Second Order Error Correction in Quantum Computing

By

Sarah Sheldon

SUBMITTED TO THE DEPARTMENT OF NUCLEAR SCIENCE AND ENGINEERING

IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE OF

BACHELOR OF SCIENCE IN NUCLEAR SCIENCE AND ENGINEERING

AT THE

MASSACHUSETTS INSTITUTE OF TECHNOLOGY

JUNE 2008

©2008 Sarah Sheldon. All Rights Reserved.

The author hereby grants to MIT permission to reproduce and to distribute publicly

paper and electronic copies of this thesis document in whole or in part.

Signature of Author:______

Sarah Sheldon

Department of Nuclear Science and Engineering

May 8, 2008

Certified by:____________________

David G. Cory

Professor of Nuclear Science and Engineering

Thesis Supervisor

Accepted by:______________

David G. Cory

Professor of Nuclear Science and Engineering

Chairman, NSE Committee for Undergraduate Students

MASSACHUSETTS INSTITUTE OF TECHNOLOGY

JUL 24 2008

LIBRARIES

ARCHIVES

Second Order Error Correction in Quantum Computing

by

Sarah Sheldon

Submitted to the Department of Nuclear Science and Engineering on May 8, 2008 in Partial Fulfillment of the Requirements for the Degree of Bachelor of Science in Nuclear Science and Engineering

Abstract

Error correction codes are necessary for the development of reliable quantum computers. Such codes can prevent the lost of information from decoherence caused by external perturbations. This thesis evaluates a five qubit code for correcting second order bit-flip errors. The code consists of encoding, decoherence, decoding, and error correction steps. This work analyzes the proposed code using geometric algebra methods and examines the state of the system after each step in the process.

Thesis Supervisor: David G. Cory
Title: Professor of Nuclear Science and Engineering

1 Introduction

1.1 Motivation for Error Correcting Codes

The goal of quantum computing is to develop methods of computation that use binary quantum states to store information. A quantum computer could execute important algorithms exponentially faster than a classical computer. In order to take advantage of this increased efficiency, we must develop robust quantum computers that will not fail due to errors. One approach is to use reliable error correction codes.

In classical computers, one bit consists of a capacitor with two states: charged and uncharged. Millions of electrons flow through these capacitors, so errors in a few electrons do not impact the performance of the computer. Alternatively, one bit in quantum computing can be any coherent state in SU(2) (a system with two orthogonal eigenstates). Here we use spin-1/2 particles as the quantum bits, or qubits. The two eigenstates of spin-1/2 particles are spin up and spin down, which we label as $|0\rangle$ and $|1\rangle$. Because a qubit is composed of a single particle, an error in the particle is more significant than in the classical case. One possible error in a system made up of such qubits is the spin flip error, where the spin of the particle flips to the opposite spin due to decoherence caused by interactions with the environment.

By correcting the error we can recover the information lost during stochastic processes.

1.2 Background: First Order Error Correction

The code for correcting single bit flip errors (first order error correction) has already been analyzed experimentally (Cory, 1998). This code requires three qubits – one data qubit and two ancillae. The ancillae are spins prepared in a state that is correlated with the data spin (but they do not contain the data themselves). For first order error corrections, we allow either zero or one spin flips to occur, either in the data spin or in an ancilla spin. It has been demonstrated that under these errors the initial states remain orthogonal, and thus the original information may be recovered (Cory, 1998)(Sharf, 2000).

The circuit for first order error correction was tested in a liquid state nuclear magnetic resonance (NMR) experiment (Cory, 1998). The code for correcting bit-flip errors consists of the following steps: encoding, which correlates the ancillae spins with the data spin; decoherence (perhaps induced by magnetic field gradients);

decoding; and error correction.

Initially the data spin is in a superposition of $|0\rangle$ and $|1\rangle$ states, so the state is $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$. The ancillae are both prepared in state $|0\rangle$. Thus the state of the whole system is

$$|\Psi\rangle = (\alpha|0\rangle + \beta|1\rangle)|00\rangle$$

The encoding circuit is a series of Controlled NOT (c-NOT) gates, and it correlates the information from the data spin with the ancillae. The c-NOT gate flips the spin of an ancilla conditional on the state of the data spin (see Fig. 1). If the data spin is in state $|1\rangle$, the c-NOT gate flips the ancilla, otherwise it does nothing. After the encoding, the total state of the system is $\Psi = \alpha|000\rangle + \beta|111\rangle$. (Here $|111\rangle$ represents the tensor product of the three spins, $|1\rangle_{data} \otimes |1\rangle_{ancilla1} \otimes |1\rangle_{ancilla2}$.)

After time decoherence allows for up to one error to occur there are four possible states:

$$\alpha|000\rangle + \beta|111\rangle$$

$$\alpha|100\rangle + \beta|011\rangle$$

$$\alpha|010\rangle + \beta|101\rangle$$

$$\alpha|001\rangle + \beta|110\rangle$$

The decoding process uses another series of c-NOT gates to transform the system into a state in which the ancillae are separable from the data spin. The four states become:

$$\alpha|000\rangle + \beta|100\rangle = (\alpha|0\rangle + \beta|1\rangle)|00\rangle$$

$$\alpha|111\rangle + \beta|011\rangle = (\alpha|1\rangle + \beta|0\rangle)|11\rangle$$

$$\alpha|010\rangle + \beta|110\rangle = (\alpha|0\rangle + \beta|1\rangle)|10\rangle$$

$$\alpha|001\rangle + \beta|101\rangle = (\alpha|0\rangle + \beta|1\rangle)|01\rangle$$

We can determine whether or not an error occured in the data spin based on the state of the ancillae. For this case there has been an error in the data spin if the ancillae are in the state $|11\rangle$.

The final step in the 3 qubit circuit is to flip the spin of the data qubit in the state with the error by implementing a Toffoli gate. The Toffoli gate flips the data spin if the ancillae are in the state $|11\rangle$ so that the initial state, $|\psi\rangle$, of the data spin is recovered (Cory, 1998; Sharf, 2000).

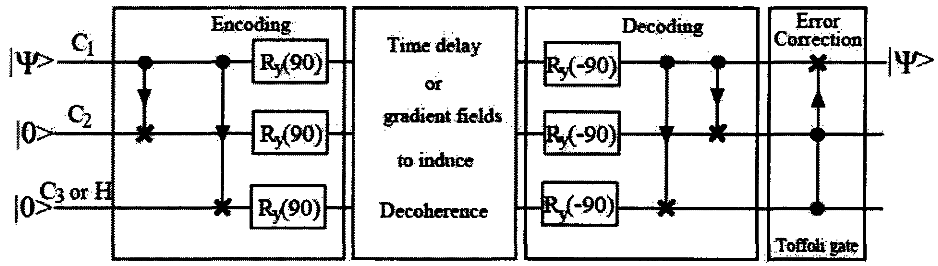

Figure 1: The circuit used for first order error correction. There is one data spin, $|\Psi\rangle$, and two ancillae correlated with the data spin. This figure is taken from (Cory, 1998).

If two spin flips occur, three qubits do not provide a large enough space to encode all possible errors in orthogonal states. For second order error correction in which we allow up to two spin flips, we need a five qubit system - one data spin and four ancillae correlated to the data spin.

2 The 5 Qubit Case

This thesis expands upon the work done in the three qubit case to investigate the possiblity of correcting second order errors in NMR. We need four ancilla spins to correct up to two errors among the data and ancilla spins. The code becomes more complicated with more spins, but the circuit must be as simple as possible to reduce the experimental difficulities associated with controlling a larger spin system.

Here we will establish the five qubit circuit and discuss the theoretical framework for second order error correction. This work could be tested using a liquid-state NMR experiment similar to the test of the three qubit case.

2.1 Encoding

In order to correct second order errors in quantum computing, we need one data spin and four ancillae. Again, we consider only spin flip errors. We begin with all ancillae in the $|0\rangle$ state and encode the data qubit by correlating the ancillae with the data spin. If the data spin is in a superposition of up and down states, then the initial

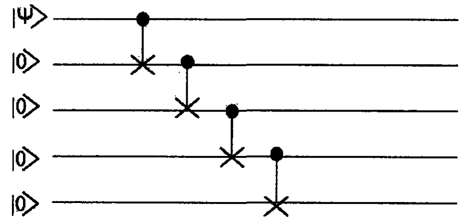

Figure 2: Encoding the data spin: we flip the spins of the ancillae conditional on the data spin $|\Psi\rangle$.

state of the system is:

$$|\Psi\rangle = \alpha\,|11111\rangle + \beta\,|00000\rangle$$

To encode the primary qubit we flip each ancilla spin if the data spin is in the state $|1\rangle$. In practice we can acheive this result by flipping each ancilla conditional on the previous spin. The encoding circuit is shown in Fig. (2). We can see from this diagram that the encoding circuit is a series of c-NOT gates which flip the spin of each qubit if the state of the previous qubit is in state $|1\rangle$. If the data spin is in a superposition of up and down states, then the initial state of the system is:

$$|\Psi\rangle = \alpha\,|11111\rangle + \beta\,|00000\rangle$$

2.2 Decoherence

If we allow up to two spin flips, then there are 16 possible states of the system after decoherence. In a physical system decoherence is due to time delays, which can be simiulated by magnetic field gradients in an NMR experiment.

After a time delay, either zero, one, or two errors have occurred. Of the sixteen possible states after decoherence, there are five states with an error in the data spin:

$$\alpha\,|01111\rangle + \beta\,|10000\rangle$$

$$\alpha\,|00111\rangle + \beta\,|11000\rangle$$

$$\alpha\,|01011\rangle + \beta\,|10100\rangle$$

$$\alpha\,|01101\rangle + \beta\,|10010\rangle$$

$$\alpha\,|01110\rangle + \beta\,|10001\rangle$$

2.3 Decoding

We can flip the ancillae again conditional on the data spin being in state $|1\rangle$. The data spin is then separable from the ancillae. For example, the states with a bit flip in the data spin become:

$$(\alpha |0\rangle + \beta |1\rangle) |1111\rangle$$
$$(\alpha |0\rangle + \beta |1\rangle) |0111\rangle$$
$$(\alpha |0\rangle + \beta |1\rangle) |1011\rangle$$
$$(\alpha |0\rangle + \beta |1\rangle) |1101\rangle$$
$$(\alpha |0\rangle + \beta |1\rangle) |1110\rangle$$

This decoding circuit is shown in Fig. (3).

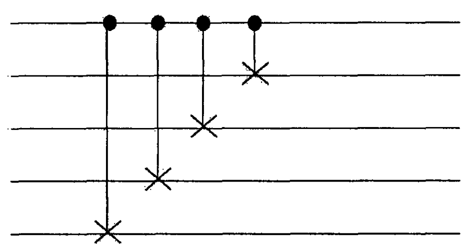

Figure 3: Basic decoding circuit: flip all ancillae condtional on the state of the data spin $|\Psi\rangle$.

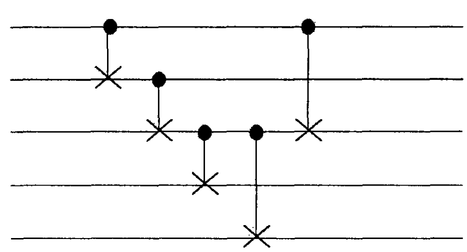

Figure 4: Alternative decoding method: instead of flipping all ancillae conditional on the data spin, we use this circuit to create an alternative set of states with the ancillae separable from the data spin.

The sixteen possible states of the system following decoding can be represented by the denisity matrix:

$$
\begin{pmatrix}
1 &&&&&&&&&&&&&&& \\
& 1 &&&&&&&&&&&&&& \\
&& 1 &&&&&&&&&&&&& \\
&&& 1 &&&&&&&&&&&& \\
&&&& 1 &&&&&&&&&&& \\
&&&&& 1 &&&&&&&&&& \\
&&&&&& 1 &&&&&&&&& \\
&&&&&&& \sigma_x &&&&&&&& \\
&&&&&&&& 1 &&&&&&& \\
&&&&&&&&& 1 &&&&&& \\
&&&&&&&&&& 1 &&&&& \\
&&&&&&&&&&& \sigma_x &&&& \\
&&&&&&&&&&&& 1 &&& \\
&&&&&&&&&&&&& \sigma_x && \\
&&&&&&&&&&&&&& \sigma_x & \\
&&&&&&&&&&&&&&& \sigma_x
\end{pmatrix}
$$

Each element is a 2×2 matrix that is the product the initial state of the primary qubit, $|\psi\rangle\langle\psi|$, with either the identity or, for states with errors in the primary qubit, the Pauli matrix, σ_x. .

The decoding circuit in Fig.(3), however, is difficult to implement experimentally. An alternative decoding circuit is shown in Fig. (4).

Using this second circuit, the states with errors in the data spin following decoding are:

$$
\begin{gathered}
(\alpha|0\rangle + \beta|1\rangle)|1011\rangle \\
(\alpha|0\rangle + \beta|1\rangle)|0100\rangle \\
(\alpha|0\rangle + \beta|1\rangle)|1100\rangle \\
(\alpha|0\rangle + \beta|1\rangle)|1001\rangle \\
(\alpha|0\rangle + \beta|1\rangle)|1010\rangle
\end{gathered}
$$

The resulting density matrix in this case is:

$$
\begin{pmatrix}
1 &&&&&&&&&&&&&&& \\
& 1 &&&&&&&&&&&&&& \\
&& 1 &&&&&&&&&&&&& \\
&&& 1 &&&&&&&&&&&& \\
&&&& \sigma_x &&&&&&&&&&& \\
&&&&& 1 &&&&&&&&&& \\
&&&&&& 1 &&&&&&&&& \\
&&&&&&& 1 &&&&&&&& \\
&&&&&&&& 1 &&&&&&& \\
&&&&&&&&& \sigma_x &&&&&& \\
&&&&&&&&&& \sigma_x &&&&& \\
&&&&&&&&&&& \sigma_x &&&& \\
&&&&&&&&&&&& \sigma_x &&& \\
&&&&&&&&&&&&& 1 && \\
&&&&&&&&&&&&&& 1 & \\
&&&&&&&&&&&&&&& 1
\end{pmatrix}
$$

2.4 Error Correction

If we use this second method of decoding, then we can apply these three operators to correct the error: $\sigma_x^1 \mathbf{E}_-^2 \mathbf{E}_+^3$, $\sigma_x^1 \mathbf{E}_+^4 \mathbf{E}_+^5$, and $\sigma_x^1 \mathbf{E}_+^2 \mathbf{E}_+^3 \mathbf{E}_+^4 \mathbf{E}_+^5$. Here, we have $\mathbf{E}_\pm = \frac{1}{2}(\mathbb{I} \pm \sigma_z)$.

Again, this circuit is difficult to implement experimentally. We can slightly improve on this error correction circuit by using the following set of operations instead: $\sigma_x^3 \mathbf{E}_+^4 \mathbf{E}_+^5$, $\sigma_x^1 \mathbf{E}_+^2 \mathbf{E}_+^3$, and $\sigma_x^1 \mathbf{E}_+^2 \mathbf{E}_+^3 \mathbf{E}_+^4 \mathbf{E}_+^5$. Note that these operations must be applied in the order that they are listed.

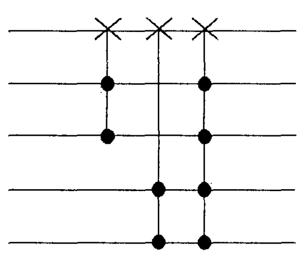

Figure 5: Circuit for correcting the spin flip error following the decoding circuit from Figure 3.

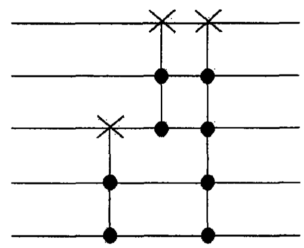

Figure 6: Alternative circuit for correcting the spin flip error following the decoding circuit from Figure 4.

