

**TABLES AND FORMULAS FOR CALCULATING
FOURIER COEFFICIENTS OF POWER-LAW DEVICES**

PAUL PENFIELD, JR.

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**MASSACHUSETTS INSTITUTE OF TECHNOLOGY
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TABLES AND FORMULAS FOR CALCULATING FOURIER COEFFICIENTS
OF POWER-LAW DEVICES

Paul Penfield, Jr.

Abstract

Tables and formulas are given to facilitate the calculation of Fourier coefficients of power-law devices that are driven from a sinusoidal source.



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INTRODUCTION

Recently, the author described a method of calculating Fourier coefficients of power-law devices driven from sinusoidal sources.¹ The integral

$$I(a, n; \beta) = \frac{1}{2\pi} \int_0^{2\pi} (1 + \beta \cos \omega t)^a \cos(n\omega t) d(\omega t)$$

was expressed in terms of a hypergeometric series, and differentiation and recursion formulas were given. In this report we give a summary of the pertinent formulas (many of them already published¹) and tables of values of $I(a, n; \beta)$.

The first set of formulas includes several examples of integrals that can be expressed in terms of the function $I(a, n; \beta)$. Many of these arise naturally in computing voltage and current coefficients and power for electrical power-law devices. The second set gives the general representation of $I(a, n; \beta)$ in terms of a hypergeometric series, with related definitions. In the third and fourth sets are recursion relations and differentiation formulas. The fifth set includes formulas for specific values of n . In the sixth set special formulas are stated for $\beta = 0$ and $\beta = 1$. The seventh set includes formulas for special values of a . The eighth set comprises some miscellaneous formulas.

The recursion relations given in the third set of formulas relate different $I(a, n; \beta)$ for values of n and a differing by integers. By repeated use of these relations, any $I(a, n; \beta)$ can be expressed in terms of $I(a', 0; \beta)$ and $I(a', 1; \beta)$, where a' lies between 0 and 1 and differs from a by some integer.

We give in Tables I and II values of $I(a, 0; \beta)$ and $I(a, 1; \beta)$ for all values of a between 0 and 1 in steps of 0.05, and for all values of β between 0 and 1 in steps of 0.02. In Table III we give values of $[I(a, 1; \beta)/\beta]$. In principle these tables, when used with the recursion relations given here, are sufficient for calculating any $I(a, n; \beta)$. In practice, however, it is sometimes easier to sum the hypergeometric series directly, rather than to use the cumbersome recursion relations.

Reference

1. P. Penfield, Jr., "Fourier Coefficients of Power-Law Devices," J. Franklin Inst. 273, 107-122 (February 1962).

FORMULAS

Set I

Integrals

$$\frac{1}{2\pi} \int_0^{2\pi} (1 \pm \beta \cos \omega t)^a e^{-jn\omega t} d(\omega t) = (\pm 1)^n I(a, n; \beta)$$

$$\frac{1}{2\pi} \int_0^{2\pi} \left[\frac{d}{dt} (1 \pm \beta \cos \omega t)^a \right] e^{-jn\omega t} d(\omega t) = (\pm 1)^n jn\omega I(a, n; \beta)$$

$$\frac{1}{2\pi} \int_0^{2\pi} \left[(1 \pm \beta \cos \omega t)^a \right]^2 e^{-jn\omega t} d(\omega t) = (\pm 1)^n I(2a, n; \beta)$$

$$\frac{1}{2\pi} \int_0^{2\pi} \left[\frac{d}{dt} (1 \pm \beta \cos \omega t)^a \right]^2 e^{-jn\omega t} d(\omega t)$$

$$= (\pm 1)^n \frac{a^2 \beta^2 \omega^2}{4} [2I(2a-2, n; \beta) - I(2a-2, n+2; \beta) - I(2a-2, n-2; \beta)]$$

$$= (\pm 1)^n \frac{a^2 \beta^2 \omega^2}{2} \left[\left(\frac{n+1}{n+2a} - \beta^2 \frac{n^2 - 2a}{n^2 - 4a} \frac{n-1+2a}{n} \right) I(2a-2, n+1; \beta) \right.$$

$$\quad \quad \quad \left. + \left(\frac{n-1}{n-2a} - \beta^2 \frac{n^2 - 2a}{n^2 - 4a} \frac{n+1-2a}{n} \right) I(2a-2, n-1; \beta) \right]$$

$$= (\pm 1)^n \frac{a^2 \beta^2 \omega^2}{2(2a-1)} [(n+1)I(2a-1, n+1; \beta) - (n-1)I(2a-1, n-1; \beta)]$$

$$= \frac{(\pm 1)^n a^2 \omega^2}{(2a-1)(2a-n)} [\beta (2a-n)^2 I(2a-1, n+1; \beta) - n(n-1) I(2a-1, n; \beta)]$$

$$= \frac{(\pm 1)^n a^2 \omega^2}{2a-1} \left[\left(1 - \frac{n^2}{2a} \right) I(2a, n; \beta) - I(2a-1, n; \beta) \right]$$

$$= (\pm 1)^n \frac{a \omega^2}{2(2a-1)} \left[\beta \frac{dI(2a, n; \beta)}{d\beta} - n^2 I(2a, n; \beta) \right]$$

$$\frac{1}{2\pi} \int_0^{2\pi} \left[\frac{d}{dt} (1 \pm \beta \cos \omega t)^a \right]^2 d(\omega t) = \frac{a^2 \beta^2 \omega^2}{2} [I(2a-2, 0; \beta) - I(2a-2, 2; \beta)]$$

$$= \frac{a \beta \omega^2}{2} [\beta I(2a-2, 0; \beta) + I(2a-2, 1; \beta)]$$

$$= \frac{a \omega^2}{2} [I(2a-1, 0; \beta) - (1-\beta^2) I(2a-2, 0; \beta)]$$

FORMULAS

Set I (continued)

$$= \frac{\alpha^2 \beta \omega^2}{2\alpha - 1} I(2\alpha - 1, 1; \beta)$$

$$= \frac{\alpha^2 \omega^2}{2\alpha - 1} [I(2\alpha, 0; \beta) - I(2\alpha - 1, 0; \beta)]$$

$$= \frac{\alpha \beta \omega^2}{2(2\alpha - 1)} \frac{dI(2\alpha, 0; \beta)}{d\beta}$$

$$\frac{1}{2\pi} \int_0^{2\pi} (1 \pm \gamma \cos^2 \omega t)^\alpha e^{-jn\omega t} d(\omega t) = \begin{cases} 0 & n \text{ odd} \\ (\pm 1)^{n/2} \left(\frac{2 \pm \gamma}{2}\right)^\alpha I\left(\alpha, \frac{n}{2}; \frac{\gamma}{2 \pm \gamma}\right) & n \text{ even} \end{cases}$$

$$\frac{1}{2\pi} \int_0^{2\pi} (1 \pm \gamma \sin^2 \omega t)^\alpha e^{-jn\omega t} d(\omega t) = \begin{cases} 0 & n \text{ odd} \\ (\mp 1)^{n/2} \left(\frac{2 \pm \gamma}{2}\right)^\alpha I\left(\alpha, \frac{n}{2}; \frac{\gamma}{2 \pm \gamma}\right) & n \text{ even} \end{cases}$$

$$\frac{1}{2\pi} \int_0^{2\pi} \frac{d}{dt} (1 \pm \gamma \cos^2 \omega t)^\alpha e^{-jn\omega t} d(\omega t) = \begin{cases} 0 & n \text{ odd} \\ jn\omega(\pm 1)^{n/2} \left(\frac{2 \pm \gamma}{2}\right)^\alpha I\left(\alpha, \frac{n}{2}; \frac{\gamma}{2 \pm \gamma}\right) & n \text{ even} \end{cases}$$

$$\frac{1}{2\pi} \int_0^{2\pi} \frac{d}{dt} (1 \pm \gamma \sin^2 \omega t)^\alpha e^{-jn\omega t} d(\omega t) = \begin{cases} 0 & n \text{ odd} \\ jn\omega(\mp 1)^{n/2} \left(\frac{2 \pm \gamma}{2}\right)^\alpha I\left(\alpha, \frac{n}{2}; \frac{\gamma}{2 \pm \gamma}\right) & n \text{ even} \end{cases}$$

$$\begin{aligned} & \frac{1}{2\pi} \int_0^{2\pi} \left[\frac{d}{dt} (1 \pm \gamma \cos^2 \omega t)^\alpha \right]^2 e^{-jn\omega t} d(\omega t) \\ &= \begin{cases} 0 & n \text{ odd} \\ (\pm 1)^{n/2} \left(\frac{2 \pm \gamma}{2}\right)^{(2\alpha-1)} \frac{\alpha^2 \gamma \omega^2}{(2\alpha-1)} \left[\left(\frac{n}{2} + 1\right) I\left(2\alpha - 1, \frac{n}{2} + 1; \frac{\gamma}{2 \pm \gamma}\right) \right. \\ & \quad \left. - \left(\frac{n}{2} - 1\right) I\left(2\alpha - 1, \frac{n}{2} - 1; \frac{\gamma}{2 \pm \gamma}\right) \right] & n \text{ even} \end{cases} \end{aligned}$$

