Experimental Determination of Damping Coefficients in the Stability of Airplanes

May, 1915

H. K. Chow

Aeronautical Engineering, XIII
EXPERIMENTAL DETERMINATION

OF

DAMPING COEFFICIENTS

IN THE STABILITY OF AEROPLANES.

May, 1915.

AERONAUTICAL ENGINEERING, XIII.
CHAPTER I.

THEORY OF SMALL OSCILLATIONS.
CHAPTER I.

THEORY OF SMALL OSCILLATIONS.

In this study we assume:

(1) that the oscillations are small;
(2) that terms higher than first powers are neglected;
(3) that there is a force of restitution proportional to the first power of the displacement;
(4) that there is a force of resistance proportional to the first power of velocity;
(5) that the above forces produce an effective force proportional to the first power of the acceleration of all the mass of the system;
(6) that the state of equilibrium is a function of the force of restitution, the force of resistance and the impressed force, and the final effect is the sum of these three forces, their products being neglected since the oscillations are small.

In symbols the above assumptions can be con-
tained in the equation
\[ \frac{d^2 x}{dt^2} + 2 a \frac{dx}{dt} + b x = 0 \quad \text{(1)} \]

when \( x \) denotes displacement, \( t \), time, \( a \) and \( b \) are constants. The first term is the effective force; the second term, the force of resistance; the third term, the force of restitution.

**Physical Derivation of Equation (1).**

Let Fig. 1 represent an oscillating apparatus. B is the support for the coil of spring, to the lower end of which is attached a weight of mass \( \frac{W}{g} \).
slugs. When the spring is deflected, the weight oscillates back and forth in a fluid. Let us analyze the motion from the instant when $W$ starts downward. $x$ gets greater and greater and $\frac{dx}{dt}$ is positive. But $\frac{d^2x}{dt^2}$ is negative, the body getting slower in its motion as $x$ increases.

If the stiffness of spring is such that a force of one pound elongates it $h$ feet, then for $x$ feet the upward force will be $\frac{x}{h}$ pounds, which is the impressed force. By D'Alembert's Principle the impressed force should be equal to effective force,

$$W \frac{d^2x}{dt^2} = \frac{x}{h}$$

since $\frac{d^2x}{dt^2}$ is negative.

$$\frac{Wd^2x}{g} + \frac{x}{h} = 0 \quad (2)$$

At the same time a force of resistance due to the friction in the fluid is acting in the same direction as the pull of the spring, being proportional to its velocity, or $k \frac{dx}{dt}$. Hence we need to add one more term to equation (2)
\[ \frac{w}{g} \frac{d^2 x}{d t^2} + k \frac{dx}{dt} + \frac{x}{h} = 0 \quad \ldots (3) \]

Equation (3) can be obtained by the direct application of D'Alembert's Principle. Thus if we put the inertia or effective force on the left hand side of the equation, and equate it to the algebraic sum of the impressed forces, we get

Effective force = \( \sum \) impressed forces

= Acting forces - resisting forces.

The acting forces in this case are zero,

\[ \therefore \text{effective force} + \text{resisting forces} = 0 \]

\[ \frac{w}{g} \frac{d^2 x}{d t^2} + k \frac{dx}{dt} + \frac{x}{h} = 0 \]

Dividing through by \( \frac{w}{g} \), we have

\[ \frac{d^2 x}{d t^2} + \left( \frac{gk}{w} \right) \frac{dx}{dt} + \left( \frac{g}{wh} \right) x = 0 \]

which is identically the same as (1) if we put

\[ 2 a = \frac{gk}{w}, \quad b = \frac{g}{wh} \]

Equation (3) is capable of being further am-
plified by introducing another term in consequence of the motion of support B itself. Suppose during time $t$ when $W$ has descended $x$ feet, support $B$ also descends a distance of $y$ feet. Then the real displacement is $x - y$ feet, and the real upward pull is $\frac{W - y}{h}$ pounds.

Hence:

$$\frac{W}{g} \frac{d^2 x}{dt^2} + k \frac{dx}{dt} \frac{x - y}{h} = 0$$

or

$$\frac{W}{g} \frac{d^2 x}{dt^2} + k \frac{dx}{dt} + \frac{x}{h} = \frac{y}{h} \quad \ldots \quad (4)$$

where $y$ may be zero or $f(t)$.

Equation (3) represents Damped Free Oscillation, (4) Damped Forced Oscillation. When $y$ in (4) is zero, the forced oscillation at once becomes Free Oscillation.
Meaning of Terms in Equation (4)

In equation (4),
\[ \frac{w}{g} \frac{d^2x}{dt^2} + k \frac{dx}{dt} + \frac{x}{h} = \frac{y}{h}, \]

beginning at the left, the first term is the inertia force due to mass; the second, the damping force due to friction; the third, the restoring force due to the pull of the spring; the fourth, the external force due to the forcing of the oscillation.

There are four terms in equation (4). Let us study what will result from omitting some of these terms either singly or in groups.

(a) \( \frac{dx}{dt} = 0, \frac{d^2x}{dt^2} = 0 \), motion of \( w = 0 \).

Spring is merely compressed. \( \frac{x}{h} = \frac{y}{h} \)

(b) \( \frac{x}{h} = 0, x = 0, \frac{dx}{dt} = 0 \), motion of a rigid body under the influence of external force.

It results in the well known formula
\[ \frac{w}{g} \frac{d^2x}{dt^2} = \frac{y}{h} = \text{force}. \]
of great stiffness, and it moves with the weight \( W \) as a rigid body by virtue of \( \frac{x}{h} \).

\[ \frac{dx}{dt} = 0, \quad \frac{y}{h} = 0 \quad \text{we have the case of} \]

S.H.M. in frictionless medium: \( \frac{w}{g} \frac{d^2x}{dt^2} + \frac{x}{h} = 0 \)

\[ \frac{dx}{dt} = 0, \quad \text{we have the case of superposed harmonic motion in a frictionless medium:} \]

\[ \frac{w}{g} \frac{d^2x}{dt^2} + \frac{x}{h} = \frac{y}{h} \]

\( (e) \) When all the terms are zero, it represents a state of rest of a rigid body.

We are, however, particularly interested in one condition, namely, \( \frac{y}{h} = 0 \). Equation \((e)\) becomes

\[ \frac{w}{g} \frac{d^2x}{dt^2} + \frac{dx}{dt} \frac{x}{h} + \frac{x}{h} = 0 \quad \ldots \quad (5) \]

which is the case of Free Oscillations.

