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A FREQUENCY-DOMAIN THEORY OF PARAMETRIC AMPLIFICATION

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MASSACHUSETTS INSTITUTE OF TECHNOLOGY
RESEARCH LABORATORY OF ELECTRONICS
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Technical Report 354

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A FREQUENCY-DOMAIN THEORY OF PARAMETRIC AMPLIFICATION

Benjamin J. Leon

This report is based on a thesis submitted to the Department of Electrical Engineering, M.I.T., April 15, 1959, in partial fulfillment of the requirements for the degree of Doctor of Science.

Abstract

In the class of high-frequency amplifiers that are known as parametric amplifiers, varactors, variable-reactance amplifiers, and so on, the active elements are variable-reactance parameters; i. e., inductances or capacitances whose values vary periodically. The power that amplifies the desired signal comes from the source which varies the parameter.

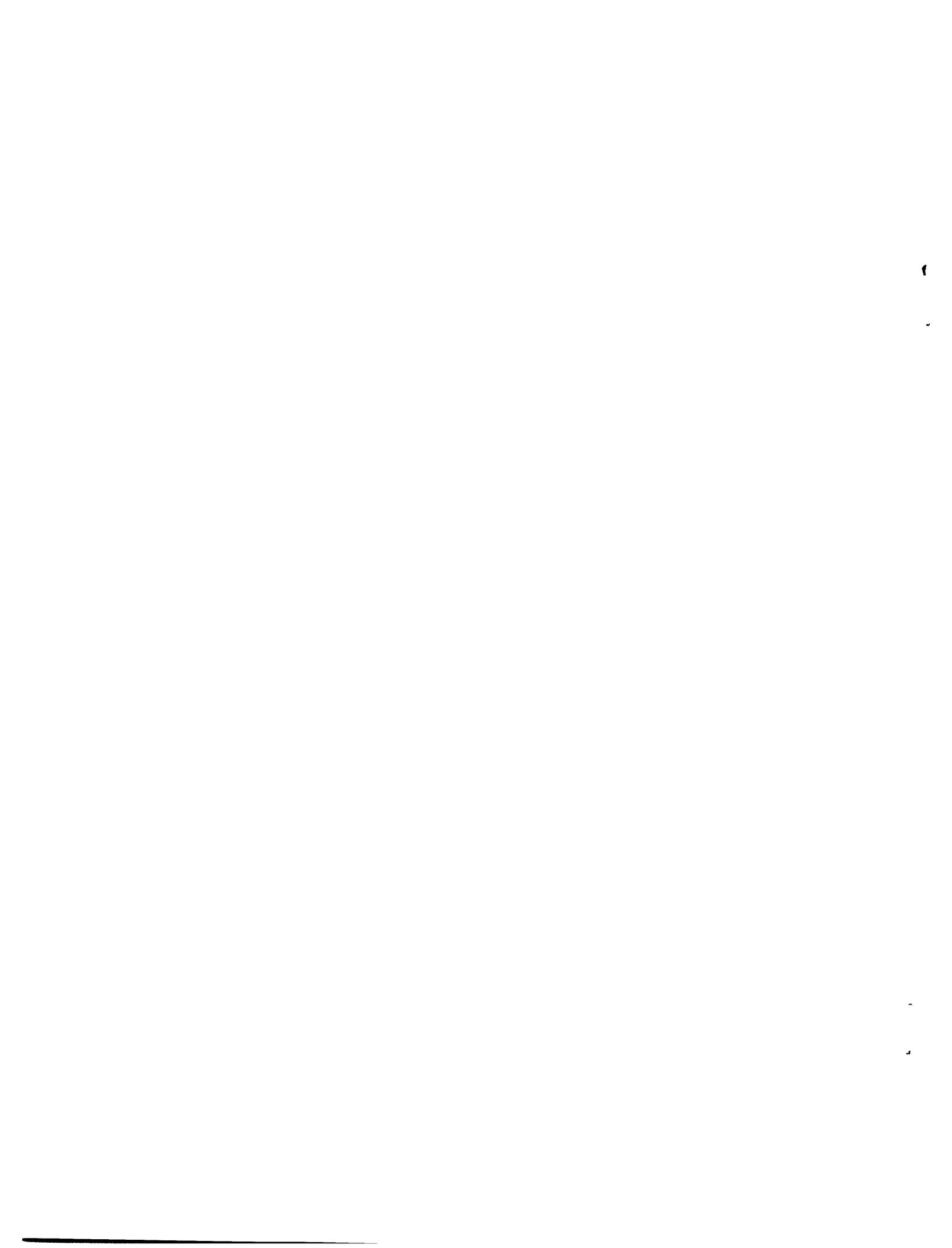
We present a general method of analysis of circuits containing a few decoupled periodic elements in a network of lumped, linear, finite, passive, bilateral, time-invariant elements. For these circuits we show that the characteristic frequency-domain equations that define the voltages and currents as functions of the complex frequency are linear difference equations with variable coefficients.

A sinusoidally varying capacitance in an arbitrary passive network with steady-state signal excitation is discussed. By using the calculus of finite differences to find exact solutions to the difference equations we are able to prove three fundamental statements concerning parametric amplifier performance: (a) The gain is independent of the phase of the signal relative to the varying parameter except in a degenerate case that can easily be avoided. (b) The passive circuit admittance at idler frequency, the difference between the signal frequency and the frequency of the parameter, is important. A circuit with a zero of admittance at the idler frequency will oscillate regardless of the admittance at the signal frequency. (c) Because of the interaction of the signal and the varying element, high-frequency voltages are produced; but the voltage amplitudes go to zero exponentially as frequency increases, if the varying parameter is positive for all time.

We discuss the mechanics of finding a solution. In mathematical literature various series methods have been presented, but there is no a priori assurance that any particular type of series will work. We show that a solution in the form of a factorial series can always be found by a routine procedure. Even for fairly complicated amplifiers the procedure is manageable as a desk calculator problem.

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I. INTRODUCTION

1.1 PARAMETRIC AMPLIFYING DEVICES

In the past few years there has been considerable interest in a new type of high-frequency amplifier that has been known by such names as parametric amplifier, varactor, variable reactance amplifier. The active elements in this class of amplifiers are variable reactance parameters; that is, inductances or capacitances whose values are made to vary periodically. The power that amplifies the desired signal comes from the source that varies the parameter.

In 1957 Suhl (1) suggested using the anomalous dispersion effect in ferrites to make a variable-inductance parametric amplifier at microwave frequencies. After this, others suggested using back-biased junction diodes as variable capacitances (2) and modulated electron beams as variable energy-storage elements (3). We shall not try to give a bibliography of the proposals or the devices that were actually constructed; the list is quite long and growing rapidly.

In amplifiers with diode capacitors or ferrite inductors the usual circuit arrangement is that shown in Fig. 1. The nonlinear element is connected to a passive network.

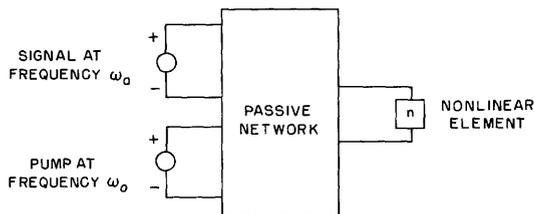


Fig. 1. Parametric amplifier with nonlinear element.

The circuit is then excited by two sources at different frequencies. One source, known as the pump, is of large amplitude and thus drives the nonlinear element over a wide range. The other source is the signal that is to be amplified. If the signal is small compared with the pump, the amplification is essentially linear, and a linear, small-signal analysis can be used. In modulated beam amplifiers the beam with its associated apparatus appears as a linear, time-

variant reactance to the signal, so that linear analysis applies directly. In this report we shall deal exclusively with linear analysis.

1.2 DEVELOPMENT OF A LINEAR-CIRCUIT MODEL

Analysis of parametric amplifiers cannot be accomplished by conventional lumped, linear, finite, passive, bilateral, time-invariant circuit theory because these amplifiers contain elements that are nonlinear and time-variant. The circuits that we wish to consider contain lumped, two-terminal elements. Associated with each element there are two time functions, the current, i , and the voltage, v . We shall also use the charge, q , and the flux linkage, λ , which are the indefinite integrals of the current and voltage, respectively. Each circuit element is characterized by a specific relation

between the current and the voltage. The elements that we shall use and the relations that characterize them are:

capacitance	$q = f_c(v, t)$
elastance	$v = f_s(q, t)$
resistance	$v = f_r(i, t)$
conductance	$i = f_g(v, t)$
inductance	$v = f_l(\lambda, t)$
reciprocal inductance	$\lambda = f_v(v, t)$
current sources	$i = i(t)$, independent of v
voltage sources	$v = v(t)$, independent of i

The characteristic relations f_c , f_s , f_r , f_g , f_l , and f_v are all functions of two variables; their partial derivatives with respect to the first variable are always positive, and consequently their inverses with respect to the first variable exist for all values of the time (4). For each element, the partial of the characteristic function with respect to the first variable will be designated by the same name as the element; for example, $\partial f_c / \partial v$ is called the capacitance. Therefore the restriction to positive partial derivatives restricts our circuits to positive elements.

In lumped, linear, finite, passive, bilateral, time-invariant circuit theory the element values – that is, the partial derivatives with respect to the first variable – are constant. Our approach to the problem of analyzing a parametric amplifier will be to extend constant-parameter circuit theory to include one, or at most, a few nonconstant elements. Throughout the development we shall use a capacitance when discussing a single nonconstant element. This is merely for definiteness, and the same analysis applies to the other elements.

Consider a variable capacitance connected to a network of constant parameters and sources as shown in Fig. 2a. If we apply Norton's theorem to the part of the network containing the constant parameters and sources, we have the circuit of Fig. 2b. Now suppose that there are two distinct sources in the network. Since the constant parameter network is linear, the driving current $i(t)$ in the Norton equivalent will be the sum of two terms: $i_1(t)$ from the first source, and $i_2(t)$ from the second. We now wish to find a method of separating the effect of the two sources on the output voltage; that is, we want to find conditions under which we can use superposition, at least in a limited sense.

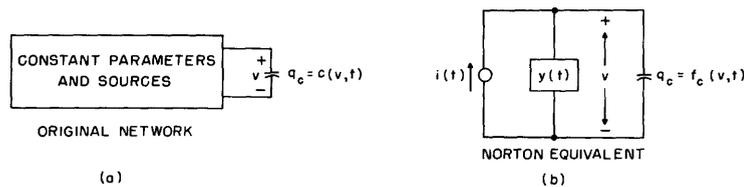


Fig. 2. Electrical network with one variable parameter.

If, in the original network, the first source is set to zero, the current drive in the Norton equivalent is $i_2(t)$. Let us call the resulting voltage $v_2(t)$. Writing Kirchhoff's current law at the one node gives

$$i_2(t) = \int_{-\infty}^{\infty} y(t-\tau) v_2(\tau) d\tau + \frac{d}{dt} f_c(v_2, t) \quad (1)$$

With both sources present, drive $i(t) = i_1(t) + i_2(t)$, and response $v(t)$, let us define

$$v_1(t) = v(t) - v_2(t) \quad (2)$$

Note that v_1 is not, in general, equal to the voltage that would appear if the first source were operated normally and the second source set to zero. If it were, superposition would apply and the network would be linear. Now with both sources Kirchhoff's law gives

$$i_1(t) + i_2(t) = \int_{-\infty}^{\infty} y(t-\tau)[v_1(\tau)+v_2(\tau)] d\tau + \frac{d}{dt} f_c(v, t) \quad (3)$$

But by the differential approximation theorem (ref. 5)

$$f_c(v, t) = f_c(v_1+v_2, t) = f_c(v_2, t) + \frac{\partial f_c}{\partial v}(v_2, t) v_1(t) + R \quad (4)$$

where

$$\lim_{v_1 \rightarrow 0} \frac{|R|}{|v_1|} = 0$$

Therefore, substituting Eq. 4 in Eq. 3, and subtracting Eq. 1 from the result gives

$$i_1(t) = \int_{-\infty}^{\infty} y(t-\tau) v_1(\tau) d\tau + \frac{d}{dt} \left[\frac{\partial f_c}{\partial v}(v_2, t) v_1(t) + R \right] \quad (5)$$

Now let us see if there are situations of interest in which R can be neglected. In Eq. 4 we have

$$\lim_{v_1 \rightarrow 0} \frac{|R|}{|v_1|} = 0$$

This means that when v_1 is small, R is very much smaller. On the other hand, the smallness of R does not guarantee the smallness of dR/dt . However, if the charge on the capacitance is composed of a sum of sinusoids, as it would be if the sources i_1 and i_2 are periodic, we expect both $\frac{\partial f_c}{\partial v}(v_2, t) v_1(t)$ and R to be sums of sinusoids. Then the ratio of the derivatives of the two terms is of the same order of magnitude as the ratio of the two original terms. Under such circumstances Eq. 5 becomes

$$i_1(t) = \int_{-\infty}^{\infty} y(t-\tau) v_1(\tau) d\tau + \frac{d}{dt} C(t) v_1(t) \quad (6)$$

where

$$C(t) = \frac{\partial f_c}{\partial v} (v_2, t)$$

But Eq. 6 is the equation of the circuit shown in Fig. 3. This circuit is linear, for there are no elements whose value depends on the electric excitation.

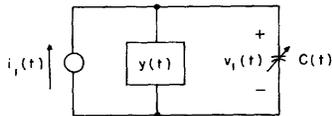


Fig. 3. Circuit with time-variant capacitance.

The mathematical restrictions that allow us to go from the circuit of Fig. 2 to Fig. 3 are exactly the physical conditions for small-signal operation of the parametric amplifier of Fig. 1. Therefore, Fig. 3 is a reasonable linearized model for analyzing a parametric amplifier for small signals. In the modulated beam amplifier, where the beam looks like a time-variant reactance to the signal, we arrive at the circuit of Fig. 3 directly.

1.3 DEFINITION OF A LINEAR PARAMETRIC AMPLIFIER

Now that we have some definite circuit models that are pertinent to parametric amplifiers we are in a position to define a class of parametric amplifiers precisely. Our definition is based on a circuit model, and we shall say that a particular device is covered by the definition if the model gives a reasonable approximation to the performance of the device.

DEFINITION. A single-stage, linear parametric amplifier consists of a single periodic time-variant reactive element – that is, a reactive element whose value is a periodic function of time, independent of the electric excitation – imbedded in a time-invariant, linear, finite, passive, bilateral network.

Since this report is devoted mainly to the analysis of the single-stage, linear parametric amplifier, the words "single-stage, linear" will be deleted. When we wish to talk about a parametric amplifier that does not quite fit the definition, we shall make special note of it. The circuit of Fig. 3 is the circuit of a parametric amplifier when $y(t)$ is the impulse response function of a time-invariant, linear, finite, passive, bilateral admittance, and $C(t)$ is a periodic function. If the time-variant reactive element is an inductance, the circuit of Fig. 4, which is the dual of Fig. 3, is appropriate. In this circuit, $z(t)$ is the impulse-response function of a time-invariant linear, finite, passive, bilateral impedance, and $L(t)$ is a periodic function. We shall call a parametric amplifier "capacitively excited" when the time-variant reactance is a capacitance and "inductively excited" when it is an inductance. Henceforth, we shall discuss only the capacitively excited case; analysis of the other case is merely the dual.