FORMULAS

Set I (continued)

$$\begin{aligned}
 & \frac{1}{2\pi} \int_0^{2\pi} \left[\frac{d}{dt} (1 \pm \gamma \sin^2 \omega t)^a \right]^2 e^{-jn\omega t} d(\omega t) \\
 &= \begin{cases} 0 & n \text{ odd} \\ (\mp 1)^{n/2} \left(\frac{2 \pm \gamma}{2} \right)^{(2a-1)} \frac{a^2 \gamma \omega^2}{2a-1} \left[\left(\frac{n}{2} + 1 \right) I \left(2a-1, \frac{n}{2} + 1; \frac{\gamma}{2 \pm \gamma} \right) \right. \\ \quad \left. - \left(\frac{n}{2} - 1 \right) I \left(2a-1, \frac{n}{2} - 1; \frac{\gamma}{2 \pm \gamma} \right) \right] & n \text{ even} \end{cases} \\
 & \frac{1}{2\pi} \int_0^{2\pi} \left[\frac{d}{dt} (1 \pm \gamma \cos^2 \omega t)^a \right]^2 d(\omega t) = \frac{1}{2\pi} \int_0^{2\pi} \left[\frac{d}{dt} (1 \pm \gamma \sin^2 \omega t)^a \right]^2 d(\omega t) \\
 &= \left(\frac{2 \pm \gamma}{2} \right)^{(2a-1)} \frac{2a^2 \gamma \omega^2}{2a-1} I \left(2a-1, 1; \frac{\gamma}{2 \pm \gamma} \right)
 \end{aligned}$$

FORMULAS

Set II

General Representation

$$\begin{aligned} I(\alpha, n; \beta) &= \frac{(-\beta)^n}{(1 + \beta^2)^\alpha} \frac{\Gamma(n - \alpha)}{\Gamma(-\alpha)\Gamma(n + 1)} F(-\alpha, n - \alpha; n + 1; \beta^2) \\ &= \frac{\beta^n}{(1 + \beta^2)^\alpha} \frac{\Gamma(1 + \alpha)}{\Gamma(1 + \alpha - n)\Gamma(n + 1)} F(-\alpha, n - \alpha; n + 1; \beta^2) \end{aligned}$$

where the hypergeometric series is

$$\begin{aligned} F(a, b; c; z) &= 1 + \frac{ab}{c} \frac{z}{1!} + \frac{ab(a+1)(b+1)}{c(c+1)} \frac{z^2}{2!} \\ &\quad + \frac{ab(a+1)(b+1)(a+2)(b+2)}{c(c+1)(c+2)} \frac{z^3}{3!} + \dots \\ &= \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \sum_{n=0}^{+\infty} \frac{\Gamma(a+n)\Gamma(b+n)}{\Gamma(c+n)} \frac{z^n}{n!} \end{aligned}$$

and the amplitude parameters are related as

$$\begin{aligned} \beta &= \frac{2p}{1 + p^2} = \frac{k^2}{2 - k^2} \\ p &= \frac{\beta}{1 + \sqrt{1 - \beta^2}} = \left[\frac{k}{1 + \sqrt{1 - k^2}} \right]^2 \\ k &= \frac{\sqrt{2}\beta}{\sqrt{1 + \beta}} = \frac{2\sqrt{p}}{1 + p} \end{aligned}$$

FORMULAS

Set III

Recursion Relations

$$\begin{aligned}
 I(\alpha, n+2; \beta) &= - \left[\frac{n+1}{n+2+\alpha} \frac{2I(\alpha, n+1; \beta)}{\beta} + \frac{n-\alpha}{n+2+\alpha} I(\alpha, n; \beta) \right] \\
 I(\alpha+2, n; \beta) &= \frac{\alpha+2}{(\alpha+2)^2 - n^2} [(3+2\alpha)I(\alpha+1, n; \beta) - (\alpha+1)(1-\beta^2)I(\alpha, n; \beta)] \\
 I(\alpha-2, n; \beta) &= \frac{1}{(\alpha-1)(1-\beta^2)} \left[(2\alpha-1)I(\alpha-1, n; \beta) + \frac{n^2-\alpha^2}{\alpha} I(\alpha, n; \beta) \right] \\
 \beta(\alpha+1)I(\alpha, n+1; \beta) + (n-1-\alpha)I(\alpha+1, n; \beta) + (\alpha+1)I(\alpha, n; \beta) &= 0 \\
 \beta(n+1+\alpha)I(\alpha, n+1; \beta) + 2nI(\alpha, n; \beta) + \beta(n-1-\alpha)I(\alpha, n-1; \beta) &= 0 \\
 \beta(n+1+\alpha)I(\alpha, n+1; \beta) + (n-\alpha)I(\alpha, n; \beta) + \alpha(1-\beta^2)I(\alpha-1, n; \beta) &= 0 \\
 (n+1+\alpha)I(\alpha+1, n; \beta) - (\alpha+1)I(\alpha, n; \beta) - \beta(\alpha+1)I(\alpha, n-1; \beta) &= 0 \\
 (n+1+\alpha)(n-1-\alpha)I(\alpha+1, n; \beta) + (\alpha+1)(1+2\alpha)I(\alpha, n; \beta) \\
 &\quad - \alpha(\alpha+1)(1-\beta^2)I(\alpha-1, n; \beta) = 0 \\
 (n+\alpha)I(\alpha, n; \beta) + \beta(n-1-\alpha)I(\alpha, n-1; \beta) - \alpha(1-\beta^2)I(\alpha-1, n; \beta) &= 0
 \end{aligned}$$

FORMULAS

Set IV

Differentiation Formulas

$$\frac{dI(\alpha, n; \beta)}{d\beta} = \frac{\alpha}{\beta} [I(\alpha, n; \beta) - I(\alpha - 1, n; \beta)]$$

$$\frac{dI(\alpha, n; \beta)}{d\beta} = \frac{n - \alpha\beta^2}{\beta(1 - \beta^2)} I(\alpha, n; \beta) + \frac{n + 1 + \alpha}{(1 - \beta^2)} I(\alpha, n + 1; \beta)$$

$$\frac{dI(\alpha, n; \beta)}{d\beta} = -\frac{1 + \alpha + \alpha\beta^2}{\beta(1 - \beta^2)} I(\alpha, n; \beta) - \frac{(n + 1 + \alpha)(n - 1 - \alpha)}{\beta(1 - \beta^2)(1 + \alpha)} I(\alpha + 1, n; \beta)$$

$$\frac{dI(\alpha, n; \beta)}{d\beta} = -\frac{n + \alpha\beta^2}{\beta(1 - \beta^2)} I(\alpha, n; \beta) - \frac{n - 1 - \alpha}{(1 - \beta^2)} I(\alpha, n - 1; \beta)$$

$$\frac{dI(\alpha, n; \beta)}{d\beta} = \frac{n}{\beta} I(\alpha, n; \beta) + \alpha I(\alpha - 1, n + 1; \beta)$$

$$\frac{dI(\alpha, n; \beta)}{d\beta} = -\frac{n}{\beta} I(\alpha, n; \beta) + \alpha I(\alpha - 1, n - 1; \beta)$$

$$\frac{dI(\alpha, n; \beta)}{d\beta} = \frac{\alpha}{2} [I(\alpha - 1, n + 1; \beta) + I(\alpha - 1, n - 1; \beta)]$$