Equation \((5)\) is exactly the same as Equation \((3)\) and can be put into familiar form like Equation \((1)\) by making suitable substitution of constants.

Thus \( \frac{d^2x}{dt^2} + 2a \frac{dx}{dt} + bx = 0 \).
Analogies between Linear Motion and Angular Motion.

There is a well known analogy between the equations for linear motion and those for angular motion.

<table>
<thead>
<tr>
<th>Linear Motion</th>
<th>Angular Motion</th>
</tr>
</thead>
<tbody>
<tr>
<td>Displacement</td>
<td>( x )</td>
</tr>
<tr>
<td>Velocity</td>
<td>( \frac{dx}{dt} = v )</td>
</tr>
<tr>
<td>Acceleration</td>
<td>( \frac{d^2x}{dt^2} = a )</td>
</tr>
<tr>
<td>Mass</td>
<td>( m )</td>
</tr>
<tr>
<td>Time</td>
<td>( t )</td>
</tr>
<tr>
<td>Displacement</td>
<td>( x = \frac{1}{2} a t^2 )</td>
</tr>
<tr>
<td>K.E.</td>
<td>( \frac{1}{2} M v^2 )</td>
</tr>
</tbody>
</table>

Hence we infer that what \( \frac{d^2x}{dt^2} + 2a\frac{dx}{dt} + bx = 0 \) is for linear motion will be \( \frac{d^2\theta}{dt^2} + 2a\frac{d\theta}{dt} + b\theta = 0 \) \( \cdots (6) \) for angular oscillations. The constants \( a \) and \( b \) contain \( m \) which will be substituted by \( I \) accordingly. That Equation (5) is correct can be proved by the direct application of D'Alembert's Principle, the procedure being the same as for the linear
We shall in future make use of Equation (6) in place of Equation (5) since the oscillations in this special investigation are of an angular nature.

**Analogy between Mechanical and Electrical Oscillations.**

Equation (4) gives for mechanical oscillations

\[ \frac{w}{\xi} \frac{d^2x}{dt^2} + b \frac{dx}{dt} + \frac{x}{h} = \frac{y}{h} \]

But it is well known in electricity that

\[ \frac{d^2v}{dt^2} + R \frac{dv}{dt} + \frac{v}{K} = \frac{e}{K} \]

Comparing the two equations, we find

The mass \( \frac{w}{\xi} \) corresponds with self-induction \( L \).

The friction per foot per second \( b \), corresponds with the resistance \( R \).

The displacement \( x \) corresponds with voltage \( v \), \( v \) being a direct function of \( q \) which is the real electrical displacement.

The want of stiffness of spring \( h \) corre-
sponds with the capacity of condenser $K$.

The forced displacement $y$ corresponds with the forced E.M.F. of an alternator.

Thus the theory of small oscillation is applicable to all forms of oscillation, be it mechanical, electrical, etc.

Solution of Equation (6).

Equation (6) is linear, and can be solved with the symbolic method. Let $D$ denote $\frac{d}{dt}$,

$$\frac{d^2}{dt^2}$$

$D^2 \frac{d^2}{dt^2}$, then Equation (6) becomes

$$(D^2 \phi + 2aD + b)\phi = 0 \quad \ldots \ldots (7)$$

The roots of (7) are

$$\lambda_1 \text{ and } \lambda_2 = -a \pm \sqrt{a^2 - b}.$$ 

The nature of the roots depends upon the character of the constants under the radical sign. There are four cases to be considered.

I. $a^2 > b$, roots all real, all negative

$$-\alpha, \quad -\beta$$

II. $a^2 = b$, roots all real, negative, equal

$$-a, \quad -a$$
III. \( a^2 < -b \) roots complex imaginary
- \( g + hi, -g - hi \)

IV. \( a^2 = 0 \) roots pure imaginary
- \( \pm \sqrt{b} i \pm \sqrt{b} i \)

In case I, our answer is:
\[
\phi = A e^{-\alpha t} + B e^{-\beta t}, \quad \cdots (8)
\]

In case II,
\[
\phi = (A + B t) e^{-\alpha t}; \quad \cdots (9)
\]

In case III,
\[
\phi = e^{-\gamma t} (\beta \sin \gamma t + \beta \cos \gamma t); \quad \cdots (10)
\]

In case IV,
\[
\phi = A \sin \sqrt{b} t + B \cos \sqrt{b} t, \quad \cdots (11)
\]

Where \( A \) and \( B \) in all cases are arbitrary constants. They can be determined by the conditions that \( \phi = 0 \) when \( t = 0 \) and \( \frac{d\phi}{dt} = \omega_0 \) when \( t = 0 \).

The graphs of equations (8), (9), (10), (11) are shown in Fig. 2.
It is evident both from the equations and from their graphs that in order that the motion may be oscillatory, the roots of (7) must be imaginary, complex or pure. That is, the quantity under the radical sign should be negative

\[ \sqrt{a^2 - b} = \sqrt{-c} \]

**Conditions of Instability of Motion.**

In the four cases considered, the motions are all stable. That this is so may be shown by an inspection of the graphs in Fig. 2. None of the graphs exhibits any tendency to increased amplitude with increase in time which is the cause of instability.
This stable condition is obtained by the assumption that \( b \) in
\[
\frac{d^2 \phi}{dt^2} + 2a \frac{d\phi}{dt} + b \phi = 0
\]
is a positive quantity, which renders the roots of
\[
D^2 + 2aD + b = 0
\]
either negative or imaginary which are criteria for stability. When we assume, and it does happen physically, that \( b \) is a negative quantity, then the quantity under the radical will be positive, and the roots are no longer imaginary. A real root means non-oscillatory motion. The roots may be both positive or one positive and the other negative. In the former case the solution of the equation becomes:
\[
\phi = A e^{\alpha_1 t} + B e^{\alpha_2 t} \quad \ldots \quad (12)
\]
where \( \alpha_1 \) and \( \alpha_2 \) are positive roots of the equation. In the latter case the solution is
\[
\phi = A e^{\beta_1 t} + B e^{-\beta_2 t} \quad \ldots \quad (13)
\]
where \( \beta_1 \) and \(-\beta_2\) are roots of the equation. \( A \) and \( B \) are arbitrary constants to be determined.

Equation (12) shows that the displacement in-
creases with time. Hence it represents unstable motion.