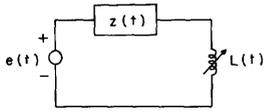


Fig. 4. Inductively excited parametric amplifier.

A parametric amplifier will be called realizable if the value of the variable element, including the parasitic capacitance (or inductance) of the passive circuit, is positive for all time. The gain of a parametric amplifier is defined as the ratio of the average power delivered to the passive circuit at the desired output frequencies to

the average power delivered by the electrical source.

To be considered an amplifier a device should have a power gain greater than one. When a parametric amplifier has a gain greater than one, power is delivered to the passive circuit by the variable reactive element. We can now see why, in our definition, we did not allow the variable element to be a resistance. To be realizable, the resistance must be positive for all time. But a positive resistance always absorbs energy; therefore, a network like that of Fig. 3, with the variable capacitance replaced by a variable resistance, cannot be an amplifier.

1.4 OBJECTIVES AND RESULTS

The primary objective of this report is the development of a general method for analyzing single-stage, linear, parametric amplifiers with steady-state signal inputs. As a secondary objective we would like our analysis technique to be suitable for circuits with more than one time-variant element and transient, as well as steady-state, inputs. Finally, we would like the analysis to yield currents and voltages that are functions of the complex frequency. Then when a parametric amplifier is used in conjunction with a linear time-invariant system the results of the amplifier analysis can be used directly in the analysis of the rest of the system without the need for laborious transforms.

In Section II we find that the performance of parametric amplifiers is characterized by linear, variable-coefficient, difference equations in the frequency domain. The order of the equation depends on the number of terms required to approximate the parameter variation with a finite Fourier series. Amplifiers with several variable parameters are characterized by simultaneous difference equations. For cascade ladder networks, such as the distributed parametric amplifier, the set of equations can be reduced to a single equation by systematic elimination. Therefore, if we can solve the single-difference equation, we shall accomplish all of our objectives, so far as amplifiers of current practical interest are concerned.

In Sections III, IV, and V a method for solving the difference equations is developed in considerable detail. In order to keep the notation within bounds the discussion is carried out for a sinusoidal capacitance in an arbitrary realizable time-invariant network. There is nothing inherent in the mathematics that requires this restriction; the extension to variable elements with more complicated variation is straightforward. In keeping with our stated objectives, the emphasis in these three sections will be on a

steady-state response. However, in developing the steady-state solution we find two different methods that lead to the transient response. The order of the discussion in Sections III-V is chosen from an engineering point of view; each step is motivated by an effort to keep our feet somewhere in the neighborhood of the physical ground.

In Section III we set out to find the voltage that appears across the amplifier when a sinusoidal signal is applied. Using the method of variation of parameters, we find a formal solution. However, this solution involves the complementary solution to the amplifier equation in the absence of a signal. By physical reasoning we are able to justify our formal solution and deduce some of the properties of the complementary solution even before we solve the equation for the undriven amplifier.

In Section IV we show mathematically that the solutions to both the driven and undriven amplifier can have the properties that are deduced physically in Section III. Then in Section V we discuss the specific procedure by which a series solution can be found. The procedure is discussed in detail, and it is shown that a convergent series can be found in all cases.

In Sections I-V we accomplish all of our objectives. However, a report on parametric amplifiers would not be complete if it did not show that these devices can, indeed, amplify. The case considered in Section VI is concerned with the parametric amplifying devices used in practice. That is, in addition to providing for the signal frequency, the network contains a resonant circuit at the idler frequency. The idler frequency is the difference between the pump and the signal frequencies. For such a device we can prove that if the idler frequency is lower than that of the pump, and the idler circuit has infinite Q , then the device has infinite gain. This applies regardless of the other characteristics of the time-invariant network; we need no ideal filters such as are required in most of the analytical literature on parametric amplifiers. Furthermore, since the gain expression is a well-behaved function because damping is added in the idler circuit, we can be sure that the amplifier still has gain when the Q is finite.

The method of analysis developed in this report differs from other methods found in the current literature on linear parametric amplifiers in that it is exact for realizable networks. In the most widely used method of analysis, Bolle's method, we must assume that the network contains ideal filters. The connection between our method and Bolle's method is discussed in Appendix B. Other methods available (6, 7) seem too cumbersome to be useful in general parametric-amplifier analysis. By using our exact method we show that

- (i) Including voltages at all frequencies is not a severe handicap, for the power carried by these voltages is finite.
- (ii) The so-called linear amplifier is indeed linear; that is, the gain is independent of the amplitude and phase of the input signal.
- (iii) The so-called parametric amplifier can indeed amplify.

II. FREQUENCY-DOMAIN EQUATIONS FOR A PARAMETRIC AMPLIFIER

In Section I we developed a circuit model appropriate to a class of variable-parameter electrical networks, and we called this model a parametric amplifier. Next we must develop a mathematical procedure for analyzing the network; that is, a procedure for finding all voltages and currents when only a few are specified by sources. In this section we shall derive the necessary equations for analyzing our parametric amplifier. The methods of solution will be discussed later.

The rules for obtaining a set of equations from a circuit model are given by Kirchhoff's laws. For all types of lumped circuits these laws lead to a set of ordinary integro-differential equations. In the case of constant-parameter networks, we find that these equations can be most easily solved by transforming to the frequency domain. The result is a set of linear, algebraic equations that can be readily solved. In fact, we normally write our equations for the circuit directly in the frequency domain, and then we solve for voltages and currents that are functions of the complex frequency, ω .

In parametric amplifiers the parameters are not all constant, and so the usual method of frequency-domain analysis by algebraic equations does not apply. However, since the nonconstant parameters are periodic functions of time, we can develop a method of frequency-domain analysis by using difference equations. The first step in writing the frequency-domain equations for circuits containing periodic parameters is to derive the frequency-domain voltage-current relations for these parameters.

2.1 VOLTAGE-CURRENT RELATIONS FOR PERIODIC CIRCUIT PARAMETERS

In order to analyze parametric amplifiers we must consider four types of variable circuit elements: capacitance, elastance, inductance, and reciprocal inductance. For completeness, we shall also include a discussion of the other two elements, resistance and conductance. The parameters that are to be discussed are periodic functions of time. Let us make another restriction that the functions are such that the Fourier-series representation converges uniformly for all time; consequently each parameter can be approximated as closely as we wish by a finite sum of exponentials.

Consider the capacitance

$$C(t) = \sum_{-k}^k C_n e^{jn\omega_0 t}$$

Since $C(t)$ is a real-time function,

$$C_{-n} = C_n^*$$

where the star denotes a complex conjugate. The time-domain voltage-charge relationship for this element is

$$q(t) = C(t) v(t) = \sum_{-k}^k C_n e^{jn\omega_0 t} v(t)$$

Multiplying both sides by $e^{-j\omega t}$, and integrating from $-\infty$ to ∞ with respect to t gives

$$Q(\omega) = \sum_{-k}^k C_n V(\omega - n\omega_0) \quad (7)$$

where $Q(\omega)$ and $V(\omega)$ are the Fourier transforms of $q(t)$ and $v(t)$, respectively. In the frequency domain, $I(\omega) = j\omega Q(\omega)$, so that

$$I(\omega) = j\omega \sum_{-k}^k C_n V(\omega - n\omega_0) \quad (8)$$

Next, consider the reciprocal inductance

$$\Gamma(t) = \sum_{-k}^k \Gamma_n C e^{jn\omega_0 t}$$

The current is then $i(t) = \Gamma(t) \lambda(t)$, and the Fourier transform is

$$I(\omega) = \sum_{-k}^k \Gamma_n \Lambda(\omega - n\omega_0)$$

Since the Fourier transform of the voltage in an inductive circuit is $V(\omega) = j\omega \Lambda(\omega)$, the voltage-current relation for the reciprocal inductance is

$$I(\omega) = \sum_{-k}^k \frac{\Gamma_n V(\omega - n\omega_0)}{j(\omega - n\omega_0)} \quad (9)$$

Similarly, for the conductance,

$$G(t) = \sum_{-k}^k G_n e^{jn\omega_0 t}$$

we have

$$I(\omega) = \sum_{-k}^k G_n V(\omega - n\omega_0) \quad (10)$$

We could also expand the reciprocals of these parameters for analysis on a loop basis. The appropriate expansions are: For elastance,

$$S(t) = \sum_{-k}^k S_n e^{jn\omega_0 t}$$

for inductance,

$$L(t) = \sum_{-k}^k L_n e^{jn\omega_0 t}$$

and for resistance,

$$R(t) = \sum_{-k}^k R_n e^{jn\omega_0 t}$$

The corresponding voltage-current relations are: For elastance,

$$V(\omega) = \sum_{-k}^k \frac{S_n I(\omega - n\omega_0)}{j(\omega - n\omega_0)} \quad (11)$$

for inductance,

$$V(\omega) = j\omega \sum_{-k}^k L_n I(\omega - n\omega_0) \quad (12)$$

and for resistance,

$$V(\omega) = \sum_{-k}^k R_n I(\omega - n\omega_0) \quad (13)$$

Choosing an appropriate method of analysis (loop or node) in a variable-parameter circuit is more involved than choosing the simpler method for a fixed-parameter circuit. In the time-invariant circuit we need only count the loops and the node pairs to see which gives the smaller number of equations. In the periodic-parameter circuit, we must also look at the number of terms in the parameter expansion. For example, if $C(t) = C_0 + e^{jt} + e^{-jt}$, with $C_0 > 2$, then

$$S(t) = \frac{1}{C_0} - \frac{e^{jt} + e^{-jt}}{C_0^2} + \frac{(e^{jt} + e^{-jt})^2}{C_0^3} + \dots$$

It might take quite a number of terms of this series to get a good approximation to the desired elastance.

2.2 EQUATIONS FOR NETWORKS CONTAINING ONE OR MORE PERIODIC PARAMETERS

Now let us consider the circuit of Fig. 5. Writing the Kirchhoff current law equation in the frequency domain gives

$$I(\omega) = Y(\omega) V(\omega) + j\omega \sum_{-k}^k C_n V(\omega - k\omega_0) \quad (14)$$

This equation is a linear, variable-coefficient difference equation of the $2k^{\text{th}}$ order. In section 2.3 we shall discuss the terminology and some of the useful properties of difference equations; then in later sections we shall investigate some methods of solving the equations. For the present, let us assume that we can solve Eq. 14 for the unknown voltage $V(\omega)$. Once we have found $V(\omega)$, we can get $I_c(\omega)$, the current flowing into the

capacitance, from Eq. 8.

We cannot use the circuit of Fig. 5 for finding the currents through and the voltages across the constant parameters in the parametric amplifier because when we make a Norton or Thévenin equivalent we lose the identity of these elements. To find these voltages and currents we use the circuit of Fig. 6, in which the capacitance is replaced by a known voltage, $V(\omega)$, and a known current, $I_c(\omega)$, in a network of constant parameters and sources. Now our known voltage or current plays the same role as any other voltage or current source in the network. The circuit can be analyzed in the usual way for constant-parameter circuits. The frequency-domain analysis can be readily used, for the equivalent source of the variable element is already specified as a frequency function.

The technique for analyzing a network with one periodic parameter that has been discussed can be extended to networks with several periodic parameters when the frequencies of all parameter variations have a common divisor. One case that can be handled occurs when the variable element is not a pure capacitance, inductance, or resistance. For example, a parallel resonant circuit whose Q , center frequency, and impedance level are all varying periodically can be represented by a parallel G, L, C

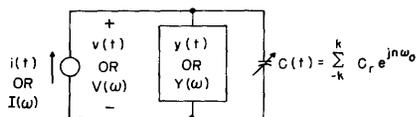


Fig. 5. Norton equivalent of a capacitively excited parametric amplifier with driving source.

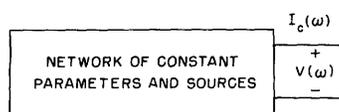


Fig. 6. Parametric-amplifier problem after solution of the Norton equivalent.

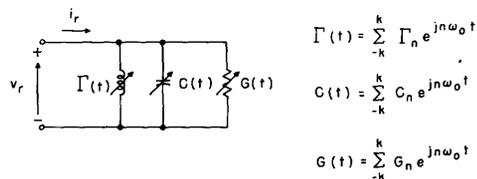


Fig. 7. Variable resonant circuit.

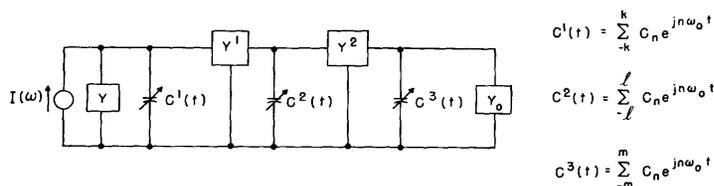


Fig. 8. Parametric ladder network.

circuit, as shown in Fig. 7. For this circuit the frequency-domain current-voltage relation is

$$I_R(\omega) = \sum_{-k}^k \left[j\omega C_n + \frac{\Gamma_n}{j(\omega - n\omega_0)} + G_n \right] V_R(\omega - n\omega_0) \quad (15)$$

When such a circuit is imbedded in a constant-parameter network, the resulting Kirchhoff current law equation is still a linear difference equation with variable coefficients.

For circuits with several variable elements that are connected across different node pairs, the resulting equations are sets of simultaneous difference equations. In general, there is no obvious way to reduce such a set of coupled equations to a single difference equation in one unknown. However, in a ladder network with the variable elements separated by constant-parameter sections, the reduction is straightforward.

Consider the network of Fig. 8. Let the constant-parameter networks Y^i have driving-point and transfer admittances Y_{11}^i , Y_{22}^i , and Y_{12}^i , and let V_i be the voltage across C^i . Then the difference equations that characterize the network are:

$$\begin{aligned} I(\omega) &= \left[Y + Y_{11}^1 \right] V_1(\omega) + j\omega \sum_{-k}^k C_n V_1(\omega - n\omega_0) - Y_{12}^1 V_2(\omega) \\ 0 &= -Y_{12}^1 V_1(\omega) + \left[Y_{22}^1 + Y_{11}^2 \right] V_2(\omega) + j\omega \sum_{-\ell}^{\ell} C_n V_2(\omega - n\omega_0) - Y_{12}^2 V_3(\omega) \\ 0 &= -Y_{12}^2 V_2(\omega) + \left[Y_{22}^2 + Y_0 \right] V_3(\omega) + j\omega \sum_{-m}^m C_n V_3(\omega - n\omega_0) \end{aligned}$$

To reduce these three simultaneous equations to a single difference equation in V_1 , we solve the first equation for V_2 in terms of V_1 , and substitute the result in the second equation. Then the resulting equation is solved for V_3 in terms of V_1 . Finally, both V_2 and V_3 in the third equation can be replaced by a linear function of V_1 , and we obtain the desired single equation in one unknown. Recently, traveling-wave parametric amplifiers have been proposed (8). These amplifiers are characterized by parametric ladder networks as shown in Fig. 8.

2.3 TERMINOLOGY AND PROPERTIES OF DIFFERENCE EQUATIONS

In the frequency domain the equations describing the behavior of parametric amplifiers are difference equations. Therefore, it is appropriate that we discuss the terminology and some of the fundamental properties of these equations. One of the best discussions of the subject has been given by L. M. Milne-Thomson (9).

