The Geometry and Topology of Quotient Varieties

by

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Abstract

Let X be a nonsingular projective variety with an algebraic action of a complex torus $(\mathbf{C}^*)^n$. We study in this thesis the symplectic quotients (reduced phase spaces) and the quotients in a more general sense. As a part of our program, we have developed a general procedure for computing the intersection homology groups of the quotient varieties. In particular, we obtained an explicit inductive formula for the intersection Poincaré polynomial of an arbitrary quotient. Also, explicit results were obtained in the case of the maximal torus actions on the flag varieties G/B.

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INTRODUCTION

0.1. The aim of this thesis is to study the geometry and the topology of the quotient varieties of torus actions in algebraic geometry.

As a part of our program, we developed a general procedure for computing the intersection homology groups of the quotient varieties. In particular, we obtained an explicit inductive formula for the intersection Poincaré polynomial of an arbitrary quotient, which involves only polynomials. (Kirwan ([Ki]) has a formula for symplectic quotients which involves power series because of the use of equivariant cohomology. However, our formulas apply not only to symplectic quotients but also to more general quotients in question and our main tool to this end is the decomposition theorem of intersection homology of Beilinson, Bernstein and Deligne [BBD]. See also [GoM3]).

Also, explicit results were obtained in the case of maximal torus actions on homogeneous space G/P, especially on the flag varieties G/B. (Historically, we first worked out the case of maximal torus actions on G/B, and generalized the results there to the general case later on.)

0.2. Let X be a projective algebraic variety with an effective action of an algebraic torus $H = (\mathbf{C}^*)^n$. Assume the torus action extends to a linear action on the ambient projective space $\mathbf{P}^N$. Choose a Kähler metric on $\mathbf{P}^N$ which is invariant under the compact torus $T = (S^1)^n \subset (\mathbf{C}^*)^n$ and let $\mu : X \to \mathbf{R}^n$ be an associated moment map, then it is known that μ is T-equivariant and $\mu(\overline{H \cdot x})$ is a convex polyhedron in $\mathbf{R}^n$ for any $x \in X$. We therefore get a decomposition of X into invariant subspaces, $X = \bigcup_{C \in \Xi} X^C$, as follows: two points x, y are in the same stratum if and only if $\mu(\overline{H \cdot x}) = \mu(\overline{H \cdot y})$. Note that Ξ is a collection of polyhedra in $\mathbf{R}^n$.

There is a natural decomposition of $\mu(X)$ into a union of convex polyhedra in $\mathbf{R}^n$

$$\mu(X) = \bigcup_{F \in \Upsilon} F$$

where Υ is the index set consisting of the following specified polyhedra: every top dimensional open polytope F in Υ is a connected component of the regular values of the moment map μ, and the other open polyhedra are just the faces of those top dimensional polyhedra.

0.3. Symplectic Quotients. Since the ordinary topological quotient (or, orbit space) of the action is non-Hausdorff, to define an appropriate "quotient" variety in the category of algebraic geometry, some "bad" orbits have to be left out. Unfortunately, there is no canonical way to do this. As a consequence, we may have many quotient varieties associated to this action. One of the classes of quotient varieties can be obtained in the following way: let p be a point in $\mu(X)$, the moment map image of X, define

$$\mathcal{U}_p = \bigcup\{X^C \mid p \in C\},$$

then $\mathcal{U}_p$ is a Zariski open subset of X and the categorical quotient $\mathcal{U}_p//H$ in the sense of Mumford's geometric invariant theory [MuF] exists. Furthermore, if p is in the interior of $\mu(X)$, then $\mathcal{U}_p//H$ has the "correct" dimension, that is, it is of dimension $dim\ X - dim\ H$. In this case, we call $\mathcal{U}_p//H$ a nondegenerate quotient. Otherwise (i.e, p is in the boundary of $\mu(X)$), then the dimension of $\mathcal{U}_p//H$ is strictly less than $dim\ X - dim\ H$. In this case, we call it a degenerate quotient. An extreme example of degenerate quotients is the case when p is a vertex of $\mu(X)$, in this case, $\mathcal{U}_p//H$ is a point variety.

$\mathcal{U}_p//H$ is often called a symplectic quotient because $\mathcal{U}_p//H$ can be naturally identified with $\mu^{-1}(p)/T$ which is a reduced phase space [MaW] when p is a regular value of μ, where $T = (S^1)^n \subset (\mathbf{C}^*)^n = H$ is the compact part of H.

Let $\mathcal{P}$ denote the set of symplectic quotients. Then there is a natural partial order $\prec$ on $\mathcal{P}$. Actually, given $F \in \Upsilon, p \in F$, then $\mathcal{U}_p$ does not depend on p, i.e, $\mathcal{U}_p \equiv \mathcal{U}_q$, if $p, q \in F$. So we also write $\mathcal{U}_F$ instead of $\mathcal{U}_p$ sometimes. Then, the partial order $\prec$ in $\mathcal{P}$ can be characterized as follows: $\mathcal{U}_G//H \prec \mathcal{U}_F//H$ if and only if G is a face of F.

Theorem (1.4, 1.5). (1). If $\mathcal{U}_q//H \prec \mathcal{U}_p//H$, then there is a canonical algebraic projective map $f : \mathcal{U}_p//H \to \mathcal{U}_q//H$ which often corresponds to a blowing up map (it may be a fibration, for example). (2). $\mathcal{P}$ together with the canonical morphisms forms a nicely connected category, i.e, any two objects in the category can be connected by a finite chain of some nice morphisms of the category. (For the definition of the nice morphisms, see chapter 1.) (3) Consequently, any two non-degenerate symplectic quotients are related by a sequence of canonical blowing-ups and blowing-downs.

We point out that for a fixed moment map μ (i.e, an equivariant embedding of X in some ambient projective space, or a metric for simplicity), $\mathcal{U}_p//H$ ($p \in \mu(X)$) do not give all symplectic quotients. So to get all of symplectic quotients, we have to vary the metric on X and to consider $\mu^{-1}(p)/T$ for various corresponding moment maps μ.

0.4. Algebraic Quotients. There is another important and interesting class of quotient varieties, "geometric" quotients and "semi-geometric" quotients, which was first defined by A. Bialynicki-Birula and J. Sommese [B-BS2]. The definition of such a quotient is, like that of a symplectic quotient, also combinatorial and depends only on the moment map. (We point out that the whole theory presented here has a moment-map-free presentation, that is, we can work out the same results without using moment maps. This indicates that many of our results are also true with appropriate modifications in characteristic $p > 0$. The trick is that the fixed point set of the torus action can be used to play the role of a moment map.) In fact, we can define the quotients in the sense of B-BS in terms of the decompositions of $\mu(X)$ into disjoint unions of moment map images of torus orbits. Since the quotients in the sense of B-BS must be non-degenerate, we give, in section 2.3, a slightly generalized version of their quotients so that the generalized quotients can be degenerate. We shall call these (generalized) quotients algebraic quotients.

Let $\mathcal{P}^*$ denote the set of all algebraic quotients, then $\mathcal{P} \subset \mathcal{P}^*$. One can define the canonical algebraic maps among the quotients in $\mathcal{P}^*$ and prove that $\mathcal{P}^*$ together with canonical morphisms forms a category. The following theorem is an analogue of the above theorem for $\mathcal{P}^*$, although its proof is combinatorially much more complex than the previous one.

Theorem (2.4, 2.6). The theorem (1.4,1.5) is also true when replacing $\mathcal{P}$ by $\mathcal{P}^*$. Moreover, $(\mathcal{P}, \prec)$ is a proper subposet of $(\mathcal{P}^*, \prec^*)$, in general.

As we shall see, the connectedness of $\mathcal{P}$ is almost obvious, but the connectedness of $\mathcal{P}^*$ is far more vague. Also given an equivariant algebraic map from one variety to another, one can "push-forward" and "pull-back" the quotients in the category of $\mathcal{P}^*$ via the equivariant morphism. But one can not "pull-back" the quotients in the category of $\mathcal{P}$, in general. In other words, the pull back of a symplectic quotient may not be symplectic in general.

Theorem(3.1, 3.2). There is a canonical "biggest quotient variety" $\mathcal{Q}$ with the following properties: (1). $\mathcal{Q}$ is a natural compactification of the space of the closures of generic orbits. (2). For any algebraic quotient $\mathcal{U}//H$, there is a natural sujective algebraic map from $\mathcal{Q}$ to $\mathcal{U}//H$.

To save space in this introduction, in the following, we shall mainly mention the properties of quotients in $\mathcal{P}$ with the understanding that similar properties also hold for quotients in $\mathcal{P}^*$. The reader should not think that it is easy to generalize results from the category of $\mathcal{P}$ to $\mathcal{P}^*$. The only reason to restrict our attention, in this introduction, to the category $\mathcal{P}$ is that the quotients in $\mathcal{P}^*$ need more terminologies and descriptions to deal with.

0.5. Stratifying Canonical Maps. The main theme of this thesis is then to investigate the algebraic maps defined in the theorems of 0.3 and 0.4 in an explicit way, and to apply the decomposition theorem (in the theory of intersection homology) to the above algebraic maps to connect intersection homologies. For this purpose, we have

Let $p \in \mu(X)$ be a general point, $q \in \mu(X)$ be in the interior of a *codim* 1 wall M. Assume also there are two $G, F \in \Upsilon$, G is a face of F such that $p \in F, q \in G$. Now let H_M = stabilizer of $\overline{X^M}$, $H^M = H/H_M$, then H^M acts effectively on $\overline{X^M}$. Let also $\mathcal{U}_q(M) = \bigcup\{X^D | \; q \in D \subset M\}$, and $B = \mathcal{U}_q(M)/H^M$, then $B = \mathcal{U}_q \cap \overline{X^M}/H^M$ can be considered as a geometric quotient of $\overline{X^M}$ with H^M action (Note that $\overline{X^M}$ is nonsingular), and $B \subset \mathcal{U}_q//H$.

Theorem (5.1.1, 2). (1) If M is a face of $\mu(X)$ (i.e, q is on the boundary of $\mu(X)$), then $B = \mathcal{U}_q//H$, and the natural projection $\pi : \mathcal{U}_p/H \rightarrow \mathcal{U}_q//H$ is a fiber bundle whose fiber is a weighted projective space.

(2) If p is an interior point, $\pi : \mathcal{U}_p/H = X \rightarrow Y = \mathcal{U}_q//H$ is the natural projection, and $A = \pi^{-1}(B)$, then $\pi|A : A \rightarrow B$ is a fiber bundle whose fiber is a weighted projective space, and π is a isomorphism off B.

(3) The fiber of π in (1) and $\pi|A$ in (2) are ordinary projective spaces if the action is quasi-free, i.e, any finite isotropy group is trivial.

The fact that $\pi|A : A \rightarrow B$ is a weighted projective bundle over B can be derived from the decomposition theorem of Bialynicki-Birula [B-B]. In fact, $dim H_M = 1$ since $dim M = n - 1$. Hence H_M gives a $(\mathbf{C}^*)$-action on X. Let μ_M be its associated moment map, then μ_M is a non-degenerate Morse function [A1], hence μ_M induces a Morse stratification $X = \bigcup_\alpha S_\alpha$ and each S_α is a cell-bundle over a certain connected component of the critical point set of μ_M (which is the same as the fixed point set of H_M). In [B-B], Bialynicki-Birula proved more, he concluded that each cell-bundle above is actually a complex vector bundle, and the induced $(\mathbf{C}^*)$-action on the fiber of the vector bundle is equivalent to a linear action. The proof of (3) is immediate.

The theorem above takes the following version when p, q are arbitrary interior points.

Theorem.(5.1.3,4). Let p, q be two interior points, and $F, G \in \Upsilon$ such that $G \prec F$ and $p \in F, q \in G$. Then there is a canonical stratification on $Y = \mathcal{U}_q//H = \bigcup_\beta C_\beta$ such that the natural projection $\pi : \mathcal{U}_p//H \rightarrow \mathcal{U}_q//H$ becomes a stratified map. More precisely, for each β, $\pi|\pi^{-1}(C_\beta) : \pi^{-1}(C_\beta) \rightarrow C_\beta$ is a fiberation tower whose fibers are all weighted projective spaces.

0.6. Small Resolutions. There are many small maps among the canonical algebraic maps of the quotient varieties. In particular, we have

Theorem (5.3, 5.4). (1). If p is a general point, then $\mathcal{U}_p//H \longrightarrow \mathcal{U}_q//H$ is a rational resolution of singularities (i.e, a resolution up to finite quotient singularities) of $\mathcal{U}_q//H$. (In fact, a resolution if the action is quasi-free.)

(2) For any interior point $r \in \mu(X)$, there exists a general point $p \in \mu(X)$ such that $\mathcal{U}_p/H \longrightarrow \mathcal{U}_q//H$ is a small map. Consequently, $H_*(\mathcal{U}_p/H) = IH_*(\mathcal{U}_q//H)$.

The following is a consequence of the above and the decomposition theorem of intersection homology theory.

Corollary. (5.6, 5.7). Let the notations be as in theorem (5.1.3,4). Then for any β, and $y \in C_\beta$, the local intersection homology groups at y are determined by the following equality:

$$IP_y(Y) = P_t(\pi^{-1}(y)) = P_t(\Pi_{i=1}^m \mathbf{P}^{d_i}) = \Pi_{i=1}^m P_t(\mathbf{P}^{d_i})$$

for some m and $d_i > 0, i = 1 \cdots m$.

In virtue of theorem (5.3,5.4) above, to calculate the intersection homology groups of quotient varieties, it is enough to focus on rationally nonsingular quotients $\mathcal{U}_p/H$ (where p are general points).

0.7. Homological Formula. Now we start to formulate our (intersection) homology formula.

Let q be a general point in the interior of $\mu(X)$ and p a point on the boundary of $\mu(X)$. Let also $\overrightarrow{p,q}$ be a piece-wise linear path from p to q such that it does not meet any *codim* ≤ 2 wall. Suppose $\overrightarrow{p,q}$ meets exactly k *codim* 1 walls $M_1, \cdots, M_k$ in the points $r_1, \cdots, r_k$ and we have

$$p \longrightarrow \epsilon(M_1)M_1 \longrightarrow \cdots \longrightarrow \epsilon(M_k)M_k \longrightarrow q$$

where $\epsilon(M_i) = \pm 1$ (depending only on the direction of $\overrightarrow{p,q}$ and M_i). We also make the following convention:

$$H_j = H/(stabilizer\ of\ \overline{X^{M_j}}\ in\ H),$$

$$T_j = T/(stabilizer\ of\ \overline{X^{M_j}}\ in\ T),$$

and $\mathcal{U}_{r_j}(M_j) = \mathcal{U}_{r_j} \cap \overline{X^{M_j}} = \bigcup\{X^D | \ r_j \in D \subset M_j\}$. (Note that $\mathcal{U}_{r_j}(M_j)/H_j = \mu^{-1}(r_j) \cap \overline{X^{M_j}}/T_j$ is a symplectic quotient of H_j-action on $\overline{X^{M_j}}$, $j = 1, \cdots, k$). Then we have

Theorem.(5.7). (An inductive homological formula.)

$$P_t(\mathcal{U}_q/H) = \Sigma_{j=1,\dots,k}\epsilon(M_j)Q_t(M_j)P_t(\mathcal{U}_{r_j}(M_j)/H_j)$$

or

$$P_t(\mu^{-1}(q)/T) = \Sigma_{j=1,\dots,k}\epsilon(M_j)Q_t(M_j)P_t(\mu^{-1}(r_j)\cap \overline{X^{M_j}}/T_j)$$

where $Q_t(M_j) = t^{2d_j+2} + \cdots + t^{2e_j}$, or 0, where $d_j \leq e_j$ are two integers depending only on M_j (They are the codimensions of certain subvarieties determined by M_j. Also in other words, the pair $(2d_j, 2e_j)$ is the signature at $\overline{X^{M_j}}$ of the Morse function μ_{M_j}.)

0.8. Vanishing Theorem and Cycle Maps. Let X be a compact complex variety, $H_i(X)$ be the ith integral homology group and $A_k(X)$ be the group generated by k-dimensional irreducible subvarieties modulo rational equivalence, then there is a canonical homomorphism (cycle map, see [Fu]):

$$Cl_X : A_i(X) \longrightarrow H_{2i}(X).$$

A variety X is said to have property (IS) if

(a) $H_i(X) = 0$ for i odd, $H_i(X)$ has no torsion for i even.

(b) $Cl_X : A_i(X) \xrightarrow{\cong} H_{2i}(X)$ for all i.

A variety X is said to have property (RS) if

(a) $H_i(X) \otimes \mathbf{Q} = 0$ for i odd.

(b) $Cl_X \otimes \mathbf{Q} \xrightarrow{\cong} H_{2i}(X) \otimes \mathbf{Q}$ for all i.

Theorem. (6.3.) Let $\mathcal{U}//H$ be an arbitrary algebraic quotient, then

(1) the rational intersection homology groups of $\mathcal{U}//H$ vanish in odd degree and have no torsion in even degree if the fixed point set has the same property.

(2) the integral intersection homology groups of $\mathcal{U}//H$ vanish in odd degree and have no torsion in even degree if the fixed point set has the same property and the action is quasi-free.

Theorem. (6.4). Let $\mathcal{U}//H$ be an arbitrary algebraic quotient, then

(1) The rational cycle maps of $\mathcal{U}//H$ are isomorphisms if the rational cycle maps of the fixed point set are isomorphisms.

(2) The cycle maps of $\mathcal{U}//H$ are isomorphisms if the cycle maps of the set of the fixed points are isomorphisms and the action of the torus H is quasi-free.

As a consequence, one can see:

Let $\mathcal{U}//H$ be an arbitrary nonsingular algebraic quotient, then (1)$\mathcal{U}//H$ has the property (RS) if the fixed point set has (RS). (2) $\mathcal{U}//H$ has property (IS) if the fixed point set has (IS) and the action of torus H is quasi-free.

0.9. Flag Manifolds. The case when $X = G/B$ is a flag variety and H is a fixed maximal torus contained in B deserves detailed study in its own right.

In this particular case, we have: The closure of X^M, where M is a wall of $\mu(X)$, can be naturally identified with P/B, where $P \supset B$ is a parabolic subgroup. Hence, all the fibrations in the theorems of 5.1 are trivial bundle because the normal bundle of P/B in G/B is trivial,

We described, in the thesis, the moment map images in the case of G/B in terms of parabolic subgroups of the Weyl group W or coxeter complexes, together we also described the torus strata closures for some interesting moment map images.

The case $G = SL(n+1, \mathbf{C})$ is particularly interesting.

Theorem. (7.2). One of the geometric quotient of maximal torus action on the variety of full flags in $\mathbf{C}^{n+1}$ can be identified (not canonically) with the variety of full flags in $\mathbf{C}^n$. As a consequence, any other geometric quotient can be derived from this flag variety by a finite sequence of blow-ups and blow-downs.

As expected, one can describe, in the case when $G = SL(n+1, \mathbf{C})$, the moment map images and their strata closures in terms of both symmetry group and Schubert conditions.

0.10. Homology of Complements of Subspaces. Naturally associated to a torus action, one can study the following three kinds of spaces: 1. The quotient varieties. 2. The torus strata. 3. The closures of torus orbits as toric varieties.

In this thesis, we mostly only study the quotient varieties. An attempt to study torus strata has led us to consider arrangements

$$\mathcal{A} = \{A_1, \cdots, A_m\}$$

in $\mathbf{R}^n$, where $A_1, \cdots, A_m$ are closed subspaces of $\mathbf{R}^n$ satisfying the following 2 conditions: (a) each A_i is either homeomorphic to Euclidean space $\mathbf{R}^k$ of dimension k or to the sphere S^k of dimension k, for some $k < n$. (b) each connected component of an arbitrary non-empty intersection $A_{i_1} \cap \cdots \cap A_{i_r}$ satisfies also condition (a).

Associated to every arrangement $\mathcal{A} = \{A_1, \cdots, A_m\}$, there is a ranked poset $\mathcal{L}(\mathcal{A}) = (\mathcal{L}, \prec, r)$ which can be constructed explicitly from the combinatorial data of the intersections of $\mathcal{A}$. Then the combinatorics of $\mathcal{L}(\mathcal{A}) = \mathcal{L}$ determines completely the homology of the complement, $M(\mathcal{A}) = \mathbf{R}^n - \cup_{i=1}^m A_i$, of $\mathcal{A}$.

Theorem. (A) (Homological formula for the complements of subspaces.)

$$H_i(\mathbf{R}^n - \cup_{i=1}^m A_i; \mathbf{Z}) = \bigoplus_{v\in\mathcal{L}} H^{n-r(v)-i-1}(K(\mathcal{L}_{>v}), K(\mathcal{L}_{(v,T)}); Z)$$

where T is the unique maximal element in $\mathcal{L}$ representing $\mathbf{R}^n$, $H^{-1}(\phi, \phi) = \mathbf{Z}$ as a convention, and $K(\mathcal{P})$ denotes the order complex of the poset $\mathcal{P}$.

When each A_i in $\mathcal{A}$ is an affine linear subspace in $\mathbf{R}^n$, our formula coincides with the one obtained by Goresky and MacPherson [GoM4]

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Contents

Chapter 1

Symplectic Quotients and Their Properties

The main object of this chapter is to collect a few well-known results. The fact that the symplectic quotients together with the canonical morphisms form a category is pointed out. Furthermore, we prove that this category is nicely connected, that is, any two objects in the category can be connected by a finite chain of some "nice" morphisms in the category.

1.1 Notation and Conventions

Let X be a complex projective variety with an action of an algebraic torus $H = (\mathbf{C}^*)^n$. We assume the torus action extends to a linear action on the ambient projective space $\mathbf{P}^N$. Choose a Kaehler metric on $\mathbf{P}^N$ which is invariant under the compact torus $T = (S^1)^n \subset (\mathbf{C}^*)^n$ and let $\mu : X \to \mathbf{R}^n$ be the associated moment map (i.e, the restriction to X of the moment map associated to the ambient projective space $\mathbf{P}^N$. So we can talk about moment maps for X even if X is singular.) Then for any x in X, it is known that $\mu(\overline{H \cdot x})$ is a convex polyhedron C in $\mathbf{R}^n$, and $H \cdot x/T$ projects homeomorphically to the interior C^0 of C under μ. ([A1], [GuSt1]).

Convention. Let Ξ denote the collection of μ*−images* of torus orbit closures. Then this is a collection of compact polyhedra in $\mathbf{R}^n$.

1.2 The Torus Stratifications

Definition. Let C be a $\mu -$ *image* of a torus orbit closure. A point $x \in X$ is in the **torus stratum** X^C if $\mu(\overline{H \cdot x}) = C$, i.e.

$$X^C = \{x \in X | \mu(\overline{H \cdot x}) = C.\}$$

Then

$$X = \bigcup_{C \in \Xi} X^C,$$

which we shall call the **torus stratification** of X. We should warn the reader that this is not a Whitney stratification but merely a decomposition of X into a union of locally compact subspaces.

Let $D, C \in \Xi$. If D is a face of C, then there is a unique algebraic map $\rho_{CD} : R^C = X^C/H \to R^D = X^D/H$ which can be characterized as follows: suppose $x \in X^C$ is a lift of $\bar{x} \in X^C/H$ and suppose $y \in X^D$ is a lift of $\bar{y} \in X^D/H$, then $\rho_{CD}(\bar{x}) = \bar{y}$ if and only if $y \in \overline{H \cdot x}$ (see [GoM1]).

1.3 The Definition of Symplectic Quotients

Definition. Let $\Box = \mu(X)$, and $p \in \Box$, define

$$\mathcal{U}_p = \bigcup\{X^C | p \in C,\}$$

then $\mathcal{U}_p$ is a zariski open subset of X, and the categorical quotient $\mathcal{U}_p//H$ in the sense of Mumford's geometric invariant theory exists. It is also very common to denote $\mathcal{U}_p$ by X_p^{ss} in accordance with geometric invariant theory. (the subscripts "ss" stand for semi-stable, hence X_p^{ss} means the collection of "semi-stable" points with respect to point p.) Moreover, if p is in the interior of $\mu(X)$, then $\mathcal{U}_p//H$ has the "correct" dimension, that is, it is of dimension $dim\ X - dim\ H$. In this case, we call $\mathcal{U}_p//H$ a nondegenerate quotient. Otherwise (i.e, p is in the boundary of $\mu(X)$), then the dimension of $\mathcal{U}_p//H$ is strictly less than $dim\ X - dim\ H$. In this case, we call it a degenerate quotient. An extreme example of degenerate quotients is the case when p is a vertex of $\mu(X)$. In this case, $\mathcal{U}_p//H$ is a point variety.

we make a convention that the interior of a polyhedron is called an open polyhedron. We shall often use D^0 to denote the interior of a polyhedron D. And if D is a face of C, then we write $D \prec C$.

There is a natural decomposition of $\Box = \mu(X)$ into a union of convex polyhedra

$$\Box = \bigcup_{F \in \Upsilon} F$$

where Υ is the index set , Such that every top dimensional open polyhedron F in Υ is a connected component of the regular values of the moment map μ, and the other open polyhedra are just open faces of those of top dimension.

Given a $F \in \Upsilon, p \in F, \mathcal{U}_p$ does not depend on p, i.e.

$$\mathcal{U}_p \equiv \mathcal{U}_q,\ if\ p, q \in F.$$

So sometimes we write $\mathcal{U}_F$ instead of $\mathcal{U}_p$. When p is in a top dimensional F, i.e, p is a regular value of μ, $\mathcal{U}_p//H$ coincides with the ordinary orbit space $\mathcal{U}_p/H$.

It is known that $\mathcal{U}_p//H$ can be identified with $\mu^{-1}(p)/T$, $T = (S^1)^n \subset (\mathbf{C}^*)^n$. If X is nonsingular, the torus action is **quasi-free** (i.e., all the finite isotropy subgroups are identity subgroup), and p is a regular value of μ, then $\mu^{-1}(p)/T$ is a nonsingular symplectic manifold, called a **reduced phase space** of the torus action. If we do not assume that the action is quasi-free, then $\mu^{-1}(p)$ may have finite quotient singularities. However, if p is not a regular value (i.e, p is in the $\mu - image$ of a torus orbit of dimension less than n), $\mu^{-1}(p)/T$ may have serious singularities in general, it is a singular symplectic space. We shall call $\mu^{-1}(p)/T$ or $\mathcal{U}_p//H$ a **symplectic quotient.**

Remark. The decomposition

$$\Box = \bigcup_{F \in \Upsilon} F$$

depends on the moment map (or the metric on X) very much . Hence for a fixed moment map μ (or a metric), $\mu^{-1}(p)/T$ $(p \in \mu(X))$ do not give all symplectic quotients. To get all of symplectic quotients, we have to vary the metric on X and consider $\mu^{-1}(p)/T$ for various corresponding moment maps μ. Note also , for two different moment maps μ_1 and μ_2, some $\mathcal{U}_p(p \in \mu_1(X))$ may be identical to $\mathcal{U}_q$ (for some $q \in \mu_2(X)$)!

1.4 What Happens When Passing Through Singular Values

Definition. A codim d $\mu - image$ of a torus orbit closure,M, is called a codim d wall if M is not contained in any other codim d $\mu - image$ of a torus orbit closure. A wall is called an interior wall if it is not a face of $\mu(X)$.

Remark. If X is nonsingular and M is a wall, then $\overline{X^M}$ is a nonsingular subvariety of X with the action of H/H_M where H_M is the isotropy subgroup of $\overline{X^M}$. This fact follows from two arguments as follows:

(1) The connected components of fixed point set of a torus action on a nonsingular variety are nonsingular.

(2) $\overline{X^M}$ is a connected component of the fixed point set of the action of H_M on X.

As pointed out in 1.3, when a point p varies within an "open" polytope of Υ, then the homeomorphic type of $\mu^{-1}(p)/T$ does not change. This fact was first observed by Duistermaat and Heckman for regular p, where they even showed that the symplectic form on $\mu^{-1}(p)/T$ changes in a simple fashion if p moves in a simple fashion. In [GuSt], it was proved that when p goes from one side of an interior codim 1 wall to the other, then the diffeotype of $\mu^{-1}(p)/T$ changes by a blowing down followed by a blowing up. These blowing up and downs are in fact canonical. This was done explicitly in [GoM1]. One of the advantages of [GoM1] is that it tells us not only what happens when passing through a codim 1 wall, but also what happens when passing through higher codim walls.

Theorem. [GoM1, GuSt3]. Let $F_1, F_2 \in \Upsilon$, and F_2 be an open face of F_1, then there is a unique map f from $\mathcal{U}_{F_1}//H$ to $\mathcal{U}_{F_2}//H$ which corresponds to a blowing up map if the both quotients are non-degenerate.

Given a $p \in Int\Box$. Let $C \in \Xi$, and $p \in C$. Define

$$\mathcal{U}_p(C) = \bigcup_{p \in D \subset C} X^D,$$

then we have

$$\begin{tikzcd}
\mathcal{U}_p(C) \arrow[r, phantom, "\subset"] \arrow[d] & \mathcal{U}_p \arrow[d] \\
\mathcal{U}_p(C)//H \arrow[r, phantom, "\subset"] & \mathcal{U}_p//H,
\end{tikzcd}$$

which is a commutative diagram. $\mathcal{U}_p(C)//H$ can be thought as a categorical quotient of the variety $\overline{X^C}$.

Convention. Two points a and b in $\mu(X)$ are said to be "close enough" to each other if there is an open polytope F in Υ whose closure contains both a and b and (at least) one of a and b is contained in F itself.

Let r be a (relatively) general point in a wall M, and p, q be two general points on two different sides of M and close enough to r (in the other words, there are two top dimensional $F_1, F_2 \in \Upsilon$ such that $p \in F_1, q \in F_2$, and $r \in F_1 \cap F_2 \subset M$). Suppose also there are two M^+, M^- in Ξ with M as their common face such that for any $E \in \Xi$, if $r \in E$, then exactly one of the following three is true: 1). E contains both of p and q; 2). $E \subset M^+$; 3). $E \subset M^-$. So without loss of generality, we assume $F_1 \subset M^+, F_2 \subset M^-$. Then the following proposition follows immediately from the definition.

Proposition. The diagram

$$\begin{tikzcd}
\mathcal{U}_p(M^+)/H \arrow[r] \arrow[d] & \mathcal{U}_r(M)/H \arrow[d] & \mathcal{U}_q(M^-)/H \arrow[l] \arrow[d] \\
\mathcal{U}_p/H \arrow[r, "f"] & \mathcal{U}_r//H & \mathcal{U}_q/H \arrow[l, "g"']
\end{tikzcd}$$

commutes,where the vertical maps are closed embeddings, and f is an isomorphism off $\mathcal{U}_p(M^+)/H$, g is an isomorphism off $\mathcal{U}_q(M^-)/H$. In fact, $\mathcal{U}_p/H - \mathcal{U}_p(M^+)/H = \mathcal{U}_r//H - \mathcal{U}_r(M)/H = \mathcal{U}_q/H - \mathcal{U}_q(M^-)/H$, so f and g are actually identities over $\mathcal{U}_p/H - \mathcal{U}_p(M^+)/H$ and $\mathcal{U}_q/H - \mathcal{U}_q(M^-)/H$, respectively.

