APPLICATION OF STOCHASTIC APPROXIMATION METHODS TO SYSTEM OPTIMIZATION

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APPLICATION OF STOCHASTIC APPROXIMATION METHODS TO SYSTEM OPTIMIZATION

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Abstract

This report is concerned with systems whose form is fixed (either by physical limitations or by the choice of the system designer), but which contain a number of variable parameters. The system input, \( v(t) \), is a sample function from a stationary ergodic random process: corresponding to \( v(t) \) there is another ergodic random process, \( d(t) \), which represents the desired output for the system. The problem of interest is how to find the setting of the variable parameters which causes the actual system output, \( q(t) \), to resemble most closely (in some desired sense) the desired output, \( d(t) \).

An iterative method of solution is the approach considered here. That is, a sequence of parameter settings is generated by selecting an initial setting and then alternately observing the system performance and altering the parameter setting. For such an approach to be useful, it must be possible to show a priori that this sequence of parameter settings converges in some meaningful sense to the setting that optimizes the system performance. We consider a particular iterative adjustment procedure and prove that, for certain forms of systems and under certain reasonable conditions on the random processes, the sequence of parameter settings generated by the adjustment procedure does converge to the optimum setting in the mean-square sense.

The reason for considering an iterative adjustment approach is twofold. (i) It allows one to take as the criterion of system performance the performance function \( E(W(d - q)) \). The error weighting function, \( W \), is required to be convex; aside from this, it can be chosen to express the purpose of the application. This is in contrast with the usual analytic approach, which requires that \( W \) be the square of its argument. (ii) No prior knowledge or measurement of the statistics of the random processes is required; the data are used directly.

The iterative method used here is essentially a gradient-seeking method; therefore we shall require that the performance function have a unique local minimum. This is the reason for requiring the weighting function on the error to be convex. It also places the principle restriction on the forms of systems to which the method is applicable.
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I. THE ITERATIVE APPROACH TO SYSTEM OPTIMIZATION

1.1 INTRODUCTION AND STATEMENT OF THE PROBLEM

This work is concerned with the design of systems that are optimum in a statistical sense. One has an input random process \( v(t) \) and a desired output random process \( d(t) \). In this report we shall consider only signals \( v(t) \) and \( d(t) \) which are the outputs of stationary ergodic sources. The problem then is to design a realizable system that operates on \( v(t) \) to produce an output \( q(t) \). The performance of this system is then measured by how closely (in some desired sense) the random variable \( q_t \) resembles the random variable \( d_t \) over the ensemble of situations encountered by the system. An example is the usual filter problem, in which \( v(t) = s(t) + n(t) \) represents a signal corrupted by additive noise; \( d(t) \) is then equal to the original signal, \( s(t) \). A second example is a situation in which we are given some fixed system (such as a loud-speaker or receiver circuit) that we are compelled to use. The problem then is to design a system to precede the fixed system in cascade so that, when the over-all system operates on the input \( v(t) \), the output of the fixed system resembles the desired output, \( d(t) \).

The usual performance criterion in such system-optimization problems is minimization of the mean-square error; that is, minimization of

\[
E\{[d_t - q_t]^2\}
\]

In some applications this performance criterion is appropriate to the purpose of the application. In many applications, however, this criterion is used solely because it is the only one for which an easy analytic solution to the problem can be obtained. Here we desire to be able to deal with a more general performance criterion – minimization of the quantity

\[
E\{W[d_t - q_t]\} 
\]

(1)

in which \( W \) may be any appropriate convex weighting function on the error, \( d - q \).

In considering the system design problem we must have a convenient method for describing the system. If one is restricted to the class of linear systems, then the system may be described by its impulse response. In attempting to solve for the system that minimizes the mean-square error, an integral equation involving the impulse response is obtained; this equation, under certain conditions, can be solved to obtain the optimum response.\(^1\)\(^2\) In the case of the more general performance criterion of expression (1), no such integral equation results; in this case the impulse response is too general a description of the system to result in a problem with a tractable solution. For this reason, we might consider a fixed form for the system but leave some of the parameters free to be varied. This approach is still applicable to nonlinear systems and is, moreover, the only approach known to the author which has met with much success in the design of optimum nonlinear systems.
We shall therefore study systems that are of certain fixed forms but which contain a number of variable parameters, $x_1, x_2, \ldots, x_k$. Thus the system output, $q(t)$, is a function of these parameters; and our problem is now to minimize the regression function

$$M(x_1, x_2, \ldots, x_k) = E \{W[d_t - q_t]\}$$

with respect to $x_1, x_2, \ldots, x_k$. Except for the case $W(e) = (e)^2$, there is no easy analytic solution to this problem. For this reason, we turn to an iterative method of solution; that is, an initial setting of the $k$ parameters is selected; and, by alternately observing the performance for a setting and then altering the setting, a sequence of adjustments of the parameters is made.

Such an iterative procedure is of doubtful value unless one can state beforehand that this sequence of parameter settings converges in some meaningful sense to the optimum parameter setting. Thus a major portion of this work is concerned with conditions that will guarantee convergence in the mean-square sense to the optimum parameter setting and with the estimation of this rate of convergence. Although the iterative procedure used here requires a fair amount of computing and yields only an approximate answer, it has two important advantages: (a) rather than being restricted to the mean-square-error criterion, it is possible for us to find the parameter setting that minimizes the mean of $W(e)$, where $W$ may be any appropriate convex function of the error; (b) the method does not require any prior knowledge or measurement of correlation functions or other statistics, instead the data are used directly.

The iterative adjustment procedure used here is essentially a gradient-seeking procedure. At each stage of the adjustment process an attempt is made to measure the gradient of $M$ for the present parameter setting. This parameter setting is then changed by an amount indicated by the gradient measurement. Such a method, when used by itself, is clearly useful only in situations for which $M$ possesses a unique minimum. For, otherwise the adjustment procedure might seek out a local minimum whose value is much greater than the true minimum of $M$. It should be noted that a simple problem such as adjusting the time constant of a filter for minimum mean-square error may not have a unique minimum, whether or not it does depends upon the spectra of the input signal and desired output signal. Most of the work here will be concerned with systems whose forms are judiciously selected so that the resulting performance function, $M$, has a unique minimum. Brief mention will be given to the multiminimum problem in section 4.6.

One may question the advantage of being able to work with any convex weighting function on the error, as opposed to being restricted to the square of the error. In particular, Sherman has shown for a particular class of processes that the optimum mean-square filter (or predictor) also minimizes the expected value of any symmetric monotone weighting function on the error. However, the only member of this class of processes for which we know how to design the optimum mean-square filter (the
conditional expectation operator) is the Gaussian process, and in a given practical
situation it might be quite difficult to verify the fact of whether or not the process in
question is Gaussian. Moreover, as the computer study discussed in section 5.1 has
brought out, the iterative method described here does not necessarily involve a greater
amount of computer time than that required to find analytically the parameter setting
that minimizes the mean-square error.

1.2 THE ITERATIVE ADJUSTMENT PROCEDURE

We now describe the iterative procedure that will be used and introduce the neces-
sary notation. It is inconvenient to work with the k-tuple of parameter settings
\([x_1, x_2, \ldots, x_k]\) as k individual quantities; hence we shall consider the parameter
setting as a k-dimensional vector \(\mathbf{x}\). The usual inner (scalar) product
\[\sum_{i=1}^{k} x_i y_i\]
is denoted by \([\mathbf{x}, \mathbf{y}]\), and the usual k-dimensional Euclidean norm (or length) of the
vector \(\mathbf{x}\),
\[\left(\sum_{i=1}^{k} x_i^2\right)^{1/2},\]
is denoted by \(\|\mathbf{x}\|\). A unit setting of the parameter \(x_i\) and zero setting of the other
k - 1 parameters will be represented by the unit vector \(\mathbf{e}_i\), \(i = 1, 2, \ldots, k\). In terms
of this notation, the problem is to find the vector (or parameter setting) that minimizes
the performance or regression function.

\[M(\mathbf{x}) = \mathbb{E}\{W[q_{x, t}]\}\]

Here, \(q_{x}(t)\) is used to denote the output of the system with the parameter setting \(x\).
The optimum parameter setting (the one that minimizes \(M\)) is denoted by \(\hat{\mathbf{q}}\).

The iterative procedure used here is essentially a gradient method; that is, at the
\(n^{th}\) step of the procedure we attempt to measure the direction in which \(M(\mathbf{x}_n)\) decreases
fastest and then change the parameter setting some amount in that direction. Such a
gradient procedure is useful only in situations in which \(M(\mathbf{x})\) has a unique minimum.
This imposes the principal restrictions on the classes of systems and weighting
functions on the error for which the method is applicable.

We first describe the iterative adjustment in terms of sampled data (or discrete
time-parameter) systems. Suppose that we have completed \(n - 1\) adjustments in the
iterative procedure; then the parameters are at the setting \(\mathbf{x}_n\). We make the \(2k\) obser-
vations \(y_{1n}, y_{2n}, \ldots, y_{2k}\), where
\[
Y_n^1 = Y_n + c_n e_1 = \frac{1}{m} \sum_{t=\tau}^{\tau+(m-1)T} W[d(t) - q_x c_n e_1(t)]
\]

\[
Y_n^2 = Y_n - c_n e_1 = \frac{1}{m} \sum_{t=\tau}^{\tau+(2m-1)T} W[d(t) - q_x c_n e_1(t)]
\]

\[
\vdots
\]

\[
Y_n^{2k} = Y_n - c_n e_k = \frac{1}{m} \sum_{t=\tau+(2k-1)T}^{\tau+2kmT} W[d(t) - q_x c_n e_k(t)].
\]

The symbol \(\tau\) denotes the time at which the \(n\)th iteration is started; \(T\) denotes the interval between sample times; and \(m\) denotes the number of samples used for a single observation. We form the \(k\)-dimensional vector \(Y_n\) whose \(i\)th component is

\[
Y_n^{2i-1} - Y_n^{2i} = Y_n + c_n e_i - Y_n - c_n e_i \quad i = 1, 2, \ldots, k.
\]

This vector \(Y_n\) is a random variable, in that different intervals of data (or different sample functions) will result in different values for the observations \(Y_n^1, i = 1, 2, \ldots, 2k\). Consider, however, the quantity obtained by taking the ensemble average of \(\frac{1}{c_n} Y_n\) over the data used for the observations \(Y_n^1, i = 1, 2, \ldots, 2k\) (here the quantity \(c_n\) is a fixed parameter). This would also be the quantity obtained if \(m\), the number of data samples used for each observation, was allowed to become infinite. This average, which will be denoted by \(M_{c_n}(x_n)\), is a difference approximation to the gradient of \(M(x)\) evaluated at the setting \(x_n\) with symmetric differences \(c_n\). The next parameter setting in the iterative procedure is, then,

\[
x_n+1 = x_n - \frac{a_n}{c_n} Y_n
\]

(2)

where \(\{a_n\}\) and \(\{c_n\}\) are sequences of positive numbers whose properties will be described later. The sense in which our procedure constitutes a gradient procedure should now be clear: \(\frac{1}{c_n} Y_n\) can be thought of as a difference approximation to the gradient of \(M(x)\) which is subject to random variation.

The adjustment procedure has been described as being carried out in real time; this would be the case if the adjustment were being carried out on an operational system or analog thereof. Now, in order to determine something about the average system performance, we need to make measurements from a large number of different intervals of the pertinent data. In this respect, we need only require that different data be used for different iterations. Thus if the system was being simulated on a digital computer, and the observations were being made with the use of a record of
data stored in the computer, the same portion of data could be used for all 2k observations needed for one iteration; the record of data would then be advanced some amount s before making the observations for the next adjustment in the sequence.

In the case of continuous time-parameter signals and systems the 2k observations used for each adjustment are made by replacing the finite sums with finite integrals. That is, assuming that the n\textsuperscript{th} adjustment again starts at time t = \tau, the 2k observations are

\[
\begin{align*}
Y_n^1 &= \frac{1}{T} \int_{\tau}^{\tau+T} W[d(t) - q_\lambda(t)] \, dt \\
Y_n^2 &= \frac{1}{T} \int_{\tau+T}^{\tau+2T} W[d(t) - q_\lambda(t)] \, dt \\
&\vdots \\
y_n^{2k} &= \frac{1}{T} \int_{\tau+(2k-1)T}^{\tau+2kT} W[d(t) - q_\lambda(t)] \, dt
\end{align*}
\]

in which

\[
X = \begin{cases} 
\frac{x_n + c_{n-1}}{2} & \tau < t < \tau + T \\
\frac{x_n - c_{n-1}}{2} & \tau + T < t < \tau + 2T \\
&\vdots \\
\frac{x_n - c_{n-k}}{2} & \tau + (2k-1) T < t < \tau + 2kT 
\end{cases}
\]

In this case, again, some time interval s is allowed to elapse before making the observations for the next adjustment.

The reader may wonder if it would not be more expedient to use only k + 1 observations for the estimate of the gradient, rather than to use 2k observations to obtain a symmetric estimate. Sacks\textsuperscript{4}, whose work is mentioned in Section II, considers this point and shows that, in the general case, when the performance function \( M(\mathbf{x}) \) is not symmetric about its minimum, the estimate of the rate of convergence is actually made faster by using 2k observations.

In Section II we are concerned with the mathematical conditions necessary for the convergence of the adjustment procedure described above. In Section III these conditions are related to physical requirements for two specific forms of systems. In Section IV we present some applications, and in Section V discuss several examples and computer studies.
II. STOCHASTIC APPROXIMATION METHODS

2.1 GENERAL BACKGROUND

The iterative procedure described in Section I is an example of a Stochastic Approximation Method. In particular, it is a multidimensional extension of the Kiefer-Wolfowitz method for locating the minimum of a one-dimensional regression function by making a sequence of observations of the random variable. Assuming certain regularity properties on the function \( M(x) \) and that the variances of the \( Y_n \) were finite, Kiefer and Wolfowitz \(^5\) proved convergence in probability of the sequence \( \{x_n - \theta\} \) to zero for suitable choice of the sequences \( \{a_n\} \) and \( \{c_n\} \). Blum \(^6\) showed convergence with probability 1 for a multidimensional case. Sacks \(^4\) made extensive estimates of the rates of convergence for both the single and multidimensional cases under a variety of restrictions on the function \( M(x) \). Dupac \(^7\) made estimates of the rate of convergence for the one-dimensional case for a variety of choices of the sequences \( \{a_n\} \) and \( \{c_n\} \).

Now let us consider the distribution of \( x_n \). We note that our basic sample space (or probability space) is the space of sequences of data \( \{v(t)\} \) and \( \{d(t)\} \), \( t = 1, 2, \ldots, T \), where \( T \) is the time at which the \((n-1)\)th iteration is completed. The parameter setting \( x_n \) depends upon both \( x_1 \) and on the data used, hence the probability distribution for \( x_n \) depends on \( x_1 \). Thus in referring to \( x_n \) we are actually speaking of a family of random variables, \( x_n(x_1) \), indexed by the initial parameter setting \( x_1 \). Now, in arriving at the quantity

\[
M_{c_n}(x_n) = \frac{1}{c_n} \text{E}\{Y_n\} \tag{3}
\]

the averaging is over all pairs of sample sequences \( v(t) \) and \( d(t) \). In the expression

\[
F(x_1, x_n) = \text{E}\{Y_n | x_n(x_1)\} \quad \tag{4}
\]

the averaging is only over those pairs of sample sequences that could have caused a transition from the parameter setting \( x_1 \) to the parameter setting \( x_n \). In both cases, \( x_n \) itself is considered as a parameter, although in the second case our conditioning depends on \( x_1 \) and \( x_n \). In the work cited above, \(^4-7\) it was assumed that

\[
M_{c_n}(x_n) = \frac{1}{c_n} F(x_1, x_n). \tag{5}
\]

This will hold true, for example, when the data used for one iteration are independent of the data used for each preceding iteration. The condition expressed by Eq. 5 is too restrictive for the problem at hand; hence we present an extension of the work of Dupac.
2.2 A STOCHASTIC APPROXIMATION THEOREM

We state some assumptions on the regression function $M(x)$ and on the processes $Y_n$ and $x_n(x_1)$ which allow us to make statements about how the sequence $\{\|x_n - \theta\|^2\}$ converges to zero in the mean. In Section III we shall translate the assumptions made here into conditions on the physical design procedure; the implication of the statements made here to the convergence of the physical design procedure will then be more or less obvious.

We make the following assumptions:

(a) $M(x)$ is decreasing in the direction $\theta - x$ for all $x$ of interest.

(b) The gradient of $M$ evaluated at $x$ [written $(\text{grad } M)(x)$] is suitably bounded in magnitude.

We must also make some assumption on how rapidly the term

$$F_n(x_1) = \left| E \left\{ \left[ x_n - \theta, Y_n(x_n) - c_n M_n(x_n) \right] \right\} \right|$$

approaches zero; that is, roughly, how much the process depends on the information about its past that is contained in $x_n - \theta$. We shall assume that

$$F_n(x_1) \leq \frac{a_n}{c_n} \quad S_1 < \infty$$

in which $S_1$ is independent of $x_1$, the initial parameter setting. We shall show in Section III that inequality (7) corresponds to an extremely broad class of physical processes.

In order to make these ideas precise we make the following assumptions:

(i) $E \{\|Y_n\|^2|x_n(x_1)\} \leq E\{\|Y_n\|^2|x_n(x_1)\} + S \quad S < \infty$;

(ii) (a) $K_0 \|x - \theta\|^2 \leq \left[ (\text{grad } M)(x), +(x - \theta) \right]$,

(b) $\| (\text{grad } M)(x) \| \leq K_1 \|x - \theta\| \quad K_1 > K_0 > 0$;

(iii) (a) $F_n(x_1) = \left| E \left\{ \left[ x_n - \theta, Y_n(x_n) - c_n M_n(x_n) \right] \right\} \right| \leq \frac{a_n}{c_n^{1/2}}$

(b) $\left| E \left\{ \|E \{Y_n|x_n(x_1)\}\|^2 - c_n^2 \|M_n(x_n)\|^2 \right\} \right| \leq S_2$,

in which $S_1 < \infty$, $S_2 < \infty$, and $S_1$ and $S_2$ are independent of $x_1$ for all $x_1 \in X$ (see (v)).

(iv) $\{a_n\}$ and $\{c_n\}$ are sequences of positive numbers satisfying

$$\sum_{n=1}^{\infty} a_n = \infty, \quad \sum_{n=1}^{\infty} a_n^2 < \infty, \quad \sum_{n=1}^{\infty} a_n c_n < \infty, \quad \sum_{n=1}^{\infty} \left( \frac{a_n}{c_n} \right)^2 < \infty, \quad \sum_{n=1}^{\infty} \frac{a_n}{c_n} \frac{a_n}{c_n^{1/2}} < \infty.$$
in which \( a_{n/2} \) and \( c_{n/2} \) are suitably interpolated for \( n \) odd.