Figures (5) and (6) depict these error correction circuits. In the second case, after the $\sigma_x^3 \mathbf{E}_+^4 \mathbf{E}_+^5$ operation the density matrix is:

$$\left(\begin{array}{cccccccc|cccccccc}
\sigma_x & & & & & & & & & & & & & & & \\
& 1 & & & & & & & & & & & & & & \\
& & 1 & & & & & & & & & & & & & \\
& & & 1 & & & & & & & & & & & & \\
& & & & 1 & & & & & & & & & & & \\
& & & & & 1 & & & & & & & & & & \\
& & & & & & 1 & & & & & & & & & \\
& & & & & & & 1 & & & & & & & & \\
\hline
& & & & & & & & \sigma_x & & & & & & & \\
& & & & & & & & & \sigma_x & & & & & & \\
& & & & & & & & & & \sigma_x & & & & & \\
& & & & & & & & & & & \sigma_x & & & & \\
& & & & & & & & & & & & 1 & & & \\
& & & & & & & & & & & & & 1 & & \\
& & & & & & & & & & & & & & 1 & \\
& & & & & & & & & & & & & & & 1
\end{array}\right)$$

We can see from this matrix that after the $\sigma_x^3 \mathbf{E}_+^4 \mathbf{E}_+^5$, we need only to flip the $\mathbf{E}_+^4 \mathbf{E}_-^5$ block and the $\mathbf{E}_+^2 \mathbf{E}_+^3 \mathbf{E}_+^4 \mathbf{E}_+^5$ element to obtain the identity. These two operations are indeed performed by $\sigma_x^1 \mathbf{E}_+^4 \mathbf{E}_+^5$ and $\sigma_x^1 \mathbf{E}_+^2 \mathbf{E}_+^3 \mathbf{E}_+^4 \mathbf{E}_+^5$.

The complete circuit using the second method of error correction is shown in Fig. (7).

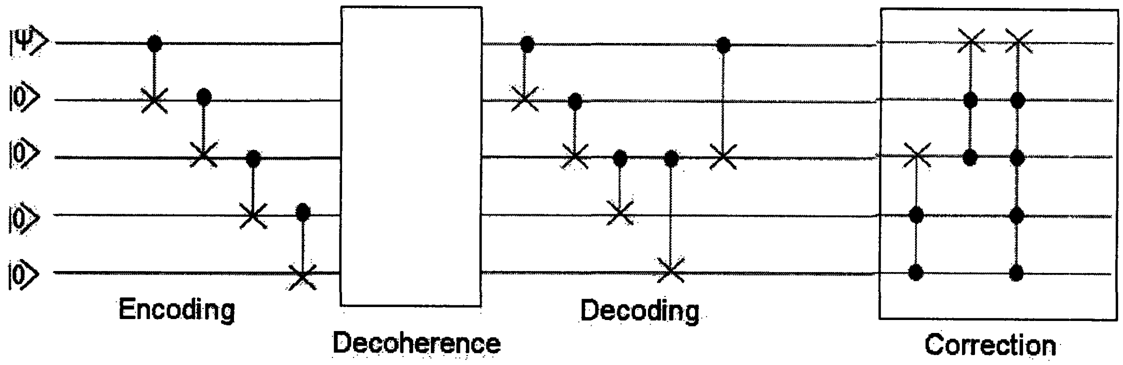

Figure 7: Complete error correction circuit.

3 Geometric Algebra Analysis

3.1 Operator Representation

We can write out the state of the five qubit system at each step in the code. Each gate has an operator representation consisting of combinations of projection operators, $\mathbf{E}^i_{\pm}$, and Pauli matrices acting on each of the five qubits.

We begin by writing out the operator form of the c-NOT gate. The c-NOT gate, $\mathbf{S}^{\mathbf{i}|\mathbf{j}}$, flips the spin of qubit i if qubit j is in state $|1\rangle$. We can, therefore, write the c-NOT gate as follows:

$$\mathbf{S}^{\mathbf{i}|\mathbf{j}} \equiv \exp(i\pi \mathbf{E}^j_-(1-2\mathbf{I}^i_x)/2) \tag{1}$$
$$= 1 - \mathbf{E}^j_-(1-2\mathbf{I}^i_x) \tag{2}$$

Here $\mathbf{I}^i_x = \frac{1}{2}\sigma^i_x$, where σ^i_x is the Pauli matrix acting on qubit i.

Similarly, the c^2-NOT gate (or Toffoli gate) flips quibit i if the state of qubits j and k is $|11\rangle$. We write the c^2-NOT gate:

$$\mathbf{T}^{\mathbf{i}|\mathbf{jk}} \equiv \exp(i\pi(1-2\mathbf{I}^i_x)\mathbf{E}^j_-\mathbf{E}^k_-) \tag{3}$$
$$= 1 - (1-2\mathbf{I}^i_x)\mathbf{E}^j_-\mathbf{E}^k_- \tag{4}$$

The final gate in the error correction code is the c^4-NOT gate, which flips quibit 1 if the state of all the ancillae is $|1111\rangle$. The c^4-NOT gate is:

$$\mathbf{V}^{\mathbf{1}|\mathbf{2345}} \equiv \exp(i\pi(1-2\mathbf{I}^1_x)\mathbf{E}^2_-\mathbf{E}^3_-\mathbf{E}^4_-\mathbf{E}^5_-) \tag{5}$$
$$= 1 - (1-2\mathbf{I}^1_x)\mathbf{E}^2_-\mathbf{E}^3_-\mathbf{E}^4_-\mathbf{E}^5_- \tag{6}$$

In all of these operations, it is not necessary for the condition for flipping a qubit to be another qubit is in the state $|1\rangle$. In cases where we want to flip a qubit if another is in state $|0\rangle$, then we substitute $\mathbf{E}_+^j$'s for the $\mathbf{E}_-^j$ terms.

3.2 Encoding

The density matrix of the initial state of the data qubit (prior to encoding) is

$$\rho_A' \equiv (\alpha\,|0\rangle + \beta\,|1\rangle)(\widetilde{\alpha}\,\langle 0| + \widetilde{\beta}\,\langle 1|) \tag{7}$$
$$= (\alpha + 2\beta\mathbf{I}_x^1)\mathbf{E}_+^1(\widetilde{\alpha} + 2\widetilde{\beta}\mathbf{I}_x^1) \tag{8}$$

The 32×32 density matrix of the five qubits is given by

$$\rho_A = (\alpha + 2\beta\mathbf{I}_x^1)\mathbf{E}_+^1(\widetilde{\alpha} + 2\widetilde{\beta}\mathbf{I}_x^1) \otimes (|0000\rangle\,\langle 0000|) \tag{9}$$

In order to encode ρ_A' with the ancillae in state $\mathbf{E}_+^2\mathbf{E}_+^3\mathbf{E}_+^4\mathbf{E}_+^5$, we act on ρ_A with the encoding operator $\mathbf{S_E}$, where $\mathbf{S_E} = S^{5|4}S^{4|3}S^{3|2}S^{2|1}$ represents the series of c-NOT gates in the encoding circuit shown in Fig. (2). The encoded state is then

$$\rho_B = \mathbf{S_E}\rho_A\mathbf{S_E'} \tag{10}$$
$$= S^{5|4}S^{4|3}S^{3|2}S^{2|1}\rho_A S^{2|1}S^{3|2}S^{4|3}S^{5|4} \tag{11}$$
$$= \mathbf{S_E}(\alpha + \beta 2\mathbf{I}_x^1)\mathbf{E}_+^1\mathbf{E}_+^2\mathbf{E}_+^3\mathbf{E}_+^4\mathbf{E}_+^5(\widetilde{\alpha} + 2\widetilde{\beta}\mathbf{I}_x^1)\mathbf{S_E'} \tag{12}$$
$$= (\alpha + \beta 32\mathbf{I}_x^1\mathbf{I}_x^2\mathbf{I}_x^3\mathbf{I}_x^4\mathbf{I}_x^5)\mathbf{E}_+^1\mathbf{E}_+^2\mathbf{E}_+^3\mathbf{E}_+^4\mathbf{E}_+^5(\widetilde{\alpha} + \widetilde{\beta}32\mathbf{I}_x^1\mathbf{I}_x^2\mathbf{I}_x^3\mathbf{I}_x^4\mathbf{I}_x^5) \tag{13}$$

We can expand $\mathbf{S_E}$ in terms of $\mathbf{I}_x^j$ and $\mathbf{E}_\pm^j$:

$$\mathbf{S_E} = (2\mathbf{I}_x^5\mathbf{E}_-^4 + \mathbf{E}_+^4)(2\mathbf{I}_x^4\mathbf{E}_-^3 + \mathbf{E}_+^3)(2\mathbf{I}_x^3\mathbf{E}_-^2 + \mathbf{E}_+^2)(2\mathbf{I}_x^2\mathbf{E}_-^1 + \mathbf{E}_+^1) \tag{14}$$
$$= 16\mathbf{I}_x^2\mathbf{I}_x^3\mathbf{I}_x^4\mathbf{I}_x^5\mathbf{E}_-^1\mathbf{E}_+^2\mathbf{E}_+^3\mathbf{E}_+^4 + 8\mathbf{I}_x^2\mathbf{I}_x^4\mathbf{I}_x^5\mathbf{E}_-^1\mathbf{E}_-^2\mathbf{E}_-^3\mathbf{E}_+^4 + 8\mathbf{I}_x^3\mathbf{I}_x^4\mathbf{I}_x^5\mathbf{E}_+^1\mathbf{E}_-^2\mathbf{E}_+^3\mathbf{E}_+^4 \tag{15}$$
$$+ 4\mathbf{I}_x^4\mathbf{I}_x^5\mathbf{E}_+^1\mathbf{E}_+^2\mathbf{E}_-^3\mathbf{E}_+^4 + 8\mathbf{I}_x^2\mathbf{I}_x^3\mathbf{I}_x^5\mathbf{E}_-^1\mathbf{E}_+^2\mathbf{E}_-^3\mathbf{E}_-^4 + 4\mathbf{I}_x^2\mathbf{I}_x^5\mathbf{E}_-^1\mathbf{E}_-^2\mathbf{E}_+^3\mathbf{E}_-^4$$
$$+ 4\mathbf{I}_x^3\mathbf{I}_x^5\mathbf{E}_+^1\mathbf{E}_-^2\mathbf{E}_-^3\mathbf{E}_-^4 + 2\mathbf{I}_x^5\mathbf{E}_+^1\mathbf{E}_+^2\mathbf{E}_+^3\mathbf{E}_-^4 + 8\mathbf{I}_x^2\mathbf{I}_x^3\mathbf{I}_x^4\mathbf{E}_-^1\mathbf{E}_+^2\mathbf{E}_+^3\mathbf{E}_-^4$$
$$+ 4\mathbf{I}_x^2\mathbf{I}_x^4\mathbf{E}_-^1\mathbf{E}_-^2\mathbf{E}_-^3\mathbf{E}_-^4 + 4\mathbf{I}_x^3\mathbf{I}_x^4\mathbf{E}_+^1\mathbf{E}_-^2\mathbf{E}_+^3\mathbf{E}_-^4 + 2\mathbf{I}_x^4\mathbf{E}_+^1\mathbf{E}_+^2\mathbf{E}_-^3\mathbf{E}_-^4$$
$$+ 4\mathbf{I}_x^2\mathbf{I}_x^3\mathbf{E}_-^1\mathbf{E}_+^2\mathbf{E}_-^3\mathbf{E}_+^4 + 2\mathbf{I}_x^2\mathbf{E}_-^1\mathbf{E}_-^2\mathbf{E}_+^3\mathbf{E}_+^4 + 2\mathbf{I}_x^3\mathbf{E}_+^1\mathbf{E}_-^2\mathbf{E}_-^3\mathbf{E}_+^4$$
$$+ \mathbf{E}_+^1\mathbf{E}_+^2\mathbf{E}_+^3\mathbf{E}_+^4$$

Because the $\mathbf{S}^{i|j}$ do commute, the order in which they are applied is important. This reflects the fact that the c-NOT gates must be applied in the correct order. The $\mathbf{I}_x^1\mathbf{I}_x^2\mathbf{I}_x^3\mathbf{I}_x^4\mathbf{I}_x^5$ term indicates the entanglement of the five spins.

3.3 Decoherence

We now allow for decoherence. Assume there are external perturbations due to random magnetic fields along the x-axis. The Hamiltonian for such a system is

$$\mathcal{H}_x = \gamma^1 B_x^1(t)\mathbf{I}_x^1 + \gamma^2 B_x^2(t)\mathbf{I}_x^2 + \gamma^3 B_x^3(t)\mathbf{I}_x^3 + \gamma^4 B_x^4(t)\mathbf{I}_x^4 + \gamma^5 B_x^5(t)\mathbf{I}_x^5 \tag{16}$$

The propagator for this Hamiltonian has the form $\exp(-i(\chi^1\mathbf{I}_x^1+\chi^2\mathbf{I}_x^2+\chi^3\mathbf{I}_x^3+\chi^4\mathbf{I}_x^4+\chi^5\mathbf{I}_x^5))$, where $\chi^i = \gamma^i \int_0^t B_x^i(\tau)d\tau$.

Applying a random rotation about the x-axis to ρ_B results in the state

$$\begin{aligned}\rho_B' = &\\ &(\alpha + \beta 32\mathbf{I}_x^1\mathbf{I}_x^2\mathbf{I}_x^3\mathbf{I}_x^4\mathbf{I}_x^5)e^{-i(\chi^1\mathbf{I}_x^1+\chi^2\mathbf{I}_x^2+\chi^3\mathbf{I}_x^3+\chi^4\mathbf{I}_x^4+\chi^5\mathbf{I}_x^5)}(\mathbf{E}_+^2\mathbf{E}_+^3\mathbf{E}_+^4\mathbf{E}_+^5)\\ &\times e^{i(\chi^1\mathbf{I}_x^1+\chi^2\mathbf{I}_x^2+\chi^3\mathbf{I}_x^3+\chi^4\mathbf{I}_x^4+\chi^5\mathbf{I}_x^5)}(\widetilde{\alpha} + \widetilde{\beta}32\mathbf{I}_x^2\mathbf{I}_x^3\mathbf{I}_x^4\mathbf{I}_x^5)\end{aligned} \tag{17}$$

To simplify the math, we can assume that $\alpha = 0$ and $\beta = 0$. So we have:

$$\begin{aligned}\rho_B' =&(e^{-i\chi^1\mathbf{I}_x^1}\mathbf{E}_+^1 e^{+i\chi^1\mathbf{I}_x^1})(e^{-i\chi^2\mathbf{I}_x^2}\mathbf{E}_+^2 e^{+i\chi^2\mathbf{I}_x^2})(e^{-i\chi^3\mathbf{I}_x^3}\mathbf{E}_+^3 e^{+i\chi^3\mathbf{I}_x^3}) \qquad (18)\\ &\times (e^{-i\chi^4\mathbf{I}_x^4}\mathbf{E}_+^4 e^{+i\chi^4\mathbf{I}_x^4})(e^{-i\chi^5\mathbf{I}_x^5}\mathbf{E}_+^5 e^{+i\chi^5\mathbf{I}_x^5})\\ =&\frac{1}{32}(1+2e^{-2i\chi^1\mathbf{I}_x^1}2\mathbf{I}_z^1)(1+2e^{-2i\chi^2\mathbf{I}_x^2}2\mathbf{I}_z^2)(1+2e^{-2i\chi^3\mathbf{I}_x^3}2\mathbf{I}_z^3) \qquad (19)\\ &\times (1+2e^{-2i\chi^4\mathbf{I}_x^4}2\mathbf{I}_z^4)(1+2e^{-2i\chi^5\mathbf{I}_x^5}2\mathbf{I}_z^5)\end{aligned}$$

We can expand this product into the sum of all terms of the form

$$\begin{aligned}&R_{\delta_1\delta_2\delta_3\delta_4\delta_5}(\chi^1,\chi^2,\chi^3,\chi^4,\chi^5)((1-\delta_1)+\delta_1 2\mathbf{I}_z^1)((1-\delta_2)+\delta_2 2\mathbf{I}_z^2)\\ &\quad\times((1-\delta_3)+\delta_3 2\mathbf{I}_z^3)((1-\delta)4)+\delta_4 2\mathbf{I}_z^4)((1-\delta_5)+\delta_5 2\mathbf{I}_z^5)\end{aligned} \tag{20}$$

where $R_{\delta_1\delta_2\delta_3\delta_4\delta_5}(\chi^1,\chi^2,\chi^3,\chi^4,\chi^5) \equiv e^{-2i(\delta_1\chi^1\mathbf{I}_x^1+\delta_2\chi^2\mathbf{I}_x^2+\delta_3\chi^3\mathbf{I}_x^3+\delta_4\chi^4\mathbf{I}_x^4+\delta_5\chi^5\mathbf{I}_x^5)}$ for $\delta_1,\delta_2,\delta_3,\delta_4,\delta_5 \in \{0,1\}$.