FORMULAS

Set V

Specific Values of n

$n = 0$

$$I(\alpha, 0; \beta) = \frac{1}{(1 + \beta^2)^\alpha} F(-\alpha, -\alpha; 1; \beta^2)$$

$$F(-\alpha, -\alpha; 1; \beta^2) = 1 + \left(\frac{\alpha\beta}{1!}\right)^2 + \left(\frac{\alpha(1-\alpha)\beta^2}{2!}\right)^2 + \left(\frac{\alpha(1-\alpha)(2-\alpha)\beta^4}{3!}\right)^2 + \dots$$

$$I(\alpha + 2, 0; \beta) = \frac{1}{2 + \alpha} [(3 + 2\alpha)I(\alpha + 1, 0; \beta) - (\alpha + 1)(1 - \beta^2)I(\alpha, 0; \beta)]$$

$$I(\alpha - 2, 0; \beta) = \frac{1}{(\alpha - 1)(1 - \beta^2)} [(2\alpha - 1)I(\alpha - 1, 0; \beta) - \alpha I(\alpha, 0; \beta)]$$

$$\frac{dI(\alpha, 0; \beta)}{d\beta} = \alpha I(\alpha - 1, 1; \beta)$$

$n = 1$

$$I(\alpha, 1; \beta) = \frac{\beta\alpha}{(1 + \beta^2)^\alpha} F(-\alpha, 1 - \alpha; 2; \beta^2)$$

$$I(\alpha, 1; \beta) = \frac{I(\alpha + 1, 0; \beta) - I(\alpha, 0; \beta)}{\beta}$$

$$I(\alpha, 1; \beta) = \frac{\alpha}{1 + \alpha} \frac{1}{\beta} [I(\alpha, 0; \beta) - (1 - \beta^2)I(\alpha - 1, 0; \beta)]$$

$n = 2$

$$I(\alpha, 2; \beta) = \frac{\beta^2\alpha(\alpha - 1)}{2(1 + \beta^2)^\alpha} F(-\alpha, 2 - \alpha; 3; \beta^2)$$

$$I(\alpha, 2; \beta) = -\frac{2}{2 + \alpha} \frac{I(\alpha, 1; \beta)}{\beta} + \frac{\alpha}{2 + \alpha} I(\alpha, 0; \beta)$$

$n = 3$

$$I(\alpha, 3; \beta) = \frac{1}{3 + \alpha} \left[\left(\frac{8}{2 + \alpha} \frac{1}{\beta^3} + \alpha - 1 \right) I(\alpha, 1; \beta) - \frac{4\alpha}{2 + \alpha} \frac{I(\alpha, 0; \beta)}{\beta} \right]$$

FORMULAS

Set VI

Specific Values of β

$$\beta = 0$$

$$I(\alpha, 0; 0) = 1$$

$$I(\alpha, n; 0) = 0 \quad n = 1, 2, 3, \dots$$

$$\frac{dI(\alpha, 1; 0)}{d\beta} = \frac{\alpha}{2}$$

$$\frac{dI(\alpha, n; 0)}{d\beta} = 0 \quad n = 0, 2, 3, 4, \dots$$

$$\beta = 1$$

$$I(\alpha, n; 1) = (-1)^n \infty \quad \alpha \leq -\frac{1}{2}$$

$$\begin{aligned} I(\alpha, n; 1) &= \frac{1}{2^\alpha} \frac{\Gamma(1+2\alpha)}{\Gamma(1+\alpha+n)\Gamma(1+\alpha-n)} \\ &= \frac{(-1)^n}{2^\alpha} \frac{\Gamma(n-\alpha)\Gamma(1+2\alpha)}{\Gamma(-\alpha)\Gamma(1+\alpha)\Gamma(n+1+\alpha)} \quad \alpha > -\frac{1}{2} \end{aligned}$$

$$I(\alpha, 0; 1) = \frac{2}{\alpha 2^\alpha} \frac{\Gamma(2\alpha)}{[\Gamma(\alpha)]^2} \quad \alpha > -\frac{1}{2}$$

$$I(\alpha, 1; 1) = \frac{2}{(1+\alpha)2^\alpha} \frac{\Gamma(2\alpha)}{[\Gamma(\alpha)]^2} \quad \alpha > -\frac{1}{2}$$

$$I(\alpha, 2; 1) = -\frac{2}{(2+\alpha)2^\alpha} \frac{1-\alpha}{1+\alpha} \frac{\Gamma(2\alpha)}{[\Gamma(\alpha)]^2} \quad \alpha > -\frac{1}{2}$$

$$I(\alpha, n+1; 1) = -\frac{n-\alpha}{n+1+\alpha} I(\alpha, n; 1) \quad \alpha > -\frac{1}{2}$$

$$I(\alpha+1, n; 1) = \frac{(1+\alpha)(1+2\alpha)}{(1+\alpha)^2 - n^2} I(\alpha, n; 1) \quad \alpha > -\frac{1}{2}$$

$$\frac{dI(\alpha, n; 1)}{d\beta} = (-1)^n \infty \quad \alpha < 0$$

$$\frac{dI(0, n; 1)}{d\beta} = 0$$

$$\frac{dI(\alpha, n; 1)}{d\beta} = (-1)^{n+1} \infty \quad 0 < \alpha \leq \frac{1}{2}$$

$$\begin{aligned} \frac{dI(\alpha, n; 1)}{d\beta} &= \frac{n^2 + \alpha^2 - \alpha}{2\alpha - 1} I(\alpha, n; 1) \\ &= \frac{n^2 + \alpha^2 - \alpha}{2\alpha - 1} \frac{1}{2^\alpha} \frac{\Gamma(1+2\alpha)}{\Gamma(1+\alpha-n)\Gamma(n+1+\alpha)} \quad \alpha > \frac{1}{2} \end{aligned}$$

$$\frac{dI(\alpha, 0; 1)}{d\beta} = \frac{\alpha-1}{2\alpha-1} \frac{2}{2^\alpha} \frac{\Gamma(2\alpha)}{[\Gamma(\alpha)]^2} \quad \alpha > \frac{1}{2}$$

$$\frac{dI(\alpha, 1; 1)}{d\beta} = \frac{\alpha^2 - \alpha + 1}{2\alpha^2 + \alpha - 1} \frac{2}{2^\alpha} \frac{\Gamma(2\alpha)}{[\Gamma(\alpha)]^2} \quad \alpha > \frac{1}{2}$$