Consider the equation

$$\begin{aligned}
& A_n V(\omega+n\omega_0) + A_{n-1} V(\omega+(n-1)\omega_0) + \dots + A_1 V(\omega+\omega_0) + A_0 V(\omega) + A_{-1} V(\omega-\omega_0) + \dots \\
& + A_{-(m-1)} V(\omega-(m-1)\omega_0) + A_{-m} V(\omega-m\omega_0) = g(\omega)
\end{aligned} \tag{16}$$

where the A_k 's are known complex-valued functions of ω and V ; V is an unknown function; ω is complex and may take on any value in the complex plane; m and n are positive integers; ω_0 is a constant, which in general may be complex, called the difference; and g is a known function of ω . Equation 16 defines a complex-valued function V for all values of the argument. The function so defined will not, in general, be unique. We shall be interested in those functions that are analytic, except for a countable number of singularities. Such a function will be called a solution to the equation.

Equation 16 will be called a linear equation if the A 's are independent of the unknown function V . A linear equation is also a constant-coefficient equation if the A_k are also independent of ω ; otherwise it is a variable-coefficient equation. The order of the equation is $(m+n)$. The equation is homogeneous if g is identically zero; if g is nonzero, the equation is called a complete equation. The solution to a homogeneous equation is called a complementary solution, and a solution to a complete equation is called a particular solution. The equation is said to be in standard form if the difference is one. Equation 16 can be put in standard form by a change of variable; $\omega = \omega_0 \underline{\omega}$. The result is a difference equation in standard form with argument $\underline{\omega}$. Two values of the argument, ω_1 and ω_2 , are said to be congruent if $\omega_1 = \omega_2 \pm \omega_0$.

In order to discuss some of the properties of the solutions to linear difference equations let us examine a second-order equation in standard form:

$$A_2(\omega) V(\omega+2) + A_1(\omega) V(\omega+1) + A_0(\omega) V(\omega) = g(\omega) \tag{17}$$

First, let us discuss the solution to the homogeneous part of Eq. 17, that is, the equation with $g(\omega)$ identically zero. Suppose that V_1 is a solution to the homogeneous equation. Then if p is a constant, pV_1 is a solution. Furthermore, if p is any arbitrary periodic function of period one, pV_1 is a solution. To show this, let us first note the definition of a periodic function. If p is a periodic function of period a , then $p(\omega \pm a) = p(\omega)$. Now let us substitute pV_1 in our original homogeneous equation (Eq. 17) with $g(\omega)$ identically zero. We have

$$A_2(\omega) p(\omega+2) V_1(\omega+2) + A_1(\omega) p(\omega+1) V_1(\omega+1) + A_0(\omega) p(\omega) V_1(\omega) \stackrel{?}{=} 0 \tag{18}$$

But, by definition, $p(\omega+2) = p(\omega+1) = p(\omega)$. Hence our questioned equation (Eq. 18) becomes

$$p(\omega)[A_2(\omega)V_1(\omega+2)+A_1(\omega)V_1(\omega+1)+A_0(\omega)V_1(\omega)] \stackrel{?}{=} 0 \tag{19}$$

But since V_1 is assumed to be a solution to the homogeneous part of Eq. 17, the expression in the bracket in the questioned Eq. 19 is zero. Therefore, we have proved the assertion that pV_1 is a solution.

Now let us turn to the complete equation (Eq. 17). If V_2 is a particular solution to

the complete equation, and V_1 is a complementary solution to the homogeneous part of the equation, then obviously $V_1 + V_2$ is also a solution to the complete equation.

Finally, we shall demonstrate the principle of superposition for the linear difference equation. Suppose in Eq. 17 that

$$g(\omega) = p_1(\omega) g_1(\omega) + p_2(\omega) g_2(\omega) \quad (20)$$

where p_1 and p_2 are periodic functions of period one, and g_1 and g_2 are arbitrary functions. Let us also suppose that V_1 is a particular solution to the equation

$$A_2(\omega) V_1(\omega+2) + A_1(\omega) V_1(\omega+1) + A_0(\omega) V_1(\omega) = g_1(\omega) \quad (21)$$

and that V_2 is a particular solution to the equation

$$A_2(\omega) V_2(\omega+2) + A_1(\omega) V_2(\omega+1) + A_0(\omega) V_2(\omega) = g_2(\omega) \quad (22)$$

Then we assert that $p_1(\omega) V_1(\omega) + p_2(\omega) V_2(\omega)$ is a particular solution to the original complete equation, Eq. 17, with the restriction of Eq. 20. To prove this assertion let us substitute in the original equation

$$\begin{aligned} & A_2[p_1(\omega+2)V_1(\omega+2)+p_2(\omega+2)V_2(\omega+2)] + A_1[p_1(\omega+1)V_1(\omega+1)+p_2(\omega+1)V_2(\omega+1)] \\ & + A_0[p_1(\omega)V_1(\omega)+p_2(\omega)V_2(\omega)] \stackrel{?}{=} p_1(\omega) g_1(\omega) + p_2(\omega) g_2(\omega) \end{aligned}$$

Since $p_1(\omega+2) = p_1(\omega+1) = p_1(\omega)$, and $p_2(\omega+2) = p_2(\omega+1) = p_2(\omega)$, we may rewrite our questioned equation, as follows:

$$\begin{aligned} & p_1(\omega)[A_2 V_1(\omega+2)+A_1 V_1(\omega+1)+A_0 V_1(\omega)] + p_2(\omega)[A_2 V_2(\omega+2)+A_1 V_2(\omega+1)+A_0 V_2(\omega)] \\ & \stackrel{?}{=} p_1(\omega) g_1(\omega) + p_2(\omega) g_2(\omega) \end{aligned}$$

But, by Eqs. 21 and 22, the expressions in the first and second brackets are equivalent to $g_1(\omega)$ and $g_2(\omega)$, respectively. Therefore, we have an identity and we can erase the question marks.

As any student of differential equations will recognize, the terminology and the properties of difference equations are often similar to those of differential equations. One significant difference is that the arbitrary multiplicative constant in differential equations finds for its counterpart in difference equations an arbitrary multiplicative periodic function. Consequently, solutions to difference equations have a higher degree of arbitrariness than those of differential equations. Analogies between difference and differential equations are often helpful; however, we must be careful because the two are not completely analogous.

III. AMPLIFIER PERFORMANCE – THE PARTICULAR SOLUTION

In order to analyze our single-stage parametric amplifier (Fig. 9), we must first solve the difference equation.

$$I(\omega) = Y(\omega) V(\omega) + \sum_{-k}^k C_n V(\omega - n\omega_0) \quad (23)$$

In order to keep the notation within bounds, we shall henceforth consider the special case with $k = 1$; that is, with sinusoidal capacitance variation. The extension to higher-order equations is straightforward for any specific problem, but very cumbersome in a general discussion. The resulting equation is

$$I(\omega) = j\omega C_1^* V(\omega + \omega_0) + Y(\omega) V(\omega) + j\omega C_1 V(\omega - \omega_0)$$

The realizability condition requires that $Y(\omega)$ contain a parasitic shunt capacitance, C_0 , greater than $2C_1$.

To further simplify the notation, we shall make the following normalizations:

- (a) Choose the time origin so that C_1 is real.
- (b) Normalize the frequency scale so that $\omega_0 = 1$.
- (c) Normalize the admittance level so that $C_1 = 1$.

These three transformations do not restrict the generality of the analysis. Equation 23 now becomes

$$I(\omega) = j\omega V(\omega+1) + Y(\omega) V(\omega) + j\omega V(\omega-1) \quad (24)$$

with

$$\lim_{\omega \rightarrow \infty} \frac{Y(\omega)}{j\omega} = C_0 > 2$$

The normalized circuit is shown in Fig. 10.

In a parametric amplifier, as in most electrical networks, we are interested in the stability, the steady-state response, and the transient response. To determine stability, we excite the network with an impulse at some time τ , and then examine the response to see if it remains bounded. In the frequency domain, a unit impulse at time τ becomes $e^{-j\omega\tau}$. Therefore, the excitation function $I(\omega)$ in Eq. 24 is $e^{-j\omega\tau}$ when stability is being investigated.

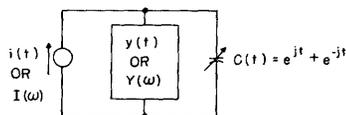


Fig. 9. Single-stage parametric amplifier.

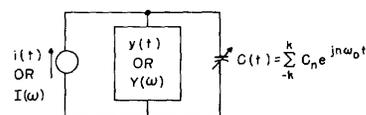


Fig. 10. Normalized amplifier.

In the real world, virtually every steady-state signal of interest can be expressed as a sum of sinusoids. Since the Fourier transform of a sinusoid is an impulse, steady-state signals are characterized by a sum of "δ-functions" in the frequency domain. Therefore, $I(\omega) = \sum I_i \delta(\omega - \omega_i)$ is the appropriate steady-state excitation for Eq. 24. Since superposition applies in this linear equation, we can consider the impulses one at a time.

When the excitation is a transient that is zero before time t_0 , the appropriate frequency function is of the form

$$I(\omega) = \sum e^{j\omega\tau_i} R_i(\omega)$$

where R_i is a rational function. In this case, also, superposition can be used to simplify the computations. To find the response for all three types of excitation, we wish to find a particular solution to Eq. 24 with the appropriate forcing function, $I(\omega)$.

3.1 METHOD OF VARIATION OF PARAMETERS

A general method that can be used for finding the particular solutions for all three excitations is the method of variation of parameters. For our second-order equation, we must first find two complementary solutions, V_1 and V_2 , to the homogeneous equation. In Sections IV and V we shall discuss these complementary solutions. For the present, let us assume that V_1 and V_2 are known, and proceed to find the desired particular solution, $V(\omega)$.

In the method of variation of parameters we assume that there are two functions, A_1 and A_2 such that

$$V(\omega) = A_1(\omega) V_1(\omega) + A_2(\omega) V_2(\omega) \quad (25)$$

Since Eq. 25 is one equation containing two unknown functions, we may arbitrarily select a second equation that the functions must satisfy. To this end we let

$$V(\omega-1) = A_1(\omega) V_1(\omega-1) + A_2(\omega) V_2(\omega-1) \quad (26)$$

To find the A's we substitute the solution, $V(\omega)$, in Eq. 24. We have Eq. 25 for $V(\omega)$, and Eq. 26 for $V(\omega-1)$. For $V(\omega+1)$ we use Eq. 25, evaluated at $(\omega+1)$ and rewritten in a more convenient form. Thus

$$\begin{aligned} V(\omega+1) &= A_1(\omega+1) V_1(\omega+1) + A_2(\omega+1) V_2(\omega+1) = V_1(\omega+1) \Delta A_1(\omega) + V_2(\omega+1) \Delta A_2(\omega) \\ &\quad + A_1(\omega) V_1(\omega+1) + A_2(\omega) V_2(\omega+1) \end{aligned} \quad (27)$$

where $\Delta A(\omega) = A(\omega+1) - A(\omega)$.

Substituting Eqs. 25, 26, and 27 in Eq. 24 yields

$$\begin{aligned}
I(\omega) &= j\omega [V_1(\omega+1)\Delta A_1(\omega) + V_2(\omega+1)\Delta A_2(\omega)] \\
&\quad + j\omega [A_1(\omega)V_1(\omega+1) + A_2(\omega)V_2(\omega+1)] \\
&\quad + Y(\omega) [A_1(\omega)V_1(\omega) + A_2(\omega)V_2(\omega)] \\
&\quad + j\omega [A_1(\omega)V_1(\omega-1) + A_2(\omega)V_2(\omega-1)]
\end{aligned} \tag{28}$$

Since V_1 and V_2 are solutions to the homogeneous equation, Eq. 28 becomes

$$I(\omega) = j\omega [V_1(\omega+1)\Delta A_1(\omega) + V_2(\omega+1)\Delta A_2(\omega)] \tag{29}$$

Equation 29 is a linear algebraic equation with two unknowns, ΔA_1 and ΔA_2 . We shall now proceed to obtain a second equation in these two unknowns and solve for ΔA_1 and ΔA_2 . This procedure results in two first-order difference equations in A_1 and A_2 , respectively. By solving these difference equations we find the functions A_1 and A_2 . Substituting these two functions in Eq. 25 gives the desired particular solution.

To find the second equation in ΔA_1 and ΔA_2 , we rewrite Eq. 26 as

$$V(\omega) = A_1(\omega+1) V_1(\omega) + A_2(\omega+1) V_2(\omega) \tag{30}$$

Subtracting Eq. 25 from Eq. 30 gives our desired result:

$$0 = V_1(\omega) \Delta A_1(\omega) + V_2(\omega) \Delta A_2(\omega) \tag{31}$$

By solving Eqs. 29 and 31 simultaneously we find that

$$\Delta A_1(\omega) = -\frac{I(\omega) V_2(\omega)}{j\omega D(\omega)} \tag{32a}$$

$$\Delta A_2(\omega) = \frac{I(\omega) V_1(\omega)}{j\omega D(\omega)} \tag{32b}$$

where

$$D(\omega) = \begin{vmatrix} V_1(\omega) & V_2(\omega) \\ V_1(\omega+1) & V_2(\omega+1) \end{vmatrix}$$

When the right-hand sides of Eqs. 32a and 32b are merimorphic functions, straightforward methods of solving for A_1 and A_2 are available in mathematical literature. However, in cases in which $I(\omega)$ is such a function – that is, when $i(t)$ is an impulse or some other transient excitation – the particular solution can be found directly without solving the homogeneous equation first (see section 5.5).

3.2 SINUSOIDAL INPUTS

In high-frequency amplifiers, the excitation that is of prime interest is the steady-state excitation. Since the variation of parameters method is best suited to the case in

which $I(\omega)$ is an impulse, we shall discuss the steady-state response in some detail. Suppose $I(\omega)$ is an impulse of complex amplitude I at frequency ω_a . Then Eqs. 32a and 32b become

$$\Delta A_1(\omega) = -\frac{I V_2(\omega_a)}{j\omega_a D(\omega_a)} \delta(\omega - \omega_a) \quad (33a)$$

$$\Delta A_2(\omega) = \frac{I V_1(\omega_a)}{j\omega_a D(\omega_a)} \delta(\omega - \omega_a) \quad (33b)$$

Consider the equation

$$A(\omega+1) - A(\omega) = f(\omega) \quad (34)$$

Formally, a possible solution to Eq. 34 is

$$A(\omega) = p(\omega) - \sum_{s=0}^{\infty} f(\omega+s) \quad (35)$$

where $p(\omega)$ is an arbitrary periodic function. This can easily be seen by substituting Eq. 35 in Eq. 34. If the series in Eq. 35 converges, the $A(\omega)$ thus defined is well defined.

A second formal solution is

$$A(\omega) = p(\omega) + \sum_{s=1}^{\infty} f(\omega-s) \quad (36)$$

For Eq. 36 to be a well-defined function, the sum, of course, must converge.

In Eqs. 33a and 33b, which we wish to solve, the function represented by $f(\omega)$ in Eq. 34 is an impulse. Consequently, solutions of both forms (Eqs. 35 and 36) are infinite sums of constant-amplitude impulses. Since the impulse is not a well-defined function in the normal sense, we shall proceed formally to find the voltage without settling the question of convergence. Then we shall select the form, Eq. 35 or Eq. 36, which results in a physically meaningful voltage.

