NONLINEAR OPERATORS FOR SYSTEM ANALYSIS

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George Zames

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Abstract

In this report the method of functional iteration is introduced as a means for solving nonlinear feedback equations. It is applied to a variety of feedback problems. Equations of feedback systems are written in terms of an operator algebra extracted from functional analysis and solved by geometric iteration. This method leads to a means of bounding the output in terms of the system loop gain, and to a procedure for synthesizing systems out of iterative physical structures. The theory is applied to the construction of an explicit model for the nonlinear distortion of a feedback amplifier, and to the proof of a theorem that states that a bandlimited signal having the width of its spectrum expanded by an invertible, nonlinear, no-memory filter can be recovered from only that part of the filtered signal which lies within the original passband; a filter for recovering the original signal is derived. An exponential iteration is introduced and applied to the study of the realizability of nonlinear feedback systems. A condition, related to certain unavoidable properties of inertia and storage, is found. Models of physical systems satisfying this condition always lead to realizable solutions in feedback problems, and avoid the impossible behavior and paradoxes that can otherwise result. Iteration is used to study the convergence of functional power-series representations, and a method of preparing tables of nonlinear transforms, based on the power-series method, is described.

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I. INTRODUCTION

The physical notion of a system is that of a device that operates on input functions of time to produce related outputs. The corresponding mathematical concept is that of the operator or functional, and functional analysis provides the rigorous mathematical foundation for system engineering. It is the purpose of this work to demonstrate how nonlinear system problems can be formulated in terms of an operator calculus distilled from functional analysis, and to develop some methods for their solution. A large part of it will be devoted to the solution of feedback equations by functional iteration.

1.1 RELATION OF THE OPERATOR METHOD TO CLASSICAL METHODS

The classical starting point for a linear system problem is an assumption concerning the actual relationship between input and output. Usually this is a differential, difference or integral equation; the superposition integral is preferred because it is an explicit relation. However, in many cases of practical interest we are concerned not with finding the output for a specific input, but rather with establishing some collective property of a whole class of inputs and outputs. It may be superfluous to make assumptions here concerning the actual input-output relation; instead, it often suffices to assume that an explicit relation exists, and has certain properties. Such an explicit relation constitutes an operator. More particularly, the operator is a generalization of the ordinary function, from which it differs only in that the dependent and independent variables are not real numbers but are themselves functions of time.

Consider, for example, the following problem: It is desired to show that, if the output of a feedback system is less than half of the input in the open loop, then it is less than the input in the closed loop. It is not necessary for this purpose to employ the superposition integral. It is enough to have a definition of "feedback system."

Such an approach might seem pedantic for dealing with linear filters, at least in engineering applications. However, in nonlinear situations the unwieldiness of available representations makes it expedient to avoid them, whenever possible, in favor of this artifice. In fact, the operator notation provides the means for making unified statements that are applicable to the various representations (differential, integral, etc.). These are relatively susceptible to physical intuition because of the explicit nature of the operator relation, the close resemblance between the definition of an operator and a description of the operation of a filter, and the possibility of reasoning by analogy with ordinary functions that exist.

1.2 BACKGROUND OF THIS RESEARCH

The notion of an operator as a "function that depends on other functions" was proposed by Volterra (1) in a paper (2) published in 1887. He arrived at a representation for operators, which is a generalization of Taylor's series, in the following manner: Consider an operator $\underline{H}$ that corresponds to the system shown in Fig. 1 that maps input

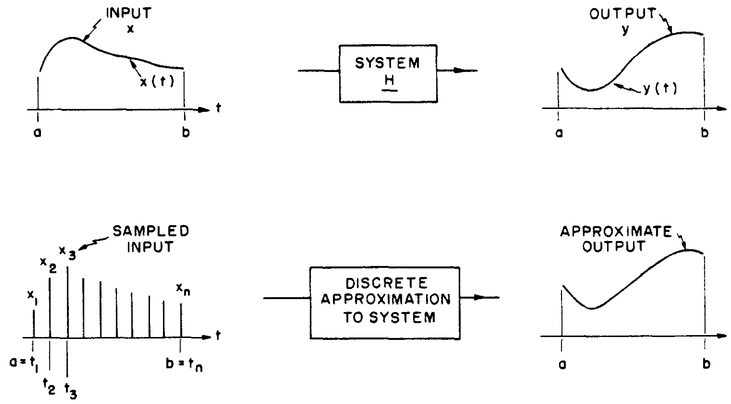


Fig. 1. Discrete approximation to a system.

functions of time, x(t), into output functions of time, y(t). Thus

$$y = \underline{H}(x) \tag{1}$$

An approximation to such an operator can be obtained by using n sample values of the input time function, x(t), and replacing the operator by an ordinary function of n variables. Such a function has a Taylor's series expansion of the type

$$y(t) = h(t) + \sum_{i=1}^{n} h_i(t)\, x_i + \sum_{i=1}^{n} \sum_{j=1}^{n} h_{ij}(t)\, x_i x_j + \dots \tag{2}$$

in which x_i is the i^{th} sample value, and $h_{ij}(t)$ is the ij^{th} Taylor coefficient, which is a function of time, t. If the number of samples, n, is increased to infinity and summations are replaced by integrations, the following representation is obtained:

$$y(t) = h(t) + \int_a^b h(t,\tau_1)\, x(\tau_1)\, d\tau_1$$

$$+ \int_a^b \int_a^b h(t,\tau_1,\tau_2)\, x(\tau_1)\, x(\tau_2)\, d\tau_1 d\tau_2 + \dots \tag{3}$$

The variables τ_1, τ_2, ... in Eq. 3 are dummy variables that have replaced the integers i, j, ... of Eq. 2.

Although Volterra formally used the representation of Eq. 1, it remained for Fréchet (3) to show, in 1910, that it was a valid representation. He established its convergence for continuous functionals of continuous functions, x, on the finite interval (a, b).

The term "functional" was coined by Hadamard. We shall designate as operators the somewhat restricted functionals that map functions of time into functions of time, which will be useful for system analysis.

Polynomial operators of the type of Eq. 3 were also studied by Liapunoff, and by

Lichtenstein, who uses them extensively in his book (4). A less general nonlinear operator has been studied by Hammerstein (5).

The interest in operators has usually stemmed from their usefulness in the boundary-value problems of physics, especially in quantum mechanics. These problems are mainly linear. As a result, nonlinear theory has had its greatest development not in functional analysis, but in the theory of differential equations, by Liapunoff and his successors. A valuable presentation of this work is that of Lefschetz (6).

Krasnoelskii (7) has made a survey of the state of nonlinear mathematics in the middle of the twentieth century.

A recent book of Wiener (8), in which he uses the concept of functionals that are orthogonal with respect to Brownian motion, is an important contribution to nonlinear functional analysis. Wiener's ideas have been pursued by several writers who have been interested in their application to communication and control theory, notably Singleton (9), Bose (10), Brilliant (11), and George (12), at the Massachusetts Institute of Technology, and Barrett (13) at Cambridge University. This report follows this sequence. This report was started concurrently with George's and the writing was finished later than his; some of the ideas elaborated here were conceived jointly by George and the writer. It introduces the method of functional iteration (14, 15), which is associated with the names of Cauchy, Lipschitz, Banach, Cacciopolli, and von Neumann.

The method of functional iteration is perhaps the most widely applicable technique for solving functional equations, and one of the very few that is useful for nonlinear equations. It is ideally suited to the study of feedback, as it can be used to solve the equations of any physically realizable feedback system (see Sec. VI). It has been employed to study nonlinear distortion in amplifiers, to formulate realizability conditions for feedback systems, and to prove a theorem concerning the inverses of certain band-limited nonlinear operators.

II. OPERATOR CALCULUS

In this section the basic language and method that are used throughout this report are presented in a heuristic manner. The operator and certain of its properties — notably those of size — are defined, and an algebra is developed for relating the properties of interconnections of systems to those of the components.

A more extensive and rigorous, but very readable, treatment of the mathematics can be found in the work of Kolmogoroff and Fomin (16).

2.1 DEFINITION OF AN OPERATOR

For our purposes, a physical system is a device that transforms functions of time into other functions of time — inputs into outputs, respectively (Fig. 2). The mathematical representation of a system is an operator, which is defined as a function whose independent and dependent variables are themselves functions of time.

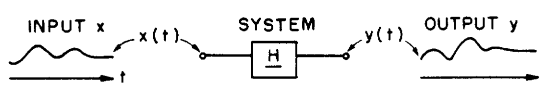


Fig. 2. A system.

It is not generally possible to draw the graph of an operator, and this makes it a more difficult concept to grasp than that of an ordinary function. Perhaps the best way to think of it is as a catalogue of pairs of functions (Fig. 3), with each output listed next to the input that generates it. This picture has its limitations because, in general, the number of functions involved is infinite and not even countable. Nevertheless it is useful, not only as an abstract model but as a means of approximate synthesis of systems, and serves as the basis for the methods of Singleton (9) and Bose (10). Both of these methods involve dividing up the collection of all possible inputs into a finite number of cells by means of a gating device. When an input that belongs to a particular cell occurs, it is "recognized" by the gate and triggers the appropriate output by means of a switching device.

However great the difficulties in representing it may be, the definition of an operator is valuable in itself, and leads to useful numerical results.

Operators are denoted by underlined capitals, such as $\underline{H}$, while functions of time are denoted by lower-case letters, for example,

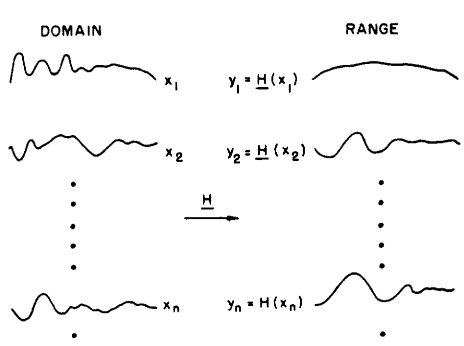


Fig. 3. An operator as a catalogue.

x and y. The fact that $\underline{H}$ operates on x to produce y is denoted by the equation

$$y = \underline{H}(x) \tag{4}$$

In referring to the function x, we mean the totality of all values assumed by x at various times, as opposed to the particular value, x(t), assumed at time t. The function x may be taken to be identical to the entire graph of x. Two functions are different if they differ at so much as a single point. Using this concept of a function, we note that a deterministic physical system can generate one, and only one, possible output in response to each input. Hence, the operator is defined to be single-valued (though several inputs may give the same output).

There are two pitfalls to be avoided: First, although it is convenient to loosely call the independent variable x in Eq. 4 the input, and the dependent variable y the output, the reverse identification is often more appropriate, for instance, when dealing with inverses. Second, there is an intuitive tendency to assign precedence in time to the input rather than to the output; this is a false move because neither has any position in time (recall that we are speaking of the whole of both) and can lead to erroneous reasoning, especially about feedback systems.

2.1.1 Domain and Range

It is customary to call the collection of all possible x in Eq. 4 the domain, and that of y the range. For example, a common domain consists of all functions of time that have finite energy (that is, finite integrated squares) and is designated L_2. We shall represent the domain and range of the operator $\underline{H}$ by Do($\underline{H}$) and Ra($\underline{H}$), respectively.

2.2 OPERATOR ALGEBRA

It is necessary, in system analysis, to relate the behavior of interconnections of subsystems – such as feedback systems – to those of the components; that is the purpose of the operator algebra.

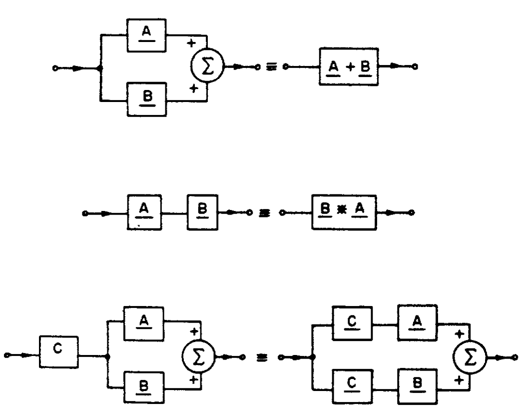


Fig. 4. Schematic representation of the sum and cascade of two operators, and of their distributive property.

The most common interconnections of systems can be decomposed into combinations of the sum and cascade, as shown in Fig. 4. Corresponding to these, we have the operator sum and cascade.

2.2.1 The Operator Sum and Cascade

The operator sum of two operators $\underline{A}$ and $\underline{B}$ is denoted $\underline{A} + \underline{B}$, and is the operator that corresponds to the interconnection in Fig. 4. The following properties of sums are obvious,

$$\underline{A} + \underline{B} = \underline{B} + \underline{A} \tag{5a}$$

$$(\underline{A}+\underline{B}) + \underline{C} = \underline{A} + (\underline{B}+\underline{C}) \tag{5b}$$

Similarly, the cascade of $\underline{B}$ following $\underline{A}$, which is denoted $\underline{B} * \underline{A}$, corresponds to the cascade of two systems. It satisfies an equation analogous to Eq. 5b,

$$(\underline{A}*\underline{B}) * \underline{C} = \underline{A} * (\underline{B}*\underline{C}) \tag{6}$$

However, cascades are not, in general, commutative, for the sequence in which two nonlinear operators occur in cascade matters. (Consider, for example, the difference between having a clipper precede or follow an amplifier.) They are commutative in the special case in which both operators are linear and time-invariant.

Sums and cascades are distributive (Fig. 4), provided that the sum follows, that is,

$$(\underline{A}+\underline{B}) * \underline{C} = (\underline{A}*\underline{C}) + (\underline{B}*\underline{C}) \tag{7}$$

2.2.2 The Zero and Identity Operators

The zero and identity operators (Fig. 5) play a central role in operator theory.

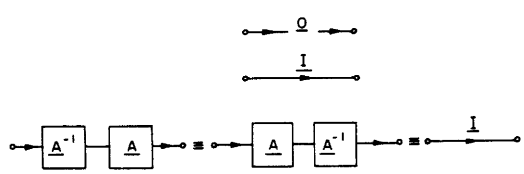


Fig. 5. The zero, identity, and inverse operators.

The zero operator, $\underline{0}$, corresponds to the open-circuit system whose output is zero whatever the input, that is,

$$\underline{0}(x) = o \tag{8}$$

so that it can be added to any operator without changing it. Thus

$$\underline{0} + \underline{A} = \underline{A}$$

The negative of any operator $\underline{A}$ is that operator $\underline{-A}$ which gives $\underline{0}$ when added to $\underline{A}$, that is,

$$\underline{A} + (-\underline{A}) = \underline{O}$$

The identity operator, $\underline{I}$, corresponds to the short-circuit system whose output always equals the input, that is,

$$\underline{I}(x) = x$$

so that it can be cascaded with any operator without changing it, that is,

$$\underline{I} * \underline{A} = \underline{A} * \underline{I} = \underline{A} \tag{9}$$

2.2.3 The Inverse

The inverse $\underline{A}^{-1}$ of an operator $\underline{A}$ "undoes," so to speak, what the operator $\underline{A}$ has done. When it is cascaded with $\underline{A}$ – either ahead of $\underline{A}$ or following it – the identity operator results, that is (Fig. 5),

$$\underline{A} * \underline{A}^{-1} = \underline{A}^{-1} * \underline{A} = \underline{I} \tag{10}$$

Any physical system that is one-to-one (that is, different inputs produce different outputs) can always be represented by an operator that has an inverse. (This condition is both necessary and sufficient.)

Inverses are often useful as series compensating systems and as demodulators. However, we are concerned with them primarily because, when they exist, they permit implicit operator equations to be written explicitly.

2.2.4 Equations of Feedback Systems

Let us use the operator algebra to relate the open-loop and closed-loop feedback

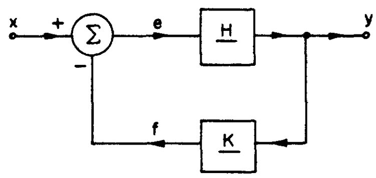


Fig. 6. A feedback system.

systems of Fig. 6. The feedback system must simultaneously satisfy the equations

$$\left.\begin{aligned} y &= \underline{H}(e) \\ f &= \underline{K}(y) \\ e &= x - f \end{aligned}\right\} \tag{11}$$

Since a relation between x and y is desired, e and f can be eliminated, and the equation

$$y = \underline{H}(x-\underline{K}(y)) \tag{12}$$

is left. If, now, x and y are operationally related, let $\underline{G}$ denote this "closed-loop" operator, so that we have

$$y = \underline{G}(x) \tag{13}$$

Substituting Eq. 13 in Eq. 12, we obtain

$$\underline{G}(x) = \underline{H}(x-\underline{K}(\underline{G}(x)))$$

Since this must hold for all inputs x, we may write it in its operational form,

$$\underline{G} = \underline{H} * (\underline{I}-\underline{K}*\underline{G}) \tag{14}$$

Equation 14 is the basic feedback equation. A similar equation can be derived for the error operator $\underline{E}$ that relates the error signal e to the input x.

The feedback equation, Eq. 14, is an implicit equation in $\underline{G}$. It can be solved for $\underline{G}$ – put into explicit form – if certain inverses exist; for, suppose that the inverse $\underline{H}^{-1}$ exists, we can cascade it with Eq. 14 to give

$$\underline{H}^{-1} * \underline{G} = \underline{I} - \underline{K} * \underline{G} \tag{15}$$

Equation 15 can be regrouped to give

$$\underline{H}^{-1} * \underline{G} + \underline{K} * \underline{G} = \underline{I}$$

Now factoring out $\underline{G}$, we obtain

$$(\underline{H}^{-1}+\underline{K}) * \underline{G} = \underline{I} \tag{16}$$

and assuming that the bracketed operator has an inverse, we arrive at the explicit form

$$\underline{G} = (\underline{H}^{-1}+\underline{K})^{-1}. \tag{17}$$

Here, we have cascaded both sides of Eq. 16 with the inverse. Equation 17 may also be written in the form,

$$\underline{G} = \underline{H} * (\underline{I}+\underline{K}*\underline{H})^{-1}$$

which has a distinct resemblance to the feedback equation of linear transform theory,

$$\underline{G}(\omega) = \frac{\underline{H}(\omega)}{1 + \underline{K}(\omega)\, \underline{H}(\omega)}, \tag{18}$$

in which all terms are spectra of the various operators.

These results can be summarized as follows:

(i) Equations for the feedback operator $\underline{G}$ (Fig. 6):

a. $\underline{G}(x) = y$

b. $\underline{G} = \underline{H} * (\underline{I}-\underline{K}*\underline{G})$

c. $\underline{G} = \underline{H} * (\underline{I}+\underline{K}*\underline{H})^{-1}$

d. $\underline{G} = (\underline{H}^{-1}+\underline{K})^{-1}$

(ii) Equations for the error operator $\underline{E}$ (Fig. 6):

a. $\underline{E}(x) = e$

b. $\underline{E} = \underline{I} - \underline{K} * \underline{H} * \underline{E}$

c. $\underline{E} = (\underline{I}+\underline{K}*\underline{H})^{-1}$

d. $\underline{E} = \underline{I} - \underline{K} * \underline{G}$

(iii) These equations are valid for unity feedback systems if $\underline{K}$ is replaced by $\underline{I}$.

Note that, in general, the feedback equation may not have any solution for $\underline{G}$, whereupon $\underline{G}$ does not exist. This question is dealt with in Section VI, in which the realizability of mathematical models of feedback systems is studied.

The necessary and sufficient condition for the existence of a solution can be shown to be the existence of the inverse $(\underline{I}+\underline{K}*\underline{H})^{-1}$; the existence of $\underline{H}^{-1}$ is not necessary.

2.3 SOME SPECIAL OPERATORS

A no-memory operator corresponds to a system whose output at any time depends entirely on the input at that time. A no-memory operator can be specified by its graph.

A linear operator is one for which cascading following summation is distributive; that is, if $\underline{H}$ is a linear operator, then we have

$$\underline{H} * (\underline{A}+\underline{B}) = (\underline{H}*\underline{A}) + (\underline{H}*\underline{B}) \tag{19}$$

in which $\underline{A}$ and $\underline{B}$ are arbitrary operators. Linear operators can be represented by superposition integrals.

A time-invariant operator represents a system whose input-output relation is independent of time; any time shift of the input produces an equal time shift of the output.

Linear, time-invariant operators are commutative in cascade; that is, we have

$$\underline{A} * \underline{B} = \underline{B} * \underline{A} \tag{20}$$

for any two such operators $\underline{A}$ and $\underline{B}$.

2.4 NORMS

The concept of integral squared error is a familiar one in engineering. It is a particular instance of the more general mathematical concept of a norm as the measure of the size of a function – not merely an undesirable error but any function. The assignment of a norm reduces a problem from an infinite number of dimensions to one, in which comparisons (better or worse) and decisions can be made.

We shall use norms for at least two purposes: First, we shall obtain relationships between the norms of the outputs and inputs of feedback systems; and second, we shall use them to estimate the errors incurred in making certain iterative approximations to feedback systems. Three different kinds of norms will be used: the least upper bound norm, the energy norm, and the impulse norm. In each case the norm of a function x is denoted $\|x\|$ with an appropriate subscript to indicate the type of norm.

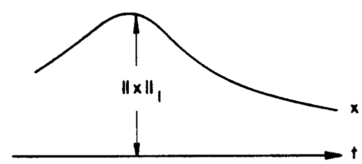


Fig. 7. Least upper bound norm of a function.

The least upper bound (l.u.b.) norm (Fig. 7) of a function is its peak absolute value. More accurately it is the least upper bound to the absolute values assumed by the function, when its independent variable ranges over the interval of definition (domain). This is denoted by the equation

$$\| x \|_1 = \underset{t}{\text{l.u.b.}} \; | x(t) | \tag{21}$$

in which the right side should be read "least upper bound of the absolute value of x(t) as t ranges over all its values." The subscript "1" will always denote the least upper bound norm, although subscripts will be omitted when the type of norm is understood or immaterial.

The energy norm is the familiar root-integral square,

$$\| x \|_2 = \left(\int_t | x(t) |^2 \, dt \right)^{1/2} \tag{22}$$

in which the integration ranges over all t in the definition interval.

The impulse norm will be defined in Section VI.

2.4.1 Properties of Norms

The usefulness of a norm depends on its judicious choice, and that is the subject of Decision Theory, which is beyond our present scope. However, all norms share three basic properties, which prevent conflicts with intuition and give their usefulness a measure of generality:

(a) Norms are positive real numbers,

$$\| x \| \geqslant 0$$

and are zero only if the function itself is zero.