We find the average state after decoherence by applying averages of the $\overline{R_{\delta_1\delta_2\delta_3\delta_4\delta_5}(\chi^1,\chi^2,\chi^3,\chi^4,\chi^5)}$. We then define the covariance matrix, $\vec{C}t$, by

$$\vec{C}t\ [c^{jk}t] \equiv \overline{\chi^k(t)\chi^l(t)}] \tag{21}$$

and find the probability density function of χ^1,χ^2,χ^3,χ^4,andχ^5,

$$P(\chi^1,\chi^2,\chi^3,\chi^4,\chi^5) = ((2\pi)^5\det(\vec{C}t))^{-1/2}e^{\frac{1}{2t}\vec{\chi}^T\vec{C}^{-1}\vec{\chi}} \tag{22}$$

Now rotate the $\vec{R}_{\delta_1\delta_2\delta_3\delta_4\delta_5}(\chi^1,\chi^2,\chi^3,\chi^4,\chi^5)$ back to the z-axis with the rotation:

$$\vec{R}' = e^{i\frac{\pi}{2}(\mathbf{I}_y^1+\mathbf{I}_y^2+\mathbf{I}_y^3+\mathbf{I}_y^4+\mathbf{I}_y^5)}\vec{R}e^{-i\frac{\pi}{2}(\mathbf{I}_y^1+\mathbf{I}_y^2+\mathbf{I}_y^3+\mathbf{I}_y^4+\mathbf{I}_y^5)} \tag{23}$$

$$= e^{-2i(\delta_1\chi^1\mathbf{I}_z^1+\delta_2\chi^2\mathbf{I}_z^2+\delta_3\chi^3\mathbf{I}_z^3+\delta_4\chi^4\mathbf{I}_z^4+\delta_5\chi^5\mathbf{I}_z^5)} \tag{24}$$

$$\vec{R}'_\delta = \vec{R}'_\delta(\mathbf{E}_+^1+\mathbf{E}_-^1)(\mathbf{E}_+^2+\mathbf{E}_-^2)(\mathbf{E}_+^3+\mathbf{E}_-^3)(\mathbf{E}_+^4+\mathbf{E}_-^4)(\mathbf{E}_+^5+\mathbf{E}_-^5) \tag{25}$$

Each term has the form $R'_\delta(\chi)\mathbf{E}_{\epsilon^1}^1\mathbf{E}_{\epsilon^2}^2\mathbf{E}_{\epsilon^3}^3\mathbf{E}_{\epsilon^4}^4\mathbf{E}_{\epsilon^5}^5 \equiv e^{-i(\epsilon^1\chi^1+\epsilon^2\chi^2+\epsilon^3\chi^3+\epsilon^4\chi^4+\epsilon^5\chi^5)}\mathbf{E}_{\epsilon^1}^1\mathbf{E}_{\epsilon^2}^2\mathbf{E}_{\epsilon^3}^3\mathbf{E}_{\epsilon^4}^4\mathbf{E}_{\epsilon^5}^5$ where $\epsilon^k \in \{-1,0,1\}$.

The average of each term is

$$\overline{e^{-i(\epsilon^1\chi^1+\epsilon^2\chi^2+\epsilon^3\chi^3+\epsilon^4\chi^4+\epsilon^5\chi^5)}} = \int_{-\infty}^{\infty} P(\vec{\chi})e^{-i\vec{\chi}\cdot\vec{\epsilon}}d\vec{\chi} = e^{-\frac{t}{2}\vec{\epsilon}^T\vec{C}\vec{\epsilon}} \tag{26}$$

The ensemble average of each $R'_\delta(\chi)$ is a sum of terms

$$e^{-\frac{t}{2}\vec{\epsilon}^T\vec{C}\vec{\epsilon}}(\mathbf{E}_{\epsilon^1}^1\mathbf{E}_{\epsilon^2}^2\mathbf{E}_{\epsilon^3}^3\mathbf{E}_{\epsilon^4}^4\mathbf{E}_{\epsilon^5}^5+\mathbf{E}_{-\epsilon^1}^1\mathbf{E}_{-\epsilon^2}^2\mathbf{E}_{-\epsilon^3}^3\mathbf{E}_{-\epsilon^4}^4\mathbf{E}_{-\epsilon^5}^5) \tag{27}$$

If all $\epsilon^j = 0$ for all j, then Eq. (27) gives 1. When one $\epsilon^j = \pm 1$ while $\epsilon^i = 0$ for all $i \neq j$, the terms in Eq. (27) become

$$\overline{e^{-2i\chi^j I_z^j}} = e^{-\frac{t}{2}c^{jj}}(\mathbf{E}_+^j+\mathbf{E}_-^j) = e^{-tc^{jj}/2} \tag{28}$$

If there are two nonzero ϵ^j, ϵ^k, Eq. 27 becomes

$$e^{-\frac{t}{2}(c^{jj}+c^{kk}+2\epsilon^j\epsilon^k c^{jk})}(E_{\epsilon^j}E_{\epsilon^k}+E_{\epsilon^{-j}}E_{\epsilon^{-k}}) \tag{29}$$

and

$$(E_{\epsilon^j}E_{\epsilon^k}+E_{\epsilon^{-j}}E_{\epsilon^{-k}}) = \frac{1}{2}(1+4\epsilon^j\epsilon^k\mathbf{I}_z^j\mathbf{I}_z^k) \equiv E_{\epsilon^j\epsilon^k}^{jk} \tag{30}$$

We can expand this result to find:

$$\begin{aligned}\overline{e^{-2i(\chi^j\mathbf{I}_z^j+\chi^k\mathbf{I}_z^k)}} &= e^{-\frac{t}{2}(c^{jj}+c^{kk})}\left(e^{-tc^{jk}}\mathbf{E}_+^{jk}+e^{tc^{jk}}\mathbf{E}_-^{jk}\right) \\ &= e^{-\frac{t}{2}(c^{jj}+c^{kk})}(\cosh(tc^{jk})-\sinh(tc^{jk})4\mathbf{I}_z^j\mathbf{I}_z^k) \\ &= e^{-\frac{t}{2}(c^{jj}+c^{kk}+8c^{jk}\mathbf{I}_z^j\mathbf{I}_z^k)}\end{aligned} \tag{31}$$

We do the same for the remaining cases in which there are three, four, or five nonzero ϵ^j terms and find the following average propagators for random rotations. For three non-zero terms:

$$\overline{e^{-2i(\chi^j\mathbf{I}_z^j+\chi^k\mathbf{I}_z^k+\chi^l\mathbf{I}_z^l)}} = e^{-\frac{t}{2}(c^{jj}+c^{kk}+c^{ll}+8c^{jk}\mathbf{I}_z^j\mathbf{I}_z^k+8c^{jl}\mathbf{I}_z^j\mathbf{I}_z^l+8c^{kl}\mathbf{I}_z^k\mathbf{I}_z^l)} \tag{32}$$

3.4 Decoding

After the time delay or magnetic field-induced decoherence, we apply the decoding circuit from Fig. (4). In the following analysis, we will use the second method of decoding and error correction presented in Section 2 (as opposed to the circuits in Figs. (3) and (5) as it is the most practical method to implement experimentally.

Applying the decoding operator $\mathbf{S_D}$ to ρ_c gives the density matrix after decoding

$$\rho_D = \mathbf{S_D}\rho_C\mathbf{S'_D} \tag{39}$$

Like the encoding operator $\mathbf{S_E}$, the decoding operator is a product of c-NOT gates,

$$\mathbf{S_D} = S^{3|1}S^{5|3}S^{4|3}S^{3|2}S^{2|1}$$

and, alternatively,

$$\mathbf{S'_D} = S^{2|1}S^{3|2}S^{4|3}S^{5|3}S^{3|1}.$$

The order of the $S^{i|j}$ gates is important as these c-NOT gates also do not commute. Therefore, we must take care to use S_D and S'_D appropriately.

We can expand S_D into a sum of products of $\mathbf{I}_x^j$ and $\mathbf{E}_\pm^j$:

$$\begin{aligned}
\mathbf{S_D} =&(2\mathbf{I}_x^3\mathbf{E}_-^1 + \mathbf{E}_+^1)(2\mathbf{I}_x^5\mathbf{E}_-^3 + \mathbf{E}_+^3)(2\mathbf{I}_x^4\mathbf{E}_-^3 + \mathbf{E}_+^3)(2\mathbf{I}_x^3\mathbf{E}_-^2 + \mathbf{E}_+^2)(2\mathbf{I}_x^1\mathbf{E}_-^1 + \mathbf{E}_+^1) && (40)\\
=&8\mathbf{I}_x^2\mathbf{I}_x^4\mathbf{I}_x^5\mathbf{E}_-^1\mathbf{E}_+^2\mathbf{E}_+^3 + 16\mathbf{I}_x^2\mathbf{I}_x^3\mathbf{I}_x^4\mathbf{I}_x^5\mathbf{E}_-^1\mathbf{E}_-^2\mathbf{E}_-^3 + 2\mathbf{I}_x^2\mathbf{E}_-^1\mathbf{E}_+^2\mathbf{E}_-^3 + 4\mathbf{I}_x^2\mathbf{I}_x^3\mathbf{E}_-^1\mathbf{E}_-^2\mathbf{E}_+^3 && (41)\\
&+ 8\mathbf{I}_x^3\mathbf{I}_x^4\mathbf{I}_x^5\mathbf{E}_+^1\mathbf{E}_-^2\mathbf{E}_+^3 + 4\mathbf{I}_x^4\mathbf{I}_x^5\mathbf{E}_+^1\mathbf{E}_+^2\mathbf{E}_-^3 + 2\mathbf{I}_x^3\mathbf{E}_+^1\mathbf{E}_-^2\mathbf{E}_-^3 + \mathbf{E}_+^1\mathbf{E}_+^2\mathbf{E}_+^3
\end{aligned}$$

We can also write S'_D as

$$\begin{aligned}
\mathbf{S'_D} =&8\mathbf{E}_-^1\mathbf{E}_+^2\mathbf{E}_+^3\mathbf{I}_x^2\mathbf{I}_x^4\mathbf{I}_x^5 + 2\mathbf{E}_-^1\mathbf{E}_+^2\mathbf{E}_-^3\mathbf{I}_x^2 + 16\mathbf{E}_-^1\mathbf{E}_-^2\mathbf{E}_-^3\mathbf{I}_x^2\mathbf{I}_x^3\mathbf{I}_x^4\mathbf{I}_x^5 + 4\mathbf{E}_-^1\mathbf{E}_-^2\mathbf{E}_+^3\mathbf{I}_x^2\mathbf{I}_x^3 && (42)\\
&+ 8\mathbf{E}_+^1\mathbf{E}_-^2\mathbf{E}_+^3\mathbf{I}_x^3\mathbf{I}_x^4\mathbf{I}_x^5 + 2\mathbf{E}_+^1\mathbf{E}_-^2\mathbf{E}_-^3\mathbf{I}_x^3 + 4\mathbf{E}_+^1\mathbf{E}_+^2\mathbf{E}_-^3\mathbf{I}_x^4\mathbf{I}_x^5 + \mathbf{E}_+^1\mathbf{E}_+^2\mathbf{E}_+^3
\end{aligned}$$

Now we have the density matrix ρ_D after decoding:

$$\begin{aligned}
\rho_D =& \mathbf{S_D}\rho_C\mathbf{S'_D}\\
=& \mathbf{S_D}(\alpha + \beta 32\mathbf{I}_x^1\mathbf{I}_x^2\mathbf{I}_x^3\mathbf{I}_x^4\mathbf{I}_x^5)\mathbf{S'_D} && (43)\\
&\times \mathbf{S_D}\mathcal{F}_C\left[\frac{1}{32}(1 + F^1 2\mathbf{I}_z^1)(1 + F^2 2\mathbf{I}_z^2)(1 + F^3 2\mathbf{I}_z^3)(1 + F^4 2\mathbf{I}_z^4)(1 + F^5 2\mathbf{I}_z^5)\right]\mathbf{S'_D}\\
&\times \mathbf{S_D}(\widetilde{\alpha} + \widetilde{\beta} 32\mathbf{I}_x^1\mathbf{I}_x^2\mathbf{I}_x^3\mathbf{I}_x^4\mathbf{I}_x^5)\mathbf{S'_D}\rho_D\\
=& (\alpha + \beta 2\mathbf{I}_x^1)\\
&\times \mathcal{F}_D\left[\frac{1}{32}\mathbf{S_D}(1 + F_C^1\mathbf{I}_z^1)(1 + F_C^2\mathbf{I}_z^2)(1 + F_C^3\mathbf{I}_z^3)(1 + F_C^4\mathbf{I}_z^4)(1 + F_C^5\mathbf{I}_z^5)\mathbf{S'_D}\right](\widetilde{\alpha} + \widetilde{\beta} 2\mathbf{I}_x^1)
\end{aligned}$$

Four non-zero terms:

$$\overline{e^{-2i(\chi^j \mathbf{I}_z^j + \chi^k \mathbf{I}_z^k + \chi^l \mathbf{I}_z^l + \chi^m \mathbf{I}_z^m)}} = e^{-\frac{t}{2}(c^{jj}+c^{kk}+c^{ll}+c^{mm}+8c^{jk}\mathbf{I}_z^j\mathbf{I}_z^k+8c^{jl}\mathbf{I}_z^j\mathbf{I}_z^l+8c^{jm}\mathbf{I}_z^j\mathbf{I}_z^m+8c^{kl}\mathbf{I}_z^k\mathbf{I}_z^l+8c^{km}\mathbf{I}_z^k\mathbf{I}_z^m+8c^{lm}\mathbf{I}_z^l\mathbf{I}_z^m)} \tag{33}$$

Five non-zero terms:

$$\overline{e^{-2i(\chi^j \mathbf{I}_z^j + \chi^k \mathbf{I}_z^k + \chi^l \mathbf{I}_z^l + \chi^m \mathbf{I}_z^m + \chi^n \mathbf{I}_z^n)}} = \exp\left(-\frac{t}{2}\left(\sum_{i=1}^{5} c^{ii} + \sum_{j\neq k=1}^{5} 8c^{jk}\mathbf{I}_z^j\mathbf{I}_z^k\right)\right) \tag{34}$$

We rotate these averages back to $\mathbf{R}_{\delta^1,\delta^2,\delta^3,\delta^4,\delta^5}(\chi^1,\chi^2,\chi^3,\chi^4,\chi^5)$ by replacing all $\mathbf{I}_z^i$ with $\mathbf{I}_x^i$.