In order to see what voltages result from the two forms of solution, let us assume $A_1(\omega)$ of the form of Eq. 35, and $A_2(\omega)$ of the form of Eq. 36 with the arbitrary periodic functions equal to zero. Thus

$$A_1(\omega) = \frac{I V_2(\omega_a)}{j\omega_a D(\omega_a)} \sum_{s=0}^{\infty} \delta(\omega+s-\omega_a) \quad (37)$$

$$A_2(\omega) = \frac{I V_1(\omega_a)}{j\omega_a D(\omega_a)} \sum_{s=1}^{\infty} \delta(\omega-s-\omega_a) \quad (38)$$

Substituting Eqs. 37 and 38 in the particular solution (Eq. 25) gives the voltage. Since the voltage depends on the complex amplitude and the frequency of the input current, we write

$$V(\omega, I, \omega_a) = \frac{I}{j\omega_a D(\omega_a)} \left\{ \begin{array}{l} V_1(\omega_a) V_2(\omega) \delta(\omega - \omega_a) \\ + \sum_{s=1}^{\infty} [V_2(\omega_a) V_1(\omega_a - s) \delta(\omega + s - \omega_a) \\ + V_1(\omega_a) V_2(\omega_a + s) \delta(\omega - s - \omega_a)] \end{array} \right\} \quad (39)$$

3.3 PHYSICAL INTERPRETATION OF THE SOLUTION

Equation 39 gives the value of the voltage that appears across a parametric amplifier when the input is an exponential of complex amplitude I at the real frequency ω_a . If the input is to be a real time function, it must contain a second exponential of complex amplitude I^* at frequency $(-\omega_a)$. With a real input we expect the resulting voltage to be a real time function. Thus if our solution (Eq. 39) is physically meaningful, we expect that

$$V(-\omega, I^*, -\omega_a) = V^*(\omega, I, \omega_a) \quad (40)$$

If Eq. 39 has this conjugate symmetry, the voltage resulting from a real sinusoidal input is a sum of real sinusoidal voltages. The power flowing in the time-invariant, passive network at each frequency is proportional to the square of the voltage amplitude at that frequency. Physically, we know that the net power flowing into storage plus that flowing out of the electrical form must be finite. In a general discussion there is no way that we can determine the phase of the various complex powers. However, if the gross power – that is, the sum of the magnitudes of the complex powers – is finite, then surely the net power is finite. Thus our solution (Eq. 39) is physically meaningful if, in addition to the conjugate symmetry (Eq. 40), it carries finite gross power. For the voltage (Eq. 39) the gross power restriction requires that

$$\sum_{s=0}^{\infty} \left\{ |V_1(\omega_a - s)|^2 + |V_2(\omega_a + s)|^2 \right\}$$

shall converge. In Section IV, after discussing some of the properties of the complementary solutions, we shall see that both conditions are satisfied.

Since the voltage expression (Eq. 39) is rather cumbersome, we can define two simpler expressions that characterize the terminal behavior. The first of these is the input impedance. This is defined as the ratio of the voltage amplitude at the applied signal frequency to the input current amplitude. Thus

$$Z_{in}(\omega_a) = \frac{V_1(\omega_a) V_2(\omega_a)}{j\omega_a D(\omega_a)} \quad (41)$$

Note that in the special case, with ω_a an integer or half-integer, there is a second term at frequency ω_a or $(-\omega_a)$. Then Eq. 41 is not the input impedance that we have

defined. For the moment, let us consider such exact synchronism between pump and signal as a degenerate case that is to be avoided. It would certainly be difficult to maintain in a physical device that is used to amplify signals that carry any information. In Appendix A we shall discuss some of the peculiarities of the degenerate case.

The second useful expression is the gain. In section 1.2 we defined the gain as the ratio of the average power delivered to the time-invariant network at the desired output frequency to the power delivered by the source. Thus the gain with output at frequency $(\omega_a + s)$ is

$$K_s(\omega_a) = \frac{|V_1(\omega_a)V_2(\omega_a + s)|^2 \text{Re}[Y(\omega_a + s)]}{\omega_a^2 |D(\omega_a)|^2 \text{Re}[Z_{in}(\omega_a)]} \quad (42a)$$

The gain with output at frequency $(\omega_a - s)$ is

$$K_{-s}(\omega_a) = \frac{|V_2(\omega_a)V_1(\omega_a - s)|^2 \text{Re}[Y(\omega_a - s)]}{\omega_a^2 |D(\omega_a)|^2 \text{Re}[Z_{in}(\omega_a)]} \quad (42b)$$

In the degenerate case mentioned above these gain expressions must be modified. We should note that Eqs. 42a and 42b are independent of the amplitude and phase of the input signal. Therefore, our linear amplifier is indeed linear.

IV. THE COMPLEMENTARY SOLUTION – GENERAL PROPERTIES

We shall now discuss some of the properties of the complementary solutions to the homogeneous difference equation for a parametric amplifier. From these properties, we shall see that, mathematically, the voltage expression (Eq. 39) satisfies the requirements that we deduced from physical reasoning in section 3.3.

4.1 THE CONJUGATE NATURE OF THE SOLUTIONS

ASSERTION 1. If $V_1(\omega)$ is a solution to the homogeneous difference equation,

$$0 = j\omega V(\omega+1) + Y(\omega) V(\omega) + j\omega V(\omega-1) \quad (43)$$

with $Y(\omega)$ a positive real admittance function, then

$$V_2(\omega) = V_1^*(-\omega) \quad (44)$$

is also a solution.

PROOF. Substitute Eq. 44 in Eq. 43 and show that the result is an equality. Then

$$0 \stackrel{?}{=} j\omega V_1^*(-\omega-1) + Y(\omega) V_1^*(-\omega) + j\omega V_1^*(-\omega-1)$$

Making the change of variables, $\underline{\omega} = -\omega$, yields

$$0 \stackrel{?}{=} -j\underline{\omega} V_1^*(\underline{\omega}-1) + Y(-\underline{\omega}) V_1^*(\underline{\omega}) - j\underline{\omega} V_1^*(\underline{\omega}+1)$$

But since Y is a positive real admittance, we have

$$Y(-\underline{\omega}) = Y^*(\underline{\omega})$$

Therefore

$$0 \stackrel{?}{=} -j\underline{\omega} V_1^*(\underline{\omega}-1) + Y^*(\underline{\omega}) V_1^*(\underline{\omega}) - j\underline{\omega} V_1^*(\underline{\omega}+1)$$

Taking the conjugate of this questioned equation gives

$$0 \stackrel{?}{=} j\underline{\omega} V_1(\underline{\omega}-1) + Y(\underline{\omega}) V_1(\underline{\omega}) + j\underline{\omega} V_1(\underline{\omega}+1)$$

But this is the original difference equation (Eq. 43) with $\underline{\omega}$ instead of ω , and by assumption V_1 is a solution. Therefore, we erase the question mark, and the assertion is proved.

When the two solutions to the homogeneous difference equation (Eq. 43) have the conjugate property (Eq. 44) the determinant $D(\omega)$, defined as

$$D(\omega) = \begin{vmatrix} V_1(\omega) & V_2(\omega) \\ V_1(\omega+1) & V_2(\omega+1) \end{vmatrix} \quad (45)$$

has conjugate symmetry about $\omega = 0$. In order to prove this statement, we need to

use Heymann's theorem (10). The proof is given completely by Milne-Thomson; therefore we shall merely restate the theorem in the notation of this report.

Heymann's theorem relates to the linear difference equation

$$A_n(\omega) V(\omega+m) + A_{n-1}(\omega) V(\omega+m-1) + \dots + A_{n-m}(\omega) V(\omega) + \dots + A_0(\omega) V(\omega+m-n) = 0 \quad (46)$$

For this n^{th} -order equation, we can find n linearly independent solutions, $V_n(\omega)$. From these solutions the determinant $D(\omega)$ is formed by

$$D(\omega) = \begin{vmatrix} V_1(\omega) & V_2(\omega) & \dots & V_n(\omega) \\ V_1(\omega+1) & V_2(\omega+1) & \dots & V_n(\omega+1) \\ \cdot & \cdot & \dots & \cdot \\ \cdot & \cdot & \dots & \cdot \\ \cdot & \cdot & \dots & \cdot \\ V_1(\omega+n-1) & V_2(\omega+n-1) & \dots & V_n(\omega+n-1) \end{vmatrix}$$

Heymann's theorem states that

$$D(\omega) = (-1)^n \frac{A_0(\omega)}{A_n(\omega)} D(\omega+1)$$

For our difference equation (Eq. 43), $n = 2$, and $A_0(\omega) = A_n(\omega) = j\omega$. Thus for our parametric amplifier,

$$D(\omega) = D(\omega+1) \quad (47)$$

ASSERTION 2. For Eq. 43 with the solutions related by Eq. 44,

$$D^*(-\omega) = D(\omega) \quad (48)$$

PROOF.

$$\begin{aligned} D(\omega) &= \begin{vmatrix} V_1(\omega) & V_1^*(-\omega) \\ V_1(\omega+1) & V_1^*(-\omega-1) \end{vmatrix} \\ D^*(-\omega) &= \begin{vmatrix} V_1^*(-\omega) & V_1(\omega) \\ V_1^*(-\omega+1) & V_1(\omega-1) \end{vmatrix} \\ &= - \begin{vmatrix} V_1(\omega) & V_1^*(-\omega) \\ V_1(\omega-1) & V_1^*(-\omega+1) \end{vmatrix} \\ &= \begin{vmatrix} V_1(\omega) & V_1^*(-(\omega-1)) \\ V_1(\omega) & V_1^*(-\omega) \end{vmatrix} = D(\omega-1) \end{aligned}$$

But by Eq. 47, $D(\omega-1) = D(\omega)$. And therefore assertion 2 is proved.

Now, with the aid of assertions 1 and 2 we can prove another assertion.

ASSERTION 3. The parametric amplifier voltage (Eq. 39) has the conjugate symmetry property (Eq. 40) if the solutions to the homogeneous equation are conjugately related by Eq. 44.

PROOF. Let us transcribe Eq. 39:

$$V(\omega, I, \omega_a) = \frac{I}{j\omega_a D(\omega_a)} \left\{ \begin{array}{l} V_1(\omega_a) V_2(\omega_a) \delta(\omega - \omega_a) \\ + \sum_{s=1}^{\infty} [V_2(\omega) V_1(\omega_a - s) \delta(\omega + s - \omega_a) \\ + V_1(\omega_a) V_2(\omega_a + s) \delta(\omega - s - \omega_a)] \end{array} \right\}$$

Substituting Eq. 44 for V_2 gives

$$V(\omega, I, \omega_a) = \frac{I}{j\omega_a D(\omega_a)} \left\{ \begin{array}{l} V_1(\omega_a) V_1^*(-\omega_a) \delta(\omega - \omega_a) \\ + \sum_{s=1}^{\infty} [V_1^*(-\omega_a) V_1(\omega_a - s) \delta(\omega + s - \omega_a) \\ + V_1(\omega_a) V_1^*(-\omega_a - s) \delta(\omega - s - \omega_a)] \end{array} \right\}$$

Now

$$V(-\omega, I^*, -\omega_a) = \frac{I^*}{-j\omega_a D(-\omega_a)} \left\{ \begin{array}{l} V_1(-\omega_a) V_1^*(\omega_a) \delta(-\omega + \omega_a) \\ + \sum_{s=1}^{\infty} [V_1^*(\omega_a) V_1(-\omega_a - s) \delta(-\omega + s + \omega_a) \\ + V_1(-\omega_a) V_1^*(\omega_a + s) \delta(-\omega - s + \omega_a)] \end{array} \right\}$$

By assertion 2, Eq. 46, $D(-\omega_a) = D^*(\omega_a)$. Furthermore,

$$\delta(-\omega + \omega_a) = \delta(\omega - \omega_a)$$

$$\delta(-\omega + s + \omega_a) = \delta(\omega - s - \omega_a)$$

and

$$\delta(-\omega - s + \omega_a) = \delta(\omega + s - \omega_a)$$

Therefore

$$V(-\omega, I^*, -\omega_a) = V^*(\omega, I, \omega_a)$$

and the assertion is proved.

4.2 POINCARÉ'S THEOREM AND ITS IMPLICATIONS

The second restriction on the voltage that we deduced in our physical interpretation was that the power remain finite (see section 3.3). We pointed out that the power was certainly finite if

$$\sum_{s=1}^{\infty} \left[|V_1(\omega_a - s)|^2 + |V_2(\omega_a + s)|^2 \right]$$

converges. If V_1 and V_2 are related by Eq. 44, we need investigate only $\sum_{s=1}^{\infty} |V_2(\omega_a + s)|^2$ because the behavior of $|V_1|$ for large negative arguments is exactly the same as the behavior of $|V_2|$ for large positive arguments. There are two published theorems on the difference equation which apply to the problem of convergence of the series $\sum_{s=1}^{\infty} |V_2(\omega_a + s)|$. The first, known as Poincaré's theorem, applies the ratio test; the second, known as Perron's theorem, applies the root test. If $\sum_{s=1}^{\infty} |V_2(\omega_a + s)|$ converges, then $\sum_{s=1}^{\infty} |V_2(\omega_a + s)|^2$ converges. We shall see that the series converges as long as the amplifier is realizable.

Poincaré's and Perron's theorems both apply to the difference equation of the Poincaré type. An equation of this type is a linear homogeneous equation whose coefficients approach constants as the argument approaches infinity. When the homogeneous difference equation (Eq. 43) for our parametric amplifier is written

$$0 = V(\omega+1) + \frac{Y(\omega)}{j\omega} V(\omega) + V(\omega-1) \quad (49)$$

the realizability condition

$$\lim_{\omega \rightarrow \infty} \frac{Y(\omega)}{j\omega} = C_0 > 2$$

makes it an equation of the Poincaré type.

Before stating the theorems we need to define the characteristic equation for a Poincaré difference equation. The characteristic equation is derived by the following procedure. Start with the constant-coefficient equation to which the original difference equation tends. In our case it is

$$V(\omega+1) + C_0 V(\omega) + V(\omega-1) = 0$$

Next, assume that $V(\omega) = \mu^\omega$, with μ a constant to be determined. Then substitute the assumed solution into the constant-coefficient equation. In our case this gives

$$\mu^{\omega+1} + C_0 \mu^\omega + \mu^{\omega-1} = 0$$

Factor the equation in the form

$$\mu^{\omega+s} p(\mu) = 0$$

in which s is a suitable integer to make $p(\mu)$ a polynomial. In our case we get

$$\mu^{(\omega+1)}(\mu^2 + c_0\mu + 1) = 0$$

Finally, the characteristic equation is $p(\mu) = 0$.

In our case the characteristic equation is $\mu^2 + c_0\mu + 1 = 0$. The roots of this equation are

$$\mu = -\frac{C_0 \pm (C_0^2 - 4)^{1/2}}{2} \quad (50)$$

When C_0 is greater than two, as it is in a realizable amplifier, the roots are real, negative, and distinct. Furthermore, the two roots are reciprocals. Henceforth, we shall use μ_1 for the larger root, that is, Eq. 50 with the plus, and use μ_2 for the other root.

We now have enough terminology to state the Poincaré and Perron theorems. The proofs, which are found in the last chapter of Milne-Thomson (11), will not be repeated here. We shall use the notation of Eq. 46.