(v) \( x \) is constrained to a bounded, closed, convex set \( X \), but is free to be varied inside \( X \). (A set \( X \) is convex if \( x_1 \in X, x_2 \in X \) implies \( ax_1 + (1-a)x_2 \in X \) for \( 0 \leq a \leq 1 \).

Assumptions (i), (ii), and (iii) need hold only for \( x \in X \). It is assumed that \( X \) is chosen sufficiently large that \( 0 \in X \).

We now make the following statements.

STATEMENT 1: Assumptions (i)-(v) imply the convergence of \( \| x_n(x_1)-\theta \| \) to zero in the mean-square sense for all \( x_1 \in X \). That is,

\[
\lim_{n \to \infty} E\{\| x_n(x_1)-\theta \|^2 \} = 0 \quad \text{for all} \quad x_1 \in X.
\]

We now set \( a_n = a/n^\alpha, c_n = c/n^\gamma \), and thus to satisfy assumption (iv) we require \( 3/4 < \alpha \leq 1, 1 - \alpha < \gamma < \alpha - 1/2 \). We also require \( a > 1/K_0 \) if \( a = 1 \).

STATEMENT 2: Assumptions (i)-(v) and the choice \( a = 1, \gamma = 1/4 \) imply that for all \( x_1 \in X \)

\[
E\{\| x_n(x_1)-\theta \|^2 \} = O(1/n^{1/2})
\]

\( f(n) = O(g(n)) \) means \( \lim_{n \to \infty} (|f(n)|/|g(n)|) < +\infty \).

Furthermore, this choice is optimum in the sense that no other choice guarantees faster convergence for all \( Y(x) \) satisfying assumptions (i)-(v), that is, for \( a \neq 1, \gamma \neq 1/4 \) there exists a \( Y(x) \) satisfying (i)-(v) so that

\[
E\{\| x_n(x_1)-\theta \|^2 \} = O(1/n^{1/2-\epsilon}) \quad \text{for some} \quad \epsilon > 0.
\]

STATEMENT 3: Under the added restriction

\[
\left| \frac{\partial^3 M(x)}{\partial x_i^3} \right| \leq Q < \infty \quad i = 1, 2, \ldots, k \text{ and } x \in X,
\]

the choice \( a = 1, \gamma = 1/6 \) implies that for all \( x_1 \in X \)

\[
E\{\| x_n(x_1)-\theta \|^2 \} = O(1/n^{2/3})
\]

and this choice is again optimum in the sense of Statement 2.

The proofs of Statements 1-3, which are given in Appendix A, follow the same general lines as those of Dupac. We only comment on the intuitive reasons for the different assumptions. The first assumption requires that the variance (as conditioned by \( x_n(x_1) \)) of the norm of \( Y_n \) must be finite. If one wishes to think of \( Y_n \) as a difference measurement of the gradient of \( M(x) \) with "noise" superimposed upon it, then assumption (i) is a restriction that the "average power of the noise" be finite. Clearly, some such restriction is necessary. Assumption (ii-a) provides that the slope of \( M(x) \)
in the direction of the optimum parameter setting never be too small; assumption (ii-b) requires that this slope never be too large in any direction. The first condition prevents the adjustment process from "sticking," while the second prohibits sustained oscillations of the adjustment. In considering assumption (iii), note that

\[ x_n(x_j) = \sum_{j=1}^{n-1} a_j \frac{1}{c_j} Y_j(x_j) + x_j \]  

(8)

and thus assumption (iii) is a restriction on how the process \( X_n \) may depend on its past. There is little that can be said intuitively concerning assumption (iv), except that we require

\[ \sum_{n=1}^{\infty} a_n = \infty \]

in order that the process be capable of reaching the optimum setting, no matter how far away from the optimum setting it may be on the \( n \)th iteration. Concerning assumption (v), the restriction that \( X \) be bounded is not necessary at this point, but will be required when we translate assumption (iii) to conditions on the physical situation. It is necessary that \( X \) be convex in order that there exist a "downhill path" from all \( x \in X \) to \( \theta \). If \( \theta \) does not lie in \( X \), then the process will converge to that setting in \( X \), say \( \theta' \), for which \( M \) is a minimum, provided that assumption (ii) still holds with \( \theta \) replaced by \( \theta' \).

2.3 SELECTION OF THE PARAMETERS OF THE ADJUSTMENT PROCEDURE

In the iterative adjustment procedure described in section 1.2 there are several quantities that may be varied: \( s \), the time interval that separates data used for succeeding iterations; \( m \), the number of data samples used for each measurement of \( Y_n \); and the coefficients \( a \) and \( c \) when \( a_n = a/n^\alpha \), \( c_n = c/n^\gamma \). The statements of the preceding section estimate the convergence of the parameter setting as

\[ b_n = E\{\|x_n(x_j) - \theta\|^2\} \leq C \frac{1}{n^\delta}. \]  

(9)

In expression (9) the rate of convergence (\( \delta \), the exponent of \( n \)) is not affected by the quantities \( s \), \( m \), \( a \), and \( c \). The coefficient \( C \) does depend on these quantities, however, and a brief investigation of this dependence is in order.

In the general case there is little that can be said because there is no reasonable assumption to make concerning the dependence of the coefficient \( S_j \) upon \( s \). For this reason, we consider the case in which the data used for each measurement of \( Y_n^i \), \( i = 1, 2, \ldots, 2k \), \( n = 1, 2, \ldots \), are independent of the data used for all other measurements. We denote by \( T \) the time taken to make a measurement of \( Y_n^i \), using one data sample for each of the 2k measurements \( Y_n^i \), \( i = 1, 2, \ldots, 2k \). Then, from
expression (A-21), we have (noting that $S_1 = S_2 = 0$ for this case)

$$F n_c^2 + D n_a^2 S(m) \quad (m)^{1/2} \quad \frac{1}{E a - 1/2} \quad \left(\frac{t}{T}\right)^{1/2}$$

in which $t = nT$ is the time required to complete $n$ iterations. Now if the average of $m$ independent observations is used as a measurement of $Y_i$, $i = 1, 2, \ldots, 2k$, then

$$S(m) = S_0/m.$$ 

If we substitute this expression for $S(m)$ in Eq.(10) and solve for the values of $a$, $c$, and $m$ which minimize the coefficient of $(t/T)^{-1/2}$, we find that one of the three relationships is redundant; and that the minimum is determined by

$$a = \frac{1}{2E} + \left[\frac{1}{(2E)^2 + \frac{1}{2E}}\right]^{1/2}$$

and

$$c^4 m F = DS_0.$$ (12)

Of the two parameters, $c$ and $m$, one may thus be fixed and the other determined by (12); the optimum value of $a$ is always determined by (11). Since the quantities $E$, $F$, and $D$ are unknown, the optimum values of $a$ and $c$ may not be determined directly. It is, however, possible to effectively estimate the values of $E$, $F$, and $D$ during the course of the adjustment procedure and change the values of $a$ and $c$ to conform to the values given by Eqs. 11 and 12.8
III. THE DESIGN OF PHYSICAL SYSTEMS

3.1 OPTIMIZATION OF A NONLINEAR FILTER

We shall now consider several system configurations whose performance can be optimized by the adjustment procedure described in Section I. The main limitation on the system configurations that we can treat is that they must yield a regression surface or performance function, \( M(x) \), which has a unique minimum satisfying assumption (ii-a). The primary result will be to establish that assumptions (i-v) of section 2.2 are satisfied for the given systems.

The first case that we consider is the design of a filter (or predictor or model) of the form shown in Fig. 1. The first reason for selecting this form is that it is general in the sense that any continuous nonlinear operator whose output depends only remotely on the distant past can be approximated arbitrarily closely by the given form if a sufficiently large number of terms is used. A second reason for selecting this form is that the output of the filter depends only on the present setting of the parameters \( x_1, x_2, \ldots, x_k \) and not on their past history. This makes the adjustment procedure much easier to analyze. It should be pointed out that if a good filter is already available, it may be placed in parallel with the filter of Fig. 1, and its output multiplied by a

\[
S_i(t) = \sum_{\tau=0}^{\infty} h_i(\tau) V(t-\tau) \quad \text{(DISCRETE TIME PARAMETER CASE)}
\]

\[
S_i(t) = \int_{0}^{\infty} h(\tau) V(t-\tau) d\tau \quad \text{(CONTINUOUS TIME PARAMETER CASE)}
\]

Fig. 1. Form of the filter to be designed.
gain $x_{k+1}$ and added at the indicated summing point. For the parameter setting $x_1 = x_2 = \ldots = x_k = 0$, $x_{k+1} = 1$, the over-all filter reduces to this original filter; and hence the performance of the over-all filter must be at least as good as that of this original filter.

We desire to use the procedure described in Section I to adjust the gain coefficients $x_1, x_2, \ldots, x_k$ of the filter in Fig. 1 so that the performance function $M(x) = E(W[d_t - q_{x,t}])$ is minimized. Subject to the restrictions stated below, $W$ should be chosen to be appropriate for the purpose that the filter is to accomplish. The following restrictions are sufficient to guarantee that the statements of section 2.2 apply to the adjustment procedure:

(a) $v(t)$ and $d(t)$ are the outputs of stationary ergodic sources and are uniformly bounded in absolute magnitude for all $t$ with probability 1.

(b) $\sum_{t=0}^{\infty} |h_i(t)| < \infty; \ i = 1, 2, \ldots, k$. The $f_i(t) = f_i[s_1(t), s_2(t), \ldots s_j(t)]$ are continuous in $s_1, s_2, \ldots, s_j$ for $i = 1, 2, \ldots k$.

(c) $\mathbb{P}\left(\sum_{i=1}^{k} (x_i - \theta_i) f_i(t) \geq D\|x - \theta\|\right) \geq \epsilon$ for $x \in X$ and $D > 0$.

Or, in terms of one sample function, there exists an $N_0$ with the property that for $N > N_0$, $\frac{n}{2N + 1} \geq \epsilon > 0$, where $n$ is the number of occurrences, $-N \leq t \leq N$, of $|\sum_{i=1}^{k} (x_i - \theta_i) f_i(t)| \geq D\|x - \theta\|$ for $x \in X$.

(d) $W(e)$ is a polynomial in $e$ of finite degree. ($W(e)$ will thus possess bounded continuous derivatives.)

(e) $W(e)$ is "strictly convex"; that is, there exists an $E > 0$ with the property that $W(\alpha a + (1-\alpha)b) \leq \alpha W(a) + (1-\alpha) W(b) - E\|a-b\|^2$ for $0 \leq \alpha \leq 1/2$.

(f) Assumption (v) is required to hold directly; that is, our parameter adjustments are confined to a bounded, closed, convex set.

(g) Consider the random processes, $d(t)$ and $f_i(t)$, $i = 1, 2, \ldots, k$; $t$ may take on either discrete or continuous values. Let $F_1[f_1(\tau), \ldots, f_k(\tau), d(\tau)]$ be any bounded continuous functional of $d(\tau)$ and the $f_i(\tau), \tau \leq t$. Let $F_2[d(\tau), f_1(\tau), \ldots, f_k(\tau)]$ be any bounded continuous functional on $d(\tau)$ and the $f_i(\tau), \tau + a \leq t + a + T$, $T$ fixed. Then we make the following requirement on the correlation between $F_1$ and $F_2$ for large $a$

$$\left| R_{F_1,F_2} \right| = \left| E\{(F_1 - \overline{F_1})(F_2 - \overline{F_2})\} \right| \leq \frac{K}{a^2} \sigma_{F_1} \sigma_{F_2}$$

in which $K$ is independent of $F_1$ and $F_2$. If $d(t)$ and the $f_i(t)$ are discrete time-parameter processes, then $F_1$ and $F_2$ are ordinary functions instead of functionals.
To gain an understanding of restriction (g) and appreciate its generality, the reader may wish to consult Fig. 2. The quantity \( F_2 \) is obtained by some operation on the data in the interval \( t + a \leq \tau \leq t + a + T \). Now let us try to predict \( F_2 \) by using some functional, \( F_1 \), on the data for \( \tau \leq t \). Note that there is a spread of \( a \) seconds between the data used for \( F_2 \) and the data used for \( F_1 \). Now the best (mean-square) linear prediction of \( F_2 - \overline{F}_2 \) given \( F_1 \), is

\[
\frac{F_2 - \overline{F}_2}{\sigma^2_{F_2}} = \frac{\frac{R_{F_1 F_2}}{\sigma^2_{F_1}}}{(F_1 - \overline{F}_1)}
\]

and the mean-square error in this prediction is
Thus our requirement on $R_{F_1F_2}$ is only a stipulation that the data $d(t)$ and $f_i(t)$, $i = 1, 2, \ldots, k$, be such that our ability to predict some function of the future data diminish as the prediction time becomes large; in particular, the error must approach its asymptotic value at a rate of $1/a^4$ as $a$ becomes large. Note that if $d(\tau)$ and the $f_i(\tau)$, $t + a \leq \tau \leq t + a + \tau$, are statistically independent of $d(t)$ and the $f_i(t)$, $\tau \leq t$ for $a \leq a_0$, then $R_{F_1F_2} \equiv 0$ for $a \geq a_0$.

Restrictions (a)-(c) provide no serious limitation on our physical situation. Restriction (a) will surely be satisfied, since the output of any physical source is always uniformly bounded. Restriction (b) requires only that the memory elements $h_1, h_2, \ldots, h_j$ be stable and that the $f_i$ be continuous, as all physical transducers are. Restriction (c) only requires that all the $h$ and all the $f$ differ from one another in the prescribed sense. An alternate statement of restriction (c) could be that the $f_i$ are all linearly independent random variables. A proof of this statement is given in Appendix A.5. Restrictions (a) through (c) are expressed in terms of the discrete time-parameter case; the alternate statement of these restrictions for the continuous case is obvious.

Restriction (d) is no real restriction in addition to (e), since any convex function is continuous and hence may be approximated arbitrarily closely of the finite domain of $W$ by a polynomial. Note that restriction (e) is stronger than the usual definition of strict convexity; it in fact requires the polynomial $W$ to contain a non-zero quadratic term. As discussed in Appendix A.5, this may be relaxed to the usual concept of strict convexity by slightly strengthening restriction (c). Restriction (e) imposes the only serious limitation on our method. Restriction (f) imposes no serious limitation on the allowable parameter settings. Restriction (g) is quite liberal for a nonperiodic random process. It should be added that the rate in restriction (g) could be reduced from $(1/\tau^2)$ to $(1/\tau^{1+\epsilon})$, $\epsilon > 0$, although this would affect the estimate of the rate of convergence as given by Statements 2 and 3. For periodic processes restriction (g) will not, in general, be satisfied. This checks with our intuition; if the frequency of the process were in synchronism with the adjustment procedure, we would expect trouble.

We shall now prove that restrictions (a)-(g) are sufficient to guarantee that the assumptions of Section II are satisfied. The proofs are carried through mainly for the sampled-data case, with comments made at the end to indicate the extension to the continuous time-parameter case.

To show that assumptions (i) and (ii) follow from restrictions (a)-(e), we use the ergodicity of the sources to write

$$M(x) = E \{ W[d_t - q_t] \} = \lim_{N \to \infty} \frac{1}{2N+1} \sum_{t=-N}^{N} W \left[ d(t) - \sum_{i=1}^{k} x_1 f_i(t) \right]$$  \hspace{1cm} (13)
Assumption (i) now follows immediately from restriction (a) on the uniform boundedness of the sources, and from restriction (b) on the form of the filter.

To show that the upper inequality in assumption (ii) is satisfied, we use a Taylor's expansion about $x = \theta$ to write

$$
\frac{\partial^2 M(x)}{\partial x_i \partial x_j} = 0 + \sum_{i=1}^{k} (x_i - \theta_i) \frac{\partial^2 M(x)}{\partial x_i \partial x_j} \frac{x = \theta + \epsilon_i}{x_i - \theta_i}.
$$

where $0 < \tau_i < 1$ for an arbitrary $j$, with $j = 1, 2, \ldots k$. Thus, the upper inequality in assumption (ii) follows with

$$
k_j = \sup_{i, j=1, 2, \ldots k} \frac{\partial^2 M(x)}{\partial x_i \partial x_j} \left|_{x \in X} \right|
$$

if all the $\frac{\partial^2 M(x)}{\partial x_i \partial x_j}$, $i, j = 1, 2, \ldots k$ are bounded for all $x \in X$. To show this, we consider

$$
S_N(x) = \frac{1}{2N + 1} \sum_{t=-N}^{N} W(d(t) - \sum_{i=1}^{k} x_i f_i(t))
$$

and

$$
S_{ij}^N(x) = \frac{1}{2N + 1} \sum_{t=-N}^{N} W^n(d(t) - \sum_{i=1}^{k} x_i f_i(t)) f_i(t) f_j(t).
$$

The continuity of $W^n$ and the uniform boundedness of the $f_i(t)$, $i = 1, 2, \ldots k$ and $d(t)$ imply the equicontinuity of $S_{ij}^N(x)$ for $x \in X$. The Arzela-Ascoli theorem then guarantees the existence of a uniformly convergent subsequence $N_{m}$, and for this subsequence

$$
\frac{\partial^2}{\partial x_i \partial x_j} \lim_{m \to \infty} S_N(x) = \lim_{m \to \infty} \frac{\partial^2}{\partial x_i \partial x_j} S_N(x) = \lim_{m \to \infty} S_{ij}^N(x)
$$

for all $x \in X$. However, by our assumption of ergodic sources, $\lim_{N \to \infty} S_{ij}^N(x)$ is unique with probability 1, and hence

$$
\frac{\partial^2}{\partial x_i \partial x_j} M(x) = \lim_{N \to \infty} \frac{1}{2N + 1} \sum_{t=-N}^{N} W^n(d(t) - \sum_{i=1}^{k} x_i f_i(t)) f_i(t) f_j(t)
$$

with probability 1 for all $x \in X$. But the right-hand side of Eq. 18 is bounded by restrictions (a) and (d), and we have established the upper inequality in assumption (ii).

We now turn to the lower inequality in assumption (ii). Using Eq. 13, restrictions (c), (e), and the convexity of $X$, we have
\[ M[ (1-a)x+\alpha \theta ] \leq (1-a) M(x) + a M(\theta) - \varepsilon D^2 E_a \| x-\theta \|^2 \quad \text{for } 0 \leq a \leq 1/2 \]  

or

\[ \frac{M[ x+\alpha (\theta-x) ] - M(x)}{\alpha \| x-\theta \|} \geq \frac{M(x) - M(\theta)}{\| x-\theta \|} + \varepsilon D^2 E \| x-\theta \| \quad \text{for } 0 \leq a \leq 1/2. \]

The right-hand side of Eq. 18 is independent of \( a \); hence, taking the limit of the left-hand side as \( a \) approaches zero, we have

\[ \left[ \frac{(\text{grad } M)(x)}{\| x-\theta \|} \right] \geq \varepsilon D^2 E \| x-\theta \| \]

as desired.