We can define the following time-independent, nonunitary operators

$$F^j = F^j(t) \equiv e^{-tc^{jj}/2} \tag{35}$$

and

$$\mathbf{F}_c^{jk} = \mathbf{F}_c^{jk}(t) \equiv e^{-tc^{jk}4\mathbf{I}_x^j\mathbf{I}_x^k} \tag{36}$$

The resulting density matrix after decoherence is

$$\begin{aligned}\rho_c \equiv{}& (\alpha + \beta 32\mathbf{I}_x^1\mathbf{I}_x^2\mathbf{I}_x^3\mathbf{I}_x^4\mathbf{I}_x^5) \\ &\times \mathcal{F}_C\left[\frac{1}{32}(1+F^1 2\mathbf{I}_z^1)(1+F^2 2\mathbf{I}_z^2)(1+F^3 2\mathbf{I}_z^3)(1+F^4 2\mathbf{I}_z^4)(1+F^5 2\mathbf{I}_z^5)\right] \\ &\times (\widetilde{\alpha} + \widetilde{\beta} 32\mathbf{I}_x^1\mathbf{I}_x^2\mathbf{I}_x^3\mathbf{I}_x^4\mathbf{I}_x^5)\end{aligned} \tag{37}$$

where $\mathcal{F}_C$ is defined as

$$\begin{aligned}\mathcal{F}_C[1] &\equiv 1 \\ \mathcal{F}_C\left[2\mathbf{I}_z^j\right] &\equiv 2\mathbf{I}_z^j \\ \mathcal{F}_C\left[4\mathbf{I}_z^j\mathbf{I}_z^k\right] &\equiv F_c^{jk}4\mathbf{I}_z^j\mathbf{I}_z^k \\ &\equiv e^{-tc^{jk}4\mathbf{I}_x^j\mathbf{I}_x^k}4\mathbf{I}_z^j\mathbf{I}_z^k \\ \mathcal{F}_C\left[8\mathbf{I}_z^j\mathbf{I}_z^k\mathbf{I}_z^l\right] &\equiv F_c^{jk}F_c^{jl}F_c^{kl}8\mathbf{I}_z^j\mathbf{I}_z^k\mathbf{I}_z^l \\ &\equiv e^{-t(c^{jk}4\mathbf{I}_x^j\mathbf{I}_x^k+c^{jl}4\mathbf{I}_x^j\mathbf{I}_x^l+c^{kl}4\mathbf{I}_x^k\mathbf{I}_x^l)}8\mathbf{I}_z^j\mathbf{I}_z^k\mathbf{I}_z^l \\ \mathcal{F}_C\left[16\mathbf{I}_z^j\mathbf{I}_z^k\mathbf{I}_z^l\mathbf{I}_z^m\right] &\equiv F_c^{jk}F_c^{jl}F_c^{jm}F_c^{kl}F_c^{km}F_c^{lm}16\mathbf{I}_z^j\mathbf{I}_z^k\mathbf{I}_z^l\mathbf{I}_z^m \\ &\equiv e^{-t(c^{jk}4\mathbf{I}_x^j\mathbf{I}_x^k+c^{jl}4\mathbf{I}_x^j\mathbf{I}_x^l+c^{jm}4\mathbf{I}_x^j\mathbf{I}_x^m+c^{kl}4\mathbf{I}_x^k\mathbf{I}_x^l c^{km}4\mathbf{I}_x^k\mathbf{I}_x^m+c^{lm}4\mathbf{I}_x^l\mathbf{I}_x^m)}16\mathbf{I}_z^j\mathbf{I}_z^k\mathbf{I}_z^l\mathbf{I}_z^m \\ \mathcal{F}_C\left[32\mathbf{I}_z^1\mathbf{I}_z^2\mathbf{I}_z^3\mathbf{I}_z^4\mathbf{I}_z^5\right] &\equiv F_c^{12}F_c^{13}F_c^{14}F_c^{15}F_c^{23}F_c^{24}F_c^{34}F_c^{35}F_c^{45}32\mathbf{I}_z^1\mathbf{I}_z^2\mathbf{I}_z^3\mathbf{I}_z^4\mathbf{I}_z^5 \\ &\equiv \exp\left(\sum_{j\neq k=1}^{5} 4\mathbf{I}_z^j\mathbf{I}_z^k\right)32\mathbf{I}_z^1\mathbf{I}_z^2\mathbf{I}_z^3\mathbf{I}_z^4\mathbf{I}_z^5\end{aligned} \tag{38}$$

where $\mathcal{F}_D\left[\mathbf{S_D X S'_D}\right] = \mathbf{S_D}\mathcal{F}_C[\mathbf{X}]\mathbf{S'_D}$. For example:

$$\mathcal{F}_D[1] = \mathcal{F}_D[\mathbf{S_D S'_D}] = \mathbf{S_D}\mathcal{F}_C[1]\mathbf{S'_D} = \mathbf{S_D S'_D} = 1 \tag{44}$$

$$\mathcal{F}_D[2\mathbf{I}_z^1] = \mathcal{F}_D[\mathbf{S_D}2\mathbf{I}_z^1\mathbf{S'_D}] = \mathbf{S_D}\mathcal{F}_C[2\mathbf{I}_z^1]\mathbf{S'_D} = \mathbf{S_D}2\mathbf{I}_z^1\mathbf{S'_D} = 2\mathbf{I}_z^1 \tag{45}$$

$$\mathcal{F}_D[4\mathbf{I}_z^1\mathbf{I}_z^2] = \mathcal{F}_D[\mathbf{S_D}2\mathbf{I}_z^2\mathbf{S'_D}] = \mathbf{S_D}\mathcal{F}_C[2\mathbf{I}_z^2]\mathbf{S'_D} = \mathbf{S_D}2\mathbf{I}_z^2\mathbf{S'_D} = 4\mathbf{I}_z^1\mathbf{I}_z^2 \tag{46}$$

The complete list of values for the $\mathcal{F}_D$ function is given in Appendix A.

We can also define $\mathbf{F}_D^{jk}$:

$$\mathbf{F}_D^{jk} \equiv \mathbf{S_D} e^{-tc^{jk}4\mathbf{I}_x^j\mathbf{I}_x^k}\mathbf{S'_D} = e^{-tc^{jk}\mathbf{S_D}4\mathbf{I}_x^j\mathbf{I}_x^k\mathbf{S'_D}} \tag{47}$$

with the following relations:

$$\begin{aligned}
\mathbf{S_D}4\mathbf{I}_x^1\mathbf{I}_x^2\mathbf{S'_D} &= 4\mathbf{I}_x^1\mathbf{I}_x^3 & \mathbf{S_D}4\mathbf{I}_x^2\mathbf{I}_x^4\mathbf{S'_D} &= 8\mathbf{I}_x^2\mathbf{I}_x^3\mathbf{I}_x^5 \\
\mathbf{S_D}4\mathbf{I}_x^1\mathbf{I}_x^3\mathbf{S'_D} &= 8\mathbf{I}_x^1\mathbf{I}_x^2\mathbf{I}_x^3 & \mathbf{S_D}4\mathbf{I}_x^2\mathbf{I}_x^5\mathbf{S'_D} &= 8\mathbf{I}_x^2\mathbf{I}_x^3\mathbf{I}_x^4 \\
\mathbf{S_D}4\mathbf{I}_x^1\mathbf{I}_x^4\mathbf{S'_D} &= 8\mathbf{I}_x^1\mathbf{I}_x^2\mathbf{I}_x^5 & \mathbf{S_D}4\mathbf{I}_x^3\mathbf{I}_x^4\mathbf{S'_D} &= 4\mathbf{I}_x^3\mathbf{I}_x^5 \\
\mathbf{S_D}4\mathbf{I}_x^1\mathbf{I}_x^5\mathbf{S'_D} &= 8\mathbf{I}_x^1\mathbf{I}_x^2\mathbf{I}_x^4 & \mathbf{S_D}4\mathbf{I}_x^3\mathbf{I}_x^5\mathbf{S'_D} &= 4\mathbf{I}_x^3\mathbf{I}_x^4 \\
\mathbf{S_D}4\mathbf{I}_x^2\mathbf{I}_x^3\mathbf{S'_D} &= 2\mathbf{I}_x^2 & \mathbf{S_D}4\mathbf{I}_x^4\mathbf{I}_x^5\mathbf{S'_D} &= 4\mathbf{I}_x^4\mathbf{I}_x^5
\end{aligned} \tag{48}$$

Applying the $\mathbf{S_D}$ and $\mathbf{S'_D}$ operators to the terms inside $\mathcal{F}_D[\,]$ in Eq. 43 produces the state:

$$\begin{aligned}
\rho_D &= (\alpha + \beta 2\mathbf{I}_x^1) \qquad (49) \\
&\times \mathcal{F}_D\left[\frac{1}{32}(1 + F_C^1 2\mathbf{I}_z^1)(1 + F_C^2 4\mathbf{I}_z^1\mathbf{I}_z^2)(1 + F_C^3 8\mathbf{I}_z^1\mathbf{I}_z^2\mathbf{I}_z^3)(1 + F_C^4 8\mathbf{I}_z^1\mathbf{I}_z^3\mathbf{I}_z^4)(1 + F_C^5 8\mathbf{I}_z^1\mathbf{I}_z^3\mathbf{I}_z^5)\right] \\
&\times (\widetilde{\alpha} + \widetilde{\beta} 2\mathbf{I}_x^1)
\end{aligned}$$

3.5 Error Correction

The last step is to apply the error correction operator, $\mathbf{S_C}$, to ρ_D. We can expand $\mathbf{S_C}$:

$$\mathbf{S_C} = \mathbf{V}^{1|2345}\mathbf{T}^{1|23}\mathbf{T}^{3|45} \tag{50}$$

$$= (1 - (1 - 2\mathbf{I}_x^1)\mathbf{E}_+^2\mathbf{E}_+^3\mathbf{E}_+^4\mathbf{E}_+^5)(1 - (1 - 2\mathbf{I}_x^1)\mathbf{E}_-^2\mathbf{E}_+^3)(1 - (1 - 2\mathbf{I}_x^3)\mathbf{E}_+^4\mathbf{E}_+^5) \tag{51}$$

$$= 1 - (1 - 2\mathbf{I}_x^3)\mathbf{E}_+^4\mathbf{E}_+^5 - (1 - 2\mathbf{I}_x^1)\mathbf{E}_-^2\mathbf{E}_+^3(1 - \mathbf{E}_+^4\mathbf{E}_+^5) - \mathbf{I}_x^3(1 - 2\mathbf{I}_x^1)\mathbf{E}_-^3\mathbf{E}_+^4\mathbf{E}_+^5 \tag{52}$$

Note that in the first line $\mathbf{T}^{1|23}$ represents the c^2-NOT gate which flips data qubit conditional on the state of the second and third qubits being $|0_2 1_3\rangle$. This corresponds to applying $(1-(1-2\mathbf{I}_x^1)\mathbf{E}_-^2\mathbf{E}_+^3)$ to the state.

As in the previous operator cases, $\mathbf{S_C} \neq \mathbf{S'_C}$. Instead,

$$\mathbf{S'_C} = \mathbf{T}^{3|45}\mathbf{T}^{1|23}\mathbf{V}^{1|2345} \tag{53}$$

$$= 1-(1-2\mathbf{I}_x^3)\mathbf{E}_+^4\mathbf{E}_+^5-(1-2\mathbf{I}_x^1)\mathbf{E}_-^2\mathbf{E}_+^3(1-\mathbf{E}_+^4\mathbf{E}_+^5)-2\mathbf{I}_x^3(1-2\mathbf{I}_x^1)\mathbf{E}_+^3\mathbf{E}_+^4\mathbf{E}_+^5 \tag{54}$$

Following the error correction step, we want to trace over the ancillae in order to reproduce the initial state of the data qubit, $\psi = \alpha|0\rangle + \beta|1\rangle$. Taking the partial trace over the ancillae is equivalent to applying the projection operators and taking the sum:

$$16\,\langle\mathcal{E}[\mathbf{X}]\rangle^{2345} \equiv 16\left\langle\sum_{\epsilon^i=\pm}\mathbf{E}_{\epsilon^1}^1\mathbf{E}_{\epsilon^2}^2\mathbf{E}_{\epsilon^3}^3\mathbf{E}_{\epsilon^4}^4\mathbf{E}_{\epsilon^5}^5\mathbf{X}\mathbf{E}_{\epsilon^1}^1\mathbf{E}_{\epsilon^2}^2\mathbf{E}_{\epsilon^3}^3\mathbf{E}_{\epsilon^4}^4\mathbf{E}_{\epsilon^5}^5\right\rangle^{2345} \tag{55}$$

Although $\mathbf{S_C}$ does not commute with $\mathbf{E}_\pm^3$, an expansion of $\mathbf{S_C}\mathcal{E}[\mathbf{X}]\mathbf{S'_C}$ produces the result:

$$\mathcal{E}[\mathbf{S_C}\mathbf{X}\mathbf{S'_C}] = \mathbf{S_C}\mathcal{E}[\mathbf{X}]\mathbf{S'_C} \tag{56}$$

Therefore, we can take $\mathcal{E}[\rho_D]$ prior to applying the error correction operation, which will simplify much of the algebra. We can also apply the trace directly to the $\mathbf{F}_D^{jk}$ from (47). For example, take the partial trace of $\mathbf{F}_D^{12}$:

$$\begin{aligned}
\mathcal{E}[\mathbf{F}_D^{12}] &= \sum_{\epsilon^i\in\pm1}\mathbf{E}_{\epsilon^2}^2\mathbf{E}_{\epsilon^3}^3\mathbf{E}_{\epsilon^4}^4\mathbf{E}_{\epsilon^5}^5\left(e^{-tc^{12}4\mathbf{I}_x^1\mathbf{I}_x^3}\right)\mathbf{E}_{\epsilon^2}^2\mathbf{E}_{\epsilon^3}^3\mathbf{E}_{\epsilon^4}^4\mathbf{E}_{\epsilon^5}^5 \\
&= \sum_{\epsilon^i\in\pm1}\mathbf{E}_{\epsilon^2}^2\mathbf{E}_{\epsilon^3}^3\mathbf{E}_{\epsilon^4}^4\mathbf{E}_{\epsilon^5}^5\left(\cosh(tc^{12})-4\mathbf{I}_x^1\mathbf{I}_x^3\sinh(tc^{12})\right)\mathbf{E}_{\epsilon^2}^2\mathbf{E}_{\epsilon^3}^3\mathbf{E}_{\epsilon^4}^4\mathbf{E}_{\epsilon^5}^5 \qquad (57) \\
&= \sum_{\epsilon^i\in\pm1}\left(\cosh(tc^{12})(\mathbf{E}_{\epsilon^2}^2\mathbf{E}_{\epsilon^3}^3\mathbf{E}_{\epsilon^4}^4\mathbf{E}_{\epsilon^5}^5)^2-4\mathbf{I}_x^1\mathbf{I}_x^3\sinh(tc^{12})(\mathbf{E}_{\epsilon^2}^2\mathbf{E}_{-\epsilon^3}^3\mathbf{E}_{\epsilon^4}^4\mathbf{E}_{\epsilon^5}^5)(\mathbf{E}_{\epsilon^2}^2\mathbf{E}_{\epsilon^3}^3\mathbf{E}_{\epsilon^4}^4\mathbf{E}_{\epsilon^5}^5)\right) \\
&= \cosh(tc^{12})\sum_{\epsilon^i\in\pm1}\mathbf{E}_{\epsilon^2}^2\mathbf{E}_{\epsilon^3}^3\mathbf{E}_{\epsilon^4}^4\mathbf{E}_{\epsilon^5}^5 = \cosh(tc^{12}) \equiv F^{12}
\end{aligned}$$

Generally, we have

$$\begin{aligned}
F^{jk} &= \mathcal{E}[\mathbf{F}_D^{jk}] = \cosh(tc^{jk}) \\
F^{ijk} &= \mathcal{E}[\mathbf{F}_D^{ij}\mathbf{F}_D^{ik}\mathbf{F}_D^{jk}] \\
&= \cosh(tc^{ij})\cosh(tc^{ik})\cosh(tc^{jk})-\sinh(tc^{ij})\sinh(tc^{ik})\sinh(tc^{jk}) \qquad (58)
\end{aligned}$$

We can also expand $F^{ijkl} = \mathcal{E}[\mathbf{F}_D^{ij}\mathbf{F}_D^{ik}\mathbf{F}_D^{il}\mathbf{F}_D^{jk}\mathbf{F}_D^{jl}\mathbf{F}_D^{kl}]$ and $F^{12345} = \mathcal{E}[\mathbf{F}_D^{12}\mathbf{F}_D^{13}\mathbf{F}_D^{14}\mathbf{F}_D^{15}\mathbf{F}_D^{23}\mathbf{F}_D^{24}\mathbf{F}_D^{25}\mathbf{F}_D^{34}\mathbf{F}_D^{35}\mathbf{F}_D^{45}]$. Because they are large sums of products of hyperbolic trigonometric functions, and are therfore not listed here.