FORMULAS

Set VII

Specific Values of α

$$\alpha = 0$$

$$I(0, 0; \beta) = 1$$

$$I(0, n; \beta) = 0 \quad n = 1, 2, 3, \dots$$

$$\alpha = 1$$

$$I(1, 0; \beta) = 1$$

$$I(1, 1; \beta) = \beta/2$$

$$I(1, n; \beta) = 0 \quad n = 2, 3, 4, \dots$$

$$\alpha = 2$$

$$I(2, 0; \beta) = 1 + \frac{\beta^2}{2}$$

$$I(2, 1; \beta) = \beta$$

$$I(2, 2; \beta) = \beta^2/4$$

$$I(2, n; \beta) = 0 \quad n = 3, 4, 5, \dots$$

$$\alpha = -1$$

$$I(-1, n; \beta) = (-\beta)^n \frac{1 + \beta^2}{1 - \beta^2}$$

$$= \frac{(-\beta)^n}{\sqrt{1 - \beta^2}[1 + \sqrt{1 - \beta^2}]^n}$$

$$\alpha = -2$$

$$I(-2, n; \beta) = \frac{(-\beta)^n}{1 - \beta^2} \left[\frac{1 + \beta^2}{1 - \beta^2} + n \right]$$

$$= \frac{(-\beta)^n}{(1 - \beta^2)[1 + \sqrt{1 - \beta^2}]^n} \left[\frac{1}{\sqrt{1 - \beta^2}} + n \right]$$

$$I(-2, 0; \beta) = \frac{1}{(1 - \beta^2)^{3/2}}$$

$$I(-2, 1; \beta) = - \frac{2\beta(1 + \beta^2)^2}{(1 - \beta^2)^3}$$

$$= - \frac{\beta}{(1 - \beta^2)^{3/2}}$$

$$\alpha = \frac{1}{2}$$

$$I(\frac{1}{2}, 0; \beta) = \frac{2}{\pi} \frac{1}{\sqrt{1 + \beta^2}} [2E(\beta) - (1 - \beta^2)K(\beta)]$$

$$= \frac{2}{\pi} \frac{1}{\sqrt{1 + \beta^2}} [E(\beta) + \beta^2 B(\beta)]$$

$$= \frac{2}{\pi} \sqrt{1 + \beta} E(k)$$

FORMULAS

Set VII (continued)

$$\begin{aligned}
 I(\frac{1}{2}, 1; \beta) &= \frac{2}{\pi} \frac{1}{3p\sqrt{1+p^2}} [(1+p^2)E(p) - (1-p^2)K(p)] \\
 &= \frac{2}{\pi} \frac{p}{3\sqrt{1+p^2}} [E(p) + B(p)] \\
 &= \frac{2}{\pi} \frac{\sqrt{1+\beta}}{3\beta} [E(k) - (1-\beta)K(k)] \\
 &= \frac{2}{\pi} \frac{\sqrt{1+\beta}}{3} [2B(k) - E(k)] \\
 I(\frac{1}{2}, n; 1) &= (-1)^{n+1} \frac{2\sqrt{2}}{\pi(4n^2-1)}
 \end{aligned}$$

$$\alpha = -\frac{1}{2}$$

$$\begin{aligned}
 I(-\frac{1}{2}, 0; \beta) &= \frac{2}{\pi} \sqrt{1+p^2} K(p) \\
 &= \frac{2}{\pi} \frac{1}{\sqrt{1+\beta}} K(k) \\
 I(-\frac{1}{2}, 1; \beta) &= -\frac{2}{\pi} \frac{\sqrt{1+p^2}}{p} [K(p) - E(p)] \\
 &= -\frac{2}{\pi} p \sqrt{1+p^2} D(p) \\
 &= -\frac{2}{\pi} \frac{1}{\beta\sqrt{1+\beta}} [K(k) - (1+\beta)E(k)] \\
 &= -\frac{2}{\pi} \frac{2\beta}{(1+\beta)^{3/2}} C(k)
 \end{aligned}$$

$$I(-\frac{1}{2}, n; 1) = (-1)^n \infty$$

$$\alpha = \frac{3}{2}$$

$$\begin{aligned}
 I(\frac{3}{2}, 0; \beta) &= \frac{2}{\pi} \frac{1}{3(1+p^2)^{3/2}} [3E(p) + 3p^4B(p) + 5p^2E(p) + 5p^2B(p)] \\
 &= \frac{2}{\pi} \frac{(1+\beta)^{3/2}}{3} [(3-2k^2)E(k) + k^2B(k)] \\
 I(\frac{3}{2}, 1; \beta) &= \frac{2}{\pi} \frac{p}{5(1+p^2)^{3/2}} [B(p) + 7E(p) + p^2E(p) + 7p^2B(p)] \\
 &= \frac{2}{\pi} \frac{2\beta\sqrt{1+\beta}}{5} [2E(k) - (1-k^2)C(k)] \\
 I(\frac{3}{2}, n; 1) &= (-1)^n \frac{2}{\pi} \frac{12\sqrt{2}}{(4n^2-1)(4n^2-9)}
 \end{aligned}$$

FORMULAS

Set VII (continued)

$$\alpha = -\frac{3}{2}$$

$$I(-\frac{3}{2}, n; \beta) = -\frac{4n^2 - 1}{1 - \beta^2} I(\frac{1}{2}, n; \beta)$$

$$I(-\frac{3}{2}, 0; \beta) = \frac{2}{\pi} \frac{(1 + p^2)^{3/2}}{(1 - p^2)^2} [E(p) + p^2 B(p)]$$

$$= \frac{2}{\pi} \frac{1}{(1 - \beta)\sqrt{1 + \beta}} E(k)$$

$$I(-\frac{3}{2}, 1; \beta) = -\frac{2}{\pi} \frac{p(1 + p^2)^{3/2}}{(1 - p^2)^2} [E(p) + B(p)]$$

$$= -\frac{2}{\pi} \frac{2\beta}{(1 + \beta)^{3/2}(1 - \beta)} [D(k) - C(k)]$$

$$= -\frac{2}{\pi} \frac{1}{(1 - \beta)\sqrt{1 + \beta}} [2B(k) - E(k)]$$

$$I(-\frac{3}{2}, n; 1) = (-1)^n \infty$$

where the elliptic integrals B, C, D, E, and K are related as

$$D(p) = \frac{K(p) - E(p)}{p^2}$$

$$B(p) = K(p) - D(p) = \frac{E(p) - (1 - p^2)K(p)}{p^2}$$

$$C(p) = \frac{D(p) - B(p)}{p^2} = \frac{(2 - p^2)K(p) - 2E(p)}{p^4}$$

FORMULAS

Set VIII

Miscellaneous

$$I(2\alpha, 0; \beta) = [I(\alpha, 0; \beta)]^2 + 2 \sum_{n=1}^{\infty} [I(\alpha, n; \beta)]^2$$

$$I(2\alpha, m; \beta) = \sum_{n=-\infty}^{\infty} I(\alpha, |n|; \beta) I(\alpha, |m - n|; \beta)$$