Remark. In the case of a homogeneous space G/P with a maximal torus action, the conditions preceding the proposition above are fulfilled automatically. (see 7.7).

1.5 Symplectic Quotients with Algebraic Maps

Let $\mathcal{P} = \mathcal{P}(X)$ denote the poset of all symplectic quotients (for various metrics on X) ordered by projection characterized in the theorem of the previous section (note also the remark before that theorem). clearly, $\mathcal{P}$ together with the canonical algebraic maps forms a finite category. We shall still use $\mathcal{P}$ to denote this category.

Definition Let $f : \mathcal{U}_p//H \to \mathcal{U}_q//H$ be a canonical morphism. f is called nice if the dimension of the isotropy subgroup of H on $\mathcal{U}_p$ is not great than the dimension of the isotropy subgroup of H on $\mathcal{U}_q$ plus 1. For example a morphism from a quotient variety to a point variety is, in general, not nice, for there is no usuful information contained in this morphism. To see "how nice" a nice morphism can be, see chapter 5.

Theorem. $\mathcal{P}$ as a category is nicely connected. In other words, any two symplectic quotients can be connected by a finite chain of some nice morphisms.

Proof. For a given metric, let μ be a moment map determined by this metric (note that any two moment maps under the same metric only differ by a constant in $\mathbf{R}^n$, so it is enough to consider only one fixed moment map for each metric), and let

$$\mathcal{P}_\mu = \{\mathcal{U}_q//H \mid q \in \mu(X)\},$$

then, it is quite clear that $\mathcal{P}_\mu$ gives a nicely connected category. Since

$$\mathcal{P} = \cup_\mu \mathcal{P}_\mu$$

and we have only finitely many different $\mathcal{P}_\mu$'s, it suffices to show that $\mathcal{P}'_{\mu_1} \cap \mathcal{P}'_{\mu_2} \neq \emptyset$. where

$$\mathcal{P}'_{\mu_1} = \{\mathcal{U}_p |\, p \in Int(\mu_1(X))\}$$
$$\mathcal{P}'_{\mu_2} = \{\mathcal{U}_q |\, q \in Int(\mu_2(X))\}$$

We want to show this by induction on the dimension of X.

If $dimX = 0$, then the assertion is trivial. Now let $dimX = N$, and assume that the assertion is true for varieties of dimension less than N. Take two codimension 1 faces σ_1 and σ_2 of $\mu_1(X)$ and $\mu_2(X)$ respectively so that $X^{\sigma_1} = X^{\sigma_2}$, i.e

$$\mu_1^{-1}(vertices\ of\ \sigma_1) = \mu_2^{-1}(vertices\ of\ \sigma_2),$$

then

$$dim\overline{X^{\sigma_1}} = dim\overline{X^{\sigma_2}} < dimX,$$

so by the induction hypothesis, there are two (relatively) general points q_1 and q_2 in σ_1 and σ_2 respectively, such that

$$\mathcal{U}_{q_1}(\sigma_1) = \mathcal{U}_{q_2}(\sigma_2)$$

Now let p_1 and p_2 be two general points in $\mu_1(X)$ and $\mu_2(X)$, close enough to q_1 and q_2 respectively, then

$$\mathcal{U}_{p_1} = \{x \in X |\, \overline{Hx} \supset (\neq) Hy for\ some\ y \in \mathcal{U}_{q_1}(\sigma_1)\}$$

$$\mathcal{U}_{p_2} = \{x \in X |\, \overline{Hx} \supset (\neq) Hy for\ some\ y \in \mathcal{U}_{q_2}(\sigma_2)\}$$

Since

$$\mathcal{U}_{q_1}(\sigma_1) = \mathcal{U}_{q_2}(\sigma_2),$$

the above implies that $\mathcal{U}_{p_1} = \mathcal{U}_{p_2}$, that is

$$\mathcal{P}'_{\mu_1} \cap \mathcal{P}'_{\mu_2} \neq \emptyset.$$

so

$$\mathcal{P}_{\mu_1} \cap \mathcal{P}_{\mu_2} \neq \emptyset.$$

Corollary. Every two symplectic quotients are connected by a sequence of algebraic maps characterized in theorem 1.4. In particular, any two non-degenerate quotients are connected by a sequence of blowing-ups and blowing-downs characterized in theorem 1.4.

Chapter 2

Algebraic Quotients and Their Properties

In this chapter, the algebraic quotients (both non-degenerate and degenerate) are defined and the canonical morphisms among these algebraic quotients are characterized. The algebraic quotients together with the canonical maps form a finite category. Furthermore, we prove that this finite category is "nicely" connected.

2.1 Admissible Polyhedral Decompositions of $\mu(X)$

Let us follow the notation in the previous chapter.

Definition. Let $Int(\Box)$ be the interior of the convex polyhedron $\Box = \mu(X)$. A decomposition of $Int(\Box)$ into a union of "open" subpolytopes is said to be admissible if it is a disjoint union of $\mu - images$ of some torus orbits.

There are various decompositions of $Int(\Box)$ into disjoint union of $\mu - images$ of some torus orbits. The number of such decompositions is finite. Similarly, we call a decomposition of $\mu(X)$ into a union of disjoint μ- images of torus orbits an **admissible polyhedral decomposition** of $\mu(X)$, or simply **admissible decomposition** of $\mu(X)$. Actually, the two concepts determine each other in a unique way, which we shall formulate in lemma 4.2. In their paper [B-B,S], A. Bialynicki-Birula and J. Sommese constructed a class of Zariski open subsets that have Hausdorff compact normal quotients. In their construction they used the terminology of moment cell complex. In what follows, we shall interpret their construction in terms of decompositions of $Int(\Box)$ into disjoint union of $\mu - image$ of torus orbits, and generalize them to some Zariski open subsets whose Hausdorff compact quotients may have ("incorrect") smaller dimensions.

2.2 Definition of Geometric Algebraic Quotients

Definition-Proposition. (An interpretation of geometric quotients in the sense of B-B,S). Let Ξ_1 be a collection of top dimensional polytopes in Ξ, and

$$\mathcal{U} = \bigcup\{X^D | D \in \Xi_1\},$$

such that the collection $\{D^0 | D \in \Xi_1\}$ meets each admissible decomposition of $Int(\Box)$ in exactly one $\mu - image$ of torus orbit. Then $\mathcal{U}$ is Zariski open and the orbit space $\mathcal{U}/H$ is Hausdorff and compact. In the case that X is nonsingular, $\mathcal{U}/H$ has (possibly) only finite quotient singularities (caused by the finite isotropy subgroups).

We point out that such a quotient must be non-degenerate.

2.3 Definition of Semi-Geometric Algebraic Quotients

Definition-Proposition. (An interpretation of semi-geometric quotients in the sense of B-B,S). Let Ξ_1 is a collection of polyhedra in Ξ and

$$\mathcal{U} = \bigcup\{X^D | D \in \Xi_1,\}$$

such that no polyhedron in Ξ_1 lies on the boundary of $\Box$ and the collection $\{D^0 | D \in \Xi_1\}$ meets every admissible decomposition of $Int(\Box)$ in exactly one open polytope, then $\mathcal{U}$ has Hausdorff compact quotient $\mathcal{U}/H$ (which may have serious singularities).

In general, $\mathcal{U}$ is not a open subset, to remedy this, we define $\tilde{\mathcal{U}} \supset \mathcal{U}$ as follows, $\tilde{\mathcal{U}} = \bigcup\{X^C | C \ has \ a \ face \ in \ \Xi_1\}$, in other words, whenever $X^D \subset \mathcal{U}$, but D is not top dimension, we add those strata X^C where D is a face of C. Then $\tilde{\mathcal{U}}$ is a Zariski open subset of X, and the categorical quotient $\tilde{\mathcal{U}}//H$ in the sense of Mumford's geometric invariant theory can be identified with $\mathcal{U}/H$. Actually, there is unique map from $\tilde{\mathcal{U}}$ to $\tilde{\mathcal{U}}//H$ whose fiber at every point is a connected union of orbits with only one closed orbit in the union. So if we write $\tilde{\mathcal{U}}$ as a union of torus strata

$$\tilde{\mathcal{U}} = \bigcup\{X^D | D \in \tilde{\Xi}_1\}$$

where $\tilde{\Xi}_1$ is a subcollection of Ξ, then we have:

(1). $\tilde{\Xi}_1$ meets every admissible decomposition of $Int(\mu(X))$.

(2). Suppose $\tilde{\Xi}_1$ meets an admissible decomposition in exactly r polytopes, say, $D_1, \cdots, D_r$, then there is a unique minimal polytope among $D_1, \cdots, D_r$ under the face relation.

Again the quotients defined above must be non-degenerate. For degenerate quotients, we have the following generalized version of 2.2 and 2.3:

Proposition. Let every assumption be as in definition-proposition 2.2 and 2.3 except that we replace the admissible decompositions of $Int(\mu(X))$ by the admissible decompositions of $\mu(X)$. Then the quotient $\mathcal{U}//H$ still exists (but may be degenerate), where the open set $\mathcal{U}$ is defined in a similar way as before.

For convenience, we shall call the Zariski open subsets in section 2.2 and $\tilde{\mathcal{U}}$ above **algebraic open subsets**, their corresponding quotients **algebraic quotients**, and the polyhedral collections $\Xi_1 \subset \Xi$ defining them **admissible polyhedral collections,** or simply **admissible collections.** Clearly, any symplectic quotient is an algebraic quotient.

Convention. We want make a useful convention here. In what follows, when we say an admissible collection of subpolyhedra we shall either mean Ξ_1 or $\tilde{\Xi}_1$, depending whether we use "*tilde*" or not. However, when we say their corresponding algebraic *open* subset, we shall only mean $\tilde{\mathcal{U}}$.

2.4 Algebraic Maps among Algebraic Quotients

Definition. Let Ξ_1, Ξ_2 be two admissible collections of polytopes in Ξ. We say $\Xi_2 \prec \Xi_1$ if for any $D \in \Xi_2$, there is $C \in \Xi_1$ such that D is a face of C (they may be equal). Note that the definition is equivalent to : for any $C \in \Xi_1$, there is $D \in \Xi_2$ such that $D \prec C$ (they may be equal).

An alternative definition is: let $\tilde{\Xi}_1$ and $\tilde{\Xi}_2$ be two admissible collections of polyhedra in Ξ. We say that $\tilde{\Xi}_2 \prec \tilde{\Xi}_1$ if $\tilde{\Xi}_1 \subset \tilde{\Xi}_2$. (Do not confuse this definition with the proposition in the next section. Consult with convention 2.3.)

Proposition. Let Ξ_1, Ξ_2 be two admissible collections of polytopes in Ξ, $\mathcal{U}_1$ and $\mathcal{U}_2$ be their corresponding open subsets. Then there is a unique algebraic map ρ_{Ξ_1,Ξ_2} from $\mathcal{U}_1/H$ to $\mathcal{U}_2/H$. ρ_{Ξ_1,Ξ_2} often corresponds to blow up map.

Proof. ρ_{Ξ_1,Ξ_2} is induced from various algebraic map ρ_{CD}. In fact, $\mathcal{U}_1$ is included in $\mathcal{U}_2$, and the map ρ_{Ξ_1,Ξ_2} is just the induced map from that inclusion.

Definition Let $f : \mathcal{U}_1//H \to \mathcal{U}_2//H$ be a canonical morphism. f is called nice if the dimension of the isotropy subgroup of H on $\mathcal{U}_1$ is not great than the dimension of the isotropy subgroup of H on $\mathcal{U}_2$ plus 1. To see "how nice" a nice morphism can be, see chapter 6.

2.5 Propositions of Admissible Collections of Subpolytopes

Now We shall give a proposition of admissible collection: Let Ξ_1 be an admissible collection and $A \in \Xi_1$. If $B \in \Xi$ and $B^0 \supset A^0$ then $B \in \Xi_1$. The proof goes as below: Assuming that $B \notin \Xi_1$. Choosing an admissible decomposition $\Im$ containing B^0, since $B \notin \Xi_1$, then there is C such that $C \in \Xi_1$, $C^0 \cap B^0 = \emptyset$, $C^0 \in \Im$, Now replace B^0 in $\Im$ by an admissible decomposition of B^0 (think of $\overline{X^B}$ as a variety) that contains A^0, then we get an admissible decomposition $\Re$ of $\mu(X)$ which contains both A^0 and C^0. Clearly $A^0 \neq C^0$, contradiction, that is, $B \in \Xi_1$.

Proposition. For any two admissible collections of polyhedra in Ξ, say Ξ_1 and Ξ_2, if $\Xi_1 \subset \Xi_2$, then $\Xi_1 = \Xi_2$.

Proof. Assuming there is a polytope C in $\Xi_2 - \Xi_1$, then there is a admissible decomposition $\Re$ of $Int(\Box)$ containing C. Take the unique D in $\Re$ that belongs to Ξ_1, then we got two distinct polyhedra C and D in Ξ_2, contradiction.

Definition. We call an admissible collection Ξ_1 of polyhedra in Ξ general if it only consists of top dimensional subpolytopes.

Theorem. Let Ξ_1 be an admissible collection of polyhedra in Ξ, then there exists an admissible collection Ξ_2 of polyhedra in Ξ such that $\Xi_1 \prec \Xi_2$, Ξ_2 is general.

Proof. Pick up all the *codim* 1 walls containing some polyhedra in Ξ_1 of non top dimension. Let M be such a *codim* 1 wall, then $span_{\mathbf{C}} M$ divides $\mathbf{R}^n$ into two half space H_M^+ and H_M^-. Clearly, $\Xi_1(M) = \Xi_1 \cap M$ is an admissible collection for $\overline{X^M}$. Here we can apply induction to assume that there is an adimissible collection $\Xi_2(M)$ for $\overline{X^M}$ consisting of polytopes of dimension $n-1$, such that $\Xi_1(M) \prec \Xi_2(M)$. Now we define Ξ_2 to be the set of top dimensional $D \in \tilde{\Xi}_1$ such that for any above selected *codim* 1 wall M, we have either $D \cap M^0 \neq \emptyset$, or D is in H_M^+ and has a face in $\Xi_2(M)$. It is obvious that $\Xi_1 \prec \Xi_2$. As we can check directly Ξ_2 must meet every admissible decomposition of $Int(\Box)$. Now assuming there is admissible decomposition $\Re$ such that two elements D_1, D_2 in Ξ_2 are contained in the union $\Re$, then by the construction of Ξ_2, there must be $C_1, C_2 \in \Xi_2(M)$ such that $C_1 \prec D_1, C_2 \prec D_2$. Then C_1 must equal to C_2 because C_1, C_2 can be in one admissible decomposition (see lemma 4.2 for an explict reason). So the fact that D_1 and D_2 are both in H_M^+ implies that they must intersect. This contradicts with the definition of an adimissible decomposition of $\mu(X)$.

Remark. In the case that X is nonsingular, then the theorem tells us that

$$\mathcal{U}_2/H \to \mathcal{U}_1//H$$

is a rational resolution (in fact, a resolution in the case the action is quasi-free).

2.6 Algebraic Quotients with Algebraic Maps

Lemma. Let $(\mathbf{C}^*)^k$ act algebraicly on a compact complex variety X, and μ be an associated moment map. Then the moment map image of any torus orbit is a face of the moment map image of a top dimensional torus orbit. In other words, for each point $x \in X$, there is a $y \in$ such that $(\mathbf{C}^*)^k \cdot y$ is of top dimension and $x \in \overline{(\mathbf{C}^*)^k \cdot y}$.

Proof. Take z to be a generic point on X. From [B-BS2] (see also §3.3), we have the following diagram of morphisms

$$\begin{tikzcd} Z_z \arrow[r, "\phi"] \arrow[d, "f"'] & X \\ Q_z & \end{tikzcd}$$

where Z_z and Q_z are compact complex spaces, ϕ is surjective, and the image under ϕ of a fiber of f is a union of top dimensional orbit closures, therefore X is the union of torus orbit closures of top dimension. The theorem hence follows because ϕ is surjective.

Let $\mathcal{P}^* = \mathcal{P}^*(X)$ be the set of all algebraic quotients (or the set all admissible collections of polytopes in Ξ, equivalently). Then $\mathcal{P}^*$ together with the canonical algebraic maps forms a finite category. We shall still use $\mathcal{P}^*$ to denote this category. Also by 2.4, the relation " $\prec$" among the quotients gives $\mathcal{P}^*$ a partially ordered structure.

Theorem. $\mathcal{P}^*$ as a category is nicely connected. As a consequence, any two algebraic quotients are connected by a sequence of canonical nice algebraic maps defined in 2.4.

Proof. By theorem 2.5, we need only to prove that two admissible collections of top dimensional polyhedra in Ξ can be connected by a successive chain of admissible collections.

So let Ξ_1 and Ξ_2 be any two admissible collections of top dimensional polyhedra in Ξ. Assume we have

$$D \in \Xi_2 - \Xi_1 \cap \Xi_2$$

Now given any admissible decomposition $\Im$ of $Int(\Box) = Int(\mu(X))$ that contains D, there should be $C \in \Xi_1$, such that C is not equal to D and C (depends on $\Im$) is also in the decomposition $\Im$. Now take two (very) general points p and q in C nad D respectively, such that the segment $\overline{p,q}$ does not meet any codimension 2 polyhedra of Ξ. Then by the lemma, we have

$$D \succ E_1 \prec E_2 \succ \cdots \succ E_k \prec C$$

where $E_i, i = 1, \cdots, k$ are some polyhedra in $\Im$ of dimension $n-1$ or dimension n such that every E_i meets $\overline{p,q}$ and $E_i \neq E_j$ if $i \neq j$. Since the number of polytopes in Ξ is finite, the maximum of k's above is also a finite number, denote it by k_0. Now extend the chain above to a chain with $k_0 + 2$ polytopes,

$$D \succ E_1 \prec E_2 \succ \cdots \succ E_k \prec E_{k+1} \cdots \prec E_{k_0} \prec C,$$

where $E_{k+1} = \cdots = E_{k_0} = C$. Then we have

$$\Xi_2 \succ \Xi_{21} \prec \Xi_{22} \succ \cdots \succ \Xi_{2k_0} \prec \Xi_3$$

where Ξ_{21} is an admissible collection of polyhedra in Ξ obtained from Ξ_2 by replacing only D by E_1's, Ξ_{22} is an admissible collection of polyhedra in Ξ obtained from Ξ_{21} by replacing $E_1's$ by E_2's, and so on. Now we have to show that $\Xi_{2i}, i = 1, \cdots, k_0$ are really admissible collections of polyhedra. We start from Ξ_{21}.

Let $\Re$ be an arbitrary admissible decomposition of $Int(\Box)$, if $\Re$ does not contain an element of $\Xi_2 - \Xi_1 \cap \Xi_2$, then the unique polytope in $\Re \cap \Xi_2$ is also the unique polyhedron in $\Re \cap \Xi_{21}$ by our construction of Ξ_{21}; If $\Re$ contains an element D of $\Xi_2 - \Xi_1 \cap \Xi_2$, then Ξ_{21} must contain a E_1 in $\Re$ with $E_1 \prec D$. This shows that Ξ_{21} meets every admissible decomposition of $Int(\Box)$. Now assuming Ξ_{21} contains two E_1 and E_1' in a single admissible decomposition $\Re$. Then $\Re$ should contain an element D of $\Xi_2 - \Xi_1 \cap \Xi_2$, from our construction of Ξ_{21}, $E_1 \prec D, E_1' \prec D$, this implies $E_1 = E_1'$. Similarly, we can show that E_{22} is also an admissible collection, and so on.

Now we have

$$\Xi_1 \cap \Xi_2 \subset \Xi_1 \cap \Xi_3, \textit{ and } |\Xi_1 \cap \Xi_2| < |\Xi_1 \cap \Xi_3|,$$

Now let Ξ_3 plays the role of Ξ_2, and do the same for Ξ_3 and Ξ_1, as we did before for Ξ_2 and Ξ_1. Since the number of subpolyhedra in Ξ is finite, and

$$|\Xi_1 \cap \Xi_2| < |\Xi_1 \cap \Xi_3| < \cdots\cdots,$$

we should finally end up with a chain

$$\Xi_2 \succ \cdots \prec \Xi_3 \succ \cdots \prec \cdots \prec \Xi_m$$

with $\Xi_m - \Xi_m \cap \Xi_1 = \emptyset$, that is, $\Xi_m = \Xi_1 \cap \Xi_m$, i.e, $\Xi_m \subset \Xi_1$. Hence, $\Xi_m = \Xi_1$ by Proposition 2.5. So the theorem is proved, as desired.

Corollary. Let Ξ_1 be an admissible collection of top dimensional polyhedra in Ξ and $\mathcal{U}_1$ be its corresponding zariski open subset. Let Ξ_2 be another admissible collection of polyhedra in Ξ and $\mathcal{U}_2$ be its corresponding zariski open subset. Suppose that Ξ_1 covers Ξ_2, that is, $\Xi_2 \prec \Xi_1$ and there is no admissible collection Ξ' so that $\Xi_2 \prec \Xi' \prec \Xi_1$, then

(1) There is a *codim* 1 wall M so that the *codim* 1 polytopes in Ξ_2 make of an admissible collection $\Xi_2(M)$ for $\overline{X^M}$.

(2) Let Ξ_1' be the subpolytopes in Ξ_1 such that they have faces in $\Xi_2(M)$. Then $\Xi_1 - \Xi_1' = \Xi_2 - \Xi_2(M)$, and Ξ_1' lies in one of half spaces divided by P_M, where P_M be the hyperplane generated by M.

(3) We have a fiber square

$$\begin{tikzcd}
A \arrow[r] \arrow[d] & \mathcal{U}_1/H \arrow[d, "f"] \\
B \arrow[r] & \mathcal{U}_2//H
\end{tikzcd}$$

where B is defined to be $(\cup\{X^D; D \in \Xi_2(M)\})/H$ which is a geometric quotient in $\overline{X^M}$, and $A = f^{-1}(B)$. Moreover, $\mathcal{U}_1//H - A$ is isomorphic to $\mathcal{U}_2//H - B$ (they are actually identical).

Proof. Let C be a *codim* 1 polyhedron in Ξ_2 and M be a *codim* 1 wall containing C, then the corollary follows from the proof of the theorem above and the fact that Ξ_1 covers Ξ_2.

Remark. This corollary is the starting point of our statement that the algebraic quotients have all the properties that symplectic quotients have. We shall make this explicit in the rest of the paper.

2.7 Counting Algebraic Quotients

In what follows, we give a method to count the number of non-degenerate elements in $\mathcal{P}^*$ if it is interesting at all.

The collection $\mathcal{A}$ of admissible decompositions of $Int(\mu(X))$ (resp, $\mu(X)$) has a partial ordering by refinement.

Lemma. Let $\Xi_1 \subset \Xi$ be an arbitrary admissible collection, $\Re$ and $\Im$ be two admissible decompositions of $Int(\mu(X))$. If $\Re \prec \Im$, then $\Xi_1 \cap \Im$ is determined uniquely by $\Xi_1 \cap \Re$.

Proof. It suffices to note that if $D \in \Xi_1 \cap \Re$, then the unique polytope C in $\Im$ containing D must belong to Ξ_1.

So if $\Im_i \cdots \Im_r$ denote the minimal elements in $\mathcal{A}$ and $|\Im_i|(1 \leq i \leq r)$ denotes the number of (open) polyhedra in $\Im_i$ which are not contained in the boundary of $\mu(X)$. Then by the lemma, we have

Proposition. The number of non-degenerate quotients is given by

$$|\Im_1| \times \cdots \times |\Im_r|$$

Example. $X = SL(3, \mathrm{C})/B$. In this case we have that the number of non-degenerate elements in $\mathcal{P}^*$ is $3 \times 3 \times 3 = 27$.

Chapter 3

The Space of the Closures of Generic Orbits

The most part of this chapter is independent of the rest of the paper and has its own interest.

From now on and henceforth, the **generic points** of X will always refer to the points in the biggest stratum $X^{\mu(X)}$ unless indicated otherwise. Accordingly, the **generic orbits** shall be the orbits of the generic points above. Note that the space of generic orbit closures equals to the space of generic orbits because both of them are just $X^{\mu(X)}/H$ by definition.

As we have known, there is no canonical compactification of the space of generic orbits, which can also be regarded as an algebraic quotient variety. However, in their papers [B-BS1,2], A.Bialynicki-Birula and J. Sommese used the work of A.Fujiki and D. Lieberman on compactness of components of the Douady space of Kahler manifolds and constructed a canonical compactification Q of the space of generic orbit closures.

In this chapter, we shall show that Q can be regarded as a "biggest quotient variety" in the sense that there is a surjective morphism from Q to any algebraic quotient. Also we shall prove that the space Q is the Chow quotient (Almost at the last moment when I prepared to submit this thesis to M.I.T, I got a copy of Kapranov's preprint [Ka]. Once I read the first few pages of his preprint, soon I realized that the space Q that I studied in this thesis is the same as the Chow quotient defined in his preprint.) The proof of this last statement leads to the following two results: (1) An alternative construction of Bialynicki-Birula and Sommese's theorem, which is much simpler and easier. (2) A generalization of B-B,S's theorem to any reductive algebraic group action. The author believes that this generalization should enable us to extend most results in [B-BS] to arbitrary algebraic group actions. Consequently, we should be able to construct many more categorical quotient varieties other than the quotients that can be identified with symplectic reduced spaces in the sense of [MaW].

3.1 A Theorem of Bialynicki-Birula and Sommese

Theorem. Let X and H be as above. There is for any $x \in X$ with $dim H \cdot x = n$ a diagram

$$\begin{tikzcd} Z_x \arrow[r, "\phi_x"] \arrow[d, "f_x"'] & X \\ Q_x & \end{tikzcd}$$

with the following properties

(a). f_x is a flat surjective morphism of connected compact complex spaces Z_x and Q_x,

(b). the restriction of ϕ_x to the fiber $f_x^{-1}(q)$ of f_x at every point q in Q is an embedding, and there is q in Q such that $\phi_x(f_x^{-1}(q)) = \overline{H \cdot x}$,

(c) there is a natural action of H on Z_x making f_x and ϕ_x equivariant with respect to the trivial action of H on Q_x and the given action of H on X,

(d). there is a dense Zariski open set $\mathcal{O}_x \subset Q_x$ such that for each $q \in \mathcal{O}_x$, $f_x^{-1}(q)$ is reduced and $\phi_x(f_x^{-1}(q))$ is the closure of a H orbit,

(e). the reduction of every fiber of f_x is pure K dimensional and for fibers $\{f_x^{-1}(q), f_x^{-1}(q')\}$ that are reduced, $\phi_x(f_x^{-1}(q)) = \phi_x(f_x^{-1}(q'))$ only if $q = q'$,

(f). given any diagram

$$\begin{tikzcd} Z' \arrow[r, "\phi'"] \arrow[d, "f'"'] & X \\ Q' & \end{tikzcd}$$

that satisfies properties (a) through (e), there is a holomorphism map:

$$c : Q' \longrightarrow Q_x$$

such that the diagram of (f) is the pullback of the diagram in the very beginning.

It should be pointed out that any two points in a single torus stratum indexed by a top dimensional subpolytope define the same diagram in the theorem. So when $x \in X^{\mu(X)}$, we will drop the subscripts of the diagram in the theorem and therefore get the following diagram:

$$\begin{tikzcd} Z \arrow[r, "\phi"] \arrow[d, "f"'] & X \\ Q & \end{tikzcd}$$

for generic points (or, for the stratum $X^{\mu(X)}$).

Clearly Q contains $\mathcal{O}$ as a zariski open subset which can be identified with

$X^{\mu(X)}/H$ by

$$o \longmapsto \phi f^{-1}(o) = \overline{H \cdot x}, o \in \mathcal{O},$$

where x is some point in $X^{\mu(X)}$.

3.2 The Space $\mathcal{Q}$ and Quotient Varieties

The following theorem says that $\mathcal{Q}$ can be regarded as the biggest "quotient" variety of X in the sense that there is natural morphism from $\mathcal{Q}$ to any of the algebraic quotients.

Theorem. Let $\mathcal{U}$ be an arbitrary algebraic open subset, then there is a natural surjective morphism h from $\mathcal{Q}$ to $\mathcal{U}//H$. This can be illustrated by the following "commutative" diagram:

$$\begin{tikzcd} Z \arrow[r, "\phi"] \arrow[d, "f"'] & X \supset \mathcal{U} \arrow[d, "\pi"] \\ \mathcal{Q} \arrow[r, "h"] & \mathcal{U}//H \end{tikzcd}$$

where π is the natural projection from $\mathcal{U}$ to $\mathcal{U}//H$. In the case that $\mathcal{U}//H$ is not degenerate, h is birational.

Proof. Given $q \in \mathcal{Q}$, then $\phi(f^{-1}(q))$ is a union of torus orbits,

$$H \cdot x_1 \cup \cdots \cup H \cdot x_m.$$

Using [B-B2], one can see that the moment map images of these orbits gives rise to a disjoint union of μ-images

$$\mu(X) = \coprod_i \mu(H \cdot x_i).$$

So by the definition of a algebraic quotient, we can define a map h by

$$q \longmapsto [\phi(f^{-1}(q)) \cap \mathcal{U}],$$

where $\phi f^{-1}(q) \cap \mathcal{U}$ is a non-empty union of orbits with a unique closed orbit in $\mathcal{U}$ and $[\phi f^{-1}(q) \cap \mathcal{U}]$ denotes the induced point on $\mathcal{U}//H$. It is easy to check that h is a well-defined morphism from $\mathcal{Q}$ to $\mathcal{U}//H$. To see the birationality and surjectivity of h, it suffices to note that h sends a zariski open subset $\mathcal{O}$ of $\mathcal{Q}$ to a zariski open subset $X^{\mu(X)}/H$ of $\mathcal{U}//H$.

Remark. (1). In the case that h is projective, then h should correspond a blow-up map from $\mathcal{Q}$ to $\mathcal{U}//H$. (2). The map $Z \xrightarrow{f} \mathcal{Q}$ gives rise to a family of algebraic variety parametrized by $\mathcal{Q}$, whose generic fiber is a toric variety, $\overline{H \cdot x}$, $x \in X^{\mu(X)}$.

The space $\mathcal{Q}$ has a canonical stratification

$$\mathcal{Q} = \bigcup_{\alpha \in \mathcal{A}} \Gamma_\alpha$$

where $\mathcal{A}$ is a finite index set, such that two points q_1 and q_2 of $\mathcal{Q}$ are in one stratum if and only if $f^{-1}(q_1)$ is isomorphic to $f^{-1}(q_2)$ as varieties, this amounts to requiring that $\mu(\phi(f^{-1}(q_1)))$ and $\mu(\phi(f^{-1}(q_2)))$ give the same decomposition of $\mu(X)$ into disjoint union of μ-images of torus orbits.

Let $\mathcal{U}$ be an algebraic open subset of X defined by admissible collection $\Xi_1 \subset \Xi$, and h be the morphism from $\mathcal{Q}$ to $\mathcal{U}//H$ characterized in theorem 3.2. Then for each stratum Γ_α of $\mathcal{Q}$, $h(\Gamma_\alpha)$ should be of form X^D/H, where $D \in \Xi_1$. So

$$\mathcal{U}//H = h(\mathcal{Q}) = \bigcup_{\alpha \in \mathcal{A}} h(\Gamma_\alpha)$$

gives a natural stratification of $\mathcal{U}//H$, which can also be induced from the torus stratification $X = \bigcup_{D \in \Xi} X^D$.