Finally, we apply the error correction operator, $\mathbf{S_C}$, to the projection of each term in (47). Before listing all possible terms of $\mathbf{S_D}\mathcal{E}[\mathbf{X}]\mathbf{S'_D}$, let us consider the result of taking the partial trace over the ancillae. This will eliminate all terms containing combinations of $\mathbf{I}_z^2$,$\mathbf{I}_z^3$,$\mathbf{I}_z^4$, and $\mathbf{I}_z^5$; or any term without a factor of $\mathbf{I}_z^1$. Therefore, we can neglect such terms, and determine $\mathbf{S_D}\mathcal{E}[\mathbf{X}]\mathbf{S'_D}$ only for $\mathbf{X}$ containing a factor of $\mathbf{I}_z^1$. Listed below are the results of the error correction operation on $\mathcal{F}_D[1]$,$\mathcal{F}_D[\mathbf{I}_z^i]$, and $\mathcal{F}_D[4\mathbf{I}_z^i\mathbf{I}_z^j]$. The full list of terms remaining after taking the partial trace is given in Appendix B.

$$\begin{aligned}
\mathbf{S_C}\mathcal{E}[\mathcal{F}_D[1]]\mathbf{S'_C} =& 1 \qquad (59)\\
\mathbf{S_C}\mathcal{E}[\mathcal{F}_D[2\mathbf{I}_z^1]]\mathbf{S'_C} =& \frac{1}{8}[6\mathbf{I}_z^1 + 12\mathbf{I}_z^1\mathbf{I}_z^2 - 4\mathbf{I}_z^1\mathbf{I}_z^3 - 4\mathbf{I}_z^1\mathbf{I}_z^4 - 4\mathbf{I}_z^1\mathbf{I}_z^5 + 24\mathbf{I}_z^1\mathbf{I}_z^2\mathbf{I}_z^3 - 8\mathbf{I}_z^1\mathbf{I}_z^2\mathbf{I}_z^4\\
&- 8\mathbf{I}_z^1\mathbf{I}_z^2\mathbf{I}_z^5 + 24\mathbf{I}_z^1\mathbf{I}_z^3\mathbf{I}_z^4 + 24\mathbf{I}_z^1\mathbf{I}_z^3\mathbf{I}_z^5 - 8\mathbf{I}_z^1\mathbf{I}_z^4\mathbf{I}_z^5 - 16\mathbf{I}_z^1\mathbf{I}_z^2\mathbf{I}_z^3\mathbf{I}_z^4\\
&- 16\mathbf{I}_z^1\mathbf{I}_z^2\mathbf{I}_z^3\mathbf{I}_z^5 - 16\mathbf{I}_z^1\mathbf{I}_z^2\mathbf{I}_z^4\mathbf{I}_z^5 + 48\mathbf{I}_z^1\mathbf{I}_z^3\mathbf{I}_z^4\mathbf{I}_z^5 - 32\mathbf{I}_z^1\mathbf{I}_z^2\mathbf{I}_z^3\mathbf{I}_z^4\mathbf{I}_z^5]\\
\mathbf{S_C}\mathcal{E}[\mathcal{F}_D[2\mathbf{I}_z^2]]\mathbf{S'_C} =& F^{12}2\mathbf{I}_z^2\\
\mathbf{S_C}\mathcal{E}[\mathcal{F}_D[2\mathbf{I}_z^3]]\mathbf{S'_C} =& F^{23}\frac{1}{2}[2\mathbf{I}_z^3 - 4\mathbf{I}_z^3\mathbf{I}_z^4 - 4\mathbf{I}_z^3\mathbf{I}_z^5 - 8\mathbf{I}_z^3\mathbf{I}_z^4\mathbf{I}_z^5]\\
\mathbf{S_C}\mathcal{E}[\mathcal{F}_D[2\mathbf{I}_z^4]]\mathbf{S'_C} =& F^{1234}2\mathbf{I}_z^4\\
\mathbf{S_C}\mathcal{E}[\mathcal{F}_D[2\mathbf{I}_z^5]]\mathbf{S'_C} =& F^{1235}2\mathbf{I}_z^5\\
\mathbf{S_C}\mathcal{E}[\mathcal{F}_D[4\mathbf{I}_z^1\mathbf{I}_z^2]]\mathbf{S'_C} =& \frac{1}{8}[6\mathbf{I}_z^1 + 12\mathbf{I}_z^1\mathbf{I}_z^2 + 12\mathbf{I}_z^1\mathbf{I}_z^3 - 4\mathbf{I}_z^1\mathbf{I}_z^4 - 4\mathbf{I}_z^1\mathbf{I}_z^5 - 8\mathbf{I}_z^1\mathbf{I}_z^2\mathbf{I}_z^3 - 8\mathbf{I}_z^1\mathbf{I}_z^2\mathbf{I}_z^4\\
&- 8\mathbf{I}_z^1\mathbf{I}_z^2\mathbf{I}_z^5 - 8\mathbf{I}_z^1\mathbf{I}_z^3\mathbf{I}_z^4 - 8\mathbf{I}_z^1\mathbf{I}_z^3\mathbf{I}_z^5 - 8\mathbf{I}_z^1\mathbf{I}_z^4\mathbf{I}_z^5 + 48\mathbf{I}_z^1\mathbf{I}_z^2\mathbf{I}_z^3\mathbf{I}_z^4\\
&+ 48\mathbf{I}_z^1\mathbf{I}_z^2\mathbf{I}_z^3\mathbf{I}_z^5 - 16\mathbf{I}_z^1\mathbf{I}_z^2\mathbf{I}_z^4\mathbf{I}_z^5 - 16\mathbf{I}_z^1\mathbf{I}_z^3\mathbf{I}_z^4\mathbf{I}_z^5 + 96\mathbf{I}_z^1\mathbf{I}_z^2\mathbf{I}_z^3\mathbf{I}_z^4\mathbf{I}_z^5]\\
\mathbf{S_C}\mathcal{E}[\mathcal{F}_D[4\mathbf{I}_z^1\mathbf{I}_z^3]]\mathbf{S'_C} =& F^{123}\frac{1}{8}[-2\mathbf{I}_z^1 + 12\mathbf{I}_z^1\mathbf{I}_z^2 + 12\mathbf{I}_z^1\mathbf{I}_z^3 + 12\mathbf{I}_z^1\mathbf{I}_z^4 + 12\mathbf{I}_z^1\mathbf{I}_z^5 + 24\mathbf{I}_z^1\mathbf{I}_z^2\mathbf{I}_z^3 - 8\mathbf{I}_z^1\mathbf{I}_z^2\mathbf{I}_z^4\\
&- 8\mathbf{I}_z^1\mathbf{I}_z^2\mathbf{I}_z^5 - 8\mathbf{I}_z^1\mathbf{I}_z^3\mathbf{I}_z^4 - 8\mathbf{I}_z^1\mathbf{I}_z^3\mathbf{I}_z^5 + 24\mathbf{I}_z^1\mathbf{I}_z^4\mathbf{I}_z^5 - 16\mathbf{I}_z^1\mathbf{I}_z^2\mathbf{I}_z^3\mathbf{I}_z^4 - 16\mathbf{I}_z^1\mathbf{I}_z^2\mathbf{I}_z^3\mathbf{I}_z^5\\
&- 16\mathbf{I}_z^1\mathbf{I}_z^2\mathbf{I}_z^4\mathbf{I}_z^5 - 16\mathbf{I}_z^1\mathbf{I}_z^3\mathbf{I}_z^4\mathbf{I}_z^5 - 32\mathbf{I}_z^1\mathbf{I}_z^2\mathbf{I}_z^3\mathbf{I}_z^4\mathbf{I}_z^5
\end{aligned}$$

$$\mathbf{S_C}\mathcal{E}[\mathcal{F}_D[4\mathbf{I}_z^1\mathbf{I}_z^4]]\mathbf{S}'_\mathbf{C} = \frac{1}{8}F^{234}[-2\mathbf{I}_z^1 - 4\mathbf{I}_z^1\mathbf{I}_z^2 + 12\mathbf{I}_z^1\mathbf{I}_z^3 + 12\mathbf{I}_z^1\mathbf{I}_z^4 - 4\mathbf{I}_z^1\mathbf{I}_z^5 - 8\mathbf{I}_z^1\mathbf{I}_z^2\mathbf{I}_z^3 + 24\mathbf{I}_z^1\mathbf{I}_z^2\mathbf{I}_z^4 \tag{60}$$

$$- 8\mathbf{I}_z^1\mathbf{I}_z^2\mathbf{I}_z^5 - 8\mathbf{I}_z^1\mathbf{I}_z^3\mathbf{I}_z^4 + 24\mathbf{I}_z^1\mathbf{I}_z^3\mathbf{I}_z^5 - 8\mathbf{I}_z^1\mathbf{I}_z^4\mathbf{I}_z^5 + 48\mathbf{I}_z^1\mathbf{I}_z^2\mathbf{I}_z^3\mathbf{I}_z^4 - 16\mathbf{I}_z^1\mathbf{I}_z^2\mathbf{I}_z^3\mathbf{I}_z^5$$
$$- 16\mathbf{I}_z^1\mathbf{I}_z^2\mathbf{I}_z^4\mathbf{I}_z^5 + 48\mathbf{I}_z^1\mathbf{I}_z^3\mathbf{I}_z^4\mathbf{I}_z^5 - 32\mathbf{I}_z^1\mathbf{I}_z^2\mathbf{I}_z^3\mathbf{I}_z^4\mathbf{I}_z^5]$$

$$\mathbf{S_C}\mathcal{E}[\mathcal{F}_D[4\mathbf{I}_z^1\mathbf{I}_z^5]]\mathbf{S}'_\mathbf{C} = \frac{1}{8}F^{235}[-2\mathbf{I}_z^1 - 4\mathbf{I}_z^1\mathbf{I}_z^2 + 12\mathbf{I}_z^1\mathbf{I}_z^3 - 4\mathbf{I}_z^1\mathbf{I}_z^4 + 12\mathbf{I}_z^1\mathbf{I}_z^5 - 8\mathbf{I}_z^1\mathbf{I}_z^2\mathbf{I}_z^3 - 8\mathbf{I}_z^1\mathbf{I}_z^2\mathbf{I}_z^4$$
$$+ 24\mathbf{I}_z^1\mathbf{I}_z^2\mathbf{I}_z^5 + 24\mathbf{I}_z^1\mathbf{I}_z^3\mathbf{I}_z^4 - 8\mathbf{I}_z^1\mathbf{I}_z^3\mathbf{I}_z^5 - 8\mathbf{I}_z^1\mathbf{I}_z^4\mathbf{I}_z^5 - 16\mathbf{I}_z^1\mathbf{I}_z^2\mathbf{I}_z^3\mathbf{I}_z^4 - 80\mathbf{I}_z^1\mathbf{I}_z^2\mathbf{I}_z^3\mathbf{I}_z^5$$
$$- 16\mathbf{I}_z^1\mathbf{I}_z^2\mathbf{I}_z^4\mathbf{I}_z^5 + 48\mathbf{I}_z^1\mathbf{I}_z^3\mathbf{I}_z^4\mathbf{I}_z^5 - 32\mathbf{I}_z^1\mathbf{I}_z^2\mathbf{I}_z^3\mathbf{I}_z^4\mathbf{I}_z^5]$$

We multiply the remaining terms by the appropriate F^i, according to (50) to find the final state after error correction:

$$\rho_E^1 \equiv \langle\rho_E\rangle^{2345} \equiv \left\langle(\alpha+\beta 2\mathbf{I}_x^1)\mathbf{S_C}\rho_D\mathbf{S}'_\mathbf{C}(\widetilde{\alpha}+\widetilde{\beta}2\mathbf{I}_x^1)\right\rangle^{2345} \tag{61}$$
$$\equiv(\alpha+\beta 2\mathbf{I}_x^1)\langle\frac{1}{32}\mathbf{S_C}\mathcal{E}[\mathcal{F}_D[1 + F^1 2\mathbf{I}_z^1 + F^2 4\mathbf{I}_z^1\mathbf{I}_z^2 + F^1F^2F^3F^{123}4\mathbf{I}_z^1\mathbf{I}_z^3$$
$$+ F^2F^3F^4F^{234}4\mathbf{I}_z^1\mathbf{I}_z^4 + F^2F^3F^5F^{235}4\mathbf{I}_z^1\mathbf{I}_z^5 + F^3 8\mathbf{I}_z^1\mathbf{I}_z^2\mathbf{I}_z^3$$
$$+ F^1F^3F^4F^{134}8\mathbf{I}_z^1\mathbf{I}_z^2\mathbf{I}_z^4 + F^1F^3F^5F^{135}8\mathbf{I}_z^1\mathbf{I}_z^2\mathbf{I}_z^5 + F^4 8\mathbf{I}_z^1\mathbf{I}_z^3\mathbf{I}_z^4 + F^5 8\mathbf{I}_z^1\mathbf{I}_z^3\mathbf{I}_z^5$$
$$+ F^1F^4F^5F^{145}8\mathbf{I}_z^1\mathbf{I}_z^4\mathbf{I}_z^5 + F^1F^2F^4F^{124}16\mathbf{I}_z^1\mathbf{I}_z^2\mathbf{I}_z^3\mathbf{I}_z^4 + F^1F^2F^5F^{125}16\mathbf{I}_z^1\mathbf{I}_z^2\mathbf{I}_z^3\mathbf{I}_z^5$$
$$+ F^2F^4F^5F^{245}16\mathbf{I}_z^1\mathbf{I}_z^2\mathbf{I}_z^4\mathbf{I}_z^5 + F^1F^2F^3F^4F^5F^{12345}16\mathbf{I}_z^1\mathbf{I}_z^3\mathbf{I}_z^4\mathbf{I}_z^5$$
$$+ F^3F^4F^5F^{345}32\mathbf{I}_z^1\mathbf{I}_z^2\mathbf{I}_z^3\mathbf{I}_z^4\mathbf{I}_z^5]]\mathbf{S}'_\mathbf{C}\rangle^{2345}(\widetilde{\alpha}+\widetilde{\beta}2\mathbf{I}_x^1)$$
$$=(\alpha+\beta 2\mathbf{I}_x^1)\frac{1}{2}(1 + 2\mathbf{I}_z^1\frac{1}{8}(3F^1 + 3F^2 + 3F^3 + 3F^4 + 3F^5 - F^1F^2F^3F^{123} \tag{62}$$
$$- F^1F^2F^4F^{124} - F^1F^2F^5F^{125} - F^1F^3F^4F^{134} - F^1F^3F^5F^{135}$$
$$- F^1F^4F^5F^{145} - F^2F^3F^4F^{234} - F^2F^3F^5F^{235} - F^2F^4F^5F^{245}$$
$$- F^3F^4F^5F^{345} + 3F^1F^2F^3F^4F^5F^{12345}))(\widetilde{\alpha}+\widetilde{\beta}2\mathbf{I}_x^1)$$
$$=\frac{1}{2}[1 + 2\Re(\widetilde{\alpha}\beta)2\mathbf{I}_x^1 + (2\Im(\widetilde{\alpha}\beta)2\mathbf{I}_y^1 + (|\alpha|^2 - |\beta|^2)\mathbf{I}_z^1)\Theta(t)]$$

where

$$\Theta(t) \equiv \frac{1}{8}[3F^1 + 3F^2 + 3F^3 + 3F^4 + 3F^5 - F^1F^2F^3F^{123} - F^1F^2F^4F^{124} \tag{63}$$
$$- F^1F^2F^5F^{125} - F^1F^3F^4F^{134} - F^1F^3F^5F^{135} - F^1F^4F^5F^{145}$$
$$- F^2F^3F^4F^{234} - F^2F^3F^5F^{235} - F^2F^4F^5F^{245} - F^3F^4F^5F^{345}$$
$$+ 3F^1F^2F^3F^4F^5F^{12345}]$$

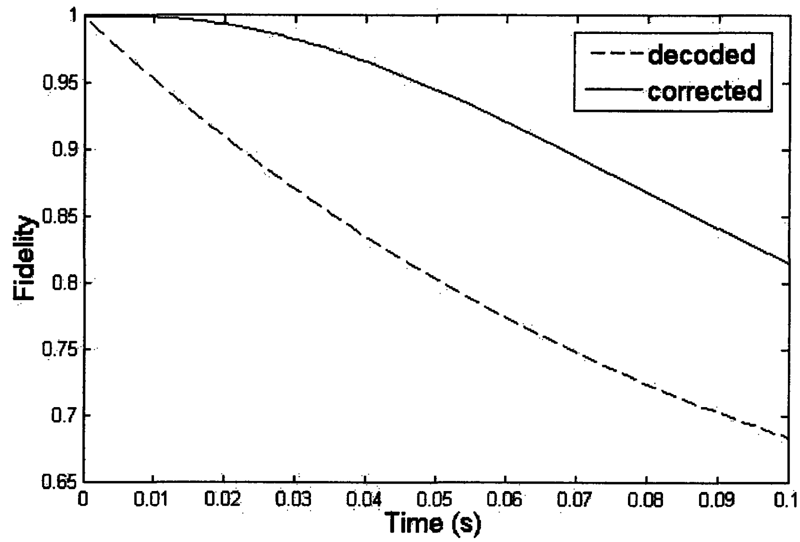

Figure 8: Plot of fidelity versus decoherence time delay for the corrected and uncorrected states.