If we desire to use the results of Statement 3, we need only use in addition the fact that \( W''(e) \) is continuous for all values of the argument which occur under restrictions (a) and (b), and the assumption that \( x \in X \). The boundedness of

\[ \frac{\partial^3 M(x)}{\partial x_i^3} \quad i = 1, 2, \ldots, k \]

under this added restriction is shown in the same manner as was the boundedness of \( \frac{\partial^2 M(x)}{\partial x_i \partial x_j} \).

In the continuous time-parameter case the performance or regression function

\[ M(x) = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} W[d(t)-q_x(t)] \, dt \]

can be written

\[ M(x) = \lim_{N \to \infty} \frac{1}{2N+1} \sum_{m=-N}^{N} \int_{mT}^{(m+1)T} W[d(t)-q_x(t)] \, dt \]

and these proofs are directly applicable to this case.

Assumption (iii-b) follows immediately from restrictions (a) and (b).

We shall now show that restrictions (a), (f) and (g) imply assumption (iii-a). We wish to bound

\[ F_n(x_i) = \left| E \left\{ \left[ x_n - \theta, Y_n(x_n) - c_n M_i (x_n) \right] \right\} \right|. \]  

The inner product in (20) consists of \( k \) terms. Consider the magnitude of the \( i \)th such term

\[ F_{i,n} = \left| E \left\{ \left[ x_i - \theta, Y_i(x_n) - c_n M_i (x_n) \right] \right\} \right|. \]

If we can show that each such term satisfies our bound, then \( F_n(x_i) \) must also. Now, \( Y_i(x) \) consists of the difference between a measurement, \( Y_{x_n}^{2i-1} \), with \( x = x_n + c_n e_i \) and
another measurement, \( Y_{n}^{2i} \), with \( x = x_{n} - c_{n} e_{1} \). We shall consider only the difference between \( Y_{n}^{2i-1} \) and \( M(x_{n} + c_{n} e_{1}) \), and neglect the terms resulting from the \( c_{n} \)'s (which will be smaller than the terms considered). We also assume that only one data sample is used for the measurement. These assumptions will not restrict the generality of the proof, but will help to simplify the cumbersome notation required. We thus wish to estimate

\[
F_{i,n}' = E \left( \frac{1}{2} \left[ W \left( d_{n} - \sum_{i=1}^{k} x_{i,n} f_{i,n} \right) - E \left[ W \left( d_{n} - \sum_{i=1}^{k} x_{i,n} f_{i,n} \right) \right] \right] \right)
\]

in which the expectation inside the brackets is taken with \( x_{n} \) as a parameter. Now, by restriction (d),

\[
W d_{n} - \sum_{i=1}^{k} x_{i,n} f_{i,n} = \sum_{j=0}^{N} \sum_{m=0}^{j} B_{jm} \left( d_{n} - \sum_{i=1}^{k} x_{i,n} f_{i,n} \right)^{m}
\]

\[
= \sum_{j=0}^{N} \sum_{m=0}^{j} B_{jm} \left( \sum_{i=1}^{N-j} x_{i,n} f_{i,n} \right)^{m}.
\]

There are a total of \( \sum_{j=0}^{N} \sum_{m=0}^{j} (k)^{m} \) terms in this sum, and hence a total of \( \sum_{j=0}^{N} \sum_{m=0}^{j} k^{m} \) terms in \( F_{i,n}' \); a typical one of these is

\[
F_{i,n}^{m} = C_{m} E \left\{ (x_{i,n} - 0) \prod_{j=1}^{k} \left( f_{j,n} \right)^{m_{j}} \left[ (d_{n})^{N-m} \prod_{j=1}^{k} (f_{j,n})^{m_{j}} - (d_{n})^{N-m} \prod_{j=1}^{k} (f_{j,n})^{m_{j}} \right] \right\}
\]

in which \( \sum_{j=1}^{k} m_{j} = m \). Thus if we can show that this term is bounded, we shall have shown that \( F_{i,n} \) is bounded. We consider only the first of the two terms in \( F_{i,n}^{m} \) (the \( x_{i,n} \) term); the second (the \( \theta_{i} \) term) can be treated in a similar manner. For this first term we have

\[
F_{i,n}^{m,1} = C_{m} x_{i,n} \prod_{j=1}^{k} (f_{j,n})^{m_{j}} \left[ (d_{n})^{N-m} \prod_{j=1}^{k} (f_{j,n})^{m_{j}} - (d_{n})^{N-m} \prod_{j=1}^{k} (f_{j,n})^{m_{j}} \right]
\]

For simplicity, let \( \rho_{n} \) denote

\[
\left[ (d_{n})^{N-m} \prod_{j=1}^{k} (f_{j,n})^{m_{j}} \right] - \left[ (d_{n})^{N-m} \prod_{j=1}^{k} (f_{j,n})^{m_{j}} \right].
\]

Also, recall that
Thus (25) can be rewritten as

\[ F_{m,n} = C_m \sum_{q=0}^{n-1} \left[ Y_{i, q} \prod_{j=1}^{k} \left( x_{j, q_j+1} \right)^m \right] \rho_n \]

or

\[ F_{m,n} = C_m \sum_{q=0}^{n-1} \left[ Y_{i, q} \prod_{j=1}^{k} \left( x_{j, q_j+1} \right)^m \right] \rho_n \]

Now, recalling that \( \frac{a}{c} Y_{i, o} = x_{i, o} \), we see that the zeroth term in each of the \( m+1 \) sums
is of the form

\[
\begin{align*}
&x_i \cdot \frac{k}{m} \sum_{j=1}^{m_k} (x_j, 1) \nu_j = x_i \cdot \frac{k}{m} \sum_{j=1}^{m_k} (x_j, 1) \nu_j = 0 \\
&\text{for all } x \in X.
\end{align*}
\]

Thus by using restriction \((g)\), each of the \(m+1\) sums in \((26)\) can be bounded in magnitude by

\[
\Sigma_1 = K_1 \sum_{j=1}^{n-1} \frac{a_j}{c_j \sup |x_j|} \frac{1}{(n-j)^2} \quad \text{for all } x \in X
\]

in which

\[
0 \leq K_1 \leq |Y| \sup |x_j| \sup |x_j|^m |c_m|.
\]

But for any sequence \(a_j/c_j\) for which \(a_j/c_j \left( \frac{1}{(n-j)^2} \right)\) is monotonically decreasing from 1 to some point \(i\) and monotonically increasing from \(i\) to \(n-1\) (which includes \(a_n = a/n^a\), \(c_n = c/n^a\), it is shown in Appendix A that the sum, \(\Sigma_1\), can be bounded by \(K_2 \left( \frac{a_n/2}{c_n/2} \right) \) (in which \(a_n/2\) is suitably interpolated for \(n\) odd). Thus we have split \(F_n(x_1)\) into a finite number of terms and shown that each term could be bounded in magnitude by

\[
K_2 \left( \frac{a_n/2}{c_n/2} \right) \quad \text{all } x \in X \quad K_2 < \infty
\]

and, since

\[
\left| \Sigma \cdot x_1 \right| \leq \Sigma \left| \cdot x_1 \right|
\]

we have

\[
F_n(x_1) \leq K_3 \left( \frac{a_n/2}{c_n/2} \right) \quad \text{all } x \in X \quad K_3 < \infty
\]

as desired.

Note that the derivation above remains valid whether \((d_i)_{i=1}^{N-m}\) and \((f_i, n)_m\), \(i = 1, 2, \ldots, k\), represent discrete time samples or finite time integrals of continuous processes, as they would when \(d(t)\) and \(v(t)\) are continuous time-parameter processes.

### 3.2 NONLINEAR COMPENSATOR FOLLOWED BY A FIXED LINEAR SYSTEM

We now consider the more general situation shown in Fig. 3. We are given some fixed linear system whose impulse response is \(h(r)\). By constraint of the design
problem, this system (which may represent a loud-speaker or circuit or other device over which the designer has no control) must terminate the over-all system. We wish to precede this fixed linear system with a nonlinear compensator in order that the over-all cascade system be optimum. The nonlinear compensator to be used will be of the form of the filter of Fig. 1, and restrictions (a)-(g) of section 3.1 will again be imposed. We now wish to establish the validity of assumptions (i)-(iii) of Section II.

Assumptions (i), (ii-b) and (iii-b) follow directly from replacing the functions $f_i(t)$, $i = 1, 2, \ldots, k$ by the functions

$$g_i(t) = \sum_{\tau=0}^{\infty} h(\tau) f_i(t-\tau) \quad i = 1, 2, \ldots, k,$$

provided that $\sum_{j=0}^{\infty} |h(j)| < \infty$. Assumption (ii-a) follows if we replace the functions $f_i(t)$ in restriction (c) by the functions $g_i(t)$, $i = 1, 2, \ldots, k$.

Establishing assumption (iii-a') is another matter, however. Since the linear system possesses memory, the output of the over-all system, $q_{x_\Delta}(t)$, will, in general, depend not only on the present value of the parameter setting but also on the past history of the parameter settings. This problem would not arise if the linear system had a finite memory of $T$ seconds, and $T$ seconds were allowed to elapse between the end of one observation, $y_n^i$, and the beginning of the next observation, $y_{n+1}^i$. This, however, is too restrictive a condition to impose on most systems.

Let us consider the source of the difficulty for a single parameter. We need to measure the quantity

$$M(x_n-c_n) = E\{W[ d_t^{-h_2}(x_n-c_n) f_t^{-h_1}(x_n-c_n) f_{t-1}^{-h_2}(x_n-c_n) f_{t-2}^{-h_3}(x_n-c_n) f_{t-3}^{-\ldots}]\},$$

but we are observing
\[
Y_{x_n-c_n} = W\{d(t)-h_o(x_n-c_n)f(t)-h_1(x_n+c_n)f(t-1)
\]
\[
- h_2(x_{n-1}-c_{n-1})f(t-2)-h_3(x_{n-1}+c_{n-1})f(t-3)-\ldots\}
\]

Now \( |x_n-x_{n-1}| \leq K\alpha/c_n \) approaches zero fast enough that these terms can be taken care of by requiring that the impulse response fall off sufficiently fast. The difference introduced by having \( x_n+c_n \) in place of \( x_n-c_n \), however, only approaches zero as fast as \( c_n \); and hence will prohibit us from proving convergence as before. These troublesome terms may be eliminated by using the adjustment scheme shown in Fig. 4 for the two-parameter case. The systems \( h^p(\tau) \) are used to simulate \( h(\tau) \). They need to resemble \( h(\tau) \), in the sense that

\[
E\left( \frac{1}{k} \left( g_i^*, m_i \right) \right) = E\left( \frac{1}{k} \left( g_i, m_i \right) \right)
\]

in which \( N \) denotes the degree of \( W \), and

\[
g_i^*(t) = \sum_{\tau=0}^{\infty} h^*(\tau) f_i(t-\tau)
\]

\[
g_i(t) = \sum_{\tau=0}^{\infty} h(\tau) f_i(t-\tau)
\]

To make the various observations used in estimating the gradient of \( M \), we no longer directly perturb the coefficients \( x_1 \) and \( x_2 \), but rather vary the gains of the amplifiers shown as 0 and \( \pm c_n \). For example, to measure \( Y_{x_n+c_n} \), we set the top amplifier at \( +c_n \) and the bottom one at 0.

As mentioned above, we need to make a restriction on the rate at which the impulse response \( h(\tau) \) falls off. In particular, we require

![Fig. 4. Arrangement for adjusting system parameters.](image-url)
We now establish assumption (iii-a). We shall restrict ourselves to considering sequences of the form
\[ a_n = a/n^\alpha \quad c_n = c/n^\beta. \]

For notational convenience, we consider only the one-dimensional case \((x = x, \text{ a single parameter})\); the method used holds true for the multidimensional case. We assume, for simplicity, that only one sample is used in making the measurement \(Y_1^n\) and the measurement \(Y_2^n\). Of the two terms \((x_n+\theta)[Y_1^n-M(x_n+c_n)]\) and \((x_n+\theta)[Y_2^n-M(x_n-c_n)]\) we consider only the first. We shall also ignore the terms that are due to \(c_n\) as being smaller than the terms considered. These assumptions do not restrict the generality of our proof, but serve to simplify the details. We thus wish to bound

\[
|F_1^n| = |(x_n-\theta) \left[ \sum_{j=0}^{n-1} h_j n^{-j} f_{n-j} \right] - E \left[ \sum_{j=0}^{n-1} h_j n^{-j} f_{n-j} \right]| \quad (30)
\]

in which \(x_n, x_{n-1}, \ldots, x_1\) appear as parameters in the expectation shown inside the brackets. We consider the first term on the right-hand side of inequality (30), first. Let

\[
a = d_n - \sum_{j=0}^{n-1} h_j n^{-j} f_{n-j} \quad \text{and} \quad b = d_n - \sum_{j=0}^{n-1} h_j n^{-j} f_{n-j},
\]

then

\[
|a - b| \leq |f| \sup_{j=0}^{n-1} |h_j| |x_n - x_{n-j}| \sup_{j=0}^{n-1} |x_{n-j}|^{\beta} \quad (31)
\]

Now, by virtue of the fact that signals \(f\) and \(d\) are bounded, \(Y_n\) is bounded in magnitude; thus

\[
|x_n - x_{n-j}| \sup \leq |y| \sup_{i=n-j}^{n-1} \frac{a_i}{c_i} \leq K_2 \sum_{i=n-j}^{n-1} \frac{a_i}{c_i}
\]

with \(K_2 < \infty\), and \(K_2\) independent of \(x_1\).

(h) \(|h(\tau)| \leq K/\tau^3 \quad K < \infty. \quad \tau > 1.\)
Thus for $a_n/c_n = a/c \cdot 1/n^{a-\gamma}$,

$$ |x_n - x_{n-j}|_{\text{sup}} \leq K_2 \frac{a}{c} \int_{n-j-1}^{n-1} \frac{1}{t^{a-\gamma}} \, dt \leq \frac{K_2 a}{\delta c} \left[(n-1)^{a-\gamma} - (n-j-1)^{a-\gamma}\right]^{(n-1)} \tag{32}$$

in which $\delta = 1 - (a-\gamma)$. Using restriction (h) and inequality (32), we show in Appendix A.4 that we can give the following bound for the sum appearing in (31):

$$ \sum_{j=0}^{n-1} |h_j||x_n - x_{n-j}|_{\text{sup}} \leq K_3 \frac{1}{n^{a-\gamma}}. $$

Thus (31) becomes

$$ \sum_{j=0}^{n-1} |a - b|_{\text{sup}} \leq K_3' n^{a-\gamma} \quad K_3' < \infty, \quad K_3' \text{ independent of } x_1. \tag{33} $$

Now, $W$ is a polynomial whose argument takes on a bounded range, hence over this range

$$ \left| \frac{dW}{de} \right|_{\text{sup}} \leq L < \infty $$

and thus

$$ |W(a) - W(b)|_{\text{sup}} \leq L |a - b|_{\text{sup}} \leq L K_3' \frac{1}{n^{a-\gamma}} \tag{34} $$

with $L K_3' < \infty$ and $L K_3'$ independent of $x_1$. Thus the first term of (30) is bounded by

$$ |x_n - \theta|_{\text{sup}} LK_3' 1/n^{a-\gamma} = K_3'' 1/n^{a-\gamma} \text{ with } K_3'' < \infty, \text{ and } K_3'' \text{ independent of } x_1.$$

We turn now to the problem of obtaining a bound for the second term in (30). This can be expressed as

$$ T = (x_n - \theta) \left[ W \left[ d_n - \sum_{j=0}^{n-1} h_j x_{n-j} f_n - j \right] \right] - E \left[ W \left[ d_n - \sum_{j=0}^{n-1} h_j x_{n-j} f_n - j \right] \right] $$

$$ = (x_n - \theta) \sum_{m=0}^{N} \sum_{q=0}^{m} \left( \frac{(d_m)^{N-m}}{q} \sum_{j=0}^{m} h_j x_{n-j} f_n - j \right) - E \left( \frac{(d_m)^{N-m}}{q} \sum_{j=0}^{m} h_j x_{n-j} f_n - j \right) \tag{36} $$

in which the expectation inside the brackets is taken with $x_n, x_{n-1}, \ldots, x_1$ considered as parameters. Now, Eq. 36 contains $N(N+1)$ terms, $\frac{N(N+1)}{2}$ multiplied by $x_n$ and $\frac{N(N+1)}{2}$ multiplied by $\theta$. We pick the $\frac{(N+1)N}{2}$th term, $T_m$, of these first $\frac{N(N+1)}{2}$ terms and derive a bound for its magnitude. The other terms can be bounded in the same fashion.
\[ |T_m| = \left[ b_{mm} \sum_{j_1=0}^{n-1} \sum_{j_2=0}^{n-1} \sum_{j_m=0}^{n-1} h_{j_1} \ldots h_{j_m} \right] \left( \sum_{j=0}^{n-1} h_{j} x_{n-j} f_{n-j} \right)^m - E \left( \sum_{j=0}^{n-1} h_{j} x_{n-j} f_{n-j} \right)^m \]

As in section 3.1, we can split the multiple summation up into \( m \) terms, the first of which is

\[ |T_{m_1}| = \left[ b_{mm} \sum_{j_1=0}^{n-1} \sum_{j_2=0}^{n-1} \sum_{j_m=0}^{n-1} h_{j_1} \ldots h_{j_m} \right] \left( \sum_{j=0}^{n-1} h_{j} x_{n-j} f_{n-j} \right)^m - E \left( \sum_{j=0}^{n-1} h_{j} x_{n-j} f_{n-j} \right)^m \]

Note that in this sum \( j_1 \geq j_k, \ k = 2, \ldots, m \); for convenience, we shall denote the term

\[ \left( \sum_{j=0}^{n-1} h_{j} x_{n-j} f_{n-j} \right)^m \]

by \( \rho_{n,n-j_1} \). Note that \( \rho_{n,n-j_1} \) depends only on data \( v(\tau), d(\tau) \), for \( n-j_1 \leq \tau \leq n \). Now let the summand in (37) be denoted by

\[ A = \left[ x_n x_{n-j_1} \ldots x_{n-j_m} \rho_{n,n-j_1} \right]. \tag{38} \]

Now note that since \( j_k \leq j_1, \ k = 2, 3, \ldots, m \), we can express \( x_n, x_{n-j_2}, \ldots, x_{n-j_m} \) as

\[ x_{n-j_k} = x_{n-j_1} + (x_{n-j_k} - x_{n-j_1}) \]

and

\[ x_n = x_{n-j_1} + (x_n - x_{n-j_1}). \]
Substituting these expressions in (38) yields $A$ as the sum of $2^m$ terms; one of these is

$$\left(\frac{x_{n-j_1}}{\rho_{n,j_1}}\right)^{m+1}$$

(39)

and the other $2^m-1$ are of the form

$$\left(\frac{x_{n-j_1}}{\rho_{n,j_1}}\right)^{m+1-s} \left(\frac{x_{n-j_1}}{\rho_{n,j_1}}\right)^{m+1-s} \cdots \left(\frac{x_{n-j_1}}{\rho_{n,j_1}}\right)^{m+1-s}$$

(40)

in which $i_1', i_2', \ldots, i_s$ denote any distinct collection of $s$ of the indices $0, j_2', j_3', \ldots, j_m'$.