Since the stratification $\bigcup_{D \in \Xi} X^D$ does not satisfy the axiom of the frontier in general, neither does $\mathcal{U}//H = \bigcup_{\alpha \in \mathcal{A}} h(\Gamma_\alpha)$ in general, this suggests that the stratification

$$\mathcal{Q} = \bigcup_{\alpha \in \mathcal{A}} \Gamma_\alpha$$

do not satisfy the axiom of the frontier in general either.

3.3 The Space $\mathcal{Q}$ and Chow Quotients

We have two aims in this section: (1) To generalize B-B,S's theorem to any reductive algebraic group action. This generalization gives automatically an alternative construction of B-B,S's theorem, which is much simpler and easier. (2) To prove that the space $\mathcal{Q}$ is the Chow quotient.

Let G be an algebraic group acting on a projective variety X. There is an invariant zariski open subset $\mathcal{U} \subset X$ of generic points such that for all points $x \in \mathcal{U}$, the varieties $\overline{G \cdot x}$ have the same dimension, say, r and represent the same homology class $\delta \in H_{2r}(X, \mathbf{Z})$. Let $C_r(X, \delta)$ be the Chow variety of all r-dimensional algebraic cycles in X which represent the homology class δ. The map $G \cdot x \mapsto \overline{G \cdot x}$ defines an embedding of $\mathcal{U}/G$ to $C_r(X, \delta)$ and the closure of the image $\overline{\mathcal{U}/G}$ in $C_r(X, \delta)$ is the Chow quotient ([Ka]). We use $\mathcal{M}$ to denote $\overline{\mathcal{U}/G}$.

Now we define

$$\mathcal{S} = \{(C,x) \in C_r(X,\delta) \times X \mid x \in C_i \mathit{\ for\ some\ } i, C = \sum_i m_i C_i \in \mathcal{M}\}$$

where C_i's are irreducible subvarieties of dim r. Then we have a diagram

$$\begin{tikzcd}
\mathcal{S} \arrow[r, "\psi"] \arrow[d, "g"'] & X \\
\mathcal{M} &
\end{tikzcd}$$

where g, ψ are the first and second projections respectively. It is straightforward to check that we have the following generalization of B-B,S's theorem.

Theorem. For the diagram above, we have:

(a). g is a flat surjective morphism of connected compact varieties $\mathcal{S}$ and $\mathcal{M}$,

(b). the restriction of ψ to the fiber $g^{-1}(m)$ of g at every point m in $\mathcal{M}$ is an embedding, and there is m in $\mathcal{M}$ such that $\psi(g^{-1}(m)) = \overline{G \cdot x}$ for some $x \in \mathcal{U}$,

(c). there is a natural action of G on $\mathcal{S}$ making g and ψ equivariant with respect to the trivial action of G on $\mathcal{M}$ and the given action of G on X,

(d). there is a dense Zariski open set $\mathcal{O} \subset \mathcal{M}$ such that for each $m \in \mathcal{O}$, $g^{-1}(m)$ is reduced and $\psi(g^{-1}(m))$ is the closure of a G orbit,

(e). the reduction of every fiber of g is pure r dimensional and for fibers $\{g^{-1}(m), g^{-1}(m')\}$ that are reduced, $\psi(g^{-1}(m)) = \psi(g^{-1}(m'))$ only if $m = m'$,

(f). given any diagram

$$\begin{tikzcd}
\mathcal{S}' \arrow[r, "\psi'"] \arrow[d, "g'"'] & X \\
\mathcal{M}' &
\end{tikzcd}$$

that satisfies properties (a) through (e), there is a holomorphism map:

$$c : \mathcal{M}' \longrightarrow \mathcal{M}$$

such that the map g' is the pullback of the map g.

Proposition. In fact, for any $x \in X$ such that $G \cdot x$ is of top dimension, we have a diagram

$$\begin{tikzcd}
\mathcal{S}_x \arrow[r] \arrow[d] & X \\
\mathcal{M}_x &
\end{tikzcd}$$

such that the theorem holds for this diagram.

The definition of $\mathcal{M}_x$ and $\mathcal{S}_x$ are similar to those of $\mathcal{M}$ and $\mathcal{S}$. Let $\mathcal{U}_x$ be a "sufficently large" invariant subset containing $G \cdot x$ such that for all $y \in \mathcal{U}_x$, $\overline{G \cdot y}$ lie in the same Chow variety $C_h(X, r)$, then define $\mathcal{M}_x$ is the closure of $\mathcal{U}_x/G$ in $C_h(X, r)$. $\mathcal{S}_x$ can be defined similarly.

Corollary. Let G be a torus. Then the diagram in the theorem above coincides with the diagram in 3.1. In particular, the space $\mathcal{Q}$ is the Chow quotient $\mathcal{M}$.

Proof. By the properties (f) in both theorems, we conclude that they must coincide.

For the action of $(\mathbf{C}^*)^{n-1}$ on $G(2, \mathbf{C}^n)$, we shall know in 9.1 that $\mathbf{P}^{n-3}$ is an algebraic quotient of this action. Since for this action $\mathcal{M}$ is isomorphic to $\overline{\mathcal{M}_{0,n}}$ of the moduli space of n-pointed stable curves of genus 0 ([Ka]). So by theorem 3.2, $\overline{\mathcal{M}_{0,n}}$ is a blow up of $\mathbf{P}^{n-3}$. (This blow up was described explicitly in [Ka].)

3.4 Special Admissible Decompositions and Some Conjectures

For any point $p \in \mathcal{Q}$, we shall call $\mu(\phi(f^{-1}(q)))$ a special admissible decomposition of $\mu(X)$.

Proposition. For any $D \in \Xi$, there is $q \in \mathcal{Q}$, such that D^0 is contained in the decomposition

$$\mu(\phi(f^{-1}(q)))$$

of $\mu(X)$ as an open polytope.

Proof. Choose an algebraic open subset $\mathcal{U}$ containing X^D, then by theorem 1.3.2 and argument above, there is stratum Γ of $\mathcal{Q}$ so that

$$h(\Gamma) = X^D/H,$$

take $q \in \Gamma$, then the proposition follows easily.

Conjecture. A collection Ξ_1 of polytopes of Ξ is admissible if and only if $\{D^0 | \ D \in \Xi_1\}$ meets every admissible decomposition of the form

$$\mu(X) = \mu(\phi(f^{-1}(q))), q \in \mathcal{Q}$$

in exactly one open subpolyhedron. (If we do not want to consider degenerate quotients, we can add the following extra condition: every polytope of Ξ_1 is not contained in the boundary of $\mu(X)$.)

The "only if" part of the conjecture is trivial. The "if" part will be an immediate consequence of the following conjecture:

Conjecture. For any decomposition Θ of $\mu(X)$ into disjoint moment map images of torus orbits , there is a point $q \in \mathcal{Q}$ such that the decomposition $\mu(X) = \mu(\phi(f^{-1}(q)))$ subscribes the decomposition Θ, that is, any open polytope in the decomposition $\mu(X) = \mu(\phi(f^{-1}(q)))$ is contained in some open polytope of Θ.

Chapter 4

Equivariant Morphisms

Given an equivariant morphism between two compact algebraic varieties with torus actions, one can push forward or pull back algebraic quotients. The point is that we have a deformation retract from the moment map image upstair to the moment map image downstair which "keeps the face relation and inclusion". However, the pull back of a symplectic quotient may no longer be symplectic.

4.1 Moment Cell Complexes

Definition Let X be an arbitrary compact algebraic variety with a torus $(\mathbf{C}^*)^n = H$ action. Let μ be an associated moment map, Ξ be the collection of all moment map images of torus orbit closures. Then the collection

$$\{Int(D) | D \in \Xi\}$$

is a collection of cells of various dimension, the moment map induces boundary maps for these cells, hence makes a cell complex, which we denote it by $\mathcal{C}(X)$, and call it **moment cell complex**. We make convention that $D^0 = Int(D)$ and the stratum indexed by D, $X^D = X^{D^0}$. We advise the reader to refer [B-BS2] for another version of the definition of $\mathcal{C}(X)$. Note that for any $D \in \mathcal{C}, x \in X^D$, $H \cdot x/T$ is homeomorphic to D^0 under μ. With this identification, we got a map λ from X to $\mathcal{C}(X)$, which is usually discontinuous. Obviously, $\mathcal{C}(X)$ is a regular cell complex, and there is a continuous surjective map m from $\mathcal{C}(X)$ to $\mu(X) \subset \mathbf{R}^n$ induced by μ such that the composition of λ and m is μ.

Remark. Contrary to the remark in 1.3, the moment cell complex associated to X does not depend upon the moment map (or the metric on X) up to cell-preserving homeomorphism, actually the moment cell complex can be defined without using the moment map (or the metric). Accordingly, the set of all admissible decompositions of $Int(\mu_1(X))$ can be identified with set of all admissible decompositions of $Int(\mu_2(X))$ for our purpose (where μ_1 and μ_2 are any two moment maps associated to the torus action). So unlike the case of symplectic quotients, we do not have to vary the metric in order to get all algebraic quotients (although the set of symplectic quotients is a subset of the set of geometric quotients), we only need to care for a fixed moment map.

Theorem. Let X, Y be compact algebraic varieties with actions of $H = (\mathbf{C}^*)^n$. Suppose $f : X \to Y$ is an equivariant morphism with respect to the actions of the torus, then there is a cell-preserving surjective map φ from $\mathcal{C}(X)$ to $\mathcal{C}(Y)$ so that the following diagram

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "\lambda_X"'] & Y \arrow[d, "\lambda_Y"] \\
\mathcal{C}(X) \arrow[r, "\varphi"] & \mathcal{C}(Y)
\end{tikzcd}$$

commutes.

Proof. Given a cell D^0 in $\mathcal{C}(X)$, let $x \in X^{D^0}$, then $H \cdot x/T$ is homeomorphic to D^0 under the moment map, then we define φ on D^0 as follows

$$D^0 \xrightarrow{\mu^{-1}} \cong H \cdot x/T \xrightarrow{\bar{f}} H \cdot f(x)/T \xrightarrow{\mu} \cong E^0$$

where $E^0 = \mu(H \cdot f(x))$ and $\bar{f}$ is induced from f. It is straightforward to check that the diagram commutes.

Corollary. For any $e' \in \mathcal{C}(Y)$,

$$f^{-1}(Y^{e'}) = \bigcup_{\varphi(e)=e'} X^e.$$

Proof. For any x in $f^{-1}(Y^{e'})$, that is, $\lambda f(x) \in e'$, or $\lambda(H \cdot f(x)) = e'$, let $e = \lambda(H \cdot x)$, then

$$\varphi(e) = \varphi\lambda(H \cdot x) = \lambda f(H \cdot x) = \lambda(H \cdot f(x)) = e'$$

hence $f^{-1}(Y^{e'}) \subset \bigcup_{\varphi(e)=e'} X^e$.

On the other hand, $\lambda f(X') = \varphi\lambda(X') = \varphi(e)$, so if $\varphi(e) = e'$, then $f(X^e) \in X^{e'}$, hence

$$f^{-1}(Y^{e'}) = \bigcup_{\varphi(e)=e'} X^e.$$

Corollary. For any $e' \in \mathcal{C}(Y)$, there is a unique distinguished cell $e \in \mathcal{C}(Y)$ such that

$$\overline{f^{-1}(Y^{e'})} = f^{-1}(\overline{Y^{e'}}) = \overline{X^{e}}.$$

Proof. By the above corollary,

$$f^{-1}(Y^{e'}) = \bigcup_{\varphi(e_1)=e'} X^{e_1},$$

among the cells e_1 with $\varphi(e_1) = e'$, there is biggest one e which can be characterized as follows: $\lambda(H \cdot x) = e$, if and only if

$$\overline{H \cdot x} \cap Fix(X,H) = f^{-1}(\overline{Y^{e_1}}) \cap Fix(X,H)$$

where $Fix(X,H)$ is the $H-fixed$ point set of X.

4.2 Deformation of Admissible Decompositions

Theorem. Let f, X, Y be as before. Let Θ be a decomposition of $\mu(X)$ in to a union od disjoint μ-images of torus orbits, then there is a deformation from Θ to a decomposition Θ' of $\mu(Y)$ into a union of disjoint μ-images of orbits, which sends finally a moment map image in Θ to a moment map image in Θ'. Moreover, the deformation keeps the face relation "$\prec$" and inclusion.

Proof. Let $Fix(Y,H) = \{b_1, \cdots b_m\}$, then

$$Fix(X,H) = S_1 \cup \cdots \cup S_m$$

where $S_i = Fix(X,H) \cap f^{-1}(b_i)$, $1 \le i \le m$.

Note that each S_i spans a face σ_i of $\mu(X)$, so to get the deformation, we simply shrink each σ_i gradually to the point b_i, and simultaneously, shrink each (homeomorphic) cell C^0 in Θ gradually to a (homeomorphic) cell D^0 in Θ', where D^0 is determined by C^0 in the following way: let $x \in X^C$, if $\overline{H \cdot x}$ intersects precisely the sets $S_{i_1}, \cdots S_{i_h}$, then D is the convex hall of $b_{i_1} \cdots b_{i_h}$. Hence, we obtain a deformation which keeps the face relation, as desired.

Remark. The theorem above also have a natural version for admissible decompositions of $Int(\mu(X))$ and $Int(\mu(Y))$ due to the following easy lemma. (The interested reader can give that natural version very easily).

Lemma. Let Θ be a decomposition of $\mu(X)$ into a union of μ-images of torus orbits, then if D^0 is in Θ, every open face of D^0 is also in Θ.

Sometimes we state the theorem above as follows:

Theorem′ . We have

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "\lambda_X"'] & Y \arrow[d, "\lambda_Y"] \\
C(X) \arrow[r, "\varphi"] \arrow[d, "m_X"'] & C(Y) \arrow[d, "m_Y"] \\
\mu(X) \arrow[r, squiggly] & \mu(Y)
\end{tikzcd}$$

where the compositions of λ and m give the moment maps, $\mu = m \circ \lambda$. That is, given an admissible decomposition Θ of $\mu(X)$, then $m_Y \circ \varphi \circ m_X^{-1}(\Theta)$ gives an admissible decomposition of $\mu(Y)$. (Note that m induces a homeomorphism from $m^{-1}(\Theta)$ to Θ.)

4.3 Pulling Back Quotients

Suppose compact algebraic varieties X and Y both have actions of a algebraic torus $H = (\mathbf{C}^*)^n$. Suppose also $f : X \to Y$ is an equivariant surjective morphism with respect to actions of H. One may want to know if f induces morphisms on quotients, furthermore if f induces fibrations if f is.

Proposition. Let $\mathcal{V}$ be an open subset of Y consisting of only torus orbits of the top dimension. Then if $\mathcal{V}$ has a compact hausdorff quotient $\mathcal{V}/H$, then so does $\mathcal{U} = f^{-1}(\mathcal{V})$. Moreover, f induces a morphism from $\mathcal{U}/H$ to $\mathcal{V}/H$.

Proof. Clearly f induces a map $\bar{f}$ from $\mathcal{U}/H$ to $\mathcal{V}/H$. Given a point $y \in Y$, let H_y be the stabilizer of H at y, then H_y acts on fiber $f^{-1}(y)$ since f is equivariant. Now let y be a lift of a point $\bar{y}$ on $\mathcal{V}/H$, then H_y should be a finite subgroup of H, it is straightforward to see the fiber of $\bar{f}$ at $\bar{y}$ is just $f^{-1}(y)/H_y$, which is compact hausdorff, hence $\mathcal{U}/H$ is hausdorff and compact.

Remark. Since the isotropy subgroups are not locally constant in general, we can not hope that $\bar{f}$ is a fibration even if f is.

Corollary. Let the notation be as in the proposition. Suppose f is an equivariant fibration with the typical fiber Z. Assume all the finite isotropy subgroups at points on Y are identity, then $\mathcal{U}/H \to \mathcal{V}/H$ is also a fibration with fiber Z.

Remark. The assumption in the corollary is satisfied for the maximal torus action on G/B when G is type of A_n.

In fact we have: Given any algebraic quotient $\mathcal{V}//H$ on Y, then $\mathcal{U} = f^{-1}(\mathcal{V})$ is an algebraic open subset on X and f induces a map $\bar{f} : \mathcal{U}//H \longrightarrow \mathcal{V}//H$. We call $\mathcal{U}//H$ is the full-back of $\mathcal{V}//H$ by f. In particular, if $\mathcal{V}//H$ is nondegenerate, then $\mathcal{U}//H$ must also be nondegenerate. Otherwise is not true.

Theorem. Let X, Y, f be the same as in the very beginning of this section, then if $\mathcal{V}$ is an algebraic open subset of Y, then $\mathcal{U} = f^{-1}(\mathcal{V})$ is an algebraic open subset of X.

Proof. Given an admissible decomposition Θ of $\mu(X)$, let Θ' be the admissible decomposition of $\mu(Y)$ deformed from Θ. If $\mathcal{U} = f^{-1}(\mathcal{V})$ misses any polytope in Θ, Θ' will also miss any polytope in Θ' by corollary 4.1. On the other hand, if $\mathcal{U}$ meets polyhedra $C_1 \cdots C_r$ in Θ, and $C_1 \cdots C_r$ have two minimums C_i and C_j under the face relation $'' \prec''$, $1 \leq i \neq j \leq r$, then by theorem 4.2, the deformation images of C_i and C_j will be two distinct minimums of deformation images of $C_1 \cdots C_r$. This contradicts with the fact that $\mathcal{V}$ is algebraic. Hence the theorem follows.

Remark. In the next chapter, we shall give an example showing that theorem 1.5.8 is not true for symplectic quotients, that is, $\mathcal{U}//H = f^{-1}(\mathcal{V})//H$ may not be, in general, a symplectic quotient, even if $\mathcal{V}//H$ is.

4.4 Pushing Forward Quotients

Now given an algebraic open subset $\mathcal{U}$ on X, one may suspect that $f(\mathcal{V})//H$ exists. Indeed, this is true.

Theorem. Let $\mathcal{U}//H$ be an algebraic quotient of X, then $f(\mathcal{V})//H$ is an algebraic quotient of Y, and f induces an algebraic map $\underline{f} : \mathcal{U}//H \rightarrow f(\mathcal{U})//H$.

Proof. Let Θ' be any admissible decomposition of $\mu(Y)$. Let Θ be an admissible decomposition of $\mu(X)$ which can be deformed to Θ'. Suppose $\mathcal{U}$ meets a polyhedron C in Θ, then $f(\mathcal{U})$ meets $m_Y \cdot \varphi \cdot m_X^{-1}(C)$ in Θ'. Assume $f(\mathcal{U})$ meets $C'_1 \cdots C'_r$ in Θ' and C'_i, C'_j are two distinct minimums. Then $\mathcal{U}$ meets $m_X \cdot \varphi^{-1} \cdot m_Y^{-1}(C'_i)$, $m_X \cdot \varphi^{-1} \cdot m_Y^{-1}(C'_j)$. Since the deformation keeps the partial order of both "inclusion" and "face relation", so a minimum in $m_X \cdot \varphi^{-1} \cdot m_Y^{-1}(C'_i)$ is uncomparable with a minimum in $m_X \cdot \varphi^{-1} \cdot m_Y^{-1}(C'_j)$, hence, contradiction.

We point out that $f(\mathcal{U})//H$ may not be nondegenerate even if $\mathcal{U}//H$ is. This is true because an interior point of $\mu(X)$ can be deformed into a boundary point of $\mu(Y)$.

But contrary to the remark for pulling back quotients, we have : if $\mathcal{U}//H$ is symplectic, then$f(\mathcal{U})//H$ must also be symplectic. In fact, if $\mathcal{U} = \mathcal{U}_p$, $p \in \mu(X)$, then $f(\mathcal{U}_p) = \mathcal{U}_{f(p)}$, $f(p) \in \mu(Y)$. To see this is true, notice that

$$\mathcal{U}_p = \{x \in X \mid p \in \mu(\overline{H \cdot x})\},$$

$$f(\mathcal{U}_p) = \{f(x) \mid f(p) \in \mu(\overline{H \cdot f(x)})\},$$

and the fact that f is surjective.

Chapter 5

The Topology of Symplectic Quotients

Historically, after the discovery of the theorems in 7.2, Professor Robert MacPherson pointed out to me that using the idea behind Atiyah's Morse theoretic arguments in [A1], we should be able to see that (suitable versions of) above-mentioned theorems also hold for arbitrary nonsingular compact algebraic varieties (or Kähler manifolds) with complex torus actions. Hence by employing the decomposition theorem, we shall be able to give an inductive cohomological formula for an arbitrary symplectic quotient. This is exactly what we are going to do in this chapter. In what follows we shall "rebuild" the theorems of 7.2 in a more general context. But instead of employing Morse theory alone, we will also apply a decomposition into subvarieties theorem of Bialynicki-Birula.

5.1 The Statements of Results

Now again like what we did in chapter 1, we assume that X is a nonsingular compact algebraic projective variety with complex torus $H = (\mathbf{C}^*)^n$ action , and $H = A \times T$ is a canonical decomposition with $A = (\mathbf{R}^>)^n$ and $T = (S^1)^n$.

We remark again that $\overline{X^M}$ is nonsingular subvariety if M is a wall in $\mu(X)$.

Convention. Let M be a wall and $\mathcal{U}$ be an algebraic open subset. Then define

$$\mathcal{U}(M) = \{x \in \mathcal{U} | \mu(H \cdot x) \subset M\}$$

Theorem 1. Let M be a *codim* 1 face of $\mu(X)$, r be a (relatively) general point on M, and p be a general point in $\mu(X)$ and clos enough to r, then the canonical projection

$$\varphi : \mathcal{U}_p/H \longrightarrow \mathcal{U}_r(M)/H (= \mathcal{U}_r//H)$$

is a fibration whose typical fiber is a weighted projective space of dimension $codim_{\mathbb{C}} X^M - 1$.

Theorem 2. Let r be a (relatively) general point on an interior codim 1 wall M and p be a general point close enough to r. Now let $B = \mathcal{U}_r(M)//H$ and $A = f^{-1}(B)$ where f is the canonical projection from $\mathcal{U}_p//H$ to $\mathcal{U}_r//H$. Then we have the following diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d] & \mathcal{U}_p//H \arrow[d] \\ B \arrow[r] & \mathcal{U}_r//H, \end{tikzcd}$$

where the horizontal maps are inclusions. Moreover $A \longrightarrow B$ is a fibration whose fiber is a weighted projective space while f is an isomorphism off B.

Using the idea above repeatedly , we shall see that the theorem above takes the following version when p, q are arbitrary interior points.

Let p, q be two interior points, and $F, G \in \Upsilon$ such that $G \prec F$ and $p \in F, q \in G$. Then there is a canonical stratification on $Y = \mathcal{U}_q//H = \bigcup_\beta C_\beta$ such that the natural projection $\pi : \mathcal{U}_p//H \rightarrow \mathcal{U}_q//H$ becomes a stratified map. More precisely, for each β, $\pi|\pi^{-1}(C_\beta) : \pi^{-1}(C_\beta) \rightarrow C_\beta$ is a fibration tower whose fibers are all weight projective spaces.

In what follows, we shall construct each stratum C_β explicitly as in the spirit of the theorem above. And one can easily read off the fibers through our construction.

Let $N_1, \cdots, N_l$ be all the *codim* 1 walls containing the point q. Then any wall containing q is of the form $N_{i_1} \cap \cdots \cap N_{i_h}$, $1 \leq h \leq l$. It should be point out that an intersection $N_{i_1} \cap \cdots \cap N_{i_h}$ may not be a "wall", that is, it may not be the moment map image of a torus orbit closure. We introduce the notation $N_{[i_1 \cdots i_h]}$ to denote $N_{i_1} \cap \cdots \cap N_{i_h}$. In fact, if $I = \{i_1 \cdots i_h\} \subset \{1, \cdots, l\}$, we set $N_I = N_{[i_1 \cdots i_h]}$. As before we have $q \in N_I$ and

$$\mathcal{U}_q(N_I) \subset \mathcal{U}_q.$$

In the case that N_I is not a "wall", we agree that $\mathcal{U}_q(N_I) = \emptyset$. Therefore we have the following subvarieties of $\mathcal{U}_q//H$,

$$\mathcal{U}_q(N_I)//H, \ For \ any \ I \subset \{1, \cdots, l\}.$$

Clearly

$$\{S_i = \mathcal{U}_q(N_i)//H, i = 1, \cdots, l\}$$

are l divisors of $\mathcal{U}_q//H$, and any $\mathcal{U}_q(N_I)//H$ is of the form $S_{i_1} \cap \cdots \cap S_{i_h} = S_{i_1 \cdots i_h}$, where $\{i_1 \cdots i_h\} = I$.

Now we can define the stratification as desired.
Define

$$C_{[1,\cdots,l]} = S_1 \cap \cdots \cap S_l;$$

For $1 \leq i \leq l$, define

$$C_{[1 \cdots \hat{i} \cdots l]} = S_1 \cap \cdots \hat{S}_i \cap \cdots \cap S_l - C_{[1,\cdots,l]};$$

For $1 \leq i < j \leq l$, define

$$C_{[1 \cdots \hat{i} \cdots \hat{j} \cdots l]} = S_1 \cap \cdots \hat{S}_i \cdots \hat{S}_j \cdots \cap S_l - \textit{the union of the previous strata};$$

....

For $1 \leq i \leq l$, define

$$C_{[i]} = S_i - \textit{the union of the previous strata};$$

And define

$$C_\emptyset = \mathcal{U}_q//H - S_1 \cup \cdots \cup S_l.$$

Then

$$\mathcal{U}_q//H = C_\emptyset \cup \coprod\nolimits_{1 \leq i_1 < \cdots < \subset_k \leq l} C_{[i_1 \cdots i_k]}.$$

And each C_I ($I \subset \{1, \cdots, l\}$) is clearly nonsingular. (In fact this is a Whitney stratification although this result is not necessary for us. To see that they satisfy Whitney conditions, the interested reader can consult with [CuSj] although the language used there appears very different from ours.)

The fact that the projection $\pi : \mathcal{U}_p//H \longrightarrow \mathcal{U}_q//H$ restricted to any C_I is a fibration tower with weighted projective spaces as fibers follows directly from the previous theorems.

Theorem 3. Let the notation be as above. Then there is a canonical stratification $\mathcal{U}_q//H = \bigcup_{I \subset \{1,\cdots,l\}} C_I$ such that the natural projection $\pi : \mathcal{U}_p \longrightarrow \mathcal{U}_q//H$ becomes a "stratified" map. More precisely, for each subset $I \subset \{1, \cdots, l\}$, the map $\pi/\pi^{-1}(C_I) : \pi^{-1}(C_I) \longrightarrow C_I$ is a fibration tower whose fibers are all weighted projective spaces.

The theorem above has the following interpretation.

Theorem 3′. Let the notations be as above. Then

$$\mathcal{U}_q//H = \bigcup_{wall\ M} \mathcal{U}_q[M]//H$$

is the the same stratification as the one in the previous theorem, where $\mathcal{U}_q[M]$ is defined as follows: $x \in \mathcal{U}_q[M]$ if and only if $x \in \mathcal{U}_q$ and there is $C \in M \cap \Xi$ such that $C \prec \mu(\overline{H \cdot x})$ and M is a minimal wall with this property. We should point out that $\mathcal{U}_q[M]$ is often empty (unless M is an intersection of $M_1 \cdots M_l$ or $\mu(X)$).

5.2 The Proofs of Some Theorems in 5.1

Our proof in this section is somehow different from the proofs in chapter 7 where the proofs are clearly direct computations by using group structures. But in this section, the proof is a combination of the idea behind Atiyah's Morse theoretic arguments [A1] and Bialynicki-Birula's "plus-decomposition" and "minus-decomposition" theorems for $\mathbf{C}^*$ (or G_m) actions. Of course, with this general proof , we can not make our conclusions so explicit as what we will have in chapter 7.

We first remark that there are " parallel wall phenomena" in $\mu(X)$ when $X = G/B$ (we shall describe this in chapter 7) which asserts that for each isotropy group in H, we have a collection of walls of **the same dimension**, which are parallel to each other and contain all of the vertices of $\mu(X)$.

In a general case (even in the case of Grassmannian), the exactly same assertion is no longer true, instead, we have

Obervation. Given a wall M, there is a collection of walls (of possibly various dimensions) such that they are parallel to each other and contain all of the vertices of $\mu(X)$.

Proof of the theorem 5.1.1. Let $M_0 = M$ be the wall in the theorem, let also β be a point on M_0 so that the line L_β through the origin o and β in $\mathbf{R}^n$ is perpendicular to M_o, (we can take the origin o to be the barycenter of $\mu(X)$ without essential loss of generality, this amounts to requiring that the integral of the moment map μ on X is zero). Then $T_\beta = \overline{\{expt\beta |\, t \in \mathbf{R}\}} \subset T$ is the isotropy subgroup of $\overline{X^{M_0}}$ in T, and in the mean time, the complexification H_β of T_β is the isotropy subgroup of $\overline{X^{M_0}}$ in H.

Let $\mu_\beta = \mu \cdot \beta$, i.e, for any $x \in X$, $\mu_\beta(x) = \mu(x) \cdot \beta$, then μ_β is actually a moment map associated to the action of H_β (or T_β). This can be described by the following picture

$$\begin{tikzcd} X \arrow[r, "\mu"] \arrow[dr, "\mu_\beta"'] & \mathbf{R}^n \arrow[d, "p"] \\ & L_\beta \cong \mathbf{R}^1, \end{tikzcd}$$

where p is the natural projection from $\mathbf{R}^n$ to L_β.

The connected components of the fixed point set of H_β are precisely the strata closures, $\overline{X^{M_0}}, \overline{X^{M_1}}, \cdots, \overline{X^{M_k}}$, where $\{M_0, M_1, \cdots, M_k\}$ are the parallel walls characterized in the observation above.

As indicated by Atiyah [A1], μ_β is a non-degenerate Morse function in the sense of Bott with the critical manifolds $\overline{X^{M_0}}, \cdots, \overline{X^{M_k}}$ since the critical set ofo μ_β is precisely the fixed point set of H_β [Ki].

Without loss of generality, we assume the wall M_k is the other one (except for M_0) which is a face of $\mu(X)$, assume for simplicity that $\overline{X^{M_0}}$ is the source of the action of H_β and $\overline{X^{M_k}}$ is the sink of the action of H_β ([B-B]). Then by Bialynicki-Birala [B-B], we have two decompositions called $(+)$ and $(-)$ decompositions as below:

$$X = X_0^+ \cup X_1^+ \cup \cdots \cup X_k^+,$$
$$X = X_0^- \cup X_1^- \cup \cdots \cup X_k^-,$$

such that for each $0 \leq i \leq k$, there is (unique) fibration $\gamma_i^+ : X_i^+ \longrightarrow \overline{X^{M_i}}$ (resp. $\gamma_i^- : X_i^- \longrightarrow \overline{X^{M_i}}$) whose fiber is isomorphic as a scheme to a vector space, the action of H_β preserve each fiber, and in fact the induced action of H_β on any fiber is equivalent to a linear action. Furthermore , X_0^+ and X_k^+ are two zariski open subsets in X.