Again, $F^i \equiv e^{-tc^{ii}}$ and $F^{ij} \equiv e^{-tc^{ij}}$ with tc^{ii} and tc^{ij} the variances and covariances between the spin phases of the five qubits. ρ_E^1 is the final, error-corrected state of the primary qubit. $\Theta(t)$ describes the the decay of the data qubit in the presence of random fields.

Now we can calculate the fidelity, f. The fidelity is a measure of the overlap of the initial and final states. It is given by

$$f = \langle \psi^{initial} | \rho^{final} | \psi^i nitial \rangle \tag{64}$$

We can find f using $|\psi\rangle = \alpha |0\rangle + \beta |1\rangle$, the intial state of the data qubit, and ρ_E^1, the corrected final state of the data qubit. We can compare this to the fidelity of ρ_D^1, the decoded state of the data qubit before error correction. $f_{corrected}$ and $f_{decoded}$ are displayed in Fig. (8). The time on the x-axis is the length of the time delay during the decoherence process.

This figure shows that the error corrected state has higher fidelity than the non-corrected state. The code does produce final states that are closer to the initial state than if there was no error correction.

4 Discussion

4.1 Results

We saw in Section 2 that the quantum error correction code displayed in Fig. 5 will correct 2nd order bit-flip errors. A detailed analysis of the operator algebra for this code produced the corrected state of the data qubit. In the prescence of decoherence, the data spin will decay according to $\Theta(t)$. If we can test the system experimentally, we can determine $\Theta(t)$ along with the variances and covariances.

We have seen that it is possible to correct for second order bit-flip errors, and we have computed the state of the five qubit system at each step in the error correction process. The code does indeed produce states with out errors in the data with higher fidelity than if the code were not used.

4.2 Further Work

The next step is to test this circuit experimentally. This would involve determining the pulse sequences required to execute each gate in the circuit. Errors correspond to decoherence from time delays, but we can simulate this decoherence with magnetic field gradients in an NMR experiment. We can apply the unitary operations to the system through RF pulses and periods of free revolution.

Such an experiment would use labeled spin-1/2 atoms. Each nucleus spin in a molecule has a different resonant frequency, which would allows us to apply pulses that flip individual spins after considering couplings between all the spins that will affect each operation.

An experiment could measure the variances and covariances between spins as well as the decay of the data spin, described theoretically by $\Theta(t)$. We could also measure the fidelities $f_{corrected}$ and $f_{decoded}$ and compare to the theoretical results.

Acknowledgments

Thank you to Professor David Cory for his guidance and encouragement throughout this work.

References

[1] D.G. Cory, et al. (1998). Experimental Quantum Error Correction. *Physical Review Letters*, 81, 2152-2155.

[2] Y. Sharf, D.G. Cory, S.S. Somaroo, T.F. Havel, E. Knill, R. Laflamme. (2000). A Study of Quantum Error Correction by Geometric Algebra and Liquid-State NMR Spectroscopy. *Molecular Physics.* 98(17), 1347-1363.

[3] E. Knill, R. Laflamme, R. martinez, C. Negreverrgne. (2001). Benchmarking Quantum Computers: The Five-Qubit Error Correcting Code. *Physical Review Letters*, 98, 5811-1814.

[4] G. Gilbert, M. Hamrick, S. Weinstein. (2007). Reliable Final Computational Results from Faulty Quantum Computation *ArXiv e-prints*, 707.

A Values for $\mathcal{F}_D$

$$
\begin{aligned}
\mathcal{F}_D[1] &= 1 \\
\mathcal{F}_D[2\mathbf{I}_z^1] &= 2\mathbf{I}_z^1 \\
\mathcal{F}_D[2\mathbf{I}_z^2] &= e^{-tc^{12}4\mathbf{I}_x^1\mathbf{I}_x^3}2\mathbf{I}_z^2 \\
\mathcal{F}_D[2\mathbf{I}_z^3] &= e^{-tc^{23}2\mathbf{I}_x^2}2\mathbf{I}_z^3 \\
\mathcal{F}_D[2\mathbf{I}_z^4] &= e^{-t(c^{12}4\mathbf{I}_x^1\mathbf{I}_x^3+c^{13}8\mathbf{I}_x^1\mathbf{I}_x^2\mathbf{I}_x^3+c^{14}8\mathbf{I}_x^1\mathbf{I}_x^2\mathbf{I}_x^5+c^{23}2\mathbf{I}_x^2+c^{24}8\mathbf{I}_x^2\mathbf{I}_x^3\mathbf{I}_x^5+c^{34}4\mathbf{I}_x^3\mathbf{I}_x^5)}2\mathbf{I}_z^4 \\
\mathcal{F}_D[2\mathbf{I}_z^5] &= e^{-t(c^{12}4\mathbf{I}_x^1\mathbf{I}_x^3+c^{13}4\mathbf{I}_x^1\mathbf{I}_x^2\mathbf{I}_x^3+c^{15}8\mathbf{I}_x^1\mathbf{I}_x^2\mathbf{I}_x^4+c^{23}2\mathbf{I}_x^2+c^{25}8\mathbf{I}_x^2\mathbf{I}_x^3\mathbf{I}_x^5+c^{35}4\mathbf{I}_x^3\mathbf{I}_x^4)}2\mathbf{I}_z^5 \\
\mathcal{F}_D[4\mathbf{I}_z^1\mathbf{I}_z^2] &= 4\mathbf{I}_z^1\mathbf{I}_z^2 \\
\mathcal{F}_D[4\mathbf{I}_z^1\mathbf{I}_z^3] &= e^{-t(c^{12}\mathbf{I}_x^1\mathbf{I}_x^3+c^{13}\mathbf{I}_x^1\mathbf{I}_x^2\mathbf{I}_x^3+c^{23}2\mathbf{I}_x^2)}4\mathbf{I}_z^1\mathbf{I}_z^3 \\
\mathcal{F}_D[4\mathbf{I}_z^1\mathbf{I}_z^4] &= e^{-t(c^{23}2\mathbf{I}_x^2+c^{24}8\mathbf{I}_x^2\mathbf{I}_x^3\mathbf{I}_x^5+c^{34}4\mathbf{I}_x^3\mathbf{I}_x^5)}4\mathbf{I}_z^1\mathbf{I}_z^4 \\
\mathcal{F}_D[4\mathbf{I}_z^1\mathbf{I}_z^5] &= e^{-t(c^{23}2\mathbf{I}_x^2+c^{25}8\mathbf{I}_x^2\mathbf{I}_x^3\mathbf{I}_x^4+c^{35}4\mathbf{I}_x^3\mathbf{I}_x^4)}4\mathbf{I}_z^1\mathbf{I}_z^5 \\
\mathcal{F}_D[4\mathbf{I}_z^2\mathbf{I}_z^3] &= e^{-tc^{13}8\mathbf{I}_x^1\mathbf{I}_x^2\mathbf{I}_x^3}4\mathbf{I}_z^2\mathbf{I}_z^3 \\
\mathcal{F}_D[4\mathbf{I}_z^2\mathbf{I}_z^4] &= e^{-tc^{34}4\mathbf{I}_x^3\mathbf{I}_x^5}4\mathbf{I}_z^2\mathbf{I}_z^4 \\
\mathcal{F}_D[4\mathbf{I}_z^2\mathbf{I}_z^5] &= e^{-tc^{35}4\mathbf{I}_x^3\mathbf{I}_x^4}4\mathbf{I}_z^2\mathbf{I}_z^5 \\
\mathcal{F}_D[4\mathbf{I}_z^3\mathbf{I}_z^4] &= e^{-tc^{14}8\mathbf{I}_x^1\mathbf{I}_x^2\mathbf{I}_x^5}4\mathbf{I}_z^3\mathbf{I}_z^4 \\
\mathcal{F}_D[4\mathbf{I}_z^3\mathbf{I}_z^5] &= e^{-tc^{15}4\mathbf{I}_x^1\mathbf{I}_x^2\mathbf{I}_x^4}4\mathbf{I}_z^3\mathbf{I}_z^5 \\
\mathcal{F}_D[4\mathbf{I}_z^4\mathbf{I}_z^5] &= e^{-tc^{45}4\mathbf{I}_x^4\mathbf{I}_x^5}4\mathbf{I}_z^4\mathbf{I}_z^5 \\
\mathcal{F}_D[8\mathbf{I}_z^1\mathbf{I}_z^2\mathbf{I}_z^3] &= 8\mathbf{I}_z^1\mathbf{I}_z^2\mathbf{I}_z^3 \\
\mathcal{F}_D[8\mathbf{I}_z^1\mathbf{I}_z^2\mathbf{I}_z^4] &= e^{-t(c^{13}8\mathbf{I}_x^1\mathbf{I}_x^2\mathbf{I}_x^3+c^{14}8\mathbf{I}_x^1\mathbf{I}_x^2\mathbf{I}_x^5+c^{34}4\mathbf{I}_x^3\mathbf{I}_x^5)}8\mathbf{I}_z^1\mathbf{I}_z^2\mathbf{I}_z^4 \\
\mathcal{F}_D[8\mathbf{I}_z^1\mathbf{I}_z^2\mathbf{I}_z^5] &= e^{-t(c^{13}8\mathbf{I}_x^1\mathbf{I}_x^2\mathbf{I}_x^3+c^{15}8\mathbf{I}_x^1\mathbf{I}_x^2\mathbf{I}_x^5+c^{35}4\mathbf{I}_x^3\mathbf{I}_x^4)}8\mathbf{I}_z^1\mathbf{I}_z^2\mathbf{I}_z^5 \\
\mathcal{F}_D[8\mathbf{I}_z^1\mathbf{I}_z^3\mathbf{I}_z^4] &= 8\mathbf{I}_z^1\mathbf{I}_z^3\mathbf{I}_z^4 \\
\mathcal{F}_D[8\mathbf{I}_z^1\mathbf{I}_z^3\mathbf{I}_z^5] &= 8\mathbf{I}_z^1\mathbf{I}_z^3\mathbf{I}_z^5 \\
\mathcal{F}_D[8\mathbf{I}_z^1\mathbf{I}_z^4\mathbf{I}_z^5] &= e^{-t(c^{14}8\mathbf{I}_x^1\mathbf{I}_x^2\mathbf{I}_x^5+c^{15}8\mathbf{I}_x^1\mathbf{I}_x^2\mathbf{I}_x^4+c^{45}4\mathbf{I}_x^4\mathbf{I}_x^5)}8\mathbf{I}_z^1\mathbf{I}_z^4\mathbf{I}_z^5 \\
\mathcal{F}_D[8\mathbf{I}_z^2\mathbf{I}_z^3\mathbf{I}_z^4] &= e^{-t(c^{24}8\mathbf{I}_x^2\mathbf{I}_x^3\mathbf{I}_x^5)}8\mathbf{I}_z^2\mathbf{I}_z^3\mathbf{I}_z^4 \\
\mathcal{F}_D[8\mathbf{I}_z^2\mathbf{I}_z^3\mathbf{I}_z^5] &= e^{-t(c^{25}8\mathbf{I}_x^2\mathbf{I}_x^3\mathbf{I}_x^4)}8\mathbf{I}_z^2\mathbf{I}_z^3\mathbf{I}_z^5 \\
\mathcal{F}_D[8\mathbf{I}_z^2\mathbf{I}_z^4\mathbf{I}_z^5] &= e^{-t(c^{12}4\mathbf{I}_x^1\mathbf{I}_x^3+c^{14}8\mathbf{I}_x^1\mathbf{I}_x^2\mathbf{I}_x^5+c^{15}8\mathbf{I}_x^1\mathbf{I}_x^2\mathbf{I}_x^4+c^{24}8\mathbf{I}_x^2\mathbf{I}_x^3\mathbf{I}_x^5+c^{25}8\mathbf{I}_x^2\mathbf{I}_x^3\mathbf{I}_x^4+c^{45}4\mathbf{I}_x^4\mathbf{I}_x^5)}8\mathbf{I}_z^2\mathbf{I}_z^4\mathbf{I}_z^5 \\
\mathcal{F}_D[8\mathbf{I}_z^3\mathbf{I}_z^4\mathbf{I}_z^5] &= e^{-t(c^{23}2\mathbf{I}_x^2+c^{24}8\mathbf{I}_x^2\mathbf{I}_x^3\mathbf{I}_x^5+c^{25}8\mathbf{I}_x^2\mathbf{I}_x^3\mathbf{I}_x^4+c^{34}4\mathbf{I}_x^3\mathbf{I}_x^5+c^{35}4\mathbf{I}_x^3\mathbf{I}_x^5+c^{45}4\mathbf{I}_x^4\mathbf{I}_x^5)}8\mathbf{I}_z^3\mathbf{I}_z^4\mathbf{I}_z^5 \\
\mathcal{F}_D[16\mathbf{I}_z^1\mathbf{I}_z^2\mathbf{I}_z^3\mathbf{I}_z^4] &= e^{-t(c^{12}4\mathbf{I}_x^1\mathbf{I}_x^3+c^{14}8\mathbf{I}_x^1\mathbf{I}_x^2\mathbf{I}_x^5+c^{24}8\mathbf{I}_x^2\mathbf{I}_x^3\mathbf{I}_x^5)}16\mathbf{I}_z^1\mathbf{I}_z^2\mathbf{I}_z^3\mathbf{I}_z^4 \\
\mathcal{F}_D[16\mathbf{I}_z^1\mathbf{I}_z^2\mathbf{I}_z^3\mathbf{I}_z^5] &= e^{-t(c^{12}4\mathbf{I}_x^1\mathbf{I}_x^3+c^{15}8\mathbf{I}_x^1\mathbf{I}_x^2\mathbf{I}_x^4+c^{25}8\mathbf{I}_x^2\mathbf{I}_x^3\mathbf{I}_x^4)}16\mathbf{I}_z^1\mathbf{I}_z^2\mathbf{I}_z^3\mathbf{I}_z^5
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{F}_D[16\mathbf{I}_z^1\mathbf{I}_z^2\mathbf{I}_z^4\mathbf{I}_z^5] &= e^{-t(c^{24}8\mathbf{I}_x^2\mathbf{I}_x^3\mathbf{I}_x^5+c^{25}8\mathbf{I}_x^2\mathbf{I}_x^3\mathbf{I}_x^4+c^{45}4\mathbf{I}_x^4\mathbf{I}_x^5)}16\mathbf{I}_z^1\mathbf{I}_z^2\mathbf{I}_z^4\mathbf{I}_z^5 \qquad (65)\\
\mathcal{F}_D[16\mathbf{I}_z^1\mathbf{I}_z^3\mathbf{I}_z^4\mathbf{I}_z^5] &= e^{-t(c^{12}4\mathbf{I}_x^1\mathbf{I}_x^3+c^{13}8\mathbf{I}_x^1\mathbf{I}_x^2\mathbf{I}_x^3+c^{14}8\mathbf{I}_x^1\mathbf{I}_x^2\mathbf{I}_x^5+c^{15}8\mathbf{I}_x^1\mathbf{I}_x^2\mathbf{I}_x^4+c^{23}2\mathbf{I}_x^2+c^{24}8\mathbf{I}_x^2\mathbf{I}_x^3\mathbf{I}_x^5+c^{25}8\mathbf{I}_x^2\mathbf{I}_x^3\mathbf{I}_x^4)}\\
&\quad\times e^{-t(c^{34}4\mathbf{I}_x^3\mathbf{I}_x^5+c^{35}4\mathbf{I}_x^3\mathbf{I}_x^4+c^{45}4\mathbf{I}_x^4\mathbf{I}_x^5)}16\mathbf{I}_z^1\mathbf{I}_z^3\mathbf{I}_z^4\mathbf{I}_z^5\\
\mathcal{F}_D[16\mathbf{I}_z^2\mathbf{I}_z^3\mathbf{I}_z^4\mathbf{I}_z^5] &= e^{-t(c^{13}8\mathbf{I}_x^1\mathbf{I}_x^2\mathbf{I}_x^3+c^{14}8\mathbf{I}_x^1\mathbf{I}_x^2\mathbf{I}_x^5+c^{15}8\mathbf{I}_x^1\mathbf{I}_x^2\mathbf{I}_x^4+c^{34}4\mathbf{I}_x^3\mathbf{I}_x^5+c^{35}4\mathbf{I}_x^3\mathbf{I}_x^4+c^{45}4\mathbf{I}_x^4\mathbf{I}_x^5)}16\mathbf{I}_z^2\mathbf{I}_z^3\mathbf{I}_z^4\mathbf{I}_z^5\\
\mathcal{F}_D[32\mathbf{I}_z^1\mathbf{I}_z^2\mathbf{I}_z^3\mathbf{I}_z^4\mathbf{I}_z^5] &= e^{-t(c^{34}4\mathbf{I}_x^3\mathbf{I}_x^5+c^{35}4\mathbf{I}_x^3\mathbf{I}_x^4+c^{45}4\mathbf{I}_x^4\mathbf{I}_x^5)}32\mathbf{I}_z^1\mathbf{I}_z^2\mathbf{I}_z^3\mathbf{I}_z^4\mathbf{I}_z^5
\end{aligned}
$$