Equation (13) may be written

$$\phi = A e^{\omega_1 t} + B e^{-\omega_2 t}.$$  

when \( t \) is large, the second term may be neglected. When \( t \) is small, the first term may be neglected. The former corresponds to unstable motion, and the latter stable motion. It is therefore a case of doubtful stability.

**Dependence of Dynamic Stability upon Static Stability.**

We have seen that when \( b \) is positive a negative quantity under the radical sign,

$$\sqrt{a^2 - b},$$

means oscillatory and stable motion; and that when \( b \) is negative, the quantity under the radical sign will always be positive which means non-oscillatory and unstable motion. It is clear then that the dynamical stability depends upon whether \( b \) is positive or negative.

Let us examine the meaning of \( b \). Referring
to the original equation
\[ \frac{d^2 \theta}{dt^2} + 2a \frac{d \theta}{dt} + b \theta = 0 \]

we find, paralleling the interpretation of the terms on page 6, \( \frac{d^2 \theta}{dt^2} \) represents angular momentum;
\( \frac{d \theta}{dt} \) damping moment; \( b \theta \), static restoring moment. As a matter of fact, \( b \theta \) does not represent static restoring moment alone; it represents the algebraic sum of all static moments which has \( \theta \) as a multiplier. The final numerical result may be positive or negative. Usually we call a static restoring moment positive. Then all static upsetting moments will be negative.

It is evident that when \( b \theta \) is negative, the sum of static upsetting moment is greater than the sum of static restoring moment, which is a condition of static instability. Hence when \( b \theta \) is positive, i.e., \( b \) is positive, the machine is statically stable. When \( b \) is negative, the machine is statically unstable. But a positive \( b \) and a negative \( b \) also account for dynamical stability.
and instability. Hence the dependence of dynamical stability upon static stability.

The preceding discussion only applies to the case of free oscillation. For forced oscillation where the forcing and natural oscillations may synchronize, there may be dynamic instability without static instability.

Summary.

The foregoing conclusions may be summarized in tabular form for reference.
### Differential Equation for Small Natural Oscillations

\[ \frac{d^2\theta}{dt^2} + 2a \frac{d\theta}{dt} + b\theta = 0 \]

<table>
<thead>
<tr>
<th>Signs of ( b )</th>
<th>Conditions</th>
<th>Roots</th>
<th>Solution of the Equation</th>
<th>Graph</th>
<th>Type of Motion</th>
<th>Type of Dynamic Stability</th>
<th>Remarks</th>
</tr>
</thead>
<tbody>
<tr>
<td>( a^2 &gt; b )</td>
<td>Both real &amp; negative (-\alpha, -\beta)</td>
<td>[ \theta = Ae^{-\alpha t} + Be^{-\beta t} ]</td>
<td><img src="image1" alt="Graph" /></td>
<td>non-periodic subsidence</td>
<td>Stable</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( a^2 = b )</td>
<td>Both equal real &amp; negative (-\alpha, -\alpha)</td>
<td>[ \theta = (A + B t)e^{-\alpha t} ]</td>
<td><img src="image2" alt="Graph" /></td>
<td>&quot;</td>
<td>&quot;</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( a^2 &lt; b )</td>
<td>Both complex imaginary (-\alpha \mp \beta i)</td>
<td>[ \theta = e^{-\alpha t}(A \sin(\beta t) + B \cos(\beta t)) ] or [ \theta = Ae^{-\alpha t}\cos(\beta t + \gamma) ]</td>
<td><img src="image3" alt="Graph" /></td>
<td>damped oscillation</td>
<td>&quot;</td>
<td>Period ( \frac{2\pi}{\beta} )</td>
<td></td>
</tr>
<tr>
<td>( b = 0 )</td>
<td>Both pure imaginary ( \pm \beta i )</td>
<td>[ \theta = A \sin(\beta t) + B \cos(\beta t) ]</td>
<td><img src="image4" alt="Graph" /></td>
<td>undamped oscillation</td>
<td>Indifferent</td>
<td>Period ( \frac{2\pi}{\sqrt{b}} )</td>
<td></td>
</tr>
<tr>
<td>( b = )</td>
<td>One zero ( 0 ), one real &amp; negative (-2\alpha )</td>
<td>[ \theta = Ae^{-2\alpha t} ]</td>
<td><img src="image5" alt="Graph" /></td>
<td>&quot;</td>
<td>&quot;</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( b = - )</td>
<td>Both real one positive (-\omega_1, \omega_2)</td>
<td>[ \theta = Ae^{-\omega_1 t} + Be^{\omega_2 t} ]</td>
<td><img src="image6" alt="Graph" /></td>
<td>Doubtful</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
CHAPTER II.

APPLICATION OF THE THEORY TO THE

STABILITY OF AEROPLANES.
CHAPTER II.

APPLICATION OF THE THEORY TO THE STABILITY OF AEROPLANES

The Object of the Investigation.

The present investigation is undertaken with a view to determining experimentally certain of the rotary coefficients necessary for the solution of differential equations as set forth in Professor G. H. Bryan's book "Stability of Aeroplanes". In particular, attention was directed towards obtaining the damping coefficient due to pure pitching. However, the methods developed and the apparatus designed are equally applicable to the determination of other rotary coefficients where the oscillations are not of a forced nature.

System of Co-ordinates and Notation.

The system of co-ordinates and the notations used are the same as that used in British Blue Book 1912 - 1913. Report No. 77, page 142*.

The usual mathematical conventions are observed as to signs. Forces are positive when acting along the positive directions of the axes, angles and moments are positive when turning occurs.
# Table of Notations

<table>
<thead>
<tr>
<th>Axis</th>
<th>Longitudinal Name of Axis</th>
<th>Symbol for Force</th>
<th>Velocity Finite</th>
<th>Name of Angle</th>
<th>Symbol for Angle</th>
<th>Velocity Very Small</th>
<th>Name of Force</th>
<th>Symbol for Force</th>
</tr>
</thead>
<tbody>
<tr>
<td>x</td>
<td>Longitudinal</td>
<td>X U V U</td>
<td>Roll</td>
<td>L P P</td>
<td>L P P</td>
<td>L P P</td>
<td></td>
<td></td>
</tr>
<tr>
<td>y</td>
<td>Lateral</td>
<td>Y V v</td>
<td>Pitch</td>
<td>M Q q</td>
<td>M Q q</td>
<td>M Q q</td>
<td></td>
<td></td>
</tr>
<tr>
<td>z</td>
<td>Normal</td>
<td>Z W w</td>
<td>Yaw</td>
<td>N R r</td>
<td>N R r</td>
<td>N R r</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

*Forces of X, Y, Z are forces of unit mass.
Moments L, M, N are moments of forces of unit mass.
For actual forces and moments, multiply these with m, the mass.*
or tends to occur from oy to oz, oz to ox, and ox to oy.