Table I. Values of $I(\alpha, 0; \beta)$ for β between 0 and 1 in

	$\alpha = .00$.05	.10	.15	.20	.25	.30	.35	.40	.45
$\beta = .00$	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000
.02	1.00000	1.00000	.99999	.99999	.99998	.99998	.99998	.99998	.99998	.99997
.04	1.00000	.99998	.99996	.99995	.99994	.99992	.99992	.99991	.99990	.99990
.06	1.00000	.99996	.99992	.99989	.99986	.99983	.99981	.99979	.99978	.99978
.08	1.00000	.99992	.99986	.99980	.99974	.99970	.99966	.99964	.99962	.99960
.10	1.00000	.99988	.99977	.99968	.99960	.99953	.99947	.99943	.99940	.99938
.12	1.00000	.99983	.99967	.99954	.99942	.99932	.99924	.99918	.99913	.99911
.14	1.00000	.99977	.99956	.99937	.99921	.99908	.99896	.99888	.99882	.99878
.16	1.00000	.99969	.99942	.99918	.99897	.99879	.99865	.99853	.99845	.99841
.18	1.00000	.99961	.99926	.99896	.99869	.99847	.99828	.99814	.99804	.99798
.20	1.00000	.99952	.99909	.99871	.99838	.99810	.99787	.99770	.99757	.99750
.22	1.00000	.99941	.99889	.99843	.99803	.99770	.99742	.99721	.99706	.99697
.24	1.00000	.99930	.99868	.99813	.99765	.99725	.99692	.99667	.99649	.99638
.26	1.00000	.99918	.99844	.99780	.99724	.99676	.99638	.99608	.99587	.99575
.28	1.00000	.99904	.99819	.99743	.99678	.99623	.99579	.99544	.99520	.99505
.30	1.00000	.99889	.99791	.99704	.99629	.99566	.99515	.99475	.99447	.99430
.32	1.00000	.99874	.99761	.99662	.99576	.99504	.99446	.99401	.99368	.99350
.34	1.00000	.99857	.99729	.99617	.99520	.99438	.99372	.99321	.99284	.99263
.36	1.00000	.99838	.99694	.99568	.99459	.99367	.99293	.99235	.99195	.99171
.38	1.00000	.99819	.99658	.99516	.99394	.99291	.99208	.99144	.99099	.99072
.40	1.00000	.99798	.99618	.99461	.99325	.99211	.99118	.99047	.98997	.98968
.42	1.00000	.99776	.99576	.99402	.99251	.99125	.99023	.98944	.98889	.98857
.44	1.00000	.99752	.99532	.99339	.99173	.99034	.98921	.98835	.98774	.98739
.46	1.00000	.99727	.99485	.99273	.99090	.98937	.98814	.98719	.98653	.98615
.48	1.00000	.99700	.99434	.99202	.99002	.98835	.98700	.98597	.98525	.98484
.50	1.00000	.99672	.99381	.99127	.98909	.98727	.98580	.98467	.98389	.98345
.52	1.00000	.99642	.99325	.99048	.98811	.98613	.98453	.98331	.98247	.98199
.54	1.00000	.99610	.99265	.98965	.98707	.98492	.98319	.98187	.98096	.98045
.56	1.00000	.99577	.99202	.98876	.98597	.98365	.98178	.98036	.97938	.97883
.58	1.00000	.99541	.99135	.98782	.98481	.98230	.98028	.97876	.97771	.97713
.60	1.00000	.99503	.99065	.98683	.98358	.98088	.97871	.97707	.97595	.97534
.62	1.00000	.99463	.98989	.98578	.98228	.97938	.97705	.97530	.97410	.97345
.64	1.00000	.99420	.98910	.98467	.98091	.97779	.97530	.97343	.97215	.97147
.66	1.00000	.99375	.98825	.98349	.97945	.97611	.97345	.97145	.97010	.96938
.68	1.00000	.99326	.98735	.98224	.97791	.97433	.97149	.96937	.96794	.96719
.70	1.00000	.99275	.98640	.98091	.97628	.97245	.96942	.96716	.96565	.96487
.72	1.00000	.99220	.98538	.97950	.97454	.97046	.96723	.96483	.96324	.96243
.74	1.00000	.99161	.98429	.97799	.97269	.96834	.96491	.96237	.96069	.95985
.76	1.00000	.99098	.98312	.97638	.97071	.96608	.96244	.95975	.95799	.95713
.78	1.00000	.99030	.98187	.97465	.96860	.96367	.95981	.95697	.95513	.95424
.80	1.00000	.98956	.98051	.97279	.96634	.96109	.95699	.95401	.95208	.95118
.82	1.00000	.98876	.97905	.97078	.96389	.95831	.95398	.95084	.94883	.94792
.84	1.00000	.98789	.97745	.96860	.96125	.95532	.95073	.94743	.94535	.94444
.86	1.00000	.98692	.97570	.96621	.95837	.95206	.94722	.94376	.94161	.94071
.88	1.00000	.98585	.97375	.96358	.95519	.94850	.94338	.93976	.93755	.93668
.90	1.00000	.98464	.97157	.96063	.95167	.94455	.93916	.93539	.93313	.93230
.92	1.00000	.98324	.96907	.95728	.94769	.94013	.93445	.93053	.92824	.92749
.94	1.00000	.98159	.96614	.95339	.94309	.93505	.92909	.92504	.92276	.92212
.96	1.00000	.97954	.96254	.94866	.93757	.92902	.92278	.91864	.91643	.91598
.98	1.00000	.97673	.95770	.94240	.93039	.92132	.91485	.91072	.90870	.90860
1.00	1.00000	.96965	.94654	.92910	.91622	.90708	.90106	.89769	.89662	.89757

steps of 0.02, and α between 0 and 1 in steps of 0.05.

.50	.55	.60	.65	.70	.75	.80	.85	.90	.95	1.00
1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000
.99997	.99997	.99998	.99998	.99998	.99998	.99998	.99999	.99999	.99999	1.00000
.99990	.99990	.99990	.99991	.99992	.99992	.99994	.99995	.99996	.99998	1.00000
.99977	.99978	.99978	.99980	.99981	.99983	.99986	.99989	.99992	.99996	1.00000
.99960	.99960	.99962	.99964	.99966	.99970	.99974	.99980	.99986	.99992	1.00000
.99937	.99938	.99940	.99943	.99947	.99953	.99960	.99968	.99977	.99988	1.00000
.99910	.99911	.99913	.99918	.99924	.99932	.99942	.99954	.99967	.99983	1.00000
.99877	.99878	.99882	.99888	.99897	.99908	.99921	.99937	.99956	.99977	1.00000
.99839	.99841	.99846	.99854	.99865	.99879	.99897	.99918	.99942	.99969	1.00000
.99796	.99798	.99804	.99814	.99829	.99847	.99870	.99896	.99927	.99961	1.00000
.99748	.99750	.99758	.99771	.99788	.99811	.99839	.99872	.99909	.99952	1.00000
.99694	.99697	.99707	.99722	.99744	.99771	.99805	.99845	.99890	.99942	1.00000
.99635	.99639	.99650	.99669	.99694	.99727	.99767	.99815	.99869	.99931	1.00000
.99571	.99575	.99588	.99610	.99640	.99679	.99726	.99782	.99846	.99919	1.00000
.99501	.99506	.99522	.99547	.99582	.99627	.99682	.99747	.99822	.99906	1.00000
.99425	.99432	.99449	.99479	.99519	.99571	.99634	.99709	.99795	.99892	1.00000
.99344	.99351	.99372	.99405	.99452	.99511	.99583	.99668	.99766	.99877	1.00000
.99257	.99265	.99289	.99327	.99379	.99447	.99528	.99625	.99735	.99861	1.00000
.99164	.99174	.99200	.99243	.99302	.99378	.99470	.99578	.99703	.99843	1.00000
.99065	.99076	.99106	.99154	.99220	.99305	.99408	.99529	.99668	.99825	1.00000
.98960	.98972	.99005	.99059	.99133	.99227	.99342	.99477	.99631	.99806	1.00000
.98848	.98862	.98899	.98959	.99041	.99146	.99272	.99421	.99592	.99785	1.00000
.98730	.98746	.98787	.98853	.98944	.99059	.99199	.99363	.99551	.99764	1.00000
.98605	.98623	.98668	.98741	.98841	.98968	.99121	.99302	.99508	.99741	1.00000
.98473	.98493	.98543	.98623	.98733	.98872	.99040	.99237	.99463	.99717	1.00000
.98334	.98357	.98412	.98499	.98619	.98771	.98954	.99169	.99415	.99692	1.00000
.98188	.98213	.98273	.98369	.98500	.98665	.98864	.99098	.99365	.99666	1.00000
.98034	.98062	.98128	.98232	.98374	.98554	.98770	.99023	.99313	.99639	1.00000
.97872	.97902	.97975	.98088	.98243	.98437	.98672	.98945	.99258	.99610	1.00000
.97701	.97735	.97814	.97938	.98105	.98315	.98568	.98864	.99201	.99580	1.00000
.97522	.97560	.97646	.97780	.97960	.98187	.98460	.98778	.99141	.99549	1.00000
.97334	.97376	.97469	.97614	.97809	.98054	.98347	.98689	.99079	.99516	1.00000
.97136	.97182	.97284	.97441	.97651	.97914	.98229	.98596	.99014	.99482	1.00000
.96929	.96979	.97090	.97259	.97485	.97768	.98106	.98499	.98946	.99447	1.00000
.96710	.96766	.96886	.97068	.97311	.97615	.97977	.98397	.98875	.99410	1.00000
.96480	.96542	.96672	.96856	.97130	.97455	.97842	.98291	.98801	.99371	1.00000
.96238	.96306	.96447	.96659	.96939	.97287	.97701	.98181	.98724	.99331	1.00000
.95982	.96058	.96211	.96439	.96740	.97112	.97554	.98065	.98644	.99289	1.00000
.95713	.95797	.95963	.96208	.96530	.96928	.97400	.97945	.98560	.99246	1.00000
.95428	.95521	.95701	.95965	.96310	.96736	.97239	.97819	.98473	.99200	1.00000
.95126	.95230	.95425	.95709	.96079	.96534	.97070	.97687	.98381	.99153	1.00000
.94806	.94920	.95133	.95439	.95836	.96321	.96893	.97548	.98286	.99104	1.00000
.94465	.94592	.94823	.95153	.95579	.96098	.96707	.97403	.98186	.99052	1.00000
.94099	.94242	.94494	.94850	.95307	.95862	.96510	.97251	.98081	.98998	1.00000
.93707	.93867	.94142	.94527	.95018	.95611	.96303	.97090	.97970	.98941	1.00000
.93282	.93462	.93764	.94181	.94710	.95345	.96083	.96921	.97854	.98881	1.00000
.92817	.93021	.93354	.93808	.94379	.95060	.95849	.96740	.97731	.98818	1.00000
.92302	.92536	.92905	.93402	.94020	.94753	.95597	.96547	.97600	.98751	1.00000
.91718	.91990	.92404	.92952	.93625	.94417	.95323	.96339	.97459	.98680	1.00000
.91025	.91350	.91824	.92436	.93177	.94040	.95019	.96109	.97304	.98602	1.00000
.90032	.90467	.91050	.91703	.92614	.93578	.94654	.95838	.97126	.98515	1.00000