B Values of $\mathbf{S_C}\mathcal{E}[X]\mathbf{S}'_\mathbf{C}$

$$\begin{aligned}
\mathbf{S_C}\mathcal{E}[\mathcal{F}_D[1]]\mathbf{S}'_\mathbf{C} =& 1 \\
\mathbf{S_C}\mathcal{E}[\mathcal{F}_D[2\mathbf{I}_z^1]]\mathbf{S}'_\mathbf{C} =& \frac{1}{8}[6\mathbf{I}_z^1 + 12\mathbf{I}_z^1\mathbf{I}_z^2 - 4\mathbf{I}_z^1\mathbf{I}_z^3 - 4\mathbf{I}_z^1\mathbf{I}_z^4 - 4\mathbf{I}_z^1\mathbf{I}_z^5 + 24\mathbf{I}_z^1\mathbf{I}_z^2\mathbf{I}_z^3 - 8\mathbf{I}_z^1\mathbf{I}_z^2\mathbf{I}_z^4 \\
& - 8\mathbf{I}_z^1\mathbf{I}_z^2\mathbf{I}_z^5 + 24\mathbf{I}_z^1\mathbf{I}_z^3\mathbf{I}_z^4 + 24\mathbf{I}_z^1\mathbf{I}_z^3\mathbf{I}_z^5 - 8\mathbf{I}_z^1\mathbf{I}_z^4\mathbf{I}_z^5 - 16\mathbf{I}_z^1\mathbf{I}_z^2\mathbf{I}_z^3\mathbf{I}_z^4 \\
& - 16\mathbf{I}_z^1\mathbf{I}_z^2\mathbf{I}_z^3\mathbf{I}_z^5 - 16\mathbf{I}_z^1\mathbf{I}_z^2\mathbf{I}_z^4\mathbf{I}_z^5 + 48\mathbf{I}_z^1\mathbf{I}_z^3\mathbf{I}_z^4\mathbf{I}_z^5 - 32\mathbf{I}_z^1\mathbf{I}_z^2\mathbf{I}_z^3\mathbf{I}_z^4\mathbf{I}_z^5] \\
\mathbf{S_C}\mathcal{E}[\mathcal{F}_D[2\mathbf{I}_z^2]]\mathbf{S}'_\mathbf{C} =& F^{12}2\mathbf{I}_z^2 \\
\mathbf{S_C}\mathcal{E}[\mathcal{F}_D[2\mathbf{I}_z^3]]\mathbf{S}'_\mathbf{C} =& F^{23}\frac{1}{2}[2\mathbf{I}_z^3 - 4\mathbf{I}_z^3\mathbf{I}_z^4 - 4\mathbf{I}_z^3\mathbf{I}_z^5 - 8\mathbf{I}_z^3\mathbf{I}_z^4\mathbf{I}_z^5] \\
\mathbf{S_C}\mathcal{E}[\mathcal{F}_D[2\mathbf{I}_z^4]]\mathbf{S}'_\mathbf{C} =& F^{1234}2\mathbf{I}_z^4 \\
\mathbf{S_C}\mathcal{E}[\mathcal{F}_D[2\mathbf{I}_z^5]]\mathbf{S}'_\mathbf{C} =& F^{1235}2\mathbf{I}_z^5 \\
\mathbf{S_C}\mathcal{E}[\mathcal{F}_D[4\mathbf{I}_z^1\mathbf{I}_z^2]]\mathbf{S}'_\mathbf{C} =& \frac{1}{8}[6\mathbf{I}_z^1 + 12\mathbf{I}_z^1\mathbf{I}_z^2 + 12\mathbf{I}_z^1\mathbf{I}_z^3 - 4\mathbf{I}_z^1\mathbf{I}_z^4 - 4\mathbf{I}_z^1\mathbf{I}_z^5 - 8\mathbf{I}_z^1\mathbf{I}_z^2\mathbf{I}_z^3 - 8\mathbf{I}_z^1\mathbf{I}_z^2\mathbf{I}_z^4 \\
& - 8\mathbf{I}_z^1\mathbf{I}_z^2\mathbf{I}_z^5 - 8\mathbf{I}_z^1\mathbf{I}_z^3\mathbf{I}_z^4 - 8\mathbf{I}_z^1\mathbf{I}_z^3\mathbf{I}_z^5 - 8\mathbf{I}_z^1\mathbf{I}_z^4\mathbf{I}_z^5 + 48\mathbf{I}_z^1\mathbf{I}_z^2\mathbf{I}_z^3\mathbf{I}_z^4 + 48\mathbf{I}_z^1\mathbf{I}_z^2\mathbf{I}_z^3\mathbf{I}_z^5 \\
& - 16\mathbf{I}_z^1\mathbf{I}_z^2\mathbf{I}_z^4\mathbf{I}_z^5 - 16\mathbf{I}_z^1\mathbf{I}_z^3\mathbf{I}_z^4\mathbf{I}_z^5 + 96\mathbf{I}_z^1\mathbf{I}_z^2\mathbf{I}_z^3\mathbf{I}_z^4\mathbf{I}_z^5] \\
\mathbf{S_C}\mathcal{E}[\mathcal{F}_D[4\mathbf{I}_z^1\mathbf{I}_z^3]]\mathbf{S}'_\mathbf{C} =& F^{123}\frac{1}{8}[-2\mathbf{I}_z^1 + 12\mathbf{I}_z^1\mathbf{I}_z^2 + 12\mathbf{I}_z^1\mathbf{I}_z^3 + 12\mathbf{I}_z^1\mathbf{I}_z^4 + 12\mathbf{I}_z^1\mathbf{I}_z^5 + 24\mathbf{I}_z^1\mathbf{I}_z^2\mathbf{I}_z^3 - 8\mathbf{I}_z^1\mathbf{I}_z^2\mathbf{I}_z^4 \\
& - 8\mathbf{I}_z^1\mathbf{I}_z^2\mathbf{I}_z^5 - 8\mathbf{I}_z^1\mathbf{I}_z^3\mathbf{I}_z^4 - 8\mathbf{I}_z^1\mathbf{I}_z^3\mathbf{I}_z^5 + 24\mathbf{I}_z^1\mathbf{I}_z^4\mathbf{I}_z^5 - 16\mathbf{I}_z^1\mathbf{I}_z^2\mathbf{I}_z^3\mathbf{I}_z^4 - 16\mathbf{I}_z^1\mathbf{I}_z^2\mathbf{I}_z^3\mathbf{I}_z^5 \\
& - 16\mathbf{I}_z^1\mathbf{I}_z^2\mathbf{I}_z^4\mathbf{I}_z^5 - 16\mathbf{I}_z^1\mathbf{I}_z^3\mathbf{I}_z^4\mathbf{I}_z^5 - 32\mathbf{I}_z^1\mathbf{I}_z^2\mathbf{I}_z^3\mathbf{I}_z^4\mathbf{I}_z^5 \\
\mathbf{S_C}\mathcal{E}[\mathcal{F}_D[4\mathbf{I}_z^1\mathbf{I}_z^4]]\mathbf{S}'_\mathbf{C} =& \frac{1}{8}F^{234}[-2\mathbf{I}_z^1 - 4\mathbf{I}_z^1\mathbf{I}_z^2 + 12\mathbf{I}_z^1\mathbf{I}_z^3 + 12\mathbf{I}_z^1\mathbf{I}_z^4 - 4\mathbf{I}_z^1\mathbf{I}_z^5 - 8\mathbf{I}_z^1\mathbf{I}_z^2\mathbf{I}_z^3 + 24\mathbf{I}_z^1\mathbf{I}_z^2\mathbf{I}_z^4 \\
& - 8\mathbf{I}_z^1\mathbf{I}_z^2\mathbf{I}_z^5 - 8\mathbf{I}_z^1\mathbf{I}_z^3\mathbf{I}_z^4 + 24\mathbf{I}_z^1\mathbf{I}_z^3\mathbf{I}_z^5 - 8\mathbf{I}_z^1\mathbf{I}_z^4\mathbf{I}_z^5 + 48\mathbf{I}_z^1\mathbf{I}_z^2\mathbf{I}_z^3\mathbf{I}_z^4 - 16\mathbf{I}_z^1\mathbf{I}_z^2\mathbf{I}_z^3\mathbf{I}_z^5 \\
& - 16\mathbf{I}_z^1\mathbf{I}_z^2\mathbf{I}_z^4\mathbf{I}_z^5 + 48\mathbf{I}_z^1\mathbf{I}_z^3\mathbf{I}_z^4\mathbf{I}_z^5 - 32\mathbf{I}_z^1\mathbf{I}_z^2\mathbf{I}_z^3\mathbf{I}_z^4\mathbf{I}_z^5] \\
\mathbf{S_C}\mathcal{E}[\mathcal{F}_D[4\mathbf{I}_z^1\mathbf{I}_z^5]]\mathbf{S}'_\mathbf{C} =& \frac{1}{8}F^{235}[-2\mathbf{I}_z^1 - 4\mathbf{I}_z^1\mathbf{I}_z^2 + 12\mathbf{I}_z^1\mathbf{I}_z^3 - 4\mathbf{I}_z^1\mathbf{I}_z^4 + 12\mathbf{I}_z^1\mathbf{I}_z^5 - 8\mathbf{I}_z^1\mathbf{I}_z^2\mathbf{I}_z^3 - 8\mathbf{I}_z^1\mathbf{I}_z^2\mathbf{I}_z^4 \\
& + 24\mathbf{I}_z^1\mathbf{I}_z^2\mathbf{I}_z^5 + 24\mathbf{I}_z^1\mathbf{I}_z^3\mathbf{I}_z^4 - 8\mathbf{I}_z^1\mathbf{I}_z^3\mathbf{I}_z^5 - 8\mathbf{I}_z^1\mathbf{I}_z^4\mathbf{I}_z^5 - 16\mathbf{I}_z^1\mathbf{I}_z^2\mathbf{I}_z^3\mathbf{I}_z^4 - 80\mathbf{I}_z^1\mathbf{I}_z^2\mathbf{I}_z^3\mathbf{I}_z^5 \\
& - 16\mathbf{I}_z^1\mathbf{I}_z^2\mathbf{I}_z^4\mathbf{I}_z^5 + 48\mathbf{I}_z^1\mathbf{I}_z^3\mathbf{I}_z^4\mathbf{I}_z^5 - 32\mathbf{I}_z^1\mathbf{I}_z^2\mathbf{I}_z^3\mathbf{I}_z^4\mathbf{I}_z^5]
\end{aligned} \tag{66}$$