(b) If a is any real number, we have,

$$\| ax \| = | a | \, \| x \|$$

(c) Norms satisfy the triangle inequality,

$$\| x+y \| \leqslant \| x \| + \| y \|$$

which asserts that the norm of a sum of functions is not greater than the sum of their norms, and is analogous to the property that the length of any side of a triangle is

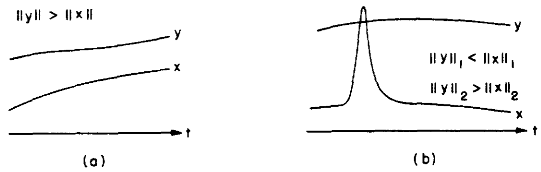


Fig. 8. Examples of normed functions.

not greater than the sum of the lengths of the other two sides.

These properties ensure, for example, that, if one function is everywhere bigger than another (Fig. 8a), then its norm is bigger than that of the other. However, they are inconclusive with regard to functions that cross, such as those in Fig. 8b; in this example one function is bigger in the energy norm but smaller in the least upper bound norm. We shall use these properties repeatedly.

2.5 GAINS

It is now desired to compute or bound the norm of the output of a system when that of the input is known. In order to do this, it is necessary to know the amplification of the system, which will differ, in general, for every input. We are especially interested in the largest incremental amplification that a system is capable of. The maximum incremental gain of an operator is therefore defined as the largest possible ratio of the norm of the difference between any two outputs to that between the inputs, that is

$$\text{incr}|||\underline{H}||| = \underset{x,y}{\text{l.u.b.}} \frac{\|\underline{H}(x)-\underline{H}(y)\|}{\|x-y\|} \tag{23}$$

in which $\text{incr}|||\underline{H}|||$ denotes the maximum incremental gain of the operator $\underline{H}$, and $\underset{x,y}{\text{l.u.b.}}$ denotes the least upper bound over all possible pairs, x and y, of time functions.

Since we shall not be using any gain other than the maximum incremental gain, it will often be referred to simply as the gain. Of course, the gain depends on the particular norm that is being used, and that will be indicated by a subscript following the gain symbol when there is any danger of ambiguity. Thus, the maximum incremental gain in the least upper bound norm is $\text{incr}|||\underline{H}|||_1$.

2.5.1 Properties of Gains

Again, it is possible to define gains in many ways; for example, the gain may be averaged over the statistics of some ensemble of inputs. However, all gains have the properties of norms, and therefore give a measure of the size of the operator. These properties are represented by the equations

$$\begin{aligned} |||\underline{H}||| &\geqslant 0 \\ |||a\underline{H}||| &= |a|\,|||\underline{H}||| \\ |||\underline{H}+\underline{K}||| &\leqslant |||\underline{H}||| + |||\underline{K}||| \end{aligned} \tag{24}$$

In addition, gains have this fourth and very important property,

$$|||\underline{H}*\underline{K}||| \leq |||\underline{H}||| \cdot |||\underline{K}||| \tag{25}$$

which asserts that the gain of a pair of operators in cascade is not greater than the product of their respective gains.

Note that the definition, Eq. 23, implies the following inequality,

$$\| \underline{H}(x)-\underline{H}(y) \| \leq \text{incr}|||\underline{H}||| \cdot \|x-y\| \tag{26}$$

which bounds the norm of the difference between any two outputs in terms of that between the corresponding inputs. An inequality of this type is called a Lipschitz condition, and is often equivalent to a bound on the slope of the operator.

2.6 COMPUTATION OF GAINS

Gains are usually easy to compute or to bound.

2.6.1 No-Memory, Linear, Time-Invariant Operators

The gain of a no-memory, time-invariant operator is the largest absolute value of its slope (Fig. 9). This is true in both the least upper bound and energy norms.

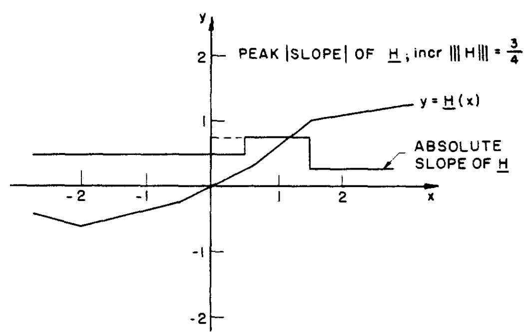


Fig. 9. Computation of the gain of the no-memory, time-invariant operator $\underline{H}$.

The least upper bound gain of a linear, time-invariant operator is the integral of the absolute value of its impulse response. Thus, if the operator has the superposition integral representation of the type

$$y(t) = \int_{-\infty}^{\infty} h(\tau)\, x(t-\tau)\, d\tau \tag{27}$$

then the following inequality holds,

$$|y(t)| \leq \left[\underset{-\infty<t<\infty}{\text{l.u.b.}} |x(t)| \right] \int_{-\infty}^{\infty} |h(\tau)|\, d\tau \tag{28}$$

which can also be written in the form

$$\|y\|_1 \leq \|x\|_1 \int_{-\infty}^{\infty} |h(\tau)|\, d\tau \tag{29}$$

It is clear from Eq. 29 that the gain is not greater than the integral on the left; in fact, it can be shown that the gain equals the integral because there is at least one x for which Eq. 28 is satisfied with the equality sign.

The energy gain of a linear, time-invariant system can be shown to be the largest value of the magnitude of its (Fourier) frequency response. To show this, Parseval's theorem is used:

$$\begin{aligned}\|y\|_2 &= \left(\int_{-\infty}^{\infty} |y(t)|^2\, dt\right)^{1/2} \\ &= \left(\int_{-\infty}^{\infty} |Y(\omega)|^2\, d\omega\right)^{1/2}\end{aligned} \tag{30}$$

in which y is any output function of time, and $Y(\omega)$ is its Fourier transform given by

$$Y(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} y(t)\, \epsilon^{-j\omega t}\, dt \tag{31}$$

If the spectrum of the system is denoted by $H(\omega)$, and that of the input by $X(\omega)$, we have

$$Y(\omega) = H(\omega)\, X(\omega) \tag{32}$$

which, when it is substituted in Eq. 30, leads to the inequality

$$\begin{aligned}\|y\|_2 &= \left(\int_{-\infty}^{\infty} |H(\omega)X(\omega)|^2\, d\omega\right)^{1/2} \\ &\leq \left[\underset{-\infty<\omega<\infty}{\text{l. u. b.}} |H(\omega)|\right] \cdot \left(\int_{-\infty}^{\infty} |X(\omega)|^2\, d\omega\right)^{1/2} \\ &= \left[\underset{-\infty<\omega<\infty}{\text{l. u. b.}} |H(\omega)|\right] \cdot \|x\|_2\end{aligned} \tag{33}$$

It is clear from Eq. 33 that the gain is not greater than the largest value (or rather the least upper bound) of $|H(\omega)|$. It can, in fact, be shown that it equals the gain because there is at least one x for which Eq. 33 is satisfied with the equality sign.

The least upper bound gain of a cascade of one linear operator and one no-memory operator, both of which are time-invariant, is the product of their respective gains.

2.6.2 Bounds on the Gains of Feedback Systems

We shall obtain bounds on the gains of the closed-loop operator $\underline{G}$, and error operator $\underline{E}$, in terms of the open-loop operator $\underline{H}$. The error operator $\underline{E}$ in Fig. 10 satisfies the equation

$$\underline{E} = \underline{I} - \underline{H} * \underline{E} \tag{34}$$

hence, the gain of $\underline{E}$ can be bounded by using the properties of gains given in

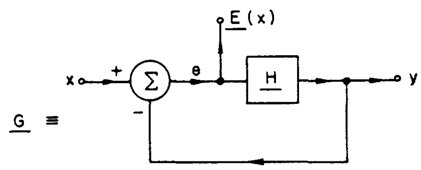


Fig. 10. A unity feedback system.

section 2.5.1. We have

$$\begin{aligned} \text{incr}|||\underline{E}||| &= \text{incr}|||\underline{I}-\underline{H}*\underline{E}||| \\ &\leq \text{incr}|||\underline{I}||| + \text{incr}|||\underline{H}*\underline{E}||| \\ &\leq 1 + [\text{incr}|||\underline{H}|||\cdot\text{incr}|||\underline{E}|||] \end{aligned} \tag{35}$$

(The gain of the identity operator is 1.) Provided that the condition $\text{incr}|||\underline{H}||| < 1$ is satisfied, Eq. 35 can be solved for $\text{incr}|||\underline{E}|||$ to give

$$\text{incr}|||\underline{E}||| \leq \frac{1}{1-\text{incr}|||\underline{H}|||} \tag{36}$$

for the required bound. The gain of $\underline{G}$ can be bounded in a similar manner,

$$\text{incr}|||\underline{G}||| \leq \frac{\text{incr}|||\underline{H}|||}{1-\text{incr}|||\underline{H}|||} \tag{37}$$

For example, if $\underline{H}$ is the no-memory operator of Fig. 9, for which $\text{incr}|||\underline{H}||| = 3/4$, then $\underline{E}$ and $\underline{G}$ are no-memory operators whose gains are $\frac{1}{1-3/4} = 4$ and $\frac{3/4}{1-3/4} = 3$, respectively.

2.6.3 Bounds on the Gains of Power-Series Operators

A very large class of operators has the power-series representation (see Brilliant (11)),

$$y(t) = \sum_{n=1}^{\infty} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} h(t, \tau_1, \tau_2, \ldots, \tau_n)\, x(\tau_1) \ldots x(\tau_n)\, d\tau_1 \ldots d\tau_n \tag{38}$$

The least upper bound gain of such an operator can be computed by forming a "majorant" no-memory operator out of the power series

$$y = \sum_{n=1}^{\infty} a_n x^n$$

in which the coefficients a_n are related to the kernels in Eq. 38 by

$$a_n = \underset{t}{\text{l.u.b.}} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} |h(t, \tau_1, \tau_2, \ldots, \tau_n)| \, d\tau_1 \ldots d\tau_n$$

The gain of the operator is then less than the greatest slope of the majorant. (It will be finite only if a restriction is placed on the greatest norm of allowable inputs, and the majorant is then computed only for values of x that are not greater than this greatest norm.)

In general this method gives only an upper bound, and often a loose upper bound, to the least upper bound gain. It gives the actual gain in the special cases of the no-memory and linear operators that were considered in section 2.6.1.

III. GEOMETRIC ITERATION THEORY AND FEEDBACK SYSTEMS

This section shows the elements of the simplest kind of iteration theory, and gives applications to the construction of models for feedback systems, nonlinear distortion, and inverses.

3.1 A HEURISTIC DISCUSSION

It has been shown in Section II that a feedback system such as that in Fig. 11a is described by the implicit operator equation

$$\underline{G} = \underline{H} * (\underline{I}-\underline{G}) \tag{39}$$

We are now faced with the problem of solving equations of the type of Eq. 39 for the unknown operator $\underline{G}$, or, what amounts to the same thing, of finding the closed-loop response of a feedback system when the open-loop input-output relation is given.

The functioning of a negative feedback system can be described in the following terms: The input produces an output; the error between input and output is then measured and produces a new output, and so on, until an equilibrium is reached. This is an inaccurate description of feedback, but a good portrayal of the iteration that will be derived. It carries an implication of time delay around the loop.

If the loop has a pure delay of length T, then iteration is indeed a valid means of solving the feedback equation, Eq. 39. Let us see why this is true. For the sake of argument, assume that the delay is in the reverse part of the loop (dotted box in Fig. 11a). If the input x commences at time zero, then there is no feedback f during

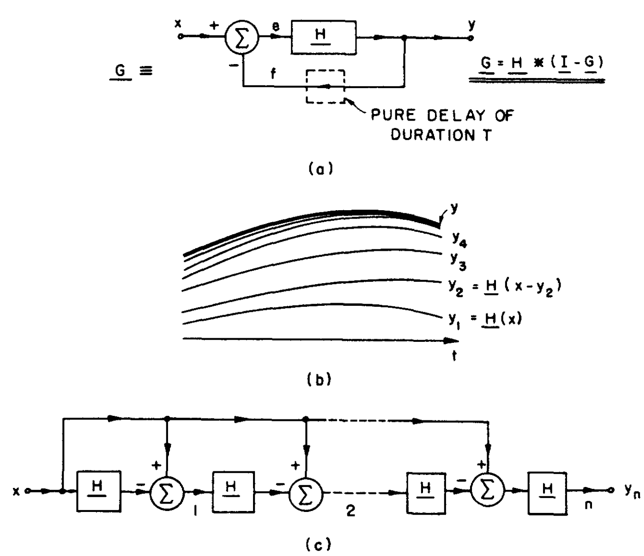


Fig. 11. (a) A unity feedback system.
(b) A sequence of approximations to the output, y.
(c) An approximate iterative structure for the feedback system of (a).

the first T seconds, so that the open-loop system $\underline{H}$ and the closed-loop system $\underline{G}$ behave identically during that time. The first approximation is, then,

$$\underline{G}_o(x) = \underline{H}(x)$$

and it gives the output y accurately for the first T seconds. Now the feedback signal f is simply y delayed by T seconds, so that f is known accurately until 2T seconds; but e and y are known accurately in any interval in which f is known accurately, so that y is now determined until 2T seconds, and so on. The n^{th} approximation is accurate until nT seconds, and is given by

$$\underline{G}_n(x) = \underline{H} * (\underline{I}(x)-\underline{G}_{n-1}(x)), \qquad n = 2, 3, \ldots \tag{40}$$

and y can be found as the limit of the sequence

$$y = \underline{G}(x) = \lim_{n\to\infty} \underline{G}_n(x)$$

Now, we do not wish to confine ourselves to systems that have a pure delay. Hence, we are tempted to seek a more general condition under which iteration is valid. The condition that will be described here is not really a generalization of delay — that will be derived in Section V — but a low loop-gain condition.

3.1.1 Convergence with a Contraction: Bounds

In order that iteration may be a valid method for solving Eq. 39, two things must happen: First, the iteration must converge; and second, it must converge to a value that satisfies Eq. 39. Now, iteration leads to a sequence of time functions such as those in Fig. 11b, which are required to converge to the true output. In order that they may converge, the separations between successive pairs of functions must keep getting smaller. This suggests that the gain of the operator $\underline{H}$ must be less than 1, since H, which will be called the loop operator, operates once in every cycle of iteration. Let us examine this more closely: The size of the difference between successive approximations is $\|y_n-y_{n-1}\|$, in which we have written y_n in place of $\underline{G}_n(x)$. This may be related to the preceding difference by using Eq. 40:

$$\|y_n-y_{n-1}\| = \|\underline{H}(x-y_{n-1})-\underline{H}(x-y_{n-2})\|$$

The result that we should like to obtain is

$$\|y_n-y_{n-1}\| \le a\|y_{n-1}-y_{n-2}\|$$

with a equal to some fraction that is less than 1. This will come about if the maximum <u>incremental</u> gain of $\underline{H}$ is a or less. Such an operator, which reduces the distances between all pairs of time functions on which it operates by some fraction that is less than a, which itself is less than 1, is called a contraction.

In this manner we get a sequence of approximations with the sizes of successive differences bounded by the geometric progression

$$\| \underline{H}(x) \| , \; a \| \underline{H}(x) \| , \; a^2 \| \underline{H}(x) \| , \; \ldots$$

The size of the output to which they converge must be bounded by the sum of these increments, that is,

$$\| y \| = \| \underline{G}(x) \| \leq \frac{1}{1 - a} \| \underline{H}(x) \|$$

and provided that $\underline{H}(o) = o$, this can be simplified.

$$\| \underline{G}(x) \| \leq \frac{a}{1 - a} \| x \| \tag{41}$$

The norm of the output of this feedback system is not greater than $a/(1-a)$ times that of the input.

The implicit feedback equation can not only be solved by iteration, but the feedback system itself can be approximated by a finite, explicit, physical, iterative structure such as that shown in Fig. 11c.

3.1.2 Transformations That Change Loop Gain: Nonlinear Distortion

The low loop-gain condition at first seems very restrictive. However, it is often possible to transform high loop-gain equations into a low loop-gain form. A typical transformation is illustrated in Fig. 12a. The operator $\underline{H}$ is split into the sum of two parts,

$$\underline{H} = \underline{H}_a + \Delta\underline{H}$$

and $\underline{H}_a$ is shifted into the forward part of the loop, giving rise to a subsystem, $\underline{G}_a$, which appears in cascade with $\Delta\underline{H}$. The resulting feedback system has $\Delta\underline{H} * \underline{G}_a$ for its

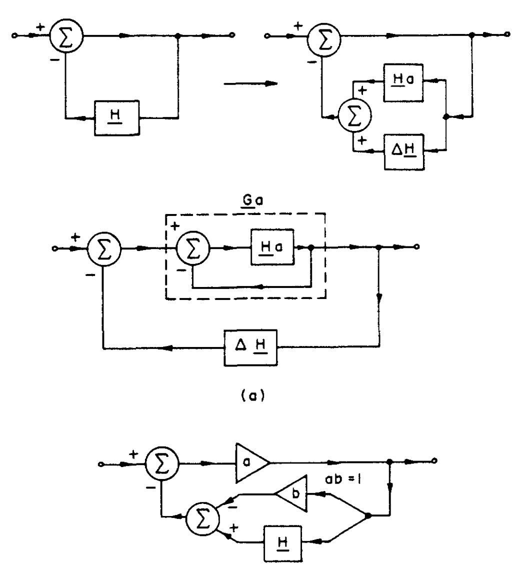


Fig. 12. (a) Example of a transformation that can change loop gain.
(b) Implementation of an inverse by means of a feedback system.

loop operator, and, if the splitting has been done cleverly, a low loop gain. Various transformations will be described and compared in sections 3.4 and 3.5.

Transformations of this type will be employed together with iterative structures such as that shown in Fig. 11c to derive explicit models for the nonlinear distortion in amplifiers (see section 3.6). The first approximation model, which avoids iteration altogether, is especially useful.

3.1.3 Inverses

It is often possible, at least theoretically, to build inverses (demodulators, for example) in the form of negative feedback systems. Accordingly, it is possible to show that they exist and to study their properties by means of an iteration. We shall first illustrate how the inverse of a linear, time-invariant system can be formally synthesized as a feedback loop. The same method will then be applied to nonlinear systems in section 3.7 and to bandlimited nonlinear systems in Section IV.

If a linear, time-invariant system with frequency response $H(\omega)$ is placed in the backward part of a feedback loop, the resulting closed-loop frequency response

$$G(\omega) = \frac{1}{1 + H(\omega)}$$

This is not quite an inverse, because of the 1 in the denominator. However, the 1 can be eliminated by taking two pure gains of magnitudes a and b, with ab = 1, and placing a in the forward part of the loop while subtracting b from H (Fig. 12b) to give

$$G(\omega) = \frac{a}{1 + a(H(\omega)-b)}$$

$$= \frac{a}{(1-ab) + aH(\omega)} = \frac{1}{H(\omega)} \tag{42}$$

Here, G is the required inverse.

In general, this is only a formal procedure. For an invertible linear system it is always mathematically valid, although the solution may not be physically realizable – at least not accurately realizable. For a nonlinear system it may even lack mathematical validity when the simultaneous feedback equations have no common solution. In Section IV an example of a nonlinear system is illustrated for which this procedure is valid.

3.2 ITERATION FOR THE ERROR OPERATOR $\underline{E}$: CONTRACTIONS

In this and in the following sections, the results described in section 3.1.1 will be elaborated and put on a rigorous basis. We shall first consider the error equation,

$$\underline{E} = \underline{I} - (\underline{H} * \underline{E}) \tag{43}$$

(to which $\underline{G}$ can be related by the transformation $\underline{G} = \underline{H} * \underline{E}$) and show that it has a unique solution for $\underline{E}$ that can be found by means of the iteration

$$\underline{E}_o = 0$$

$$\underline{E}_n = \underline{I} - (\underline{H} * \underline{E}_{n-1}) \tag{44}$$

provided that $\underline{H}$, which will be referred to as the "loop operator," has a maximum incremental gain α that is less than 1; that is, $\alpha = \text{incr}|||\underline{H}||| < 1$. An operator having this property is called a contraction because it reduces the distances between all pairs of time functions x, y, according to the inequality

$$\| \underline{H}(x)-\underline{H}(y) \| \leq \alpha \| x-y \| \tag{45}$$

It is emphasized that α may not equal 1.

It is assumed that the domains and ranges of the given operators, $\underline{I}$ and $\underline{H}$ are all equal to each other, and that each is a complete linear, normed space (that is, a Banach space; the set of all functions of time that possess the least upper bound norm and on which addition and scalar multiplication are defined, is such a space; the same holds true for the energy norm). We then have

$$\text{Do}(\underline{E}) = \text{Ra}(\underline{E}) = \text{Do}(\underline{H}) = \text{Ra}(\underline{H}) = \text{Do}(\underline{I}) = \text{Ra}(\underline{I}) \tag{46}$$

in which Do and Ra represent domain and range, respectively.

The norm in Eq. 45 has been left arbitrary, and convergence of the iteration, Eq. 44, is not uniform but functionwise.

3.2.1 Proof of Convergence

We first establish the convergence of the sequence in Eq. 44. To this end, a bound is obtained on the distance $\| \underline{E}_m(x)-\underline{E}_n(x) \|$, where m and n are any two integers, and x is any time function that belongs to $\text{Do}(\underline{H})$. Assume that we have $m \geq n$. Using the triangle inequality, we obtain

$$\begin{aligned} \| \underline{E}_m(x)-\underline{E}_n(x) \| &= \| \underline{E}_m(x)-\underline{E}_{m-1}(x)+\underline{E}_{m-1}(x)+\ldots+\underline{E}_{n+1}(x)-\underline{E}_n(x) \| \\ &\leq \| \underline{E}_m(x)-\underline{E}_{m-1}(x) \| + \| \underline{E}_{m-1}(x)-\underline{E}_{m-2}(x) \| + \| \underline{E}_{n+1}(x)-\underline{E}_n(x) \|, \\ & \qquad m \geq n;\ n = 1, 2, \ldots \end{aligned} \tag{47}$$

Now, using the property of contractions, Eq. 45, we can write

$$\begin{aligned} \| \underline{E}_m(x)-\underline{E}_{m-1}(x) \| &= \| \underline{I}-(\underline{H}*\underline{E}_{m-1})-\underline{I}+(\underline{H}*\underline{E}_{m-2}) \| \\ &= \| (\underline{H}*\underline{E}_{m-1})-(\underline{H}*\underline{E}_{m-2}) \| \\ &\leq \alpha \| \underline{E}_{m-1}-\underline{E}_{m-2} \|, \qquad m = 2, 3, \ldots \end{aligned} \tag{48}$$

in which we have used Eq. 44 for $E_m(x)$. Using Eq. 48 (m-1) times, we obtain

$$\begin{aligned} \| \underline{E}_m(x)-\underline{E}_{m-1}(x) \| &\leq \alpha^{m-1} \| \underline{E}_1(x)-\underline{E}_o(x) \| \\ &= \alpha^{m-1} \| \underline{E}_1(x) \| \end{aligned} \tag{49}$$

Substituting Eq. 49 for each term on the right-hand side of Eq. 47, we get

$$\| \underline{E}_m(x)-\underline{E}_n(x) \| \leq (a^{m-1}+a^{m-2}+\ldots+a^n) \| \underline{E}_1(x) \|$$

$$\leq a^n \frac{1-a^{m-n}}{1-a} \| \underline{E}_1(x) \| ; \qquad m, n = 1, 2, \ldots \tag{50}$$

in which the geometric progression has been summed and bounded.