By the reasoning used in section 3.1 and restriction (g), it is possible to show that the term given by (39) is bounded by

$$\frac{a_{n-j_1}}{K'_m} \leq \frac{a_{n-j_1}}{K'_m}$$

or

$$K_4 \frac{1}{(n-j_1)^{a-\gamma}}$$

(41)

with $K_4 < \infty$, and $K_4$ independent of $x_1$. Next consider the terms of the form given by expression (40). Let

$$B = \sup_{x_1 \in X} |x_1 - x_2|, \quad \rho^* = |\rho_{n,j_1}| \sup_{x_2 \in X} Y^* = |Y_n|_{\sup}.$$  

Then, since

$$|x_n - x_{n-j_1}| \leq \sum_{q=n-j_1}^{n-1} \frac{a_q}{c_q} Y^*,$$

it follows that

$$|x_n - x_{n-j_1}|_{\sup} \geq \left|\frac{x_{n-j_1}}{x_{n-j_1}} - x_{n-j_1}\right|_{\sup} \quad k = 2, 3, \ldots, m$$

and thus we can bound each of the $2^m-1$ terms of the form given by (40) by

$$B^m Y^* \rho^* \sum_{q=n-j_1}^{n-1} \frac{a_q}{c_q}$$

for all $x_1 \in X$.

Now for $a_q/c_q = a/c 1/q^{a-\gamma}$, we have shown that

$$\sum_{q=n-j_1}^{n-1} \frac{a_q}{c_q} \leq \frac{a}{c} \left\lfloor \frac{1}{\delta} \right\rfloor \left(\frac{n-1}{\delta} - (n-1-j_1)\delta\right)$$

25
in which $\delta = 1 - (a-\gamma)$. Thus it follows that the $2^{m-1}$ terms of (38) that are of the form (40) can be bounded by

$$K \frac{1}{\delta} \left[ (n-1)^\delta - (n-j_1-1)^\delta \right] \quad \text{for all } x_1 \in X. \quad (42)$$

Now comparing expressions (41) and (42), we see that each of the $2^m$ terms making up the quantity $A$ can be bounded in magnitude by

$$ \begin{cases} 
K \frac{1}{\delta} \left[ (n-1)^\delta - (n-j_1-1)^\delta \right] & j_1 \neq 0 \quad \text{for all } m \quad -M \leq m \leq M \\
K \frac{1}{n^{a-\gamma}} & j_1 = 0 \quad \text{for all } x_1 \in X.
\end{cases} $$

Substituting this bound in (37), we obtain

$$|T_m| \leq b_m K (h_0)^{m+1} \frac{1}{n^{a-\gamma}} + b_m K 2^m \sum_{j_1=1}^{n-1} \sum_{j_2=0}^{j_1} \sum_{j_m=0}^{j_1} |h_{j_1}| \cdots |h_{j_m}| \left[ (n-1)^\delta - (n-j_1-1)^\delta \right]$$

$$\leq b_m K (h_0)^{m+1} \frac{1}{n^{a-\gamma}} + b_m K 2^m \left( \sum_{j=0}^{\infty} |h_j| \right) \sum_{j=1}^{m-1} \sum_{j_1=1}^{n-1} |h_{j_1}| \frac{1}{\delta} \left[ (n-1)^\delta - (n-1-j_1)^\delta \right] \quad (43)$$

for all $x_1 \in X$. It is shown in Appendix A.4 that the sum in inequality (43) can be bounded by $K_7 1/n^{a-\gamma}$, and thus $|T_m| \leq K_8 1/n^{a-\gamma}$, for all $x_1 \in X$.

We have thus split $F_n(x_1)$ up into a number of terms (the total number of which depends only on the degree of $W$) and shown that each term can be bounded by $K_9 1/n^{a-\gamma}$, for all $x_1 \in X$, and thus $F_n(x_1) \leq K_10 1/n^{a-\gamma}$, for all $x_1 \in X$, as desired.

### 3.3 BOSE FILTER FOLLOWED BY A NONLINEAR NO-MEMORY TRANSDUCER

We consider the system shown in Fig. 5. A Bose nonlinear filter \({}^{11}\) is followed by a nonlinear no-memory transducer that is described by the equation

$$q(t) = g[m(t)]. \quad (44)$$

The functional expansion of the Bose filter may be of any form for which the functions $f_1(t)$ are orthonormal in the sense that

$$f_1(t) = \begin{cases} 
0 \\
1
\end{cases} \quad (45)$$

26
and

$$f_i(t) f_j(t) = 0 \quad i, j = 1, 2, \ldots, k \quad i \neq j.$$  \hspace{1cm} (46)

We assume that the nonlinear transducer is monotonic and satisfies the restriction

$$0 < K_1 \leq \frac{\partial g(m)}{\partial m} \leq K_2 < \infty$$  \hspace{1cm} (47)

for all $$m \quad -M \leq m \leq M, \text{ with } M = \max |x_i| \quad i = 1, 2, \ldots, k \quad \nabla_i \in \mathcal{X}.$$

Only a slight modification of the proof given in section 3.1 is necessary, in this case, to include a nonlinear transducer that satisfies (47). However, because of the orthogonality requirement expressed by Eq. (46), the present case has two properties that deserve mention. First, we note that $$M(x)$$ may be expressed as

$$M(x) = \sum_{i=1}^{k} M(x_i) = \sum_{i=1}^{k} P(f_i \neq 0) \sum_{i=1}^{k} P(f_i \neq 0) F \{W \left[ d-g(x_i) \right] \} f_{i} \neq 0 \},$$  \hspace{1cm} (48)

and hence the $$x_i$$'s may be adjusted independently of one another. The situation is thus reduced from one minimization problem in $$k$$ variables to $$k$$ minimization problems in one variable. Second, because of the orthogonality, restriction (c) reduces to

$$P \{|f_i| > D\} \geq \epsilon > 0, \quad D > 0 \quad i = 1, 2, \ldots, k,$$  \hspace{1cm} (49)

or in view of Eq. 45, to

$$P \{f_i \neq 0\} \geq \epsilon > 0 \quad i = 1, 2, \ldots, k.$$

Inequality (49) serves to give us more insight into the meaning of restriction (c).
Note that restrictions (d) and (e) both prohibit the use of the error function \( W(e) = |e| \). It is true that, for all practical purposes, \( W(e) = |e| \) can be approximated by a polynomial of finite degree. To actually do this, however, would undesirably complicate carrying out the iterative procedure. The only reason for requiring \( W \) to be a polynomial was to be able to show that assumption (iii-a) followed from restriction (g). If we are concerned with the case in which independent data are used for each succeeding iteration, then \( F_n(\alpha) = 0 \), and it is no longer necessary to restrict \( W \) to a polynomial. It will, instead, only be necessary to require that \( W \) possess continuous first and second derivatives and be strictly convex in order that assumptions (ii-a) and (ii-b) be satisfied. Thus, forgetting the convexity requirement for the moment, if we were interested in the criterion \( W(e) = |e| \) (and had independent data for each iteration) we could use some transducer that behaved as \(|e|\) for all \( e \) other than those near zero and which possessed continuous first and second derivatives for \( e = 0 \) (for example, a rectifier).

The convexity requirement, however, is more troublesome. Although it would be easy to construct a device that behaved as \(|e|\) for large values of \( e \), it would be difficult to build a device that approximates \(|e|\) but is still strictly convex. We note that, other than restriction (g), we have not placed restrictions on the signal \( d(t) \) except that it be uniformly bounded in magnitude. Although the function \( W(e) = |e| \) is not strictly convex, it is still convex, that is,

\[
W[\alpha a + (1-\alpha)b] \leq \alpha W[a] + (1-\alpha) W[b] \quad 0 \leq \alpha \leq 1.
\]

Now, if \( \text{sgn}(a) = -\text{sgn}(b) \), then for any \( W(e) \) satisfying (50) there exists an \( E > 0 \) so that, for \( \min \{|a|,|b|\} \geq \epsilon > 0 \),

\[
W[\alpha a + (1-\alpha)b] \leq \alpha W(a) + (1-\alpha) W(b) - \alpha E|a-b| \quad 0 \leq \alpha \leq 1/2
\]

Hence, if we replace restriction (e) by the weaker condition expressed by Eq. 50, and restriction (c) by the stronger condition that there exist a \( D > 0 \) with the property that for all \( x \in X \)

\[
P\left\{ \text{sgn}\left( \sum_{i=1}^{k} x_{f_{i},t} - d_{t} \right) = -\text{sgn}\left( \sum_{i=1}^{k} \theta_{f_{i},t} - d_{t} \right) \right\} > D\|\mathbf{x}-\theta\|,
\]

then we again obtain inequality (17), and assumption (ii-a) is still satisfied.

The condition expressed by (51) is quite untractable; it would be extremely difficult in a practical situation to ascertain whether or not it is satisfied. Nevertheless, the condition is reasonable enough that one might carry out the procedure for \( W(e) = |e| \).
Fig. 6. Plots of system performance for a simple example.
with a fair amount of confidence that the procedure would converge.

For the Bose filter mentioned in section 3.3, (51) reduces to

$$P\{\text{sgn}[x_i - d_t] = -\text{sgn}[0_i - d_t], \min[|x_i - d_t|, |0_i - d_t|] \geq \epsilon\} \geq D|x_i - 0| \quad i = 1, 2, \ldots, k.$$  

3. 5 CONVEXITY OF THE ERROR WEIGHTING FUNCTION AND UNIQUENESS OF THE MINIMUM OF $M(x)$

In section 3.1 we required that the function $W(e)$ be strictly convex. The reason for this was to guarantee that our performance function, $M(x) = E\{W[d_t - q_x(t)]\}$, have only a unique minimum. The connection between the convexity of $W$ and the uniqueness of the minimum of $M(x)$ may be somewhat obscure. To clarify this point, we present a simple example. Consider two input signals $s_1(t)$ and $s_2(t)$ and a desired output signal $d(t)$. We choose to estimate $d(t)$ by $q(t) = x_1s_1(t) + y_2s_2(t)$.

If we assume the following probability distribution for $s_1, s_2, d$,

$$P(s_1, s_2, d) = \begin{cases} 
1/4 & s_1 = 0, s_2 = 1, d = 0 \\
1/4 & s_1 = 0, s_2 = 1, d = 1 \\
1/4 & s_1 = 1, s_2 = 0, d = 0 \\
1/4 & s_1 = 1, s_2 = 0, d = 1 \\
0 & \text{otherwise}
\end{cases}$$

then

$$M(x, y) = 4E\{W[d - qx_1s_1 - y_2s_2]\} = W[x] + W[1-x] + W[y] + W[1-y].$$

Figure 6 shows lines of equal average weighted error drawn in the $x-y$ plane for the three error criteria $W[e] = e^2$, $W[e] = |e|$, and $W[e] = +[|e|]^{1/2}$, respectively. Now consider $(x, y)$ as being restrained to a convex set $X$, in the $x-y$ plane. For $W(e) = e^2$ (which is strictly convex), there is clearly a unique minimum for any convex set, $X$, of parameter settings. The case $W(e) = |e|$ (which is only convex) is the dividing line; although for some choices of $X$, the minimum is not unique; nevertheless, all minima are connected and result in the same error. For $W[e] = +[|e|]^{1/2}$, there are clearly choices of $X$ for which there are separate minima. The four local minima in Fig. 6c happen to have the same average weighted error only because of the symmetry of $P(s_1, s_2, d)$. 

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4.1 A MORE GENERAL PERFORMANCE CRITERION

We have considered the optimum parameter setting to be that setting which mini-
mized the performance function

\[ M(x) = E\{W[d_t - q_{x,t}]\} \]

in which \( W \) may be any non-negative strictly convex weighting function on the error. This implies that the error is equally harmful under all situations (we are referring to situations that are distinct from the value of the error). In certain applications the error will be more harmful when certain conditions on the input and desired output exist. An example of this is found in the transmission of pulses as shown by the waveforms of Fig. 7. The signal \( s(t) \) consists of a train of periodic pulses that have been corrupted by a continuous noise signal to form the signal \( v(t) \). The actual purpose of the filter is to recover the height of the pulses; however, minimization of the performance function \( M(x) \) requires that the filter also do a good job of smoothing out the noise between pulses. As pointed out by Bose, \(^{11}\) this requirement may impair the filter's ability to recover the pulses.

![Fig. 7. Pulse-transmission waveforms.](image)

In cases of this kind it would be more meaningful to minimize the performance function
in which $W_1$ is again a non-negative strictly convex function, and $W_2$ is an appropriate bounded non-negative function. The use of this type of performance function does not greatly complicate the iterative adjustment procedure. In the example discussed above, it would only be necessary to carry out the adjustments at those times when a pulse is being transmitted in order to minimize the performance function

$$M(x) = E\left\{W_1[d_t-q_x, t] W_2[v_{t-T_1}, \ldots, v_{t-T_m}, d_{t-T_1}, \ldots, d_{t-T_m}] \right\}$$

$$0 \leq \tau_1 < \tau_2 < \ldots < \tau_m < \infty$$

(52)

However, we must still show that the adjustment procedure converges for performance criteria of the type expressed by Eq. 52. In establishing assumptions (i) and (ii), we estimated upper and lower bounds on certain averages of $W_1[d(t)-q_x(t)]$. Thus if

$$0 < K_1 \leq W_2 \leq K_2 < \infty,$$

(53)

then these estimates could still be obtained by taking $W_2$ outside of the averaging operation. If

$$0 \leq W_2 \leq K_2 < \infty,$$

(54)

then it is only necessary to strengthen restriction (c) to read

$$(c') \quad P\left\{W_2[v_{t-T_1}, \ldots, v_{t-T_m}, d_{t-T_1}, \ldots, d_{t-T_m}] \left| \sum_{i=1}^{k} (x_i-\theta_i) f_1, t \right| \geq D \|x-\theta\| \right\}$$

$$\geq \epsilon > 0 \quad \text{for all } x \in X \quad D > 0$$

in order for the preceding work establishing assumption (ii-a) to still be valid.

The work of sections 3.1 and 3.2 establishing the validity of assumption (iii-a') also remains valid, with the exception that the estimate of $F_n(x_1)$ now becomes

$$F_n(x_1) \leq K \left[ \frac{a(1/2)(n-\tau_m)}{c(1/2)(n-\tau_m)} \right]^{1/2}$$

(55)

4.2 DESIGN OF A FILTER WHEN A SECOND INDEPENDENT CHANNEL IS AVAILABLE

In the design of a filter the uncorrupted signal, $d(t)$, is generally not available. One way of circumventing this problem is to use a record of the data $v(t)$ and $d(t)$. If a
second independent channel is available, another method is possible. This situation is shown in Fig. 8 in which the noise source \( n_2(t) \) is statistically independent of the signal, \( d(t) \), and also of the other noise source, \( n_1(t) \).

What we desire to minimize is the performance function

\[
M(x) = E\{W[d - q_x, t]\}
\]

in which \( d(t) \) is the result of some functional operation on \( s(t) \), the uncorrupted signal. If \( W \) is a polynomial of degree \( N \) and the first \( N-1 \) moments of \( n_2(t) \) are known, it is quite easy to find the \( N \)th degree polynomial \( W_0 \) that is such that

\[
M(x) = E\{W_0[d + n_2, t - q_x, t]\} - K
\]

in which \( K \) is some non-negative constant. Hence, by replacing \( d - q_x \) and \( W \) in our procedure by \( d + n_2 - q_x \) and \( W_0 \), we can again minimize the desired performance function.

4.3 COMPENSATION OF A FEEDBACK SYSTEM CONTAINING A FIXED LINEAR ELEMENT

We shall now consider the problem of designing compensation for a feedback system. The situation is shown in Fig. 9a. A disturbance, \( y(t) \), is added at the output of the fixed linear system (or at some point internal to the linear system), and noise, \( n(t) \), is added to the feedback path of the over-all system. The purpose of the feedback path is to minimize the effect of the disturbance, and the primary purpose of the compensator is to minimize the effects of the noise. If the fixed element is minimum phase (and hence possesses a stable inverse), this problem can be transformed to an equivalent filter problem. If the fixed element is non-minimum phase, this approach is not possible. It is this situation that we consider here.

By making the compensator of the form shown in Fig. 9b, the over-all system reduces to the form shown in Fig. 9c. If the impulse response of the linear system \( L \)
Fig. 9. Compensation of a feedback system containing a linear fixed element. 
(a) Over-all system. (b) Form of the compensator. (c) Compensation of a feedback system containing a linear fixed element.
falls off as rapidly as $1/\tau^3$, then the situation is covered by the work of section 3.2; and the adjustment procedure described there may be applied. In some cases, the desired output, $d(t)$, is the same as the original input, $v(t)$. In this instance, we see that the purpose of the system of Fig. 9c is to remove the noise, $n(t)$, from the signal $v(t) - y(t)$.

It should be noted that the compensator of Fig. 9b may well be unstable. As long as the over-all feedback system is stable, however, this is no cause for concern. The feedback system will be stable as long as the simulator, $L^*$, is a sufficiently accurate reproduction of the fixed system $L$. In the general case, we cannot make this statement precise; in the special case, when the filter of Fig. 9b is linear and represented by the transfer function $G_x$, we require

$$G_x[L^* - L] \neq 1 \quad \text{or} \quad \text{Re} \left[ G_x(L^* - L) \right] < 1$$

for all $x \in X$, that is, for all allowable parameter settings.

It should be emphasized that the convergence of the adjustment procedure requires that the impulse response of the fixed element fall off as rapidly as $(1/\tau^3)$. This requirement makes the procedure inapplicable to control problems, in which the fixed elements usually contain an integrator and hence possess impulse responses that do not fall off to zero.

4.4 SIMPLE EXAMPLE — A SINGLE GAIN SETTING

The proofs involved in establishing statements 1-3 and in showing the sufficiency of the physical restrictions are somewhat involved and tedious for the reader to pursue. For this reason, we shall present a simple example for which we may directly estimate the rate of convergence. The basic method involves the same idea used in the proofs;

![Fig. 10. Single gain adjustment.](image-url)
the difference is that we are able to calculate the pertinent terms rather than having to make the involved estimates required in the general case.