Table II. Values of $I(\alpha, 1; \beta)$ for β between 0 and 1 in

	$\alpha = .00$.05	.10	.15	.20	.25	.30	.35	.40	.45
$\beta = .00$.00000	.00000	.00000	.00000	.00000	.00000	.00000	.00000	.00000	.00000
.02	.00000	.00050	.00100	.00150	.00200	.00250	.00300	.00350	.00400	.00450
.04	.00000	.00100	.00200	.00300	.00400	.00500	.00600	.00700	.00800	.00900
.06	.00000	.00150	.00300	.00450	.00600	.00750	.00900	.01051	.01201	.01351
.08	.00000	.00200	.00401	.00601	.00801	.01001	.01201	.01401	.01601	.01801
.10	.00000	.00251	.00501	.00751	.01002	.01252	.01502	.01752	.02002	.02252
.12	.00000	.00301	.00602	.00903	.01203	.01504	.01804	.02104	.02404	.02704
.14	.00000	.00352	.00703	.01054	.01405	.01756	.02106	.02456	.02807	.03157
.16	.00000	.00402	.00804	.01206	.01607	.02008	.02409	.02810	.03210	.03610
.18	.00000	.00453	.00906	.01359	.01811	.02262	.02713	.03164	.03614	.04064
.20	.00000	.00505	.01009	.01512	.02015	.02517	.03018	.03519	.04020	.04519
.22	.00000	.00556	.01112	.01666	.02220	.02772	.03324	.03875	.04426	.04976
.24	.00000	.00608	.01215	.01821	.02426	.03029	.03632	.04233	.04834	.05434
.26	.00000	.00661	.01319	.01977	.02633	.03287	.03940	.04592	.05243	.05893
.28	.00000	.00713	.01424	.02134	.02841	.03547	.04251	.04953	.05654	.06354
.30	.00000	.00766	.01530	.02292	.03051	.03808	.04563	.05316	.06067	.06817
.32	.00000	.00820	.01637	.02451	.03262	.04070	.04876	.05680	.06482	.07282
.34	.00000	.00874	.01744	.02611	.03475	.04335	.05192	.06047	.06899	.07749
.36	.00000	.00929	.01853	.02773	.03689	.04601	.05510	.06416	.07318	.08218
.38	.00000	.00984	.01963	.02937	.03906	.04870	.05830	.06737	.07740	.08689
.40	.00000	.01040	.02074	.03102	.04124	.05141	.06153	.07161	.08164	.09164
.42	.00000	.01097	.02187	.03269	.04345	.05415	.06479	.07538	.08591	.09641
.44	.00000	.01155	.02300	.03438	.04568	.05691	.06807	.07917	.09022	.10121
.46	.00000	.01213	.02416	.03609	.04794	.05970	.07139	.08300	.09455	.10604
.48	.00000	.01272	.02533	.03783	.05023	.06253	.07474	.08687	.09893	.11091
.50	.00000	.01333	.02652	.03959	.05254	.06539	.07813	.09078	.10334	.11582
.52	.00000	.01394	.02773	.04138	.05489	.06828	.08156	.09472	.10779	.12077
.54	.00000	.01457	.02897	.04320	.05728	.07122	.08503	.09871	.11229	.12577
.56	.00000	.01521	.03022	.04505	.05970	.07420	.08854	.10275	.11684	.13081
.58	.00000	.01586	.03150	.04693	.06217	.07722	.09211	.10684	.12144	.13591
.60	.00000	.01653	.03281	.04886	.06468	.08030	.09573	.11099	.12609	.14105
.62	.00000	.01722	.03415	.05082	.06724	.08343	.09941	.11520	.13081	.14626
.64	.00000	.01793	.03553	.05283	.06986	.08663	.10316	.11948	.13560	.15154
.66	.00000	.01865	.03694	.05489	.07253	.08989	.10698	.12382	.14045	.15689
.68	.00000	.01940	.03840	.05701	.07528	.09322	.11087	.12825	.14539	.16231
.70	.00000	.02018	.03989	.05919	.07809	.09663	.11485	.13277	.15041	.16782
.72	.00000	.02098	.04144	.06143	.08099	.10014	.11892	.13738	.15553	.17342
.74	.00000	.02182	.04305	.06376	.08397	.10374	.12310	.14210	.16076	.17912
.76	.00000	.02269	.04472	.06616	.08706	.10745	.12740	.14693	.16610	.18494
.78	.00000	.02360	.04647	.06867	.09026	.11129	.13182	.15190	.17157	.19088
.80	.00000	.02456	.04830	.07129	.09359	.11528	.13640	.15702	.17719	.19696
.82	.00000	.02558	.05023	.07404	.09703	.11943	.14115	.16232	.18298	.20321
.84	.00000	.02667	.05228	.07694	.10074	.12377	.14610	.16781	.18897	.20964
.86	.00000	.02783	.05447	.08003	.10462	.12834	.15128	.17354	.19518	.21629
.88	.00000	.02910	.05683	.08334	.10875	.13319	.15675	.17955	.20167	.22319
.90	.00000	.03049	.05942	.08694	.11321	.13838	.16257	.18590	.20848	.23041
.92	.00000	.03206	.06229	.09091	.11809	.14402	.16884	.19269	.21571	.23801
.94	.00000	.03387	.06558	.09540	.12355	.15026	.17571	.20008	.22350	.24612
.96	.00000	.03606	.06951	.10068	.12990	.15742	.18349	.20831	.23208	.25495
.98	.00000	.03900	.07465	.10745	.13784	.16619	.19282	.21801	.24199	.26496
1.00	.00000	.04617	.08605	.12119	.15270	.18142	.20794	.23274	.25618	.27856

steps of 0.02, and α between 0 and 1 in steps of 0.05.