In the mean while, the non-degenerate Morse function μ_β in the sense of Bott gives a Morse stratification

$$X = S_0 \cup S_1 \cup \cdots \cup S_k$$

where each S_i is the "unstable' manifold" of μ_β at the critical manifold $\overline{X^{M_i}}$, that is, if we let ϕ_t denote the gradient flow μ_β, then we have

(1) For any $x \in M$, the gradient flow $\phi_t(x)$ has a unique limit point $\phi_\infty(x)$ in the critical set of μ_β as $t \to \infty$.

(2) $S_i = \{x \in M | \ \phi_\infty(x) \in \overline{X^{M_i}}\}$, $0 \leq i \leq k$.

By the uniqueness of B-B's $(+)$-decomposition theorem, we conclude that the decomposition

$$X = S_0 \cup S_1 \cup \cdots \cup S_k$$

coincides with the decomposition

$$X = X_0^+ \cup X_1^+ \cup \cdots \cup X_k^+$$

in an apparent way.

(We remark that the $(-)$-decomposition can be obtained from the $(+)$- decomposition by considering the $(\mathbf{C}^*)$ – *action* induced from the group isomorphism $\lambda \to \lambda^{-1}$, $\lambda \in \mathbf{C}^*$. So similarly, we have also a comparison between a Morse stratification and the $(-)$-decomposition, but we do not need this for our purpose.)

Note that the gradient flow ϕ_t of μ_β commutes with the torus action, so each Morse stratum S_i $(0 \leq i \leq k)$ is H- equivariant, hence so is each X_i^+, $(0 \leq i \leq k)$.

Now as in the theorem, let r be a (relatively) general point on M_0, and p be a general point close enough to r, then $\mathcal{U}_p \subset \mathcal{U}_r \subset X_0^+$. In fact,

$$\mathcal{U}_p = \mathcal{U}_r - \mathcal{U}_r \cap \overline{X^{M_0}}$$

where $\overline{X^{M_0}}$ can be regarded as the zero section of the vector bundle γ_0^+. It is not hard to see that

$$\mathcal{U}_r \longrightarrow \mathcal{U}_r \cap \overline{X^{M_0}}$$

is also a vector bundle. (The serious reader can refer to [BH]. [BH] contains a proof for an arbitrary $\mathbf{C}^*$-stable subvariety, not only $\mathcal{U}_r$). Now the fiber of

$$\mathcal{U}_p/H \longrightarrow (\mathcal{U}_r \cap \overline{X^{M_0}})//H = \mathcal{U}_r//H$$

is just the fiber of the morphism γ_0^+ modulo the induced action of H_β, hence it is a weighted projective space since the induced action is equivalent to a linear action (on a vector space).

It is clear that the dimension of the fibers is $codim_{\mathbf{C}} X^M - 1$.

Proof of theorem 5.1.2. The same argument as above except that we replace the morphism γ_0^+ by γ_i^+ (or γ_i^-) for some i.

Convention. Given any wall M_i as above, then M_i separates $\mu(X)$ into two regions. We denote the that meets $\mu(X_i^+)$ by M_i^+, and the other one by M_i^-. Sometimes, we also use $M_i^<$ and $M_i^>$ to denote these two regions.

5.3 Small Resolutions: the Simple Cases

Definition . A proper surjective algebraic map $f : Y \to Z$ between irreducible complex n-dimensional algebraic varieties is small if Y is (rationally) nonsingular and for all $r > 0$,

$$codim_{\mathbf{C}}\{x \in Z | dim_{\mathbf{C}} f^{-1}(x) \geq r\} > 2r$$

A small resolution $f : Y \to Z$ is a resolution of singularities which is a small map.

In the case that Y is rationally nonsingular, we shall say f is a rational resolution.

An alternative definition of a small map goes like this: An algebraic map $f : Y \longrightarrow Z$ is small if there exists a stratification of Z by locally, closed, smooth subvarieties $(Z_i) 1 \leq i \leq m$ such that for any $z \in Z_i$, we have

$$dim f^{-1}(z) < \frac{1}{2}(dim Z - dim Z_i).$$

Let M be a wall, $r \in M$ be a relatively general point, and p, q be two general point close enough to r. Let f be the projection from $\mathcal{U}_p//H$ to $\mathcal{U}_r//H$, and g be the projection from $\mathcal{U}_q//H$ to $\mathcal{U}_r//H$. Let also $B = \mathcal{U}_r(M)//H$, $A = f^{-1}(B), A' = g^{-1}(B)$. Then by the facts that we have got before, $A \longrightarrow B$ is a (rational) $\mathbf{P}^d$-bundle,while $A' \longrightarrow B$ is a (rational) $\mathbf{P}^e$- bundle. We have the following fact about small maps.

Proposition. Suppose $d \leq e$ without the loss of the generality, then

$$f : \mathcal{U}_p//H \rightarrow \mathcal{U}_r//H$$

is a (rationally) small resolution.

Proof. We follow the notations in the previous section. We assume our wall M is the wall M_i there, i.e, $M = M_i$

Let a be a point in $\overline{X^{M_i}}$, then by theorem 4.1 [B-B], there are two subspaces $T_a(X)^+$ and $T_a(X)^-$ of the tangent space $T_a(X)$ at a, such that

$$T_a(X_i^+) = T_a(\overline{X^{M_i}}) \oplus T_a(X)^+,$$

$$T_a(X_i^-) = T_a(\overline{X^{M_i}}) \oplus T_a(X)^-,$$

and

$$T_a(X) \oplus T_a(\overline{X^{M_i}}) = T_a(X)^+ \oplus T_a(X)^-.$$

Thus

$$dim X_i^+ + dim X_i^- = dim X + dim X^{M_i}.$$

So we have

$$(dim X_i^+ - dim X^{M_i} - 1) + (dim X_i^- - dim X^{M_i} - 1)$$

$$= dim X - dim X^{M_i} - 2.$$

From the previous section, we know that one of d and e is $dim\ X_i^+ - dim\ X^M - 1$

and the other one is $dim\ X_i^- - dim\ X^M - 1$. So we have $d + e = dim\ X - dim\ X^M - 2$. Now $\mathcal{U}_r//H = B \cup (\mathcal{U}_r//H - B)$ is a stratification by smooth subvarieties. To check the smallness of f, we only need to focus on stratum B since f is an isomorphism over $\mathcal{U}_r//H - B$. Now suppose $x \in B$, then

$$\begin{aligned} dim\ f^{-1}(x) = dim\ \mathbf{P}^d = d &\leq 1/2(d+e) \\ &= 1/2(dim\ X - dim\ X^M - 2) \\ &< 1/2(dim\ X - dim\ X^M - 1) \\ &= 1/2(dim\ \mathcal{U}_r//H - dim\ B). \end{aligned}$$

So in an explicit way, we have

(1). If $dim\ X_i^+ < dim\ X_i^-$, then $\mathcal{U}_p/H \to \mathcal{U}_r//H$ is a small (rational) resolution.

(2). If $dim\ X_i^- < dim\ X_i^+$, then $\mathcal{U}_q/H \to \mathcal{U}_r//H$ is a small (rational) resolution.

(3). If $dim\ X_i^+ = dim\ X_i^-$, then both $\mathcal{U}_p/H \to \mathcal{U}_r//H$ and $\mathcal{U}_q/H \to \mathcal{U}_r//H$ are small (rational) resolutions.

As an immediate consequence we have:

$IH_*(\mathcal{U}_r//H)$ is isomorphic to $H_*(\mathcal{U}_p/H)$ if $d \leq e$ (i.e, $dim\ X_i^+ \leq dim\ X_i^-$), or isomorphic to $H_*(\mathcal{U}_q/H)$ if $e \leq d$ (i.e, $dim\ X_i^- \leq dim\ X_i^+$).

5.4 Small Resolutions: the General Case

In general, we have,

Theorem. For every singular quotient $\mathcal{U}_q//H(q \in \mu(X))$, there exists a general point p in $\mu(X)$ such that p is close enough to q , and $\mathcal{U}_p/H \to \mathcal{U}_q//H$ is a (rationally) small resolution .

Proof. We shall first present a detailed proof for v where v is general in a codim 2 wall and then give a general proof without too much detail so that the reader can grasp the point hiden behind the technique. As one may see already, the proposition 5.3 is our very first step.

So let $v \in N = M_1 \cap M_2$ where N is a codim 2 wall, M_1, M_2 are codim 1 walls.

Now for each M_i, one of $X^{M_i^<}$ and $X^{M_i^>}$ has a smaller dimension, denote it by C_i, then

Take u to be a general point in $C_1 \cap C_2$, so that u is close enough to v. Let also $p_1 \in M_1, p_2 \in M_2$ be two relatively general points in $M_1 \cap C_2$ and $M_2 \cap C_1$ respectively, and close enough to v, then we will show now that

$$f : \mathcal{U}_u/H \to \mathcal{U}_v//H$$

is a small resolution.

The (serious) singular locus of X_v is

$$\Sigma = \mathcal{U}_v(M_1)/H \bigcup \mathcal{U}_v(M_2)/H$$

and

$$\Sigma_1 = \mathcal{U}_v(M_1)/H \cap \mathcal{U}_v(M_2)/H = \mathcal{U}_v(M_1 \cap M_2)/H$$

Note that f is an isomorphism off Σ and $f^{-1}(\Sigma)$.

$$f^{-1}(\Sigma) = f^{-1}(\mathcal{U}_v(M_1)/H) \bigcup f^{-1}(\mathcal{U}_v(M_1)/H)$$

So

$$dim f^{-1}(x) = \begin{cases} d_1 & x \in \mathcal{U}_v(M_1)/H - \Sigma_1 \\ d_2 & x \in \mathcal{U}_v(M_2)/H - \Sigma_1 \\ d_1 + d_2 & x \in \Sigma_1 \end{cases}$$

where d_i is the dimension of the fiber of

$$\mathcal{U}_u(C_i)/H \to \mathcal{U}_{p_i}(M_i)/H.$$

Now we have

$$\mathcal{U}_v//H = (\mathcal{U}_v//H - \Sigma) \cup (\mathcal{U}_v(M_1)//H - \Sigma_1) \cup (\mathcal{U}_v(M_2)//H - \Sigma_1) \cup \Sigma_1$$

$$= S_0 \cup S_1 \cup S_2 \cup S_{12}$$

Now pick up u_1, u_2 general in M_1, M_2 resp. such that u, u_1, U_2, v are all close enough to each other. Then

$$\mathcal{U}_u//H \to \mathcal{U}_v//H$$

is the composite

$$\mathcal{U}_u//H \xrightarrow{f_1} \mathcal{U}_{u_1}//H \xrightarrow{f_{12}} \mathcal{U}_v//H.$$

Similarly, it also the composite

$$\mathcal{U}_u//H \xrightarrow{f_2} \mathcal{U}_{u_2}//H \xrightarrow{f_{21}} \mathcal{U}_v//H.$$

It is fairly clear that f_1, f_{12}, f_2, f_{21} are small maps. We can regard S_1 as a subvariety of $\mathcal{U}_{u_1}(M_1)//H$. So if $y \in S_1$,

$$dim\ f^{-1}(y) = dim\ f_1^{-1}(y) < 1/2(dim\mathcal{U}_{u_1}//H - dim\mathcal{U}_{u_1}(M_1)//H)$$

$$= 1/2(dim\ \mathcal{U}_v//H - dim\ S_1).$$

Similarly, for $y \in S_2$, there exists $y_2 \in \mathcal{U}_{u_2}(M_2)//H$,

$$dim\ f^{-1}(y) = dim\ f_2^{-1}(y_2) < 1/2(dim\ \mathcal{U}_v//H - dim\ S_2).$$

If $y \in S_{12} = \sum_1$, let g be $\mathcal{U}_{u_1}(M_1)//H \to \mathcal{U}_v(M_1)//H$ and $y_1 \in \mathcal{U}_{U_1}(M_1)//H$, then

$$dim\ f^{-1}(y) = dim\ g^{-1}(y) + dim\ f_1^{-1}(y_1)$$

$$< 1/2(dim\ \mathcal{U}_v(M_1)//H - dim\ \mathcal{U}_v(M_1 \cap M_2)//H)$$

$$+1/2(dim\ \mathcal{U}_{u_1}//H - dim\ \mathcal{U}_{u_1}(M_1)//H)$$

$$= 1/2(dim\ \mathcal{U}_{v_1}//H - dim\ S_{12}).$$

Hence we conclude that f is small.

The proof of smallness in general.

We follow the notation as in 5.1. Let $N_1, \cdots, N_l$ be all the *codim* 1 walls containing the point q. Then $N_1, \cdots, N_l$ divide $\mu(X)$ into many connected components. We pick up a connected component of $\mu(X) - \bigcup_{i=1}^{l} N_i$ such that it has the following property: if u is a point in this component, v is a point in N_i (for any i), and u, v are close enough to each other, then $\mathcal{U}_u//H \longrightarrow \mathcal{U}_v//H$ is a small map. This can be done by using 5.3. We start with N_1, $\mu(X) - N_1$ has two components, one of them has the following property: if u is in this component, $v \in N_1$ and u, v are close enough to each other, then $\mathcal{U}_u//H \longrightarrow \mathcal{U}_v//H$ is a small map. Now fix this connected component of $\mu(X) - N_1$, then N_2 divides it into two components, we apply the above procedure again, and get a desired region. Repeat this procedure, we finally end up with a connected component of $\mu(X) - \bigcup N_i$ with the desired property.

Now we pick up a point p in the selected component of $\mu(X) - \bigcup N_i$ such that p and q are close enough to each other. We claim now that

$$f : \mathcal{U}_p//H \longrightarrow \mathcal{U}_q//H$$

is a small map.

To prove our assertion, we recall that there is a stratification $\bigcup_{I \subset \{1, \ldots, l\}} C_I$ of $\mathcal{U}_q//H$ (see 5.1 for the definition of C_I) such that f restricted to $f^{-1}(C_I)$ is a fibration tower over C_I whose fibers are rationally projective spaces.

In fact, if we pick up points $r_J \in N_J, J \subset \{1, \cdots, l\}$, $(r_\emptyset = p, r_{\{1,\ldots,l\}} = q)$ such that all r_J are close enough to each other, then the map

$$f : \mathcal{U}_p//H \longrightarrow \mathcal{U}_q//H$$

is the composite:

$$\mathcal{U}_p//H \xrightarrow{f_1} \mathcal{U}_{r_{\{i_1\}}}//H \xrightarrow{f_2} \mathcal{U}_{r_{\{i_1 i_2\}}}//H \rightarrow \cdots \xrightarrow{f_l} \mathcal{U}_q//H$$

where $\{i_1, \cdots, i_l\}$ is any permutation of $\{1, \cdots, l\}$. Let $I = \{i_1, \cdots i_k\} \subset \{1, \cdots, l\}$, then $f|f^{-1}(C_I) : f^{-1}(C_I) \rightarrow C_I$ is the composite:

$$f_1^{-1} \cdots f_k^{-1}(C_I) \xrightarrow{\bar{f}_1} \cdots \rightarrow f_k^{-1}(C_I) \xrightarrow{\bar{f}_k} C_I$$

where each $\bar{f}_h$ is induced from $f_h (1 \leq h \leq k)$ and each $\bar{f}_h$ is a projective bundle. We assume that the fiber of $\bar{f}_h$ is of dimension d_h. Then by the smallness of f_h (this is an implication of proposition 5.3), we have

$$d_h < 1/2(dim\, \mathcal{U}_{r_{\{i_1, \cdots, i_{h-1}\}}}(N_{\{1, \cdots, i_{h-1}\}})//H - dim\, \mathcal{U}_{r_{\{i_1, \cdots, i_h\}}}(N_{\{i_1, \cdots, i_h\}})//H),$$

for $1 \leq h \leq k$. Now for any point $y \in C_I$,

$$dim\ f^{-1}(y) = d_1 + \cdots + d_k < 1/2(dim\ X - dim\ C_I)$$

Since $dim\, \mathcal{U}_{r_\emptyset}(N_\emptyset)//H = dim\ X$ and $dim\, \mathcal{U}_{r_{\{i_1, \cdots, i_k\}}}(N_{\{i_1, \cdots, i_k\}})//H = dim\ C_I$.

Remark. In fact we have proved that

$$dim\ f^{-1}(y) \leq 1/2(dim\ X - dim\ C_I) + k$$

or

$$codim\ C_I \geq 2 dim\ f^{-1}(y) - 2k$$

which shows that f is "very" small.

Remark. As one can see from this section that there are many small maps in the canonical maps among symplectic quotients which can be told explicitly in practice. Proposition 5.3 is the key to tell small maps. The same comments are also true for algebraic quotients.

5.5 The Decomposition Theorem

In this section we recall the decomposition theorem of intersection homology theory. This powerful theorem was conjectured by S.Gelfand and R. MacPherson, and proved by Beilinson, Beinstein and Deligne. Thoroughout this thesis, we restrict our attention to (co)homology over rational numbers unless indicated otherwise.

Theorem. (The decomposition theorem.)

Let $f : X \rightarrow Y$ be a projective algebraic map. Then there exists:

(1) A stratification $Y = \bigcup_\alpha Y_\alpha$ of Y,

(2) A list of enriched strata $E_\beta = (Y_\beta, L_\beta)$ where Y_β is a stratum of Y and L_β is a local system over Y_β, and

(3) For each enriched stratum E_β, a polynomial in t, $\phi^\beta = \sum_j \emptyset_j^\beta t^j$ such that for any open subset $\mathcal{U} \subset Y$

$$IH_k(f^{-1}(\mathcal{U})) = \bigoplus_\beta \bigoplus_{i+j=k} IH_i(\mathcal{U} \cap \overline{Y_\beta}; L_\beta) \otimes Q\phi_j^\beta.$$

In particular,

$$IH_k(X) = \bigoplus_\beta \bigoplus_{i+j=k} IH_i(\overline{Y_\beta}; L_\beta) \otimes Q\phi_j^\beta$$

if we take $\mathcal{U} = Y$.

We shall next present a popularized version of the decomposition theorem for some special cases, which is useful for us in the latter calculation of the intersection Poincarè polynomials of quotients. This popularized version is taken from some lectures given by R. MacPherson in the intersection homology seminar at MIT in 1989.

Theorem. (A special version of the decomposition theorem.)

Let $f : X \rightarrow Y$ be a projective algebraic map, and X is a nonsingular variety. We follow the notation in the theorem above. Assume that every local system L_β in the theorem above is trivial, then there exists a collection of polynomials ϕ_β for all strata such that

$$P(X) = \sum_\beta IP(\overline{Y_\beta}) \cdot \phi_\beta,$$

and for each $y \in Y$

$$IPf^{-1}(y) = \sum_\beta IP_Y(\overline{Y_\beta}) \cdot \phi_\beta.$$

Futhermore, ϕ_β shares the properties of $IH(V)$ where V is a projective variety of dimension $dim_{\mathbb{C}}\, X - dim_{\mathbb{C}}\, Y_\beta$ (e.g, Hard Lefschetz, Poincarè duality).

5.6 The Formulas for Intersection Homology: the Simple Cases

Let M be an interior *codim* 1 wall in $\mu(X)$. Let $r \in M$ be (relatively) general, $p, q \in \mu(X)$ be general in $\mu(X)$ close enough to r, but in different sides of M, then we have as before

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f_1"'] & X_p \arrow[d, "f"] \\ B \arrow[r] & X_r, \end{tikzcd}$$

where $X_p = \mathcal{U}_p/H$, $X_r = \mathcal{U}_r//H$, $B = \mathcal{U}_r(M)/H$, $A = f^{-1}(B)$. It is known that

$$A \xrightarrow{f_1} B$$

is a fibration whose fiber is a rationally homological projective space of dimension d.

Let $S_1 = B$, $S_0 = X_r - B$, then $X_r = S_0 \cup S_1$. Now we can apply the special version of the decomposition theorem because any weighted projective bundle has no monodromy over a field.

By decomposition theorem, there exist two polynomials φ_{S_0} and φ_{S_1}, such that

$$P(f^{-1}(y)) = \varphi_{S_0} IP_y(\overline{S_0}) + \varphi_{S_1} IP_y(\overline{S_1}), \forall y \in X_r,$$

and

$$P(X_p) = \varphi_{S_0} IP(\overline{S_0}) + \varphi_{S_1} IP(\overline{S_1})$$

i.e,

$$P(X_p) = \varphi_{S_0} IP(X_r) + \varphi_{S_1} IP(B).$$

Now we want to determine φ_{S_0} and φ_{S_1}.

Take $y_0 \in S_0$, then $f^{-1}(y_0)$ is a single point, hence

$$1 = \varphi_{S_0} \cdot 1 + \varphi_{S_1} \cdot 0$$

because $IP_{y_0}(\overline{S_0}) = 1$ (since y_0 is a regular point in $\overline{S_0} = X_r$, that is, S_0 is regular) and $IP_{y_0}(S_0) = 0$ (since $y_0 \notin S_1$). So $\varphi_{S_0} = 1$.

Take $y_1 \in S_1$, then $f^{-1}(y_1)$ is a weighted projective space of dimension d, hence

$$1 + t^2 + \cdots + t^{2d} = IP_{y_1}(\overline{S_0}) + \varphi_{S_1} \cdot 1$$

that is

$$\varphi_{S_1} = 1 + t^2 + \cdots + t^{2d} - IP_{y_1}(X_r).$$

Similarly, we have

$$\begin{tikzcd} A' \arrow[r] \arrow[d, "g_1"'] & X_q \arrow[d, "g"] \\ B \arrow[r] & X_r \end{tikzcd}$$

where $X_q = \mathcal{U}_q/H$, $A' = g^{-1}(B)$, and

$$A' \xrightarrow{g_1} B$$

is a fibration whose fiber is a rationally projective space of dimension e, then, again by decomposition theorem, there are two polynomials φ'_{S_0} and φ'_{S_1} so that

$$P(g^{-1}(y)) = \varphi'_{S_0} IP_y(\overline{S_0}) + \varphi'_{S_1} IP_y(\overline{S_1}), \forall y \in X_r$$

and

$$P(X_q) = \varphi'_{S_0} IP(X_r) + \varphi'_{S_1} IP(B).$$

So repeat the calculation presented above with the same choice of y_0 and y_1, we have $\varphi'_{S_0} = 1$, and

$$\varphi'_{S_1} = 1 + t^2 + \cdots + t^{2e} - IP_{y_1}(X_r)$$

Thus subtract the two equations below

$$P(X_q) = IP(X_r) + \varphi'_{S_1} IP(B)$$

$$P(X_p) = IP(X_r) + \varphi_{S_1} IP(B)$$

we have

$$\begin{aligned} P(X_q) - P(X_p) &= (\varphi'_{S_1} - \varphi_{S_1}) IP(B) \\ &= \epsilon(M) Q_t(M) P(B) \end{aligned}$$

where $Q_t(M)$ and $\epsilon(M)$, as before, are defined by

$$Q_t(M) = \begin{cases} t^{2(d+1)} + \cdots + t^{2e} & \text{if } d < e \\ t^{2(e+1)} + \cdots + t^{2d} & \text{if } d > e \\ 0 & \text{if } d = e \end{cases}$$

$$\epsilon(M) = \begin{cases} 1 & \text{if } d_i < e_i \\ -1 & \text{if } d_i > e_i \\ 0 & \text{if } d_i = e_i \end{cases}$$

We summerize above results as follows

Theorem. Let y be any point in B. Then,

(1) $P(X_p) = IP(X_r) + (1 + t^2 + \cdots + t^{2d} - IP_{y_1}(X_r))P(B)$
(2) $P(X_p) = IP(X_r) + (1 + t^2 + \cdots + t^{2e} - IP_{y_1}(X_r))P(B)$
(3) $P(X_q) = P(X_p) + \epsilon(M)Q(M)P(B)$.

Corollary. Let the notations be as in the beginning. Let y be a point in $\mathcal{U}_r(M)/H$, then

$$IP_y(\mathcal{U}_r//H) = \begin{cases} 1 + t^2 + \cdots + t^{2d}, & \text{if } d \leq e \\ 1 + t^2 + \cdots + t^{2e}, & \text{if } e \leq d \end{cases}$$

Now if M is a *codim* 1 face of $\mu(X)$, $\theta \in M$ is a relatively general point in M, and a is is a general point in $\mu(X)$ and close enough to θ, then

$$\mathcal{U}_a/H \rightarrow \mathcal{U}_\theta(M)/H (= \mathcal{U}_\theta//H)$$

is a fibration whose fiber is a rationally homological projective space of dimension $m = codim_{\mathbf{C}} X^M - 1$. So We know by the decomposition theorem:

Lemma. $P(\mathcal{U}_a/H) = P(\mathrm{P}^m) \cdot P(\mathcal{U}_\theta(M)/H)$.

Like before we define a polynomial $Q_t(M)$ for the face M by

$$Q_t(M) = P(\mathrm{P}^m) = 1 + t^2 + \cdots + t^{2m}$$

and agree that $\epsilon(M) = 1$.

5.7 The Formulae for Intersection Homology: the General Case

So now let $q \in Int(\mu(X))$ be general, and let $M_0 \prec \mu(X)$ be a *codim* 1 face of $\mu(X)$. Take a point $r_0 \in M$, such that r_0 is (relatively) general in M_0 and the vector $\overline{r_0, q}$ from r_0 to q does not meet any *codim* ≤ 2 wall (this assumption is for technical reason, and is not necessary). Also we pick up a general point p in $\mu(X)$ so that p is close enough to r_0. Then as before, we assume that the change from r_0 to q is described as follows

$$r_0 \in M_0 \rightarrow p \rightarrow \epsilon(M_1)M_1 \rightarrow \cdots \rightarrow \epsilon(M_k)M_k \rightarrow q,$$

where $M_1, \cdots, M_k$ are exactly the walls that $\overline{p, q}$ meets. Then apply theorem 5.6. (3), and lemma 5.6, we get

Theorem. (An inductive homological formula)

$$P(\mathcal{U}_q/H) = \Sigma_{j=0\ldots k}\epsilon(M_j)Q(M_j)P(\mathcal{U}_{r_j}(M_j)/H_j)$$

or

$$P(\mu^{-1}(q)/T) = \Sigma_{j=0\ldots k}\epsilon(M_j)Q(M_j)P(\mu^{-1}(r_j) \cap \overline{X^{m_j}}/T_j)$$

where $r_j = \overline{p,q} \cap M_j$,

$$H_j = H/(\mathit{stabilizer\ of}\ \overline{X^{M_j}}\ \mathit{in}\ H),$$

$$T_j = T/(\mathit{stabilizer\ of}\ \overline{X^{M_j}}\ \mathit{in}\ T).$$

Remark. Note again that for any wall M, $\overline{X^M}$ is a nonsingular compact projective variety with the action of torus $H/(\mathit{stabilizer\ of}\ \overline{X^M})$. Hence induction applies indeed.

Then we have the following three essential situations.

(1) p and q are general. Then

$$P(X_q) = P(X_p) + \sum_{j=1,\cdots,k}\epsilon(M_j)Q(M_j)P(B_j)$$

where $B_j = \mathcal{U}_{o_j}(M_j)/H$; $o_j = M_j \cap \overline{p,q}$.

(2) p general, q is on a wall , then we take q' general so that p, q, q' are colinear and we have the following situation,

$$p \rightarrow \epsilon(M_1)M_1 \rightarrow \cdots \rightarrow \epsilon(M_k)M_k \ni q \rightarrow q'(general),$$

then,

$$IP(X_q) = \begin{cases} P(X_p) + \Sigma_{j=1,\cdots,k-1}\epsilon(M_j)Q(M_j)P(B_j) & \text{if } \epsilon(M_k) = 1 \\ P(X_p) + \Sigma_{j=1,\cdots,k}\epsilon(M_j)Q(M_j)P(B_j) & \text{if } \epsilon(M_k) = -1 \end{cases}$$

(3)p, q both on walls, then we take p', q' general so that p', p, q, q' are colinear and we have the following situation,

$$p' \rightarrow p \in \epsilon(M_1)M_1 \rightarrow \epsilon(M_2)M_2 \rightarrow \cdots \rightarrow \epsilon(M_k)M_k \ni q \rightarrow q'$$

Then,

$$IP(X_q) = \begin{cases} IP(X_p) + \sum_{j=1,\cdots,k} \epsilon(M_j)Q(M_j)P(B_j) & \text{if } \epsilon(M_1) < 0, \epsilon(M_k) < 0 \\ IP(X_p) + \sum_{j=1,\cdots,k-1} \epsilon(M_j)Q(M_j)P(B_j) & \text{if } \epsilon(M_1) < 0, \epsilon(M_k) > 0 \\ IP(X_p) + \sum_{j=2,\cdots,k} \epsilon(M_j)Q(M_j)P(B_j) & \text{if } \epsilon(M_1) > 0, \epsilon(M_k) < 0 \\ IP(X_p) + \sum_{j=2,\cdots,k-1} \epsilon(M_j)Q(M_j)P(B_j) & \text{if } \epsilon(M_1) > 0, \epsilon(M_k) > 0 \end{cases}$$

Proof. They are all simple applications of propositions in the beginning of this section.

Remark. In (2) of the theorem , (a) if $\epsilon(M_k) = 0$, the two formulae in the expression are the same.
(b) If q general, p is on a wall, just reverse the diagram

$$(general)p' \leftarrow p \leftarrow (-\epsilon(M_1))M_1 \leftarrow \cdots \leftarrow (-\epsilon(M_k))M_k \leftarrow q$$

and apply the existing formula , we shall get a desired one.

We shall do some examples in chapter 7.

5.8 Comments on Kirwan's Formula

In her book [Ki], Kirwan was able to present a cohomological formula for $\mu^{-1}(o)/T$ (actually her formula applies for general compact reductive Lie group) by employing Morse theory. The basic idea is to view the norm square of the moment map, $||\mu||^2$, as a Morse function in an appropriate sense, and therefore get a Morse stratification (which is equivariantly perfect)

$$X = \bigcup_{\beta \in B} S_\beta$$

with index set B = *set of connected components of critical subsets of* $||\mu||$. It is observed that, in the case of torus actions, her arguments only valids for $\mu^{-1}(p)/T$, where p is the barycenter of $\mu(X)$ and is in general position.

So let o be the barycenter of $\mu(X)$ and be general. Then B is the set of the barycenters of the various walls in $\mu(X)$ so that if $\beta \in B$, then $\overline{o,\beta}$ is perpendicular to the wall that β belongs to. Then Kirwan's formula can be restated as follows

$$P_t(\mu^{-1}(o)/T) = P(X)P(BT) - \Sigma_{\beta \in B} t^{d(\beta)} P_t^{T_\beta}(\overline{X^{M_\beta}} \cap \mu^{-1}(\beta))$$

where $d(\beta)$ is the codimension of the stratum S_β, M_β is the wall that β belongs to, and $T_\beta = T/(\textit{stabilizer of } \overline{X^{M_\beta}} \textit{ in } T)$.

We point out that In Kirwan's formula, there involves equivariant Poincare "polynomials" (that is, power serious). But, in our formula, there are only polynomials. Also our formula applies for arbitrary algebraic quotients. Moreover, in order to apply her formula to singular quotients, considerable efforts were made by Kirwan on their desingularization. However, these efforts are not required in order to apply our formula.