The case to be considered is the iterative adjustment of the single gain coefficient shown in Fig. 10. The sampled-data signal \( v(t) \) may be the result of any bounded operation on the actual input. The error criterion will be \( W(e) = e^2 \); that is, we wish to minimize the mean-square error. We shall assume that the two measurements,

\[
Y_{x_n + c_n} = W[d(n)-(x_n + c_n)v(n)]
\]

and

\[
Y_{x_n - c_n} = W[d(n)-(x_n - c_n)v(n)].
\]

are performed simultaneously with the same data. Thus

\[
Y_n = Y_{x_n + c_n} - Y_{x_n - c_n} = \left[ (x_n + c_n)v(n) - d(n) \right]^2 - \left[ (x_n - c_n)v(n) - d(n) \right]^2
\]

\[
= 4c_n \left[ x_n v_n^2 - d_n v_n \right]. \tag{56}
\]

Now

\[
x_{n+1} = x_n - \frac{a_n}{c_n} Y_n = x_n \left( 1 - 4a_n v_n^2 \right) + 4a_n v_n d_n \tag{57}
\]

and hence, taking the average and noting that \( x_n \) is independent of \( v_n \), we obtain

\[
\bar{x}_{n+1} = \bar{x}_n \left( 1 - 4a_n \bar{v}_n^2 \right) + 4a_n \bar{v}_n d_n. \tag{58}
\]

Using Eq. 58 recursively, we obtain

\[
\bar{x}_{n+1} = \bar{x}_1 \prod_{j=1}^{n} \left( 1 - 4a_j \bar{v}_j^2 \right) + \sum_{j=1}^{n} 4a_n \bar{v}_n d_n \prod_{i=j+1}^{n} \left( 1 - 4a_i \bar{v}_i^2 \right). \tag{59}
\]

Setting

\[
a_n = \frac{A}{4v^2(n+1)} \tag{60}
\]

we have

\[
\prod_{j=1}^{n} \left( 1 - 4a_j \bar{v}_j^2 \right) = \frac{\Gamma(n+2-A)}{(n+1)! \Gamma(2-A)}. \tag{61}
\]

If we apply Stirling's approximation to this expression, the result, for \( n \gg 1 \) and \( n \gg A \), is
\[
\frac{n}{j=1} \left( 1 - \frac{A}{j+1} \right) = \frac{1}{\Gamma(2-A)(n+1)^A}.
\]  

(61)

Similarly,
\[
\frac{n}{j=m} \left( 1 - \frac{A}{j+1} \right) = \frac{m^A}{(n+1)^A}.
\]  

(62)

We can show by solving the optimization equation for the system that the optimum gain setting is given by \( \theta = \frac{\nu d}{\nu^2} \).

Combining Eqs. 59, 61, and 62, we thus obtain
\[
\bar{x}_{n+1} = \frac{x_1}{\Gamma(2-A)(n+1)^A} + \theta \frac{1}{(n+1)^A} \sum_{j=1}^{n} \frac{A(j+1)^A}{(j+1)^A}
\]
or
\[
\bar{x}_n = \frac{x_1}{\Gamma(2-A)n^A} + \theta \left( 1 + \frac{1}{n^A} \right).
\]  

(63)

We now wish to estimate \((x_n)^2\).

\[
(x_{n+1})^2 = (x_n)^2 - 2x_n \frac{a_n}{c_n} Y_n + \frac{a_n^2}{c_n^2} (Y_n)^2
\]

\[
= (x_n)^2 \left[ 1 - 8a_n \nu_n^2 + 16a_n^2 \nu_n^4 \right]
\]

\[
+ 8x_n \left[ a_n \bar{d}_n \nu_n - 4a_n^2 \nu_n \bar{d}_n \right] + 16a_n^2 \nu_n^2 \bar{d}_n.
\]

Now \( \nu_n \) and \( \bar{d}_n \) are independent of \( x_n \), and hence
\[
(x_{n+1})^2 = (x_n)^2 \left[ 1 - 8a_n \bar{\nu}^2 + 16a_n^2 \bar{\nu}^4 \right]
\]

\[
+ 8x_n \left[ a_n \bar{d} \nu - 4a_n^2 \nu \bar{d} \right] + 16a_n^2 \bar{d} \nu^2.
\]  

(64)

Setting
\[
d_n = 8a_n \frac{\nu}{\nu^2} - 16a_n^2 \frac{\nu^4}{\nu^2}
\]
and
\[
w_n = 8x_n \left[ a_n \bar{d} \nu - 4a_n^2 \nu \bar{d} \right] + 16a_n^2 \bar{d} \nu^2
\]
and solving Eq. 64 recursively, we obtain
\[(x_{n+1})^2 = x_1^2 \frac{n}{\prod_{j=1}^{n} (1-d_j)} + \sum_{j=1}^{n} w_j \left[ \frac{n}{\prod_{l=1}^{n} (1-d_l)} \right] \]  

(65)

For \( a_n \) as determined by Eq. 60, we have

\[(1-d_n) = (1-A_1/n+1)(1-A_2/n+1)\]

in which

\[A_1 = 1 - \frac{1}{\sqrt{\pi}} \left[ (v^2)^{2-v^4} \right]^{1/2}, A_2 = 1 + \frac{1}{\sqrt{\pi}} \left[ (v^2)^{2-v^4} \right]^{1/2}.\]  

(66)

It is thus possible to obtain

\[\prod_{j=1}^{n} (1-d_j) = \frac{1}{\Gamma(2-A_1) \Gamma(2-A_2)(n+1)^{2A}} \]  

(67)

and

\[\prod_{j=m}^{n} (1-d_j) = \frac{m^{2A}}{(n+1)^{2A}}, \]  

(68)

Combining Eqs. 68, 67, 65, and 63, carrying out the summations, and dropping higher order terms in \( n \), we have

\[\frac{(x_n)^2}{n^2A} \frac{1}{\Gamma(2-A_1) \Gamma(2-A_2)} + \theta^2 \left( 1 - \frac{1}{n^{2A}} \right) \]

\[+ \frac{2x_1 \theta A}{\Gamma(2-A) (2A-1) n} + \frac{A^2}{2A-1} \left[ \frac{d^2 \nu^2 - \theta^2 \nu^3 d}{\nu^2 \theta^2} \right] \left[ \frac{1}{n} - \frac{1}{n^{2A}} \right] \]  

(69)

and hence

\[E\{(x_n-\theta)^2\} = \frac{x_1^2}{n^2A} \frac{1}{\Gamma(2-A_1) \Gamma(2-A_2)} + \frac{2x_1 \theta A}{\Gamma(2-A) (2A-1) n} \left[ \frac{A^2}{2A-1} \left[ \frac{d^2 \nu^2 - \theta^2 \nu^3 d}{\nu^2 \theta^2} \right] \left[ \frac{1}{n} - \frac{1}{n^{2A}} \right] \right] \]

\[+ \frac{2\theta^2}{n^A} - \frac{\theta^2}{n^{2A}} + \frac{A^2}{2A-1} \left[ \frac{d^2 \nu^2 - \theta^2 \nu^3 d}{\nu^2 \theta^2} \right] \left[ \frac{1}{n} - \frac{1}{n^{2A}} \right]. \]  

(70)

Now \( a_n \) is determined as

\[a_n = \frac{A}{\theta^2(n+1)} \]

and hence, for a known \( a_1 \), \( A \) can only be estimated. It is, nevertheless, instructive
to set $A$ so that the dominant terms in Eq. 70 are minimized. This occurs for $A = 1$. In order to obtain the dominant term more accurately, we substitute $A = 1$ in expressions (63), (67), and (68). Substituting these results in Eq. 65, we can now evaluate some of the resulting terms more accurately to obtain

$$\left( x_n \right)^2 \leq \frac{x_1^2}{n^2} \left( \frac{1}{\Gamma(2-A_1)} \right) \left( \frac{1}{\Gamma(2-A_2)} \right) + \theta^2 \frac{p^{(n-1)}}{n} + \frac{2x_1 \theta(n-1)}{n^2} + \left[ \frac{d^2 v^2 - 2\theta v^2}{v^2} \right] \frac{(n-1)}{n^2}.$$ 

Hence

$$E\left\{ (x_n - \theta)^2 \right\} \leq \frac{1}{n^2} \left[ \frac{x_1^2}{\Gamma(2-A_1)} \right] \left( \frac{1}{\Gamma(2-A_2)} \right) - 2x_1 \theta + \frac{d^2 v^2 - 2\theta v^2}{v^2} \frac{(n-1)}{n^2}.$$ 

$$= \frac{1}{n} \left[ \frac{x_1^2}{\Gamma(2-A_1)} \right] \left( \frac{1}{\Gamma(2-A_2)} \right) + \theta^2 + \frac{d^2 v^2 - 2\theta v^2}{v^2} \frac{(n-1)}{n^2}.$$ 

(71)

For large $n$, the dominant term in this expression is

$$R_n = \frac{1}{n} \left[ \frac{x_1^2}{\Gamma(2-A_1)} \right] \left( \frac{1}{\Gamma(2-A_2)} \right).$$ 

(72)

4.5 MINIMIZATION OF A FINITE TIME-AVERAGE REGRESSION FUNCTION

In some situations one might be restricted to working with a limited record of data containing only $N$ samples. In this case the finite time-average regression function

$$M_N(x) = \frac{1}{N} \sum_{n=1}^{N} W(d(n)-q_\lambda(n))$$ 

(73)

could be evaluated for any $\lambda$. $M_N(x)$ is again a convex function for the class of error-weighting functions previously described, and one could use some analytic or iterative method to find the vector $\theta^*$ which minimized $M_N(x)$.

One way to compare this method with the iterative method previously described would be to compare $E\{\|\theta - \theta^*\|^2\}$ with $E\{\|\theta - x_N\|^2\}$, in which $x_N$ represents the final iterate obtained by the previous method.

One might intuitively feel that finding the vector which minimized $M_N(x)$ would make greater use of the given data, and hence that $E\{\|\theta^* - \theta\|^2\}$ would always be the smaller quantity. To disprove this idea, we consider the example in section 4.4. We assume
that \( N \) is fairly large, so that the dominant term of \( E\{ (x_N - \theta)^2 \} \) is given approximately by Eq. 72 as

\[
E\{ (x_N - \theta)^2 \} = \frac{1}{N} \left[ \theta^2 + \frac{d^2 v^2}{v^2} - 2\theta v^3 d \right].
\]  

(74)

Now \( \theta' \) is given as

\[
\theta' = \frac{1}{N} \sum_{i=1}^{N} d_i v_i
\]

\[
\frac{1}{N} \sum_{i=1}^{N} \frac{v_i^2}{N}
\]

and hence

\[
E\{ (\theta - \theta')^2 \} = \theta'^2 - 2\theta E \left\{ \left( \sum_{i=1}^{N} d_i v_i \right) \right\} + E \left\{ \left( \sum_{i=1}^{N} \frac{d_i v_i}{N} \right)^2 \right\}.
\]  

(75)

Now consider the quantity

\[
E \left\{ \left( \sum_{i=1}^{N} \frac{d_i v_i}{N} \right)^2 \right\}.
\]

The term \( d_i v_i \) is statistically independent of \( (N-1) \) of the \( N \) terms of \( \sum_{i=1}^{N} (v_i)^2 \) and hence we may write as an approximation

\[
E \left\{ \left( \sum_{i=1}^{N} \frac{d_i v_i}{N} \right)^2 \right\} \approx \frac{E \left\{ \sum_{i=1}^{N} (d_i v_i)^2 \right\}}{E \left\{ \sum_{i=1}^{N} \frac{v_i^2}{N} \right\}}.
\]

Applying similar reasoning to the last term of Eq. 75, we have

\[
E\{ (\theta - \theta')^2 \} \approx \theta'^2 - 2\theta \frac{v d}{v^2} + \frac{N \frac{d^2 v^2}{v^2} + (N^2 - N) \frac{(dv)^2}{(v^2)^2}}{N v^4 + (N^2 - N) (v^2)^2}
\]

\[
\approx \theta'^2 - 2\theta^2 + \frac{N \frac{d^2 v^2}{v^2} + (N^2 - N) \frac{(dv)^2}{(v^2)^2}}{N^2 (v^2)^2} \approx \frac{1}{N} \left[ \frac{v^2 d^2}{(v^2)^2} - \theta^2 \right].
\]  

(76)
Thus both \( E \{ (x_N^N - \theta)^2 \} \) and \( E \{ (\theta - \theta')^2 \} \) decay as \( \frac{1}{N} \) for our example. Subtracting the coefficient of \( E \{ (\theta - \theta')^2 \} \) from that of \( E \{ (x_N^N - \theta)^2 \} \), we have

\[
D = 2\theta^2 - 2 \frac{v^3 d}{(v^2)^2} \cdot \frac{2}{((vd)^2 - v^3 d)} \tag{77}
\]

which can be of either sign. Thus it is possible for the distribution of \( v \) and \( d \) to be such that

\[
E \{ (x_N^N - \theta)^2 \} < E \{ (\theta - \theta')^2 \} \quad \text{for large } N.
\]

Unfortunately, no statement can be made in the general case that compares the two. We can always estimate

\[
E \{ \| x_N^N(x_1) - \theta \|^2 \}
\]

for the iterative procedure and

\[
E \{ (M(\theta') - M(\theta))^2 \}
\]

for the time-average regression case. However, if we assume nothing more about \( M(x) \) than restriction (ii) of section 2.2, it is not possible to make direct comparisons between the two methods (either comparing

\[
E \{ \| x_n^N(x_1) - \theta \|^2 \} \quad \text{with} \quad E \{ \| \theta - \theta' \|^2 \}
\]

or

\[
E \{ (M(\theta) - M(\theta'))^2 \} \quad \text{with} \quad E \{ (M(x_n^N(x_1)) - M(\theta))^2 \}
\]

If we neglect the mean-square error case (which has an analytic solution), the computational advantage would seem to be with the stochastic approximation method. We could show that for \( N \) sufficiently large

\[
[(\text{grad } M_N^N(x), x - \theta)] \geq K^I \| x - \theta \|^2 \quad \text{(78)}
\]

\[
\| (\text{grad } M_N^N(x)) \| \leq K^I \| x - \theta' \|^2 \quad \text{(79)}
\]

for some \( K^I \geq K^I_0 > 0 \), so that a search routine or iterative method could be used to locate \( \theta' \). Either method would require a large number of passes through the data. The Newton-Raphson method, although an efficient iterative method, requires estimates of second derivatives and would take \( 2k(k+1) \) passes through the data for each iteration. More suitable might be a gradient method in which we set

\[
x_{n+1} = x_n - A_n (\text{grad } M_N^N(x_n)) \quad \text{(80)}
\]

and use \( 2k \) passes through the data to estimate \( (\text{grad } M_N^N(x_n)) \) for each iteration. Combining Eq. 80 with inequalities (78) and (79), we may make the estimate
\[ \|x_{n+1} - \theta'\|^2 \leq \|x_n - \theta'\|^2 \left[ 1 - 2A_nK_0 + A_n^2K_1^2 \right] \]  

whence

\[ \|x_{n+1} - \theta'\|^2 \leq \sum_{j=1}^{j_0} \left[ 1 - 2K_0jA_j + K_1^2A_j^2 \right] \|x_1 - \theta'\|^2 \]  

which has the fastest asymptotic decay (geometric) for \( A_j = A = \text{constant} \). The decay is most rapid for \( A = \frac{K_0}{K_1^2} \), in which case

\[ \|x_n - \theta'\|^2 \leq \|x_1 - \theta'\|^2 \left[ 1 - \left( \frac{K_0}{K_1^2} \right)^n \right] \]  

This, however, would be efficient only if \( \frac{K_0}{K_1^2} \) could be estimated accurately and \( \frac{K_0}{K_1^2} \) was not small compared with one.

4.6 SITUATIONS IN WHICH \( M(x) \) POSSESSES MORE THAN A SINGLE LOCAL MINIMUM

In all of the situations discussed above, the true minimum of the function

\[ M(x) = E \left\{ W_1[ d_{1-t-q_x} t ] W_2[ d_{t-\tau_1} \ldots d_{t-\tau_m}, v_{t-\tau_1} \ldots v_{t-\tau_m} ] \right\} \]

was the only local minimum. In many situations, such as in the coding-decoding problem or the compensation of a feedback system with a nonlinear fixed element, there will exist more than one local minimum. In such situations the iterative procedure described above will not suffice by itself because it may converge to a local minimum that is not the true minimum, and hence must be combined with some more general form of search routine. One such method would be to run two of the gradient-seeking methods described above, simultaneously: at specified intervals, the performances of the two different resulting settings are compared, the inferior setting is discarded, and a new iteration procedure with random initial setting is initiated in its place. Unfortunately, one cannot prove anything concerning the rate of convergence to the true minimum in this case. Successful employment of such techniques would seem, for the most part, to depend on experience and engineering judgment.
V. COMPUTER STUDIES AND EXAMPLES

5.1 DESIGN OF A SIMPLE PREDICTOR

A great deal of attention has been given to proving that the iterative procedure described in Section I converges under certain suitable conditions. Having proved convergence of the adjustment scheme, our problem is to demonstrate that the procedure is feasible; that is, that a solution can be obtained by using the adjustment procedure in a reasonable amount of computer time. To establish this point, a computer study was made of the design of a predictor for the extrapolation of sampled radar position measurements of a missile flight.

An interval of 760 samples was available on magnetic tape. First, very low frequency components of the data were removed by calculating the average of each subinterval of 35 samples. A linear interpolation of these averages was then subtracted from the data.

The prediction scheme is described by the equation

\[ v^*(t+a) = \sum_{i=0}^{9} x_i v(t-i) \]  \hspace{1cm} (84)

in which \( v^*(t+a) \) is the predicted value of \( v(t+a) \), and \( e(t) = v(t+a) - v^*(t+a) \). The reason for selecting this form of predictor was that it was only necessary to measure the correlation function \( \overline{v(t) v(t+\tau)} \) for \( \tau = 0, 1, \ldots, 9 \), and \( a \) and solve the appropriate set of 10 linear equations in order to find the values of \( x_0, x_1, \ldots, x_9 \) which yield the minimum mean-square error. This provided a convenient check case. Moreover, the predictor described by Eq. 84 is easy to simulate on a digital computer. For the example considered, \( a \) was taken to be two sample times, and the error criteria \( W(e) = |e|, W(e) = e^2, \) and \( W(e) = e^4 \) were considered.

A second form of prediction was briefly investigated. This had the form

\[ v^*(t+a) = x_1 f(t) + x_3 [f(t)^3] \] \hspace{1cm} (85)

with

\[ f(t) = \sum_{i=0}^{9} z_i v(t-i), \]

the \( z_i \) being the optimum coefficients found for the predictor of Eq. 84. This addition of a nonlinear term resulted in a reduction of less than one-half per cent in the mean-square error. For this reason, the study of such forms was not extensive and will not be discussed further.