Poincaré's theorem may be stated: An n^{th} -order difference equation of the Poincaré type with (a) distinct moduli for the roots, μ_i , of the characteristic equation; (b) the ratio of the first coefficient to the last coefficient (A_0/A_n in Eq. 46) nonzero for the argument $(\omega_a + s)$, with s an integer and ω_a a constant, possesses an n solution V_1, V_2, \dots, V_n such that

$$\lim_{s \rightarrow \infty} \frac{|V_i(\omega_a + s + 1)|}{|V_i(\omega_a + s)|} = |\mu_i|$$

The difference equation (Eq. 49) clearly satisfies conditions (a) and (b) so that Poincaré's theorem applies. If we associate the solution V_2 to Eq. 49 with the root μ_2 of Eq. 50, which is less than one, the series $\sum_{s=1}^{\infty} |V_2(\omega_a + s)|$ converges by the ratio test.

Perron's theorem is a much more general theorem than Poincaré's. It applies for equations in which the roots of the characteristic equation are not distinct. However, in our case these more general conditions are unnecessary. For higher-order amplifiers – amplifiers whose variable elements are not pure sinusoids – we might need Perron's theorem.

Perron's theorem, for our purposes, may be stated: For the n^{th} -order difference equation discussed under Poincaré's theorem the solutions have the property:

$$\lim_{s \rightarrow \infty} \sqrt[s]{|V_i(\omega_a + s)|} = |\mu_i|$$

As we have stated, Perron's theorem shows that $\sum_{s=1}^{\infty} |V_2(\omega_a + s)|$ converges by the

root test. From Perron's theorem we deduce that for large values of ω , $V_2(\omega) = \mu_2^\omega U(\omega)$, where $U(\omega)$ goes to infinity no faster than a polynomial as ω becomes infinite. In Section V we shall find our solution in this form in the entire plane, not merely for large ω .

V. BOOLE'S METHOD OF SYMBOLIC OPERATORS

In order to analyze a specific parametric amplifier and obtain the amplitudes of the various voltages numerically we must still find one solution to the homogeneous difference equation. To find a solution for a variable-coefficient difference equation is not a simple task. In fact, there does not seem to be a method that will solve our equation in closed form. Most of the books on difference equations merely point out that a solution in the form of a factorial series can be found in much the same way as a power-series solution is found for a variable-coefficient differential equation. Milne-Thomson (12) discusses in great detail a specific procedure for finding the series. The method is known as "Boole's method of symbolic operators."

We shall start here by defining the operators and stating those properties that are needed in the solution of the homogeneous equation for the parametric amplifier. We shall then go through the procedure for the second-order equation. Generalizations to higher-order equations are straightforward. In the discussion of the procedure Milne-Thomson mentions two points at which the method may fail. However, we shall see that in our case a solution can always be found. The method can also be used to solve the complete equation for the parametric amplifier with transient inputs directly without first finding a solution to the homogeneous equation.

5.1 DEFINITION AND PROPERTIES OF THE OPERATORS

With Boole's operational method, as with any operational method, we wish to convert our original equation into an operational equation. More specifically, we start with the homogeneous equation. If we call V_c the complementary solution, this equation has the form

$$f(\omega, V_c) = 0$$

We then convert it to the form

$$F(\text{operators}) V_c(\omega) = 0$$

Finally, we assume a series for $V_c(\omega)$; and by knowing how the operators operate on the individual terms in the series, we can evaluate the coefficients.

To apply Boole's method to our equation we need two operators:

(i) ρ^m defined by

$$\rho^m V(\omega) = \frac{\Gamma(\omega+1)}{\Gamma(\omega+1-m)} V(\omega-m)$$

where m is any complex constant, and Γ is the gamma function.

(ii) π defined by

$$\pi V(\omega) = \omega [V(\omega) - V(\omega-1)]$$

The operators ρ and π as defined in Milne-Thomson are somewhat more general in that they may also incorporate a linear change of variable. However, this extra generality is not needed in our solutions and is therefore omitted.

Now let us look at some of the properties of the operators ρ and π which enable us to reduce the difference equation to operational form and then evaluate the coefficients in the assumed series solution. The properties are merely stated here; the proofs are found in Milne-Thomson (9). Throughout this discussion we shall use k and m for complex constants, s for positive integers, and n for positive or negative integers.

The operator ρ with its Γ -functions looks quite formidable, but a well-known property of the Γ -function renders the operator quite manageable. That property is $\Gamma(\omega+1) = \omega\Gamma(\omega)$. Thus for integer exponents ρ does not involve Γ -functions at all. That is,

$$\rho^s V(\omega) = \omega(\omega-1)(\omega-2) \dots (\omega-s+1) V(\omega-s) \quad (51a)$$

$$\rho^{-s} V(\omega) = \frac{1}{(\omega+1)(\omega+2) \dots (\omega+s)} V(\omega+s) \quad (51b)$$

Since factorial expressions like those in Eqs. 51a and 51b appear quite often in the solution of difference equations, we shall use the simplifying notation

$$\omega^{(s)} = \omega(\omega-1)(\omega-2) \dots (\omega-s+1) \quad (52a)$$

$$\omega^{(-s)} = \frac{1}{(\omega+1)(\omega+2) \dots (\omega+s)} \quad (52b)$$

We shall refer to Eq. 52a as a factorial expression, and to Eq. 52b as an inverse factorial expression. Using the simplifying notation, we can rewrite Eqs. 51a and 51b compactly for any positive or negative integer. Thus

$$\rho^n V(\omega) = \omega^{(n)} V(\omega-n) \quad (53)$$

The operator ρ is defined for arbitrary complex exponents. It obeys the normal exponent law

$$\rho^k [\rho^m V(\omega)] = \rho^{k+m} V(\omega)$$

Operation with π can also be repeated; and hence for positive-integer exponents π^s is well-defined. From the definitions of ρ and π , we find that

$$\omega V(\omega) = (\pi+\rho) V(\omega)$$

Furthermore, since operation with $(\pi+\rho)$ can be repeated, we have

$$\omega^s V(\omega) = (\pi+\rho)^s V(\omega) \quad (54)$$

When expanding the expression $(\pi+\rho)^s$ we must be careful, because π and ρ do not commute. However, there is a theorem (13) that allows us to separate the two operators. It states: If F is a polynomial of order s , then

$$F(\pi+\rho) V(\omega) = F(\pi) V(\omega) + F_1(\pi) \rho V(\omega) + \frac{1}{2!} F_2(\pi) \rho^2 V(\omega) + \dots + \frac{1}{s!} F_s(\pi) \rho^s V(\omega) \quad (55)$$

The polynomials F_i are formed by

$$F_i(x) = F_{i-1}(x) - F_{i-1}(x-1) = \Delta_{-1} F_{i-1}(x)$$

If we use the symbol Δ_{-1}^i to mean the operation Δ_{-1} applied i times in succession, we see that

$$F_i = \Delta_{-1}^i F$$

By using Eqs. 53, 54, and 55 we are able to transform the normalized equation for any parametric amplifier to operational form. Before going through the detailed procedure for transforming the equation, let us continue with the properties of the operators which enable us to find a series solution. When ρ operates on the constant, one, the result is a ratio of Γ -functions. When the operand is one, we shall omit it. Thus, without the operand,

$$\rho^m = \frac{\Gamma(\omega+1)}{\Gamma(\omega+1-m)}$$

The series solutions that we shall assume for our difference equations are power series in ρ . With negative exponents the series is called a series of inverse factorials, or a factorial series of the first kind. The series is

$$\sum_{s=0}^{\infty} a_s \rho^{k-s} = \frac{\Gamma(\omega+1)}{\Gamma(\omega+1-k)} \sum_{s=0}^{\infty} a_s (\omega-k)^{(-s)} \quad (56)$$

With positive exponents the series is called a "Newton series," or a factorial series of the second kind. This series is

$$\sum_{s=0}^{\infty} b_s \rho^{k+s} = \frac{\Gamma(\omega+1)}{\Gamma(\omega+1-k)} \sum_{s=0}^{\infty} b_s (\omega-k)^{(s)} \quad (57)$$

Once we have assumed a solution in the form of a power series in ρ , we shall have to operate on it with π . Again we turn to a theorem (14) concerning the operators ρ and π . It states that if F is a polynomial, then

$$F(\pi) \rho^m = F(m) \rho^m \quad (58)$$

5.2 REDUCTION OF THE HOMOGENEOUS EQUATION TO OPERATIONAL FORM

Now let us see how the operators ρ and π can be used for solving the homogeneous difference equation for a parametric amplifier. We shall proceed in detail for the case of sinusoidal parameter variation, just as we did in the two previous chapters. The extension to higher-order equations for more complicated parameter variation is the

same concept except that it is more complex.

The difference equation to be solved is

$$j\omega V_c(\omega+1) + Y(\omega) V_c(\omega) + j\omega V_c(\omega-1) = 0 \quad (59)$$

Since $Y(\omega)$ is a positive real admittance function with parasitic shunt capacitance C_o , we can write

$$Y(\omega) = \frac{P(\omega)}{Q(\omega)} = \frac{C_o \left[(j\omega)^n + A_1(j\omega)^{n-1} + \dots + A_n \right]}{(j\omega)^{n-1} + B_2(j\omega)^{n-2} + \dots + B_n} \quad (60)$$

Substituting Eq. 60 in Eq. 59 and multiplying by $Q(\omega)$ gives the difference equation with polynomial coefficients:

$$j\omega Q(\omega) V_c(\omega+1) + P(\omega) V_c(\omega) + j\omega Q(\omega) V_c(\omega-1) = 0 \quad (61)$$

Before introducing the operators into Eq. 61 we shall find it convenient to introduce a free constant, μ , by the substitution

$$V_c(\omega) = \mu^\omega U(\omega) \quad (62)$$

Making this substitution in Eq. 61 and multiplying by $\mu^{-\omega+1}$, we obtain

$$j\omega Q(\omega) \mu^2 U(\omega+1) + \mu P(\omega) U(\omega) + j\omega Q(\omega) U(\omega-1) = 0 \quad (63)$$

For the present, μ is a constant to be determined. However, when we determine it, we shall find that it has the same value as the constant discussed in section 4.3. The solution (Eq. 62) is then in the form predicted by Perron's theorem.

To reduce Eq. 63 to operational form, we proceed as follows: First we use Eq. 53 with n equal to $(+1)$ and (-1) to eliminate $U(\omega-1)$ and $U(\omega+1)$. The result is

$$[j\omega(\omega+1)Q(\omega)\mu^2\rho^{-1} + P(\omega)\mu + jQ(\omega)\rho] U(\omega) = 0$$

Next, we use Eq. 54 to eliminate all the ω 's in the bracket. Thus

$$[(\pi+\rho)(\pi+\rho+1)Q(\pi+\rho)\mu^2\rho^{-1} + P(\pi+\rho)\mu + jQ(\pi+\rho)\rho] U(\omega) = 0$$

Finally, we use Eq. 55 to put the equation in the form

$$\left[F_{-1}(\pi)\rho^{-1} + F_0(\pi) + \dots + F_n(\pi)\rho^{n+1} \right] U(\omega) = 0 \quad (64)$$

The polynomials F_i in Eq. 64 are constructed as follows:

$$F_{-1}(\pi) = j\pi(\pi+1) Q(\pi) \mu^2 = j\mu^2 N(\pi)$$

$$F_0(\pi) = j\mu^2 \Delta_{-1} N(\pi) + \mu P(\pi) \quad (65)$$

$$F_i(\pi) = \frac{j\mu^2}{(i+1)!} \Delta_{-1}^{i+1} N(\pi) + \frac{\mu}{i!} \Delta_{-1}^i P(\pi) + \frac{j}{(i-1)!} \Delta_{-1}^{i-1} Q(\pi) \quad \text{for } i \geq 1$$

To learn more about the polynomials F_i we must investigate the operator Δ_{-1} . Let us examine

$$\begin{aligned}\Delta_{-1} X^m &= X^m - (X-1)^m \\ &= X^m - X^m + mX^{m-1} - \binom{m}{2} X^{m-2} + \dots + (-1)^m \\ &= mX^{m-1} - \binom{m}{2} X^{m-2} + \dots + (-1)^m\end{aligned}$$

Consequently, when Δ_{-1} operates on a polynomial, the result is a polynomial of one order lower. Therefore, the order of the F_i of Eq. 65 is $(n-i)$, n being the order of the admittance (Eq. 60). For $i > n$, the F_i are zero. As the operation with Δ_{-1} is repeated, we see that for $k < m$

$$\Delta_{-1}^k X^m = m^{(k)} X^{m-k} + (\text{terms of lower order}) \quad (66)$$

The polynomial F_n is a polynomial of zeroth order; that is, it is a constant. In Eq. 65 the leading term of $N(\pi)$ is $\pi^2(j\pi)^{n-1}$, that of $P(\pi)$ is $C_o(j\pi)^n$, and that of $Q(\pi)$ is $(j\pi)^{n-1}$. Thus by using Eq. 66 in Eq. 65, we have

$$F_n(\pi) = \frac{j^n \mu^2 (n+1)!}{(n+1)!} + \frac{j^n C_o \mu n!}{n!} + \frac{j^n (n-1)!}{(n-1)!} = j^n [\mu^2 + C_o \mu + 1]$$

We now select μ so that $F_n(\pi)$ is zero, and thus simplify Eq. 64. Consequently,

$$\mu = -\frac{C_o \pm (C_o^2 - 4)^{1/2}}{2} \quad (67)$$

As in section 4.2, we shall use μ_1 for Eq. 67 with the plus sign, and μ_2 for the root with the minus sign. With μ determined, Eq. 64 becomes

$$\left[F_{-1}(\pi) \rho^{-1} + F_0(\pi) + \dots + F_{n-1}(\pi) \rho^{n-1} \right] U(\omega) = 0 \quad (68)$$

5.3 SOLUTION IN FACTORIAL-SERIES FORM

In order to solve Eq. 68 we shall assume a factorial-series (Eq. 56) form for $U(\omega)$. We shall begin by assuming a series of the first kind for two reasons. First, the evaluation of the coefficient is somewhat more straightforward than for a series of the second kind, and the behavior of the inverse series at infinity is easier to ascertain. Thus we can easily see how this solution fits in with the Poincaré and Perron theorems. We assume that

$$U(\omega) = \sum_{s=0}^{\infty} a_s \rho^{k-s} \quad (69)$$

Substituting this series for $U(\omega)$ in Eq. 68 gives

$$F_{-1}(\pi) \sum_{s=1}^{\infty} a_{s-1} \rho^{k-s} + F_0(\pi) \sum_{s=0}^{\infty} a_s \rho^{k-s} + \dots + F_{n-1}(\pi) \sum_{s=1-n}^{\infty} a_{s+n-1} \rho^{k-s} = 0 \quad (70)$$

By using Eq. 58 we eliminate the operator π , and Eq. 70 becomes

$$\sum_{s=1}^{\infty} a_{s-1} F_{-1}(k-s) \rho^{k-s} + \sum_{s=0}^{\infty} a_s F_0(k-s) \rho^{k-s} + \dots + \sum_{s=1-n}^{\infty} a_{s+n-1} F_{n-1}(k-s) \rho^{k-s} = 0 \quad (71)$$

If we set the coefficient of each power of ρ in Eq. 71 equal to zero, the equality is surely satisfied. The result is a set of algebraic equations from which we can evaluate k and the a_i . Thus

$$\left. \begin{aligned} a_0 F_{n-1}(k-1+n) &= 0 \\ a_0 F_{n-2}(k-2+n) + a_1 F_{n-1}(k-2+n) &= 0 \\ \cdot &\quad \cdot \\ \cdot &\quad \cdot \\ \cdot &\quad \cdot \\ a_0 F_{-1}(k-1) + a_1 F_0(k-1) + \dots + a_{n-1} F_{n-2}(k-1) + a_n F_{n-1}(k-1) &= 0 \\ a_1 F_{-1}(k-2) + a_2 F_0(k-2) + \dots + a_n F_{n-2}(k-2) + a_{n+1} F_{n-1}(k-2) &= 0 \\ \cdot &\quad \cdot \\ \cdot &\quad \cdot \\ \cdot &\quad \cdot \\ a_{s-n} F_{-1}(k+n-s-1) + a_{s-n+1} F_0(k+n-s-1) + \dots + a_{s-1} F_{n-2}(k+n-s-1) + a_s F_{n-1}(k+n-s-1) &= 0 \end{aligned} \right\} \quad (72)$$

The first equation of Eqs. 72 requires that

$$F_{n-1}(k-1+n) = 0$$

Since F_{n-1} is a first-order polynomial, this equation determines k uniquely. In terms of the constants of the admittance (Eq. 60), we find that

$$k = j \frac{(\mu^2 + 1)}{(\mu^2 - 1)} (B_2 - A_1) \quad (73)$$

The details of the evaluation of k are given in Appendix C. The only significant point for the present discussion is that k is an imaginary number.