It must now be shown that the sequence $\underline{E}_n$ converges. A well-known theorem in Analysis asserts that a sequence must be a Cauchy sequence in order to converge. A sequence is said to be Cauchy if, given any arbitrarily small number ϵ, it is possible to find an integer N, with the property that the distance between any pair of terms lying beyond N in the sequence is less than ϵ. $\underline{E}_n$ has this property, since for any integer N for which $m > n \geq N$ holds, we have

$$\| \underline{E}_m(x)-\underline{E}_n(x) \| \leq \frac{a^N}{1-a} \| \underline{E}_1(x) \| ; \qquad m > n \geq N$$

$$N = 0, 1, 2, \ldots$$

3.2.2 Completeness

The mere fact that a sequence $\underline{E}_n(x)$ is a Cauchy sequence does not ensure the existence of an e for which we can write

$$e = \lim_{n \to \infty} \underline{E}_n(x) \tag{51}$$

In order for it to imply Eq. 51, $\underline{E}_n(x)$ must belong to a "complete metric space," whereupon e also belongs to the space. This is the case wherever the space of all time functions having the least upper bound norm or the energy norm is used. (The Riesz-Fisher theorem is the proof of the completeness of the space of all functions having an energy norm.)

We now define $\underline{E}$ as that operator whose domain and range equal Do($\underline{H}$) and which satisfies

$$\underline{E}(x) = e$$

in which e is given by Eq. 51, for all x in Do($\underline{E}$).

3.2.3 Sufficiency and Uniqueness

It must now be shown that the $\underline{E}$ that has been determined is in fact a solution of Eq. 43, and the only solution. Consider the norm of the difference between the two sides of Eq. 43. By using our value for $\underline{E}$, it can thus be bounded:

$$\| [\underline{I}-(\underline{H}*\underline{E})](x)-\underline{E}(x) \|$$

$$\leq \| [\underline{I}-(\underline{H}*\underline{E})](x)-[\underline{I}-(\underline{H}*\underline{E}_n)](x) \| + \| [\underline{I}-(\underline{H}*\underline{E}_n)](x)-\underline{E}(x) \| \tag{52}$$

in which we have added and subtracted $[\underline{I}-(\underline{H}*\underline{E}_n)](x)$, n being any integer. Using the contraction condition for $\underline{H}$ to bound the first term on the right-hand side of Eq. 52, and the iteration formula, Eq. 44 to rewrite the second term, we have the bound,

$$\| [\underline{I}-(\underline{H}*\underline{E})](x)-\underline{E}(x) \| \leq a \| \underline{E}(x)-\underline{E}_n(x) \| + \| \underline{E}_{n+1}(x)-\underline{E}(x) \| \tag{53}$$

Now the right-hand side of Eq. 53 can be made arbitrarily small, since $\underline{E}_n$ converges to $\underline{E}$. The left-hand side must therefore be zero; this implies the desired result,

$$\underline{E} = \underline{I} - (\underline{H}*\underline{E})$$

so that $\underline{E}$ is truly a solution of Eq. 43.

Suppose, next, that another solution, $\underline{E}'$, exists, that is,

$$\underline{E}' = \underline{I} - (\underline{H}*\underline{E}')$$

The difference $\| (\underline{E}-\underline{E}')(x) \|$ must be zero for all x, showing that $\underline{E}$ and $\underline{E}'$ are equal; this follows from the inequality

$$\begin{aligned} \| (\underline{E}-\underline{E}')(x) \| &= \| [\underline{I}-(\underline{H}*\underline{E})](x)-[\underline{I}-(\underline{H}*\underline{E}')](x) \| \\ &\leq a \| (\underline{E}-\underline{E}')(x) \| \end{aligned}$$

which can only be satisfied with $\| (\underline{E}-\underline{E}')(x) \| = 0$, since

$$\| (\underline{E}-\underline{E}')(x) \| \geq 0 \text{ and } 0 \leq a < 1.$$

3.2.4 Bounds on $\| \underline{E}(x) \|$

A bound on $\| \underline{E}(x) \|$ can be obtained from Eq. 50, which gives the inequality

$$\| \underline{E}_m(x)-\underline{E}_n(x) \| \leq \frac{a^n}{1-a} \| \underline{E}_1(x) \| \tag{54}$$

Letting n = 0 in Eq. 54, we get

$$\| \underline{E}_m(x) \| \leq \frac{1}{1-a} \| \underline{I}(x)-\underline{H}(0) \| \tag{55}$$

Since Eq. 55 holds for all $\underline{E}_m$, it must hold for $\underline{E}$, and the sought-for bound is obtained,

$$\| \underline{E}(x) \| \leq \frac{1}{1-a} \| x-\underline{H}(0) \| \tag{56}$$

The gains of $\underline{E}$ can also be thus bounded: Since $\underline{E}(x)$ exists for all x in Do($\underline{H}$), Eq. 36 is applicable and gives a bound on the maximum incremental gain of $\underline{E}$,

$$\text{incr} \||\underline{E}\|| \leq \frac{1}{1-a} \tag{57}$$

3.2.5 Bounds on Truncation Error

A bound on the error resulting from truncating the iteration in its n^{th} cycle can be obtained directly from inequality 50. Since that inequality holds for all $\underline{E}_m$, it must also hold for $\underline{E}$ and gives

$$\| \underline{E}(x)-\underline{E}_n(x) \| \leq \frac{a^n}{1-a} \| x-\underline{H}(0) \|$$

as the required bound. Thus, the error drops off as a^n, a being less than 1.

3.3 ITERATION FOR THE CLOSED-LOOP OPERATOR $\underline{G}$

The closed-loop unity feedback operator $\underline{G}$, that satisfies Eq. 39, $\underline{G} = \underline{H} * (\underline{I}-\underline{G})$, can be determined by the very similar iteration

$$\begin{aligned} &\underline{G}_o = 0 \\ &\underline{G}_n = \underline{H} * (\underline{I}-\underline{G}_{n-1}), \qquad n = 1, 2, \ldots \\ &\underline{G}(x) = \lim_{n\to\infty} \underline{G}_n(x) \end{aligned} \tag{58}$$

This iteration also converges to a unique solution, provided that $\underline{H}$ is a contraction. The following bounds apply:

$$\begin{aligned} \|\underline{G}(x)\| &\leq \frac{\|\underline{H}(x)\|}{1-a} \\ &\leq \frac{a}{1-a}\|x\| + \frac{1}{1-a}\|\underline{H}(0)\| \end{aligned} \tag{59}$$

$$\begin{aligned} \|\underline{G}(x)-\underline{G}_n(x)\| &\leq \frac{a^{n-1}}{1-a}\|\underline{H}(x)\| \\ &\leq \frac{a^n}{1-a}\|x\| + \frac{a^{n-1}}{1-a}\|\underline{H}(0)\| \qquad n = 1, 2, \ldots \end{aligned} \tag{60}$$

3.4 TRANSFORMATIONS UNDER WHICH THE RATE OF CONVERGENCE IS MAINTAINED FIXED

A variety of feedback equations may have the same loop operator and be essentially equivalent to each other, at least insofar as their solution by iteration is concerned.

3.4.1 Translations

The addition of a known operator to the unknown transforms the feedback equations but leaves the loop operator invariant. For example, the error equation (Eq. 43),

$$\underline{E} = \underline{I} - (\underline{H}*\underline{E})$$

can be transformed by means of the translation,

$$\underline{E} = \underline{I} - \underline{G}$$

into the feedback equation, Eq. 39, $\underline{G} = \underline{H} * (\underline{I}-\underline{G})$, both equations having the loop operator $\underline{H}$. Similarly, the more general equation

$$\underline{G}' = \underline{A} \pm \underline{H} * (\underline{B}\pm\underline{G}')$$

in which $\underline{A}$ and $\underline{B}$ are given operators, can be translated into either Eqs. 43 or 39.

3.4.2 Scalar Unitary Transformation

The transformation $\underline{G} = c\underline{G}'$, in which c is a real constant, transforms Eq. 39 into

$G' = c^{-1}\underline{H}c * (\underline{I}-\underline{G}')$.

The loop operator is thus changed (if $\underline{H}$ is not linear), but its gain can be shown to remain invariant.

3.5 TRANSFORMATIONS THAT CHANGE THE RATE OF CONVERGENCE

The usefulness of the iteration, Eq. 58, is contingent on having a small enough contraction for the loop operator, since its rate of convergence depends inversely on the maximum incremental gain, incr$|||\underline{H}||| = a$, of the loop operator $\underline{H}$. It is often possible to transform a given iteration into one having a smaller loop gain, and the following transformations have been found useful for this purpose:

(a) direct perturbation

(b) inverse perturbation

(c) loop rotation, $\underline{A}^{-1} * \underline{H} * \underline{A}$

(d) compression — forming a single operator out of several cycles of iteration.

The smallness of the gain a is a measure of the usefulness of a transformation, and a meaningful problem that arises is that of finding the best possible transformation for a given equation. This problem has been solved in certain simple cases, but requires further study in general.

3.5.1 Direct Perturbation

A perturbed operator is one that differs by a "small" amount from a given operator. It is often possible to transform a feedback equation into an improved form (having a smaller loop gain) by perturbing the loop operator about another for which the feedback equation has a known solution.

Thus $\underline{G} + \underline{H} * (\underline{I}-\underline{G})$ can be transformed by splitting the loop operator $\underline{H}$ into the sum

$$\underline{H} = \underline{H}_a + \Delta\underline{H} \tag{61}$$

in which $\underline{H}_a$ is a linear operator, and the corresponding equation

$$\underline{G}_a = \underline{H}_a * (\underline{I}-\underline{G}_a) \tag{62}$$

has a known solution,

$$\underline{G}_a = (\underline{I}+\underline{H}_a)^{-1} * \underline{H}_a \tag{63}$$

That is, $\underline{H}$ is a perturbation of $\underline{H}_a$, and conversely.

The transformed equation is obtained by substituting Eq. 61 in Eq. 39. This operation gives

$$\begin{aligned}\underline{G} &= (\underline{H}_a+\Delta\underline{H}) * (\underline{I}-\underline{G}) \\ &= [\underline{H}_a*(\underline{I}-\underline{G})] + [\Delta\underline{H}*(\underline{I}-\underline{G})] \\ &= (\underline{H}_a*\underline{I}) - (\underline{H}_a*\underline{G}) + [\Delta\underline{H}*(\underline{I}-\underline{G})]\end{aligned} \tag{64}$$

in which we have made use of the distributive property of the cascade with the sum, for the linear operator $\underline{H}_a$. Equation 64 may be regrouped to give

$$(\underline{I}+\underline{H}_a) * \underline{G} = \underline{H}_a + [\Delta\underline{H}*(\underline{I}-\underline{G})]$$

whence

$$\underline{G} = \left[(\underline{I}+\underline{H}_a)^{-1}*\underline{H}_a\right] + \left[(\underline{I}+\underline{H}_a)^{-1}*\Delta\underline{H}*(\underline{I}-\underline{G})\right] \tag{65}$$

is obtained as the transformed equation. (The inverse $(\underline{I}+\underline{H}_a)^{-1}$ has been assumed to exist.) This transformation is illustrated schematically in Fig. 13.

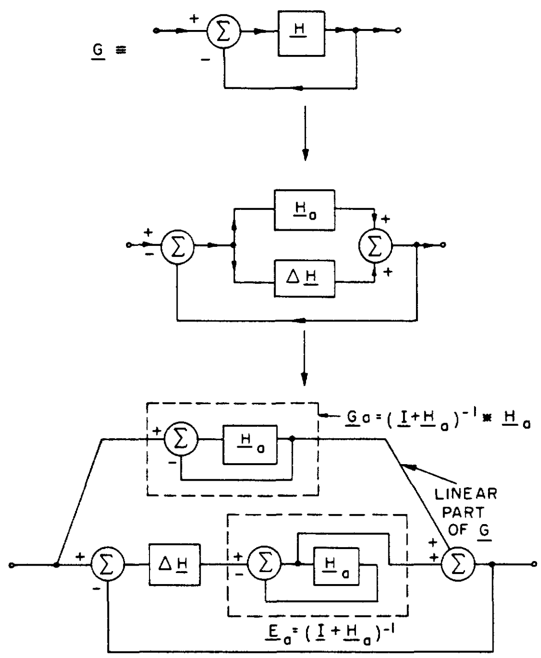


Fig. 13. Schematic illustration of perturbation.

Equation 65 can be expressed in a more convenient form. Since $(\underline{I}+\underline{H}_a)^{-1} * \underline{H}_a$ is the closed-loop operator of a linear feedback system having $\underline{H}_a$ for its loop operator, let us represent it by

$$\underline{G}_a = (\underline{I}+\underline{H}_a)^{-1} * \underline{H}_a$$

Similarly, let $\underline{E}_a$ represent the corresponding error operator,

$$\underline{E}_a = (\underline{I}+\underline{H}_a)^{-1}$$

Equation 65 can now be written,

$$\underline{G} = \underline{G}_a + [\underline{E}_a*\Delta\underline{H}*(\underline{I}-\underline{G})] \tag{66}$$

3.5.2 Example of Direct Perturbation: Splitting a No-Memory Time-Invariant Loop Operator

When $\underline{H}$ is a no-memory time-invariant operator $\underline{N}$ (Fig. 14) the splitting must take the form

$$\underline{N} = \underline{N}_a + \Delta\underline{N}$$

$$= c(\underline{I}+\Delta\underline{N}) \tag{67}$$

in which c is a real constant. (A linear, time-invariant, no-memory operator always has the form of a "pure gain," $c\underline{I}$.)

The transformed feedback equation is, therefore,

$$\underline{G} = \frac{c}{1+c}\underline{I} + \left[\left(\frac{c}{1+c}\Delta\underline{N}\right) * (\underline{I}-\underline{G})\right] \tag{68}$$

Here, c is chosen so as to minimize the loop gain, $\text{incr}\left|\left|\left|\frac{c}{1+c}\Delta\underline{N}\right|\right|\right|$. We shall find that this is smallest and is always less than 1 if c is set equal to the average of the maximum and minimum slopes of the graph of $\underline{N}$, provided that these exist and are both greater than zero. Let n be the graph of $\underline{N}$ (Fig. 14), and let us assume that it is differentiable except, at most, at a countable number of points. Let $\beta = \text{g.l.b. } n'$ be the greatest lower bound and $\gamma = \text{l.u.b. } n'$ the least upper bound on the slope n' of n, over those points in Do(n) at which n is differentiable. Assume that we have $\beta > 0$, so that we can write

$$0 < \beta \leqslant n' \leqslant \gamma < \infty \tag{69}$$

Now, the loop gain for the transformed equation, Eq. 68, is given by

$$\text{incr}\left|\left|\left|\frac{c}{1+c}\Delta\underline{N}\right|\right|\right| = \frac{c}{1+c}\,\text{incr}|||\Delta\underline{N}|||$$

$$= \frac{c}{1+c}\,\text{incr}\left|\left|\left|\frac{1}{c}\Delta\underline{N}-\underline{I}\right|\right|\right| \tag{70}$$

in which we have used Eq. 67 for $\Delta\underline{N}$. It can be shown that the gain on the right-hand side of Eq. 70 can be expressed in terms of β and γ. Thus

$$\text{incr}\left|\left|\left|\frac{1}{c}\underline{N}-\underline{I}\right|\right|\right| = \frac{1}{c}\text{incr}|||\Delta\underline{N}-c\underline{I}|||$$

$$= \frac{1}{c}\max(|\gamma-c|, |\beta-c|)$$

$$= \frac{1}{c}\left[\frac{1}{2}(\gamma-\beta)+\left|c-\frac{1}{2}(\gamma+\beta)\right|\right] \tag{71}$$

Substituting Eq. 71 in Eq. 70, we obtain

$$\text{incr}\left|\left|\left|\frac{c}{1+c}\Delta\underline{N}\right|\right|\right| = \frac{\frac{1}{2}(\gamma-\beta) + |\delta|\; c-\frac{1}{2}(\gamma+\beta)|}{1+c} \tag{72}$$

In order to minimize the gain in Eq. 72, it is convenient to introduce the term δ given by $\delta = c - \frac{1}{2}(\gamma+\beta)$. Equation 72 now becomes

$$\mathrm{incr}\left|\left|\left| \frac{c}{1+c} \Delta N \right|\right|\right| = \frac{\frac{1}{2}(\gamma-\beta) + |\delta|}{1 + \frac{1}{2}(\gamma+\beta) + \delta} \tag{72'}$$

Now inequality 59 implies

$$0 \leq \frac{1}{2}(\gamma-\beta) < \frac{1}{2}(\gamma+\beta) < 1 + \frac{1}{2}(\gamma+\beta),$$

from which it follows that Eq. 72' is minimum when $\delta = 0$, and we have the results

$$c = \frac{1}{2}(\gamma+\beta) \tag{72a}$$

$$\mathrm{incr}\left|\left|\left| \frac{c}{1+c} \Delta \underline{N} \right|\right|\right| = \frac{\gamma - \beta}{\gamma + \beta + 2} < 1 \tag{72b}$$

Hence, under our assumptions, this choice of c always produces a contraction. It also minimizes $\mathrm{incr}|||\Delta \underline{N}|||$, which becomes a contraction whose gain is

$$\mathrm{incr}|||\Delta \underline{N}||| = \frac{\gamma - \beta}{\gamma + \beta}$$

Note that this transformation is useful only for negative feedback, where closed-loop gain is less than open-loop gain, that is

$$\left|\frac{c}{1+c}\right| \leq |c|$$

and does not lead to a reduction in loop gain if feedback is positive. This procedure

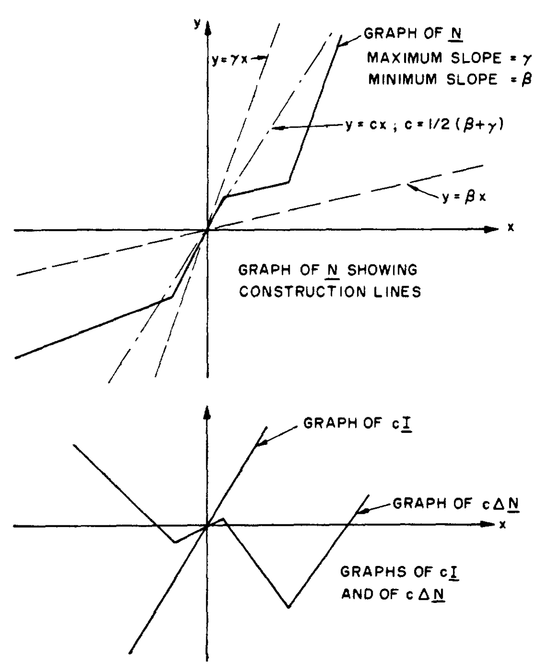


Fig. 14. Splitting the no-memory, time-invariant operator $\underline{N}$.

is illustrated in Fig. 14.

3.5.3 Inverse Perturbation

If the loop operator $\underline{H}$ has an inverse, $\underline{H}^{-1}$, then the feedback equation (Eq. 39) can be transformed into an equation having $\underline{H}^{-1}$ as its loop operator,

$$\underline{G} = \underline{I} - (\underline{H}^{-1} * \underline{G}) \tag{72c}$$

The maximum incremental gain of $\underline{H}^{-1}$ is related to that of $\underline{H}$. In fact it equals the reciprocal of the minimum incremental gain of $\underline{H}$. Hence it may be convenient to use Eq. 72c instead of Eq. 39 and to split $\underline{H}^{-1}$ instead of $\underline{H}$, to get

$$\underline{H}^{-1} = (\underline{H}^{-1})_a + \Delta(\underline{H}^{-1})$$

which, on being combined with Eq. 72c, leads to the inversely perturbed feedback equation,

$$\underline{G} = \left[\underline{I} + (\underline{H}^{-1})_a\right]^{-1} - \left(\left[\underline{I} + (\underline{H}^{-1})_a\right]^{-1} * \Delta(\underline{H}^{-1}) * \underline{G}\right) \tag{73}$$

3.5.4 Inverse Perturbation with a No-Memory, Time-Invariant Loop Operator: Comparison with Direct Perturbation

If the loop operator is the operator $\underline{N}$ of section 3.5.2, then inverse perturbation leads to the equations

$$\begin{aligned} \underline{N}^{-1} &= (\underline{N}^{-1})_a + \Delta(\underline{N}^{-1})' \\ &= d\underline{I} + d\Delta(\underline{N}^{-1}) \end{aligned}$$

in which d is a real number, and

$$\underline{G} = \frac{1}{1+d}\underline{I} - \left[\frac{d}{1+d}\Delta(\underline{N}^{-1}) * \underline{G}\right] \tag{74}$$

Using the same reasoning as in the case of direct perturbation, the loop gain $\text{incr}\left|\left|\left|\frac{d}{1+d}\Delta(\underline{N}^{-1})\right|\right|\right|$ can be shown to be least when d equals the average of the maximum and minimum slopes of $\underline{N}^{-1}$, which are $1/\beta$ and $1/\gamma$, respectively. Hence, we have

$$d = \frac{1}{2}\left(\frac{1}{\beta} + \frac{1}{\gamma}\right) \tag{75a}$$

$$\begin{aligned} \text{incr}|||\Delta(\underline{N}^{-1})||| &= \frac{\gamma - \beta}{\gamma + \beta} \\ &= \text{incr}|||\Delta\underline{N}||| \end{aligned} \tag{75b}$$

$$\text{incr}\left|\left|\left|\frac{d}{1+d}\Delta(\underline{N}^{-1})\right|\right|\right| = \frac{\gamma - \beta}{\gamma + \beta + 2\beta\gamma} \tag{75c}$$

Comparing Eqs. 75c and 72b, we find that inverse perturbation gives a loop gain that is smaller than that of direct perturbation if the inequality $\beta\gamma > 1$ is satisfied; the two are equal for $\beta\gamma = 1$, and the former is greater than the latter if we have $\beta\gamma < 1$.

