we gave a sufficient condition for global asymptotic stability that allows the Lyapunov functions to increase locally while guaranteeing their convergence to zero in the limit. The conditions of our main theorem were convex. Therefore, all the techniques developed for finding Lyapunov functions based on convex programming can readily be applied to our new formulation. We showed that whenever a non-monotonic Lyapunov function is found, one can construct a standard Lyapunov function from it. However, the standard Lyapunov function will have a more complicated structure. The nature of this additional complexity depends on the dynamics itself. We demonstrated the advantages of our methodology over standard Lyapunov theory through examples from polynomial systems, and linear systems with constrained and arbitrary switching. As an application, we discussed how non-monotonic Lyapunov functions can be used to give upper bounds on the joint spectral radius of a finite set of matrices.

In continuous time, we presented conditions invoking higher derivatives of Lyapunov functions that allow the Lyapunov function to increase but bound the rate at which the increase can happen. Here, we built on previous work in [12] that provides a nonconvex sufficient condition for asymptotic stability using the first three derivatives of Lyapunov functions. We gave a convex condition for asymptotic stability that includes the condition in [12] as a special case. Once again, we demonstrated the connection to standard Lyapunov functions. An example of a polynomial vector field was given to show the potential advantages of using higher order derivatives over standard Lyapunov theory. We also discussed a theorem in [44] that imposes minor conditions on the first and second derivatives to reject existence of periodic orbits, limit cycles, or chaotic attractors. We gave some simple convex conditions that imply the requirement in [44] and we compared them with those given in [13].

Our work leaves three future directions to be explored. First, it would be interesting to classify when searching for non-monotonic Lyapunov functions is guaranteed to be advantageous. As we discussed in this thesis, our non-monotonic Lyapunov functions can be interpreted as standard Lyapunov functions of a specific structure. Since our formulations include Lypunov's theorem as a special case, they will always perform at least as good as standard Lyapunov theory. The interesting question is for what type of dynamical systems are they guaranteed to do strictly better? In this thesis, we gave many examples that showed one can get strict improvement using our methodology, but we have not specified the exact situations when this must happen. Is it the case, for instance, that for a particular type of dynamical system the existence of a non-monotonic quadratic Lyapunov function, which has the first three derivatives in its structure *necessary*? Second, the connection of our methodology to vector Lyapunov functions (e.g. [22], [24]) needs to be clarified. Since our convex theorems map the state space into multiple Lyapunov functions instead of one, we suspect that our non-monotonic functions may be related to the concept of vector Lyapunov functions. Finally, other control applications such as synthesis, or robustness and performance analysis can be explored using non-monotonic Lyapunov functions.

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