**Preliminary Discussion of the Apparatus.**

![Diagram]

**Fig. 4.**

.Angle $\theta$ is exaggerated for the sake of clear-

Let Fig. 4 represent an apparatus for produc-
ing small oscillations. The model of an aero-
plane is mounted on the top of an upright piece which is rigidly attached to the horizontal rod. The whole rests on a knife edge, held down by two powerful helical springs.

Let the horizontal and vertical pieces be axes of $x$ and $z$, and the angular displacement be called $\theta$; $C$ be the distance of c.g. of the whole apparatus from the axis of rotation; $a$, $a$, the moment arm of springs $s$, $s$, $l$ the distance of force $X$ from axis of rotation.

Let the apparatus be deflected one way by giving it a slight push. A strong restoring moment $s a$ will bring it back to its original position. The inertia of the apparatus, however, carries it further than the original position. It is brought back again by $s a$, and carried farther by the inertia and so on. Thus we obtain a motion of oscillation.

It has been noted that a powerful spring was used. This is necessary in order that an oscillatory motion can be produced. For consider again the original equation
\[
\frac{d^2 \phi}{dt^2} + 2a \frac{d \phi}{dt} + b \phi = 0
\]

or
\[
D^2 + 2aD + b = 0
\]

the roots of which are \(-a \pm \sqrt{a^2 - b}\).

We must make \(b\) large so that the roots will be imaginary and the motion oscillatory.

A powerful spring gives a large \(b\). With this arrangement we expect the solution of Equation (6) to come under Case III, page 11, viz.:

\[
\phi = e^{-\xi t} (A \sin \omega t + B \cos \omega t)
\]
or what is the same thing

\[
\phi = C e^{-\xi t} \cos (\omega t + \gamma)
\]

where \(C\) and \(\gamma\) are arbitrary constants.

**Derivation of Equation Similar to Equation (6).**

The motion of an aeroplane in space is three-dimensional. Whatever effect a small variation of one quantity in one direction has will also set up variations of different amount in other directions.
Hence $X, Y, Z, L, M, N$, are functions of $u, v, w, p, q, r, \Theta, \phi$ and $\Psi$, which, however, can not be expressed in any simple form.

For the study of small oscillation such as the problem in hand, the general expression can be reduced to the form:

$$X = g \sin \theta + X_0 + uX_u + vX_v + wX_w + \rho X_p + qX_q + rX_r \ldots \quad (14)$$

$$Y = -g \cos \theta \sin \phi + Y_0 + uY_u + vY_v + wY_w + pY_p + qY_q + rY_r \ldots \quad (15)$$

$$Z = -g \cos \theta \cos \phi + Z_0 + uZ_u + vZ_v + wZ_w + pZ_p + qZ_q + rZ_r \ldots \quad (16)$$

$$L = L_0 + uL_u + vL_v + wL_w + pL_p + qL_q + rL_r \ldots \quad (17)$$

$$M = M_0 + uM_u + vM_v + wM_w + pM_p + qM_q + rM_r \ldots \quad (18)$$

$$N = N_0 + uN_u + vN_v + wN_w + pN_p + qN_q + rN_r \ldots \quad (19)$$

Equations (14), (15), (16), (17), (18), and (19) are obtained by expansion into Maclaurin series to terms of the first order, all terms higher than first power and all products of first powers are
neglected. This assumption is fundamental in Differential Calculus.

In the present investigation, \( Y = 0, \ L = 0, \ N = 0, \ g = 0. \) The aerodynamic force and moment acting on the model are respectively \( X \) and \( M. \) Again with this special arrangement of the apparatus, \( p = 0, \ r = 0, \ v = 0, \ w = 0. \) Hence

\[
X = X_0 + uX_u + qX_q \quad \cdots \cdots \cdots \cdots \quad (20)
\]

\[
M = M_0 + uM_u + qM_q \quad \cdots \cdots \cdots \cdots \quad (21)
\]

Since drift = \( mX \)

\[
\text{Drift} = mX_0 + muX_u + mqX_q \quad \cdots \cdots \cdots \cdots \quad (22)
\]

Again since pitching couple = \( mM \)

\[
\text{Pitching couple} = mM_0 + muM_u + mqM_q \quad (23)
\]

Having simplified so far, let us analyze the forces and couples acting on the model in a new position of equilibrium after being displaced \( \theta \) to the right, Fig. 4. The force of the wind can be resolved into a force of drift and a couple of pitching. They are moments acting about the pivot. To this must be added the moment due to the down pull of gravity at c.g. The
resisting moment is the moment of spring. By
D'Alembert's Principle,

Effective Moment = Acting Moments - Resisting Moments

or in symbols

\[ \frac{d^2 \theta}{dt^2} = D l + G + c W \sin \theta - S a \quad \ldots \quad (24) \]

In (24) I stands for Moment of Inertia in
slug-ft\(^2\), D, for drift; l, the distance from
c.g. of model to pivot; G, the pitching couple;
c, the distance of C.G. of whole apparatus to
the pivot; W, the mass of the whole apparatus;
S, the pull in the spring; a, the moment arm of
the spring. Making substitution of (22) for D,
(23) for G, \( \theta \) for \( \sin \theta \) since \( \theta \) is very small,
and \( S = S_0 + K a \theta \) where \( K \) is the stiffness of
the spring, we obtain

\[ \frac{d^2 \theta}{dt^2} = (m X_0 + m u X_u + m q X_q) l + (m M_0 + m M_u + m M_q) \]

\[ + c W \theta - S_0 a - K a^2 \theta \quad \ldots \quad (25) \]

For initial condition

\[ m X_0 l + m M_0 = S_0 a \]

or \[ (m X_0 l + m M_0) - S_0 a = 0 \quad \ldots \quad (26) \]
Subtracting (26) from (25), there remains
\[ \frac{d^2 \theta}{dt^2} = (m u x_u + m q x_q) l + m u x_u + m q M_q \]
\[ + c w \theta - k a^2 \theta \ldots \ldots (27) \]