It is convenient to express these results in terms of the parameters $c = \frac{1}{2}(\beta+\gamma)$ and γ/β, which give a measure of the average gain and nonlinearity of $\underline{N}$, respectively. Curves showing the loop gain as a function of these quantities are shown in Fig. 15, while Fig. 16 shows the regions in the c versus γ/β plane in which one or the other method gives a smaller loop gain. It is clear that direct perturbation is better for low average slopes, c, (always if $c < 1$) and large nonlinearity $\left(\frac{\gamma}{\beta} \gg 1\right)$, whereas inverse

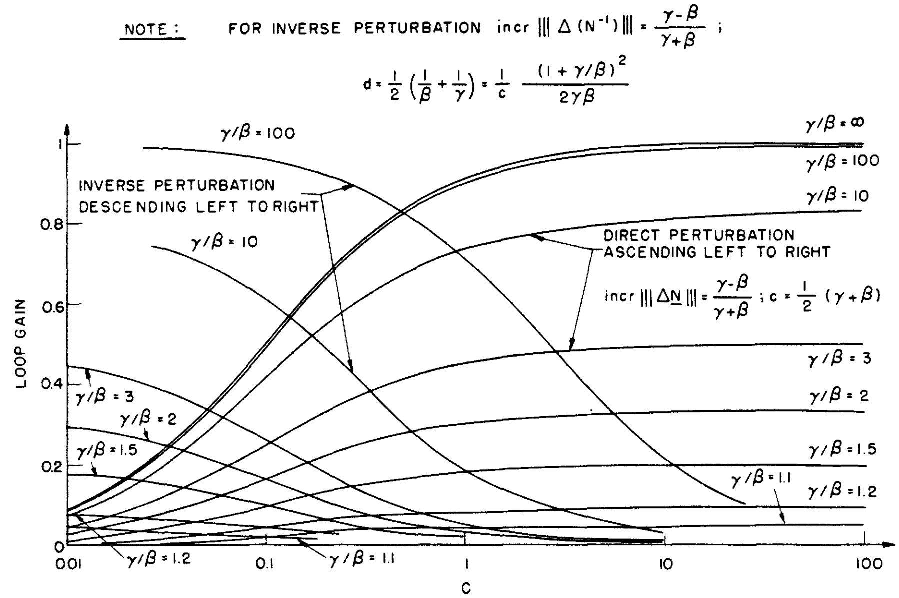


Fig. 15. Curves of loop gain as a function of c and γ/β for direct and inverse perturbation.

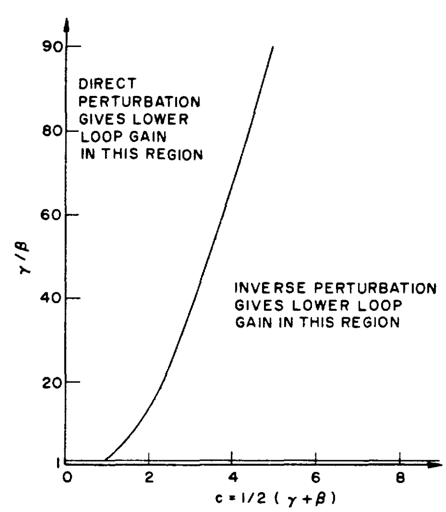


Fig. 16. Regions of preference for the two types of perturbation.

perturbation is better when c is large and γ/β is not.

3.5.5 The Loop Rotation $\underline{A}^{-1} * \underline{H} * \underline{A}$

The substitution $\underline{G} = \underline{A} * \underline{G}'$, in which $\underline{A}$ is any linear operator possessing an inverse, in Eq. 39 leads to the transformed equation (see Fig. 17a),

$$\underline{G}' = \underline{A}^{-1} * \underline{H} * \underline{A} * (\underline{A}^{-1} - \underline{G}') \tag{76}$$

Provided that $\underline{A}$ and $\underline{A}^{-1}$ do not commute in cascade with $\underline{H}$, the new loop operator is different from the old, and the loop gain may be smaller. We shall employ this transformation in section 3.6.1 (Eq. 81).

3.5.6 Compression

The substitution of Eq. 39 back into itself results in

$$\underline{G} = \underline{H} * (\underline{I} - \underline{H} * (\underline{I} - \underline{G})) \tag{77}$$

It may be advantageous to iterate Eq. 77 instead of Eq. 39. This is analogous to summing a series by pairs.

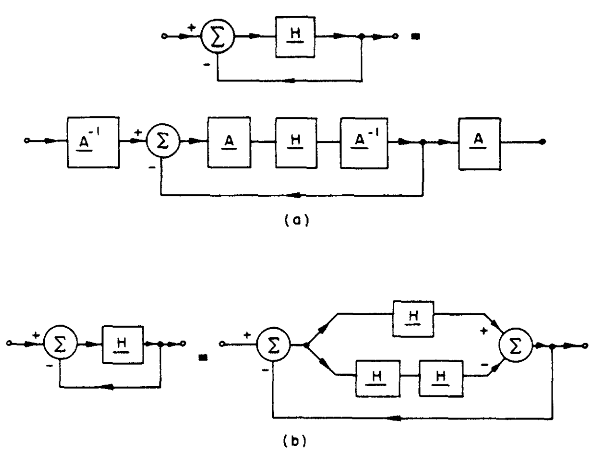


Fig. 17. (a) The loop rotation $\underline{A}^{-1} * \underline{H} * \underline{A}$.
(b) Compression.

Suppose, for example (Fig. 17b), that $\underline{H}$ is a linear operator having the spectrum $H(s) = 1/(s+1)$. Its gain in the energy norm is the maximum absolute value of its spectrum, which is $\text{incr}|||\underline{H}|||_2 = 1$, so that $\underline{H}$ is not a contraction. However, the transformed equation $\underline{G} = (\underline{H} - \underline{H} * \underline{H}) * (\underline{I} - \underline{G})$, has the loop operator $\underline{H}' = \underline{H} - \underline{H} * \underline{H}$ which is a contraction because its spectrum is

$$\begin{aligned} H'(s) &= \frac{1}{s+1} - \left[\frac{1}{s+1}\right]^2 \\ &= \frac{s}{(s+1)^2} \end{aligned}$$

Equation 78 is maximum when $|s| = |j\omega| = 1$, so that its loop gain is $\text{incr}|||\underline{H}'(s)||| = \frac{1}{4}$.

3.6 MODEL FOR THE NONLINEAR DISTORTION OF A NEGATIVE FEEDBACK SYSTEM

Negative feedback systems typically operate at high loop gains ($\text{incr}\,|||\underline{H}||| \gg 1$), and cannot be studied by direct iteration, which requires a contraction. However, perturbation about the linear part of the loop operator usually leads to a contraction, at least for small nonlinear distortions. This is true because high loop-gain, linear, negative feedback systems tend to behave like identity operators, since they have over-all gains of the order of 1.

3.6.1 Direct-Perturbation Model for a Bandpass System

The open-loop system $\underline{H}$ is assumed to consist of a linear part, $\underline{H}_a$, and a small additive, nonlinear distortion, $\Delta\underline{H}$ that is $\underline{H} = \underline{H}_a + \Delta\underline{H}$. The linear part $\underline{H}_a$ by itself

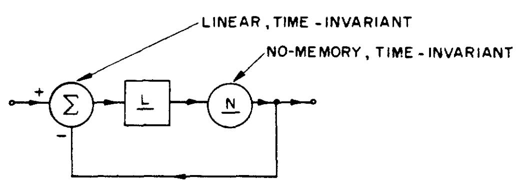


Fig. 18. A feedback system.

gives rise to the linear feedback system (assumed known), $\underline{G}_a = (\underline{I}+\underline{H}_a)^{-1} * \underline{H}_a$, and we wish to find the additive nonlinear distortion $\Delta\underline{G}$ arising from the presence of $\Delta\underline{H}$; that is, we shall have

$$\underline{G} = \underline{G}_a + \Delta\underline{G} \tag{79}$$

where $\underline{G}$ is the over-all feedback system resulting from $\underline{H}$, and satisfies Eq. 39.

The directly perturbed equation for $\underline{G}$ is Eq. 66. If we substitute Eq. 79 for $\underline{G}$ in Eq. 66, we have for the nonlinear distortion

$$\begin{aligned}\Delta\underline{G} &= \underline{E}_a * \Delta\underline{H} * (\underline{I}-\underline{G}_a-\Delta\underline{G}) \\ &= \underline{E}_a * \Delta\underline{H} * (\underline{E}_a-\Delta\underline{G})\end{aligned} \tag{80}$$

Consider, for example, the system in Fig. 18, consisting of the no-memory operator $\underline{N}$ following the linear operator $\underline{L}$, both of which are time-invariant, for which

$$\begin{aligned}\underline{H} &= \underline{N} * \underline{L} \\ &= \underline{I}c + (\Delta\underline{N}*c\underline{L})\end{aligned}$$

The constant c has been included in order to study the effect of changing the loop gain. For this system Eq. 80 becomes $\Delta\underline{G} = \underline{E}_a * \Delta\underline{N} * c\underline{L} * (\underline{E}_a-\Delta\underline{G})$, and can be transformed into a form that has lower loop gain by means of the transformation

$$\Delta\underline{G} = \underline{E}_a * \underline{Q} \tag{81}$$

which rotates the loop operator and yields

$$\underline{Q} = \Delta\underline{N} * c\underline{L} * \left[\underline{E}_a - (\underline{E}_a * \underline{Q}) \right]$$

$$= \Delta\underline{N} * \underline{G}_a * (\underline{I} - \underline{Q}) \qquad (82)$$

The transformed distortion $\underline{Q}$ can be found by means of the iteration

$$\underline{Q}_o = \underline{0}$$

$$\underline{Q}_n = \Delta\underline{N} * \underline{G}_a * (\underline{I} - \underline{Q}_{n-1}), \qquad n = 1, 2, \ldots \qquad (83)$$

which leads to the model for nonlinear distortion, $\Delta\underline{G} = \underline{E}_a * \underline{Q}$, shown in Fig. 19. The first approximation model is the realization of $\underline{Q}_1$. An inversely perturbed model is

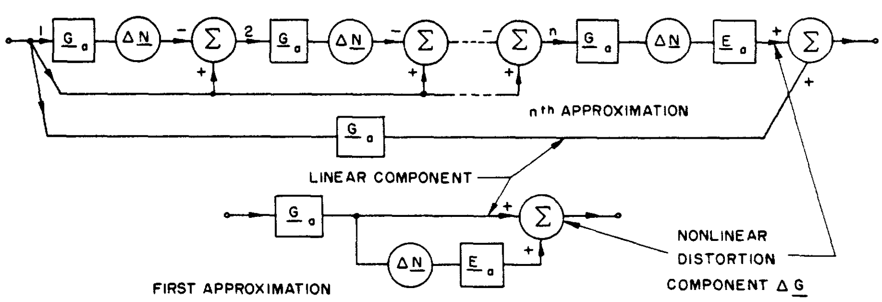


Fig. 19. Models for the nonlinear distortion in $\underline{G}$ of Fig. 18; first and n^{th} approximate iterative structures are shown.

described by the author in another paper.[17]

3.6.2 Determination of c

The iterative model is valid only as long as the loop operator $\Delta\underline{N} * \underline{G}_a$ is a contraction, that is

$$\alpha = \text{incr} \||| \Delta\underline{N} * \underline{G}_a \||| < 1$$

If $\underline{L}$ is the bandpass system of Fig. 20 having the frequency response of an RC coupling network, with breakpoints at a and b, and unity midband frequency response, it may be shown that, for b » a, we have quite accurately

$$\text{incr} \||| \underline{G}_a \||| = \frac{2c}{1 + c}$$

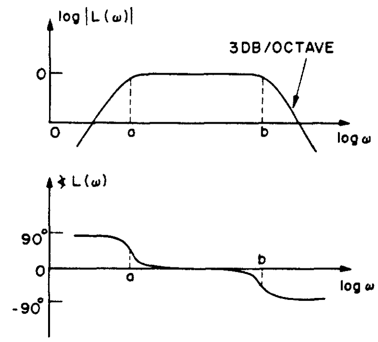


Fig. 20. Frequency response of the bandpass operator $\underline{L}$.

This is the same function of c that we encountered in section 3.5.2, except for a factor of 2. Since a is given by

$$a = \text{incr}|||\Delta\underline{N}||| \cdot \text{incr}|||\Delta\underline{G}|||$$

it is minimized by the same choice of c as in section 3.5.2, namely $c = \frac{1}{2}(\gamma+\beta)$. Refer to Fig. 14 in which γ and β are the greatest upper and least lower bounds to the slope of $\underline{N}$, and satisfy $0 < \beta \leq \gamma < \infty$. Then a is given by

$$\begin{aligned} a &= \frac{2c}{1+c} \cdot \frac{\gamma+\beta}{\gamma-\beta} \\ &= 2\frac{\gamma-\beta}{\gamma+\beta+2} \end{aligned} \tag{84}$$

and a contraction is produced whenever the degree of nonlinearity, γ/β, is less than 3, or if c is less than 1.

3.6.3 Bound on Nonlinear Distortion

The nonlinear distortion for any input x may be bounded as follows:

$$\begin{aligned} \|\Delta\underline{G}(x)\| &= \|(\underline{E}_a * \underline{Q})(x)\| \\ &\leq \text{incr}|||\underline{E}_a||| \cdot \|\underline{Q}(x)\| \end{aligned} \tag{85}$$

in which we have assumed that we have $\underline{E}_a(0) = \underline{Q}(0) = 0$.

The gain $\text{incr}|||\underline{E}_a|||$ can be bounded as follows:

$$\begin{aligned} \text{incr}|||\underline{E}_a||| &= 1 + \text{incr}|||\underline{G}_a||| \\ &\leq \frac{1+3c}{1+c} \end{aligned} \tag{86}$$

To bound $\| \underline{Q}(x) \|$ we observe that $\underline{Q}$ satisfies

$$\underline{Q} = \Delta \underline{N} * \underline{G}_a * (\underline{I}-\underline{Q})$$

which has the form of Eq. 39, so that Eq. 59 may be used to give the bound,

$$\| \underline{Q}(x) \| \leq \frac{\alpha}{1 - \alpha} \| x \| \tag{87}$$

Since we have shown in Eq. 84 that α is given by

$$\alpha = \frac{2c}{1 + c} \cdot \sigma$$

(we have written σ for $\mathrm{incr} \| \Delta \underline{N} \|$), Eq. 87 can be rewritten in terms of c, and we have

$$\| \underline{Q}(x) \| \leq \frac{2c\sigma}{1 + c(1-2\sigma)} \| x \| \tag{88}$$

Substituting Eqs. 88 and 86 in Eq. 85, we obtain the following bound on the nonlinear distortion:

$$\| \Delta \underline{G}(x) \| \leq \frac{1 + 3c}{1 + c} \cdot \frac{2c\sigma}{1 + c(1-2\sigma)} \| x \|$$

$$\leq 3 \| x \| \tag{89}$$

since σ is always less than 1/2 and c is non-negative.

3.6.4 Truncation Error

The error that is introduced by terminating the iterative structure in Fig. 19 can be bounded by applying Eq. 60 to Eq. 83, and we obtain

$$\| \Delta \underline{G}(x) - \Delta \underline{G}_n(x) \| \leq \frac{\alpha^n}{1 - \alpha} \mathrm{incr} \| \underline{E}_a \| \cdot \| x \| \tag{90}$$

as the bound on the error in the n^{th} approximation, $\Delta \underline{G}_n(x)$.

3.6.5 Effect of Increasing Loop Gain on Nonlinear Distortion

The purpose of using high-gain negative feedback is to reduce distortion. In fact, if the linear gain c in the loop is increased in our example (while $\Delta \underline{N}$ is maintained invariant), then the linear distortion, $\underline{E}_a(x)$, drops off roughly as $\frac{1}{(1+c)}$ for signals x, within the passband. However, the nonlinear distortion $\Delta \underline{G}$ may remain large, even though it is filtered by $\underline{E}_a$, because it may have large energy outside the passband, where $\underline{E}_a$ can act as an amplifier. Nonlinear distortion is reduced only for those signals x for which the spectrum of $\underline{Q}(x)$ is small wherever $\underline{E}_a(x)$ is not.

3.7 INVERSES BY PERTURBATION

Perturbation offers a means for showing the existence and determining the inverses of certain operators. An important theorem in functional analysis asserts that, if an operator has an inverse, then any other operator that is "close" to it also has an inverse, where both operators map a complete, normed, linear space into itself.

The inverse of an operator $\underline{H}$ that maps any space into itself is identical to the preinverse, and the existence of either implies that of the other. Hence, by definition of a preinverse, the existence of $\underline{H}^{-1}$ is equivalent to that of the solution for $\underline{G}$ in the equation

$$\underline{H} * \underline{G} = \underline{I} \tag{91}$$

This can be found by perturbation as follows: $\underline{H}$ is split into the sum

$$\underline{H} = \underline{H}_a + \Delta\underline{H} \tag{92}$$

Substituting Eq. 92 for $\underline{H}$ in Eq. 91, we obtain

$$(\underline{H}_a + \Delta\underline{H}) * \underline{G} = \underline{I}$$

which can be rewritten as the feedback equation

$$\underline{G} = (\underline{H}_a)^{-1} * \left[\underline{I} - (\Delta\underline{H} * \underline{G})\right] \tag{92a}$$

Since the range of $(\underline{H}_a)^{-1}$ is a complete, normed, linear space, it follows from iteration theory that Eq. 92a has a solution for $\underline{G}$, and hence that $\underline{H}^{-1} = \underline{G}$ exists, provided only that the condition

$$\text{incr}\,|||\,(\underline{H}_a)^{-1}\,|||\;\cdot\;\text{incr}\,|||\,\Delta\underline{H}\,|||\; < 1$$

is satisfied. If $\underline{H}_a$ is linear, then the weaker condition

$$\text{incr}\,|||\,(\underline{H}_a)^{-1} * \Delta\underline{H}\,|||\; < 1$$

suffices.

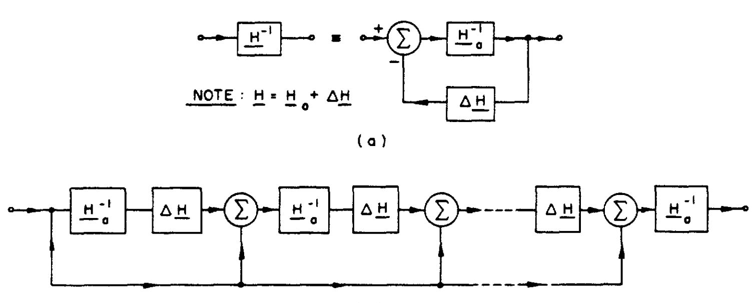


Fig. 21. (a) Realization of an inverse by means of a feedback system.
(b) Iterative structure for the inverse of (a).

Under these conditions, an inverse can be implemented as a (not necessarily realizable) feedback system (Fig. 21a) that, under the assumed conditions, is equivalent to the iteration scheme (Fig. 21b),

$$\underline{G}_o = \underline{0}$$

$$\underline{G}_n = (\underline{H}_1)^{-1} * \left[\underline{I} - (\Delta\underline{H} * \underline{G}_{n-1}) \right], \qquad n = 1, 2, \ldots$$

$$\underline{H}^{-1} = \lim_{n \to \infty} \underline{G}_n \tag{93}$$

IV. A THEOREM CONCERNING THE INVERSES OF CERTAIN BANDLIMITED NONLINEAR OPERATORS

The results described here were first published by the author (18) in 1959. A very similar result was published later by Landau (19).

When a bandlimited signal is filtered nonlinearly, the width of the spectrum of the resulting signal, in general, has no bounds. Expansion of the spectrum is characteristic of nonlinear operations, and hinders their use in communication channels whose bandwidth must usually be constrained.

However, filtering does not, loosely speaking, add any new degrees of freedom to a signal. For example, when a bandlimited signal is filtered by a nonlinear operator without memory that has an inverse, a sampling of the output signal at the Nyquist rate obviously specifies the situation completely even though the output is not bandlimited; the inverse operation can be performed on the samples instead of on the signal, and yields, in effect, a Nyquist sampling of the original signal, which is completely specified by the samples. It seems plausible, therefore, that some of the spectrum added by invertible nonlinear filtering is redundant, and hence that it could be discarded without affecting the possibility of recovering the original signal.

We shall discuss a situation (Fig. 22) in which a function of time, x(t), whose bandwidth is $2\omega_0$, centered about zero, and whose energy is finite, is filtered by any nonlinear operator without memory, N, that has an inverse and has both a finite maximum slope and a minimum slope that is greater than zero. The spectrum of y(t), the output of N, is then narrowed down to the original passband by means of a bandlimiter, B. It will be shown that the cascade combination of B following N has an inverse. This is equivalent to our stated hypothesis concerning recoverability. In a sense, then, the amount of bandwidth needed to describe the signal is conserved.

It will be found that the original signal can be recovered by means of a feedback system (Fig. 26) which can be approximated with arbitrary accuracy by a realizable iterative structure (Fig. 27), provided that a delay in recovering the original signal can be tolerated. This structure is not affected critically by small inaccuracies in its components, or by the presence of a small additive noise at the input.

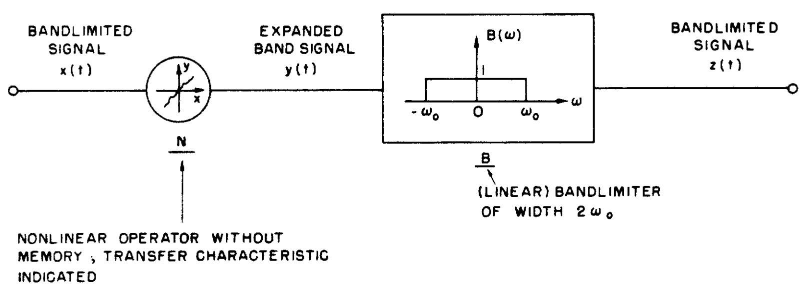


Fig. 22. Bandlimiter following a nonlinear operator without memory.

4.1 OUTLINE OF THE METHOD OF INVERSION

It is required to show that $\underline{B} * \underline{N}$ has an inverse and to find it. The essential difficulty is that the bandlimiter $\underline{B}$ has no inverse by itself, since its spectrum is identically zero outside the passband. It would have an inverse (simply the identity operator) if the input to it lay entirely in the passband, but y is not such a signal. To circumvent this difficulty, we resort to the following device: $\underline{N}$ is split into the sum of two parts (Fig. 23)

$$\underline{N} = \underline{N}_a + \underline{N}_b \tag{94}$$

of which $\underline{N}_a$ is linear (hence, $\underline{N}_a$ is a pure gain). Thus,

$$\underline{B} * \underline{N} = \underline{B} * (\underline{N}_a + \underline{N}_b) \tag{95}$$

and, since $\underline{B}$ is linear and hence distributive,

$$\underline{B} * \underline{N} = \underline{B} * \underline{N}_a + \underline{B} * \underline{N}_b \tag{96}$$

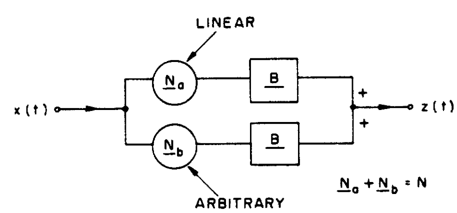


Fig. 23. An equivalent split form for $\underline{B} * \underline{N}$.