.50	.55	.60	.65	.70	.75	.80	.85	.90	.95	1.00
.00000	.00000	.00000	.00000	.00000	.00000	.00000	.00000	.00000	.00000	.00000
.00500	.00550	.00600	.00650	.00700	.00750	.00800	.00850	.00900	.00950	.01000
.01000	.01100	.01200	.01300	.01400	.01500	.01600	.01700	.01800	.01900	.02000
.01500	.01650	.01800	.01950	.02100	.02250	.02400	.02550	.02700	.02850	.03000
.02001	.02201	.02401	.02601	.02801	.03001	.03201	.03400	.03600	.03800	.04000
.02502	.02752	.03002	.03252	.03502	.03751	.04001	.04251	.04501	.04750	.05000
.03004	.03304	.03604	.03903	.04203	.04503	.04802	.05102	.05401	.05701	.06000
.03506	.03856	.04206	.04555	.04905	.05254	.05603	.05953	.06302	.06651	.07000
.04010	.04409	.04809	.05208	.05607	.06006	.06405	.06804	.07203	.07601	.08000
.04514	.04963	.05412	.05861	.06310	.06759	.07207	.07655	.08104	.08552	.09000
.05019	.05518	.06017	.06516	.07014	.07512	.08010	.08507	.09005	.09503	.10000
.05525	.06074	.06623	.07171	.07718	.08266	.08813	.09360	.09907	.10453	.11000
.06033	.06632	.07230	.07827	.08424	.09021	.09617	.10213	.10809	.11404	.12000
.06542	.07190	.07838	.08485	.09131	.09776	.10422	.11066	.11711	.12356	.13000
.07053	.07751	.08447	.09143	.09838	.10533	.11227	.11921	.12614	.13307	.14000
.07565	.08313	.09059	.09803	.10547	.11291	.12033	.12775	.13517	.14259	.15000
.08080	.08876	.09671	.10465	.11258	.12050	.12841	.13631	.14421	.15210	.16000
.08596	.09442	.10286	.11128	.11970	.12810	.13649	.14487	.15325	.16163	.17000
.09115	.10010	.10903	.11794	.12683	.13571	.14458	.15344	.16230	.17115	.18000
.09636	.10580	.11521	.12461	.13398	.14334	.15269	.16202	.17135	.18068	.19000
.10160	.11152	.12142	.13130	.14115	.15099	.16081	.17062	.18041	.19021	.20000
.10686	.11728	.12766	.13801	.14834	.15865	.16894	.17922	.18948	.19974	.21000
.11215	.12306	.13392	.14475	.15555	.16633	.17709	.18783	.19856	.20928	.22000
.11748	.12887	.14021	.15151	.16278	.17403	.18525	.19645	.20764	.21882	.23000
.12284	.13471	.14653	.15830	.17004	.18175	.19343	.20509	.21673	.22837	.24000
.12824	.14059	.15288	.16512	.17732	.18949	.20162	.21374	.22583	.23792	.25000
.13367	.14650	.15927	.17198	.18464	.19725	.20984	.22240	.23494	.24747	.26000
.13915	.15246	.16569	.17886	.19198	.20504	.21808	.23108	.24406	.25703	.27000
.14468	.15846	.17216	.18578	.19935	.21286	.22633	.23978	.25319	.26660	.28000
.15026	.16451	.17866	.19274	.20675	.22071	.23462	.24849	.26234	.27617	.29000
.15589	.17060	.18522	.19974	.21420	.22858	.24292	.25722	.27149	.28575	.30000
.16157	.17675	.19182	.20679	.22168	.23649	.25125	.26597	.28066	.29533	.31000
.16732	.18296	.19848	.21389	.22920	.24444	.25962	.27475	.28984	.30492	.32000
.17314	.18924	.20519	.22103	.23677	.25242	.26801	.28354	.29904	.31452	.33000
.17903	.19558	.21197	.22824	.24439	.26045	.27643	.29236	.30826	.32413	.34000
.18500	.20200	.21882	.23550	.25206	.26852	.28489	.30121	.31749	.33374	.35000
.19106	.20849	.22574	.24283	.25979	.27663	.29339	.31009	.32674	.34337	.36000
.19722	.21508	.23275	.25023	.26758	.28480	.30193	.31899	.33601	.35300	.37000
.20348	.22177	.23984	.25772	.27544	.29303	.31052	.32793	.34530	.36265	.38000
.20986	.22857	.24703	.26529	.28337	.30132	.31915	.33691	.35462	.37231	.39000
.21638	.23549	.25434	.27296	.29139	.30967	.32784	.34593	.36396	.38198	.40000
.22305	.24255	.26176	.28073	.29950	.31811	.33659	.35499	.37333	.39166	.41000
.22989	.24977	.26933	.28863	.30772	.32663	.34541	.36410	.38274	.40136	.42000
.23693	.25717	.27707	.29668	.31605	.33525	.35430	.37327	.39218	.41108	.43000
.24421	.26478	.28499	.30488	.32453	.34398	.36328	.38250	.40166	.42081	.44000
.25177	.27265	.29313	.31328	.33316	.35284	.37237	.39180	.41119	.43057	.45000
.25968	.28083	.30155	.32191	.34199	.36186	.38157	.40119	.42077	.44036	.46000
.26805	.28942	.31032	.33084	.35107	.37107	.39092	.41068	.43042	.45017	.47000
.27707	.29857	.31957	.34017	.36047	.38054	.40047	.42031	.44015	.46003	.48000
.28712	.30861	.32957	.35011	.37036	.39039	.41029	.43014	.45000	.46993	.49000
.30011	.32101	.34144	.36125	.38135	.40105	.42069	.44034	.46007	.47995	.50000

Table III. Values of $\frac{I(\alpha, 1; \beta)}{\beta}$ for β between 0 and 1 in

	$\alpha = .00$.05	.10	.15	.20	.25	.30	.35	.40	.45
$\beta = .00$.00000	.02500	.05000	.07500	.10000	.12500	.15000	.17500	.20000	.22500
.02	.00000	.02500	.05000	.07501	.10001	.12501	.15001	.17501	.20001	.22501
.04	.00000	.02501	.05002	.07502	.10003	.12503	.15004	.17504	.20004	.22504
.06	.00000	.02502	.05004	.07505	.10006	.12507	.15008	.17508	.20009	.22509
.08	.00000	.02504	.05007	.07509	.10012	.12513	.15014	.17515	.20015	.22515
.10	.00000	.02506	.05011	.07515	.10018	.12521	.15022	.17524	.20024	.22524
.12	.00000	.02508	.05015	.07521	.10026	.12530	.15032	.17534	.20035	.22535
.14	.00000	.02511	.05021	.07529	.10036	.12541	.15044	.17546	.20047	.22547
.16	.00000	.02515	.05028	.07538	.10047	.12553	.15058	.17561	.20062	.22562
.18	.00000	.02519	.05035	.07548	.10059	.12567	.15073	.17577	.20079	.22579
.20	.00000	.02524	.05044	.07560	.10073	.12583	.15091	.17595	.20098	.22597
.22	.00000	.02529	.05053	.07573	.10089	.12601	.15110	.17616	.20118	.22618
.24	.00000	.02534	.05063	.07587	.10106	.12621	.15132	.17638	.20141	.22641
.26	.00000	.02540	.05075	.07603	.10125	.12643	.15155	.17663	.20167	.22666
.28	.00000	.02547	.05087	.07620	.10146	.12666	.15181	.17690	.20194	.22694
.30	.00000	.02555	.05100	.07638	.10169	.12692	.15209	.17719	.20224	.22723
.32	.00000	.02562	.05115	.07658	.10193	.12720	.15239	.17751	.20256	.22756
.34	.00000	.02571	.05131	.07680	.10220	.12750	.15271	.17785	.20291	.22790
.36	.00000	.02580	.05148	.07703	.10248	.12782	.15306	.17821	.20328	.22827
.38	.00000	.02590	.05166	.07728	.10278	.12816	.15343	.17860	.20368	.22867
.40	.00000	.02601	.05185	.07755	.10311	.12853	.15383	.17902	.20410	.22909
.42	.00000	.02612	.05206	.07784	.10345	.12892	.15426	.17946	.20456	.22954
.44	.00000	.02624	.05228	.07814	.10382	.12934	.15471	.17994	.20504	.23002
.46	.00000	.02637	.05252	.07847	.10422	.12979	.15519	.18044	.20555	.23053
.48	.00000	.02651	.05277	.07881	.10464	.13027	.15571	.18098	.20610	.23107
.50	.00000	.02666	.05304	.07918	.10509	.13077	.15626	.18155	.20668	.23165
.52	.00000	.02681	.05333	.07958	.10556	.13131	.15684	.18216	.20729	.23226
.54	.00000	.02698	.05364	.08000	.10607	.13188	.15745	.18280	.20795	.23291
.56	.00000	.02716	.05397	.08044	.10661	.13249	.15811	.18349	.20864	.23359
.58	.00000	.02735	.05432	.08092	.10719	.13314	.15881	.18421	.20938	.23432
.60	.00000	.02756	.05469	.08143	.10780	.13383	.15955	.18499	.21016	.23509
.62	.00000	.02778	.05509	.08197	.10845	.13457	.16034	.18581	.21099	.23591
.64	.00000	.02801	.05551	.08255	.10915	.13535	.16119	.18668	.21187	.23678
.66	.00000	.02826	.05597	.08317	.10990	.13619	.16209	.18761	.21281	.23771
.68	.00000	.02853	.05646	.08384	.11070	.13709	.16304	.18861	.21381	.23869
.70	.00000	.02882	.05699	.08455	.11156	.13805	.16407	.18967	.21488	.23974
.72	.00000	.02914	.05756	.08532	.11248	.13908	.16517	.19080	.21602	.24086
.74	.00000	.02948	.05818	.08616	.11347	.14019	.16635	.19202	.21724	.24205
.76	.00000	.02985	.05885	.08706	.11455	.14139	.16763	.19333	.21855	.24334
.78	.00000	.03026	.05958	.08804	.11572	.14269	.16901	.19475	.21996	.24472
.80	.00000	.03070	.06038	.08911	.11699	.14410	.17050	.19628	.22149	.24620
.82	.00000	.03120	.06126	.09029	.11839	.14564	.17214	.19795	.22315	.24782
.84	.00000	.03175	.06224	.09160	.11993	.14734	.17393	.19977	.22496	.24957
.86	.00000	.03236	.06334	.09306	.12165	.14923	.17591	.20179	.22696	.25150
.88	.00000	.03307	.06458	.09471	.12358	.15135	.17813	.20403	.22917	.25363
.90	.00000	.03388	.06602	.09660	.12579	.15375	.18063	.20656	.23164	.25601
.92	.00000	.03485	.06771	.09881	.12836	.15654	.18352	.20945	.23447	.25870
.94	.00000	.03603	.06976	.10148	.13144	.15985	.18693	.21285	.23776	.26183
.96	.00000	.03756	.07240	.10488	.13531	.16398	.19113	.21699	.24175	.26557
.98	.00000	.03980	.07617	.10964	.14065	.16958	.19676	.22246	.24693	.27037
1.00	.00000	.04617	.08605	.12119	.15270	.18142	.20794	.23274	.25618	.27856