$$
\begin{aligned}
\mathbf{S_C}\mathcal{E}[\mathcal{F}_D[8\mathbf{I}_z^1\mathbf{I}_z^2\mathbf{I}_z^3]]\mathbf{S}'_\mathbf{C} =& \frac{1}{8}[6\mathbf{I}_z^1 - 4\mathbf{I}_z^1\mathbf{I}_z^2 + 12\mathbf{I}_z^1\mathbf{I}_z^3 - 4\mathbf{I}_z^1\mathbf{I}_z^4 - 4\mathbf{I}_z^1\mathbf{I}_z^5 + 24\mathbf{I}_z^1\mathbf{I}_z^2\mathbf{I}_z^3 + 24\mathbf{I}_z^1\mathbf{I}_z^2\mathbf{I}_z^4 \\
&+ 24\mathbf{I}_z^1\mathbf{I}_z^2\mathbf{I}_z^5 - 8\mathbf{I}_z^1\mathbf{I}_z^3\mathbf{I}_z^4 - 8\mathbf{I}_z^1\mathbf{I}_z^3\mathbf{I}_z^5 - 8\mathbf{I}_z^1\mathbf{I}_z^4\mathbf{I}_z^5 - 16\mathbf{I}_z^1\mathbf{I}_z^2\mathbf{I}_z^3\mathbf{I}_z^4 \\
&- 16\mathbf{I}_z^1\mathbf{I}_z^2\mathbf{I}_z^3\mathbf{I}_z^5 + 48\mathbf{I}_z^1\mathbf{I}_z^2\mathbf{I}_z^4\mathbf{I}_z^5 - 16\mathbf{I}_z^1\mathbf{I}_z^3\mathbf{I}_z^4\mathbf{I}_z^5 - 32\mathbf{I}_z^1\mathbf{I}_z^2\mathbf{I}_z^3\mathbf{I}_z^4\mathbf{I}_z^5 \\
\mathbf{S_C}\mathcal{E}[\mathcal{F}_D[8\mathbf{I}_z^1\mathbf{I}_z^2\mathbf{I}_z^4]]\mathbf{S}'_\mathbf{C} =& \frac{1}{8}F^{134}[-2\mathbf{I}_z^1 - 4\mathbf{I}_z^1\mathbf{I}_z^2 + 12\mathbf{I}_z^1\mathbf{I}_z^3 + 12\mathbf{I}_z^1\mathbf{I}_z^4 - 4\mathbf{I}_z^1\mathbf{I}_z^5 - 8\mathbf{I}_z^1\mathbf{I}_z^2\mathbf{I}_z^3 + 24\mathbf{I}_z^1\mathbf{I}_z^2\mathbf{I}_z^4 \\
&- 8\mathbf{I}_z^1\mathbf{I}_z^2\mathbf{I}_z^5 - 8\mathbf{I}_z^1\mathbf{I}_z^3\mathbf{I}_z^4 + 24\mathbf{I}_z^1\mathbf{I}_z^3\mathbf{I}_z^5 - 8\mathbf{I}_z^1\mathbf{I}_z^4\mathbf{I}_z^5 + 48\mathbf{I}_z^1\mathbf{I}_z^2\mathbf{I}_z^3\mathbf{I}_z^4 \\
&- 16\mathbf{I}_z^1\mathbf{I}_z^2\mathbf{I}_z^3\mathbf{I}_z^5 - 16\mathbf{I}_z^1\mathbf{I}_z^2\mathbf{I}_z^4\mathbf{I}_z^5 + 48\mathbf{I}_z^1\mathbf{I}_z^3\mathbf{I}_z^4\mathbf{I}_z^5 - 32\mathbf{I}_z^1\mathbf{I}_z^2\mathbf{I}_z^3\mathbf{I}_z^4\mathbf{I}_z^5] \\
\mathbf{S_C}\mathcal{E}[\mathcal{F}_D[8\mathbf{I}_z^1\mathbf{I}_z^2\mathbf{I}_z^5]]\mathbf{S}'_\mathbf{C} =& \frac{1}{8}F^{135}[-2\mathbf{I}_z^1 - 4\mathbf{I}_z^1\mathbf{I}_z^2 + 12\mathbf{I}_z^1\mathbf{I}_z^3 - 4\mathbf{I}_z^1\mathbf{I}_z^4 + 12\mathbf{I}_z^1\mathbf{I}_z^5 - 8\mathbf{I}_z^1\mathbf{I}_z^2\mathbf{I}_z^3 - 8\mathbf{I}_z^1\mathbf{I}_z^2\mathbf{I}_z^4 \\
&+ 12\mathbf{I}_z^1\mathbf{I}_z^2\mathbf{I}_z^5 + 48\mathbf{I}_z^1\mathbf{I}_z^3\mathbf{I}_z^4 - 8\mathbf{I}_z^1\mathbf{I}_z^3\mathbf{I}_z^5 - 8\mathbf{I}_z^1\mathbf{I}_z^4\mathbf{I}_z^5 - 16\mathbf{I}_z^1\mathbf{I}_z^2\mathbf{I}_z^3\mathbf{I}_z^4 \\
&+ 48\mathbf{I}_z^1\mathbf{I}_z^2\mathbf{I}_z^3\mathbf{I}_z^5 - 16\mathbf{I}_z^1\mathbf{I}_z^2\mathbf{I}_z^4\mathbf{I}_z^5 + 48\mathbf{I}_z^1\mathbf{I}_z^3\mathbf{I}_z^4\mathbf{I}_z^5 - 32\mathbf{I}_z^1\mathbf{I}_z^2\mathbf{I}_z^3\mathbf{I}_z^4\mathbf{I}_z^5] \\
\mathbf{S_C}\mathcal{E}[\mathcal{F}_D[8\mathbf{I}_z^1\mathbf{I}_z^3\mathbf{I}_z^4]]\mathbf{S}'_\mathbf{C} =& \frac{1}{8}[6\mathbf{I}_z^1 - 4\mathbf{I}_z^1\mathbf{I}_z^2 - 4\mathbf{I}_z^1\mathbf{I}_z^3 - 4\mathbf{I}_z^1\mathbf{I}_z^4 + 12\mathbf{I}_z^1\mathbf{I}_z^5 - 8\mathbf{I}_z^1\mathbf{I}_z^2\mathbf{I}_z^3 + 24\mathbf{I}_z^1\mathbf{I}_z^2\mathbf{I}_z^4 \\
&- 8\mathbf{I}_z^1\mathbf{I}_z^2\mathbf{I}_z^5 + 24\mathbf{I}_z^1\mathbf{I}_z^3\mathbf{I}_z^4 - 16\mathbf{I}_z^1\mathbf{I}_z^3\mathbf{I}_z^5 + 24\mathbf{I}_z^1\mathbf{I}_z^4\mathbf{I}_z^5 + 48\mathbf{I}_z^1\mathbf{I}_z^2\mathbf{I}_z^3\mathbf{I}_z^4 \\
&- 16\mathbf{I}_z^1\mathbf{I}_z^2\mathbf{I}_z^3\mathbf{I}_z^5 - 16\mathbf{I}_z^1\mathbf{I}_z^2\mathbf{I}_z^4\mathbf{I}_z^5 - 16\mathbf{I}_z^1\mathbf{I}_z^3\mathbf{I}_z^4\mathbf{I}_z^5 - 32\mathbf{I}_z^1\mathbf{I}_z^2\mathbf{I}_z^3\mathbf{I}_z^4\mathbf{I}_z^5 \\
\mathbf{S_C}\mathcal{E}[\mathcal{F}_D[8\mathbf{I}_z^1\mathbf{I}_z^3\mathbf{I}_z^5]]\mathbf{S}'_\mathbf{C} =& \frac{1}{8}[6\mathbf{I}_z^1 - 4\mathbf{I}_z^1\mathbf{I}_z^2 - 4\mathbf{I}_z^1\mathbf{I}_z^3 + 12\mathbf{I}_z^1\mathbf{I}_z^4 - 4\mathbf{I}_z^1\mathbf{I}_z^5 - 8\mathbf{I}_z^1\mathbf{I}_z^2\mathbf{I}_z^3 - 8\mathbf{I}_z^1\mathbf{I}_z^2\mathbf{I}_z^4 \\
&+ 24\mathbf{I}_z^1\mathbf{I}_z^2\mathbf{I}_z^5 - 16\mathbf{I}_z^1\mathbf{I}_z^3\mathbf{I}_z^4 + 48\mathbf{I}_z^1\mathbf{I}_z^3\mathbf{I}_z^5 + 48\mathbf{I}_z^1\mathbf{I}_z^4\mathbf{I}_z^5 - 16\mathbf{I}_z^1\mathbf{I}_z^2\mathbf{I}_z^3\mathbf{I}_z^4 \\
&+ 48\mathbf{I}_z^1\mathbf{I}_z^2\mathbf{I}_z^3\mathbf{I}_z^5 - 16\mathbf{I}_z^1\mathbf{I}_z^2\mathbf{I}_z^4\mathbf{I}_z^5 - 16\mathbf{I}_z^1\mathbf{I}_z^3\mathbf{I}_z^4\mathbf{I}_z^5 - 32\mathbf{I}_z^1\mathbf{I}_z^2\mathbf{I}_z^3\mathbf{I}_z^4\mathbf{I}_z^5
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{S_C}\mathcal{E}[\mathcal{F}_D[8\mathbf{I}_z^1\mathbf{I}_z^4\mathbf{I}_z^5]]\mathbf{S}'_\mathbf{C} =& \frac{1}{8}F^{145}[-2\mathbf{I}_z^1 - 4\mathbf{I}_z^1\mathbf{I}_z^2 + 24\mathbf{I}_z^1\mathbf{I}_z^3 - 4\mathbf{I}_z^1\mathbf{I}_z^4 - 4\mathbf{I}_z^1\mathbf{I}_z^5 - 8\mathbf{I}_z^1\mathbf{I}_z^2\mathbf{I}_z^3 - 8\mathbf{I}_z^1\mathbf{I}_z^2\mathbf{I}_z^4 \\
&- 8\mathbf{I}_z^1\mathbf{I}_z^2\mathbf{I}_z^5 + 24\mathbf{I}_z^1\mathbf{I}_z^3\mathbf{I}_z^4 + 24\mathbf{I}_z^1\mathbf{I}_z^3\mathbf{I}_z^5 + 24\mathbf{I}_z^1\mathbf{I}_z^4\mathbf{I}_z^5 - 16\mathbf{I}_z^1\mathbf{I}_z^2\mathbf{I}_z^3\mathbf{I}_z^4 \\
&- 16\mathbf{I}_z^1\mathbf{I}_z^2\mathbf{I}_z^3\mathbf{I}_z^5 + 48\mathbf{I}_z^1\mathbf{I}_z^2\mathbf{I}_z^4\mathbf{I}_z^5 - 16\mathbf{I}_z^1\mathbf{I}_z^3\mathbf{I}_z^4\mathbf{I}_z^5 + 96\mathbf{I}_z^1\mathbf{I}_z^2\mathbf{I}_z^3\mathbf{I}_z^4\mathbf{I}_z^5] \\
\mathbf{S_C}\mathcal{E}[\mathcal{F}_D[16\mathbf{I}_z^1\mathbf{I}_z^2\mathbf{I}_z^3\mathbf{I}_z^4]]\mathbf{S}'_\mathbf{C} =& \frac{1}{8}F^{124}[-2\mathbf{I}_z^1 + 12\mathbf{I}_z^1\mathbf{I}_z^2 - 4\mathbf{I}_z^1\mathbf{I}_z^3 + 12\mathbf{I}_z^1\mathbf{I}_z^4 - 4\mathbf{I}_z^1\mathbf{I}_z^5 - 8\mathbf{I}_z^1\mathbf{I}_z^2\mathbf{I}_z^3 - 8\mathbf{I}_z^1\mathbf{I}_z^2\mathbf{I}_z^4 \\
&+ 24\mathbf{I}_z^1\mathbf{I}_z^2\mathbf{I}_z^5 + 24\mathbf{I}_z^1\mathbf{I}_z^3\mathbf{I}_z^4 - 8\mathbf{I}_z^1\mathbf{I}_z^3\mathbf{I}_z^5 - 8\mathbf{I}_z^1\mathbf{I}_z^4\mathbf{I}_z^5 + 48\mathbf{I}_z^1\mathbf{I}_z^2\mathbf{I}_z^3\mathbf{I}_z^4 \\
&- 16\mathbf{I}_z^1\mathbf{I}_z^2\mathbf{I}_z^3\mathbf{I}_z^5 + 48\mathbf{I}_z^1\mathbf{I}_z^2\mathbf{I}_z^4\mathbf{I}_z^5 - 16\mathbf{I}_z^1\mathbf{I}_z^3\mathbf{I}_z^4\mathbf{I}_z^5 - 32\mathbf{I}_z^1\mathbf{I}_z^2\mathbf{I}_z^3\mathbf{I}_z^4\mathbf{I}_z^5 \\
\mathbf{S_C}\mathcal{E}[\mathcal{F}_D[16\mathbf{I}_z^1\mathbf{I}_z^2\mathbf{I}_z^3\mathbf{I}_z^5]]\mathbf{S}'_\mathbf{C} =& \frac{1}{8}F^{125}[-2\mathbf{I}_z^1 + 12\mathbf{I}_z^1\mathbf{I}_z^2 - 4\mathbf{I}_z^1\mathbf{I}_z^3 - 4\mathbf{I}_z^1\mathbf{I}_z^4 + 12\mathbf{I}_z^1\mathbf{I}_z^5 - 8\mathbf{I}_z^1\mathbf{I}_z^2\mathbf{I}_z^3 - 8\mathbf{I}_z^1\mathbf{I}_z^3\mathbf{I}_z^4 \\
&+ 24\mathbf{I}_z^1\mathbf{I}_z^3\mathbf{I}_z^5 - 8\mathbf{I}_z^1\mathbf{I}_z^4\mathbf{I}_z^5 - 16\mathbf{I}_z^1\mathbf{I}_z^2\mathbf{I}_z^3\mathbf{I}_z^4 + 48\mathbf{I}_z^1\mathbf{I}_z^2\mathbf{I}_z^3\mathbf{I}_z^5 + 48\mathbf{I}_z^1\mathbf{I}_z^2\mathbf{I}_z^4\mathbf{I}_z^5 \\
&- 16\mathbf{I}_z^1\mathbf{I}_z^3\mathbf{I}_z^4\mathbf{I}_z^5 - 32\mathbf{I}_z^1\mathbf{I}_z^2\mathbf{I}_z^3\mathbf{I}_z^4\mathbf{I}_z^5
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{S_C}\mathcal{E}[\mathcal{F}_D[16\mathbf{I}_z^1\mathbf{I}_z^2\mathbf{I}_z^4\mathbf{I}_z^5]]\mathbf{S}'_\mathbf{C} =& \frac{1}{8}F^{245}[-2\mathbf{I}_z^1 - 4\mathbf{I}_z^1\mathbf{I}_z^2 - 4\mathbf{I}_z^1\mathbf{I}_z^3 - 4\mathbf{I}_z^1\mathbf{I}_z^4 - 4\mathbf{I}_z^1\mathbf{I}_z^5 + 24\mathbf{I}_z^1\mathbf{I}_z^2\mathbf{I}_z^3 \\
&+ 24\mathbf{I}_z^1\mathbf{I}_z^2\mathbf{I}_z^4 + 24\mathbf{I}_z^1\mathbf{I}_z^2\mathbf{I}_z^5 - 8\mathbf{I}_z^1\mathbf{I}_z^3\mathbf{I}_z^4 - 8\mathbf{I}_z^1\mathbf{I}_z^3\mathbf{I}_z^5 + 24\mathbf{I}_z^1\mathbf{I}_z^4\mathbf{I}_z^5 \\
&= 16\mathbf{I}_z^1\mathbf{I}_z^2\mathbf{I}_z^3\mathbf{I}_z^4 - 16\mathbf{I}_z^1\mathbf{I}_z^2\mathbf{I}_z^3\mathbf{I}_z^5 + 48\mathbf{I}_z^1\mathbf{I}_z^2\mathbf{I}_z^4\mathbf{I}_z^5 + 48\mathbf{I}_z^1\mathbf{I}_z^3\mathbf{I}_z^4\mathbf{I}_z^5 \\
&- 32\mathbf{I}_z^1\mathbf{I}_z^2\mathbf{I}_z^3\mathbf{I}_z^4\mathbf{I}_z^5] \\
\mathbf{S_C}\mathcal{E}[\mathcal{F}_D[16\mathbf{I}_z^1\mathbf{I}_z^3\mathbf{I}_z^4\mathbf{I}_z^5]]\mathbf{S}'_\mathbf{C} =& \frac{1}{8}F^{12345}[6\mathbf{I}_z^1 - 4\mathbf{I}_z^1\mathbf{I}_z^2 - 8\mathbf{I}_z^1\mathbf{I}_z^3 + 12\mathbf{I}_z^1\mathbf{I}_z^4 + 12\mathbf{I}_z^1\mathbf{I}_z^5 - 8\mathbf{I}_z^1\mathbf{I}_z^2\mathbf{I}_z^3 \\
&- 8\mathbf{I}_z^1\mathbf{I}_z^2\mathbf{I}_z^4 - 8\mathbf{I}_z^1\mathbf{I}_z^2\mathbf{I}_z^5 - 8\mathbf{I}_z^1\mathbf{I}_z^3\mathbf{I}_z^4 - 8\mathbf{I}_z^1\mathbf{I}_z^3\mathbf{I}_z^5 - 8\mathbf{I}_z^1\mathbf{I}_z^4\mathbf{I}_z^5 \\
&- 16\mathbf{I}_z^1\mathbf{I}_z^2\mathbf{I}_z^3\mathbf{I}_z^4 - 16\mathbf{I}_z^1\mathbf{I}_z^2\mathbf{I}_z^3\mathbf{I}_z^5 + 48\mathbf{I}_z^1\mathbf{I}_z^2\mathbf{I}_z^4\mathbf{I}_z^5 + 48\mathbf{I}_z^1\mathbf{I}_z^3\mathbf{I}_z^4\mathbf{I}_z^5 \\
&+ 96\mathbf{I}_z^1\mathbf{I}_z^2\mathbf{I}_z^3\mathbf{I}_z^4\mathbf{I}_z^5 \\
\mathbf{S_C}\mathcal{E}[\mathcal{F}_D[32\mathbf{I}_z^1\mathbf{I}_z^2\mathbf{I}_z^3\mathbf{I}_z^4\mathbf{I}_z^5]]\mathbf{S}'_\mathbf{C} =& \frac{1}{8}F^{345}[-2\mathbf{I}_z^1 + 12\mathbf{I}_z^1\mathbf{I}_z^2 - 4\mathbf{I}_z^1\mathbf{I}_z^3 - 4\mathbf{I}_z^1\mathbf{I}_z^4 - 4\mathbf{I}_z^1\mathbf{I}_z^5 - 8\mathbf{I}_z^1\mathbf{I}_z^2\mathbf{I}_z^3 \\
&+ 24\mathbf{I}_z^1\mathbf{I}_z^2\mathbf{I}_z^4 + 24\mathbf{I}_z^1\mathbf{I}_z^2\mathbf{I}_z^5 - 8\mathbf{I}_z^1\mathbf{I}_z^3\mathbf{I}_z^4 - 8\mathbf{I}_z^1\mathbf{I}_z^3\mathbf{I}_z^5 + 24\mathbf{I}_z^1\mathbf{I}_z^4\mathbf{I}_z^5 \\
&- 16\mathbf{I}_z^1\mathbf{I}_z^2\mathbf{I}_z^3\mathbf{I}_z^4 - 16\mathbf{I}_z^1\mathbf{I}_z^2\mathbf{I}_z^3\mathbf{I}_z^5 - 16\mathbf{I}_z^1\mathbf{I}_z^2\mathbf{I}_z^4\mathbf{I}_z^5 + 48\mathbf{I}_z^1\mathbf{I}_z^3\mathbf{I}_z^4\mathbf{I}_z^5 \\
&+ 96\mathbf{I}_z^1\mathbf{I}_z^2\mathbf{I}_z^3\mathbf{I}_z^4\mathbf{I}_z^5
\end{aligned}
$$