We can now find the inverse of the part $\underline{B} * \underline{N}_a$ because the spectrum of the output of $\underline{N}_a$ lies within the passband of $\underline{B}$. However, $\underline{B} * \underline{N}_b$ suffers from the original difficulty. Nevertheless, we can find the inverse of $\underline{B} * \underline{N}$ by perturbing $\underline{B} * \underline{N}$ about $\underline{B} * \underline{N}_a$, invoking the principle of section 3.7, which asserts that if any operator has an inverse, then any other operator that is close to it also has an inverse that can be found by iteration. Both operators are assumed to map the same complete, normed, linear, (Banach) space into itself.

Accordingly, it will first be established that $\underline{B} * \underline{N}$ and $\underline{B} * \underline{N}_a$ are operators that map a complete, normed, linear space into itself. The proof of the theorem will then merely entail establishing that $\underline{N}$ can be split in such a way that $\underline{B} * \underline{N}$ is close to $\underline{B} * \underline{N}_a$, in the sense that

$$\text{incr} \left|\left|\left| (\underline{B} * \underline{N}_a)^{-1} * (\underline{B} * \underline{N}_b) \right|\right|\right| = \alpha < 1 \tag{97}$$

is satisfied. This can be accomplished by splitting $\underline{N}$ along a line whose slope is the

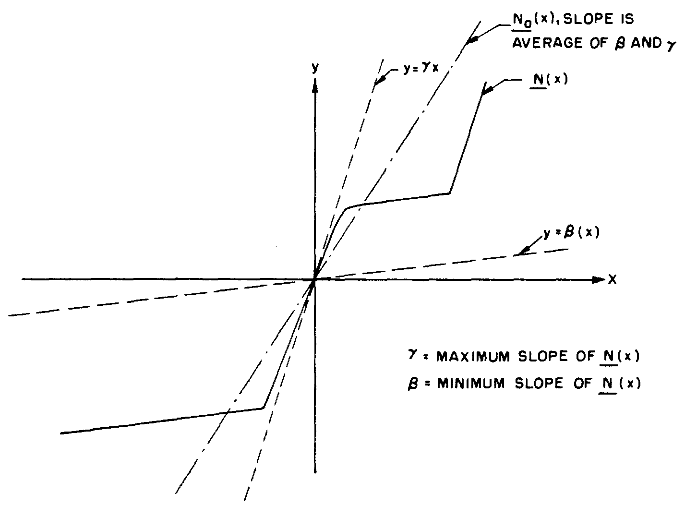


Fig. 24. Graphs of the operator $\underline{N}$, and its linear part, $\underline{N}_a$.

average of the greatest and least slopes of the graph of $\underline{N}$ (Fig. 24) and by realizing that a bandlimiter never increases the energy of a signal that passes through it.

4.2 DEFINITIONS OF THE RELEVANT SPACES AND OPERATORS

4.2.1 The Space L_2

The space L_2 is the set of all functions x(t) of time t, defined on the infinite interval, $-\infty < t < \infty$, with the property that each x has the energy norm $\|x\|$ (we shall omit the subscript "2" in this section).

$$\|x\| = \left(\int_{-\infty}^{\infty} |x(t)|^2 \, dt\right)^{1/2} < \infty \qquad (98)$$

Here, x is defined to be equal to zero if $\|x\| = 0$, so that all "null" functions are equal to zero.

4.2.2 The Bandlimiter, $\underline{B}$

The bandlimiter $\underline{B}$ is defined as an operator whose domain and range are L_2. If y is any time function in L_2 and $z = \underline{B}(y)$, then z is related to y by the integrals

$$z(t) = \int_{-\omega_o}^{\omega_o} Y(\omega)\, e^{j\omega t}\, d\omega \qquad (99)$$

$$Y(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} y(t)\, e^{-j\omega t}\, dt \qquad (100)$$

whose existence in the Lebesque sense is ensured, since y is in L_2.

For our purposes, the energy-reducing property of bandlimiters,

$$\|\underline{B}(y)\| \leq \|y\| \tag{101}$$

for all y in L_2, is necessary. It is derived by using Parseval's theorem:

$$\begin{aligned} \|\underline{B}(y)\| &= \int_{-\infty}^{\infty} |z(t)|^2\, dt = \int_{-\omega_o}^{\omega_o} |Y(\omega)|^2\, d\omega \\ &\leq \int_{-\infty}^{\infty} |Y(\omega)|^2\, d\omega = \|y\| \end{aligned} \tag{102}$$

This property is equivalent to the statement,

$$\text{incr}\,|||\underline{B}||| < 1 \tag{103}$$

4.2.3 The Space $\underline{B}(L_2)$

The space $\underline{B}(L_2)$ is that subspace of L_2 whose elements x have spectra that vanish outside the passband interval, $-\omega_o \leq \omega \leq \omega_o$; that is,

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt = 0, \qquad -\omega_o > \omega > \omega_o \tag{104}$$

It can be demonstrated that $\underline{B}(L_2)$ is a linear space; for, it is closed under addition and scalar multiplication, and the zero function belongs to it. It is a normed space, being a subset of L_2 on which a norm has been defined. It is a complete space, being a closed subset of L_2 whose completeness follows from the Riesz-Fisher theorem.

4.3 THEOREM 1

The operator $\underline{B} * \underline{N}$, consisting of a time-invariant operator without memory $\underline{N}$, that maps $\underline{B}(L_2)$ (inputs to $\underline{N}$ have finite energy and spectra that vanish outside the passband of $\underline{B}$) into L_2, followed by a bandlimiter $\underline{B}$, has an inverse that can be found as the functionwise limit in the mean of an iteration, provided that the following conditions are satisfied:

(a) $\underline{N}$ satisfies the dual Lipschitz conditions:

$$\begin{aligned} \beta\|x_2(t)-x_1(t)\| &\leq \|\underline{N}(x_2(t))-\underline{N}(x_1(t))\| \\ &\leq \gamma\|x_2(t)-x_1(t)\| \qquad 0<\beta\leq\gamma<\infty \end{aligned} \tag{105}$$

for all real $x_1(t)$ and $x_2(t)$.

(b) For simplicity only, the passband of $\underline{B}$ is confined to $-\omega_o \leq \omega \leq \omega_o$ and $\underline{N}(0) = 0$.

The inverse $(\underline{B}*\underline{N})^{-1}$ will be denoted by $\underline{G}$, and $\underline{G}$ can be found by the iteration formula

$$\underline{G}(z) = \lim_{n\to\infty} \underline{G}_n(z)$$

$$\underline{G}_o = \underline{0} \tag{106}$$

$$\underline{G}_n = \underline{N}_a^{-1} - \left(\underline{B} * \underline{N}_a^{-1} * \underline{N}_b * \underline{G}_{n-1}\right), \qquad n = 2, 3, \ldots$$

In formula 106, z is any member of $\underline{B}(L_2)$ and, from Eq. 94, $\underline{N}_a$ is an operator whose domain and range are $\underline{B}(L_2)$ and which maps each x in its domain according to

$$\underline{N}_a(x) = y; \;\; y(t) = \frac{1}{2}(\beta + \gamma)\, x(t)$$

and $\underline{N}_b$ has $\underline{B}(L_2)$ for its domain, L_2 for its range, and is defined as $\underline{N}_b = \underline{N} - \underline{N}_a$.

4.3.1 Proof of Theorem 1

Under the hypothesis of Theorem 1, $\underline{B} * \underline{N}$ and $\underline{B} * \underline{N}_a$ both map $\underline{B}(L_2)$, which is a complete, normed, linear space into itself. This theorem, therefore, follows from the principle of section 3.7 if it can be shown that $\underline{B} * \underline{N}$ is close to $\underline{B} * \underline{N}_a$; that is,

$$\text{incr} ||| (\underline{B} * \underline{N}_a)^{-1} * (\underline{B} * \underline{N}_b) ||| < 1 \tag{107}$$

The loop operator on the left of expression 107 can be simplified by using the facts that

$$\underline{B} * \underline{N}_a = \underline{N}_a \tag{108}$$

since $\underline{N}_a$ is restricted to the passband of $\underline{B}$, and

$$\underline{N}_a^{-1} * \underline{B} = \underline{B} * \underline{N}_a^{-1} \tag{109}$$

Using Eqs. 108 and 109, we get

$$\begin{aligned} \text{incr} ||| (\underline{B} * \underline{N}_a)^{-1} * (\underline{B} * \underline{N}_b) ||| &= \text{incr} ||| \underline{B} * \underline{N}_a^{-1} * \underline{N}_b ||| \\ &\leq \text{incr} ||| \underline{B} ||| \cdot \text{incr} ||| \underline{N}_a^{-1} * \underline{N}_b ||| \\ &\leq \text{incr} ||| \underline{N}_a^{-1} * \underline{N}_b ||| \end{aligned} \tag{110}$$

The last inequality was obtained by using the energy-reducing property (Eq. 101) of $\underline{B}$.

We shall now conclude the proof by showing that $\underline{N}$ can be split so that $\underline{N}_a^{-1} * \underline{N}_b$ is a contraction; that is,

$$\text{incr} ||| \underline{N}_a^{-1} * \underline{N}_b ||| < 1 \tag{111}$$

whence Eq. 107 follows. The iteration formula 106, which is a particular case of Eq. 93, is then valid.

4.3.2 Splitting the Nonlinear Operator

A nonlinear operator without memory $\underline{N}$ defined on L_2 by

$\underline{N}(x) = y; \quad \underline{N}(x(t)) = y(t)$

which satisfies the dual Lipschitz conditions

$$\beta \| x_2 - x_1 \| \leq \| \underline{N}(x_2) - \underline{N}(x_1) \| \leq \gamma \| x_2 - x_1 \| \qquad 0 < \beta \leq \gamma < \infty \tag{112}$$

for all x_1 and x_2 in L_2 and is monotonic, can be split into the sum of two operators (Eq. 94), $\underline{N}_a$ and $\underline{N}_b$, of which $\underline{N}_a$ is linear, and has the property that $\underline{N}_a^{-1} * \underline{N}_b$ is a contraction. For simplicity, we assume that $\underline{N}(0) = \underline{N}_a(0) = \underline{N}_b(0) = 0$.

PROOF: Let $\underline{N}_a$ be the linear operator defined as

$$\underline{N}_a(x) = y; \quad y(t) = \frac{1}{2}(\gamma + \beta)\, x(t) \tag{113}$$

and let $\underline{N}_b$ be defined as

$$\underline{N}_b(x) = y; \quad y(t) = \underline{N}(x(t)) - \frac{1}{2}(\gamma + \beta)\, x(t) \tag{114}$$

so that Eq. 94 is satisfied by construction. From Eqs. 113 and 114, it follows that

$$\underline{N}_a^{-1} * \underline{N}_b(x) = y; \quad \underline{N}_a^{-1} * \underline{N}_b(x(t)) = \frac{2}{\gamma + \beta}\left[\underline{N}(x(t)) - \frac{1}{2}(\gamma + \beta)\, x(t)\right]$$

$$= \left[\frac{2}{\gamma + \beta}\underline{N}(x(t))\right] - x(t) \tag{115}$$

The operator $\underline{N}_a^{-1} * \underline{N}_b$ will be shown to be a contraction. Let $x_1(t)$ and $x_2(t)$ be any two real numbers, and suppose that $x_2(t) > x_1(t)$. Using Eq. 115, we obtain

$$\underline{N}_a^{-1} * \underline{N}_b(x_2(t)) - \underline{N}_a^{-1} * \underline{N}_b(x_1(t)) = \frac{2}{\gamma + \beta}\left[\underline{N}(x_2(t)) - \underline{N}(x_1(t))\right] - \left[x_2(t) - x_1(t)\right] \tag{116}$$

The right-hand side of Eq. 116 is the difference of two expressions, each of which is positive because $x_2(t) > x_1(t)$, and $\underline{N}$ is assumed to be monotonic. Suppose, first, that the difference is positive or zero. Then, by using the upper Lipschitz condition of Eq. 112, we obtain

$$\underline{N}_a^{-1} * \underline{N}_b(x_2(t)) - \underline{N}_a^{-1} * \underline{N}_b(x_1(t)) \leq \frac{2}{\gamma + \beta}\gamma\left[x_2(t) - x_1(t)\right]$$

$$- \left[x_2(t) - x_1(t)\right]$$

$$= \frac{\gamma - \beta}{\gamma + \beta}\left[x_2(t) - x_1(t)\right] \tag{117}$$

When the difference in Eq. 116 is negative, the lower Lipschitz condition may be used instead to give Eq. 117 with the inequality and sign reversed. Hence,

$$|\underline{N}_a^{-1}*\underline{N}_b(x_2(t)) - \underline{N}_a^{-1}*\underline{N}_b(x_1(t))| \leq a|x_2(t) - x_1(t)|$$

$$0 \leq a = \frac{\gamma - \beta}{\gamma + \beta} < 1$$

which implies that

$$\|\underline{N}_a^{-1}*\underline{N}_b(x_2) - \underline{N}_a^{-1}*\underline{N}_b(x_1)\| \leq a\|x_2 - x_1\| \qquad 0 \leq a < 1$$

and, therefore, $\underline{N}_a^{-1} * \underline{N}_b$ is a contraction; that is,

$$\text{incr}\,|||\underline{N}_a^{-1}*\underline{N}_b||| < 1$$

(Recall from section 3.5.2 that this manner of splitting $\underline{N}$ minimizes the gain in Eq. 111.)

4.4 ADDITIONAL CONSIDERATIONS

4.4.1 Interpretation of Theorem 1

When $\underline{N}$ has a derivative the theorem is applicable, provided that this derivative has an upper bound, γ, and a lower bound, β, that is greater than zero. The graph of $\underline{N}$ can then be bounded by two straight lines, as in Fig. 24. The graph of $\underline{N}_a(x)$ is a straight line, whose slope is the average of the slopes of the bounding lines.

The iteration has the schematic form, shown in Fig. 25, and may be realized, at least formally, as the feedback system, shown in Fig. 26.

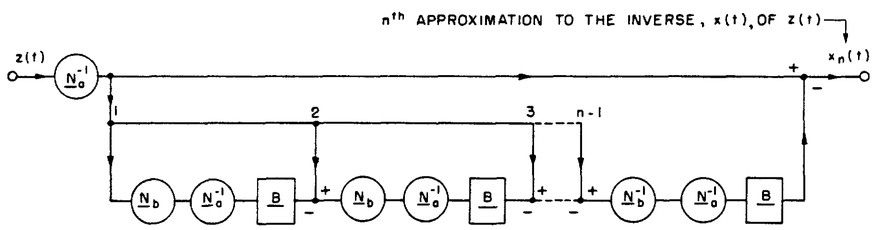


Fig. 25. Schematic form for the n^{th} approximation to the inverse of $\underline{B} * \underline{N}$. (Note that $\underline{N}_a^{-1}$ is a pure gain.)

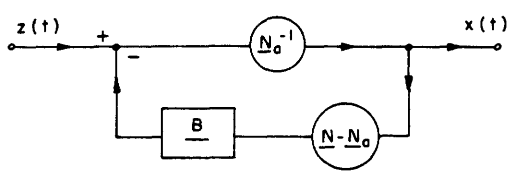


Fig. 26. Realization of the inverse of $\underline{B} * \underline{N}$ by means of a feedback system.

The rapidity of convergence of the iteration is greater than that of the geometric series, $1 + a + a^2 + \ldots$, in which $a = (\gamma - \beta)/(\gamma + \beta)$. For rapid convergence it is therefore desirable that the difference between maximum and minimum slopes be small.

4.4.2 Extensions of Theorem 1

With slight modifications Theorem 1 is valid when $\underline{N}(0) \neq 0$ and for arbitrary passbands.

The no-memory condition is not essential — any monotonic operator that meets the slope conditions will do. In fact, the theorem is valid for any operator that is close (in the specified sense) to a linear operator.

4.4.3 Imperfect Bandlimiting

An ideal bandlimiter is not realizable. However, it can be approximated, for example, by a Butterworth filter. We shall compute a bound on the error in the inverse signal, x(t), that results from this approximation.

We have shown that x(t) satisfies without error the relation

$$x = (\underline{B}*\underline{N})^{-1}(z) = \underline{N}_a^{-1}(z) - \underline{B} * \underline{N}_a^{-1} * \underline{N}_b(x) \tag{118}$$

In place of $\underline{B}$ we use an approximate bandlimiter, $\underline{B}'$, which satisfies the restriction

$$|\underline{B}'(\omega)| \leq 1 \qquad -\infty < \omega < \infty$$

whence, just as with $\underline{B}$ (Eq. 101), we have, for y in L_2

$$\|\underline{B}'(y)\| \leq \|y\| \tag{119}$$

This ensures that the iteration converges when $\underline{B}'$ is used in place of $\underline{B}$. That is, there exists some x' with the property that

$$x' = \underline{N}_a^{-1}(z) - \left[\underline{B}' * \underline{N}_a^{-1} * \underline{N}_b(x') \right] \tag{120}$$

Subtracting Eq. 119 from Eq. 120, we get for the error in x,

$$\begin{aligned} \|x'-x\| &= \|\underline{B}*\underline{N}_a^{-1}*\underline{N}_b(x) - \underline{B}'*\underline{N}_a^{-1}*\underline{N}_b(x')\| \\ &= \Big\| \left(\underline{B}*\underline{N}_a^{-1}*\underline{N}_b(x) - \underline{B}'*\underline{N}_a^{-1}*\underline{N}_b(x)\right) \\ &\quad + \left(\underline{B}'*\underline{N}_a^{-1}*\underline{N}_b(x) - \underline{B}'*\underline{N}_a^{-1}*\underline{N}_b(x')\right) \Big\| \end{aligned} \tag{121}$$

Applying the triangle inequality, we get

$$\|x'-x\| \leq \|(\underline{B}-\underline{B}')*\underline{N}_a^{-1}*\underline{N}_b(x)\| + \|\underline{B}'*\underline{N}_a^{-1}*\underline{N}_b(x) - \underline{B}'*\underline{N}_a^{-1}*\underline{N}_b(x')\| \tag{122}$$

Since the operator $\underline{B}' * \underline{N}_a^{-1} * \underline{N}_b$ in the second term on the right-hand side is a contraction, we may write

$$\|x' - x\| \leq \|(\underline{B} - \underline{B}')*\underline{N}_a^{-1} * \underline{N}_b(x)\| + \alpha\|x' - x\| \tag{123}$$

whence, we get

$$\|x' - x\| \leq \frac{1}{1-\alpha} \|(\underline{B} - \underline{B}')*\underline{N}_a^{-1}*\underline{N}_b(x)\|$$

$$= d\|(\underline{B} - \underline{B}')*\underline{N}_b(x)\| \tag{124}$$

in which we have combined the constants $1/(1-\alpha)$ and $\underline{N}_a^{-1}$ into d.

It is clear, then, that the error in determining the original signal, x, is proportional to the error with which $\underline{B}'$ operates on $\underline{N}_b(x)$, and becomes small as $\underline{B}'$ approaches $\underline{B}$.

If the approximate bandlimiter, $\underline{B}'$, is realizable, there is an irreducible error. However, if a delay in the inversion is tolerable, x can be recovered with any desired accuracy by combining the bandlimiter with a delay and delaying the signal z by a corresponding length of time in each iteration cycle (Fig. 27).

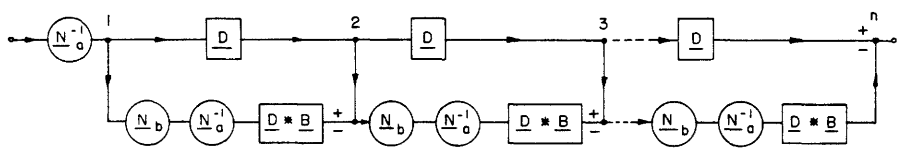


Fig. 27. Schematic form for the n^{th} approximation to the delayed inverse; total delay is (n-1) T seconds. (D is a delay of T sec; $\underline{D} * \underline{B}$ is a delayed bandlimiter approximated as a unit by a realizable filter.)

4.4.4 Bound on $|\underline{G}_n(z)|$; Convergence in the Least-Upper-Bound Norm

In order to ensure that the iteration can be implemented, it is necessary to establish that $\underline{N}_b$ does not have to handle infinite inputs; this will be true if $|\underline{G}_n(z)|$ is bounded.

Since $\underline{G}_n(z)$ satisfies the iteration

$$\underline{G}_o(z) = \underline{N}_a^{-1}(z) \tag{125}$$

$$\underline{G}_n(z) = \underline{N}_a^{-1}(z) - \left(\underline{B}*\underline{N}_a^{-1}*\underline{N}_b*\underline{G}_{n-1}\right)(z)$$

It follows from Eq. 56 of iteration theory that $\|\underline{G}_n(z)\|$ has the bound

$$\|\underline{G}_n(z)\| \leq \frac{1}{c(1-\alpha)}\|z\| \tag{126}$$

in which we have denoted the gain of the linear gain operator $\underline{N}_a$ by c, and that of the loop operator $\underline{B} * \underline{N}_a^{-1} * \underline{N}_b$ by α.

Now the absolute value $|\underline{G}_n(z)|$ can be bounded in terms of the norm $\|\underline{G}_n(z)\|$, as a consequence of the fact that $\underline{G}_n(z)$ is bandlimited; for, if x is any bandlimited function, we may write

$$|x(t)| = \left| \int_{-\infty}^{\infty} x(t-\tau) \frac{\sin \omega_o \tau}{\pi \tau} \, d\tau \right|$$

$$\leq \frac{1}{\pi} \left(\int_{-\infty}^{\infty} |x(t-\tau)|^2 \, d\tau \cdot \int_{-\infty}^{\infty} \left| \frac{\sin \omega_o \tau}{\tau} \right|^2 d\tau \right)^{1/2}$$

$$= \left[\frac{\omega_o}{\pi} \right]^{1/2} \cdot \|x\| \qquad (127)$$

Applying Eq. 127 to Eq. 126, we obtain a bound on $|\underline{G}_n(z)|$,

$$|\underline{G}_n(z)| \leq \frac{(\pi/\omega_o)^{1/2}}{c(1-a)} \|z\|, \qquad c = \|\underline{N}_b\|$$

Moreover, Eq. 127 implies

$$|\underline{G}_n(z) - \underline{G}_m(z)| \leq \left(\frac{\omega_o}{\pi}\right)^{1/2} \|\underline{G}_n(z) - \underline{G}_m(z)\|$$

which shows that $\underline{G}_n(z)$ converges uniformly in the least upper bound norm (Eq. 30), as well as in the energy norm.