steps of 0.02, and α between 0 and 1 in steps of 0.05.

.50	.55	.60	.65	.70	.75	.80	.85	.90	.95	1.00
.25000	.27500	.30000	.32500	.35000	.37500	.40000	.42500	.45000	.47500	.50000
.25001	.27501	.30001	.32501	.35001	.37501	.40000	.42500	.45000	.47500	.50000
.25004	.27504	.30003	.32503	.35003	.37502	.40002	.42501	.45001	.47500	.50000
.25008	.27508	.30008	.32507	.35006	.37505	.40004	.42503	.45002	.47501	.50000
.25015	.27514	.30013	.32512	.35011	.37509	.40008	.42506	.45004	.47502	.50000
.25024	.27522	.30021	.32519	.35017	.37515	.40012	.42509	.45006	.47503	.50000
.25034	.27532	.30030	.32528	.35025	.37521	.40017	.42513	.45009	.47504	.50000
.25046	.27544	.30041	.32538	.35034	.37529	.40024	.42518	.45012	.47506	.50000
.25061	.27558	.30054	.32550	.35044	.37538	.40031	.42524	.45016	.47508	.50000
.25077	.27574	.30069	.32563	.35056	.37548	.40039	.42530	.45020	.47510	.50000
.25095	.27591	.30085	.32578	.35069	.37559	.40049	.42537	.45025	.47513	.50000
.25115	.27610	.30103	.32594	.35084	.37572	.40059	.42545	.45030	.47515	.50000
.25138	.27632	.30123	.32613	.35100	.37586	.40070	.42554	.45036	.47518	.50000
.25162	.27655	.30145	.32633	.35118	.37601	.40083	.42563	.45043	.47521	.50000
.25189	.27681	.30169	.32655	.35137	.37618	.40096	.42573	.45050	.47525	.50000
.25218	.27709	.30195	.32678	.35158	.37636	.40111	.42585	.45057	.47529	.50000
.25249	.27738	.30223	.32703	.35181	.37655	.40127	.42597	.45065	.47533	.50000
.25283	.27770	.30253	.32731	.35205	.37676	.40144	.42610	.45074	.47537	.50000
.25319	.27805	.30285	.32760	.35231	.37698	.40162	.42623	.45083	.47542	.50000
.25358	.27842	.30319	.32791	.35258	.37721	.40181	.42638	.45093	.47547	.50000
.25399	.27881	.30356	.32825	.35288	.37747	.40202	.42654	.45104	.47552	.50000
.25443	.27923	.30395	.32860	.35319	.37774	.40224	.42670	.45115	.47558	.50000
.25489	.27967	.30436	.32898	.35353	.37802	.40247	.42688	.45127	.47564	.50000
.25539	.28014	.30480	.32938	.35388	.37832	.40271	.42707	.45139	.47570	.50000
.25592	.28064	.30527	.32980	.35425	.37864	.40297	.42726	.45152	.47577	.50000
.25647	.28117	.30576	.33025	.35465	.37898	.40325	.42747	.45166	.47584	.50000
.25707	.28174	.30628	.33072	.35507	.37934	.40354	.42769	.45181	.47591	.50000
.25769	.28233	.30684	.33122	.35551	.37971	.40385	.42793	.45197	.47599	.50000
.25836	.28296	.30742	.33175	.35598	.38011	.40417	.42817	.45213	.47607	.50000
.25906	.28363	.30804	.33232	.35647	.38053	.40451	.42843	.45231	.47616	.50000
.25981	.28434	.30870	.33291	.35699	.38097	.40487	.42870	.45249	.47625	.50000
.26060	.28509	.30939	.33353	.35754	.38144	.40525	.42899	.45268	.47634	.50000
.26144	.28588	.31012	.33420	.35813	.38194	.40565	.42929	.45288	.47644	.50000
.26233	.28672	.31090	.33490	.35874	.38246	.40607	.42961	.45309	.47655	.50000
.26328	.28762	.31173	.33564	.35939	.38301	.40652	.42995	.45332	.47666	.50000
.26429	.28857	.31260	.33643	.36008	.38359	.40699	.43030	.45355	.47678	.50000
.26536	.28958	.31353	.33727	.36081	.38421	.40749	.43067	.45380	.47690	.50000
.26651	.29065	.31452	.33816	.36159	.38487	.40802	.43107	.45407	.47703	.50000
.26774	.29180	.31558	.33910	.36242	.38556	.40857	.43149	.45434	.47717	.50000
.26906	.29304	.31671	.34011	.36330	.38630	.40917	.43194	.45464	.47732	.50000
.27048	.29436	.31792	.34120	.36424	.38709	.40980	.43241	.45495	.47747	.50000
.27201	.29579	.31923	.34236	.36525	.38794	.41048	.43291	.45529	.47763	.50000
.27368	.29734	.32064	.34361	.36633	.38884	.41120	.43345	.45564	.47781	.50000
.27550	.29903	.32217	.34497	.36750	.38982	.41198	.43403	.45602	.47800	.50000
.27751	.30089	.32385	.34646	.36878	.39088	.41282	.43466	.45643	.47820	.50000
.27974	.30295	.32570	.34809	.37018	.39204	.41374	.43533	.45687	.47841	.50000
.28226	.30525	.32777	.34991	.37173	.39332	.41475	.43608	.45736	.47865	.50000
.28516	.30790	.33013	.35196	.37348	.39476	.41587	.43690	.45789	.47891	.50000
.28861	.31101	.33289	.35435	.37549	.39640	.41715	.43783	.45849	.47919	.50000
.29298	.31491	.33629	.35726	.37791	.39835	.41866	.43892	.45918	.47952	.50000
.30011	.32101	.34144	.36125	.38135	.40105	.42069	.44034	.46007	.47995	.50000

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