5.9 Comments on Ordinary Homology

In this section, our goal is to understand the ordinary homology groups of singular quotients (so to this end, "small maps" do not give any help!). Our argument will depend on the following observation: if p, q are two points in the interior of $\mu(X)$, p is general in a *codim* $r-1$ wall N, and p, q are "close enough" to each other, then we have the following commutative diagram:

$$\begin{tikzcd}
A \arrow[r, hook] \arrow[d] & \mathcal{U}_p//H \arrow[d, "f"] \\
B \arrow[r, hook] & \mathcal{U}_q//H
\end{tikzcd}$$

where $B = \mathcal{U}_q(N)/H,\ A = f^{-1}(B)$.

Lemma. Suppose we have a diagram of algebraic varieties

$$\begin{tikzcd}
A \arrow[r, hook] \arrow[d] & X \arrow[d, "f"] \\
B \arrow[r, hook] & Y
\end{tikzcd}$$

such that $A \rightarrow B$ is rationally a projective bundle and f is an isomorphism off B. Assuming B,$Y - B$ have vanishing homology in odd degrees, then

$$H_*(X) \oplus H_*(B) = H_*(Y) \oplus H_*(A).$$

In particular, $P(X) + P(B) = P(Y) + P(A)$.

Proof. By [Fu] 19.1. (6), we have a long exact sequence

$$H_{i+1}(X-A)(\otimes\mathbf{Q}) \rightarrow H_iA(\otimes\mathbf{Q}) \rightarrow H_iX(\otimes\mathbf{Q}) \rightarrow H_i(X-A)(\otimes\mathbf{Q}) \rightarrow H_{i-1}(A).$$

So by the vanishing assumptions, we have

$$H_*(X) = H_*(X-A) \oplus H_*(A).$$

Similarly,

$$H_*(Y) = H_*(Y - B) \oplus H_*(B).$$

Hence,

$$H_*(X) \oplus H_*(B) = H_*(Y) \oplus H_*(A).$$

because $X - A$ is isomorphic to $Y - B$.

From now on, we assume that the fixed point set of H has vanishing homology in odd degrees. Now let q be an interior point in $\mu(X)$, and general in a *codim* r wall N. We take a sequence of interior points $q_{r-1}, \cdots, q_1, q_0$ in $\mu(X)$, such that q_i is general in a *codim* i wall N_i $(0 \leq i \leq r-1)$,

$$N \subset N_{r-1} \subset \cdots \subset N_1 \subset N_0 = \mu(X),$$

and q_i, q_j are close enough to each other for any $0 \leq i, j \leq r$, where we agree $N = N_r, q = q_r$. Hence for each $1 \leq i \leq r$, we have

$$\begin{tikzcd} A_i \arrow[r, hook] \arrow[d] & X_{i-1} \arrow[d, "f_i"] \\ B_i \arrow[r, hook] & X_i \end{tikzcd}$$

where $X_{i-1} = \mathcal{U}_{q_{i-1}}//H$, $X_i = \mathcal{U}_{q_i}//H$, $B_i = \mathcal{U}_{q_i}(N_i)/H = \mu^{-1}(q_i) \cap \overline{X^{N_i}}/H$, $A_i = f_i^{-1}(B_i)$ which is a fibration over B_i whose fiber is a weighted projective space of dimension d_i. If we assume that all $X_i - B_i$ has vanishing homology in odd degrees, then by the lemma and induction on the walls of $\mu(X)$, we have

$$P(X_{i-1}) - P(X_i) = P(B_i)(t^2 + \cdots + t^{2d_i}), i = 1, \cdots, r$$

Add these r equations together, we get

$$P(X_0) - P(X_r) = \sum_{i=1}^{r} P(B_i)(t^2 + \cdots + t^{2d_i}).$$

Hence, we have

Proposition. (An ordinary homological formula.) Let the assumptions be as above. Then

$$P(X_r) = P(X_0) - \sum_{i=1}^{r} P(B_i)(t^2 + \cdots + t^{2d_i})$$

Combine this proposition with theorem 5.7, we have

Proposition. (An inductive ordinary homological formula for singular quotients.) Let assumptions be as in above, and let

$$r_0 \in M_0 \to \epsilon(M_1)M_1 \to \cdots \to \epsilon(M_k)M_k \to q_0$$

be a diagram for q_0 as defined in 5.7. Then

$$P(X_r) = \sum_{j=0}^{k} \epsilon(M_j)Q(M_j)P(\mathcal{U}_{r_j}/H_j) - \sum_{i=1}^{r} P(B_i)(t^2 + \cdots + t^{2d_i})$$

or

$$P(\mu^{-1}(q_r)/T) = \sum_{j=0}^{k} \epsilon(M_j)Q(M_j)P(\mu^{-1}(r_j) \cap \overline{X^{M_j}}/T)$$

$$-\sum_{i=1}^{r} P(\mu^{-1}(q_i) \cap \overline{X^{N_i}}/T)(t^2 + \cdots + t^{2d_i})$$

where unspecified notations are same as in theorem 5.7.

Chapter 6

The Topology of Algebraic Quotients

We reformulate the theorems of chapter 5 for algebraic quotients. We shall find that all quotient varieties enjoy the property that their cycle maps are all isomorphisms.

6.1 Statements of Results

Let M be *codim* 1 face of $\mu(X)$. Let $\Xi(M)$ be an admissible collection of top dimensional polyhedron in M, and $\mathcal{U}(M) = \bigcup_{C \in \Xi(M)} X^C$, then $\mathcal{U}(M)//H$ is a (rationally) nonsingular quotient of $\overline{X^M}$. Define

$$\Xi_1 = \{C \in \Xi \mid \exists D \in \Xi(M) \textit{ such that } D \prec C\}$$

$$\Xi_2 = \Xi_1 - \Xi(M)$$

then clearly we have $\Xi_1 \prec \Xi_2$. Let $\mathcal{U}_1, \mathcal{U}_2$ be the corresponding algebraic open subsets resp. Then it is not hard to see that $\mathcal{U}_1//H = \mathcal{U}(M)//H$.

Theorem 1. The natural map

$$\varphi : \mathcal{U}_2//H \longrightarrow \mathcal{U}_1//H$$

is a fibration whose typical fiber is a weighted projective space of dimension

$$codim_c X^M - 1.$$

Now we follow the notation in corollary 2.6. Suppose we have two admissible collection of polyhedra in Ξ, say Ξ_1 and Ξ_2, and Ξ_2 covers Ξ_1. We have also

(1) Ξ_2 consists of top dimensional polyhedra.

(2) The collection of *codim* 1 polyhedra in Ξ_1 forms an admissible collection $\Xi_1(M)$ for $\overline{X^M}$.

Let $\mathcal{U}_1, \mathcal{U}_2, \mathcal{U}_1(M)$ be corresponding "open" subsets of Ξ_1, Ξ_2 and $\Xi_1(M)$ resp. We have

Theorem 2. Let $B = \mathcal{U}_1(M)//H$ and $A = f^{-1}(B)$ where f is $\mathcal{U}_2//H \to \mathcal{U}_1//H$. Then

$$\begin{tikzcd} A \arrow[r] \arrow[d] & \mathcal{U}_2//H \arrow[d] \\ B \arrow[r] & \mathcal{U}_1//H \end{tikzcd}$$

is a fiber square where $A \to B$ is a fibration whose fiber is a weighted projective space and f is an isomorphism off B.

For two arbitrary algebraic quotients, we have : let $\mathcal{U}_1, \mathcal{U}_2$ be two arbitrary algebraic open subsets such that there is a canonical nice map f from $\mathcal{U}_2//H$ to $\mathcal{U}_1//H$. Then there is a canonical stratification $\mathcal{U}_1//H = \bigcup_\beta C_\beta$ of $\mathcal{U}_1//H$ such that over every C_β, f is a fibration tower whose fibers are all weighted projective spaces.

It would be very tedious to give an explicit construction of strata C_β as what we did in 5.1. Nevertheless, in practice, given any quotient, we will be able to obtain such construction using the same idea as we did before. Conceptually, however, we still have

Theorem 3. Let the notations be as before. Then

$$\mathcal{U}_1//H = \bigcup_{Wall\ M} \mathcal{U}_1[M]//H$$

is a Whitney stratification such that over each stratum $\mathcal{U}_1[M]//H$ (if $\mathcal{U}_1[M]$ is not empty), f is a fibration tower with weighted projective spaces as fibers, where $\mathcal{U}_1[M]$ is defined as follows: a point x is in $\mathcal{U}_1[M]$ if and only if $x \in \mathcal{U}_1$ and there exists $C \subset M \cap \Xi$ such that $C \prec \mu(\overline{H \cdot x})$ and M is a minimal wall with this property.

All the proofs in this section are essentially the same as in chapter 5. It is fairly straightforward to write down these proofs once one carefully reads section 2.6 and the proofs in chapter 5. So we omit this unnecessary duplicate to save time and space.

6.2 Small Resolutions

We continue to follow the notation in corollary 2.6 and the notations in the previous section. Suppose Ξ_1' lies in M^+. Then

Theorem 1. Suppose *dim* $X_M^+ \leq$ *dim* X_M^-, then

$$f : \mathcal{U}_1//H \longrightarrow \mathcal{U}_2//H$$

is a (rationally) small resolution.

Proof. same as that of proposition 5.3.

In general we have

Theorem 2. For every singular quotient $\mathcal{U}_1//H$, there is a (rationally) nonsingular quotient $\mathcal{U}_2//H$ such that the canonical map $\mathcal{U}_2//H \rightarrow \mathcal{U}_1//H$ is a small map.

Proof. The idea to find out $\mathcal{U}_2$ is essentially the same as in section 5.4. The key is that: given a *codim* l wall N and a *codim* $l-1$ wall M such that $N \subset M$, then N divides M into two regions, and (at least) one of these two regions gives a "small map". So let $\tilde{\Xi}_1$ be the admissible collection defining $\mathcal{U}_1$. Let $N_1 \cdots N_l$ be all the *codim* 1 walls containing some polyhedra in $\tilde{\Xi}_1$. Recall the construction in the proof of theorem 2.5. The definition of Ξ_2 there involves three kinds of data: *codim* 1 walls containing some polyhedra in Ξ_1, some admissible collections of polyhedra on these walls (chosen by induction), and some selected half spaces divided by these *codim* 1 walls. Now we construct our Ξ_2 here in the same way as we did in 2.5 except that for any *codim* 1 wall above, we choose the half space that gives small maps (see the theorem above) and when we use induction we put an additional "smallness" hypothesis. Let $\mathcal{U}_2$ be the corresponding open subset of Ξ_2. The proof of the fact that $\mathcal{U}_2//H \rightarrow \mathcal{U}_1//H$ is small should be completely an analogy of the proof for symplectic quotients. It will be fairly apparent once one grasps the idea behind the previous proof.

6.3 The Vanishing of Homology in Odd Degrees

Let X be an algebraic variety, we denote by $A_k(X)$ the group generated by k-dimensional irreducible subvarieties modulo rational equivalence (see [F],1.3.) Let

$H_i^{BM}(X)$ be the (Borel-Moore) integral homology of X; this is the singular homology of X if X is compact. There is a canonical homomorphism ("cycle map", see [Fu], 19.1):

$$cl_X : A_i(X) \longrightarrow H_{2i}^{BM}(X).$$

Definition. A variety X is said to have property (IS) if
(a) $H_i^{BM}(X) = 0$ for i odd, $H_i(X)$ has no torsion for i even,
(b) $cl_X : A_i(X) \xrightarrow{\cong} H_{2i}(X)$ for all i.
A variety X is said to have property (RS) if
(a) $H_i^{BM}(X) \otimes \mathbf{Q} = 0$ for i odd,
(b) $cl_X \otimes \mathbf{Q} : A_i(X) \otimes \mathbf{Q} \xrightarrow{\cong} H_{2i}(X) \otimes \mathbf{Q}$ for all i.

Obviously, (IS) implies (RS) since (RS) is just a rational version of (IS).
We now formulate a known result (see [DeLP], for example).

Theorem. ([DeLP]). Let X be a smooth projective variety with an action of a complex torus H. Then X has property (IS) (resp. (RS)) if X^H has property (IS) (resp. (RS)).

The proof is essentially based on B-B's decomposition theorem and the lemma below.

Lemma. ([DeLP], 1.8). If X has α-partition into pieces which have property (IS) (resp. (RS)), then X has property (IS) (resp. (RS)).

Recall that a finite partition of a variety X into subsets is said to be an α-partition if the subsets in the partition can be indexed $X_1 \cdots X_m$ in such a way that $X_1 \cup \cdots \cup X_i$ is closed in X for $i = 1, \cdots, k$.
Now the theorem follows easily since B-B's decomposition is an α-partition.

The question in which we are interested is whether a quotient variety has property (IS) (resp. (RS)) or not. To answer this question partially, we have

Proposition. (The vanishing of intersection homology in odd degrees). Let X be a smooth algebraic projective variety with an action of a complex torus H. Then the rational intersection homology groups of an arbitrary categorical quotient vanish in odd degrees. Moreover, if the action of the torus is quasi-free (i.e, there are no non-trivial finite stabilizers), then, the integral intersection homology groups of an arbitrary categorical quotient vanish in odd degrees and have no torsion in even degrees.

Proof. It follows straightforwardly from our inductive homological formula before.

6.4 Cycle Maps

Lemma. Let $Z \subset X$ be a closed embedding, and $\mathcal{U}$ be the complement of Z. Suppose also that Z has property (IS) (resp. (RS)), then cl_X (resp. $cl_X \otimes \mathbf{Q}$) is isomorphism if and only if $cl_{\mathcal{U}}$ (resp. $cl_{\mathcal{U}} \otimes \mathbf{Q}$) is isomorphism.

Proof. Combine [Fu], 1.8 and 19.1(6), we have the following commutative diagram,

$$\begin{tikzcd}
 & A_iZ(\otimes\mathbf{Q}) \arrow[r] \arrow[d] & A_iX(\otimes\mathbf{Q}) \arrow[r] \arrow[d] & A_i\mathcal{U}(\otimes\mathbf{Q}) \arrow[r] \arrow[d] & 0 \arrow[d] \\
 {} \arrow[r] & H_{2i}Z(\otimes\mathbf{Q}) \arrow[r] & H_{2i}X(\otimes\mathbf{Q}) \arrow[r] & H_{2i}\mathcal{U}(\otimes\mathbf{Q}) \arrow[r] & 0,
\end{tikzcd}$$

hence the lemma follows.

Now we can state our main theorem in this section.

Theorem. Let $\mathcal{U}//H$ be an arbitrary categorical quotient. Then,

(a) the rational cycle map of $\mathcal{U}//H$ is an isomorphism if the rational cycle map of X^H has the same property.

(b) the cycle map of $\mathcal{U}//H$ is an isomorphism if the cycle map of X^H has the same property and the action is quasi-free.

Proof. We need to show that the cycle map

$$cl_{\mathcal{U}//H} \otimes \mathbf{Q} : A_i(\mathcal{U}//H) \otimes \mathbf{Q} \longrightarrow H_{2i}(\mathcal{U}//H) \otimes \mathbf{Q}$$

is isomorphism for all i, or

$$cl_{\mathcal{U}//H} : A_i(\mathcal{U}//H) \longrightarrow H_{2i}(\mathcal{U}//H)$$

is isomorphism for all i if the action is quasi-free.

First all of , let $\mathcal{U} = \mathcal{U}_p$ where p is a general point close enough to a relatively general point r in a *codim* 1 face M of $\mu(X)$. Then we have that

$$\mathcal{U}_p/H \longrightarrow \mathcal{U}_r(M)/H$$

is a weighted projective (resp. projective, if the action is quasi-free) bundle over $\mathcal{U}_r(M)/H$. Using induction (the trivial case is X^H), we can assume that $\mathcal{U}_r(M)/H$

has the desired property. Hence $\mathcal{U}_p/H$ has also the desired property (see [Fu], 19.1).

Now let M be an interior wall, and r, p be as before, then we have a fiber square

$$\begin{tikzcd}
A \arrow[r] \arrow[d] & \mathcal{U}_p/H = X \arrow[d] \\
B = \mathcal{U}_r(M)/H \arrow[r] & \mathcal{U}_r//H = Y
\end{tikzcd}$$

where A is a weighted projective (resp. projective, if the action is quasi- free) bundle over B, and $X - A$ is isomorphic to $Y - B$.

Hence by the lemma and induction, X has the desired property if and only if Y has. Since any two quotient varieties are connected by a sequence of fiber squares like above, so the theorem follows our assertion in the beginning.

As a corollary of proposition 6.3 and theorem 6.4

Corollary. Let $\mathcal{U}//H$ be an arbitrary nonsingular algebraic quotient. Then $\mathcal{U}//H$ has property (RS) if X^H has (RS).

Chapter 7

The Case of Flag Varieties

Historically, we first worked out the results in this chapter.

7.1 Weighted Projective Spaces

Definition. Let $Q = \{q_0, \cdots, q_d\}$ be a finite collection of positive integers, $S(Q)$ the polynomial algebra $k[T_0, \cdots, T_d]$ over the complex number $\mathbf{C}$, graded by the condition

$$deg(T_i) = q_i, i = 0, \cdots, d,$$

then the space $\mathsf{P}(Q) = Proj(S(Q))$ is called the **weighted projective space** with weight $Q = \{q_0, \cdots, q_d\}$.

Alternatively, $\mathsf{P}(Q)$ can be defined in a geometric way. Let $\mathbf{C}^*$ acts on $\mathbf{C}^{d+1}$ by

$$\lambda \cdot (z_0, \cdots, z_d) = (\lambda^{q_0} z_0, \cdots, \lambda^{q_r} z_d),$$

then $\mathsf{P}(Q)$ is just the orbit space of $\mathbf{C}^{d+1} - 0$ under this $\mathbf{C}^*$ action. Clearly for any positive integer a,

$$\mathsf{P}(aq_0, \cdots, aq_d) \cong \mathsf{P}(q_0, \cdots, q_d)$$

and $\mathsf{P}(1, \cdots, 1)$ is just the ordinary projective space P^d.

We remark that a weighted projective space is a rational manifold with (possibly) only finite cyclic quotient singularities. It is also a compact toric variety whose associated cone decomposition is combinatorially isomorphic to the cone decomposition associated to the ordinary projective space of the same dimension, in particular, the rational (co)homology groups of a weighted projective space of dimension d are isomorphic to the (co)homology groups of P^d

7.2 Statements of Some Results

Let G be a reductive algebraic group over complex number, H a Cartan subgroup. Let also Φ be a root system of G with respect to H, $\Pi = \{\alpha_1, \cdots, \alpha_n\}$ a simple root system, and Φ^+ the set of positive roots. We use B to denote the Borel subgroup containing H with respect to Φ^+ . In this section we shall deal with the left action of H on the flag manifold G/B.

We will prove in a later section that every wall M (including face) of $\mu(X)$ is determined by a unique standard parabolic subgroup W_J of the Weyl group W of H, where J is a subset of $\{1, \cdots, n\}$, and W_J is generated by simple reflections s_{α_i}, $i \in J$. And the stratum closure $\overline{X^M}$ is H-equivariantly isomorphic to $P_J \cdot [B] \cong P_J/B$, where P_J is the standard subgroup of G corresponding to W_J, $[B]$ is the base point of G/B representing the B orbit through the identity element. In above case, we shall call M a wall of **type** J.

We use Φ_J to denote the roots in Φ which can be written as linear combinations of simple roots α_i, $i \in J$.

Theorem 1. Let M be a codim 1 face of $\mu(G/B)$ of type J. We set $J^c = \{1, \cdots, n\} - J = r$, and let $\beta_0, \cdots, \beta_\nu$ be all the positive roots whose α_r coefficients in their linear combinations of simple roots, $n_{\beta_0}, \cdots, n_{\beta_\nu}$, are nonzero. Let r be a relatively general point on M, and $p \in \mu(G/B)$ is general and close enough to r, then we have (a)

$$\varphi : \mathcal{U}_p/H \longrightarrow \mathcal{U}_r(M)/H (= \mathcal{U}_r//H)$$

is a fiber bundle whose fiber is the weighted projective space $\mathbf{P}(n_{\beta_0}, \cdots, n_{\beta_\nu})$ of dimention ν, and $\nu = dim\ X - dim\ X^M - 1$.

(b). Let $\mathcal{L}_w$ be the weight lattice of G, H_J be the hyperplane generated by $\{\alpha_i | i \in J\}$, then φ is an ordinary projective bundle if and only if the lattice $\mathcal{L}_w \cap H_J$ is generated by $\{\alpha_i | i \in J\}$.

Remark. Moreover, we have the following precise results listed according to the type of the group G (see the next page for the Dynkin diagrams).

A_n. φ is an ordinary projective $\mathbf{P}^\nu$-bundle for any codim 1 face of any type.

B_n. φ is an ordinary projective bundle if and only if the face M is of type J, where $J^c = \{1\}$.

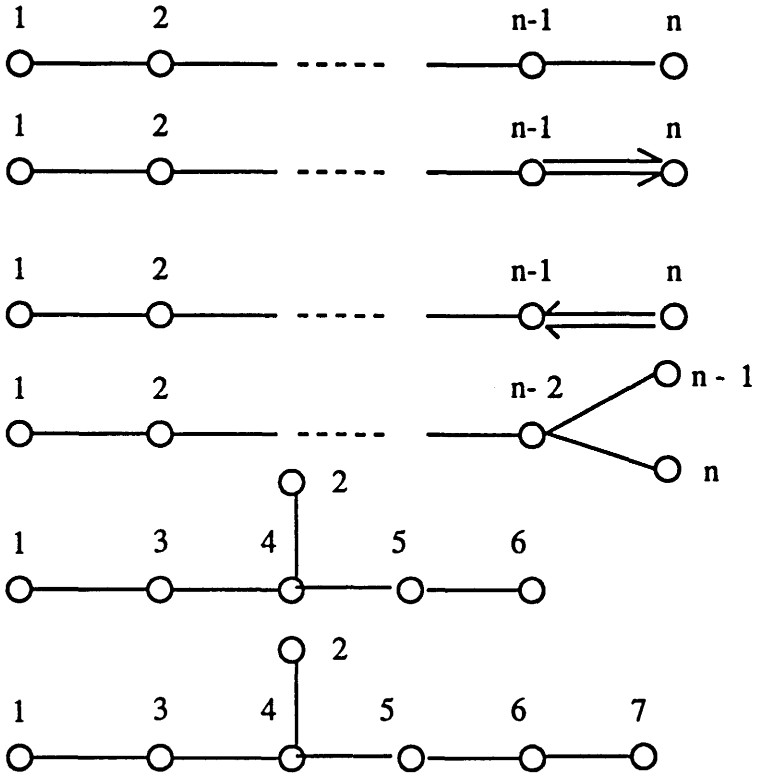
C_n. φ is an ordinary projective bundle if and only if the face is of the type J, where $J^c = \{n\}$.

D_n. φ is an ordinary projective bundle if and only if the face is of the type J, where $J^C = \{1\}$, or $\{n-1\}$, or $\{n\}$.

E_6. φ is an ordinary projective bundle if and only if the face M is of type J, where $J^c = \{1\}$, or $\{6\}$.

E_7. φ is an ordinary projective bundle if and only if the face M is of type J, where $J^c = \{7\}$.

For E_8, F_4, and G_2 , φ is not an ordinary projective bundle for any codim 1 face.



Let M be an interior wall of $\mu(G/B)$, then we shall see later that M defines two moment map images of torus orbit closures, say $M^>$ and $M^<$, which satisfies the conditions in proposition 1.4, that is, $M^> \cap M^< = M$, and if r is a relatively general point on M, and $p \in M^<, q \in M^>$ are two general points and close enough to r, then any $\mu - image$ of torus orbit closure with a face on M and containing r is either contained in $M^<$ or in $M^>$, it all depends on if the $\mu - image$ contains p or q.

Theorem 2. We have the following commutative diagram

$$\begin{tikzcd}
\mathcal{U}_p(M^<)/H \arrow[r, "f_1"] \arrow[d] & \mathcal{U}_0(M)/H \arrow[d] & \mathcal{U}_q(M^>)/H \arrow[l, "g_1"'] \arrow[d] \\
\mathcal{U}_p/H \arrow[r, "f"] & \mathcal{U}_0//H & \mathcal{U}_q/H \arrow[l, "g"']
\end{tikzcd}$$

where f_1 and g_1 are restrictions of f and g, respectively. Moreover, f_1 is a fiber bundle over $\mathcal{U}_0(M)/H$ whose fiber is a weighted projective space of dimension d, and g_1 is a fiber bunndle over $\mathcal{U}_0(M)/H$ whose fiber is a weighted projective space of dimension e, where $d = dimX^{M^<} - dimX^M - 1$, $e = dimX^{M^>} - dimX^M - 1$, and $d + e = dim\ X - dimX^M - 2$. Furthermore, the weights of the weighted projective spaces above are induced from the coefficients in the linear combinations of simple roots for positive roots. In particular, if the group G is of the type A_n, then all the weighted projective spaces above coincide with some ordinary projective spaces.

Remark. We shall see that both $\overline{X^{M^<}}$ and $\overline{X^{M^>}}$ are "nice" Schubert varieties.

Let G be $SL(n+1, \mathbf{C})$, then the flag manifold G/B can be identified with the space of flags of vector subspaces,

$$V^1 \subset \cdots \subset V^n \subset \mathbf{C}^{n+1},$$

in $\mathbf{C}^{n+1}$ or the space of flags of projective linear subspaces,

$$P^0 \subset P^1 \subset \cdots \subset P^{n-1} \subset \mathbf{P}^n,$$

in $\mathbf{P}^n$. We will use the two interpretations of $SL(n+1)/B$ alternatively, whichever is convenient. Chosen a coordinate system $\{e_1 \cdots e_{n+1}\}$ in $\mathbf{C}^{n+1}$ (or in $\mathbf{P}^n$). We say a subspace V^n is in general position if V^n does not contain any e_i $(i = 1, \cdots, n+1)$, a subspace V^1 is general if V^1 is not contained in any $n-dim$ coordinate subspace. It is known that all the general $n - dimensional$ (respectively, $1 - dimensional$) subspaces make of a single torus orbit.

Theorem 3. Let

$$\mathcal{U}(1) = \{flags\ in\ \mathbf{C}^{n+1} |\ V^1\ is\ general\},$$

$$\mathcal{U}(n) = \{flags\ \in\ \mathbf{C}^{n+1} |\ V^n\ is\ general\},$$

then both $\mathcal{U}(1)$ and $\mathcal{U}(n)$ are geometric open subsets, and their quotient spaces can be identified with the flag variety of flags in $\mathbf{C}^n$, $SL(n)/B$.

Proof. The fact that $\mathcal{U}(1)$ and $\mathcal{U}(n)$ are geometric is an immediate consequence of the theorem 4.3 by considering the following projections

$$G/B \longrightarrow \mathbf{P}^n = \{V^1 \subset \mathbf{C}^{n+1}\}$$

and

$$G/B \longrightarrow \mathbf{P}^n = \{V^n \subset \mathbf{C}^{n+1}\}.$$

However, we can also show directly that they are geometric quotients after we describe the moment map imges of $SL(n+1,\mathbf{C})/B$ (see section 3.4). To prove that $\mathcal{U}(n)/H$ is isomorphic to $SL(n,\mathbf{C})/B$, we fix a general n-space V_0^n. Consider the projection

$$\begin{tikzcd} SL(n+1)/B = \{V^1 \subset \cdots \subset V^n \subset \mathbf{C}^{n+1}\} \arrow[d, "f"] \\ \mathbf{P}^n = \{V^n \subset \mathbf{C}^{n+1}\} \end{tikzcd}$$

then the following map defined for flags $(V^1 \subset \cdots \subset V^n \subset \mathbf{C}^{n+1})$ where V^n are general:

$$\begin{tikzcd} H \cdot \{V^1 \subset \cdots \subset V^n \subset \mathbf{C}^{n+1}\} \arrow[d] \\ H \cdot \{V^1 \subset \cdots \subset V^n \subset \mathbf{C}^{n+1}\} \bigcap f^{-1}(V_0^n) \end{tikzcd}$$

identifies $\mathcal{U}(n)/H$ with the space of complete flags in V_0^n since $H \cdot (V_0^n) =$ the set of all general n-spaces and the finite isotropy subgroups of H are identity subgroup. Similarly, $\mathcal{U}(1)/H$ can be identified with the space of flags in $\mathbf{C}^{n+1}$ that their first subspace are a fixed general 1-space V_0^1, or the space of flags in $\mathbf{C}^{n+1}/V_0^1$.

7.3 Moment Map Images of G/B

$X = G/B$ thorough out this section till section 16 (although many results hold for other homogeneous spaces G/P with appropriate modifications. We shall indicate this whenever the situation applies).

Let $N(H)$ be the normalizer of H in G, then $W = N(H)/H$ is by definition the Weyl group of G with respect to H. Let Φ be the root system associated to H and $\pi = \{\alpha_1 \cdots \alpha_n\}$ be a fundamental system in Φ (it amounts to choosing a fundmental Weyl Chamber C^+). Using the Killing form of *Lie G* , we identify Lie H with its dual space $(Lie\ H)^*$.

It is known that the fixed point set of H, X^H can be identified with W. Suppose K is the maximal compact subgroup of G, then we know that the *K − invariant* Kähler metrics on G/B are in one-to-one correspondence with the elements in interior of $C^+, Int(C^+)$.

Proposition. Fix a point $\tau \in Int(C^+)$, hence a Kähler metric on G/B, let μ be a moment map associated to H action under this metric, then after a translation if necessary, $\mu(X) =$ *convex hall of* $\{w \cdot \tau | w \in W\}$.

7.4 Parabolic Subgroups of W

Definition. Let $J \subset \{1 \cdots n\}$, we shall call the subgroup of W, W_J, generated by the fundamental reflections s_{α_r} with $r \in J$, a **standard parabolic subgroup** of W.

The subgroups W_J and their conjugates in W are all called **parabolic subgroups** of W.

We shall quote the following propositions from [Carter], which are useful for us later.

Proposition. Given $J \subset \{1 \cdots n\}$. Let D_J be the set of elements $w \in W$ such that $w \cdot s_{\alpha_r} \in \Phi^+$ for all $r \in J$, then

1) $W = \amalg_{d \in D_J} dW_J$.

2) $W = \amalg_{c \in D_J^{-1}} W_J c$.

3) d (or c) is the smallest element in dW_J (or $W_J c$) under the usual Bruhat order on W.

The Weyl Chambers give rise to a rational cone decomposition of $\mathbf{R}^n$. Given a subset $J \subset \{1 \cdots n\}$,

$$C_J = \{v; (v, \alpha_r) = 0 \; for \; r \in J; (v, \alpha_r) > 0 \; for \; r \in \{1 \cdots n\} - J = J^C\}$$

is a codim $|J|$ face of the fundamental Weyl Chamber, and all the faces the fundamental Weyl Chamber are of this form.

Proposition. The stabilizer of C_J in W is W_J.

7.5 Parallel Walls and Faces of $\mu(G/B)$

Given $J \subset \{1 \cdots n\}$, there is (induced) Bruhat order on the cosets $\{dW_J \mid d \in D_J\}$ (or $\{W_J c \mid c \in D_J^{-1}\}$) where $d_1 W_J \leq d_2 W_J$ ($W_J c_1 \leq W_J c_2$) if and only if $d_1 \leq d_2$ ($c_1 \leq c_2$). And there are always a unique minimal element $eW_J = W_J$ $(or, W_J e = W_J)$ and a unique maximal element $d_0 W_J$ $(or, W_J c_0)$.