4.4.5 Effect of Noise

Suppose that, instead of the signal z, we have $z + \Delta z$ contaminated by the additive noise Δz; the error in the output of the iteration scheme is $\underline{G}(z + \Delta z) - \underline{G}(z)$. Now the maximum incremental gain of $\underline{G}$ is $\frac{1}{c(1-a)}$ (with the use of Eq. 57), whence

$$\|\underline{G}(z + \Delta z) - \underline{G}(z)\| \leq \frac{1}{c(1-a)} \|\Delta z\|$$

or, if we use Eq. 127,

$$|\underline{G}(z + \Delta z) - \underline{G}(z)| \leq \frac{(\omega_o/\pi)^{1/2}}{c(1-a)} \|\Delta z\|$$

The same upper bound holds for $|\underline{G}_n(z + \Delta z) - \underline{G}_n(z)|$.

V. EXPONENTIAL ITERATION THEORY

Although the geometric iteration of Section III has some very desirable qualities – notably, a rapid rate of convergence that is uniform over the infinite time interval – its usefulness is limited by the low loop-gain condition, which restricts its use to systems that are highly damped and only slightly nonlinear.

The exponential iteration that will be described here leads to a weaker type of convergence, but one that has much wider application, and is useful for most, if not all, physical situations. (This section contains the mathematical background for the study of physical realizability, which will be dealt with in Section VI.)

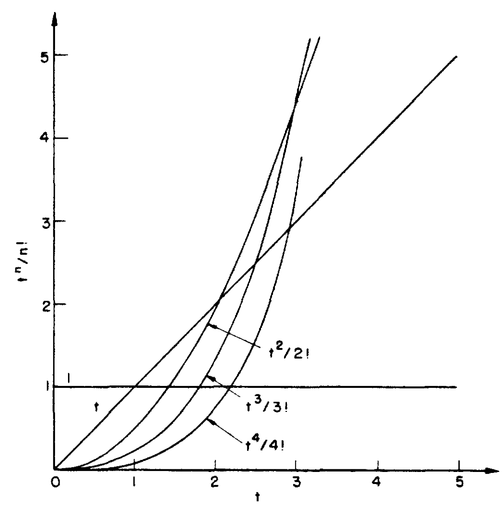


Fig. 28. The functions $\frac{t^n}{n!}$.

The basis for the exponential iteration is the ability of a repeated integration to reduce the size of the integrand. For example, if a unit step function is repeatedly integrated from 0 to T, the resulting sequence of functions (Fig. 28)

$$1,\ t,\ \frac{t^2}{2!},\ \dots \frac{t^n}{n!},\ \dots \tag{128}$$

converges to 0 if t is held fixed while n is increased without limit. (Note that this is true only if the original step begins at the origin or some other finite time, not at minus infinity.) Correspondingly, a feedback system that contains an integrator in the loop can often be solved by iteration, even if its loop gain is high, because the repeated integration produces a contraction after a finite number of cycles. (This may be viewed as an instance of the compression transformation of section 3.5.6.) The resulting method is useful for the treatment of physical systems, all of which have inertia, with the consequence that they can usually be represented by the cascade of an integrator with other operators. For example, the nonlinear distortion model that we found in section 3.6 to be valid for low degrees of nonlinearity ($\gamma/\beta < 3$) can be represented in such an integral form, and is universally valid (as long as none of the slopes is infinite).

The exponential iteration is general enough to be useful for unstable systems. Since these tend to "blow up" toward infinity, attention must necessarily be confined to a finite interval that has a beginning and an end. We have chosen to make the beginning coincide with the origin, as this choice does not lead to any loss of generality. Accordingly, all of the operators concerned will be of the type whose domain and range consist of time functions that are restricted to the finite interval $[0,T]$ (the set of all t that satisfy $0 < t < T$). It will be logically necessary to assume that a different operator is produced by each new choice of interval, even though all such operators pertain to a single physical system. Moreover, the definitions of norm and gain will have to be extended to make them functions of the interval length. This is to be contrasted with the geometric theory, in which it was never necessary to focus on the behavior of an operator at a particular time.

The condition under which convergence will be established is a "Lipschitz condition" of an integral type; it is more general than that of having an integrator in the loop, and can be applied in any norm. Systems that have integrators in the loop will be dealt with in section 5.5 as a special case.

5.1 DEFINITIONS

5.1.1 Functions and Operators with Restricted Domains

A function x', whose domain $[0,T]$ is a subset of the domain of some other function x, but which is identical to x on $[0,T]$, will be referred to as x restricted to $[0,T]$.

Similarly, an operator $\underline{H}'$ will be referred to as $\underline{H}$ restricted to $[0,T]$ if the domain and range of $\underline{H}'$ consists of all the functions in the domain and range of $\underline{H}$ restricted to $[0,T]$; and whenever we have $y = \underline{H}(x)$, then we have $y' = \underline{H}(x')$, where x' and y' are x and y restricted to $0, T$.

5.1.2 Norm-Time Functions

The norm-time function of any function x is denoted $\|x, t\|_i$, and is defined to be the greatest of the norms $\|x\|_i$ obtained by restricting x to various intervals $[0, t_1]$ that lie

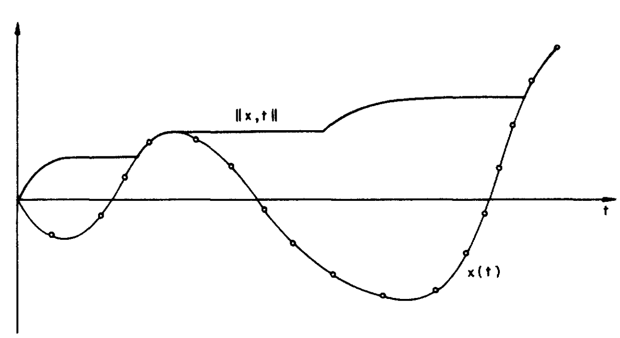


Fig. 29. Example showing the norm-time function, $\|x, t\|$ of the time function x.

within the interval $[0, t]$; for example (see Fig. 29),

$$\|x, t\|_1 = \underset{0 \leq t' \leq t}{\text{l.u.b.}} |x(t')| \tag{129}$$

(Norm subscripts will be omitted hereafter whenever the type of norm is arbitrary, or a single specified norm will be used.)

With the time parameter fixed, norm-time functions are ordinary norms and share all of the properties of norms (see section 2.4.1). They also have the property – which we shall use often – that they are nondecreasing functions of their time parameter t.

5.1.3 Gain-Time Functions

In an analogous fashion the gain-time function of any operator $\underline{H}$ is denoted $|||\underline{H}, t|||_i$ and is defined to be the least upper bound to the gains $|||\underline{H}|||_i$ obtained by restricting $\underline{H}$ to various intervals $[0, t']$ that lie within $[0, t]$.

Gain-time functions are nondecreasing functions of time, and have all of the properties of gains (see section 2.3.4) for fixed time.

5.1.4 The Space $B_i(T)$

The space $B_i(T)$ is defined to be the space of all time functions x defined on the interval $[0, T]$ that have the norm $\|x\|_i$. (The subscript i is omitted when it is clear which norm is being used.)

5.1.5 Notation, $\underline{H}(x, t)$

The value assumed by $\underline{H}(x)$ at time t, $\underline{H}$ being any operator and x any time function, is denoted $\underline{H}(x, t)$.

5.2 THEOREM 2: EXPONENTIAL ITERATION OF THE FEEDBACK ERROR EQUATION

The feedback error equation (Eq. 43),

$$\underline{E} = \underline{I} - \underline{H} * \underline{E}$$

in which $\underline{I}$ and $\underline{E}$ are operators that map B(T) into itself, has a unique solution for E that may be found as the limit of the iteration.

$$\underline{E}_o = \underline{0}$$

$$\underline{E}_n = \underline{I} - \underline{H} * \underline{E}_{n-1}, \qquad n = 1, 2, \ldots \tag{130}$$

$$\underline{E}(x) = \lim_{n \to \infty} \underline{E}_n(x)$$

which converges uniformly on $[0, T]$, provided that the following conditions are fulfilled:

(a) $\underline{H}$ satisfies the integral Lipschitz condition

$$\|\underline{H}(x)-\underline{H}(y), t\| \leq h(t) \int_0^t \|x-y, t_1\| \, dt_1 \tag{131}$$

in which x and y are any pair of functions in B(T), t is any point in the interval [0, T], and h(t) is a real-valued function on [0, T].

Note that Eq. 131 also implies

$$\|\underline{H}(x)-\underline{H}(y), t\| \leq th(t) \|x-y, t\| \tag{132}$$

(b) The space B(T) is a complete, normed, linear space; but the norm is left unspecified here.

5.2.1 Proof of Theorem 2

As usual, we commence by showing that the sequence is Cauchy; and, for that purpose, we bound the distance between any pair of terms in it. We use the triangle inequality,

$$\begin{aligned}\|\underline{E}_m(x)-\underline{E}_n(x), t\| \leq \|\underline{E}_m(x)-\underline{E}_{m-1}(x), t\| + \dots \\ + \|\underline{E}_{m-1}(x)-\underline{E}_{m-2}(x), t\| \\ + \|\underline{E}_{n+1}(x)-\underline{E}_n(x), t\|, \qquad m \geq n = 1, 2, \dots\end{aligned} \tag{133}$$

Each term on the right-hand side of Eq. 133 can be bounded:

$$\begin{aligned}&\|\underline{E}_m(x)-\underline{E}_{m-1}(x), t\| \\ &= \|(\underline{I}-\underline{H}*\underline{E}_{m-1})(x)-(\underline{I}-\underline{H}*\underline{E}_{m-2})(x), t\| \\ &\leq h(t) \int_0^t \|\underline{E}_{m-1}(x)-\underline{E}_{m-2}(x), t_1\| \, dt_1\end{aligned} \tag{134}$$

Here, Eqs. 130 and 131 have been used. Substituting Eq. 134 in itself m - 1 times, and recalling that $\underline{E}_o = 0$, we get

$$\begin{aligned}&\|\underline{E}_m(x)-\underline{E}_{m-1}(x)\| \\ &\leq (h(t))^{m-1} \int_0^t \dots \int_0^t \|\underline{E}_1(x), t_{m-1}\| \, dt_{m-1} \dots dt_1 \\ &\leq \frac{(th(t))^{m-1}}{(m-1)!} \|\underline{E}_1(x), t\|, \qquad m = 1, 2, \dots\end{aligned} \tag{135}$$

Substituting Eq. 135 for each term on the right-hand side of Eq. 133, we get a series that is bounded by an exponential,

$$\|\underline{E}_m(x)-\underline{E}_n(x), t\|$$

$$\leq \frac{(th(t))^n}{n!}\left(1+\frac{(th(t))}{(n+1)}+\ldots+\frac{(th(t))^{m-n-1}}{(n+1)\ldots(m-1)}\right)\|\underline{E}_1(x), t\|$$

$$\leq \frac{(th(t))^N}{N!}\epsilon^{th(t)}\|\underline{E}_1(x), t\|, \qquad m \geq n \geq N = 0, 1, \ldots \tag{136}$$

Since the right-hand side of Eq. 136 can be made arbitrarily small by choosing N to be large enough, the sequence is a Cauchy sequence. Since B(T) is complete, the sequence must be uniformly convergent, and we are permitted to write

$$e = \lim_{n\to\infty} \underline{E}_n(x)$$

whereupon $\underline{E}$ is defined to be an operator that maps $\underline{B}$(T) into itself, and for which $\underline{E}(x) = e$.

5.2.2 Sufficiency

The operator $\underline{E}$ that has been thus determined is a solution of Eq. 43 for, consider the difference between the two sides of this equation, which we shall bound:

$$\|(\underline{I}-\underline{H}*\underline{E})(x)-\underline{E}(x), t\|$$

$$\leq \|(\underline{I}-\underline{H}*\underline{E})(x)-(\underline{I}-\underline{H}*\underline{E}_n)(x), t\|$$

$$+ \|(\underline{I}-\underline{H}*\underline{E}_n)(x)-\underline{E}(x), t\|, \qquad n = 1, 2, \ldots \tag{137}$$

Here, we have used the triangle inequality. Using the Lipschitz condition, Eq. 132, to bound the first term on the right-hand side of Eq. 137 and the iteration formula, Eq. 130, to express the second term, we have

$$\|(\underline{I}-\underline{H}*\underline{E})(x)-\underline{E}(x), t\|$$

$$\leq th(t)\|\underline{E}(x)-\underline{E}_n(x), t\| + \|\underline{E}_{n+1}(x)-\underline{E}(x), t\| \tag{138}$$

Since $\underline{E}_n(x)$ converges to $\underline{E}(x)$, and t is constant, the right-hand side of Eq. 138 can be made arbitrarily small by choosing n to be large enough; hence the left-hand side must be zero, and the result, Eq. 43, follows.

5.2.3 Uniqueness

In a similar manner, it can be shown that, if $\underline{E}'$ is any other solution of Eq. 43 – that is, $\underline{E}' = \underline{I} - \underline{H} * \underline{E}'$, then $\underline{E} = \underline{E}'$, so that $\underline{E}$ is unique. This is shown by the following sequence of manipulations:

$$\|\underline{E}(x)-\underline{E}'(x), t\| = \|\underline{H}*\underline{E}(x)-\underline{H}*\underline{E}'(x), t\|$$

$$\leq h(t) \int_0^t \|\underline{E}(x)-\underline{E}'(x), t_1\| \, dt_1$$

$$\leq \frac{(th(t))^n}{n!} \|\underline{E}(x)-\underline{E}'(x), t\|, \qquad n = 1, 2, \ldots \tag{139}$$

The right-hand side of Eq. 139 can be made arbitrarily small.

5.3 BOUNDS ON $\underline{E}$

5.3.1 Bound on $\|\underline{E}(x), t\|$

Using the triangle inequality, we have (note that $\underline{E}_o(x) = 0$)

$$\|\underline{E}(x), t\| \leq \|\underline{E}(x)-\underline{E}_n(x), t\| + \|\underline{E}_n(x)-\underline{E}_o(x), t\| \tag{140}$$

Since the first term on the right side of Eq. 140 can be made as small as desired, this equation is valid when it is set equal to zero; moreover, the second term is bounded by Eq. 136. Hence,

$$\|\underline{E}(x), t\| \leq \epsilon^{th(t)} \|\underline{E}_1(x), t\|$$

$$\leq \epsilon^{th(t)} [\|x(t), t\| + \underline{H}(0)] \tag{141}$$

is the required bound.

5.3.2 Bound on incr$|||\underline{E}, t|||$

Proceeding in a manner analogous to that of section 5.3.1, we obtain the Lipschitz bound

$$\|\underline{E}(x)-\underline{E}(y), t_1\| \leq \epsilon^{t_1 h(t_1)} \|x-y, t_1\|$$

where x and y are any pair of functions in B(t), and t_1 is any point on the interval $[0, T]$. Consequently, we have

$$\text{incr}|||\underline{E}, t||| \leq \epsilon^{th(t)} \tag{142}$$

Note that $\underline{E}$ does not satisfy an integral Lipschitz condition such as Eq. 131 because it behaves like an identity operator near the origin.

5.3.3 Bound on Truncation Error

Equation 136 supplies a truncation error bound, when it is realized that it must hold in the limit as m approaches infinity, whereupon we get

$$\| \underline{E}(x)-\underline{E}_n(x), t \| \leq \frac{(th(t))^n}{n!} \epsilon^{th(t)} \| \underline{E}_1(x), t \| \tag{143}$$

where

$$\| \underline{E}_1(x), t \| = \| (\underline{I}-\underline{H}(0))(x), t \|$$

$$\leq \| x, t \| + \underline{H}(0)$$

5.4 ITERATION FOR THE FEEDBACK OPERATOR $\underline{G}$

Entirely analogous results are obtainable for the operator $\underline{G}$ that satisfies the feedback equation (Eq. 39), $\underline{G} = \underline{H} * (\underline{I}-\underline{G})$ and is related to $\underline{E}$ by $\underline{G} = \underline{H} * \underline{E}$. The operator $\underline{G}$ can be found by means of the iteration

$$\underline{G}_o = \underline{0}$$

$$\underline{G}_n = \underline{H} * (\underline{I}-\underline{G}_{n-1}), \qquad n = 1, 2, \ldots \tag{144}$$

$$\underline{G}(x) = \lim_{n\to\infty} \underline{G}_n(x)$$

and has the following bounds:

$$\| \underline{G}(x), t \| \leq th(t)\, e^{th(t)} \| x, t \| + \underline{H}(0) \tag{144a}$$

$$\text{incr} \||\underline{G}, t \|| \leq th(t)\, e^{th(t)} \tag{144b}$$

$$\| \underline{G}(x)-\underline{G}_n(x), t \| \leq \frac{(th(t))^{n+1}}{n!} e^{th(t)} \| x, t \| + e^{th(t)} \underline{H}(0), \qquad n = 0, 1, \ldots \tag{144c}$$

The operator $\underline{G}$ itself satisfies the integral Lipschitz inequality,

$$\| \underline{G}(x)-\underline{G}(y), t \| = \| \underline{H}*\underline{E}(x)-\underline{H}*\underline{E}(y), t \|$$

$$\leq h(t) \int_0^t \| \underline{E}(x)-\underline{E}(y), t_1 \| \, dt_1$$

$$\leq h(t)\, \epsilon^{th(t)} \int_0^t \| x-y, t_1 \| \, dt_1 \tag{145}$$

in which we have used $\underline{G} = \underline{H} * \underline{E}$ to express $\underline{G}$, and Eq. 142 for $\text{incr} \||\underline{E}, t\||$.

5.5 INTEGRAL OPERATORS

An integral operator is any operator that can be put in the form of a cascade of finite gain operators, at least one of which is an integrator (Fig. 30). Such an

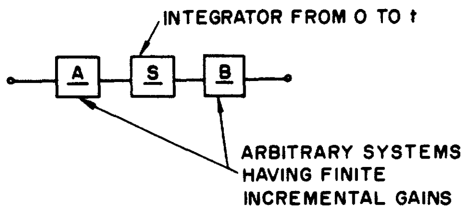


Fig. 30. An integral system.

operator can always be put in the form of an integrator in between two operators, $\underline{A}$ and $\underline{B}$, whose gains are finite. (All of the operators concerned map B(T) into itself, and the gain is maximum incremental in the least upper bound norm.)

5.5.1 Application to Feedback Systems

The exponential iteration theory is always applicable to feedback equations whose loop operator $\underline{H}$ is an integral operator; for, $\underline{H}$ then satisfies the integral Lipschitz condition, Eq. 131, in the least upper bound norm. To show this, let us assume that we have

$$\underline{H} = \underline{B} * \underline{S} * \underline{A} \tag{146}$$

in which $\underline{A}$ and $\underline{B}$ are any two operators having maximum incremental gains a and b, respectively, which are finite, and $\underline{S}$ is an integrator on $\underline{B}(t)$; that is,

$$\underline{S}(x, t) = \int_0^t x(t')\, dt' \tag{147}$$

We now have, for any t in [0, T],

$$\begin{aligned} |\underline{H}(x, t)-\underline{H}(y, t)| &= |\underline{B}*\underline{S}*\underline{A}(x, t)-\underline{B}*\underline{S}*\underline{A}(y, t)| \\ &\leq b \int_0^t |\underline{A}(x, t')-\underline{A}(y, t')|\, dt' \leq ba \int_0^t \|x-y, t'\|\, dt' \end{aligned} \tag{148}$$

in which a and b are any pair of functions in $\underline{B}$(T). The integral Lipschitz inequality follows:

$$\|\underline{H}(x)-\underline{H}(y), t\|_1 \leq ab \int_0^t \|x-y, t'\|_1\, dt' \tag{149}$$

5.5.2 Linear, Time-Invariant, One-Sided, Integral Operators

A linear, time-invariant, one-sided operator can be shown to be integral if its impulse response has a finite variation on every interval [0, t], and vanishes for negative time. Consider the operator $\underline{H}$ represented by the convolution integral

$$y(t) = \int_0^t x(\tau)\, h(t-\tau)\, d\tau \tag{150}$$

We shall show that the operator $\underline{H}'$, given by

$$y'(t) = \frac{d}{dt} \int_0^t x(\tau)\, h(t-\tau)\, d\tau \tag{151}$$

exists and has a finite gain, whereupon $\underline{H}$ can be written in the integral form $\underline{H} = \underline{S} * \underline{H}'$. To show this, we evaluate

$$\begin{aligned}
y'(t) &= \frac{d}{dt} \int_0^t x(\tau)\, h(t-\tau)\, d\tau \\
&= \lim_{\Delta t \to \infty} \frac{1}{\Delta t} \left(\int_0^t x(\tau)\, h(t-\tau)\, d\tau - \int_0^t x(\tau)\, h(t+\Delta t-\tau)\, d\tau \right) \\
&= \lim_{\Delta t \to \infty} \frac{1}{\Delta t} \int_0^t x(\tau)(h(t-\tau)-h(t-\tau+\Delta t))\, d\tau \\
&\leq \|x, t\|_1 \lim_{\Delta t \to \infty} \frac{1}{\Delta t} \int_0^t |h(t-\tau)-h(t-\tau+\Delta t)|\, d\tau \\
&= \|x, t\|_1\, v(h, t)
\end{aligned} \tag{152}$$

in which $v(h, t)$ represents the variation of the kernel $\underline{H}$, on the interval $[0, t]$ and equals the gain function of $\underline{H}'$ by virtue of Eq. 152. Thus, $\underline{H}$ can be represented by a finite gain operator followed by an integrator, and is therefore integral. (It is enough for the variation to be finite on every finite interval; it may be infinite on the infinite interval.)

Note that an integral operator of this type remains an integral operator if it is cascaded with no-memory, time-invariant operators (in any order), provided that they all have finite gain (finite slope).

5.5.3 Application of Exponential Iteration to a Model for the Nonlinear Distortion of a Feedback System

The iterative model that was derived in section 3.6 failed to converge geometrically for large degrees of nonlinearity $\left(\frac{\gamma}{\beta} > 3\right)$. Nevertheless, it is a valid model for all finite ratios, γ/β (whenever the no-memory operator has an upper bound to its slope) and converges exponentially because its open-loop system is integral; this follows from section 5.5.2.

VI. REALIZABILITY OF SYSTEMS

We shall now attempt to answer the question, What is a good enough mathematical model of a physical system — a model that does not lead to impossible results when it is used in a feedback problem (20)?