Now we know, using \( \dot{\theta} = \frac{d\theta}{dt} \),

\[ \dot{\theta} = \dot{x}_1, \quad u = \dot{x}_1, \quad q = \dot{x}_2. \]

Making the substitution, we have

\[ \ddot{\theta} = (m \dot{x}_1 x_u + m \dot{x}_2 x_q) l + m \dot{x}_1 \dot{M}_u + m \dot{x}_2 \dot{M}_q \]
\[ + c w \theta - k a^2 \theta \ldots \ldots (28) \]

Collecting terms of \( \dot{x}_1 \) and \( \theta \), we obtain

\[ \ddot{\theta} = (m l^2 x_u + m l x_q + m l M_u + m M_q) \dot{x}_1 \]
\[ + (c w - k a^2) \theta \ldots \ldots (29) \]

Previous experiment* shows that the value for \( x_q \) is negligible when compared with \( M_q \).

Also from the consideration of horizontal steady flight, \( \dot{M}_u = 0 \). Hence (29) may be simplified further to

\[ I \dot{\theta} = (m M_q + m l^2 x_u) \dot{x}_1 + (c w - k a^2) \theta \ldots \ldots (30) \]
or

\[ \ddot{\theta} - \left( \frac{m M_q + m l^2 x_u}{I} \right) \dot{\theta} + \left( \frac{k a^2 - c w}{I} \right) \theta = 0 \ldots \ldots (31) \]

\( x_u = -0.14; \quad x_q = \pm 0.05; \quad M_q = -210. \)
Let \( \mu_1 \ddot{\theta} + \mu_2 \dot{\theta} = -\left(\alpha M_1 + m l^2 \frac{\ddot{\theta}}{l}\right) \dot{\theta} \), where \( \mu_0 \ddot{\theta} \) and \( \mu_2 \dot{\theta} \) stand for damping moments of the apparatus without the model and of the model alone respectively. Therefore (31) becomes,

\[
\frac{d^2 \theta}{dt^2} + \frac{\mu_0 + \mu}{I} \frac{d\theta}{dt} + \frac{K l^2 - C W}{I} \theta = 0 \quad \text{(32)}
\]

which is exactly similar to equation (6) page 8. The solution of (32) falls under Case III, page 11. It is to be remembered that the apparatus was designed to make the solution of the differential equation representing the motion of the system, come under Case III. Hence the solution of (32)

\[
\theta = C e^{-\frac{\mu_0 + \mu}{2 l} t} \cos \left\{ \sqrt{\frac{K l^2 - C W}{I} - \left(\frac{\mu_0 + \mu}{2 I}\right)^2} t + \gamma \right\}
\]

(33)

where \( C \) and \( \gamma \) are arbitrary constants, the first depending upon initial displacement the second depending upon initial velocity. At the moment when \( \theta \) becomes \( \theta_0 \), the amplitude of the swing
velocity is zero. Rewriting (33), we obtain

\[ \theta = \theta_0 \frac{\mu_0 + \mu}{2I} t \cos \left\{ \sqrt{\frac{K a^2 - CW}{I}} \left[ \left( \frac{\mu_0 + \mu}{2I} \right)^2 + o \right] \right\} \]  

(34)

**Meaning of Terms.**

The expression \( \frac{\mu_0 + \mu}{2I} \) is called logarithmic decrement. The expression

\[ \sqrt{\frac{K a^2 - CW}{I}} \left[ \left( \frac{\mu_0 + \mu}{2I} \right)^2 \right] \]

divided into \( 2\pi \)
gives period, i.e.,

\[ p = \frac{2\pi}{\sqrt{\frac{K a^2 - CW}{I}} \left[ \left( \frac{\mu_0 + \mu}{2I} \right)^2 \right]} \]  

(35)

Which one to Measure, Logarithmic Decrement or Period.

It is very easy to find \( p \) and it looks as though this is the way to find the coefficients \( x_u \) and \( x_q \) which are connected with \( \mu_0 \) and \( \mu \) by the relation \( \mu_0 + \mu = m x_q + m x_u \).
assuming that the moment of inertia of the whole apparatus is known. But the expression involves quantities like $K$, the stiffness of the spring, and $C$, the distance of c.g. of whole apparatus from the pivot, both of which are difficult to get, especially $K$.

Turning now to the expression for logarithmic decrement, we find that it contains coefficient of $\frac{d\theta}{dt}$ only which does not involve $K$ and $C$. It contains $I$ which must be determined just as $I$ in the expression for period must be determined. We see, therefore, there is every advantage in making use of the expression for logarithmic decrement for the determination of $mX_u$ and $m\dot{X}_u$. It is to be noted that $X_u$ can be determined by a static test of a model, and therefore the only unknown quantities in the expression are $m\dot{X}_u$ and $I$.

Referring to equation (33), if we draw a curve through the points of oscillation corresponding with
\[
\cos \sqrt{\frac{K a^2 - C W}{I} - \left(\frac{e_0 + \mu}{2 I}\right)^2} t = 1/2 \quad \cdots \quad (36)
\]
i.e., through the highest points, its equation is
\[
e = e_0 e^{-\frac{\mu}{2 I} t} \quad \cdots \quad (37)
\]
or
\[
\log_e \frac{\theta}{\theta_0} = -\frac{\mu_0 + \mu}{2 I} t \quad \cdots \quad (38)
\]

If we plot a curve of \( \log_e \frac{\theta}{\theta_0} \) on the

time \( t \) as base, the slope of the curve which is

a straight line will give \( \frac{\mu_0 + \mu}{2 I} \), from which,

when \( I \) is known, \( \mu \) can be obtained.

This decision of observing logarithmic decrement instead of period suggests the design of

an apparatus such as above outlined with the additional auxiliary apparatus to make such observation

possible. In the next chapter, we shall de-

scribe the apparatus and the manner in which it

was handled to perform an experiment.
CHAPTER III

DESCRIPTION OF THE APPARATUS

AND METHODS OF

PERFORMING THE EXPERIMENTS.
CHAPTER III.

DESCRIPTION OF APPARATUS

AND METHOD OF EXPERIMENT.

Requisites of a Good Apparatus.