The second equation of Eqs. 72 can be used to evaluate a_1 as a constant times a_0 . Then a_2 can be evaluated as another constant times a_0 from the third equation, and so on. The coefficient, a_0 , remains arbitrary because any solution to the homogeneous equation can be multiplied by a constant. Therefore, we may as well choose a_0 equal to one.

Thus we have evaluated all of the constants in the assumed solution $U(\omega)$, Eq. 69.

Before we can state that $U(\omega)$ is a solution to the difference equation (Eq. 63) we must be sure that the series converges. To investigate convergence, we shall use the series in more conventional terminology from Eq. 56. Thus

$$\begin{aligned} U(\omega) &= \frac{\Gamma(\omega+1)}{\Gamma(\omega+1-k)} \sum_{s=0}^{\infty} a_s (\omega-k)^{(-s)} \\ &= \frac{\Gamma(\omega+1)}{\Gamma(\omega+1-k)} \left[1 + \frac{a_1}{(\omega-k+1)} + \frac{a_2}{(\omega-k+1)(\omega-k+2)} + \dots \right] \end{aligned} \quad (74)$$

In this series the ratio of the s^{th} term to the $(s-1)^{\text{th}}$ term is

$$\frac{t_s}{t_{s-1}} = \frac{a_s (\omega-k)^{(-s)}}{a_{s-1} (\omega-k)^{(-s+1)}} = \frac{a_s}{a_{s-1} (\omega-k+s)}$$

We are interested in the limit of this ratio as s approaches infinity. Therefore, we may replace $(\omega-k+s)$ by s .

For large values of s , the recursion formula (Eq. 72) for the a_s is

$$a_{s-n} F_{-1}(k+n-s-1) + a_{s-n+1} F_0(k+n-s-1) + \dots + a_s F_{n-1}(k+n-s-1) = 0$$

We have found that the F_i are polynomials of order $(n-i)$. Therefore, for large values of s we may replace $F_i(k+n-s-1)$ by $\phi_i s^{n-i}$, where the ϕ_i are constants. Thus as s approaches infinity the recursion formula becomes

$$a_{s-n} \phi_{-1} (-s)^{n+1} + a_{s-n+1} \phi_0 (-s)^n + \dots + a_{s-1} \phi_{n-2} s^2 = a_s \phi_{n-1} s$$

Dividing this equation by $(a_{s-1} s^2 \phi_{n-1})$, we obtain

$$\frac{t_s}{t_{s-1}} = \frac{a_s}{a_{s-1} s} = \frac{a_{s-n} \phi_{-1} (-s)^{n-1} + a_{s-n+1} \phi_0 (-s)^{n-2} + \dots + a_{s-1} \phi_{n-2}}{a_{s-1} \phi_{n-1}} \quad (75)$$

In order to test for convergence of our series (Eq. 74), we must find $\lim_{s \rightarrow \infty} \left| \frac{t_s}{t_{s-1}} \right|$.

Let us assume that the limit exists, and that it is equal to T . On the right-hand side of Eq. 75 we have a sum of terms of the form

$$(-1)^{(h-1)} \frac{\phi_{n-h-1} a_{s-h} s^{h-1}}{\phi_{n-1} a_{s-1}}$$

where h is an integer greater than one and less than or equal to n . Now

$$\frac{a_{s-h} s^{h-1}}{a_{s-1}} = \frac{a_{s-h} s}{a_{s-h-1}} \cdot \frac{a_{s-h-1} s}{a_{s-h-2}} \dots \frac{a_{s-2} s}{a_{s-1}}$$

Therefore

$$\lim_{s \rightarrow \infty} \frac{a_s h^s}{a_{s-1}} = \frac{1}{T^{h-1}}$$

Thus, if we take the limit of Eq. 75, we obtain

$$T = \frac{1}{\phi_{n-1}} \left[\frac{(-1)^{n-1} \phi_{-1}}{T^{n-1}} + \dots - \frac{\phi_{n-3}}{T} + \phi_{n-2} \right]$$

Multiplying by $\phi_{n-1} T^{n-1}$ gives a polynomial, the roots of which determine the possible values of T . The polynomial is

$$\phi_{n-1} T^n - \phi_{n-2} T^{n-1} + \dots + (-1)^{n-1} \phi_{-1} \quad (76)$$

In Appendix D we show that this polynomial can be factored as

$$(T-1)^{n-1} \left(T - \frac{\mu}{2\mu + C_o} \right)$$

The polynomial, Eq. 76, tells us that the only possible values for the limit, T , are one and $\mu/(2\mu + C_o)$. Of course, for a particular amplifier there is no such ambiguity, since the coefficients, a_s , are uniquely determined. However, the interesting thing is that we can investigate all possible amplifier configurations by studying only two cases. When

$$T = \frac{\mu}{2\mu + C_o} \quad (77)$$

we get a convergent series if $|T|$ is less than one. Substituting Eq. 67 with the plus sign for μ , we find that our series converges if and only if

$$C_o < \frac{3}{\sqrt{2}} \quad (78)$$

The series diverges if Eq. 67 with the minus sign is used.

For the second case, $T = 1$, the simple ratio test does not give any information about convergence. In this case we use Weierstrass' criterion (15). This criterion states: A series, $\sum_{s=0}^{\infty} t_s$, of complex terms for which $\frac{t_s}{t_{s-1}} = 1 - \frac{a}{s} + o\left(\frac{1}{s}\right)$, where $\lambda > 1$, and a is independent of s , is absolutely convergent if and only if $R_e(a) < 1$. For $R_e(a) < 0$, the series is invariably divergent. If $0 < R_e(a) < 1$, each of the series

$$\sum_{s=0}^{\infty} |t_s - t_{s+1}| \quad \text{and} \quad \sum_{s=0}^{\infty} (-1)^s t_s \quad \text{is convergent.}$$

For our series (Eq. 74),

$$\frac{t_s}{t_{s-1}} = \frac{a_s}{a_{s-1}(\omega-k+s)} = \frac{a_s}{a_{s-1}s} \left[1 - \frac{\omega-k}{s} + \frac{(\omega-k)^2}{s^2} - \dots \right]$$

Applying Weierstrass' criterion, we see that the series converges in the half-plane, $\omega > 1$. In section 5.4 we shall show that once a solution is known in a half-plane, it can be extended to the entire plane. Therefore, when $T = 1$, we can always find a solution $U(\omega)$ in the form of Eq. 74.

Once a convergent series for $U(\omega)$ has been found we return to Eq. 62 to find $V_c(\omega)$, the complementary solution to the homogeneous difference equation. Next, we should like to ascertain whether this solution is of the form V_1 or V_2 for use in the voltage expression (Eq. 39).

If in Eq. 67 we use the minus sign for μ , our solution $V_c(\omega) = \mu_2^\omega U(\omega)$ is $V_2(\omega)$ if

$$\lim_{\omega \rightarrow \infty} \frac{|U(\omega+1)|}{|U(\omega)|} = 1$$

In expression 74 for $U(\omega)$ the series approaches one as ω approaches infinity. Thus for large ω ,

$$\frac{U(\omega+1)}{U(\omega)} = \frac{\Gamma(\omega+2)}{\Gamma(\omega+2-k)} \times \frac{\Gamma(\omega+1-k)}{\Gamma(\omega+1)} = \frac{(\omega+1) \Gamma(\omega+1)}{(\omega+1-k) \Gamma(\omega+1-k)} \times \frac{\Gamma(\omega+1-k)}{\Gamma(\omega+1)}$$

Obviously, this last expression approaches unity as ω approaches infinity, and the solution is V_2 .

The difference equation (Eq. 68) could also be solved by assuming a factorial series of the second kind for $U(\omega)$, that is,

$$U(\omega) = \sum_{s=0}^{\infty} b_s \rho^{m+s} \tag{79}$$

Substituting this series for $U(\omega)$ in Eq. 68 gives

$$\begin{aligned} F_{-1}(\pi) \sum_{s=-1}^{\infty} b_{s+1} \rho^{m+s} + F_0(\pi) \sum_{s=0}^{\infty} b_s \rho^{m+s} + \dots \\ + F_{n-1}(\pi) \sum_{s=n-1}^{\infty} b_{s+1-n} \rho^{m+s} = 0 \end{aligned}$$

By using Eq. 58, we eliminate the operator π . Thus

$$\begin{aligned} \sum_{s=-1}^{\infty} b_{s+1} F_{-1}(m+s) \rho^{m+s} + \sum_{s=0}^{\infty} b_s F_0(m+s) \rho^{m+s} + \dots \\ + \sum_{s=n-1}^{\infty} b_{s+1-n} F_{n-1}(m+s) \rho^{m+s} = 0 \end{aligned}$$

By setting the coefficients of each power of ρ equal to zero, we get the recursion formulas for the b_i :

$$\left. \begin{aligned}
b_0 F_{-1}(m-1) &= 0 \\
b_1 F_{-1}(m) + b_0 F_0(m) &= 0 \\
\cdot & \quad \cdot \\
\cdot & \quad \cdot \\
\cdot & \quad \cdot \\
b_n F_{-1}(m+n-1) + b_{n-1} F_0(m+n-1) + \dots + b_0 F_{n-1}(m+n-1) &= 0 \\
\cdot & \quad \cdot \quad \cdot \\
\cdot & \quad \cdot \quad \cdot \\
\cdot & \quad \cdot \quad \cdot \\
b_{n+1} F_{-1}(m+n) + b_n F_0(m+n) + \dots + b_1 F_{n-1}(m+n) &= 0 \\
\cdot & \quad \cdot \quad \cdot \\
\cdot & \quad \cdot \quad \cdot \\
\cdot & \quad \cdot \quad \cdot \\
b_s F_{-1}(m+s-1) + b_{s-1} F_0(m+s-1) + \dots + b_{s-n} F_{n-1}(m+s-1) &= 0
\end{aligned} \right\} \quad (80)$$

The first equation of Eqs. 80 requires that

$$F_{-1}(m-1) = 0$$

Since F_{-1} is a polynomial of order $(n+1)$, it has $(n+1)$ roots. Thus there are $(n+1)$ possible values for the constant m .

From the first equation of Eqs. 65, we see that

$$F_{-1}(m-1) = j(m-1)(m) Q(m-1) \mu^2$$

where Q is the denominator polynomial of the admittance (Eq. 60). Therefore, we can proceed with our general discussion by choosing m equal to zero or one. If we choose m equal to zero, $F_1(m)$ is zero as well as $F_1(m-1)$. Consequently, we are unable to evaluate b_1 from the second equation of Eqs. 80. Therefore, we choose m equal to one. With this choice of m , the evaluation of the b_i from Eqs. 80 proceeds exactly as the evaluation of the a_i from Eqs. 72.

Since m is an integer, the Γ -functions drop out of the solution. Therefore

$$U(\omega) = \sum_{s=0}^{\infty} b_s(\omega)^{s+1} = \omega + b_1 \omega(\omega-1) + b_2 \omega(\omega-1)(\omega-2) + \dots \quad (81)$$

For this series the ratio of the s^{th} term to the $(s-1)^{\text{th}}$ term is

$$\frac{t_s}{t_{s-1}} = \frac{b_s(\omega-s)}{b_{s-1}}$$

Let

$$T = \lim_{s \rightarrow \infty} \frac{t_s}{t_{s-1}} = \lim_{s \rightarrow \infty} \frac{-b_s s}{b_{s-1}}$$

Then, by the same procedure used in connection with the series of the first kind, we find that for large values of s the recursion relation, Eq. 80, can be written

$$(T+1)^{n-1} \left(T + \frac{2\mu + C_0}{\mu} \right) = 0$$

For the series of the second kind (as with the first kind) we can investigate convergence for all possible admittances by studying only two cases. For the first case, we require that

$$|T| = \left| \frac{2\mu + C_0}{\mu} \right| < 1 \quad (82)$$

Since Eq. 82 is the reciprocal of Eq. 77, the case 1 series of the second kind converges when the case 1 series of the first kind diverges, and vice versa.

For the second case, we cannot use Weierstrass' criterion directly, because T is equal to minus one. Therefore we examine the series

$$\sum_{s=0}^{\infty} \tau_s = \sum_{s=0}^{\infty} (-1)^s t_s$$

By applying the criterion to this series we see that $U(\omega)$, Eq. 81, is well defined for ω greater than zero.

Once a convergent solution has been found, we use Eq. 62 to find the complementary solution to the original difference equation. Since the series of the second kind can always be found with the use of μ_1 , we can write

$$V_c(\omega) = \mu_1^\omega U(\omega) \quad (83)$$

This solution is most probably V_1 in the voltage expression (Eq. 39). However, we can not be absolutely sure from the general series form that $U(\omega)$ does not behave like $\mu_2^{2\omega}$ and that the solution is really V_2 .

5.4 SOLUTION IN THE ENTIRE PLANE

With solutions in series of both the first and second kinds there is the possibility that the series converges only in a half-plane on the right. To extend the solution to the entire plane, we use the original homogeneous equation. For example, suppose we have a solution, V_c , that is meromorphic for $\omega > b$, and unknown for $\omega \leq b$. Let us

write the homogeneous difference equation Eq. 59 in the form

$$V_c(\omega-1) = -V_c(\omega+1) - \frac{Y(\omega)}{j\omega} V_c(\omega) \quad (84)$$

Clearly, this equation evaluates the function $V_c(x)$ for $(b-1) < x \leq b$ as the sum of two known meromorphic functions. By continuing the process we can find the function V_c in the entire plane.