The program for iteratively adjusting the coefficients \( x_i \) is now described. At the
start of the $n^{th}$ iteration we have the coefficients $x_0(n), x_1(n), \ldots, x_9(n)$. To carry out the $n^{th}$ iteration, the quantities $v[(n-1)s+1], v[(n-1)s+2], \ldots, v[(n-1)s+10]$ and $v[(n-1)s+10+a]$ are stored, $s$ being the parameter that controls the rate at which the data are used. The calculations

$$Z_1(n) = \left| [x_0(n)+c_n]v[(n-1)s+10] + x_1(n)v[(n-1)s+9] + \ldots + x_9(n)v[(n-1)s+1] - v[(n-1)s+10+a] \right|$$

$$Z_2(n) = \left| [x_0(n)-c_n]v[(n-1)s+10] + x_1(n)v[(n-1)s+9] + \ldots + x_9(n)v[(n-1)s+1] - v[(n-1)s+10+a] \right|$$

are then carried out. We next set

$$Y_n^i = Z_i(n) \quad \text{if } W(e) = |e|$$

$$Y_n^i = [Z_1(n)]^2 \quad \text{if } W(e) = (e)^2$$

$$Y_n^i = [Z_1(n)]^4 \quad \text{if } W(e) = (e)^4 \quad i = 1, 2, \ldots, 20$$

and complete the iteration by calculating

$$x_0(n+1) = x_0(n) - \frac{a_n}{c_n} \left[ Y_n^{19} - Y_n^{20} \right]$$

$$x_9(n+1) = x_9(n) - \frac{a_n}{c_n} \left[ Y_n^{19} - Y_n^{20} \right].$$

At the end of the iterative process the final coefficients were used to compute the average prediction error over the entire interval. The quantities computed were

$$\epsilon_A = \frac{1}{750-a} \sum_{t=10}^{760-a} |v(t+a) - v*(t+a)|$$

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for \( W(e) = |e| \),

\[
\epsilon_2 = \frac{1}{750 - a} \left[ \sum_{t=10}^{760-a} [v(t+a)-v^*(t+a)]^2 \right]^{1/2}
\]

for \( W(e) = e^2 \), and

\[
\epsilon_4 = \frac{1}{750 - a} \left[ \sum_{t=10}^{760-a} [v(t+a)-v^*(t+a)^4] \right]^{1/4}
\]

for \( W(e) = e^4 \), with \( v^*(t+a) = \sum_{i=0}^{9} x_i(N) v(t-i) \). Here, \( N \) is the number of the final iteration.

Throughout the study \( c_n \) was determined as

\[
c_n = \frac{0.04}{n^{1/6}},
\]

and \( a_n \) as

\[
a_n = \frac{A}{n},
\]

\( A \) being a variable parameter. The behavior of the process depends critically on the parameter \( A \); hence each case considered was always carried out for a number of values of \( A \). For \( W(e) = e^4 \) the dependence was particularly critical. As \( A \) was increased, the convergence of the process increased steadily until a point was reached for which the process would break into violent oscillations that took a long time to die out.

The parameter \( s \) controls the separation between the intervals of data for successive iterations. For \( W(e) = |e| \) and \( W(e) = e^2 \), \( s \) was taken between 4 and 12. For \( W(e) = e^4 \), only the iterations for which the prediction error was relatively large resulted in non-negligible changes in the coefficients. Thus it was necessary to take \( s \) between 2 and 4 sample times, in order to have a sufficient number of iterations.

Examples of the behavior of the coefficient \( x_0 \) during the iterative process are shown for \( W(e) = |e| \) and \( W(e) = e^4 \) in Figs. 11 and 12, respectively. In each case, several values of the parameter \( A \) were used.

Because the interval of data was relatively short, one application of the iterative process over the given interval of data did not result in a vector \( x_n \) for which the error was near minimum. For this reason, it was necessary to apply the process repetitively, and to use the final iterate, \( x_{n'} \), of one run as the starting point for a new iterative sequence. This repetitive process is shown in Tables I, II, and III for the error criteria \( W(e) = |e| \), \( W(e) = e^2 \), and \( W(e) = e^4 \), respectively. Each iterative run was carried out for several values of the parameter \( A \). Generally, of the several resulting
Fig. 11. Behavior of the iterative process for $W(e) = |e|$, $s = 3$.

For $A = 12.0$ the process broke into violent oscillations.

Fig. 12. Behavior of the iterative process for $W(e) = (e)^4$, $s = 2$.  
Table I. Adjustment of mean-square coefficients.

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\[ c_A = \frac{1}{N} \sum_{i=1}^{N} |e_i| \]
Table II. Adjustment of mean absolute coefficients.

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\[
\epsilon_2 = \frac{1}{N} \left[ \sum_{i=1}^{N} (\epsilon_i)^2 \right]^{1/2}
\]
Table III. Adjustment of mean fourth coefficients.

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\[ c_4 = \frac{1}{N} \left[ \sum_{i=1}^{N} (e_i)^4 \right]^{1/4} \]
sets of coefficients, the one that resulted in the lowest value of $E$ was used for the start of the new iterative sequence. If two sets resulted in nearly equal $E$'s, the one for which $A$ was larger was favored. The exact results of one iterative process were not always used for the start of the next. If, for example, one of the coefficients was changed to an abnormal value at the outset of the iterative process and then decayed steadily toward a more normal value throughout the remainder of the process, this change and decay was taken into account in selecting the coefficients to start the next iterative sequence.

The process for the mean-square case was started at the arbitrary setting $x_0 = x_1 = x_2 = \ldots = x_9 = 0.1$. The absolute and fourth-power cases were started with coefficients that resulted after two successive applications of the mean-square iterative procedure to the coefficients mentioned above. From 5 to 7 applications of the iterative process were required, each of which took the given interval of data to reach the point at which $E$, the average prediction error over the interval of data, was not decreased appreciably.

For the criterion $W(e) = |e|$, initial convergence was rapid; but as $x$ came near 0, the convergence became quite slow. For $W(e) = e^4$ the situation was opposite; initial convergence was slow, but the process continued converging nicely as $x$ came near 0.

As a check case, the optimum (mean-square sense) values of the coefficients $x_i$ ($i=0, 1, 2, \ldots 9$) were found by correlation techniques, by averaging the necessary values of the correlation function $v(n) v(m)$ over an interval of 650 samples. This set of coefficients is compared in Table IV with the sets of coefficients obtained by the approximation method for the three error criteria chosen. The average prediction error as measured by all three error criteria over the 760 sample interval is also given for the set of correlation coefficients and the three sets of approximation coefficients which were optimum under their own criteria.

As evidenced by Table IV, the predictor designs for the three different error criteria are equivalent in the sense that all three result in nearly equal values of average prediction error for any given one of the three error criteria. The average predictor error for the predictors designed by approximation methods was only from 1/2 to 3 per cent greater than that for the predictor designed by correlation techniques. In view of the limitation on the length of data available, this performance is entirely satisfactory.

The important point to be brought out is that the total computing time required by the adjustment procedure for finding the optimum set of coefficients for any one of the three error criteria was no greater than the computing time required to measure the necessary correlation function and solve the associated set of simultaneous equations for the minimum mean-square error coefficients. Thus the adjustment method is no more trouble to apply than correlation techniques, yet it allows consideration of a variety of error criteria.

Although the computer study described in this section proved the feasibility of the adjustment procedure, it did not point out the distinction between different error criteria; indeed, all three error criteria that were used were nearly equivalent. For this reason, the computer study described in the following section was undertaken.
Table IV. Summary of the performance of the three sets of coefficients, Example 1.

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<tr>
<td></td>
<td>2</td>
<td>.1301</td>
<td>.1679</td>
<td>.2307</td>
<td>1.159</td>
<td>-.459</td>
<td>-.145</td>
<td>-.047</td>
<td>.007</td>
<td>-.083</td>
<td>-.004</td>
<td>.014</td>
<td>.038</td>
<td>-.178</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>.1314</td>
<td>.1289</td>
<td>.2285</td>
<td>1.116</td>
<td>-.453</td>
<td>-.124</td>
<td>-.079</td>
<td>-.048</td>
<td>-.083</td>
<td>-.009</td>
<td>.014</td>
<td>.034</td>
<td>-.199</td>
</tr>
</tbody>
</table>

\[
\begin{align*}
\epsilon_A &= \frac{1}{N} \sum_{i=1}^{N} |e_i| \\
\epsilon_2 &= \frac{1}{N} \left[ \sum_{i=1}^{N} (e_i)^2 \right]^{1/2} \\
\epsilon_4 &= \frac{1}{N} \left[ \sum_{i=1}^{N} (e_i)^4 \right]^{1/4} \\
\end{align*}
\]

For the data

\[
\begin{align*}
v_A &= \frac{1}{N} \sum_{i=1}^{N} |v_i| = .1846 \\
v_2 &= \frac{1}{N} \left[ \sum_{i=1}^{N} (v_i)^2 \right]^{1/2} = .2481 \\
v_4 &= \frac{1}{N} \left[ \sum_{i=1}^{N} (v_i)^4 \right]^{1/4} = .3549
\end{align*}
\]
5.2 DESIGN OF A NONLINEAR NO-MEMORY FILTER

In the second computer study, the data were not taken from a physical source but were generated from the Rand tables\textsuperscript{13} according to an assumed probability distribution. The problem was the design of a nonlinear no-memory filter for the recovery of a sampled-data signal, $s(t)$, from a corrupted signal $\xi(t) = s(t) + n(t)$, $n(t)$ being sampled-data additive noise. It was assumed for both the signal and noise that the value of the data at one sample time was independent of the values of the data at other sample times. The signal and noise were also assumed to be statistically independent with distributions

$$P(s) = \begin{cases} 
1 & -0.5 \leq s < 0.5 \\
0 & \text{otherwise}
\end{cases} \quad (86)$$

$$P(n) = \begin{cases} 
0.2 & n = 0.5 \\
0.2 & n = -0.5 \\
0.6 & n = 0 \\
0 & \text{otherwise}.
\end{cases} \quad (87)$$

For these assumed distributions, it is easy to show that the minimum mean-square-error filter is given by the equation

$$s^*(t) = E\{s_t | \xi_t\} = \begin{cases} 
\xi_t - 0.5 & -0.5 \leq \xi_t < 1.0 \\
\frac{1}{4}(\xi_t - 0.5) + \frac{3}{4} \xi_t & 0 \leq \xi_t < 0.5 \\
\frac{1}{4}(\xi_t + 0.5) + \frac{3}{4} \xi_t & -0.5 \leq \xi_t < 0 \\
\xi_t + 0.5 & -1 \leq \xi_t < -0.5
\end{cases} \quad (88)$$

The graph of this transducer is shown in Fig. 13. It should be noted that for $-0.5 \leq \xi < 1.0$ and $-0.1 \leq \xi < -0.5$ the filtering is perfect; that is, there is zero error. There is thus no reason to change the filter characteristic in these regions for any weighting function on the error. Further, for symmetric weighting functions on the error, the symmetry of both the distributions dictates that our filter characteristic be symmetric. Thus, for symmetric weighting functions on the error, we need only determine the filter characteristic for $0 \leq \xi < 0.5$. Since the minimum mean-square-error characteristic is linear in this region, it is reasonable to consider a filter characteristic of the form
The problem is now to determine the values of \( x_1 \) and \( x_2 \) which optimize the performance of the filter for different weighting functions on the error.

The Rand tables were used to artificially generate a sequence of data according to the distributions given by Eqs. 86 and 87. Of this sequence of data, those samples were sorted out for which \( 0 < \zeta(t) < 0.5 \). These samples numbered slightly more than 500 and, together with the corresponding samples of \( s(t) \), were stored on magnetic tape.

![Fig. 13. Filter characteristic for nonlinear no-memory filter.](image)

The weighting functions on the error which were considered were \( W(e) = |e| \), \( W(e) = (e)^2 \), and \( W(e) = (e)^4 \). The iterative procedure was carried out by calculating

\[
\begin{align*}
z_1(n) &= |-[x_1(n)+c_n^1]+x_2(n)\xi(n)-s(n)| \\
z_2(n) &= |-[x_1(n)-c_n^1]+x_2(n)\xi(n)-s(n)| \\
z_3(n) &= |x_1(n)+[x_2(n)+c_n^2]\xi(n)-s(n)|
\end{align*}
\]

53
\[ z_4(n) = | -x_1(n) + [x_2(n) - c_n^2] \xi(n) - s(n) | \]

at each step of the procedure. The next parameter setting in the procedure was then determined by

\[ x_1(n+1) = x_1(n) - a_n \frac{1}{c_n} \left[ Y_{1n} - Y_{2n} \right] \]

\[ x_2(n+1) = x_2(n) - a_n \frac{2}{c_n} \left[ Y_{3n} - Y_{4n} \right] \quad n = 1, 2, \ldots, 500, \quad (90) \]

in which

\[ Y_{1n}^i = z_i(n) \quad i = 1, 2, 3, 4 \quad \text{for } W(e) = |e|; \]

\[ Y_{2n}^i = [z_i(n)]^2 \quad i = 1, 2, 3, 4 \quad \text{for } W(e) = (e)^2; \]

and

\[ Y_{3n}^i = [z_i(n)]^4 \quad i = 1, 2, 3, 4 \quad \text{for } W(e) = (e)^4. \]

The quantities \( a_n^1, a_n^2, c_n^1, \) and \( c_n^2 \) were taken to be

\[ a_n^1 = A_1/n \]

\[ a_n^2 = A_2/n \]

\[ c_n^1 = 0.005/n^{1/6} \]

\[ c_n^2 = 0.01/n^{1/6}. \]

For each of these three error criteria, the procedure converged rather well; only 2 or 3 passes through the record of data were necessary to reach a point for which further adjustment resulted in only minor improvement in performance. Again, it was necessary to do some exploring in each case to find values of \( A_1 \) and \( A_2 \) for which the procedure converged rapidly.

For each of the three resulting sets of parameters, the quantities

\[ \epsilon_1 = \sum_{t=1}^{500} \left| -x_1 + x_2 \xi(t) - s(t) \right| \]

\[ \epsilon_2 = \left[ \sum_{t=1}^{500} \left( -x_1 + x_2 \xi(t) - s(t) \right)^2 \right]^{1/2} \]
and

\[ \varepsilon_4 = \left[ \sum_{t=1}^{500} \left(-x_1 - x_2 t(t) - s(t)\right)^4 \right]^{1/4} \]

were evaluated in order to demonstrate that the parameter setting that gave optimum performance for one criterion might do very poorly under another criterion. The results, shown in Table V, are in accordance with intuition. When no noise is present the error is small and is approximately \( x_1 \); when noise is present the error is large and is approximately \( 0.5 - x_1 \). Thus weighting large errors more heavily tends to increase \( x_1 \). It should be noted that the parameters obtained by the adjustment procedure in the mean-square case differ somewhat from those given by Eq. 88 [calculated from the distributions (86) and (87)]. There are two reasons for this. First, the minimum of the mean-square error is broad in terms of the two parameters \( x_1 \) and \( x_2 \), as evidenced by the nearly equal values of \( \varepsilon_2 \) for the two sets of parameters. Second, there is always some departure from the statistical average for quantities measured over a finite interval of data.

Table V. Summary of the performance of the three sets of coefficients, Example 2.

<table>
<thead>
<tr>
<th>Parameters</th>
<th>( x_1 )</th>
<th>( x_2 )</th>
<th>( \varepsilon_1 )</th>
<th>( \varepsilon_2 )</th>
<th>( \varepsilon_4 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Absolute</td>
<td>.001923</td>
<td>1.003</td>
<td>.1316</td>
<td>.2554</td>
<td>.3569</td>
</tr>
<tr>
<td>Square Parameters (Adjustment Procedure)</td>
<td>( .1070 )</td>
<td>( .8673 )</td>
<td>( .1969 )</td>
<td>( .2197 )</td>
<td>( .2612 )</td>
</tr>
<tr>
<td>Square Parameters (Calculated)</td>
<td>( .125 )</td>
<td>1.000</td>
<td>( .2199 )</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Fourth-Power Parameters</td>
<td>( .1732 )</td>
<td>( .8842 )</td>
<td>( .2265 )</td>
<td>( .2309 )</td>
<td>( .2401 )</td>
</tr>
</tbody>
</table>

5.3 EXAMPLE: USE OF THE PERFORMANCE FUNCTION

\[ M(x) = E\{w_1[d_t - q_{x,t}]w_2[d_t]\} \]

This final example demonstrates the improvement in performance which may be obtained by proper use of the performance criterion of section 4.1. Consider a pulse-transmission system that transmits one pulse every other second. The pulse signal
to be transmitted has zero mean and autocorrelation function

\[
        s(t) s(t+\tau) = \begin{cases} 
        1 & \tau = 0 \quad \tau \text{ an integer} \\
        0.4 & \tau = \pm 2 \quad t \text{ an even integer} \\
        0 & \text{otherwise} \\
        0.5 & \tau = 0 \quad \tau \text{ an integer} \\
        0.2 & \tau = \pm 2 \quad t \text{ an integer} \\
        0 & \text{otherwise}
    \end{cases}
\] (91)

The pulses are transmitted over a channel. In the process of transmission the pulses are corrupted by continuous correlated noise, the noise being independent of the signal transmitted. In order to take advantage of the correlation of the noise in trying to recover the transmitted signal, the received signal is sampled once a second. The additive noise has zero mean and autocorrelation function

\[
        n(t) n(t+\tau) = \begin{cases} 
        1 & \tau = 0 \quad t \text{ and } \tau \text{ both integers} \\
        0.4 & \tau = \pm 1 \\
        0 & \text{otherwise}
    \end{cases}
\] (92)

It will be assumed that the error incurred in recovering a pulse amplitude is to be weighted as the square of the error (this allows an analytic solution to the example instead of requiring a computer solution). The filter is to be of the form

\[
        s^*(t) = \sum_{i=0}^{2} x_i \xi(t) \quad t \text{ an integer}
\] (93)

in which \(\xi(t)\) denotes the received signal. Now if we take as our performance criterion

\[
        M(x) = E\{[s(t) - s^*_t]^2\} \quad t \text{ an integer},
\] (94)

the optimization equations are

\[
        \sum_{i=0}^{2} x_i [s(t-j) s(t-i) + n(t-j)n(t-i)] = s(t) s(t-j) \quad j = 0, 1, 2
\] (95)

or, in light of Eqs. 91 and 92,

\[
        1.5x_0 + 0.4x_1 + 0.2x_2 = 0.5
\]

\[
        0.4x_0 + 1.5x_1 + 0.4x_2 = 0
\]

\[
        0.2x_0 + 0.4x_1 + 1.5x_2 = 0.2
\] (96)
with the solution $x_0 = 0.3508$, $x_1 = -0.1256$, $x_2 = 0.1201$.