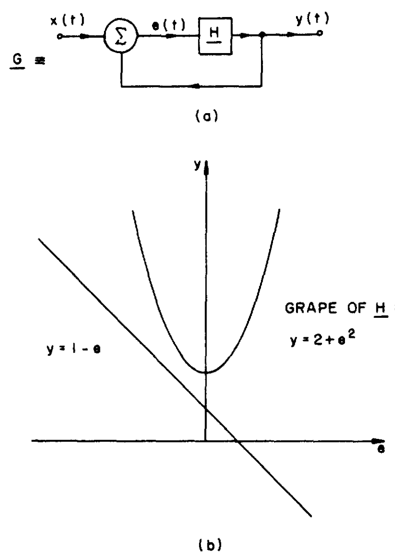


Fig. 31. (a) A feedback system. (b) Example of feedback equations that have no solution; x = 1.

A feedback system is the embodiment of the solution of a pair of simultaneous equations. For example, the system $\underline{G}$, which is shown in Fig. 31a, is described by the equations

$$\begin{aligned} y &= \underline{H}(e) \\ e &= x - y \end{aligned} \tag{153}$$

A solution eliminates one of the three functions of time — x, y, or e — and leaves an explicit relation, for example,

$$y = \underline{G}(x) \tag{154}$$

in which the operator $\underline{G}$ must satisfy the operator equation (Eq. 39)

$$\underline{G} = \underline{H} * (\underline{I}-\underline{G})$$

Equation 39 is the mathematical model of the system.

In a physical system — we are not restricting ourselves to stable systems — every input produces a well-defined and unique output. Hence, if the mathematical model is to match a real system, it must at least have a unique solution. The solution must be one that can be approximated by a physical system. This limits the class of useful models; for, many idealizations that might be convenient to use yield impossible solutions or have no solution, even if they satisfy the well-known realizability criterion of "no response before excitation."

6.1 EXAMPLES OF UNREALIZABLE SYSTEMS

Consider the no-memory system that results if H is a time-invariant, no-memory device whose graph is the parabola shown in Fig. 31b. If the input x is a unit step, then the output y is the solution of the simultaneous equations

$$y = e^2 + 2$$

$$e = 1 - y$$

But these equations have no solution, since their graphs do not intersect. Hence, the model is not realizable.

Consider, next, the linear system obtained by letting H be a pure gain of magnitude - 2. Applying the conventional feedback equation, we get

$$Y(\omega) = \frac{H(\omega)}{1 + H(\omega)} X(\omega)$$

$$= 2X(\omega) \tag{155}$$

in which $X(\omega)$, $Y(\omega)$, and $H(\omega)$ are the frequency spectra of the input, output, and the operator H, respectively. The response to a unit step is a step of amplitude 2. It is easy enough to verify, by substituting the value that we have found for y in Eq. 153, that this is indeed a solution. However, it is one that the physical system never exhibits because the inevitable delay at the highest frequencies results in instability; the output becomes infinite instantaneously if the delay is zero. Hence this model is useless, although it can be made useful by including an arbitrarily small delay.

Finally, consider the servomechanism illustrated in Fig. 32, consisting of a relay-controlled motor. The relay is idealized as a no-memory device with two states, and the motor is assumed to be linear and time-invariant. The response to a small pulse does not exist, even though we might be tempted to describe its behavior as hunting of zero amplitude and infinite frequency. If we attempt to find the output by iteration, the sequence does not converge. The system, then, is not realizable unless the slope in the transition region is made finite or a time delay is included.

In all of these examples the open-loop model is useful because it can be realized in a limiting sense, while the closed-loop model cannot be realized at all. It is essential to take the limit after the loop is closed, not before.

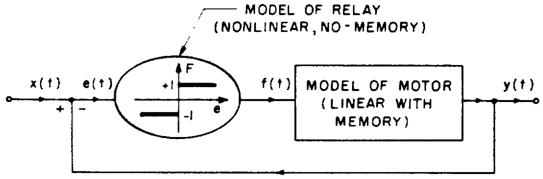


Fig. 32. Unrealizable model of a relay-controlled motor servomechanism.

6.2 PROPERTIES THAT DETERMINE A REALIZABILITY CLASS OF SYSTEMS

A class of systems which has sufficient conditions for realizability must be capable of giving valid results in problems involving addition, cascading, feedback, or any combination of these; it must never be subject to any of the difficulties that we have described. The following properties determine such a class, which we shall refer to as a realizability class:

(a) Every system belonging to the class can be approximated by a physical device. (The meaning of this will be explored more carefully below.)

(b) When a system belonging to the class is placed in a feedback loop, the feedback equations have a unique solution so that the derived system exists. It is a trivial assertion that the same must hold true for the sum or cascade of two systems. (This property eliminates the difficulty of the first and third examples.)

(c) When a system is derived by summing or cascading two systems belonging to the class, or by placing a system that belongs to the class in a feedback loop, an arbitrarily close approximation to the derived system can be obtained by making physical approximations that are close enough to the original system or pair, and placing the approximations in the feedback loop, sum, or cascade.

For example, assume that the system $\underline{H}$ in Eq. 39 belongs to this class, so that property (b) ensures that that equation has a solution for $\underline{G}$. If, now, $\underline{H}'$ represents a physical device, then it gives rise to a physical feedback system $\underline{G}'$ which satisfies the corresponding relation, $\underline{G}' = \underline{G}' * (\underline{I} - \underline{G}')$. The error in approximating $\underline{G}$ by $\underline{G}'$ is given by

$$\underline{G} - \underline{G}' = \underline{H} * (\underline{I} - \underline{G}) - \underline{H}' * (\underline{I} - \underline{G}')$$

Property (c) asserts that $\underline{G} - \underline{G}'$ can be made as small as desired by making $\underline{H} - \underline{H}'$ small enough. (Property (a) ensures that it is, in fact, possible to build $\underline{H} - \underline{H}'$ arbitrarily small.)

This property disposes of the difficulty which the second example exhibits; for this class of systems, it is immaterial whether limits are taken before or after closing the loop.

(d) The system derived by summing or cascading a pair of systems that belong to this class, or by placing a system that belongs to the class in a feedback loop, itself belongs to this class. This property ensures the usefulness of the model in problems involving arbitrary combinations of summation, cascading, and feedback.

6.3 THE REALIZABILITY CLASS OF SYSTEMS

Physical systems never exhibit the paradoxical behavior that has just been described because they invariably have inertial and storage elements, which delay or smooth the output, and because their amplification is always finite, even in an incremental sense.

The mathematical realizability condition must therefore express the fact that the output is a smoothed version of the input which has been subjected to a finite incremental amplification; or, in other words, that the system is not too explosive and that it attenuates high frequencies. The conditions that we shall define (Eqs. 156 and 157) introduce smoothing in the form of an integration. Each states that the norm of the output is less than a constant times the integral of the input, and that this is true incrementally; that is, for the differences between all possible pairs of inputs and the corresponding differences between pairs of outputs. (A condition of this kind is called an integral Lipschitz condition.) Actually norm-time functions are used instead of norms, and the condition must hold for every positive time. Moreover, it is required that the system satisfy two such conditions, in two different norms (two different ways of measuring the size of functions) simultaneously. The first of these is the least upper bound norm (subscript "1"), which is simply the largest value that the function attains on the given interval. The second is the "impulse norm" (subscript "3"), which we shall leave undefined for the moment in order to approach it heuristically.

6.3.1 Definition of the Realizability Class

The realizability class of systems consists of all of those systems $\underline{H}$ that satisfy the following integral Lipschitz conditions:

$$\| \underline{H}(x)-\underline{H}(y), t \|_1 \leq h(t) \int_0^t \| x-y, t' \|_1 \, dt' \tag{156}$$

$$\| \underline{H}(x)-\underline{H}(y), t \|_3 \leq h(t) \int_0^t \| x-y, t' \|_3 \, dt' \tag{157}$$

for all possible times t in the interval $[0, T]$, and for all possible pairs of input time functions, x and y. It is assumed that $\underline{H}$ is an operator that maps the space $B_1(T)$ into itself. The subscripts "1" and "3" refer to the least upper bound and impulse norms, respectively.

The realizability class has all of the properties described in section 6.2. Property (a) is discussed in section 6.5, and properties (b), (c), and (d) are established by Theorem 3. That theorem also establishes the fact that any feedback problem involving a member of the class can be solved by (exponential) iteration.

Each of inequalities 156 and 157 implies that the system may not have any response before excitation; that is, inputs that are identical until a certain time produce outputs that are identical until that time.

6.4 APPLICATION OF REALIZABLE SYSTEMS

Before establishing the properties that have been claimed for realizable systems, let us state without proof what the realizability conditions amount to for some common systems.

A linear, time-invariant system is realizable if its impulse response has a finite variation on every finite interval and vanishes for negative time. If the impulse response h is differentiable except at a countable number of points, t_i, then the variation v(x, t) is given by the equation

$$v(x,t) = \int_0^t \left|\frac{d}{dt} h(t')\right| dt' + \sum_i \left(h\left(t_i^+\right) - h\left(t_i^-\right)\right)$$

Thus, a system whose impulse response has impulses, doublets, or other singularity functions is not realizable. However, almost every bounded impulse response that is encountered in practice is realizable. In general, the realizability conditions give results consistent with intuition, at least after some deliberation.

A no-memory device is never realizable by itself. If the absolute value of its slope has an upper bound, then it is realizable if it is preceded by any realizable system in cascade. (Integral systems (see section 5.5) need not be realizable.)

For the purpose of determining whether or not a given system is realizable, the two Lipschitz conditions may often be replaced by the single condition,

$$\left|\underline{H}(x,t)-\underline{H}(y,t)\right| \leq \underset{0\leq t'\leq t}{\text{l. u. b.}} \int_0^t (x(t')-y(t'))\, dt' \tag{158}$$

It is proved in section 6.10 that inequality 158 implies inequalities 156 and 157 (although the converse is not true).

6.5 METHOD OF APPROXIMATION

When we say that a system can be approximated by a physical device over a finite time interval, we mean that it is possible to construct a device whose output in response to any input does not differ by more than some arbitrarily small error from that of the given system in response to the same input, over the time interval.

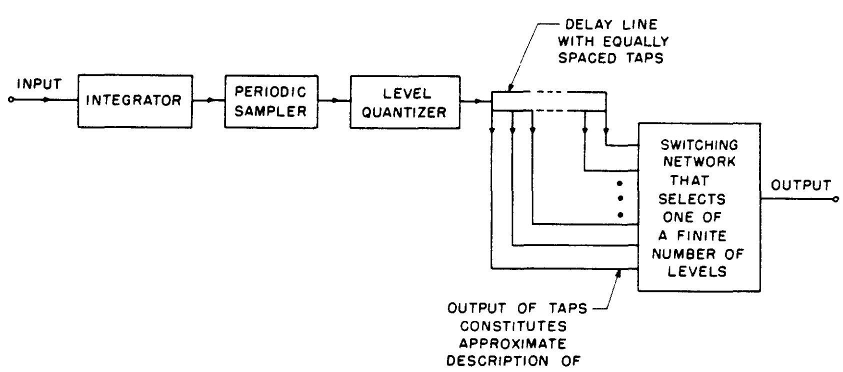


Fig. 33. Synthesis of a physical approximation to a filter.

To prove that a system can be approximated by a physical device, we shall show how to synthesize it out of a finite number of elements that are assumed to be realizable, not ideally, but with tolerances on accuracy and restrictions on range of operation. The apparatus (Fig. 33) consists of a gating circuit that divides the collection of possible inputs into a number of small "cells," in each of which the inputs do not differ from each other by more than a small amount. The apparatus does not distinguish between the various inputs in a single cell, giving the same output for all of them. This leads to a small approximation error.

The gating is accomplished by sampling the inputs at regular time intervals, quantizing the samples into a finite number of equal levels, and delaying the samples by means of a series of delay lines as suggested by Singleton (9). Thus, at any time, the entire past of the input is approximately determined by the outputs of the delay lines. These outputs operate a level selector through a switching device, approximately reproducing the output, one increment at a time. The integrator that precedes the system has the purpose of changing inputs of bounded height into inputs of bounded slope.

This clearly impractical scheme is useful for showing realizability. It is possible to prove that any system that satisfies the realizability conditions of section 6.3 can be approximated with any desired accuracy by such an apparatus for any input whose height does not exceed some bound specified in advance, over some finite interval [0, T]. The impulse norm, defined in section 6.8, is the measure of error. The proof is long and tedious, and is therefore omitted.

6.6 REQUIREMENTS FOR APPROXIMATION: COMPACTNESS AND UNIFORM CONTINUITY

In order that this, or in fact any, meaningful approximation procedure may be possible, several requirements must be met.

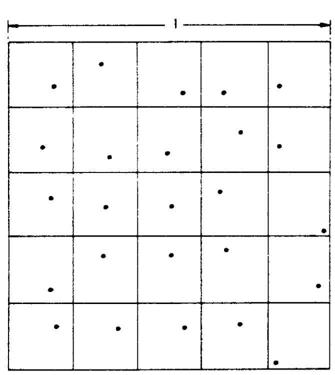


Fig. 34. Example of a compact space; each square cell has a single point that approximates all of the other points in that cell within the length of a diagonal, $\sqrt{2}/n$; n is equal to 5 here.

The first requirement is that the collection of all possible inputs be compact, which is a technical way of saying that it is possible to divide it into a finite number of cells. For example, the unit square (Fig. 34) is a compact collection of points because a division of it into n^2 smaller squares always leads to cells in which any two points are not more than $\sqrt{2}/n$ apart, so that any point serves as an approximation to all the others. If a tolerance t is required, then any finite n greater than $t/\sqrt{2}$ is adequate.

Unfortunately, the collection of all time functions is not compact, even if they are restricted to a finite time interval, and never exceed some maximum height. It is necessary to impose some sort of limitation on the high-frequency content of inputs and thus to restrict their fluctuations in order to achieve compactness.

The second requirement that must be met is that the system $\underline{H}$ be uniformly continuous; that is, that a small error in the input produce a small error in the output. This is necessary if the scheme of division into cells is to be used on $\underline{H}$.

The third requirement is that, since the open-loop system operates on the error signals in the feedback loop, not on the inputs themselves, the compact collection of inputs be mapped into a compact collection of errors. This will come about if the error operator $\underline{E}$ is uniformly continuous, in accord with a well-known theorem in analysis.

Finally, the closed-loop system $\underline{G}$ must be uniformly continuous, so that it maps compact collections of functions into compact collections when it is part of some larger interconnection of systems.

6.7 MATHEMATICAL CONSIDERATIONS: A WEAKER REALIZABILITY CONDITION

All of the desired properties for realizable systems, with the exception of the approximation property (a), can be achieved by imposing a single condition of the integral Lipschitz type on $\underline{H}$, for example, Eq. 156. The resulting operators $\underline{G}$ and $\underline{E} = \underline{I} - \underline{G}$ have unique uniformly continuous (in the relevant norm) solutions, $\underline{G}$ itself satisfies the corresponding Lipschitz condition, and a small error in $\underline{H}$ leads to a small error in $\underline{G}$. By imposing a suitable high-frequency restriction on all inputs, such as an upper bound on the absolute values of their derivatives, it is possible to achieve the approximation property too. Hence, Eq. 156 is proposed as a weaker type of realizability condition.

An alternative approach adopted here obviates any restriction other than that on maximum amplitude of inputs, by the subterfuge of using a (rather unusual) norm suggested by Brilliant (11b). We shall call it the "impulse" norm. It de-emphasizes the high-frequency components of the signal by means of an integration. Brilliant has shown that the collection of all integrable time functions on the finite interval $[0, T]$ that are uniformly bounded on $[0, T]$ is compact in this norm.

6.8 THE IMPULSE NORM

The impulse norm is the least upper bound norm of the integrated function. Thus, if $\underline{S}$ denotes an integrator (from 0 to t), then the impulse norm $\|x\|_3$ of any function x is given by,

$$\|x\|_3 = \|\underline{S}(x)\|_1 \tag{159}$$

Since the effect of our integration is to weigh the frequency components of a function inversely with frequency (in the terminology of Fourier transforms an integration is a multiplication by $1/j\omega$), we have a norm that de-emphasizes high frequencies.

As usual, we are interested in the norm-time function, defined as

$$\|x, t\|_3 = \|\underline{S}(x), t\|_1 = \underset{0 \leq t' \leq t}{\text{l.u.b.}} \int_0^t x(t')\,dt' \tag{160}$$

If the least upper bound norm-time function is known, then the impulse norm-time function may be bounded by using the following inequality, which is derived from Eq. 160,

$$\|x, t\|_3 \leq \int_0^t \|x, t'\|_1\,dt' \leq t\,\|x, t\|_1 \tag{161}$$

6.9 THEOREM 3: REALIZABLE FEEDBACK SYSTEMS

(a) The feedback error equation (Eq. 43),

$$\underline{E} = \underline{I} - \underline{H} * \underline{E}$$

and the feedback equation (Eq. 39),

$$\underline{G} = \underline{H} * (\underline{I} - \underline{G})$$

in which all of the operators concerned map $B_1(T)$ into itself, have unique solutions for $\underline{E}$ and $\underline{G}$ which may be found by means of the respective iterations,

$$\underline{E}_o = \underline{0}$$

$$\underline{E}_n = \underline{I} - \underline{H} * \underline{E}_{n-1}, \qquad n = 1, 2, \ldots$$

$$\underline{E}(x) = \lim_{n\to\infty} \underline{E}_n(x)$$

and

$$\underline{G}_o = \underline{0}$$

$$\underline{G}_n = \underline{H} * (\underline{I} - \underline{G}_{n-1}), \qquad n = 1, 2, \ldots$$

$$\underline{G}(x) = \lim_{n\to\infty} \underline{G}_n(x)$$

provided that the realizability conditions,

$$\|\underline{H}(x) - \underline{H}(y), t\|_1 \leq h(t) \int_0^t \|x - y, t'\|_1\,dt' \tag{162}$$

$$\|\underline{H}(x) - \underline{H}(y), t\|_3 \leq h(t) \int_0^t \|x - y, t'\|_3\,dt' \tag{163}$$

are fulfilled for each pair of functions x and y belonging to $B_1(T)$, and for each t in $[0, t]$. The convergence is uniform on the interval $|0, T|$ for each x in $B_1(T)$ in both least upper bound and impulse norms. For simplicity, assume that $\underline{H}(0) = \underline{E}(0) = 0$.

(b) $\underline{E}$ and $\underline{G}$ are uniformly continuous on $B_1(T)$ in both norms.

(c) $\underline{G}$ (but not $\underline{E}$) satisfies the realizability conditions, Eqs. 162 and 163.

(d) An arbitrarily close approximation, $\underline{G}'$, can be made to $\underline{G}$, by making a close enough approximation, $\underline{H}'$, to $\underline{H}$, which satisfies the realizability conditions in the following sense. Since $\underline{H}'$ is realizable, the feedback equation

$$\underline{G}' = \underline{H}' * (\underline{I}-\underline{G}') \tag{164}$$

has a unique solution for $\underline{G}'$. We assert that if any two positive, real numbers b and r are given, it is possible to find two other positive real numbers, B and R, with the property that if $\underline{H}'$ satisfies the realizability conditions (with a constant $\underline{H}'(t)$), and if we have

$$\| \underline{H}'(x)-\underline{H}(x), T \|_3 < R \tag{165}$$

for all x in $B_1(T)$ for which $\|x_1, T\|_1 < B$, then we have

$$\| \underline{G}'(x)-\underline{G}(x), T \|_3 < r \tag{166}$$

for all x in $B_1(T)$ for which $\|x_1, T\|_1 < b$. The constants B and R are given by the equations

$$B = b\epsilon^{Th(T)} \tag{167}$$

$$R = r\epsilon^{-Th'(T)} \tag{168}$$

6.9.1 Proof of Theorem 3

This theorem is a direct application of the theorems on exponential iteration of sections 5.2 and 5.4. Part 1 of the hypothesis follows directly. Part 2, concerning uniform continuity of $\underline{E}$ and $\underline{G}$, is implied by Eqs. 142 and 144b, which show that the maximum incremental gains of both operators are finite. (Uniform continuity follows from the gain definition, Eqs. 23.) Part 3 is implied by Eq. 145.

To prove Part 4, we shall evaluate the closed-loop difference, $\| \underline{G}'(x)-\underline{G}(x), t \|_3$, which we shall designate by the symbol d(x, t), for any t in [0, T]. Substituting Eqs. 39 and 164 for $\underline{G}$ and $\underline{G}'$, we obtain

$$\begin{aligned} d(x,t) &= \| \underline{G}'(x)-\underline{G}(x), t \|_3 \\ &= \| \underline{H}'*(\underline{I}-\underline{G}')(x) - \underline{H}*(\underline{I}-\underline{G}(x), t \|_3 \end{aligned} \tag{169}$$

Adding and subtracting a term, and using the triangle inequality, we get

$$\begin{aligned} d(x,t) \leq\ & \| \underline{H}'*(\underline{I}-\underline{G}')(x) - \underline{H}'*(\underline{I}-\underline{G})(x), t \|_3 \\ &+ \| \underline{H}'*(\underline{I}-\underline{G})(x) - \underline{H}*(\underline{I}-\underline{G})(x), t \|_3 \end{aligned} \tag{170}$$

Applying the realizability condition for $\underline{H}'$, represented by Eq. 163, to bound the first term on the right side of Eq. 170, and the notation $\Delta \underline{H} = \underline{H} - \underline{H}'$ to express the second, we obtain the inequality

$$d(x,t) \leq h'(t) \int_0^t \|(\underline{G}-\underline{G}')(x), t'\|_3 \, dt' + \|\Delta \underline{H}*(\underline{I}-\underline{G})(x), t\|_3 \tag{171}$$

Now, let us assume that $\|x, t\|_1 < b$ holds. Therefore, we have $\|(\underline{I}-G)(x), t\|_1 < b\epsilon^{th(t)}$, by virtue of Eq. 141 of our exponential iteration theory. But it has also been assumed that $\|\Delta\underline{H}(e), t\|_3 < R$ for all e that satisfy $\|e, t\|_1 < b\epsilon^{th(t)}$. Hence, for all x that satisfy $\|x, t\|_1 < b$ the second term on the right of Eq. 171 can be bounded:

$$d(x,t) < h'(t) \int_0^t d(x,t')\, dt' + R \tag{172}$$

Equation 172 is an integral inequality, which may be solved by repeatedly substituting it into itself. After bounding the resulting integrals, inequalities of the following type are obtained:

$$d(x,t) < R\left(1+(th'(t)) + \ldots + \frac{(th'(t))^{n+1}}{(n+1)!}\right) + \frac{(th'(t))^n}{n!} d(x,t), \qquad n = 1, 2, \ldots \tag{173}$$

Since d(x, t) must be finite, the last term on the right side of Eq. 173 must vanish as n approaches infinity, and we are left with the bound

$$d(x,t) < R\epsilon^{th'(t)} < r \tag{174}$$

in which we have used Eq. 168 and the fact that $h'(t) \leq h'(T)$ to obtain the second inequality. The proof is now complete.