The value of \( (m \mu_q + m l^2 x_u) \) obtained from the logarithmic decrement gives the total damping effect of the apparatus and the model, i.e., \( \mu_0 + \mu \). In order to find the damping of the model, we have to subtract \( \mu_0 \) from the observed values of the total damping. It is clear then, that the smaller the apparatus damping the greater the accuracy of the result for the model damping.

Apparatus damping is itself composed of two items, viz.: friction damping and wind damping. To reduce the former, all pivots were point contact pivots; to reduce the latter, parts of the apparatus were shielded.

The apparatus must have means for adding a known moment of inertia to the system without disturbing the position of c.g. either horizontally or vertically. This adding or subtracting of
moment of inertia from the system permits the adjustment of long or short period which is very necessary in view of the fact the oscillations are to be observed with eyes instead of a photographic plate.

The apparatus must have means to indicate the oscillation in a magnified degree. A mechanical magnifier, such as a long stick will offer too much apparatus damping by the air, and it is also impossible to obtain a stick of small mass rigid enough. We are to use a beam of light in place of a stick to do the magnification.

The apparatus must have means to produce the oscillation.
Fig. 5b  End View
The sketch shows in two views the arrangement of the apparatus for pitching experiments in the middle portion of the 4-foot wind tunnel in the Aerodynamic Laboratory of the Massachusetts Institute of Technology.

On the trumpet of the balance which forms the vertical axis is screwed a special fitting which gives the support to two pivot points. Above the points and across the tunnel rests a horizontal bar in well shaped shallow cavities. The line joining the two points is the axis of rotation for the system. A horizontal rod at right angles to the horizontal bar is secured to the latter, on which are made a number of cavities for the placing of the pointed ends of hooks attached to the vertical springs which maintain the oscillation. The tension in springs can be adjusted by turning the butterfly nuts at the lower ends of the spring.

To the two horizontal pieces is rigidly secured a vertical piece with such attachment at its top as to permit the mounting of an aero-plane model.
The transverse horizontal bar has a total reflecting mirror at its end. A beam of light coming from outside the window through a double convex spectacle lens from a long filament incandescent lamp is reflected upwards to the glass ceiling of the tunnel and is received on a piece of fine ground glass. When experiments are performed during day time, it is necessary to cover the glass window and ceiling with black cloth, leaving openings for the artificial light to pass through. The ground glass screen is also enclosed in a dark box. When the apparatus rocks about the lateral axis, i.e., $\omega y$, the reflected light is made to deviate from its normal or zero position. The oscillations are thus exactly reproduced and indicated on the ground glass screen. The angular motion is transformed to linear motion on the screen which can be observed and recorded. For this purpose a scale is devised whereby the linear displacements can be measured.

We have referred to the fact that the apparatus should include a device for producing the
motion. This is accomplished by the bell crank lever and cord underneath the floor of the tunnel, whereby the spring is given a slight jerk through the chord by the observer on the top of the tunnel. The spring reacts upon the apparatus and, upon releasing, sets the system to oscillation, which is a free or natural one since the supports for the springs are fixed during the motion.
Method of Experiment

How to Obtain Logarithmic Decrement.

The model used in this experiment is the model of a biplane hydroplane supplied by a company. It has a single pontoon, and the engine and passengers are enclosed in a short, blunt, sudden-terminating body, to the rear of which is placed a propeller which is, however, absent on the model. The tail and elevator are carried at the end of four sticks attached to the rear struts, sufficiently separated to allow the turning of the propeller. The ends of the sticks are collected two and two vertically, which, when covered, form a horizontal knife edge, but which is not covered in this case. The distance of the tail from c.g. of the model is about 6.5 inches. The machine has a small tail and a short lever arm for the tail.

The model is first mounted on the top of the upright piece with the nose towards the wind and with correct altitude. This arrangement is good for experiments in pitching only. For
yawing, with the nose heading the wind as before, the wings will be placed vertical. For rolling, the presentation and altitude of the model being the same as for pitching; the whole apparatus has to be turned 90° in the horizontal plane, so that the rolling will take place about an axis in the direction of the wind. It is to be noted that this shifting of apparatus necessitates a corresponding change in the arrangement of spring attachments and in the optical part of the apparatus.

The model having been mounted in the manner above described, the springs are then hooked on to the horizontal rods. If the period is too rapid, we can do one of the following things or the combination of them.

(a) Slacken up the spring.
(b) Put the hooks closer to the axis of rotation.
(c) Move out the weights on the horizontal rods.

(a) and (b) diminish the restoring moment of the spring, i.e., make $K a^2$ small in Equation (35).
resulting in a larger $p$. (c) increases the moment of inertia $I$ in the same equation which affects the period directly.

The zero is seen to have changed when the wind is on. This is due to the drift of the wind upon the model. He should not shift the box to adjust the new zero. He should signal to the other person to tighten or slacken the spring by turning the butterfly nuts. While his assistant is doing that he should watch the movement of the light till it comes back to the original zero, and then signal to the assistant to stop. This insures a true zero angle of incidence of the planes with the wind.

The damping effect of the tail is to be investigated at different speeds of the wind, and the wider the range of speeds the more reliable will be the result. This tunnel gives a wind of forty miles per hour as the highest speed. Let us say the speeds are to be $\frac{1}{2}0$, 

\[
p = \frac{2\pi}{\sqrt{\frac{kq^2 - CW}{I} - \left(\frac{\omega_0 + \mu}{2I}\right)^2}}
\]
35, 30, 25, 20 and 10 miles per hour. It is preferable that we use the high speed first and then slow it down.

The observer now signals to his assistant to give him a wind of forty miles per hour to begin with. The assistant makes the necessary adjustment for the new zero. The observer then gives a pull on the cord to set the apparatus to oscillate, counts the beats, and jots them down. He repeats the same process for at least a dozen times while his assistant is keeping the wind constant, and gets a dozen sets of numbers. If they come out pretty uniform, he need not repeat any further. He now takes up a stop watch and gets the period of the oscillation at that speed of the wind. (See Equation (35).)

He now signals to his assistant to slow down the wind to a speed previously agreed upon; tells him to make the adjustment of the new zero since the drift is changed with change of wind speed. Then he does the pulling of the cord, counting of the beats, and the taking of the period as described above. The whole thing is repeated for other wind speeds.
The Determination of Moment of Inertia 
of the Whole Apparatus.

Thus far we have dealt with the problem of finding the logarithmic decrement. In order that and can be determined it is necessary to eliminate by direct experiment or by calculation. The latter method is impracticable inasmuch as the aeroplane model is a complicated structure and the work is laborious as well as inaccurate.