Both of the series forms Eqs. 74 and 81 are analytic in the right half-plane. For Eq. 81 the analyticity is obvious. We can see that Eq. 74 is analytic in the right half-plane if we rewrite it in the form

$$U(\omega) = \Gamma(\omega+1) \sum_{s=0}^{\infty} \frac{a_s}{\Gamma(\omega+1-k+s)} \quad (85)$$

The reciprocal Γ -function is analytic everywhere, and the Γ -function has poles only at zero and all negative integers. Thus $U(\omega)$ has poles at all integer points on the real axis in the left half-plane, but it is analytic everywhere else. Since multiplication by μ^ω does not disturb the analyticity, both forms of solution fit the discussion of the preceding paragraph.

As we extend the solution $V_c(\omega)$ to the left by Eq. 84 it remains analytic until we evaluate $V_c(\omega-1)$ at a point where $Y(\omega)/j\omega$ has a pole. If $V_c(\omega)$ has a zero, $V_c(\omega-1)$ is analytic, and we proceed. However, if $V_c(\omega)$ is nonzero, then $V_c(\omega-1)$ will also have a pole. As we continue evaluating V_c to the left, the pole will propagate to all points congruent with the pole of $Y(\omega)/j\omega$.

One situation in which such a string of poles occurs is found in case 1 of the series of the first kind. Then Eq. 85 with its left half-plane poles applies in the entire plane. There may well be other situations like this when we extend a case 2 solution in either series form. However, these poles can be eliminated by utilizing one of the properties of the complementary solution discussed in section 2.3. That is, if $V_c(\omega)$ is a complementary solution, and $p(\omega)$ is a period function, then $p(\omega) V_c(\omega)$ is also a complementary solution. If we choose a $p(\omega)$ that has zeros at the points where the original $V_c(\omega)$ has poles, we get a new complementary solution that is an entire function. When $U(\omega)$ is given by Eq. 85, $p(\omega)$ can be chosen as $\sin \omega$.

Consequently, for the homogeneous difference equation (Eq. 59), a solution, V_c , that is an entire function can always be found.

5.5 THE COMPLETE EQUATION WITH TRANSIENT INPUT

Boole's method of symbolic operators can also be used to solve the complete equation for the parametric amplifier with transient inputs.

In section 3.1 we saw that the general transient case could be analyzed if we could solve the difference equation (Eq. 23) with

$$I(\omega) = e^{-j\tau} \frac{E(\omega)}{F(\omega)} \quad (86)$$

where E and F are polynomials.

For definiteness, let us look at the second-order difference equation with input $I(\omega)$, Eq. 86.

$$e^{-j\tau} \frac{E(\omega)}{F(\omega)} = j\omega V(\omega-1) + \frac{P(\omega)}{Q(\omega)} V(\omega) + j\omega V(\omega+1)$$

Multiplying this equation by $F(\omega) Q(\omega)$ gives

$$e^{-j\tau} E(\omega) Q(\omega) = j\omega Q(\omega) F(\omega) V(\omega-1) + P(\omega) F(\omega) V(\omega) + j\omega Q(\omega) F(\omega) V(\omega+1) \quad (87)$$

The left-hand side of this equation is an entire function, and therefore it can most probably be expanded in a factorial series, at least in a half-plane (16). This series can be written as powers of ρ .

The right-hand side of Eq. 87 can be reduced to operational form, just as the homogeneous equation was reduced. Then a series of the same kind as that used for the left-hand side of Eq. 87 can be assumed for $V(\omega)$. The evaluation of the coefficients in the assumed series then proceeds essentially as it did for the homogeneous equation. Now, however, the recursion relations similar to Eqs. 72 and 80 have constants on the right instead of zeros. Consequently, the coefficient a_0 in the series for $V(\omega)$ is not arbitrary.

VI. A PARAMETRIC AMPLIFIER WITH GAIN

Throughout this report we have used the term "parametric amplifier" to designate a class of electrical networks with one or more periodic parameters. However, we have not yet shown that such a circuit could amplify a signal. In the literature on parametric amplifiers we find four distinct reasons given to justify the claim that such a device can amplify. Essentially, these arguments may be summarized, as follows:

(i) The second-order differential equation with periodic coefficients (Mathieu's equation (17)), has unstable solutions, and thus a periodic circuit element can supply electric energy to a network.

(ii) The power-flow relations for a nonlinear reactance (Manley-Rowe relations (18)) show that a nonlinear reactance can convert energy from a pumping source to a signal at a different frequency, provided that the associated linear network allows power flow at only a few frequencies.

(iii) Bolle's method (see Appendix B) shows that a network with one period reactance and two ideal filters can amplify.

(iv) Parametric amplifiers have been built and they work.

Of these four arguments, only the last is very satisfying. We shall proceed to show that a sinusoidal capacitance in a particular realizable electric circuit can have infinite gain. The network that we select is prompted by the devices that have been built. Thus we assume a passive admittance, $Y(\omega)$, which has a resonance at the idler frequency; that is, the difference between the signal and pump frequencies. As we shall see, the idler resonance is sufficient for amplification, the other circuit properties need not be specified.

Consider the normalized parametric amplifier shown in Fig. 11. The admittance

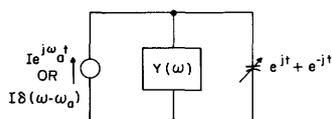


Fig. 11. Normalized parametric amplifier.

function $Y(\omega)$ is a positive real admittance, subject to two further restrictions. The first is realizability:

$$\lim_{\omega \rightarrow \infty} \frac{Y(\omega)}{j\omega} = C_0 > 2$$

The second is the presence of the idler:

$$Y(\omega_b) = 0 \quad \text{for } |1 - \omega_a| = \omega_b < 1$$

The difference equation for this amplifier is

$$I \delta(\omega - \omega_a) = j\omega V(\omega - 1) + Y(\omega) V(\omega) + j\omega V(\omega + 1) \quad (88)$$

In section 3.3 we found (Eq. 41) that the driving-point impedance as seen by the source at frequency ω_a is

$$Z_{in}(\omega_a) = \frac{V_1(\omega_a) V_2(\omega_a)}{j\omega_a D(\omega_a)} \quad (89)$$

Here V_1 and V_2 are complementary solutions to the homogeneous difference equation for the amplifier, and $D(\omega_a)$ is their determinant. Furthermore, we found that

$$V_1(\omega) = V_2^*(-\omega_a) \quad (90)$$

$$D(\omega_a) = D^*(-\omega_a) \quad (91)$$

and

$$D(\omega_a \pm 1) = D(\omega_a) \quad (92)$$

When we examine the circuit of Fig. 11 with excitation first at $\omega_a = 1 + \omega_b$, and then at $\omega_a = 1 - \omega_b$, we shall find that the real parts of these two impedances have opposite signs. Therefore, at one of the two frequencies the network appears as a negative resistance to the source. We shall discuss the implications of this after the proof has been given.

Let us write the homogeneous part of Eq. 88 in the form

$$V_c(\omega - 1) = -\frac{Y(\omega)}{j\omega} V_c(\omega) - V_c(\omega + 1) \quad (93)$$

Then since $Y(\omega_b)$ is zero, we have

$$V_1(\omega_b - 1) = -V_1(\omega_b + 1) \quad (94)$$

and

$$V_2(\omega_b - 1) = -V_2(\omega_b + 1) \quad (95)$$

Now

$$\begin{aligned} Z_{in}(\omega_b + 1) &= \left[\frac{V_1(\omega_b + 1) V_2(\omega_b + 1)}{j(\omega_b + 1) D(\omega_b + 1)} \right] \\ &= \left[\frac{Ae^{j\alpha} B e^{j\beta}}{C e^{j\pi/2} D e^{j\gamma}} \right] = \left[\frac{AB}{CD} \right] e^{j\left(\alpha + \beta - \gamma - \frac{\pi}{2}\right)} \end{aligned}$$

By using Eqs. 90-93, we find

$$\begin{aligned}
Z_{\text{in}}(1-\omega_b) &= \left[\frac{V_1(1-\omega_b) V_2(1-\omega_b)}{j(1-\omega_b) D(1-\omega_b)} \right] \\
&= \left[\frac{V_2^*(\omega_b-1) V_1^*(\omega_b-1)}{j(1-\omega_b) D^*(\omega_b-1)} \right] \\
&= \left[\frac{V_2^*(\omega_b+1) V_1^*(\omega_b+1)}{j(1-\omega_b) D^*(\omega_b+1)} \right] \\
&= \left[\frac{Ae^{-j\alpha} Be^{-j\beta}}{Ee^{j\pi/2} De^{-j\gamma}} \right] = \left[\frac{AB}{ED} \right] e^{-j\left(\alpha+\beta-\gamma+\frac{\pi}{2}\right)}
\end{aligned}$$

Thus if we let $\lambda = \alpha + \beta - \gamma$, a real constant, and $H = (AB)/D$, a positive constant, we have

$$R_e[Z_{\text{in}}(1+\omega_b)] = \frac{H}{1+\omega_b} \sin \lambda$$

and

$$R_e[Z_{\text{in}}(1-\omega_b)] = -\frac{H}{1-\omega_b} \sin \lambda$$

That is,

$$(1+\omega_b) R_e[Z_{\text{in}}(1+\omega_b)] = -(1-\omega_b) R_e[Z_{\text{in}}(1-\omega_b)] \quad (96)$$

From the derivation of Eq. 96 there is no way of knowing at which of the two frequencies, $(1-\omega_b)$ or $(1+\omega_b)$, the negative input resistance occurs. For all parametric amplifiers for which conclusive data have been published amplification occurs when the signal is below the pumping frequency. In those circuits in which the admittance function $Y(\omega)$ is almost the admittance of the parasitic capacitance for ω above (ω_b+1) , we can use an analytical plausibility argument to show that the negative resistance probably occurs at $(1-\omega_b)$. The argument goes as follows. At very high frequencies the input impedance (Eq. 89) is essentially the input impedance for a capacitive circuit (see Appendix A). This impedance is purely reactive and well behaved. As we move to lower frequencies, finding V_1 and V_2 by Eq. 93, we do not expect any strange behavior to occur until after the point where $Y(\omega)$ goes through some gyrations. Thus we do not expect a negative input impedance to occur at frequencies above the zero of $Y(\omega)$.

Thus far, our analysis has shown that a network with an infinite-Q resonance at a frequency ω_b is a stable oscillator at frequency $(1-\omega_b)$ or $(1+\omega_b)$. Since an infinite-Q network is about as hard to realize as an ideal filter, our argument is still no more convincing than arguments (ii) and (iii) above. However, those arguments break down

completely when the ideal-filter restriction is relaxed. On the other hand, the present argument allows us to lower the Q and keep some of the gain.

To show that infinite Q is not required, we examine the components of the input impedance (Eq. 89). In section 5.4 we pointed out that solutions V_1 and V_2 that are analytic everywhere can be found. Furthermore, from the process by which these solutions were generated we see that the solutions vary continuously as the coefficients in the admittance are varied. Thus the product $V_1(\omega_a) V_2(\omega_a)$ varies continuously as the zero is moved away from the axis. The determinant $D(\omega)$ also varies continuously as the Q is lowered, for D is constructed from the V_c 's. Moreover, a $Y(\omega)$ can be found for which $D(\omega_a)$ is nonzero. This is obvious from Cassoratti's theorem (19), which, for our amplifier, states that $D(\omega)$ is nonzero everywhere, except possibly at points congruent with the singularities of $Y(\omega)/j\omega$. Thus, as the Q of the idler is decreased from infinity, $Z_{in}(\omega_a)$ is continuous. Therefore, as the zero moves from the axis out into the plane, the amplifier varies continuously from a stable oscillator, to an infinite gain amplifier, to a stable amplifier, and then to a strictly passive device.

The preceding discussion implies one very important practical application for parametric amplifiers. It appears that the frequency characteristics of the amplifier are determined primarily by the idler circuit. Thus for a fixed idler circuit we can vary the pumping frequency, and thus vary the signal frequency while we are keeping the band-pass constant. Such a device certainly seems easier to construct than good tunable filters. This property, as well as the many other useful properties of parametric amplifiers found in published works, warrants more investigation, both analytical and experimental.

Acknowledgment

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APPENDIX A. THE ONE ELEMENT KIND NETWORK

One class of parametric circuits that can be handled without the complexity of Boole's method is the one element kind circuit. By one element kind we mean a circuit all of whose elements, both constant and time-variant, are of the same kind; that is, all resistances, all capacitances, or all inductances. The simplification occurs because the appropriate difference equation is a constant-coefficient equation. Besides that of Milne-Thomson, who devotes a whole chapter to the constant-coefficient case, there are popular applied mathematics texts (20) that discuss these equations.

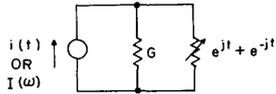


Fig. A-1. Normalized resistive parametric circuit.

In practical construction one element kind networks are most likely to appear as resistive networks. Examples include circuits containing motor-driven potentiometers and resistive diode mixers. Consequently, we shall consider a resistive network that contains a sinusoidal resistance. The current-voltage relations for resistive networks are identical with the charge-voltage relations for capacitive networks and with the flux linkage-current relations for inductive networks. Thus the analysis here applies directly to the other single element kind networks.

Consider the circuit of Fig. A-1. The current-voltage equation for this circuit is

$$I(\omega) = V(\omega+1) + GV(\omega) + V(\omega-1) \quad (\text{A-1})$$

If the circuit is to be realizable we must require G to be greater than two.

To get the complementary solutions to the homogeneous part of Eq. A-1, we proceed exactly as in section 4.2. That is, we assume that

$$V_c(\omega) = \mu^\omega$$

Substitution of this solution in the homogeneous part of Eq. A-1 gives

$$0 = \mu^{\omega+1} + G\mu^\omega + \mu^{\omega-1}$$

Multiplying by $\mu^{-\omega+1}$ gives the algebraic equation in μ :

$$0 = \mu^2 + G\mu + 1$$

This equation is satisfied if we choose

$$\mu = -\frac{G \pm (G^2 - 4)^{1/2}}{2}$$

For each of the two values of μ , one with the plus sign and one with the minus sign, we obtain a complementary solution. To keep the notation consistent with that of

Sections III, IV, and V, we use the following:

$$|\mu| = \frac{G + (G^2 - 4)^{1/2}}{2} \quad (\text{A-2})$$

$$V_1(\omega) = e^{j\pi\omega} |\mu|^\omega \quad (\text{A-3a})$$

$$V_2(\omega) = V_1^*(-\omega) = e^{j\pi\omega} |\mu|^{-\omega} \quad (\text{A-3b})$$

$$D(\omega) = \begin{vmatrix} V_1(\omega) & V_2(\omega) \\ V_1(\omega+1) & V_2(\omega+1) \end{vmatrix} = e^{j2\pi\omega} (G^2 - 4)^{1/2} \quad (\text{A-4})$$

Since G is greater than two, μ is real and negative, and $|\mu|$ is greater than one.