The performance criterion of Eq. 94 gives a measure not only of how well the pulses are recovered from the noise but also of how well the noise pulses are smoothed out at the instants between the signal pulses. What we really wish to minimize is the performance criterion

$$M(x) = E\{[s_t - s_t^*]^2 W_2[s_t]\}$$

in which

$$W_2[s(t)] = \begin{cases} 1 & s \neq 0 \\ 0 & s = 0. \end{cases}$$

Alternatively,

$$M(x) = E\{W[s_t - s_t^*]\}$$

The minimization (Eqs. 95) for this performance function is

$$2.0x_0 + 0.4x_1 + 0.4x_2 = 1.0$$

$$0.4x_0 + 2.0x_1 + 0.4x_2 = 0$$

$$0.4x_0 + 0.4x_1 + 2.0x_2 = 0.4$$

with the solution $x_0 = 0.5$, $x_1 = -0.125$, $x_2 = 0.125$.

We now wish to make a comparison of how well the two sets of filter coefficients recover the transmitted pulses. In particular, we wish to evaluate $M(x)$ of Eq. 97 for

(i) The solutions of Eqs. 96; that is, the coefficients calculated to remove the noise from the received signal; and

(ii) The solutions of Eqs. 98; that is, the coefficients calculated to remove the noise from the received signal only when a pulse has been transmitted.

After some routine calculation, we find for the first set of coefficients

$$M(x) = 0.4678$$

and for the second set of coefficients,

$$M(x) = 0.3375.$$
VI. SUMMARY

Section I of this report described the basic iterative adjustment procedure, and the second discussed certain mathematical aspects of this procedure. The main results were established in Section III. In section 3.1, it was shown that, under rather reasonable restrictions on the random processes involved, the coefficients of the nonlinear network shown in Fig. 1 could be adjusted by the iterative method described in Section I. Actually, the portion of nonlinear network preceding the gain coefficients could be completely general as long as the functions $f_i(t)$, $i = 1, 2, \ldots, k$, satisfied the restrictions laid down in section 3.1. This result has direct significance for the design of such systems as filters, predictors, models, and detection circuits; for, in such cases, the designer is free to choose a system of the form given in Fig. 1. In section 3.2 it was shown that the nonlinear network of Fig. 1 could be used preceding a fixed linear system in cascade, provided that the impulse response of the fixed linear system drops off fast enough, and that suitable modifications are made in the adjustment procedure. This has applications to systems in which the network to be designed must precede a fixed transmission system (or some other fixed linear system), and the signal that is of interest is the output of the fixed linear system.

The estimates made in Section II of the rate of convergence of the adjustment procedure are applicable to the situation discussed in sections 3.1 and 3.2. The rate of convergence was given in terms of the rate at which the quantity $E\{\|x_n - \theta\|^2\}$ approached zero. Here, $x_n$ denotes the parameter setting after $n - 1$ adjustments; $\|x\|$ denotes the Euclidean norm of the $k$-dimensional vector $x$; and $\theta$ is the parameter setting that minimizes the performance function $M(x) = E\{W[d_i - x_t]t\}$. Note that the rate of convergence is in terms of how fast the parameter setting approaches the optimum parameter setting, and not in terms of how fast the system performance approaches the optimum performance. Although there is obviously some connection between these rates, it is not, in general, possible to establish a relationship between the two.

Unfortunately, the results of sections 3.1 and 3.2 do not allow the adjustment procedure to be used directly in the solution of problems, such as the coding-decoding problem or feedback control problem, in which the performance function has more than one local minimum. As we pointed out in section 4.6, the adjustment procedure can be combined with a more general search routine to handle such cases, although no pat statement can be made regarding the length of time that it takes for the adjustment procedure to locate the true optimum parameter setting.

In Section V two computer studies are presented establishing the feasibility of the adjustment procedure.
APPENDIX

A.1 PROOF OF STATEMENT 1

To show the mean-square convergence of $\|x_n - \theta\|$ to zero, we estimate $\|x_n - \theta\|^2$. It should be remembered that $x_n$ and $x_{n+1}$ are families of random variables indexed by $x_1$ even when not specifically indicated. Inequalities are understood to hold for all $x_1 \in X$ and inequalities between random variables are understood to be with probability one. We have

$$x_{n+1} = x_n - \frac{a_n}{c_n} Y_n$$

or

$$x_{n+1} - \theta = (x_n - \theta) - \frac{a_n}{c_n} Y_n.$$  

Taking the inner product of each side of the equation with itself, we have

$$\|x_{n+1} - \theta\|^2 = \|x_n - \theta\|^2 - 2 \frac{a_n}{c_n} [x_n - \theta, Y_n] + \frac{a_n^2}{c_n} \|Y_n\|^2$$

hence, using assumption (i), we obtain

$$E \left\{\|x_{n+1} - \theta\|^2 \mid x_n(x_1)\right\} < \|x_n - \theta\|^2 + \frac{a_n^2}{c_n} E\{Y_n \mid x_n(x_1)\}^2$$

$$+ \frac{a_n^2}{c_n} S - 2 \frac{a_n}{c_n} [x_n - \theta, E\{Y_n \mid x_n(x_1)\}]$$

$$\leq \|x_n - \theta\|^2 - 2 a_n \left[ x_n - \theta, E\{Y_n \mid x_n(x_1)\} \right]$$

$$+ a_n^2 \left[ M_{c_n}(x_n) \right]$$

$$- 2 \frac{a_n}{c_n} \left[ x_n - \theta, E\{Y_n \mid x_n(x_1)\} - c_n M_{c_n}(x_n) \right]$$

$$+ \frac{a_n^2}{c_n} \left[ S + E\{Y_n \mid x_n(x_1)\}^2 - c_n^2 \left[ M_{c_n}(x_n) \right]^2 \right].$$  \hspace{1cm} (A-1)

We now derive bounds for the second and third terms in inequality (A-1):
\[ M_c(x) = \frac{1}{c} \left[ \ldots, M(x + c\varepsilon_i), \ldots \right] \]
\[ = \left[ \ldots, M_1(x + c\tau \varepsilon_i), \ldots \right] + M_1(x - c\tau \varepsilon_i) \]
\[ = (\text{grad } M)(x + c\tau) + (\text{grad } M)(x - c\tau') \]  
where \(0 < \tau_i < 1, 0 < \tau_i' < 1\), for \(i = 1, 2, \ldots k\). Therefore

\[ \|M_c(x)\|^2 = \|(\text{grad } M)(x + c\tau) + (\text{grad } M)(x - c\tau')\|^2 \]
but

\[ \|a + b\|^2 = \|a\|^2 + \|b\|^2 + 2[a, b] \]
\[ \leq \|a\|^2 + \|b\|^2 + 2\|a\| \|b\| \]
\[ \leq 2[\|a\|^2 + \|b\|^2] \]

and, using assumption (ii-b), we have

\[ \|M_c(x)\|^2 < 2[\|(\text{grad } M)(x + c\tau)\|^2 + \|(\text{grad } M)(x - c\tau')\|^2] \]
\[ < 2K_1^2[\|x - \theta + c\tau\|^2 + \|x - \theta - c\tau'\|^2] \]
\[ < 8K_1^2[\|x - \theta\|^2 + c^2 k]. \]  
(A-3)

Now it is possible to write

\[ (\text{grad } M)(cx + c\tau) = D(x + c\tau - \theta) + A \]

in which, by assumption (ii-a),

\[ K_0 \leq D \leq K_1, \quad \|A\| \leq K_1 \|x + c\tau - \theta\| \]

and

\[ [x - \theta + c\tau, A] = 0 \quad \text{or} \quad [x - \theta, A] = -c[A, \tau] \]

hence

\[ -[x - \theta, (\text{grad } M)(x + c\tau)] = -D[x - \theta, x - \theta + c\tau] - [x - \theta, A] \]
\[ = -D \|x - \theta\|^2 - Dc \|x - \theta, \tau\| \]
\[ + c[A, \tau]. \]

However,

\[ |[x - \theta, \tau]| \leq \|x - \theta\| \|\tau\| \leq k^{1/2} \|x - \theta\| \]

and furthermore
\[
\begin{align*}
|\tau, A| & \leq \|\tau\| K_1 \|x+c\tau-\theta\| \\
& \leq K_1 \|x-\theta\| \|\tau\| + K_1 c \|\tau\|^2 \\
& \leq K_1 k^{1/2} \|x-\theta\| + K_1 c k.
\end{align*}
\]

Now \(K_0 \leq D \leq K_1\), and we finally obtain

\[
-[x-\theta, (\text{grad } M)(x+c\tau)] \leq -K_0 \|x-\theta\|^2 + 2K_1 c k^{1/2} \|x-\theta\| + c^2 k K_1. \tag{A-4}
\]

Applying inequality (A-4) to the expression

\[
-[x-\theta, M_c(x)] = -[x-\theta, (\text{grad } M)(x+c\tau)] - [x-\theta, (\text{grad } M)(x-c\tau')],
\]

we obtain

\[
-[x-\theta, M_c(x)] \leq -2K_0 \|x-\theta\|^2 + 4K_1 k^{1/2} c \|x-\theta\| + 2c^2 K_1 k. \tag{A-5}
\]

If we use the bounds of inequalities (A-3) and (A-5) in Eq. (A-1), noting that \(\|x_n-\theta\|\) and \(M_{c_n}(x_n)\) are constants with respect to \(E\{x_n(x_1)\}\), we have

\[
E \left\{ \|x_{n+1}-\theta\|^2 | x_n(x_1) \right\} \leq \|x_n-\theta\|^2 + 8K_0^2 a_n^2 \|x_n-\theta\|^2 + 8K_1^2 k c_n^2 a_n^2 \\
- 4K_0 a_n \|x_n-\theta\|^2 + 8a_n c_n k^{1/2} K_1 \|x_n-\theta\| \\
+ 4a_n c_n k K_1 \\
+ \frac{a_n^2}{c_n^2} \left[ S + E \left\{ Y_{n+1} \right\}^2 - c_n^2 \|M_{c_n}(x_n)\|^2 \right] \\
- 2 a_n c_n E \left\{ x_n-\theta, Y_n(x_n) - c_n M_{c_n}(x_n) \right\} \left[ x_n(x_1) \right]. \tag{A-6}
\]

Now let

\[
B = \sup_{x_1 \in X, x_2 \in X} \|x_1-x_2\|. \tag{A-7}
\]

Applying this inequality to inequality (A-6) and taking expected values, we have

\[
E \left\{ \|x_{n+1}-\theta\|^2 \right\} \leq E \left\{ \|x_n-\theta\|^2 \right\} \left[ 1 - 4K_0 a_n \right]
\]

\[
+ 8a_n c_n k^{1/2} K_1 E \{\|x_n-\theta\|\} + 8K_1^2 k a_n^2 c_n^2 \\
+ 8K_1^2 B a_n^2 + 4a_n c_n^2 k K_1.
\]
Using assumption (iii-a) on the second to last term of inequality (A-8) and assumption (iii-b) on the last term, we obtain

\[
E \left\{ \|X_{n+1} - \theta\|^2 \right\} \leq E \left\{ \|X_n - \theta\|^2 \right\} \left[ 1 - 4K_0a_n \right] + 8a_n c_n k^{1/2} K_1 E \{ \|x_n - \theta\| \} + 8K_1^2 k a_n^2 + 8K_1^2 B^2 a_n^2 + 4a_n c_n^2 k K_1
\]

\[
+ \frac{a_n^2}{c_n} \left[ S + 2BS \right].
\]  

(A-9)

In equality (A-9) we have assumed for convenience that

\[
S_1 \frac{a_n/2}{c_n/2} \leq S_1 \frac{a_n}{c_n}.
\]

This is indeed true for \( a_n = A/n^c \), \( c_n = C/n^\gamma \), the case for which we wish to find estimates of the rate of convergence. In the general case we still obtain convergence because of our assumption (iv) that

\[
\sum_{n=1}^{\infty} \frac{a_n}{c_n} \frac{a_n/2}{c_n/2} < \infty,
\]

as well as

\[
\sum_{n=1}^{\infty} \left( \frac{a_n}{c_n} \right)^2 < \infty.
\]

Now, denoting \( S + 2BS_1 + S_2 \) by \( S_3 \) and \( E \{ \|x_n - \theta\| \} \) by \( b_n \), etc., and using the bound (A-7) on the second term in Eq. (A-9), we have

\[
\begin{align*}
b_{n+1} &\leq b_n \left[ 1 - 4K_0 a_n \right] + 4a_n c_n K_1 k^{1/2} \left( 2B + k^{1/2} c_n \right) + \frac{a_n^2}{c_n} S_3 \\
&\quad + 8K_1^2 B^2 a_n^2 + 8K_1^2 k a_n^2 c_n^2
\end{align*}
\]
or

\[ b_{n+1} \leq b_n d_n + W_n \quad n = 1, 2, \ldots. \]

Now \( a_n \to 0 \); therefore, choose an \( n_0 \) s.t. \( d_n > 0 \) for \( n \geq n_0 \) and

\[
b_{n+1} \leq b_{n_0} \prod_{n=n_0}^{n-1} d_k + \sum_{k=n_0}^{n-1} W_k \prod_{j=k+1}^{n} d_j \quad n = n_0 + 1, n_0 + 2, \ldots.
\]

since \( \sum a_n = \infty \), \( \prod_{k=n_0}^{n} d_k \) diverges to zero as \( n \to \infty \); from assumption (iv) and Kronecker's Lemma

\[
\sum_{k=n_0}^{n-1} W_k \prod_{j=k+1}^{n} d_j \to 0 \quad \text{as } n \to \infty.
\]

Hence \( \lim_{n \to \infty} b_n = 0 \). Q.E.D. Statement 1.

**A.2 STATEMENT 2**

To consider Statement 2 we set

\[
a_n = \frac{a}{n} \quad c_n = \frac{c}{n^y} \quad (a, c > 0) \quad (A-10)
\]

in which, in order to satisfy assumption (iv), we require

\[
\frac{3}{4} \leq a \leq 1, \quad 1 - a < \gamma < a - \frac{1}{2} \quad (A-11)
\]

If \( a = 1 \), we further require

\[
a > \frac{1}{4 k_0} \quad (A-12)
\]

We now state two of Chung's lemmas.\(^{14}\)

**LEMMA 1**: if for all \( n \geq n_0 \)

\[
b_{n+1} \leq \left(1 - \frac{c}{n^s}\right) b_n + \frac{c^t}{n^t} \quad 0 < s < 1, s < t, c, c^t > 0, \quad (A-13)
\]

then

\[
\lim_{n \to \infty} n^{t-s} b_n \leq \frac{c^t}{c} \quad (A-14)
\]
LEMMA 2: If for all $n > n_0$

$$b_n \leq \left(1 - \frac{c}{n}\right)b_n + \frac{c'}{n^{p+1}} \quad c > p > 0, \ c' > 0 \quad (A-15)$$

then

$$b_n \leq \frac{c'}{c - p} \cdot \frac{1}{n^p} + o\left(\frac{1}{n^{p+1}} + \frac{1}{n^c}\right) \quad (A-16).$$

Now to prove the positive part of Statement 2, we show that assumptions (i)-(v) and expressions (A-10), (A-11), and (A-12) imply

$$b_n = \begin{cases} 
0\left(\frac{1}{n^2}\right) & \gamma < \frac{a}{4} \\
0\left(\frac{1}{n^{a-2\gamma}}\right) & \gamma \geq \frac{a}{4}.
\end{cases}$$

(Note that $\gamma < \frac{a}{4}$ can occur only when $\frac{a}{2} < a < 1$.) The negative part of the statement is proved by using two more of Chung's lemmas and the one-dimensional example

$$M(x) = \begin{cases} 
(x-\theta)^2 & x < \theta \\
\frac{1}{2}(x-\theta)^2 & x > \theta 
\end{cases}.$$ 

For a proof of this, the reader is referred to Dupac.\textsuperscript{7}

From expressions (A-9) and (A-10), we have

$$b_{n+1} \leq b_n\left(1 - \frac{2K_1\frac{a}{n}}{a+\gamma} x_{n+1} \frac{1}{n^{2(a-\gamma)}} \right) + 8\frac{ac}{n^{a+\gamma}} K_1 k^{1/2} E\{\|x_n - \theta\|^2\}$$

$$+ \frac{a^2}{c^2} S_3 \frac{1}{n^{2(a-\gamma)}} + 8K_1^2 B^2 \frac{a^2}{n^{2a}}$$

$$+ 8K_1^2 k \frac{a^2}{n^{2(a+\gamma)}} + \frac{4kK_1 a^2}{n^{a+2\gamma}} \quad (A-17).$$

For arbitrary $\epsilon_n > 0$ we have

$$E\{\|x_n - \theta\|^2\} \leq \int_{\|x_n - \theta\| < \epsilon_n} \|x_n - \theta\|^2 \ dP + \int_{\|x_n - \theta\| \geq \epsilon_n} \|x_n - \theta\|^2 \ dP$$

$$\leq \epsilon_n P\{\|x_n - \theta\| < \epsilon_n\} + \frac{1}{\epsilon_n} \int_{\|x_n - \theta\|^2} \ dP$$

$$\leq \epsilon_n + \frac{b_n}{\epsilon_n}. \quad (A-18)$$
If we let $0 < \epsilon < 4$ and

$$\epsilon_n = \frac{8ck^{1/2}K_1}{\epsilon K_0 n^\gamma},$$

inequality (A-17) becomes

$$b_{n+1} \leq b_n \left[ 1 - \frac{4K_1 a}{n^\alpha} + \frac{\epsilon K_0 a}{n^\alpha} \right] + \frac{ak(8cK_1)^2}{K_0 \epsilon n^{\alpha+2\gamma}} + \frac{4kK_1 a^2 c^2}{n^{\alpha+2\gamma}}$$

$$+ S_3 \frac{a^2}{c^2} \frac{1}{n^{2(a-\gamma)}} + \frac{8K_1 B^2 a^2}{n^2a} + \frac{8K_1 ka^2 c^2}{n^2(a+\gamma)}$$

$$\leq b_n \left[ 1 - \frac{(4-\epsilon) K_0 a}{n^\alpha} \right] + \left( 1 + \frac{16K_1}{K_0 \epsilon} \right) 4K_1 a^2 \frac{1}{n^{\alpha+2\alpha}}$$

$$+ S_3 \frac{a^2}{c^2} \frac{1}{n^{2(a-\gamma)}} \left[ 1 + \frac{8K_1 Bc^2}{S_3 n^{2\gamma}} + \frac{8kK_1^2 c^4}{S_3 n^4\gamma} \right].$$

(A-19)

Take an arbitrary $\eta > 0$; then there exists an $n_0 = n_0(\eta)$ that is such that

$$b_{n+1} \leq b_n \left[ 1 - \frac{(4-\epsilon) K_0 a}{n^\alpha} \right] + \left( 1 + \frac{16K_1}{K_0 \epsilon} \right) 4K_1 a^2 \frac{1}{n^{\alpha+2\alpha}} + \frac{(1+\eta) S_3 a^2}{c^2 n^{2(a+\gamma)}}$$

$$\leq b_n \left[ 1 - \frac{(4-\epsilon) K_0 a}{n^\alpha} \right] + \frac{1}{t} \left[ 1 + \frac{16K_1}{K_0 \epsilon} \right] 4K_1 a^2 \frac{(1+\eta) S_3 a^2}{c^2}$$