Before we state our theorem we need a lemma which is an easy consequence of [C].

Lemma. The Lie algebras of isotropy subgroups of H are precisely the subspaces generated by the faces of Weyl Chambers.

Theorem. Given $J \subset \{1 \cdots n\}$ and W_J a parabolic subgroup of W, then we have

(1) For each $c \in D_J^{-1}$, the convex hall M_c of $\{w \cdot c \cdot \tau | w \in W_J\}$ is the moment map image of a torus orbit closure of dim $|J|$. And for any two $c_1, c_2 \in D_J^{-1}$, M_{c_1} and M_{c_2} are parallel. Moreover, such convex halls give rise to all walls.

(2) For each $d \in D_J$, the convex hall F_d of $\{d \cdot w \cdot \tau | w \in W_J\}$ is the moment map image of a torus orbit closure of dim $|J|$, it is actually a face of $\mu(X)$. Such convex halls give rise to all faces of $\mu(X)$.

Roughly, the theorem says that the right cosets of W_J give rise to parallel walls of the same type, while the left cosets give rise to faces of the same type.

Proof (1) We denote G/B by X. Then the fixed points set of H, X^H, is equal to W. Let P_J be the parabolic subgroup of G associated to W_J, then for any $c \in D_J^{-1}$,

$$(P_J c \cdot [B]) \cap X^H = W_J c \cdot [B]$$

where $[B]$ is the base point of $X = G/B$, since $P_J \cdot [B]$ is a $H-invariant$ closed subvariety, and $\mu(P_J c \cdot [B]) = convex\ hall\ of\ W_J c \cdot \tau$, hence M_c is the moment map image of a torus orbit closure (in $P_J \cdot c[B]$). We remark that $P_J \cdot c[B]$ is actually $\overline{X^{M_c}}$ (which will be a consequence of some results later). To prove that M_{c_1} and M_{c_2} are parallel for any $c_1, c_2 \in D_J^{-1}$, it suffices by the consideration of dimensions to show that M_c (of dim $|J|$) is perpendicular to the face of a Weyl Chamber, C_J (of codim $|J|$).

The linear subspace V parallel to M_c is given by

$$V = span\{wc \cdot \tau - c \cdot \tau | w \in W_J\},$$

but for any $v \in C_J, w \in W_J$,

$$(wc \cdot \tau - c \cdot \tau, v) = (wc \cdot \tau, v) - (c \cdot \tau, v) = (c \cdot \tau, w^{-1} \cdot v) - (c \cdot \tau, v) = 0$$

Since W_J is the stabilizer of C_J. Hence we proved that M_c is perpendicular to C_J.

The last statement should be clear by the lemma above and a basic property of moment map.

(2) Similarly, we notice that for any $d \in D_J$,

$$(dP_J \cdot [B]) \cap X^H = dW_J \cdot [B]$$

hence, the convex hall of $dW_J \cdot \tau = \mu(dP_J \cdot [B])$ is the moment map image of a generic torus orbit closure in $dP_J \cdot [B]$. To show that it is a face of $\mu(X)$, it suffices to consider the convex hall of $W_J \cdot \tau$ since for any $w \in W$, the map $a \mapsto w \cdot a$,(for any $a \in\ Lie\ H$),gives an isometry form Lie H to itself. Hence if the convex hall of $W_J \cdot \tau$ ia a face of $\mu(X)$, so is that of $dW_J \cdot \tau$. It is also enough to consider the cases of maximal parabolic subgroups because for any J, there are maximal

$J_1 \cdots J_r$ such that $J = J_1 \cap \cdots \cap J_r$ and

$$(dW_{J_1}) \cap \cdots \cap (dW_{J_r}) = d(W_{J_1} \cap \cdots \cap W_{J_r}) = dW_{J_1 \cap \cdots \cap J_r} = dW_J,$$

and because an intersection of faces of a polytope is still a face. To complete the proof we claim here that it will be an immediate consequence of the assertion below.

Claim. Let $c_1, c_2, c_3 \in D_J^{-1}$ where J is a subset of $\{1 \cdots n\}$ with $n-1$ elements. If $c_1 \prec c_2 \prec c_3$ under the Bruhat order, then the wall M_{c_2} is in between the wall M_{c_1} and M_{c_3}.

There are many ways to prove this. One simple proof will appear in section 7.7.

I suspect the claim holds under a even weaker assumption on c_1, c_2, c_3 (but I could not prove it), it is stated as follows.

Define the rank function r on W with the Bruhat partial order by setting $r(w) = l(w)$, the length of w, then r induces a rank function (also denoted by r) on the cosets $\{W_J c | c \in D_J^{-1}\}$ (or $\{dW_J | d \in D_J\}$) with the property that $r(W_J c) = r(c)$ $(r(dW_J) = r(d))$.

Conjecture. Let $c_1, c_2, c_3 \in D_J^{-1}$, where J is maximal, If $r(c_1) < r(c_2) < r(c_3)$, then the wall M_{c_2} is in between the wall M_{c_1} and M_{c_3}.

7.6 More Properties of Parallel Walls and Faces of $\mu(X)$

We observe by theorem 7.5 that

Corollary.. Let $J \subset \{1 \cdots n\}$, and $f : G/B \to G/P_J$ be the natural projection. Then for any fixed point $\overline{d}$ on G/P_J $(d \in D_J)$, $f^{-1}(\overline{d}) = dP_J \cdot [B]$ is the stratum closure whose moment map image is the convex hall of $dW_J \cdot \tau$. And all faces of $\mu(G/B)$ can be described in this way.

Sometimes we will say that the convex hall of $W_J c \cdot \tau$, a wall of type J, and the convex hall $dW_J \cdot \tau$, a face of $\mu(X)$ of type J.

Given $J \subset \{1 \cdots n\}$, let $W_J c_0$ be the maximal element of $\{W_J c | c \in D_J^{-1}\}$, then $W_J \cdot c_0$ is a face of $\mu(X)$, hence there is $K \subset \{1 \cdots n\}$ with $|K| = |J|$ and a $d \in D_K$ such that $W_J c_0 = dW_K$. Because both c_0 and d are the smallest element in $W_J c_0$ and dW_K, respectively, hence $d = c_0$ by uniqueness. $W_K = c_0^{-1} W_J c_0$. On the other hand, if $c^{-1} W_J c = W_K$, then $W_J c = cW_K$ gives a face, hence $c = c_0$. This shows,

Corollary. Among the conjugates of W_J, only W_J and $c_0^{-1}W_Jc_0$ are **standard** parabolic subgroups.

Finally, we remark some intersection properties of walls to close this section.

Proposition. Given $J, K \subset \{1 \cdots n\}$, and $d_i \in D_J, d' \in D_K$, $c \in D_J^{-1}$, $c' \in D_K^{-1}$, then

(1) $dW_J \cap d'W_K \neq \emptyset$ if and only if $d = d' \in D_J \cap D_K$. In this case

$$dW_J \cap d'W_K = d(W_J \cap W_K) = dW_{J\cap K}.$$

(2) $W_Jc \cap W_Kc' \neq \emptyset$ if and only if $c = c' \in D_J^{-1} \cap D_K^{-1}$. In this case

$$W_Jc \cap W_Kc' = (W_J \cap W_K)c = W_{J\cap K}c.$$

(3) $dW_J \cap W_Kc \neq \emptyset$ if and only if $d = c \in D_J \cap D_K^{-1}$. In this case

$$dW_J \cap W_Kc = (cW_Jc^{-1} \cap W_K)c.$$

Proof. By the uniqueness of the smallest element in each coset.

7.7 Half Regions and Their Torus Strata

Let $W_J, j \subset \{1 \cdots n\}$, be a parabolic subgroup of W. The induced Bruhat order on the posets $\{W_Jc | c \in D_J^{-1}\}$ is actually the same as the Bruhat order on the Schubert varieties of G/P_J.

For any $c_1, c_2 \in D_J^{-1}$, we will say that the wall M_{c_1} defined by W_Jc_1 is less than the wall M_{c_2} defined by W_Jc_2 if $c_1 < c_2$. Clearly, there is no element in W_Jc_1 is "greater" than some element in W_Jc_2 and the maximal element of W_Jc_2 is "greater" than any element in W_Jc_1 if $M_{c_1} < M_{c_2}$.

Definition. A wall M defines two regions $M^>$ and $M^<$ in $\mathbf{R}^n$ as follows,

$$M^> = Convex\ hall\ of\ \{wall\ M' \mid M' \geq M\}$$

$$= Convex\ hall\ of\ vertices\ of\ walls\ M'\ with\ M' \geq M,$$

$$M^< = Convex\ hall\ of\ \{wall\ M' \mid M' \leq M\}$$

$$= Convex\ hall\ of\ vertices\ of\ walls\ M'\ with\ M' \leq M.$$

We shall call them the **half regions** defined by M.

Proposition. $M^<$ and $M^>$ are the moment map images of some torus orbit closures. Furthermore, if $M =$ *convex hall of* W_Jc, let u, v be the maximal element and the smallest element in W_Jc, respectively, then

$$\overline{X^{M^<}} = \overline{S_u} = \overline{BuB/B},$$

$$\overline{X^{M^>}} = \overline{S_v^*} = \overline{B^*vB/B},$$

where B^* is the oposite Borel subgroup of B.

Proof. Since $\overline{S_u} \cap X^H = \{w \in W | w \leq u\}$, hence $\mu(\overline{S_u}) \subset M^<$. On the other hand, if $x \in \overline{X^{M^<}}$, but x is not in $\overline{S_u}$, then there exist $w \in W$ with w is not less than u such that $x \in S_w$, a direct computation shows that $w \in \overline{H \cdot x}$, this is impossible since $\overline{H \cdot x} \subset \overline{X^{M^<}}$. Therefore $\overline{X^{M^<}} \subset \overline{S_u}$. Hence, $\overline{S_u} = \overline{X^{M^<}}$.
Similarly, $\overline{X^{M^>}} = \overline{S_v^*}$.

We remark, as a consequence of the proof above, we have

Corollary. Every Schubert variety is a union of torus strata.

7.8 Intersections of Half Regions

Let M, N be two parallel walls with $N < M$, define

$$C_N^M = \textit{convex hall of walls } Q \textit{ with } N \leq Q \leq M.$$

Then clearly, $C_N^M = M^< \cap N^>$.
As a consequence of the proof proposition 7.7, we have

Corollary. C_N^M is the moment map image of a torus orbit closure. And if $M = W_Jc_1, N = W_Jc_2$, and u is the largest element of W_Jc_1, v is the smallest element of W_Jc_2, then

$$\overline{X^{M^> \cap N^<}} = \overline{S_u} \cap \overline{S_v^*} = \overline{X^{M^<}} \cap \overline{X^{N^>}}.$$

Remark. In the case of $G = SL(n+1, \mathbf{C})$, we shall describe $M^<, M^>, C_N^M$ in terms of Schubert conditions on flags.

Proof of the claim in 7.5. Let $c_1 \prec c_2 \prec c_3$ be as in theorem 5.2, by corollary 3, $M_{c_2} \subset M_{c_3}^{\leq} \cap M_{c_1}^{\geq}$, this shows that M_{c_2} must be in between M_{c_1} and M_{c_3} since $M^{<} \cap M^{>} = M$ for any wall M.

Lemma. The moment map image of any orbit closure is contained in a wall of the same dimension.

Proof. This is a consequence of the classification of the isotropy subgroups of H (lemma 5.1) and a basic fact of a moment map.

Proposition. The moment map image of every orbit closure is an intersection of half regions.

Proof. Let C be the moment map image of a torus orbit closure. Without loss of generalities, we assume that C is of top dim. Let $\sigma_1 \cdots \sigma_l$ be the exactly the codim 1 faces of C that are not on original faces of $\mu(X)$, this is, they are contained in interior walls $M_1, \cdots, M_l$, respectively. Now each M_i defines two half regions, and by their properties, exactly one of them contains C, say E_i, then it is an easy fact of polytopes that

$$C = \bigcap\nolimits_{i=1,\cdots,l} E_i$$

Remark. Follow the notation above, it is clear

$$\overline{X^C} \subset \bigcap\nolimits_{i=1,\cdots,l} \overline{X^{E_i}}$$

We have known that for every half region, its stratum closure is union of strata, hence an intersection of strata closure defined by half regions is also a union of some strata. But it is possible that $\overline{X^C}$ is not a union of strata, so $\overline{X^C}$ could be a proper subset of $\bigcap_{i=1,\cdots,l} \overline{X^{E_i}}$. If this happens, $\bigcap_{i=1,\cdots,l} \overline{X^{E_i}}$ should not be irreducible since both $\overline{X^C}$ and $\bigcap_{i=1,\cdots,l} \overline{X^{E_i}}$ contain an open subset X^C.

Definition. A Schubert variety is call of first class if its moment map image is a half region defined before.

As a consequence of the corollary 7.7 and the proposition above, we have

Corollary. Every Schubert variety is an intersection of some Schubert varieties of first class.

7.9 Regions Defined by Faces of $\mu(X)$

Let W_J ($J \subset \{1, \cdots, n\}$) be a parabolic subgroup of W. Then the convex hall of dW_J ($d \in D_J$) gives a face of $\mu(X)$ of type J. Then the Bruhat order on $\{dW_J | d \in W_J\}$ induces a poset structure on the faces of $\mu(X)$ of type J.

As in section 6, we define two regions associated to a face F of type J.

Definition . Let F be a face of $\mu(X)$ of type J, we define

$$F^- = convex\ hall\ of\ faces\ F'\ of\ type\ J\ with\ F' \leq F$$

$$F^+ = convex\ hall\ of\ faces\ F'\ of\ type\ J\ with\ F' \geq F$$

Clearly, $F^- \cap F^+ = F$ by the definition.

Proposition. Suppose F is the convex hall of $dW_J(d \in D_J)$. Let $f : G/B \to G/P_J$ be the natural projection. f maps $dW_J(d \in D_J)$ to a fixed point $\bar{d}$ of H on G/P_J. Let u, v be the largest and smallest element of dW_J, respectively, (in fact $v = d$), then we have

(1) F^- and F^+ are the moment map images of some torus orbit closures, respectively.

(2)

$$\overline{X^{F^-}} = f^{-1}(\overline{B\bar{d}P_J/P_J}) = \overline{BuB/B}$$

where $B\bar{d}P_J/P_J$ is a Schubert cell on G/P_J indexed by $\bar{d}(= f(dW_J))$. and

$$\overline{X^{F^+}} = f^{-1}(\overline{B^*\bar{d}P_J/P_J}) = \overline{B^*vB/B}$$

where B^* is the opposite Borel subgroup of B.

Proof. The proposition follows immediately by the proof in 7.7 if

$$f^{-1}(\overline{B\bar{d}P_J/P_J}) = \overline{BuB/B}\ and$$

$$f^{-1}(B^*\bar{d}P_J/P_J) = \overline{B^*vB/B}$$

are proved. But it is straightforward to check that

$$f^{-1}(\overline{B\bar{d}P_J/P_J}) \cap X^H = \overline{BuB/B} \cap X^H,$$

and

$$f^{-1}(B^*\bar{d}P_J/P_J) \cap X^H = \overline{B^*vB/B} \cap X^H.$$

Since they are all Schubert varieties on G/B, so the two equalities hold.

Let $d \in D_J$, $\bar{d}$ be the corresponding image of f on G/B, we use $S_{\bar{d}}$ to denote the Schubert cell $B\bar{d}P_J/P_J$, and $S_{\bar{d}}^*$ to denote the Schubert cell $B^*\bar{d}P_J/P_J$. Now as a consequence of the above, we have

Corollary. Let $d_1, d_2 \in D_J$, and $d_1 < d_2$, let $F_1(F_2)$ be the convex hall of $d_1 W_J \cdot x(d_2 W_J \cdot x)$, then the convex hall of faces F' with $F_1 \leq F' \leq F_2$, say C, is the moment map image of some torus orbit closure. In fact,

$$C = F_1^+ \cap F_2^-$$

$$\overline{X^C} = f^{-1}(\overline{S_{\bar{d}_1}^*} \cap \overline{S_{\bar{d}_2}}).$$

Remark. Although the use of regions defined by faces in this section is not so clear as the regions defined by parallel walls, I suspect the two are equally useful, in other words, we may substitute the half regions of 7.7 by the regions defined in this section so that the result in section 7.8 still hold.

7.10 The Star Constructions and Their Applications

Let M be an interior wall of codim 1, r be a relatively general point on M. Let also p be a general point in $M^<$, and q be a general point in $M^>$, then theorem 7.2.2 states that the following diagram

$$\begin{tikzcd}
\mathcal{U}(M^<)/H \arrow[r, "f_1"] \arrow[d] & \mathcal{U}_0(M)/H \arrow[d] & \mathcal{U}_q(M^>)/H \arrow[l, "g_1"'] \arrow[d] \\
\mathcal{U}_p/H \arrow[r, "f"] & \mathcal{U}_0//H & \mathcal{U}_q/H \arrow[l, "g"']
\end{tikzcd}$$

commutes, where the vertical maps are closed embeddings, and f_1 is a fiber bundle whose fiber is a weighted projective space of *dim* d. g_1 is a fiber bundle whose fiber is a weighted projective space of *dim* e. $d + e$ does not depend on parallel walls. Moreover (and clearly), f is identity off $\mathcal{U}_p(M^<)/H$ and g is identity off $\mathcal{U}_q(M^>)/H$.

Now we shall develop some notations and observe some fact in order to prove the theorem.

Definition . Let a be a vertex of $\mu(X)$, define

$$star(a) = \bigcup\{X^C | a \in C\}.$$

Lemma. (1) $Star(a)$ is a biggest Schubert cell for any a.

(2) If $a \in C$, the moment map image of torus orbit closure, then $star(a) \cap \overline{X^C}$ is contractible.

(3) Let $\overline{S_w}$ be a Schubert variety, $C = \mu(\overline{S_w})$, and $a = \mu(w)$. then

$$star(a) \cap \overline{X^C} = star(a) \cap \overline{S_w} = S_w.$$

Proof. (1) Let $a = \mu(w), w \in W$. Choose Borel subgroup B' containing H such that $\overline{B'wB/B} = G/B$. Denote $B'wB/B = S'_w$, then we have to show that $star(a) = S'_w$.

Now for any $x \in S'_w$, since w is the only fixed point of the torus action in S'_w, an easy argument of Morse theory or a simple direct calculation shows that $\overline{H \cdot x} \ni w$, hence $a \in \mu(\overline{H \cdot x})$, that is, $x \in star(a)$, we got $S'_w \subset star(a)$.

Now if $y \in star(a)$, that is $a \in \mu(\overline{Hy})$ or $w \in \overline{Hy}$. Assume that $y \notin S'_w$, then $\exists u \in W, u <' w$, and $y \in S'_u = B'uB/B$, so $\overline{Hy} \cap W \subset \overline{S'_u} \cap W = \{v \leq' u | v \in W\}$, which contradicts with that $w \in \overline{Hy}$.

(2) This is a consequence of [B-B] or a simple application of Morse theory.

(3) The same argument as in (1) shows that

$$S_w \subset star(a) \cap \overline{S_w}.$$

For the other direction of inclusion, if $x \in star(a) \cap \overline{S_w}$, then $w \in \overline{H \cdot x}$, assuming $x \notin S_w$, since $x \in \overline{S_w}$, then as in (1), we have $\overline{H \cdot x} \subset \overline{S_u}$, for some $u < w$, contradiction.

Remark. The lemma above holds for every homogeneous space G/P without any modification.

Lemma. Let C be a half region defined by a wall M, and $M_1 \subset C$ is a wall which is parallel and closest to M. Then we have

(1) Let $p \in C$ and is in between M_1 and M, that is, $p \in M_1^> \cap M^<$ if $M_1 < M$, or $p \in M^> \cap M_1^<$ if $M < M_1$, then $\mathcal{U}_p(C)/H$ is a rationally nonsingular compact variety with only finite quotient singularities. Particularly, it is nonsingular in the case that G is of type A_n.

(2) Let r be a relatively general point on M_1, p is a general point in C which is in between M_1 and M and close enough to r, then the algebraic map

$$f : \mathcal{U}_p(C)/H \to \mathcal{U}_r(C)/H$$

described before is an isomorphism

(3) Let q be general point close enough to r as pictured above, then $\mathcal{U}_q(C)/H$

is also rationally nonsingular or nonsingular in the case that the torus action is quasi-free.

Proof. (1) By lemma 1, for each vertex a of C, $star(a) \cap \overline{X^C}$ is a Schubert cell with respect to a suitable Borel subgroup containing H, whose closure is just $\overline{X^C}$. In other words, $star(a) \cap \overline{X^C}$ is a smooth open cell of $\overline{X^C}$. Now by the position of p, it is clear that

$$\mathcal{U}_p(C) \subset \bigcup\{star(a) \cap \overline{X^C} | a \in \text{ vertex set of } C\},$$

th right side is a smooth open subset of $\overline{X^C}$ since it is a

union of smooth open subsets, now $\mathcal{U}_p(C)$ is an open subset of the right side, hence is also smooth, therefore we conclude that $\mathcal{U}_p(C)/H$ is nonsingular.

(2) This is because for each moment image D between M_1 and M, the algebraic map

$$\rho_{D,D\cap M} : X^D/H \to X^{D\cap M}/H$$

has to be isomorphism since it is birational and finite.

(3) The assertion is justified if we notice that $\mathcal{U}_q(C)/H$ is the blow up of the nonsingular variety $\mathcal{U}_r(C)//H$ along the nonsingular variety $\mathcal{U}_r(M)/H$.

7.11 A Direct Proof of Theorem 7.2.1

We follow the notation in sections 7.3,4,and 5. We can assume that the face M is the convex hall of $W_J \cdot \tau$ without the loss of the generality, where$J \subset \{1, \cdots n\}$, and $|J| = n-1$. As we have known

$$\overline{X^M} = P_J \cdot [B]$$

Now let $r \in M$ be a relatively general point, and $p \in \mu(X)$ be a general point close enough to r. Then we have clearly that

$$\mathcal{U}_p \subset \bigcup\{star(a) | a \in \text{vertex set of } M\}$$

$$\mathcal{U}_r(M) \subset \bigcup\{star(a) \cap \overline{X^M} | a \in \text{vertex set of } M\} (= \overline{X^M}).$$

Now we will work on open sunsets $star(a) \cap \mathcal{U}_p$ of $\mathcal{U}_p$ and $\mathcal{U}_r(M) \cap star(a)$ of $\mathcal{U}_r(M)$ for each vertex a individually. We shall consider the restriction,φ_a, of the projection

$$\varphi : \mathcal{U}_p/H \to \mathcal{U}_r(M)/H$$

to the open subset $star(a) \cap \mathcal{U}_p/H$ of $\mathcal{U}_p/H$, and prove that

$$\varphi_a : (star(a) \cap \mathcal{U}_p)/H \to (star(a) \cap \mathcal{U}_r(M))/H$$

is a weighted projective bundle of a fixed type for every vertex a of M. If this is proved, so is the first part of the theorem.

Now each Schubert cell $star(a)$ is a top Schubert cell with respect to some suitable Borel subgroup. For simplicity, we can assume that $star(a)$ is the top Schubert cell with respect to the action of the Borel subgroup B (that is, we take $a = x$). Then $star(a)$ is equivariantly isomorphic to the unipotent radical $\mathcal{U}$ of B equivariantly with respect to the action of Maximal torus.

Let $Lie\ G = Lie\ H + \sum_{\alpha \in \Phi} \mathbf{g}^\alpha$ where $\mathbf{g}^\alpha$ are eigenspaces.

Now

$$\mathcal{U} \cong Lie\,\mathcal{U} \cong \Pi_{\alpha \in \Phi^+} \mathbf{g}^\alpha = \Pi_{\alpha \in \Phi^+ \cap \Phi_J} \mathbf{g}^\alpha \times \Pi_{\beta \in \Phi^+ - \Phi_J} \mathbf{g}^\alpha$$

where Φ_J is the set of roots that can be written as linear combinations of $\alpha_i, i \in J$.

Let H_J be the subgroup of H whose Lie algebra is generated by $\alpha_i, i \in J$. Let H_1 be the subgroup of H whose Lie algebra is $\bigcap_{i \in J} \alpha_i^\perp$, then H_1 is the stabilizer of $\overline{X^M} = P_J \cdot [B]$ and

$$Lie\ H = Lie\ H_J \bigoplus Lie\ H_1$$

$$H = H_J \times H_1$$

Now for any element h of H, h can be written as

$$h = exp\theta \cdot exp\lambda = exp(\theta + \lambda)$$

where $\theta \in Lie\ H_J, \lambda \in Lie\ H_1$.

Let

$$u = \Pi_{\alpha \in \Phi^+ \cap \Phi_J} g_\alpha \times \Pi_{\beta \in \Phi^+ - \Phi_J} g_\beta$$

be an element in $\mathcal{U}$, where $g_\alpha \in \mathbf{g}^\alpha$, $g_\beta \in \mathbf{g}^\beta$, then $h \cdot u$

$$= \Pi_{\alpha \in \Phi^+ \cap \Phi_J} e^{(\theta+\lambda,\alpha)} g_\alpha \times \Pi_{\beta \in \Phi^+ - \Phi_J} e^{(\theta+\lambda,\beta)} g_\beta$$

$$= \Pi_{\alpha \in \Phi^+ \cap \Phi_J} e^{(\theta,\alpha)} g_\alpha \times \Pi_{\beta \in \Phi^+ - \Phi_J} e^{(\theta,\beta)} e^{(\lambda,\alpha_r) n_\beta} g_\beta$$

where n_β is the coefficients of $\alpha_r, r \in J^C$, in the expression of g_β.

Under the equivariant isomorphism from $star(a)$ to $\mathcal{U}$, the projection φ_a from $star(a) \cap \mathcal{U}_p / H$ to $star(a) \cap \mathcal{U}_r(M)/H$ is equivalent to the projection given by

$$\begin{tikzcd} \Pi_{\alpha \in \Phi^+ \cap \Phi_J} g_\alpha \times \Pi_{\beta \in \Phi^+ - \Phi_J} g_\beta \arrow[d] \\ \Pi_{\alpha \in \Phi^+ \cap \Phi_J} g_\alpha \end{tikzcd}$$

So given a $H (or\ H_J)$ orbit in $\mathcal{U}_r(M) \cap star(a)$,

$$H_J \cdot (\Pi_{\alpha \in \Phi^+ \cap \Phi_J} g_\alpha).$$

The fiber of the projection φ_a is the orbit space of $\Pi_{\beta\in\Phi^+-\Phi_J} g_\beta$ by the torus H_1 with the action given by

$$exp\lambda \cdot \Pi_{\beta\in\Phi^+-\Phi_J} g_\beta$$
$$= \Pi_\beta e^{(\lambda,\alpha_r)n_\beta}\, g_\beta,\ for\ any\ \lambda \in LieH_1.$$

So the fiber of φ_a is the weighted projective space $\mathsf{P}(n_{\beta_0},\cdots,n_{\beta_\nu})$. By the definition of $(n_{\beta_0},\cdots,n_{\beta_\nu})$, it is easy to see that the sequence of the integers does not depends on the choice of each vertex on M, it depends only on the type of the wall. Hence ,we proved that φ is a fiber bundle with the typical fiber $\mathsf{P}(n_{\beta_0},\cdots,n_{\beta_\nu})$.

To see the rest of the theorem is true, it suffices to look at the coefficients of the maximal long root for G of every type. The maximal long roots for all types of group G are listed below.
$A_n.\alpha_1 + \cdots + \alpha_n$
$B_n.\alpha_1 + 2\alpha_2 + \cdots + 2\alpha_n$
$C_n.2\alpha_1 + \cdots + 2\alpha_{n-1} + \alpha_n$
$D_n.\alpha_1 + 2\alpha_2 + \cdots + 2\alpha_{n-2} + \alpha_{n-1} + \alpha_n$
$E_6.\alpha_1 + 2\alpha_2 + \cdots + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6$
$E_7.2\alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4 + 3\alpha_5 + 2\alpha_6 + \alpha_7$
$E_8.2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 6\alpha_4 + 5\alpha_5 + 4\alpha_6 + 3\alpha_7 + 2\alpha_8$
$F_4.2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 2\alpha_4$
$G_2.3\alpha_1 + 2\alpha_2$

7.12 The Triviality of Some Canonical Bundles

As one can see from the proof in the previous section, we have:
(1) $\varphi_a : \mathcal{U}_p \cap star(a)/H \to \mathcal{U}_r(M) \cap star(a)/H$ is a trivial bundle.
(2) In particular, if both r and p are close enough to a vertex of M then $\varphi : \mathcal{U}_p/H \to \mathcal{U}_r/H$ is a trivial bundle.

(1) is clear from the proof above. (2) is because if p and r are close enough to a vertex a of M, then $\mathcal{U}_p \subset star(a)$, $\mathcal{U}_r(M) \subset star(a)$, so $\varphi = \varphi_a$.
In fact, this triviality is not coincident: the reason is:

Proposition. The normal bundle of P/B in G/B is trivial where B is a Borel subgroup of a reductive group G and P is a parabolic subgroup of G containing B.

Proof. The normal bundle of P/B in G/B is

$$P \times_B \mathfrak{g}/\mathfrak{p}$$

where **g** and **p** are the Lie algebras of G and P respectively. Note that P acts on **g**/**p**. Now take a basis of **g**/**p**, then this basis generates, by applying the action of P, a group of global sections which will trivialize the normal bundle.

The same statement is false for the tangent bundle. The tangent bundle of P/B in G/B is

$$P \times_B \mathbf{p}/\mathbf{b}$$

where **p** and **b** are the Lie algebras of P and B respectively. But P does not act on **p**/**b**.

Consequently, we have

Corollary. All the fibrations characterized in 7.2 are trivial.

Remark. It seems for me that the corollary is true for any smooth variety with the torus action. But I have no proof.

Now we are ready to prove the theorem 7.2.2. We would like to present two very different proofs . The first proof is totally analogous to the proof of the theorem 7.2.1, although it needs a little more effort. This proof is valid for G of every type. However, we shall also provide an alternative proof for the case that $G = SL(n+1, \mathbb{C})$ without using the group structure of G, and therefore it may fit some general context (if there is a similar situation there).

7.13 The First Proof of Theorem 7.2.2.

As assumed $r \in M$, $p \in M^{<}$, we only prove the statement concerning p because the other half is totally analogous, and the fact that $d + e = constant$ is just a simple calculation of dimensions of some Schubert varieties. In fact

$$d = dimX^{M^{<}} - dimH - (dimX^{M} - (dimH - 1))$$

$$e = dimX^{M^{>}} - dimH - (dimX^{M} - (dimH - 1))$$

so,

$$\begin{aligned} d + e &= dimX^{M^{<}} + dimX^{M^{>}} - 2dimX^{M} - 2 \\ &= dimG/B - dimX^{M} - 2 \end{aligned}$$

since

$$dimG/B = dimX^{M^{<}} + dimX^{M^{>}} - dimX^{M}$$

by the results in §6.