6.10 A SUFFICIENT CONDITION FOR REALIZABILITY

The two realizability conditions are implied by the single condition,

$$\|\underline{H}(x)-\underline{H}(y), t\|_1 \leq h(t) \; \|x-y, t\|_3 \tag{175}$$

(although the converse is not true).

This can be shown in the least upper bound norm, by using inequality 161.

$$\|\underline{H}(x)-\underline{H}(y), t\|_1 \leq h(t) \; \|x-y, t\|_3 \leq h(t) \int_0^t \|x-y, t'\|_1 \, dt' \tag{176}$$

Equation 156 is therefore valid.

In the impulse norm, using inequality 161, we have

$$\|\underline{H}(x)-\underline{H}(y), t\|_3 \leq \int_0^t \|\underline{H}(x)-\underline{H}(y), t'\|_1 \, dt'$$

$$\leq h(t) \int_0^t \|x-y, t'\|_3 \, dt'$$

in which the last inequality was obtained by using Eq. 175, and the fact that h(t) is an increasing function of time.

VII. APPLICATIONS OF ITERATION THEORY TO POWER-SERIES OPERATORS

Certain results have been established concerning operators and feedback systems without regard to the manner of their representation. We shall now use some of these results to study operators that are represented by power series of convolution integrals of the type

$$z(t) = \int_{-\infty}^{\infty} k_1(t-\tau)\, y(\tau)\, d\tau + \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} k_2(t-\tau_1, t-\tau_2)\, y(\tau_1)\, y(\tau_2)\, d\tau_1 d\tau_2 + \dots \tag{177}$$

in which the input and output are y(t) and z(t), respectively, and the k's are kernels that completely specify the operator. Operators of this type have been used by Brilliant (11) and Barrett (13) as the basis of a transform calculus for the analysis of nonlinear feedback systems. Their method will be described here, and certain additional results will be established:

(a) An iterative method of solving feedback equations which "converges in degree" will be described. This method, which involves perturbation about the linear part of the loop operator, yields all terms in Eq. 177 whose degree is not greater than n without error at the n^{th} cycle.

(b) It will be shown how this iteration can be used to study the convergence of the power series for the closed-loop operator.

(c) A method of constructing tables of transform formulas for the solution of feedback equations will be described.

7.1 NONLINEAR TRANSFORM CALCULUS

Nonlinear operator equations can be solved if each of the operators concerned can be expressed as a power series. If the unknown operator is expressed as a power series with undetermined kernels, then an integral equation is obtained relating the kernels of the unknown operator to the kernels of the known operators. By grouping all terms appearing in this equation according to degree, a sequence of homogeneous integral

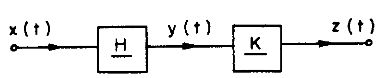


Fig. 35. A cascade.

equations is obtained which can be solved for the unknown kernels, one at a time. For example, suppose that two operators $\underline{K}$ and $\underline{H}$ are given, and it is required to find their cascade $\underline{K} * \underline{H}$ (Fig. 35). All three operators are expressed in series form, as in Eq. 177; that is,

$$y(t) = \int_{-\infty}^{\infty} h_1(t-\tau)\, x(\tau)\, d\tau + \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h_2(t-\tau_1, t-\tau_2)\, x(\tau_1)\, x(\tau_2)\, d\tau_1 d\tau_2 + \dots \tag{178}$$

and

$$z(t) = \int_{-\infty}^{\infty} [k*h]_1(t-\tau)\, x(\tau)\, d\tau + \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} [k*h]_2(t-\tau_1, t-\tau_2)\, x(\tau_1)\, x(\tau_2)\, d\tau_1 d\tau_2 + \dots \qquad (179)$$

Now Eq. 178 is substituted for y in Eq. 177, and yields an expression for z that must represent the cascaded operator $\underline{K} * \underline{H}$. Hence, the derived kernels – which are convolutions of the original ones – must equal the unknown kernels $[k*h]_n$ of corresponding degree. For example, the first two kernels are given by

$$[k*h]_1(\tau) = \int_{-\infty}^{\infty} k_1(\sigma)\, h_1(\tau-\sigma)\, d\sigma \qquad (180)$$

$$[k*h]_2(\tau_1, \tau_2) = \int_{-\infty}^{\infty} k_1(\sigma)\, h_2(\tau_1 - \sigma, \tau_2 - \sigma)\, d\sigma$$

$$+ \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} k_2(\sigma_1, \sigma_2)\, h_1(\tau_1 - \sigma_1)\, h_2(\tau_2 - \sigma_2)\, d\sigma_1 d\sigma_2 \qquad (181)$$

The convolutions in Eqs. 180 and 181 can be replaced by multiplications by taking Fourier transforms of several variables. Thus, we get

$$[K*H]_1(\omega) = \int_{-\infty}^{\infty} [k*h]_1(\tau)\, \exp(-j\omega\tau)\, d\tau$$

$$= K_1(\omega)\, H_1(\omega) \qquad (182)$$

for the spectrum of the cascaded kernel of degree 1, and

$$[K*H]_2(\omega_1, \omega_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} [k*h]_2(\tau_1, \tau_2)\, \exp[j(\omega_1\tau_1 + \omega_2\tau_2)]\, d\tau_1 d\tau_2$$

$$= K_1(\omega_1 + \omega_2)\, H_1(\omega_1, \omega_2) + K_2(\omega_1, \omega_2)\, H_1(\omega_1)\, H_1(\omega_2) \qquad (183)$$

for the cascaded spectrum of degree 2, and so on. The spectrum of the cascaded kernel of degree 2 (and of every degree greater than 2) is therefore a sum of products of spectra, instead of the product that is encountered in linear theory.

A very similar procedure can be followed with other equations, for example, the inverse equation

$$\underline{K} * \underline{H} = \underline{I} \qquad (184)$$

and the feedback equation (Eq. 39), $\underline{G} = \underline{H} * (\underline{I} - \underline{G})$ which describes the system in Fig. 36. Tables have been prepared for the spectra of the unknown kernels of various degrees for

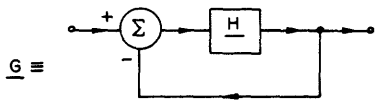


Fig. 36. A feedback system.

Table 1. Formulas for the spectra of cascaded kernels.

$$[K*H]_1(\omega) = K_1(\omega)\, H_1(\omega)$$

$$[K*H]_2(\omega_1,\omega_2) = K_1(\omega_1+\omega_2)\, H_2(\omega_1,\omega_2) + K_2(\omega_1,\omega_2)\, H_1(\omega_1)\, H_1(\omega_2)$$

$$[K*H]_3(\omega_1,\omega_2,\omega_3) = K_1(\omega_1+\omega_2+\omega_3) H_3(\omega_1,\omega_2,\omega_3) + 2K_2(\omega_1,\omega_2+\omega_3) H_1(\omega_1) H_2(\omega_2+\omega_3)$$
$$+ K_3(\omega_1,\omega_2,\omega_3)\, H_1(\omega_1)\, H_1(\omega_2)\, H_1(\omega_3)$$

$$[K*H]_4(\omega_1,\omega_2,\omega_3,\omega_4) = K_1(\omega_1+\omega_2+\omega_3+\omega_4)\, H_4(\omega_1,\omega_2,\omega_3,\omega_4)$$
$$+ 2K_2(\omega_1,\omega_2+\omega_3+\omega_4)\, H_1(\omega_1)\, H_3(\omega_2,\omega_3,\omega_4)$$
$$+ K_2(\omega_1+\omega_2,\omega_3+\omega_4)\, H_2(\omega_1,\omega_2)\, H_2(\omega_3,\omega_4)$$
$$+ 3K_3(\omega_1,\omega_2,\omega_3+\omega_4)\, H_1(\omega_1)\, H_1(\omega_2)\, H_2(\omega_3,\omega_4)$$
$$+ K_4(\omega_1+\omega_2+\omega_3+\omega_4)\, H_1(\omega_1)\, H_1(\omega_2)\, H_1(\omega_3)\, H_1(\omega_4)$$

Table 2. Formulas for the spectra of inverse kernels.

$$[H^{-1}]_1(\omega) = \frac{1}{H_1(\omega)}$$

$$[H^{-1}]_2(\omega_1,\omega_2) = \frac{H_2(\omega_1,\omega_2)}{H_1(\omega_1) H_1(\omega_2) H_1(\omega_1+\omega_2)}$$

$$[H^{-1}]_3(\omega_1,\omega_2,\omega_3) = \frac{1}{H_1(\omega_1+\omega_2+\omega_3)\prod_{i=1}^{3} H_i(\omega_i)}\left\{-H_3(\omega_1,\omega_2,\omega_3) + \frac{2H_2(\omega_1,\omega_2+\omega_3)\, H_2(\omega_2,\omega_3)}{H_1(\omega_2+\omega_3)}\right\}$$

$$[H^{-1}]_4(\omega_1,\omega_2,\omega_3,\omega_4) = \frac{1}{H_1(\omega_1+\omega_2,\omega_3+\omega_4)\prod_{i=1}^{4} H_i(\omega_i)}\left\{-H_4(\omega_1,\omega_2,\omega_3,\omega_4)\right.$$
$$+\frac{2H_2(\omega_1,\omega_2+\omega_3+\omega_4)\, H_3(\omega_2,\omega_3,\omega_4)}{H_1(\omega_2+\omega_3+\omega_4)}$$
$$-\frac{H_2(\omega_1+\omega_2,\omega_3+\omega_4)\, H_2(\omega_1,\omega_2)\, H_2(\omega_3,\omega_4)}{H_1(\omega_1+\omega_2)\, H_1(\omega_3+\omega_4)}$$
$$+\frac{3H_3(\omega_1,\omega_2,\omega_3+\omega_4)\, H_2(\omega_3,\omega_4)}{H_1(\omega_3+\omega_4)}$$
$$\left.-\frac{4H_2(\omega_1,\omega_2+\omega_3+\omega_4)\, H_2(\omega_2,\omega_3+\omega_4)\, H_2(\omega_3,\omega_4)}{H_1(\omega_2+\omega_3+\omega_4)\, H_1(\omega_3+\omega_4)}\right\}$$

Table 3. Formulas for the spectra of feedback kernels.

$$G_1(\omega) = \frac{H_1(\omega)}{1 + H_1(\omega)}$$

$G_n(\omega_1, \ldots \omega_n)$, $n \geq 2$ is derived from $[H^{-1}]_n(\omega_1, \ldots \omega_n)$ in Table 2 by replacing H_1 with $1 + H_1$ and changing signs.

these three types of equations (Tables 1, 2, and 3), from which the unknown kernels can be found by an inverse Fourier transformation.

These, then, are the rudiments of the transform method for the solution of nonlinear operator equations.

7.2 AN ITERATION THAT CONVERGES IN DEGREE

It has been shown that feedback equations such as Eq. 39 can be solved for $\underline{G}$ by iteration. If the power-series representation of $\underline{G}$ is of interest, then it is desirable to be able to find $\underline{G}$ without any error in the first few terms of its expansion after only a finite number of cycles. For example, we might wish to find a model for the nonlinear distortion of a feedback system (as in Fig. 19) that has no error in the second-degree term. It is possible to accomplish this by perturbing Eq. 39 about its linear part. That is, $\underline{H}$ is split into a sum of two parts,

$$\underline{H} = \underline{H}^1 + \Delta\underline{H} \tag{185}$$

in which $\underline{H}^1$ is the linear term in the power-series expansion, and $\Delta\underline{H}$ is the sum of all the other terms.

$$\Delta\underline{H} = \sum_{n=2}^{\infty} \underline{H}^n$$

in which $\underline{H}^n$ denotes the term of degree n in the series. The feedback equation, Eq. 39, is now perturbed according to Eq. 185, and gives (see Eq. 66)

$$\underline{G} = \underline{G}^n + (\underline{E}^1 * \Delta\underline{H} * (\underline{I} - \underline{G})) \tag{186}$$

for the transformed feedback equation. Here, $\underline{G}^1$ and $\underline{E}^1$ are both linear and are given by

$$\underline{G}^1 = (\underline{I} + \underline{H}^1)^{-1} * \underline{H}^1$$

$$\underline{E}^1 = (\underline{I} + \underline{H}^1)^{-1}$$

The transformed equation has $\underline{E}^1 * \Delta\underline{H}$ for its loop operator. Since $\underline{E}^1$ is linear and $\Delta\underline{H}$ has no linear term, their cascade has no linear term either. (This is a fairly obvious property of polynomials; for example, if $z = y$ is cascaded with $y = ax^2 + bx^3 + \ldots$, the result is $z = ax^2 + bx^3 + \ldots$ which has no linear term.) Now, if the loop operator of a feedback equation has no linear term, then the successive differences, $\underline{\Delta}_n$, between iterations, $\underline{G}_n$, which satisfy

$$\begin{aligned} \underline{\Delta}_n &= \underline{G}_n - \underline{G}_{n-1} \\ &= \underline{E}^1 * \Delta\underline{H} * (\underline{I}-\underline{G}_{n-1}) - \underline{E}^1 * \Delta\underline{H} * (\underline{I}-\underline{G}_{n-2}) \end{aligned} \tag{187}$$

have the degree of their terms of lowest degree increased by one in each cycle; that is, if we indicate the lowest degree of any term in $\underline{\Delta}_n$ by $N(\underline{\Delta}_n)$, then

$$N(\underline{\Delta}_n) \geq 1 + N(\underline{\Delta}_{n-1}) \tag{188}$$

We shall not prove this property; it results because each of the two terms on the right-hand side of Eq. 187 has the same term of degree $N(\underline{\Delta}_{n-1})$, so that they cancel, leaving N for the remainder higher by 1.

As a result, the sequence of iterations has the form illustrated by the following no-memory example:

$$\begin{array}{lcccccccccc} \underline{G}_1(x) & = & ax & + & px^2 & + & gx^3 & + & rx^4 & + & \ldots \\ \underline{G}_2(x) & = & ax & + & bx^2 & + & sx^3 & + & tx^4 & + & \ldots \\ \underline{G}_3(x) & = & ax & + & bx^2 & + & cx^3 & + & ux^4 & + & \ldots \\ \underline{G}_4(x) & = & ax & + & bx^2 & + & cx^3 & + & dx^4 & + & \ldots \\ \vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \end{array}$$

Terms below the diagonal remain unchanged by the iteration, while those above it change in every cycle.

7.3 ITERATION AS A MEANS FOR STUDYING THE CONVERGENCE OF POWER-SERIES SOLUTIONS OF FEEDBACK EQUATIONS

The power-series method of solving feedback equations is valid only under fairly restricted conditions that are often not satisfied by models of physical systems. Hence, the precise formulation of these conditions is of great practical interest – not just an exercise in rigor.

Implicit in the power-series method are two assumptions: First, that the operator whose solution is being sought – say, the feedback operator $\underline{G}$ – exists; and second, that it has a unique power-series representation. The validity of these assumptions can often be established by iteration.

In order to verify the fact that $\underline{G}$ has a power-series representation, it must be established that the sequence of partial sums of $\underline{G}$ converges to $\underline{G}$. The n^{th} partial sum, $\underline{S}_n$, is given by

$$\underline{S}_n = \sum_{m=1}^{n} \underline{G}^m$$

in which $\underline{G}^m$ is the term of degree m in the expansion (and must not be confused with $\underline{G}_m$, the m^{th} iteration).

We should like to draw conclusions about the convergence of the partial sums $\underline{S}_n(x)$ from that of the iterations $\underline{G}_n(x)$. It was established in section 7.2 that the sum of the first n terms in the expansion for $\underline{G}_n(x)$ is identical to $\underline{S}_n(x)$, so that we can write

$$\underline{G}_n(x) = \underline{S}_n(x) + \sum_{m=n+1}^{\infty} \underline{G}^m(x) \tag{189}$$

Under these circumstances, the convergence of $\underline{G}_n(x)$ would imply that of $\underline{S}_n(x)$ if the terms $\underline{G}^m(x)$ were all positive, which, in general, they are not. (If negative terms occur on the right-hand side of Eq. 189, then $\underline{G}_n(x)$ might be the difference between two large terms, $\underline{S}_n(x)$ and the sum, both of which might diverge while $\underline{G}_n(x)$ converged. Instead of looking at $\underline{G}_n(x)$, therefore, we shall examine the sequence $\{\text{abs } \underline{G}_n(|x|)\}$, consisting of the same sums of convolutions as $\underline{G}_n(x)$, but with the kernels and the input x replaced by their absolute values. Thus, we have a sequence of positive terms, abs $\underline{G}_n(|x|)$, each of which is a power series containing a subseries that bounds $\underline{S}_n(x)$, so that the convergence of $\underline{G}_n(|x|)$ must imply that of $\underline{S}_n(x)$. But the convergence of abs $\underline{G}_n(|x|)$ can be studied with reference to the modified feedback equation

$$\text{abs } \underline{G} = \text{abs } \underline{H} * (\underline{I} + \text{abs } \underline{G}) \tag{190}$$

in which abs $\underline{H}$ is identical to $\underline{H}$, except that kernels have been replaced by their absolute values. Equation 190 is confined to positive inputs. (Note that negative feedback has been replaced by positive feedback.)

Thus we have a tool for studying the convergence of power series for $\underline{G}$; it is implied by the convergence of the iteration for the modified feedback equation, Eq. 190, for positive inputs x, and we can apply our knowledge concerning geometric and exponential iteration to it.

7.3.1 Geometric Convergence

If abs $\underline{H}$ is a contraction, then the geometric theory of Section II is applicable, the iteration for Eq. 190 converges, and so does the power series for $\underline{G}$. There is a difficulty here; the gain of a power series of positive terms such as abs $\underline{H}$ increases without limit for large inputs, so that it can never be a true contraction. However, it is enough if abs $\underline{H}$ is a contraction for small inputs only; it can be shown that the iteration for G converges for all inputs to G whose norm is less than some constant b, provided that the maximum incremental gain of H is not greater than α (which is less than 1) for all inputs to H whose norm is less than $b/(1-\alpha)$. It can be shown, furthermore, that this condition is always met with some small enough b. Thus we arrive at Brilliant's (11a) result that the power series for G is always valid for small enough inputs to G.

7.3.2 Exponential Convergence for Realizable Systems

Similarly, it can be shown that if H is realizable for small enough inputs (or if it satisfies any one integral Lipschitz condition of the type that realizability involves), then the power series for G converges for arbitrary inputs to G for some short enough time after starting.

Unfortunately, even this condition fails to achieve the ideal convergence for all inputs and all times.

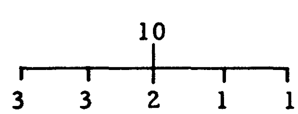
7.4 CONSTRUCTION OF TABLES OF CASCADING, INVERSION, AND FEEDBACK FORMULAS

The cascading, inversion, and feedback formulas listed in Tables 1, 2, and 3 can be derived by straightforward computation, using the method outlined in section 7.1. However, this method is tedious in the extreme, and it is therefore desirable to find general relations for the structure of these formulas, relations that will permit us to write directly formulas of arbitrary degree. A procedure that has been found useful for this purpose relies on the one-to-one correspondence that exists between various terms appearing in the formulas and certain partitions and trees.

7.4.1 Partitions and Trees

The structures of the cascading and inversion polynomials can be related to the structures of trees and partitions. This provides a relatively easy means for writing them, and offers some insight into the behavior that they describe.

A partition of degree n is a division of n identical objects into s cells. The representation (10:3, 3, 2, 1, 1), or



denotes a partition of 10 objects into 5 cells containing 3, 3, 2, 1, and 1 objects, respectively. This partition has two "repetitions" of 3, and two of 1. We associate a coefficient, a, with each partition, which is the number of its rearrangements and is given by

$$a = \frac{s!}{r_1!\ r_2!\ \cdots}$$

where $r_1, r_2, \ldots$ are the numbers of repetitions. In our example,

$$a = \frac{5!}{2!\ 2!} = 30$$

A tree of degree n is a sequence of partitions, the first of which has degree n terminating in cells containing only ones. It is denoted by the graph shown in Fig. 37. The

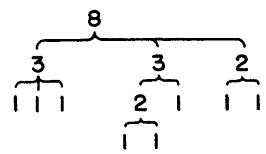


Fig. 37. A tree.

"multiplicity" m of a tree is the number of its partitions or nodes (m = 5 in Fig. 37). The number of rearrangements of a tree, b, is the product of the coefficients a for all of the partitions. It is

$$b = \frac{3!}{2!} \times \frac{3!}{3!} \times 2! \times \frac{2!}{2!} = 6 .$$

7.4.2 Structure of the Cascade Spectrum

The cascade spectrum of degree n, $[k{*}h]_n$, is a sum of terms of positive sign. There is one term corresponding to every possible partition without regard to arrangement of degree n, multiplied by the associated coefficient a. The correspondence is most easily illustrated by an example.

The spectrum $[k{*}h]_{10}$ has a term corresponding to the partition (10:3, 3, 2, 1, 1) and that term is

$$30K_5(\omega_1 + \omega_2 + \omega_3, \omega_4 + \omega_5 + \omega_6, \omega_7 + \omega_8, \omega_9, \omega_{10})\ H_3(\omega_1, \omega_2, \omega_3)\ H_3(\omega_4, \omega_5, \omega_6)$$
$$H_2(\omega_7, \omega_8)\ H_1(\omega_9)\ H_1(\omega_{10}) .$$

There is a single K-factor of order s (5, in this example) whose independent variables are sums of frequencies occurring in cells of 3, 3, 2, 1, and 1. There are s H-factors, each with the degree and variables of one of these cells. The arrangement of the cells and the assignment of the subscripts are immaterial, as long as the indicated pairing is maintained.

7.4.3 Structure of the Inverse Spectrum

There is a similar correspondence between inverse spectra and trees of the same degree. An inverse spectrum of order n is a sum of terms, one for each partition of order n, with a coefficient $(-1)^m b$. Thus, $[H^{-1}]_8$ has a term corresponding to the tree in Fig. 37 and that term is

$$\frac{(-1)^5 \times 6 \times H_3(\omega_1+\omega_2+\omega_3, \omega_4+\omega_5+\omega_6, \omega_7+\omega_8) H_3(\omega_1, \omega_2, \omega_3) H_2(\omega_4+\omega_5, \omega_6) H_2(\omega_7, \omega_8) H_2(\omega_4, \omega_5)}{\prod_{i=1}^{8} H_i(\omega_i) H_1(\omega_1+\omega_2+\omega_3+\omega_4+\omega_5+\omega_6+\omega_7+\omega_8) H_1(\omega_1+\omega_2+\omega_3) H_1(\omega_4+\omega_5+\omega_6) H_1(\omega_7+\omega_8) H_1(\omega_4+\omega_5)}$$

The numerator is constructed by inspection of the tree, one partition at a time. The denominator has an H_1 factor for every H factor in the numerator, with a sum of the variables that occur. The subscripts correspond to downward paths in the tree.

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