To analyze the circuit of Fig. A-1 for the steady-state behavior we consider $I(\omega) = I \delta(\omega - \omega_a)$, where the amplitude, I , on the right is, in general, complex. The steady-state voltage expression (Eq. 39) applies to the present case if we recall that the current, $I(\omega)$, in a resistive circuit corresponds to the charge $I(\omega)/j\omega$ in the capacitive circuit. Thus the voltage is

$$\begin{aligned} V(\omega, I, \omega_a) &= \frac{I}{D(\omega_a)} \left\{ \begin{aligned} &V_1(\omega_a) V_2(\omega_a) \delta(\omega - \omega_a) + \sum_{s=1}^{\infty} [V_1(\omega_a) V_2(\omega_a + s) \delta(\omega - \omega_a - s) \\ &+ V_2(\omega_a) V_1(\omega_a - s) \delta(\omega - \omega_a + s)] \end{aligned} \right\} \\ &= \frac{I}{(G^2 - 4)^{1/2}} \left\{ \delta(\omega - \omega_a) + \sum_{s=1}^{\infty} (-1)^s |\mu|^{-s} [\delta(\omega - \omega_a - s) + \delta(\omega - \omega_a + s)] \right\} \quad (\text{A-5}) \end{aligned}$$

For the simple circuit of Fig. A-1 we could solve for the voltage in the time domain by elementary algebraic means. Let us carry out the solution and compare the results with Eq. A-5. By Ohm's law,

$$v(t) = \frac{i(t)}{G + e^{jt} + e^{-jt}} = \frac{i(t)}{G} \left[\frac{1}{1 + \frac{e^{jt} + e^{-jt}}{G}} \right]$$

Since G is greater than two, $(e^{jt} + e^{-jt})/G$ is less than one, and we can expand the term in the bracket in a power series. We obtain

$$v(t) = \frac{i(t)}{G} \left[1 - \frac{e^{jt} + e^{-jt}}{G} + \frac{(e^{jt} + e^{-jt})^2}{G} - \dots \right] \quad (\text{A-6})$$

Now let us look at the case in which $i(t)$ is an exponential of complex amplitude, I , at frequency ω_a . That is,

$$i(t) = I e^{j\omega_a t}$$

When the exponential $i(t)$ is inserted in Eq. A-6, the binomials in $(e^{jt} + e^{-jt})$ are expanded, and the terms regrouped frequency by frequency, the resulting form is

$$v(t) = I \sum_{-\infty}^{\infty} A_s e^{j(\omega_a + s)t} \quad (\text{A-7})$$

The A_s in this expression are expressed as infinite series, the terms of which involve binomial coefficients and powers of $1/G$.

To compare Eq. A-7 with our frequency domain result, Eq. A-5, we note, first, that the functional forms of the two equations constitute a Fourier-transform pair. If the two equations actually form such a pair, we have

$$A_s = \frac{1}{G^2 - 4} (-1)^s |\mu|^{-s} \quad (\text{A-8})$$

Let us check Eq. A-8 for A_0 . The terms that contribute to A_0 are the constant terms in the bracket of Eq. A-6 after expansion of the binomials. There is one such term in each of the even powers of $(e^{jt} + e^{-jt})$. Thus

$$A_0 = \frac{1}{G} + \frac{2}{G^3} + \frac{6}{G^5} + \dots + \frac{1}{G^{2n+1}} \binom{2n}{n} \dots$$

But this is exactly the expansion of $(G^2 - 4)^{-1/2}$.

Before leaving the sinusoidal steady state, let us examine the input impedance. Except in the degenerate case, when ω_a is an integer or half-integer,

$$Z_{in}(\omega_a) = \frac{V(\omega_a, I, \omega_a)}{I} = \frac{1}{(G^2 - 4)^{1/2}}$$

Thus for our resistive circuit, Z_{in} is real as long as the circuit is realizable. For large shunt constant conductance, G , the impedance approaches $1/G$, as we would expect. As G decreases toward two, the impedance increases toward infinity. The additional power that this large impedance takes from the source is dissipated at the other frequencies.

Now let us look at the degenerate case, when ω_a is $1/2$. For this case,

$$\begin{aligned} Z_{in}\left(\frac{1}{2}\right) &= \frac{V(1/2, I, 1/2) + V^*(-1/2, I, 1/2)}{I} \\ &= \frac{1}{(G^2 - 4)^{1/2}} \left[1 - \frac{I^*}{I|\mu|} \right] \end{aligned}$$

If we write the complex I as $|I| e^{j\theta}$, we get

$$Z_{in}\left(\frac{1}{2}\right) = \frac{1}{(G^2-4)^{1/2}} \left[1 - \frac{G - (G^2-4)^{1/2}}{2} e^{j2\theta} \right]$$

Since $|\mu|$ is greater than one, $\text{Re} \left[Z_{in}\left(\frac{1}{2}\right) \right]$ is still positive, but Z_{in} now has an imaginary part. Furthermore, $Z_{in}\left(\frac{1}{2}\right)$ is a function of the phase of the applied signal with respect to the time variation of the parameter.

In the one element kind reactive circuit, the same analysis shows that the input impedance in the nondegenerate case is purely reactive. The sign of the reactance is the same as that of the passive circuit. In the degenerate case, a resistive term appears. This resistive term can be positive or negative, as determined by the phase.

Next, let us look at the transient analysis of our one element kind parametric circuit. As an example, suppose we find the time-variant impulse response. For a unit impulse applied at time τ , the appropriate frequency-domain forcing function is

$$I(\omega) = e^{-j\tau\omega}$$

Many methods for finding a particular solution to the complete constant-coefficient equation have been published. Since we already have the solution to the homogeneous equation, and we have developed the variation of parameters method most of the way in section 2.1, we shall use the variation-of-parameters method. In section 2.1, we found a particular solution in the form

$$V(\omega) = A_1(\omega) V_1(\omega) + A_2(\omega) V_2(\omega) \quad (\text{A-9})$$

The functions A_1 and A_2 are determined by the first-order difference equations

$$\Delta A_1(\omega) = \frac{I(\omega) V_2(\omega)}{D(\omega)} = \frac{e^{-[\ln|\mu| + j(\tau+\pi)]\omega}}{(G^2-4)^{1/2}}$$

$$\Delta A_2(\omega) = -\frac{e^{[\ln|\mu| - j(\tau+\pi)]\omega}}{(G^2-4)^{1/2}}$$

The solution (21) to the equation $\Delta A(\omega) = k^\omega$ is

$$A(\omega) = \frac{k^\omega}{k-1}$$

To show that this is true, we examine the equation

$$\Delta A(\omega) = A(\omega+1) - A(\omega) = \frac{k^{\omega+1} - k^\omega}{k-1} = k^\omega$$

Thus the voltage, Eq. A-9, becomes

$$\begin{aligned}
V(\omega) &= \frac{e^{-j\tau\omega}}{(G^2-4)^{1/2}} \left[\frac{1}{-e^{-\ln|\mu|-j\tau} - 1} + \frac{1}{e^{\ln|\mu|-j\tau} + 1} \right] \\
&= \frac{e^{-j\tau\omega}}{(G^2-4)^{1/2}} \left[\frac{1}{|\mu| e^{-j\tau} + 1} - \frac{1}{\frac{e^{-j\tau}}{|\mu|} + 1} \right] \\
&= \frac{e^{-j\tau\omega}}{G + e^{j\tau} + e^{-j\tau}}
\end{aligned}$$

This is obviously the right answer.

APPENDIX B. BOLLE'S METHOD

In most of the literature on parametric amplifiers an approximate method for finding the amplitudes of the voltages at the frequencies of interest is used. This method was first discussed in detail by Bolle (22, 23). Recently Duinker (24) has extended this method to include all types of periodic variable elements in a general n-mesh or n-node network. The Duinker paper also contains a thorough discussion of linearization, which is somewhat different from that given in section 1.2, and a complete bibliography. In this section we shall show how Bolle's method fits in with our difference-equation approach. A more thorough discussion of Bolle's method, with an example, has been given elsewhere (25).

The complete difference equation for the normalized parametric amplifier with sinusoidal capacitance variation and sinusoidal input is

$$I \delta(\omega - \omega_a) = j\omega V(\omega+1) + Y(\omega) V(\omega) + j\omega V(\omega-1) \quad (\text{B-1})$$

In section 3.2 we found that the particular solution to this equation is given in the form

$$V(\omega) = \sum_{-\infty}^{\infty} V_s \delta(\omega - \omega_a - s) \quad (\text{B-2})$$

where the V_s are complex constants.

For Bolle's method we substitute the formal solution (Eq. B-2) in Eq. (B-1) and examine the resulting equation at each of the frequencies where impulses occur. For the equation to hold for all frequencies, the amplitudes of the impulses at each frequency where impulses occur must satisfy the equality. Thus the difference equation reduces to the infinite set of coupled algebraic equations.

$$\begin{aligned} I &= j\omega_a V_1 + Y(\omega_a) V_0 + j\omega_a V_{-1} \\ 0 &= j(\omega_a + s) V_{s+1} + Y(\omega_a + s) V_s + j(\omega_a + s) V_{s-1} \quad \text{for } s \neq 0 \end{aligned} \quad (\text{B-3})$$

Since each of these equations contains three of the unknown voltages, we must obtain two of the voltages by some other means. Then we can solve for all the voltages with Eqs. B-3. The usual assumption that is made is that the tuned circuits that make up the admittance $Y(\omega)$ are good approximations to ideal filters. Then all except a finite number of V_s are zero, and we can solve for the nonzero voltages in the passbands of the filters.

APPENDIX C

EVALUATION OF THE CONSTANT k IN THE INVERSE FACTORIAL-SERIES COMPLEMENTARY SOLUTION

In section 5.3 we mentioned that the constant k for the inverse factorial series is to be evaluated from Eqs. 72. The first of these equations requires that

$$F_{n-1}^{(k+n-1)} = 0$$

The first-order polynomial F_{n-1} is given by

$$F_{n-1} = \frac{j\mu^2}{n!} \Delta_{-1}^n [O] + \frac{\mu}{(n-1)!} \Delta_{-1}^{n-1} [P] + \frac{j}{(n-2)!} \Delta_{-1}^{n-2} [Q] \quad (C-1)$$

The polynomial, O , is of order $(n+1)$; P is of order n ; and Q is of order $(n-1)$. Therefore, each of the operations with Δ_{-1} is of the form

$$\Delta_{-1}^{m-1} (\text{polynomial of order } m)$$

In connection with the evaluation of the constant, μ , we also saw that

$$\Delta_{-1}^r X^s = \begin{cases} 0 & \text{if } s < r \\ m! & \text{if } s = r \end{cases}$$

To evaluate F_{n-1} we must still investigate $\Delta_{-1}^{m-1} X^m$. Now

$$\begin{aligned} \Delta_{-1}^{m-1} X^m &= \Delta_{-1}^{m-2} \left[\Delta_{-1} X^m \right] = \Delta_{-1}^{m-2} \left[mX^{m-1} - \binom{m}{2} X^{m-2} + \text{terms of lower order} \right] \\ &= m \Delta_{-1}^{m-2} X^{m-1} - \frac{m(m-1)(m-2)!}{2} = m \Delta_{-1}^{m-2} X^{m-1} - \frac{m!}{2} \\ &= m(m-1) \Delta_{-1}^{m-3} X^{m-2} - \frac{m(m-1)(m-2)(m-3)!}{2} - \frac{m!}{2} \end{aligned}$$

Continuation of this process through $(m-1)$ operations yields

$$\Delta_{-1}^{m-1} X^m = m! X - \frac{(m-1)m!}{2}$$

From the definitions of the polynomials O , P , and Q in terms of the admittance $Y(\omega)$, Eq. 60, we obtain

$$\Delta_{-1}^n O(\omega) = \Delta_{-1}^n \left[j^{n-1} \omega^{n+1} + (j^{n-1} + B_2 j^{n-2}) \omega^n \right]$$

$$\Delta_{-1}^{n-1} P(\omega) = \Delta_{-1}^{n-1} \left[C_0 (j^n \omega^n + A_1 j^{n-1} \omega^{n-1}) \right]$$

$$\Delta_{-1}^{n-2} Q(\omega) = \Delta_{-1}^{n-2} \left[j^{n-1} \omega^{n-1} + B_2 j^{n-2} \omega^{n-2} \right]$$

Thus

$$\begin{aligned} F_{n-1}(\omega) &= j^n \left\{ \mu^2 \left[(n+1)\omega - \frac{(n+1)n}{2} + 1 - jB_2 \right] + C_0 \mu \left[n\omega - \frac{n(n-1)}{2} - jA_1 \right] \right. \\ &\quad \left. + (n-1)\omega - \frac{(n-1)(n-2)}{2} - jB_2 \right\} \\ &= j^n \left\{ (\mu^2 + C_0 \mu + 1) \left(n\omega - \frac{n^2}{2} \right) + \omega(\mu^2 - 1) - \frac{n}{2} (\mu^2 - C_0 \mu - 3) \right. \\ &\quad \left. + \mu^2 - 1 - j(\mu^2 + 1) B_2 - jC_0 \mu A_1 \right\} \end{aligned}$$

But since $\mu^2 + C_0 \mu + 1 = 0$, we have

$$F_{n-1}(\omega) = j^n \left[(\omega + 1 - n)(\mu^2 - 1) - j(\mu^2 + 1)(B_2 - A_1) \right]$$

Thus if we set $F_{n-1}(k+n-1)$ equal to zero, we obtain

$$(k+n-1+1-n)(\mu^2 - 1) - j(\mu^2 + 1)(B_2 - A_1) = 0$$

Therefore

$$k = j \frac{\mu^2 + 1}{\mu^2 - 1} (B_2 - A_1)$$

APPENDIX D

FACTORIZATION OF THE POLYNOMIAL (EQ. 76)

We wish to show that the polynomial $[X^n \phi_{n-1} - X^{n-1} \phi_{n-2} + \dots + (-1)^{n-1} \phi_{-1}]$, where the ϕ_i are given by

$$\phi_i = j^n \left[\mu^2 \binom{n+1}{i+1} + C_o \mu \binom{n}{i} + \binom{n-1}{i-1} \right]$$

can be factored in the form $(X-1)^{n-1} \left(X - \frac{\mu}{2\mu + C_o} \right)$. Since j^n appears in all the ϕ_i we can divide it out. Then we proceed by induction.

For $n = 1$, we have

$$(2\mu^2 + C_o \mu) X - \mu^2 = \mu \left(X - \frac{\mu}{2\mu + C_o} \right)$$

For the induction step we must show that multiplying the polynomial for $n = h$ by $(X-1)$ gives the polynomial for $n = h + 1$. Thus we must show that

$$\phi_i^{h+1} = \phi_i^h + \phi_{i-1}^h$$

where the superscript indicates the value of n .

Now

$$\phi_i^h + \phi_{i-1}^h = j^n \left\{ \mu^2 \left[\binom{h+1}{i+1} + \binom{h+1}{i} \right] + C_o \mu \left[\binom{h}{i} + \binom{h}{i-1} \right] + \binom{h-1}{i-1} + \binom{h-1}{i-2} \right\}$$

But

$$\begin{aligned} \binom{h}{i} + \binom{h}{i-1} &= h! \left[\frac{1}{i!(h-i)!} + \frac{1}{(i-1)!(h-i+1)!} \right] \\ &= \frac{h!}{(i-1)!(h-i)!} \left[\frac{h-i+1+i}{i(h-1+1)} \right] = \frac{(h+1)!}{i!(h+1-i)!} = \binom{h+1}{i} \end{aligned}$$

Similarly,

$$\binom{h+1}{i+1} + \binom{h+1}{i} = \binom{h+2}{i+1}$$

and

$$\binom{h-1}{i-1} + \binom{h-1}{i-2} = \binom{h}{i-1}$$

Therefore

$$\phi_i^{h+1} = \phi_i^h + \phi_{i-1}^h$$

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