(A-20)

where $t = \min (a+2\gamma, 2a-2\gamma) = \frac{3}{2} a - \frac{1}{2} |4\gamma - a| \geq 1 \geq a = s$. Therefore for $a < 1$, the conditions of Lemma 1 are satisfied with

$$t - s = \frac{1}{2} (a - |4\gamma - a|) = \begin{cases} 2\gamma & \gamma < a/4 \\ a - 2\gamma & \gamma \geq a/4 \end{cases}$$

and

$$\lim_{n \to \infty} n^{t-s} b_n \leq \frac{c'}{c} < \infty \text{ or } b_n = O\left( \frac{1}{n^{t-s}} \right).$$

For $a = 1$ we set

$$P = t - 1 = \min (1+2\gamma, 2-2\gamma) - 1$$

$$= \min (2\gamma, 1-2\gamma).$$

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Since $0 < \gamma \leq \frac{1}{2}$,

$$0 < P \leq \max (2\gamma, 1-2\gamma) \leq 1 - \frac{\epsilon}{4} \quad \text{for } 0 < \epsilon < 4;$$

$$\leq \left(1 - \frac{\epsilon}{4}\right) 4K_0a, \text{ by inequality (A-12)};$$

$$= (4-\epsilon) K_0a = c.$$

Hence, for $a > \frac{1}{4K_0}$, the conditions of Lemma 2 hold and

$$b_n \leq \frac{c^4}{c-p} 0 \left(\frac{1}{n-1}\right) + 0 \left(\frac{1}{n^2} + \frac{1}{n^c}\right) = 0 \left(\frac{1}{n-1}\right).$$

It should be noted that $b_n$ falls off fastest for $a = 1$, $\gamma = \frac{1}{4}$, in which case $b_n = 0 \left(\frac{1}{n^{1/2}}\right)$; specifically,

$$b_n \sim \frac{16K_0^4}{(4-\epsilon) K_0 a} \frac{a}{4K_1} \frac{c^2}{c-p} \frac{1}{(4-\epsilon) K_0 a - \frac{1}{2}} \frac{a}{4K_1} \frac{c^2}{c-p} \frac{1}{n^{1/2}} \quad (A-21).$$

A. 3 STATEMENT 3

Using a multidimensional Taylor's expansion, we have

$$[\ldots, M(x+c\varepsilon_i)-M(x-c\varepsilon_i), \ldots]$$

$$= \ldots, 2c \frac{\partial M(\xi)}{\partial x_i} \bigg|_{\xi=x} + \frac{c^3}{6} \frac{\partial^3 M(\xi)}{\partial x_i^3} \bigg|_{\xi=x+c\tau_1e_i}$$

$$- \frac{c^3}{6} \frac{\partial^3 M(\xi)}{\partial x_i^3} \bigg|_{\xi=x-c\tau_1e_i}, \ldots$$

where $0 < \tau; < 1$, $0 < \tau_i < 1$, for $i = 1, 2, \ldots k$. Thus

$$M_c(x) = 2(\grad M)(x) + \frac{c^2}{6} \left[ \ldots, \frac{\partial^3 M(\xi)}{\partial x_i^3} \bigg|_{\xi=x+c\tau_1e_i} \right.$$

$$- \frac{\partial^3 M(\xi)}{\partial x_i^3} \bigg|_{\xi=x-c\tau_1e_i}, \ldots \bigg].$$
Hence, by using inequality (A-4) with c = 0,

\[-[x^0, M_c(x)] \leq -2K_0 \|x^0\|^2 + \frac{c^2}{3} - k^{1/2}Q \|x^0\|.

(A-22)

Now, using assumptions (i) and (iii) and inequalities (A-3) and (A-22) in inequality (A-1), we have

\[E\left\{\|x_{n+1} - x^0\|^2 : x_n \right\} \leq \|x^0 - x_n\|^2 - 4a_nK_0 \|x_{n+1} - x_n\|^2 + a_n^2k_1 \|x^0\|^2 + \frac{2}{3}a_nk^{1/2}Qc_n^2 \|x^0\|\]

\[+ \frac{2}{n^2} \left[ S + E\{x_n(x_n(x_n)) \} C^2_n \|M_{c_n}(x_n)\|^2 \right] - 2a_nS_n E\left\{x_n - x_n(x_n) - c_nM_{c_n}(x_n) \mid x_n \right\}. \]

Taking expected values, using assumption (iii), substituting an = a/n, and cn = c/n, and substituting the bound (A-7) for the third term, we have

\[b_{n+1} \leq b_n \left[ 1 - 4 \frac{aK_o}{a} \right] + \frac{2}{3}k^{1/2}Qac_n^2 \frac{ac_n^2}{n^2+2} \frac{E\{x_n - x_n\}}{\|x_n - x_n\|} \]

\[+ \frac{a_n^2}{c^2} \frac{S_3}{n^2(a-\gamma)} \left[ 1 + \frac{8k_1^2B_2c_4}{S_3n^2 \gamma} + \frac{8k_1^2c_4^4}{S_3n^4 \gamma} \right] \]

so that

\[b_n \leq b_n \left[ 1 - 4 \frac{aK_o}{a} \right] + \frac{2k^{1/2}Qac_n^2}{3n^2+2a} \frac{E\{x_n - x_n\}}{\|x_n - x_n\|} + \frac{(1+\eta) S_3a_n^2}{c_n^2n^2(a-\gamma)} \]

(A-23)

for all n \( \geq n_0(\eta) \), where \( \eta > 0 \) is arbitrary. Now if we set

\[\epsilon_n = \frac{2k^{1/2}Qc_n^2}{3\epsilon K_0 n^2} \quad 0 < \epsilon < 4 \]

(A-25)

in equality (A-18) and replace \( E\{\|x_n - x_n\|\} \) with the resulting bound, inequality (A-24) becomes

\[b_{n+1} \leq b_n \left[ 1 - \frac{(4-\epsilon) K_o a}{a} \right] + \frac{4}{9} \frac{kQ^2ac_n^4}{\epsilon K_0 a^2 n^4 \gamma} + \frac{(1+\eta) S_3a_n^2}{c_n^2n^2(a-\gamma)} \]

\[\leq b_n \left[ 1 - \frac{(4-\epsilon) K_o a}{a^3} \right] + \frac{1}{n^2} \frac{kQ^2ac_n^4}{\epsilon K_0 a^2 + \frac{(1+\eta) S_3a_n^2}{c_n^2n^2(a-\gamma)}} \]

(A-26)
where \( t = \min(\alpha + 4\gamma, 2\alpha - 2\gamma) = \frac{3}{2}\alpha + \gamma - \frac{1}{2}\left| 6\gamma - \alpha \right| \geq 1 \geq \alpha = S \). At this point we can again use Lemmas 1 and 2 to establish the positive side of Statement 3. The negative side is established by a one-dimensional example for which the reader is referred to Dupac.

### A.4 DERIVATION OF BOUNDS FOR CERTAIN SUMS

We first wish to estimate the sum

\[
\Sigma_1(n) = \sum_{i=1}^{n-1} \frac{a_i/c_1}{(n-i)^2}.
\]

(A-27)

We have assumed that \( a_i/c_1 \) is monotonically decreasing from 1 to some point \( j \) and monotone increasing from \( j \) to \( n-1 \); hence

\[
\Sigma_1 \leq a_1/c_1 \frac{1}{(n-1)^2} + \frac{a_{n-1}}{c_{n-1}} + \int_1^n (a/c)(t) \frac{1}{(n-t)^2} dt + \int_1^{n-1} (a/c)(t) \frac{1}{(n-t)^2} dt
\]

(A-28)

in which \( (a/c)(t) \) is some appropriate interpolation of \( a_i/c_i \). We note, in particular, that for \( a_n/c_n = \frac{1}{c} \frac{1}{n^\gamma} \), the function

\[
f(t) = \frac{a}{c} \frac{1}{t^{\alpha-\gamma}} \frac{1}{(n-t)^2}
\]

has the derivative

\[
f'(t) = \frac{a}{c} \frac{1}{t^{\alpha-\gamma}} \frac{1}{(n-t)^3} - (\alpha-\gamma) \frac{1}{t^{\alpha-\gamma+1}} \frac{1}{(n-t)^2}.
\]

The equation \( f'(T) = 0 \) is satisfied for \( T \) satisfying the equation

\[2T = (\alpha-\gamma+1)(n-T) \text{ or } (3+\alpha-\gamma) T = n\]

which has only one solution for \( 1 < T < n-1 \). Hence the choice \( a_n/c_n = \frac{a}{c} \frac{1}{n^\gamma} \) does satisfy our hypothesis.

We return to inequality (A-28) and rewrite it as

\[
\Sigma_1 \leq a_1/c_1 \frac{1}{(n-1)^2} + \frac{a_{n-1}}{c_{n-1}} + \int_1^{n/2} a/c(t) \frac{1}{(n-t)^2} dt + \int_{n/2}^{n-1} a/c(t) \frac{1}{(n-t)^2} dt
\]

\[\leq a_1/c_1 \frac{1}{(n-1)^2} + \frac{a_{n-1}}{c_{n-1}} + a_1/c_1 \int_1^{n/2} \frac{1}{(n-t)^2} dt + \frac{a_{n/2}}{c_{n/2}} \int_{n/2}^{n-1} \frac{1}{(n-t)^2} dt
\]

\[\leq a_1/c_1 \frac{1}{(n-1)^2} + \frac{a_{n-1}}{c_{n-1}} - a_1/c_1 \frac{2}{n} + a_1/c_1 \frac{1}{(n-1)}
\]

\[+ \frac{a_{n/2}}{c_{n/2}} \left[ \frac{2}{n-1} \right].
\]

(A-29)
By picking out the dominant term in inequality (A-29), we may thus bound $\Sigma_1$ by

$$\Sigma_1 \leq K \frac{a_{n/2}}{c_{n/2}}$$  \hspace{1cm} (A-30)

We now wish to estimate the sum

$$\Sigma_2 = \sum_{j=1}^{n-1} |h_j| [(n-1)\delta - (n-1-j)\delta].$$  \hspace{1cm} (A-31)

Now

$$(n-1)\delta - (n-1-t)\delta = (n-1-t+t)\delta - (n-1-t)\delta$$

$$= - (n-1-t)\delta + (n-1-t)\delta + \frac{\delta t}{1!(n-1-t)^{1-\delta}} + \frac{\delta(\delta-1) t^2}{2!(n-1-t)^{2-\delta}} + \ldots$$

$$= \frac{\delta t}{1!(n-1-t)^{1-\delta}} + \frac{\delta(\delta-1) t^2}{2!(n-1-t)^{2-\delta}} + \frac{\delta(\delta-1)(\delta-2) t^3}{3!(n-1-t)^{3-\delta}} + \ldots$$

This series is an alternating series whose terms are monotonically decreasing in magnitude for $0 < \delta < 1$ and $0 < t < n - 1$. The magnitude of this series is thus bounded by the magnitude of the first term, and

$$(n-1)\delta - (n-1-t)\delta \leq \frac{t\delta}{(n-1-t)^{1-\delta}}.$$  \hspace{1cm} (A-32)

Now bounding $|h_j|$ by $\frac{1}{j^3}$ for $j \geq 1$ and using the bound of inequality (A-32) in inequality (A-31), we have

$$\Sigma_2 \leq \delta \sum_{j=1}^{n-2} \frac{1}{(n-1-j)^{1-\delta}} + \frac{(n-1)\delta}{(n-1)^3}.$$  

But

$$\sum_{j=1}^{n-2} \frac{1}{j^2} \frac{1}{(n-1-j)^{1-\delta}} = \sum_{j=1}^{n-2} \frac{1}{j^{(1-\delta)}(n-1-j)^2} = \Sigma_1(n-1)$$

and hence

$$\Sigma_2 \leq K_2 \frac{1}{n^{(1-\delta)}}.$$
A.5 ALTERNATIVE RESTRICTIONS TO REPLACE RESTRICTIONS (c) AND (d)

In section 3.1 we stated that restriction (e) was more restrictive than the usual definition of strict convexity but that this could be relaxed at the expense of slightly strengthening restriction (c). Here we shall consider alternative restrictions (c') and (e') which yield the same end result as (c) and (e). We shall also elaborate on the meaning of restrictions (c) and (c'). Suppose that we replace restriction (e) by (e') for all a and b in the domain of W there exists an \( \epsilon, \epsilon_0, \) and \( \epsilon_1 \) greater than zero so that

\[
W[a(a + (1 - a)b)] \leq \begin{cases} 
    aW[a] + (1-a) W[b] & 0 \leq a \leq 1 \\
    aW[a] + (1-a) W[b] - E_0(a-b)^2 & \text{if } |a|, |b| \geq \epsilon_0 \text{ and } 0 \leq a \leq \epsilon_1
\end{cases}
\]

and simultaneously replace restriction (c) with

\[
(c') \ P \left( \left\{ \sum_{i=1}^{k} (x_i - \theta_i)(f_{i,t} - \theta_i) \right\} \geq D \|x - \theta\|, \left| d_t - \sum_{i=1}^{k} x_i f_{i,t} \right| \geq \epsilon_0, \left| d_t - \sum_{i=1}^{k} \theta_i f_{i,t} \right| \geq \epsilon_0 \right) \geq \epsilon
\]

for \( x \in X \) and \( D, \epsilon, \epsilon_0 > 0 \). It is easily shown that Eq. 17 will also follow from these two restrictions; hence, they will be sufficient to replace (c) and (e).

Let us first consider restriction (e'). It can be shown that, if

\[
0 < M_0 \leq W^e(e) \leq M_1 < \infty
\]

for

\[
0 < \epsilon_0 \leq |e| \leq K < \infty,
\]

then restriction (e') will be satisfied. Note that this will include all functions of the form \( W[e] = |e|^p, \ p > 1 \); restriction (e') is thus essentially in accord with the usual definition of strict convexity.

Next, consider restriction (c'); it requires nonzero probability of the simultaneous occurrence of the three events \( \left| d_t - \sum_{i=1}^{k} x_i f_{i,t} \right| \geq \epsilon_0, \left| d_t - \sum_{i=1}^{k} \theta f_{i,t} \right| \geq \epsilon_0 \), and

\[
\sum_{i=1}^{k} (x_i - \theta_i)(f_{i,t} - \theta_i) \geq D^2 \|x - \theta\|^2 \text{ for all } x \in X.
\]

Now there is no simple interpretation of the simultaneous occurrence of these three events, since they are not independent. However, no one of these events is of such a nature that it tends to exclude the occurrence of the other two. Thus, to examine under what conditions restriction (c') will be satisfied, we shall consider under what conditions each event occurs separately with nonzero probability. Note that nonzero probability of the occurrence of the third event constitutes restriction (c).

Since the random variables involved are bounded, it is easily shown that
\[
\begin{align*}
E \left\{ \left[ d_t - \sum_{i=1}^{k} x_i f_{i,t} \right]^2 \right\} & \geq 2 \epsilon_o^2 \\
\implies 
\Pr \left\{ \left[ d_t - \sum_{i=1}^{k} x_i f_{i,t} \right]^2 > \epsilon_o^2 \right\} & \geq \frac{\epsilon_o^2}{B}
\end{align*}
\]

in which
\[
B = \sup_{\text{all } f_i, \text{ all } d, \text{ all } x} \left\{ \left[ d_t - \sum_{i=1}^{k} x_i f_{i,t} \right]^2 \right\}
\]

Thus the nonzero probability of occurrence of the first two events is guaranteed if the mean-square error is nonzero for all parameter settings. Similarly, it can be shown that
\[
\begin{align*}
E \left\{ \left[ \sum_{i=1}^{k} (x_i - \theta_i) f_{i,t} \right]^2 \right\} & \geq 2D^2 \|x - \theta\|^2 \\
\implies 
\Pr \left\{ \left[ \sum_{i=1}^{k} (x_i - \theta_i) f_{i,t} \right]^2 > D^2 \|x - \theta\|^2 \right\} & \geq \frac{D^2}{k^2 F^2}
\end{align*}
\]

in which \( F = \sup_{\text{all } f_i} |f_i| \).

We now show that this condition on \( E \left\{ \left[ \sum_{i=1}^{k} (x_i - \theta_i) f_{i,t} \right]^2 \right\} \) follows if the \( f_{i,t} \) are linearly independent random variables. For notational convenience we will drop the \( t \) subscripts and denote \( (x_i - \theta_i) \) by \( z_i \), and \( \left[ \sum_{i=1}^{k} z_i f_{i} \right]^2 \) by \( \sum_{k=1}^{k} z_k^2 \). Since \( \left( \sum z_i f_{i} \right)^2 \geq \left[ \sum z_i (f_{i} - \bar{f}_{i}) \right]^2 \), we shall assume that the \( f_i \) have zero mean. Then
\[
\begin{align*}
\left( \sum_{i=1}^{k} z_i f_{i} \right)^2 & \leq \sum_{i=1}^{k} z_i f_{i}^2 + \sum_{i=1}^{k} z_i f_{i}^2 \\
& = \sum_{k=1}^{k} z_k^2 + 2 \rho_k \sum_{k=1}^{k} \sigma_{f_k} z_k + \sigma_{f_k}^2 z_k^2 \\
& = 1/2(1-\rho_k)(\Sigma_{k=1}^{k-1} - \sigma_{f_k} z_k)^2 + 1/2(1+\rho_k)(\Sigma_{k=1}^{k-1} + \sigma_{f_k} z_k)^2
\end{align*}
\]
in which \( \rho_k \) denotes the correlation coefficient between \( f_k \) and \( \sum_{i=1}^{k-1} z_i f_i \). But, by the assumption of linear independence of the \( f_i \),

\[
|1 - \rho_k| \geq \epsilon_k, \quad |1 + \rho_k| \geq \epsilon_k, \quad \epsilon_k > 0
\]

for all \( z_i \). Thus

\[
\left( \sum_{i=1}^{k} z_i f_i \right)^2 \geq \frac{1}{2} \epsilon_k \left( \sum_{i=1}^{k-1} \sigma_i \sigma_{f_i} z_k \right)^2 + \frac{1}{2} \epsilon_k \left( \sum_{i=1}^{k-1} + \sigma_i \sigma_{f_i} z_k \right)^2
\]

\[
\geq \epsilon_k \left[ \sum_{i=1}^{k} \sigma_i^2 + \sigma_{f_i}^2 z_k^2 \right]
\]

Repeating this procedure \( k - 1 \) more times, we obtain the desired result:

\[
\left( \sum_{i=1}^{k} z_i f_i \right)^2 \geq \sum_{i=1}^{k} \epsilon_i \sigma_i^2 z_i^2 \geq 2D^2 \sum_{i=1}^{k} z_i^2
\]

in which \( 2D^2 = \min_{i=1, 2, \ldots, k} \left( \epsilon_i \sigma_i^2 \right) > 0 \).
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