So, to prove the theorem, it suffices to show the following claim:

$$\mathcal{U}_p(M^<)/H \longrightarrow \mathcal{U}_r(M)/H$$

is a fiber bundle whose fiber is a weighted projective space.

We remark that $\mathcal{U}_p(M^<)/H$ is a quotient on $\overline{X^{M^<}}$ where $\overline{X^{M^<}}$ is a "nice" Schubert variety whose moment map image is just $M^<$, a half region.

As before

$$\mathcal{U}_p \subset \{star(a) \cap \overline{X^{M^<}} | a \in vertex\ set\ of\ M\}$$

$$\mathcal{U}_r(M) \subset \{star(a) \cap \overline{X^{M}} | a \in vertex\ set\ of\ M\}$$

since no vertex of M is more special than the other vertices of M, it is sufficient to show that

$$\psi_a : (star(a) \cap \overline{X^{M^<}} \cap \mathcal{U}_p)/H \rightarrow (star(a) \cap \mathcal{U}_r(M))/H$$

is a weighted projective space-bundle. But $star(a) \cap \overline{X^{M^<}}$ is a Schubert cell S_y for some $y \in W$, and it is well-known that S_y is $H-$quivariantly isomorphic to $\mathcal{U} \cap y\mathcal{U}^* y^{-1}$ where $\mathcal{U}, \mathcal{U}^*$ denote the unipotent radical of B and B^* (see [KL]).

Let

$$\Phi_y = \{\gamma \in \Phi^+ \mid y \cdot \gamma \in \Phi^-\},$$

then $\mathcal{U} \cap y\mathcal{U}^* y^{-1}$ is $H-$equivariantly isomorphic to

$$\Pi_{\gamma\in\Phi_y}\mathfrak{g}^\gamma = \Pi_{\alpha\in\Phi_y\cap\Phi_J}\mathfrak{g}^\alpha \times \Pi_{\beta\in\Phi_y-\Phi_J}\mathfrak{g}^\beta$$

In what follows, we just need to translate the corresponding part of the proof of theorem 7.2.1 word for word. So we omit them.

7.14 The Second Proof of Theorem 7.2.2 when $G = SL(n+1, C)$

Let the notations be as before. Now let $M < M_1 < \cdots < M_k$ be a maximal chain of parallel walls, that is, M_k is a face of $\mu(X)$, and $r(M_{i+1}) = r(M_i) - 1, i = 0, \cdots, k-1$. We agree that $M_o = M$.

Let

$$A_i = \mathcal{U}_p(M_i^<)/H, i = 0, \cdots, k$$

$$B_i = \mathcal{U}_r(M_i^<)//H, i = 0, \cdots, k.$$

Then we have the following diagram of varieties

$$\begin{tikzcd}
\mathcal{U}_p(M^{<})/H = A_0 \arrow[r, hook] \arrow[d] & A_1 \cdots \arrow[r, hook] \arrow[d] & \arrow[d] \cdots & A_k = \mathcal{U}_p/H \arrow[d] \\
\mathcal{U}_r(M)/H = B_0 \arrow[r, hook] & B_1 \cdots \arrow[r, hook] & & B_k = \mathcal{U}_p/H,
\end{tikzcd}$$

which commutes. Furthermore, each square

$$\begin{tikzcd}
A_i \arrow[r, hook] \arrow[d] & A_{i+1} \arrow[d] \\
B_i \arrow[r, hook] & B_{i+1}
\end{tikzcd}$$

is the diagram of a blow up in the sense that B_i is the center of the blow up and A_i is the exceptional divisor. Now we only need the first square for our purpose.

By lemma 2, we conclude

$$\begin{tikzcd}
A_0 \arrow[r, hook] \arrow[d] & A_1 \arrow[d] \\
B_0 \arrow[r, hook] & B_1
\end{tikzcd}$$

is a blow-up diagram of non-singular varieties. Now because $B_0 \hookrightarrow B_1$ is a regular embedding, by a fact of algebraic geometry, $A_0 \rightarrow B_0$ has to be a projective $\mathbf{P}^d$ – *bundle*. (see [ES-B].)

Remark. In the case that G is not of type A_n, the varieties in the last diagram may not be nonsingular varieties due to the existence of the nontrivial finite isotropy subgroups. What we can say is that the varieties have only finite quotient singularities. In this case (with possibly a non-quasi-free torus action), I do not know if we can deduce from algebraic geometric arguments that $A_0 \rightarrow B_0$ is fiber bundle whose fiber is a weighted projective space.

7.15 The Singular Loci of Singular Quotients

Corollary. Let M be an interior wall, and r be a point in the interior M. Then

(1) if M is next to the boundary of $\mu(X)$, and r is relatively general on M, then $\mathcal{U}_r//H$ is rationally nonsingular or nonsingular when $G = SL(n+1, \mathbf{C})$.

(2) Otherwise, $\mathcal{U}_r//H$ is seriously singular. In fact, the singular locus of $\mathcal{U}_r//H$ is just $\mathcal{U}_r(M)/H$ if r is a relatively general point on M.

From the proof in this chapter, we can see easily that most results in this chapter hold for any Schubert variety of first class (see section 7.8 for the definition).

7.16 Intersection Homology of Symplectic Quotients

Example 1. $X = Sp(\mathbb{C}^4)/B$.

$$P(\mathcal{U}_p/H) = (1+t^2+t^4)+(t^2)+(t^2)+(t^2) = 1+4t^2+t^4.$$

Similarly,

$$IP(\mathcal{U}_{r_1}/H) = P(\mathcal{U}_{r_1}/H) = 1+3t^2+t^4,$$
$$IP(\mathcal{U}_{r_2}/H) = P(\mathcal{U}_{r_2}/H) = 1+2t^2+t^4.$$

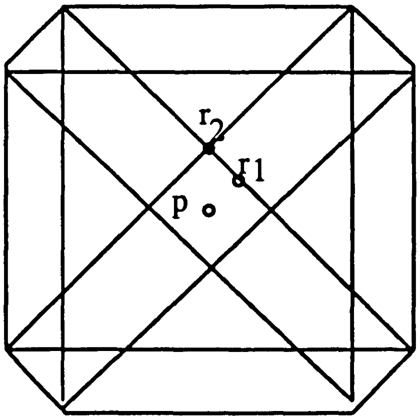


Figure 7.1: The moment map image of $Sp(\mathbb{C}^4)/B$.

Example 2. G_2. $X = G_2/B$.

$$P(\mathcal{U}_p/H) = (1+t^2+t^4+t^6+t^8)+(t^2+t^4+t^6)+(t^4)$$
$$+(t^2+t^4+t^6)+(t^2+t^4+t^6)+(t^4)+(t^4)+(t^4)+(t^4)$$
$$= 1+4t^2+9t^4+4t^6+t^8$$

Similarly,

$$IP(\mathcal{U}_{r_1}/H) = 1+4t^2+8t^4+4t^6+t^8,$$
$$IP(\mathcal{U}_{r_2}/H) = 1+4t^2+7t^4+4t^6+t^8.$$

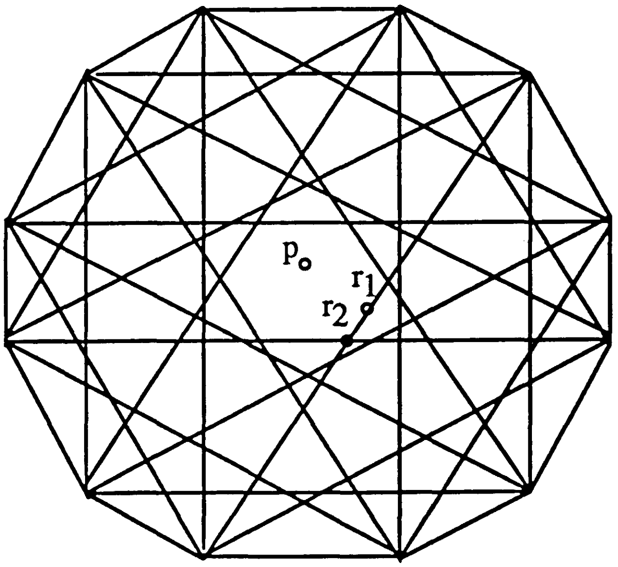


Figure 7.2: The moment map image of G_2/B.

Chapter 8

Explicit Results for G/B, $G = SL(n+1, \mathbb{C})$

The Schubert-like conditions are frequently used in this chapter.

8.1 Parallel Walls in Terms of Symmetry Groups

We use two interpretations of the flag manifold $SL(n+1, \mathbf{C})/B$ as follows.

(1) The space of all flags in $\mathbf{C}^{n+1}$

$$0 \subset V^1 \subset V^2 \subset \cdots \subset V^n \subset V^{n+1}$$

where V^i is a dimension i linear subspace of $\mathbf{C}^{n+1}$.

(2) The space of all flags in $\mathbf{P}^n$

$$P^0 \subset P^1 \subset \cdots \subset P^{n-1} \subset P^n$$

where P^i is a dimension i linear projective subspace of $\mathbf{P}^n$.

Choose a basis $\{e_1, \cdots, e_{n+1}\}$ in $\mathbf{C}^{n+1}$, then a coordinate flag is a flag where each subspace is spanned by some of basis vectors. When working in the projective space $\mathbf{P}^n$, a basis amounts to choosing $n+1$ points $\{a_1, \cdots, a_{n+1}\}$ that span $\mathbf{P}^n$, and a coordinate flag in $\mathbf{P}^n$ is a flag when each subspace is spanned by some of base points. We shall work on $\mathbf{C}^{n+1}$ or $\mathbf{P}^n$, alternatively, it depends on whichever is convenient for us.

The $H - fixed$ points can now be identified with the coordinate flags, which, in turn, is identified with elements of symmetry group S_{n+1} (Weyl group). The identification is indicated as follows.

$$\mathbf{C} \cdot \{e_{i_1}\} \subset \mathbf{C} \cdot \{e_{i_1}, e_{i_2}\} \subset \cdots \subset \mathbf{C} \cdot \{e_{i_1}, \cdots, e_{i_{n+1}}\}$$

$$\updownarrow$$

$$(i_1, \cdots, i_{n+1}) \in S_{n+1}$$

where $1 \leq i_k \leq n+1$, and $\mathbf{C}\{x, y, z, \cdots\}$ denote the subspace spanned by $x, y, z, \cdots$.

We still need more conventions.

Definition. Let $S, T \subset \{1, 2, \cdots, n\}$, with $|S| = |T|$, define M_T^S = set of permutations $(h_1 \cdots h_{n+1})$ in S_{n+1} such that elements of S only occur in the positions indexed by elements in T.

That is, if $S = \{i_1, \cdots, i_k\}$, $T = \{j_1, \cdots, j_k\}$, then M_T^S consists of those permutations $(h_1 \cdots h_{n+1})$ where $i_1, \cdots, i_k$ occur only in the $j_1th, \cdots, j_kth$ positions.

Now we can interpret our results in chapter 7.

Theorem. M_T^S (precisely, the moment map images of the corresponding flags) is the vertex set of a wall in $\mu(G/B)$. To abuse the notations, we use M_T^S to denote the wall also. Then for any $S, T, T' \subset \{1, \cdots, n+1\}$ with $|S| = |T| = |T'|$, M_T^S and $M_{T'}^S$ are parallel. Moreover, every wall is of this form.

We shall call M_T^S a wall of **type** S. Note by our convention, $M_T^S = M_{T^C}^{S^C}$.

Example. $G/B = SL(4, \mathbf{C})/B$. Then the boundary of the moment map images of G/B consists of 6 square faces and 8 hexagonal faces. Hence every the codim 1 wall is either square or hexagonal polytope. The following gives all hexagonal walls: $1 \leq a \leq 4$,

$$M_{\{1\}}^{\{a\}} = \{(a * * *)\}$$

$$M_{\{2\}}^{\{a\}} = \{(* a * *)\}$$

$$M_{\{3\}}^{\{a\}} = \{(* * a *)\}$$

$$M_{\{4\}}^{\{a\}} = \{(* * * a)\}$$

The induced Bruhat diagram on the walls is

$$M_{\{1\}}^{\{a\}} \rightarrow M_{\{2\}}^{\{a\}} \rightarrow M_{\{3\}}^{\{a\}} \rightarrow M_{\{4\}}^{\{a\}}$$

And all square faces are given by

$$M_{\{1\,2\}}^{\{a\,b\}} = \{(a\,b * *), (b\,a * *)\}$$

$$M_{\{1\,3\}}^{\{a\,b\}} = \{(a * b *), (b * a *)\}$$

$$M^{\{a\,b\}}_{\{1\,4\}} = \{(a \ast \ast b), (b \ast \ast a)\}$$

$$M^{\{a\,b\}}_{\{2\,3\}} = \{(\ast\, a\, b\, \ast), (\ast\, b\, a\, \ast)\}$$

$$M^{\{a\,b\}}_{\{2\,4\}} = \{(\ast\, a \ast b), (\ast\, b \ast a)\}$$

$$M^{\{a\,b\}}_{\{3\,4\}} = \{(\ast \ast a\, b), (\ast \ast b\, a)\}$$

$1 \le a \ne b \le 4$, and the Bruhat diagram on the walls is

$$\begin{tikzcd}
 & & M^{\{ab\}}_{\{23\}} \arrow[dr, dash] & & \\
M^{\{ab\}}_{\{12\}} \arrow[r, dash] & M^{\{ab\}}_{\{13\}} \arrow[ur, dash] \arrow[dr, dash] & & M^{\{ab\}}_{\{24\}} \arrow[r, dash] & M^{\{ab\}}_{\{34\}} \\
 & & M^{\{ab\}}_{\{14\}} \arrow[ur, dash] & &
\end{tikzcd}$$

We remark that

Corollary. M^S_T is a face of $\mu(X)$ if and only if $T = \{1, 2, \cdots, k\}$ or $T = \{1, 2, \cdots, k\}^C = \{k+1, \cdots, n+1\}$.

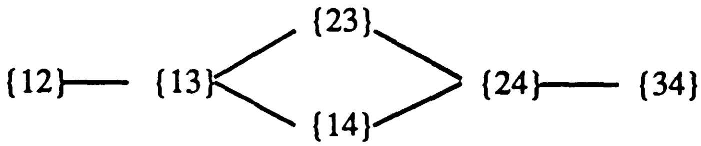
Therefore, the following is a face-diagram

$$\begin{tikzcd}
 & & M^{\{23\}}_{\{12\}} \arrow[dr, dash] & & \\
M^{\{12\}}_{\{12\}} \arrow[r, dash] & M^{\{13\}}_{\{12\}} \arrow[ur, dash] \arrow[dr, dash] & & M^{\{24\}}_{\{12\}} \arrow[r, dash] & M^{\{34\}}_{\{12\}} \\
 & & M^{\{14\}}_{\{12\}} \arrow[ur, dash] & &
\end{tikzcd}$$

reverse this diagram we get

$$\begin{tikzcd}
 & & M^{\{14\}}_{\{34\}} \arrow[dr, dash] & & \\
M^{\{12\}}_{\{34\}} \arrow[r, dash] & M^{\{13\}}_{\{34\}} \arrow[ur, dash] \arrow[dr, dash] & & M^{\{24\}}_{\{34\}} \arrow[r, dash] & M^{\{34\}}_{\{34\}} \\
 & & M^{\{23\}}_{\{34\}} \arrow[ur, dash] & &
\end{tikzcd}$$

Use the Canonical projection from $GL(4, \mathbf{C})/B$ to $G_2(\mathbf{C}^4)$, we can see that the above two diagrams descend to



which is exactly the Bruhat diagram of Schubert varieties on $G_2(\mathbf{C}^4)$.

8.2 Schubert Conditions and Strata Indexed by Parallel Walls

Now we want to construct $\overline{X^{M_T^S}}$ in terms of flags in $\mathbf{P}^n$.

Given a wall of type S, with $|S| = k$. Then S determines a coordinate linear subspace $\mathbf{C}^S$ spanned by $\{e_i | i \in S\}$ and a coordinate subspace $\mathbf{C}^{S^c}$ spanned by $\{a_j | j \notin S\}$ with $\mathbf{C}^S \cap \mathbf{C}^{S^c} = 0$. Then our following arguments will show that any flags in a given stratum closure $\overline{X^{M_T^S}}$ (S fixed) can be constructed from two flags in $\mathbf{C}^S$ and $\mathbf{C}^{S^c}$ by a specific method determined by T.

Now given arbitrary two flags in $\mathbf{C}^S$ and $\mathbf{C}^{S^c}$, respectively,

(1) $\xi^1 \subset, \cdots, \subset \xi^{k-1} \subset \xi^k = \mathbf{C}^S$

(2) $\eta^1 \subset, \cdots, \subset \eta^{n-k} \subset \eta^{n-k+1} = \mathbf{C}^{S^c}$

To construct a new flag

$$\zeta^1 \subset, \cdots, \subset \zeta^n \subset \mathbf{C}^{n+1}$$

from flag (1) and flag (2), we have the following choices:

$\zeta^1 = \xi^1, or\ \eta^1,$

$\zeta^2 = \xi^2, or\ span\ of\ \xi^1 and\ \eta^1, or\ \eta^2,$

$\zeta^3 = \xi^3, or\ span\ of\ \xi^2 and\ \eta^1, or\ span\ of\ \xi^1 and\ \eta^2, or\ \eta^3,$

...

$\zeta^n = span\ of\ \xi^k and\ \eta^{n-k}, or\ span\ of\ \xi^{k-1} and\ \eta^{n-k+1}.$

Theorem. Suppose now $T = \{j_1, \cdots, j_k\} \subset \{1, \cdots, n+1\}$, then any flag in $\overline{X^{M_T^S}}$,

$$\zeta^1 \subset, \cdots, \subset \zeta^n \subset \mathbf{C}^{n+1},$$

can be constructed from a flag in $\mathbf{C}^S$,

$$\xi^1 \subset, \cdots, \subset \xi^{k-1} \subset \xi^k = \mathbf{C}^S,$$

and a flag in $\mathbf{C}^{S^c}$,

$$\eta^1 \subset, \cdots, \subset \eta^{n-k} \subset \eta^{n-k+1} = \mathbf{C}^{S^c},$$

as follows:

$\zeta^1 = \eta^1, \cdots, \zeta^{j_1-1} = \eta^{j_1-1}$
$\zeta^{j_1} = span\ of\ \xi^1 and\ \eta^{j_1-1}, \cdots, \zeta^{j_2-1} = span\ of\ \xi^1 and\ \eta^{j_2-2}$
$\zeta^{j_2} = span\ of\ \xi^2 and\ \eta^{j_2-2}, \cdots, \zeta^{j_3-1} = span\ of\ \xi^2 and\ \eta^{j_3-3}$
$\cdots$
$\zeta^{j_{k-1}} = span\ of\ \xi^{k-1} and\ \eta^{j_{k-1}-k+1}, \cdots, \zeta^{j_k-1} = span\ of\ \xi^{k-1} and\ \eta^{j_k-1}$
$\zeta^{j_k} = span\ of\ \xi^k and\ \eta^{j_k-k}, \cdots, \zeta^n = span\ of\ \xi^k and\ \eta^{n-k}$

Obviously by our construction,the spaces of flags constructed in this way for various T all have the same isotropy subgroup (depending only on S). Hence by a basic property of moment map, their moment map images are parallel, therefore they exhaust all strata closures indexed by parallel walls of type S according to section 7.5.

In fact , the above method of constructing new flags are the only method that can make $\zeta^1, \zeta^2, \cdots, \zeta^n$ flags.

Let $\mathbf{C}^S$ be the coordinate subspace spanned by $\{e_i | i \in S\}$. Then in terms of Schubert conditions, $\overline{X^{M_T^S}}$ is described as follows

Theorem. $\overline{X^{M_T^S}}$ = all flags $V^1 \subset, \cdots, \subset V^n \subset \mathbf{C}^{n+1}$ satisfying the following conditions;

(1) $dimV^\mu \cap \mathbf{C}^S = 0, 1 \leq \mu \leq j_1$, $dimV^{j_1} \cap \mathbf{C}^S = 1$
$dimV^\mu \cap \mathbf{C}^S = 1, j_1 \leq \mu \leq j_2$, $dimV^{j_2} \cap \mathbf{C}^S = 2$
$\cdots$
$dimV^\mu \cap \mathbf{C}^S = k-1, j_{k-1} \leq \mu \leq j_k$, $dimV^{j_k} \cap \mathbf{C}^S = k$

(2) $dimV^\mu \cap \mathbf{D}^S + dimV^\mu \cap \mathbf{C}^{S^C} = \mu$, $1 \leq \mu \leq n+1$. ($\Leftrightarrow V^\mu = V^\mu \cap \mathbf{C}^S \bigoplus V^\mu \cap \mathbf{C}^{S^C}$)

As a corollary of our construction, we observe that $\overline{X^{M_T^S}}$ is equivariantly isomorphic to the product of the space of flags in $\mathbf{C}^k$ and the space of flags in $\mathbf{C}^{n-k+1}$ with respect the obvious actions of the torus H.

8.3 Schubert Conditions and Strata Indexed by Half Regions

As before, let $S \subset \{1, \cdots n+1\}$ with $|S| = k$.

It can be shown by (for example) checking their vertex sets that we have the following:

Theorem. (a). Let $C = (M^S_{\{j_1, \cdots, j_k\}})^<$ be bounded by $M^S_{\{1, \cdots, k\}}$ and $M^S_{\{j_1, \cdots, j_k\}}$, then $\overline{X^C}$ consists of flags

$$V^1 \subset \cdots \subset V^n \subset \mathbf{C}^{n+1}$$

satisfying the following conditions

$$dim V^{j_1} \cap \mathbf{C}^S \geq 1$$

$$\cdots$$

$$dim V^{j_k} \cap \mathbf{C}^S \geq k$$

(b). Let $D = (M^S_{\{i_1,\cdots,i_k\}})^>$ be bounded by $M^S_{\{i_1,\cdots,i_k\}}$ and $M^S_{\{n+2-k\cdots n+1\}}$, then by $M^S_T = M^{S^C}_{T^C}$, we have $\overline{X^D}$ consists of flags

$$V^1 \subset \cdots \subset V^n \subset \mathbf{C}^{n+1}$$

satisfying the following Schubert conditions

$$dim V^{h_1} \cap \mathbf{C}^{S^C} \geq 1$$

$$\cdots$$

$$dim V^{h_{n+1-k}} \cap \mathbf{C}^{S^C} \geq n+1-k$$

where $\{h_1 \cdots h_{n+1-k}\} = \{i_1 \cdots i_k\}^C$.

(c). Now if $(i_1, \cdots, i_k) \leq (j_1, \cdots, j_k)$ under the Bruhat order on S_{n+1} (this means $i_1 \leq j_1, \cdots, i_k \leq j_k$), then $C \cap D$ is the region bounded by $M^S_{\{i_1,\cdots,i_k\}}$ and $M^S_{\{j_1,\cdots,j_k\}}$, and $\overline{X^{C\cap D}}$ consists of flags

$$V^1 \subset \cdots \subset V^n \subset \mathbf{C}^{n+1}$$

satisfying

$$dim V^{j_1} \cap \mathbf{C}^S \geq 1; \qquad dim V^{h_1} \cap \mathbf{C}^{S^C} \geq 1$$

$$\cdots$$

$$dim V^{j_k} \cap \mathbf{C}^S \geq k; \qquad dim V^{h_{n+1-k}} \cap \mathbf{C}^{S^C} \geq n+1-k.$$

We have shown that the moment map image B of a orbit closure is an intersection of half regions, so $\overline{X^B}$ is contained in an intersection of strata closures indexed by half regions, however it is very difficult to characterize when a set of Schubert-like conditions gives nonempty set of flags satisfying those Schubert-like conditions.

A simple case (corresponding to intersection of two half regions) presented below seems already requiring a lot of effect to solve it, that is, Is there any flag satisfying

$$dim V^{j_1} \cap \mathbf{C}^S \geq 1; \qquad dim V^{i_1} \cap \mathbf{C}^T \geq 1$$

$$\cdots$$

$$dim V^{j_k} \cap \mathbf{C}^S \geq k; \qquad dim V^{i_m} \cap \mathbf{C}^T \geq m$$

where $S, T \subset \{1, \cdots, n+1\}$ with $|S| = k, |T| = m$. Of course, when $k + m = n + 1, T = S^C$ and $(i_1, \cdots, i_m)^C \prec (j_1, \cdots, j_k)$, this is just case (c) in the theorem 4.3.1.

8.4 On the Zariski Open Subsets $\mathcal{U}(1)$ and $\mathcal{U}(n)$

Recall that

$$\mathcal{U}(1) = \{flags\ in\ \mathbf{C}^{n+1} | the\ first\ subspace\ V^1\ is\ general\}$$

$$\mathcal{U}(n) = \{flags\ in\ \mathbf{C}^{n+1} | the\ last\ subspace\ V^n\ is\ general\}$$

In this section we shall prove $\mathcal{U}(1)$ and $\mathcal{U}(n)$ are geometric open subsets, explore some other quotients on G/B and give some relations among these quotients, especially in the case when $G = SL(4, \mathbf{C})$.

Let f_k be the projection form G/B to $G(k, \mathbf{C}^{n+1})$ defined as follows $f_k : V^1 \subset \cdots \subset V^k \subset \cdots \subset V^n \subset \mathbf{C}^{n+1} \mapsto V^k$.

We have known that all the codim 1 faces of $\mu(G/B)$ are given by the moment map images of f_k^{-1} (*a coordinate* k *− space*), $1 \leq k \leq n$. We shall call $\mu(f_k^{-1}(a\ coordinate\ k - space))$ a face of type k.

Theorem. Under the convention above.

$$\mathcal{U}(1) = \bigcup \{X^D | D\ meets\ every\ codim\ 1\ face\ of\ type\ 1\}$$

$$\mathcal{U}(n) = \bigcup \{X^D | D\ meets\ every\ codim\ 1\ face\ of\ type\ n\}$$

Clearly, $\mathcal{U}(1)$ and $\mathcal{U}(n)$ consist only of strata indexed by top dimensional polytopes.

Proof. We only prove the statement for $\mathcal{U}(1)$. For any $x \in \mathcal{U}(1)$, let $D = \mu(\overline{H \cdot x})$. The moment map image of $\mathbf{P}^n = \{V^1 \subset \mathbf{C}^n\}$ can be thought as obtained from $\mu(G/B)$ by collapsing each type 1 face to a vertex. Consequently, if D misses a type 1 face of $\mu(G/B)$, then $\mu(\overline{H \cdot f_1(x)})$ will miss a vertex of $\mu(\mathbf{P}^n)$, but $f_1(x)$ is general. So $\mu(\overline{H \cdot f_1(x)}) = \mu(\mathbf{P}^n)$. Hence D meets every type 1 face of $\mu(X)$. On the other hand, if D meets every type 1 face, and $x \in X^D$, the same argument shows $\mu(\overline{H \cdot f_1(x)}) = \mu(\mathbf{P}^n)$, hence $f_1(x)$ is general. This completes the proof.

A direct proof of the first part of theorem 7.2.3 First of all, by the consideration of parallel walls, it is impossible that an admissible decomposition of $\mu(G/B)$ contains two open polytopes which meet every codim 1 face of type 1 (resp. n). On the other hand, given any admissible decomposition $\Im$ of $\mu(G/B)$, since there is only one top dimensional polytope Δ in any admissible decomposition of the moment map image of $\mathbf{P}^{n+1}$, there should be (at least) one polytope in $\Im$ which can be collapsed to Δ, so this polytope must meet every codim 1 face of type 1 (resp. n).

Remark. In fact, the proof can be made more explicit in terms of symmetry group $\mathcal{S}_{n+1}$. That is , if $D \in \Xi$ contains vertices

$$\mathcal{A} = \{some(i_1, \cdots, i_{n+1})\}$$

(where we identify each element of $\mathcal{S}_{n+1}$ with a vertex of $\mu(G/B)$), then the i_1's that appear in $\mathcal{A}$ should range all over from 1 to $n+1$ if D meets every face of type 1.

It is quite obvious that the Weyl group W sends a face to a face of the same type , hence $\mathcal{U}(1)$ and $\mathcal{U}(n)$ are both $W-invariant$ since $w \cdot X^D = X^{w \cdot D}$ for each $w \in W$, and therefore W acts on $\mathcal{U}(1)/H$ and $\mathcal{U}(n)/H$.

Now let us assume a general metric on G/B is taken so that the barycenter 0 of $\mu(G/B)$ is a general point in $\mu(G/B)$, then $\mathcal{U}_0$ is $W - invariant$ since $W \cdot 0 = 0$, hence W acts on $\mathcal{U}_0/H$ also. I don't know if $\mathcal{U}_0, \mathcal{U}(1), \mathcal{U}(n)$ are all the $W - invariant$ geometric open subsets.

Proposition. Let $G = SL(4, \mathbf{C})$.

(1) $\mathcal{U}_p/H$ is isomorphic to $\mathbf{P}^1 \times \mathbf{P}^2$ if p is general and close enough to a hexagonal face.

(2) $\mathcal{U}_p/H$ is isomorphic to $\mathbf{P}^3$ if p is general and close enough to a square face.

Chapter 9

Miscellaneous

We have a little discussion on torus strata in 9.4.

9.1 On the Grassmannian $G(k, \mathbf{C}^{n+1})$

By [G-G-MacP-S], under a specific moment map μ, the image of $G(k, \mathbf{C}^{n+1})$ is a hypersimplex defined below:

$$\Delta_{n+1,k} = \{(x_1, \cdots, x_{n+1}) \in R^{n+1} | \sum x_i = k\}.$$

There are various descriptions of faces of $\Delta_{n+1,k}$. (in terms matroids, for example). Below we will give descriptions for their strata closures in terms of subspaces and show that they are all Grassmannians, as expected. We observe

Proposition. Suppose a coordinate system on $\mathbf{C}^{n+1}$ is chosen, and $V_0^i \subset V_0^j$ are two coordinate subspaces with $i \leq k \leq j$. Let

$$\Gamma = \{V^k \in G_k(\mathbf{C}^{n+1}) | V_0^i \subset V^k \subset V_0^j\}$$

Then the moment map image of Γ is a face of $\Delta_{n+1,k}$, which is isomorphic to $\Delta_{j-i,k-i}$. In fact, Γ is the closure of the stratum indexed by that face. Clearly Γ can be identified with

$$G_{k-i}(\mathbf{C}^{j-i}) = \{0 \subset V^k / V_0^i \subset V_0^j / V_0^i\}$$

Moreover, every face of $G_k(\mathbf{C}^{n+1})$ comes from this way.

Now we want to say something about the quotients on Grassmannians.

Conjecture. There is zariski open subset $\mathcal{U}$ on $G_k(\mathbf{C}^{n+1})$ such that $\mathcal{U}$ has a h sdorff quotient $\mathcal{U}/H$ which can be identified with $G_{k-1}(\mathbf{C}^{n-1})$.

This conjecture is trivial for $G_2(\mathbf{C}^4)$ since every quotient in this space has to be $\mathbf{P}^1$. In fact, the conjecture is true for any $G(2, \mathbf{C}^{n+1})$. This can be seen from theorem 5.1.1 and the proposition above. Indeed, we can actually construct such quotients.