There is a well known experimental method of finding the moment of inertia of a body that does not admit of geometrical treatment. The body is mounted on some oscillating apparatus and the period of oscillation is observed. The quantities, period and moment of inertia, are connected according to Equation (35)

\[ p = \frac{2 \pi}{\sqrt{\frac{K a^2 - C W}{I} - \left(\frac{\omega + \mu}{2 I}\right)^2}} \]

The correction damping term \( \left(\frac{\omega + \mu}{2 I}\right)^2 \) can be
neglected since the experiment is always performed in still air and apparatus friction is either zero or very small. Hence,

\[ p = \frac{2 \pi}{\sqrt{\frac{K a^2 - C W}{I}}} \]  

(39)

By adding a known moment of inertia \( I_1 \), we get a new period,

\[ p_1 = \frac{2 \pi}{\sqrt{\frac{K a^2 - C W}{I + I_1}}} \]  

(40)

From (39) and (40) we can find \( I \) provided \( K a^2 - C W \) does not change in the two experiments, and is equal to some unknown constant \( K \) which can be eliminated since we have two equations for two unknowns \( K \) and \( I \). From (39),

\[ p^2 = \frac{\frac{4 \pi^2 I}{K}}{1} \]  

(41)

From (40)

\[ \frac{p^2}{1} = \frac{\frac{4 \pi^2 (I + I_1)}{K}}{1} \]  

(42)

Dividing (42) by (41), we have
\[ \left( \frac{p_1}{p} \right)^2 = \frac{I + I_1}{I}, \quad \ldots \ldots \quad (\text{45}) \]

From which we obtain

\[ I = I_1 \left\{ \left( \frac{p_1}{p} \right)^2 - 1 \right\}, \quad \ldots \ldots \quad (\text{44}) \]

The condition that \( k a^2 - g \) must not change during the two experiments implies the fact that the setting of the spring must not change both as to its tension and moment arm; and further that the adding of the known moment of inertia \( I_1 \) must not change the mass of the apparatus nor the vertical position of the c.g.

The first condition is easily met. The second condition suggests that we must not introduce any extra mass from outside into the system, and that in order not to change the vertical c.g., the shifting weights must lie in the horizontal plane containing the axis of rotation. Both are satisfied by shifting the weights on the horizontal rods outwards preferably in equal amounts although not necessary.

The precaution taken for the constant c.g. is not absolutely necessary in view of the fact
that the influence of gravitational moment is small compared with spring and aerodynamic moments for small oscillations.

From Equations (41) and (42) we see that the period is affected by the square root of moment of inertia. It is evident, therefore, that in order to obtain an appreciable difference in period in the second experiment, we need to add a large known moment of inertia. This is done by substituting long arms for the short ones used in the experiments for logarithmic decrement, taking care that in doing so we have not introduced any extra mass into the system. By shifting the original weights on the new long rods to a good distance outward, we add a large moment of inertia. We must remember that $I \propto m x^2$, $p \propto \sqrt{I} \propto \sqrt{m} x$, and a change in $x$ is much more powerful than a change in $m$ as far as its influence on the period is concerned."

*The statement is only general. $m$ is not permitted to change in this case in accordance with the restriction stated on page 43."
Elimination of $\mu$

We have on page 28 separated the apparatus damping and the model damping by calling the former $\mu$ and the latter $\mu'$. It remains for us to determine the numerical value of $\mu$ with a view to determining the numerical value of $\mu'$ which may be used for stability calculations after proper correction for a scale of model and speed of wind. To do so, we must take away the model and repeat the experiments of logarithmic decrement and those of moment of inertia, the parts exposed to the wind as well as the mass of the apparatus being different from before. The equation expressing the motion is similar to Equation (32), page 28.

\[
\frac{d^2\theta}{dt^2} + \frac{d\theta}{dt} + \left(\frac{K a'^2 - CW'}{I'}\right) \theta = 0 \quad \ldots (45)
\]

the primes denoting new values of the quantities when the model is off the apparatus. The difference of $(\mu_0 + \mu)$, obtained from the first set of experiments with the model, and $\mu_0$ obtained
from the second set of experiments without the model, gives net for the model.
CHAPTER IV.

METHOD OF COMPUTATION FROM OBSERVED DATA AND ANALYSIS OF RESULTS.
CHAPTER IV.

METHOD OF COMPUTATION FROM

OBSERVED DATA AND ANALYSIS OF RESULTS.

Data for Logarithmic Decrement

when the Model is on.

---

### TABLE I

<table>
<thead>
<tr>
<th>Common Log Scale</th>
<th>0.000</th>
<th>0.301</th>
<th>0.699</th>
<th>0.1077</th>
</tr>
</thead>
<tbody>
<tr>
<td>log e, e = 7</td>
<td>0.846</td>
<td>0.699</td>
<td>0.477</td>
<td>0.301</td>
</tr>
<tr>
<td></td>
<td>0.000</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

| Period of Complete Swing in Seconds. | 0.834 0.820 0.794 0.774 0.747 0.734 0.727 |

---

*Interpolated.*
† in Column two are half amplitudes.
‡‡ Complete Swing =  A → B

Note:— The values of number of swings are the average values of a number of observations as shown under each wind speed. Common logs are used for convenience. They should be multiplied by 2.30 to get loge. Period in seconds was obtained with a stop-watch.
On page 29 we find the solution of the differential equation to be

$$\theta = \theta_0 e^{-\frac{\omega_0 + \mu}{2 I} t} \cos \sqrt{\frac{K a^2 - CW}{I} - \left(\frac{-\omega + \mu}{2 I}\right)^2} t$$