Proposition. Define a Zariski open subset $\mathcal{U} \subset G(1, \mathbf{P}^n) = G(2, \mathbf{C}^{n+1})$ as follows: Pick up a coordinate system in $\mathbf{P}^n$ (i.e, $n+1$ linearly independent points, called vertices). Choose two coordinate hyperplanes P_1, P_2 of $\mathbf{P}^n$. Let P_1^0 denote the generic part of P_1. Define

$$\mathcal{U} = \{span\ of\ p\ and\ q\ |p \in P_1^0, q \in P_2 - P_1 \cap P_2 - coordinate\ verteces\}$$

Then the ordinary orbit space $\mathcal{U}/(\mathbf{C}^*)^n$ of $\mathcal{U}$ is isomorphic to $\mathbf{P}^{n-2}$.

Proof. Pick up a generic hyperplane in P_2. It can be checked that $\mathcal{U}/(\mathbf{C}^*)^n$ can be identified with this hyperplane.

9.2 Homogeneous Spaces that Project to $\mathbf{P}^n$

In this section, we consider the space of partial flags $\{V^{i_1} \subset \cdots \subset V^{i_k} \subset \mathbf{C}^{n+1}\}$ such that either $i_1 = 1$ or $i_k = n$.

Just as what we did for the space of complete flags $\{V^1 \subset \cdots \subset V^n \subset \mathbf{C}^{n+1}\}$, we have

(1) Let

$$G/P = \{V^{i_1} \subset \cdots \subset V^{i_k} \subset V^n \subset \mathbf{C}^{n+1}\}$$
$$\mathcal{U} = \{V^{i_1} \subset \cdots \subset V^{i_k} \subset V^n | V^n is\ general\}$$

then

$$\mathcal{U}/H \cong \{V^{i_1} \subset \cdots \subset V^{i_k} \subset V_0^n | V_0^n is\ fixed\}$$

(2) Let

$$G/P = \{V^1 \subset V^{i_1} \subset \cdots \subset V^{i_k} \subset \mathbf{C}^{n+1}\}$$
$$\mathcal{U} = \{V^1 \subset V^{i_1} \subset \cdots \subset V^{i_k} | V^1 is\ general\}$$

n

$$\{V_0^1 \subset V^{i_1} \subset \cdots \subset V^{i_k} \subset \mathbf{C}^{n+1} | V_0^1 is\ fixed\}$$
$$\cong \{V^{i_1}/V_0^1 \subset \cdots \subset V^{i_k}/V_0^1 \subset \mathbf{C}^{n+1}/V_0^1\}.$$

hese homogeneous spaces share many results with the complete flag manifold, as parallel wall phenomena and hence a belowing up and belowing down ures when crossing a wall, etc. Since there is no need to develop new tools to prove those results, we will not list them explicitly here.

9.3 Fibrations $G/P_J \to G/P_I$ and Weight Diagrams.

Let $P = P_J \subset G$ be a standard parabolic subgroup. Consider a finite-dimensional irreducible representation ρ of G on a vector space V with highest weight

$$\lambda = \sum_{i \notin J} n_i \lambda_i, n_i > 0.$$

where $\lambda_1, \cdots, \lambda_n$ are fundamental weights.

Then the homogeneous space $G/P = Y$ can be identified with the orbit of a principal vector v_λ (in V_λ) in the projectivization $\mathbf{P}(v)$ of V. Sometimes, we call $J^C = \{1, \cdots, n\} - J$ the support of λ.

Choose an $K - \textit{invariant}$ (recall K is a maximal compact subgroup of G) Hermitian metric on V. It induces a Kähler metric on $P(V)$, hence on $G \cdot v_\lambda \cong G/P$. It can be proved that under this metric the moment map image $\mu = \mu_J$ of G/P is the convex hall of $W \cdot \lambda$ in $\eta^* \cong \mathbf{R}^n$.

Now let $\pi(\lambda)$ be the set of weights of the representation ρ of G. Let $P' = P_I$ be a parabolic subgroup of G containing P, that is, $J \subset I$. We are trying to compare the moment map image of G/P and the moment map image of G/P'. Then we have the following observations.

Choose $\lambda = \sum_{i \in J^C} n_i \lambda_i$ general enough (i.e, with large enough coefficients) so that there is a positive weight $v \in \pi(\lambda)$ such that $v = \sum_{j \in I^C} m_j \lambda_j$ with $m_j > 0$, then

(1) Convex hall $W \cdot v$ is the moment map image of G/P_I with respect to the chosen metric on V, which is contained in the moment map image of G/P_J as an interior subpolytope.

(2) The closure of a connected component, (hence its faces), of regular values of μ is spanned by some weights in $\pi(\lambda)$.

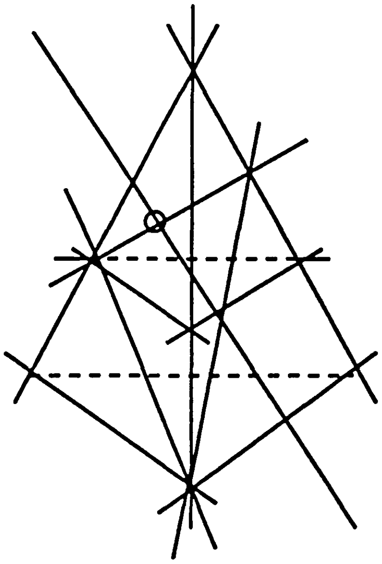
In fact let $G/P_1 \to G/P_2 \to \cdots \to G/P_k$ be a sequence of fibration, i.e, $P_1 \subset P_2 \subset \cdots \subset P_k$ are all standard parabolic subgroup, then we can make a similar statement as the one above, that is, for some choice of λ, and with respect to the chosen metric on V, we have the moment map image of G/P_i contains the moment map image of $G/P_{i+1} (1 \leq i \leq k-1)$. To do so, we have to make λ to be more general.

9.4 Some Examples on Torus Strata of $G = SL(n+1, \mathrm{C})/B$

A. In [G-G-MacP-S], an example is given to demonstrate that the closure of a torus stratum on $G_6(\mathrm{C}^9)$ is not a union of torus strata. Use this example, the natural fibration $SL(9, \mathrm{C})/B$ to $G_6(\mathrm{C}^9)$, and corollary 13.5, it is straightforward to pull back the counterexample on $G_6(\mathrm{C}^9)$ to $SL(9, \mathrm{C})/B$.

B. Let $f : X = SL(n_1, \mathrm{C})/B \to \mathrm{P}^n = Y$ be the natural projection. Given a stratum Γ on X, clearly $f(\Gamma)$ is a torus stratum on P^n. Since every stratum on P^n is a single torus orbit, the restriction of f on Γ, $\Gamma \to f(\Gamma)$, is a fibration . It is wished that a torus action can be introduced on the fiber Z of f according to the coordinate system on C^{n+1} so that the fiber of $\Gamma \to f(\Gamma)$ is dense open in a torus stratum of Z. If this could be done, by the induction on the dimension of G/B, we would be able to show that every stratum on G/B were nonsingular. However, the following example shows that we can not do that (see the next page for a picture).

This is a picture for a flag $P^0 \subset P^1 \subset P^2 \subset P^3$ in P^4 visualized in this way: the big tetrahedron stands for a general P^3 in P^4, the four triangle face of the big tetrahedron and the smaller horizontal triangle are the intersections of P^3 with the five coordinate $3 - space$ in P^4, the slopy triangle is P^2, the long line is P^1, and the blank dot is P^0.



C. Let X = the space of partial flags $P^2 \subset P^3$ in P^6 and $f : X \to G_2(\mathrm{P}^6)$ be the natural projection. The following example shows that the restriction of f to a torus stratum on X may not be a fibration over a stratum on $G_2(\mathbf{P}^6)$.

Definition ([HM]) $Y^p_q = \{\mathit{configurations\ of}\ p+q+1\ \mathit{points\ in}\ \mathbf{P}^{p-1}.\}$
Now let

$$\Gamma = \{P^2 \subset \mathbf{P}^6 | P^2 \subset \mathbf{P}^5, P^2\ \mathit{avoids}\ 2\ \mathit{faces\ of}\ \mathbf{P}^5\}$$

be a stratum in $G_2(\mathbf{P}^6)$, where $\mathbf{P}^5$ is a fixed coordinate $5-\mathit{space}$ in $\mathbf{P}^6$. Then

$$\Gamma \cong (\mathbf{C}^*)^5 \times Y^3_2.$$

Let

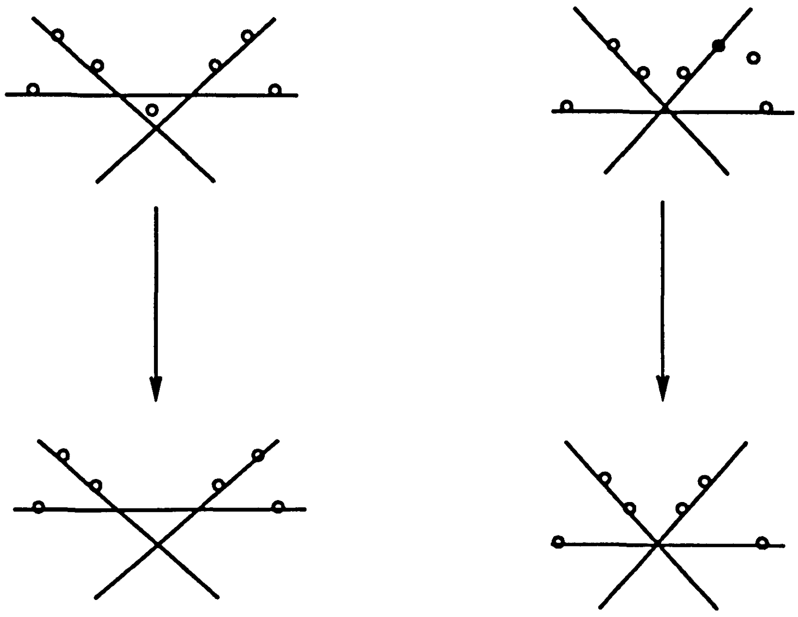
$$l(\Gamma) = \{P^2 \subset P^3 \subset \mathbf{P}^6 | P^2 \subset \mathbf{P}^5, \mathit{avoids}\ 2\ \mathit{faces\ of}\ \mathbf{P}^5, P^3\ \mathit{avoids}\ 2\ \mathit{faces\ of}\ \mathbf{P}^6\}$$

$$\cong \{P^3 \subset \mathbf{P}^6 | P^3\ \mathit{avoids}\ 2\ \mathit{faces\ of}\ \mathbf{P}^6\}$$

Then,

$$l(\Gamma) \cong (\mathbf{C}^*)^6 \times Y^3_3.$$

So $l(\Gamma) \to \Gamma$ is not a fibration because $Y^3_3 \to Y^3_2$ is known not a fibration as demonstrated below ([HM])



D. Sometimes, a torus stratum can be embedded into some $\mathbf{C}^N$ as the complement of some hypersurfaces. In many case, these hypersurfaces are homeomorphic to some flat hyperplanes, especially, when the space under the consideration has small dimension, for instance, $SL(3,\mathbf{C})/B$. In a separate paper, we will give a homological formula for the complement of such an arrangement in $\mathbf{R}^n$ in terms of the combinatorial data of the arrangement in [Hu]. (see also appendix A.)

9.5 Real Parts of Symplectic Quotients and Real Moment Maps

Let X be a complex algebraic variety. The real part $X_{\mathbf{R}}$ of X, if it ever exists, is a real algebraic variety, such that

$$X = X_{\mathbf{R}} \times_{spec(\mathbf{R})} Spec(\mathbf{C})$$

Now a complex torus $H = (\mathbf{C}^*)^n$ can be decomposed into the product of $A = (\mathbf{R}^>)^n$ and compact part $T = (S^1)^n$:

$$(\mathbf{C}^*)^n = (\mathbf{R}^>)^n \times (S^1)^n.$$

Note that the real part of $(\mathbf{C}^*)^n$ is not merely $(\mathbf{R}^>)^n$, but

$$\begin{aligned} &(\mathbf{R}^>)^n \times (\{+1\},\{-1\})^n \\ &= (\mathbf{R}^> \times (\{+1\},\{-1\}))^n \\ &= (\mathbf{R}^*)^n \\ &\cong (\mathbf{R}^>)^n \times (\mathbf{Z}_2)^n \\ &= A \times \Gamma \end{aligned}$$

where $\{+1\},\{-1\}$ are the only two real points of S^1, and $\Gamma = (\mathbf{Z}_2)^n \subset (S^1)^n$.

We say a H-action on X is compactible with the real structure of X if the real part $A \times \Gamma$ of H preserves the real part $X_{\mathbf{R}}$ of X. Now suppose X is endowed with such an action of H, then we have an induced $(\mathbf{R}^*)^n = A \times \Gamma$ action on $X_{\mathbf{R}}$. Let μ be an associated moment map of H,

Definition. The real moment map $\mu_{\mathbf{R}}$

$$\mu_{\mathbf{R}} : X_{\mathbf{R}} \longrightarrow \mathbf{R}^n$$

of $(\mathbf{R}^*)^n$ action on $X_{\mathbf{R}}$ is the restriction of μ to $X_{\mathbf{R}}$.

Since $\Gamma = (\mathbf{Z}_2)^n = (\mathbf{R}^*)^n \cap (S^1)^n$, we have

Proposition.

(1) $\mu_{\mathbf{R}}$ is Γ equivariant.

(2) $\mu_{\mathbf{R}}(\overline{(\mathbf{R}^*)^n \cdot x})$ is a convex polyhedron in $\mathbf{R}^n$ whose vertices are the images of $(\mathbf{R}^*)^n$-fixed points in $\overline{(\mathbf{R}^*)^n \cdot x}$, for any $x \in X_{\mathbf{R}}$. In particular, $\mu_{\mathbf{R}}(X_{\mathbf{R}}) = \mu(X)$ is a convex polyhedron in $\mathbf{R}^n$.

(3) $\mu^{-1}(p)$ has a real part $\mu^{-1}(p) \cap X_{\mathbf{R}} = \mu_{\mathbf{R}}^{-1}(p)$ for any $p \in \mathbf{R}^n$.

(4) The symplectic quotient $\mu^{-1}(p)/T$ has a real part $\mu_{\mathbf{R}}^{-1}(p)/\Gamma$ for any $p \in \mathbf{R}^n$.

Still as in section 1.1, we have

$$\mu_{\mathbf{R}}(X_{\mathbf{R}}) = \mu(X) = \bigcup_{F \in \Upsilon} F$$

Theorem. Let F_1, F_2 be two polyhedra in Υ, and F_2 be an open face of F_1. Let $p \in F_1, r \in F_2$, then the unique algebraic map f from $\mu^{-1}(p)/T$ to $\mu^{-1}(r)/T$ restricts to a real algebraic map $f_{\mathbf{R}}$ from $\mu_{\mathbf{R}}^{-1}(p)/\Gamma$ to $\mu_{\mathbf{R}}^{-1}(r)/\Gamma$ such that $f_{\mathbf{R}}$ corresponds to a real blowing up map. The statement can be illustrated as follows:

$$\begin{tikzcd}
\mu_{\mathbf{R}}^{-1}(p)/T \arrow[r] \arrow[d, "f_{\mathbf{R}}"'] & \mu^{-1}(p)/T \arrow[d, "f"'] \\
\mu_{\mathbf{R}}^{-1}(r)/\Gamma \arrow[r] & \mu^{-1}(r)/\Gamma
\end{tikzcd}$$

where the vertical maps are natural projections.

In the case that $X = G/B$ with the action of a maximal torus H. The above theorem applies since the action is compactible with the real structure of G/B. In fact, almost all theorems in chapter 5 and 6 concerning symplectic quotients have word-for-word translations for their real parts – the real "symplectic" quotients. Of course, it is harder to study the topology of real quotients since there are few theorems for real algebraic varieties.

Appendix A

The Homology of the Complements of Subspaces

Spaces Associated to Torus Actions. Naturally associated to a torus action, one can study the following three kinds of spaces: 1. The quotient varieties. 2. The torus strata. 3. The closures of torus orbits as toric varieties.

Complements of Subspaces. In this thesis, we mostly only study the quotient varieties. An attempt to study torus strata has led us to consider arrangements: $\mathcal{A} = \{A_1, \cdots, A_m\}$ in $\mathbf{R}^n$, where $A_1, \cdots, A_m$ are closed subspaces of $\mathbf{R}^n$ satisfying the following 2 conditions: (a) each A_i is either **homeomorphic** to Euclidean space $\mathbf{R}^k$ of dimension k or **homeomorphic** to the sphere S^k of dimension k, for some $k < n$. (b) each connected component of an arbitrary non-empty intersection $A_{i_1} \cap \cdots \cap A_{i_r}$ also satisfies condition (a).

Associated to every arrangement $\mathcal{A} = \{A_1, \cdots, A_m\}$, there is a ranked poset $\mathcal{L}(\mathcal{A}) = (\mathcal{L}, \prec, r)$ which can be constructed explicitly from the combinatorial data of the intersections of $\mathcal{A}$. Then the combinatorics of $\mathcal{L}(\mathcal{A}) = \mathcal{L}$ determines completely the homology of the complement, $M(\mathcal{A}) = \mathbf{R}^n - \cup_{i=1}^m A_i$, of $\mathcal{A}$. We have the following **homological formula for the complements of subspaces:**

$$H_i(\mathbf{R}^n - \cup_{i=1}^m A_i; \mathbf{Z}) = \bigoplus_{v \in \mathcal{L}} H^{n-r(v)-i-1}(K(\mathcal{L}_{>v}), K(\mathcal{L}_{(v,V)}); \mathbf{Z})$$

where V is the unique maximal element in $\mathcal{L}$ representing $\mathbf{R}^n$, $H^{-1}(\emptyset, \emptyset) = \mathbf{Z}$ as a convention, and $K(\mathcal{J})$ denotes the order complex of a poset $\mathcal{J}$.

When each A_i in $\mathcal{A}$ is an affine linear subspace in $\mathbf{R}^n$, our formula coincides with the one obtained by Goresky and MacPherson [GM4]. In fact, in this particular case, we have proved that the formula still holds even if we consider the arrangements of the acyclic subspaces in an ambient acyclic space provided that every space in consideration satisfied Lefschetz duality with compact support. We shall not give proofs in this thesis. The proof will appear elsewhere.

Appendix B

Extention to General Group Actions

Symplectic Category. Since my work is restricted in the category of algebraic geometry, one may ask whether it works in the symplectic category. There will be no essential difficulty when a symplectic manifold carries a Kahler structure, because in this case a compact Lie group action can always be extended to a complex Lie group action [GuSt2]. Now, what can we say if we do not know whether X carries a Kahler form or not.

General Group Actions. Geometric invariant theory assigns projective "quotient" varieties to any linear action of a complex reductive algebraic group G on a projective variety X. In my thesis, we restricted to the case when G is a complex torus $(\mathbf{C}^*)^n$ and studied the geometry and the topology of quotient varieties of this $(\mathbf{C}^*)^n$ action. It is now very natural to ask: To what extend does the method we developed in the case $G = (\mathbf{C}^*)^n$ apply to the case of a general linear algebraic group action and what can we say beyond?

It is observed that in some nice cases, the K (a compact Lie group) reductions can be reduced to T (a maximal compact torus of K) reductions. The condition is that the moment map image $\mu(X)$ does not touch the walls of the Weyl chambers. In this case there is a decomposition of X in the level of symplectic category: $X = K \times_T \mu^{-1}(D)$, where $D = \mu(X) \cap (a\ fixed\ Weyl\ chamber)$. The K reductions on X can be reduced to T reductions on $\mu^{-1}(D)$. In the case X has a Kahler structure, $\mu^{-1}(D)$ has also a Kahler structure (very hopefully), therefore T action on $\mu^{-1}(D)$ can be extended to $(\mathbf{C}^*)^n$ action on $\mu^{-1}(D)$, hence by my thesis, the question for these particular actions can be solved completely. So the real question is now: what can we do when $\mu(X)$ touches the walls of Weyl chambers?

My speculation is that in this case, the K reductions on X should be reduced to T reductions on a singular space $(\mu^{-1}(D))$, while the singularities of this singular space do not provide serious trouble when considering quotients. Nevertheless, much more efforts must be made when studying general group actions.

Appendix C

Combinatorics of the Posets $\mathcal{P}$ and $\mathcal{P}^*$

In this section we shall give more combinatorial properties of $\mathcal{P}$ and $\mathcal{P}^*$ defined in the previous two sections.

For any partially ordered set $\mathcal{L}$, we may consider its **order complex** $K(\mathcal{L})$ whose vertices are the elements of $\mathcal{L}$, whose simplexes are the linearly ordered subsets of $\mathcal{L}$, $v_0 < \cdots < v_q$. We have known that $K(\mathcal{P})$ and $K(\mathcal{P}^*)$ are connected simplicial complexes. It would be interesting to know more topologies about $K(\mathcal{P})$ and $K(\mathcal{P}^*)$. In the case that $X = SL(3, \mathrm{C})/B$ with the action of a maximal complex torus in $SL(3, \mathrm{C})$, we have $\mathcal{P} = \mathcal{P}^*$, and $K(\mathcal{P}) = K(\mathcal{P}^*)$ is homeomorphic to a closed solid 3-ball.

We say a finite poset $\mathcal{L}$ satisfies the Jordan-Dedekind chain condition if all maximal chains between elements a and b have the same length, for all pair of elements a and b. An absolutely maximal chain is a chain which is not expendable. one may expect that $\mathcal{P}$ and $\mathcal{P}^*$ satisfy the J-D condition. In fact, this is not true. Take $X = SL(4, \mathrm{C})/B$, $H =$ *a maximal complex torus*, then one can check that the (absolutely) maximal chains of $\mathcal{P}$ (or $\mathcal{P}^*$) do not have the same length.

Definition. A **pseudo-lattice** is a poset $\mathcal{L}$ such that for any two elements u, v of $\mathcal{L}$:

(a) the subset $\{w : w \geq u, w \geq v\}$ is either empty or has a unique minimal element, denoted by $u \vee v$ and called join of u and v.

(b) the subset $\{w : w \leq u, w \leq v\}$ is either empty or has a unique maximal element, denoted by $u \wedge v$ and called meet of u and v.

If we further require that $\{w : w \geq u, w \geq v\}$ and $\{w : w \leq u, w \leq v\}$ are non-empty, then $\mathcal{L}$ will be called a **lattice**.

Clearly, a lattice has a unique minimal element o and a unique maximal element Ω. Moreover, if a poset $\mathcal{L}$ has a *unique minimal element* o, *then the height* $h(v)$ of an element $v \in \mathcal{L}$ is defined to be the least upper bound of lengths of chains $o = v_0 < v_1 < \cdots < v_k = v$ between o and v.

Theorem. $\mathcal{P}$ and $\mathcal{P}^*$ are pseudo-lattices.

Proof. We first prove our statement for $\mathcal{P}$. Clearly, we do not have to consider two comparable elements.

So let u, v be two incomparable elements of $\mathcal{P}$, if

$$S = \{w : u \leq w, v \leq w\} \neq \emptyset$$

then there should be a metric on X and hence a moment map μ such that u, v correspond to polytopes F_1 and F_2 of collection Υ (where $\mu(X) = \bigcup_{F \in Y} F$ defined in 1.1) and there is (at least) a polytope of Υ having F_1 and F_2 as its faces. It is an easy fact from polyhedron theory that among the polytopes of Υ having F_1 and F_2 as their faces, there is a unique minimal one under the order $'' \prec''$ (i.e, $D \prec C$ if and only if D is a face of C). In other words, S has a unique minimal element under the order induced from $\mathcal{P}$.

Now we consider

$$T = \{w : u \geq w, v \geq w\}$$

if $T \neq \emptyset$, then there should be a moment map μ such that u, v correspond to polytopes F_1 and F_2 of Υ associated to μ and F_1, F_2 have a common face. Clearly, $F_1 \cap F_2$ is the unique maximal one among their common faces, that is, T has a unique maximal element.

It will be a little harder to show the statement for $\mathcal{P}^*$.

Take any two incomparable elements u, v of $\mathcal{P}^*$ abd assume that they correspond to admissible collection Ξ_1 and Ξ_2. Now if $S = \{w : w \geq u, w \geq v\} \neq \emptyset$, then there will be an admissible collection Ξ' such that

$$\Xi_1 \prec \Xi', \Xi_2 \prec \Xi'.$$

Therefore for any admissible decomposition $\Re$ of $Int(\Box)$, there should be $C \in \Xi'$, $D_1 \in \Xi_1$, and $D_2 \in \Xi_2$, such that C, D_1 and D_2 are all in $\Re$. By the properties that $\Xi_1 \prec \Xi', \Xi_2 \prec \Xi'$ and the admissibilities of Ξ_1 and Ξ_2, we have

$$D_1 \prec C \textit{ and } D_2 \prec C$$

In other words, elements of Ξ_1 and Ξ_2 can be paired, $(D_1, D_2) \in \Xi_1 \times \Xi_2$, by the property that D_1, D_2 are faces of exactly one polytope C in an admissible decomposition for such $\Re$, we select E to be the unique smallest polytope in $\Re$ that has D_1 and D_2 as their faces, then clearly, th collection Ξ_{12} of such E's is admissible, and corresponds to the unique minimal element of S.

Now assume that

$$T = \{w : w \leq u, w \leq v\} \neq \emptyset,$$

then there should be an admissible collection Ξ^* such that

$$\Xi^* \prec \Xi_1, \Xi^* \prec \Xi_2.$$

So for any admissible decomposition $\Re$, there should be $B \in \Xi^*$,$D_1 \in \Xi_1, D_2 \in \Xi_2$ such that B, D_1, D_2 are all in $\Re$. Then as before , we have

$$B \prec D_1 \textit{ and } B \prec D_2.$$

Now take F to be the unique largest polytope in $\Re$ which is a common face of D_1 and D_2, then clearly, the collection Ξ^*_{12} of such F's is admissible, and corresponds to the unique maximal element of T. Hence the theorem is proved, as desired.

From 3.2, we know that the space Q of generic closed orbits and their limits projects to any geometric quotient variety, therefore we define

$$\tilde{\mathcal{P}} = \begin{cases} \mathcal{P} \bigcup \{Q\}, & \text{if } \mathcal{P} \text{ has a unique minimal element} \\ \mathcal{P} \bigcup \{Q\} \bigcup \{o\}, & \text{if } \mathcal{P} \text{ does not have a unique minimal element.} \end{cases}$$

where o is *spec*(**C**) as a scheme. Then $\tilde{\mathcal{P}}$ is a poset ordered by "projection" and contains $\mathcal{P}$ as poset.

Similarly, we define

$$\tilde{\mathcal{P}^*} = \begin{cases} \mathcal{P}^* \bigcup \{Q\}, & \text{if } \mathcal{P}^* \text{ has a unique minimal element} \\ \mathcal{P}^* \bigcup \{Q\} \bigcup \{o\}, & \text{if } \mathcal{P}^* \text{ does not have a unique minimal element.} \end{cases}$$

By the theorem in this section, we have,

Corollary. $\tilde{\mathcal{P}}$ and $\tilde{\mathcal{P}^*}$ are lattices.

Let $\mathcal{L}$ be a poset with a unique minimal element o, then an **atom** is an element which covers o (we say u covers v is $u > v$, and if $u \geq w \geq v$, then either $u = w$, or $w = v$).

Convention. Suppose $\mathcal{L}$ is a poset, let $\mathcal{L}^{-1}$ denote the poset obtained from $\mathcal{L}$ by reversing the order.

Proposition. Every element of $(\mathcal{P}\cup\{Q\})^{-1}$ or $(\mathcal{P}^{*}\cup\{Q\})^{-1}$ is a join of atoms That is, every element of $\mathcal{P}^{-1}$ or $(\mathcal{P}^{*})^{-1}$ is a join of minimal elements.

Proof. For $(\mathcal{P}\cup\{Q\})^{-1}$, let $u \in \mathcal{P}^{-1}$. Then there is a metric on X, and hence a moment map μ, such that there is a polytope F in the Υ associated to μ (where $\mu(X) = \bigcup_{F\in\Upsilon} F$) so that F corresponds to u. Now $\bigcup_{F\in\Upsilon} F$ is an n-circuit of polytopes, and it is easy fact of the theory of n-circuit of polytopes that F is the intersection of some n-polytopes in Υ and F is the unique common face of these n-polytopes. In other words, this implies that u is the join of some atoms because n-polytopes of Υ correspond to atoms of $(\mathcal{P}\cup\{Q\})^{-1}$.

For $(\mathcal{P}^{*}\cup\{Q\})^{-1}$, let $w \in (\mathcal{P}^{*})^{-1}$. We fixed a moment map μ. Then for each admissible decomposition $\mathfrak{R}$ of $Int(\mu(X))$, there should be exactly one polytope D in $\mathfrak{R}$ belonging to the admissible collection Ξ_1 of polytopes that corresponds to $w \in (\mathcal{P}^{*})^{-1}$. Similar to the argument as before, we conclude that D is an intersection of n-polytopes in $\mathfrak{R}$ because $\mathfrak{R}$ can also be regarded as an n-circuit of polytopes. It is fairly clear that from this we can deduce that w is a join of atoms since admissible collections of n-polytopes of Ξ correspond to atoms of $(\mathcal{P}^{*}\cup\{Q\})^{-1}$.

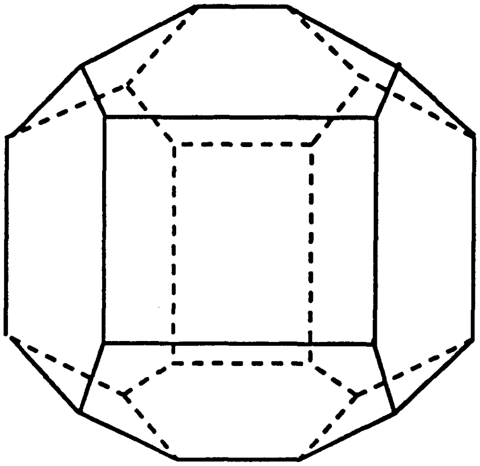


Figure C.1: The moment map image of $GL(4, \mathbf{C})/B$

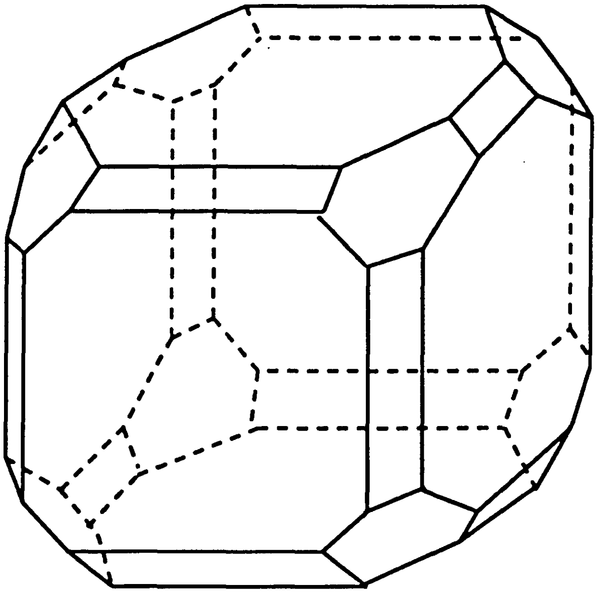


Figure C.2: The moment map image of $Sp(\mathbf{C}^6)/B$

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