If we draw a curve through the points of oscillation corresponding with

$$\cos \sqrt{\frac{K a^2 - CW}{I} - \left(\frac{-\omega + \mu}{2 I}\right)^2 t} = 1,$$

i.e., through practically the highest points its equation is

$$\theta = \theta_0 e^{-\frac{\omega_0 + \mu}{2 I} t}$$

Taking logarithms on both sides, we have

$$\log_e \theta = -\frac{\omega_0 + \mu}{2 I} t + \log_e \theta_0$$

If we plot $\log_e \theta$ on a base of $t$, the graph will be a straight line and $\frac{\omega_0 + \mu}{2 I}$ will be the slope of the line. Chart I shows the general disposition of the curves. The ordinates represent the common log of the half amplitude, and the abscissae the number of complete swings which is a
function of time. A point A on the chart has the meaning that it takes little over 13 complete swings to reduce the half amplitude 7 on the scale to the half amplitude 2 on the same scale. Referring to Table I, we find the period of oscillation for 40 miles per hour to be .727 second. Therefore, it takes $13 \times .727 = 9.45$ seconds for the wind to damp the motion to $5/7$ of what it was. The rate of damping is:

$$\text{rate of damping} = \frac{\log_e 5/7 - \log_e 7}{9.45} = \frac{2.3(\log_e 5 - \log_e 7)}{9.45}$$

$$= \frac{2.3(0.7 - 0.845)}{9.45} = \frac{-0.334}{9.45}$$

$$= -0.034 \text{ per second.}$$

The following table shows the rate of damping in $\text{O per second}$ at different wind speeds, which is evidently the tangent of the angles made by the straight lines with the axis of abscissae in Chart I. In calculating the tangent, the sides of the largest triangle made by the inclined line with the
axes have been used. For instance, for \( \tan \alpha \),
(see Chart I) \( \frac{B\ C}{C\ A} \) is just as good as \( \frac{B\ E}{E\ D} \).

But the latter is preferred for the reason that we can measure \( B\ E, E\ D \) much more accurately than \( B\ C, C\ A \).
<table>
<thead>
<tr>
<th>Speed of Wind Miles per Hr.</th>
<th>Tan $\alpha$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0 ........................ 2.30 x .8</td>
<td>$\frac{2.30 \times 0.8}{72 \times 0.834}$ = 0.0540</td>
</tr>
<tr>
<td>10 ........................ 2.30 x .8</td>
<td>$\frac{2.30 \times 0.8}{45 \times 0.820}$ = 0.0599</td>
</tr>
<tr>
<td>20 ........................ 2.30 x 0.85</td>
<td>$\frac{2.30 \times 0.85}{36.5 \times 0.794}$ = 0.0674</td>
</tr>
<tr>
<td>25 ........................ 2.30 x 0.85</td>
<td>$\frac{2.30 \times 0.85}{33.5 \times 0.774}$ = 0.0754</td>
</tr>
<tr>
<td>30 ........................ 2.30 x 0.82</td>
<td>$\frac{2.30 \times 0.82}{30 \times 0.747}$ = 0.0840</td>
</tr>
<tr>
<td>35 ........................ 2.30 x 0.80</td>
<td>$\frac{2.30 \times 0.80}{26.5 \times 0.734}$ = 0.0946</td>
</tr>
<tr>
<td>40 ........................ 2.30 x 0.80</td>
<td>$\frac{2.30 \times 0.80}{21 \times 0.727}$ = 0.1177</td>
</tr>
</tbody>
</table>

Chart II shows the relation between the rate of damping and the speed of wind.

In saying that $\tan \alpha$ is the rate of damping expressed in per cent per second, we are as-
signing a unit to \( \frac{\mu - 0 + \mu}{2} \), which is equal to \( \tan \alpha \), and which is hitherto undefined. That the unit is correct may be seen from the expression \( \log \frac{5/7}{9.45} \) on page 51, which is similar to those under column of \( \tan \alpha \) in Table II. The numerator expresses the ratio of the half amplitudes and the denominator expresses the time in seconds during which this change takes place. From this it is obvious that we are interested only in the ratio of the amplitudes and not in the absolute value of the individual amplitudes, and that it is immaterial as to the unit of the scale,—inches, feet, radians, degrees, etc. This fact permits changing the scale at will. For example, the scale for the experiment without model was different from the one with the model.

Reference has been made to the fact that we were taking the half amplitudes. It is possible that the two halves are not symmetrical. In order to eliminate this error, the above experiment was repeated and the other half amplitude was taken.
The following tables are identical with Table I, page 49, and Table II on page 53.

**TABLE III.**

<table>
<thead>
<tr>
<th>Common $\theta$</th>
<th>No. of Complete Swings at Wind of Speed</th>
<th>$\Phi_0=7$</th>
</tr>
</thead>
<tbody>
<tr>
<td>log $\theta$</td>
<td>Scale</td>
<td>0 10 20 25 30 35 40</td>
</tr>
<tr>
<td>$\Phi_0=7$</td>
<td></td>
<td>mi/hr mi/hr mi/hr mi/hr mi/hr</td>
</tr>
<tr>
<td>0.000</td>
<td>1</td>
<td>--- --- --- --- --- ---</td>
</tr>
<tr>
<td>0.301</td>
<td>2</td>
<td>4.7 --- 12.6 1.1.0 9.9 9.9 10.0</td>
</tr>
<tr>
<td>0.477</td>
<td>3</td>
<td>2.8 --- 8.5 7.8 6.7 6.1 6.7</td>
</tr>
<tr>
<td>0.699</td>
<td>5</td>
<td>3.1 --- 9.5 8.8 6.7 6.7 7.0</td>
</tr>
<tr>
<td>0.836</td>
<td>7</td>
<td>2.1 --- 7.5 7.0 5.7 5.0 5.0</td>
</tr>
</tbody>
</table>

Period of Complete Swing in Seconds:

|                | 0.830 | 0.775 | 0.755 | 0.740 | 0.700 | 0.678 |

*Interpolated.*
<table>
<thead>
<tr>
<th>Speed of Wind</th>
<th>Tan</th>
<th>$\mu_0 + \mu$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>116 x 0.83</td>
<td>0.0202</td>
</tr>
<tr>
<td>20</td>
<td>2.3 x 0.845</td>
<td>0.0650</td>
</tr>
<tr>
<td>25</td>
<td>38.5 x 0.775</td>
<td>0.0694</td>
</tr>
<tr>
<td>30</td>
<td>2.3 x 0.845</td>
<td>0.0920</td>
</tr>
<tr>
<td>35</td>
<td>28.5 x 0.745</td>
<td>0.1010</td>
</tr>
<tr>
<td>40</td>
<td>27 x 0.678</td>
<td>0.1060</td>
</tr>
</tbody>
</table>

It is necessary to calculate the moment of inertia of the apparatus and the model together as a unit to eliminate $I$ in $\frac{\mu_0 + \mu}{2 I}$ shown in Table II and IV.
The distribution of mass is shown in Fig. 6, and the values of known moment of inertia are given in Table V.
# TABLE V.

<table>
<thead>
<tr>
<th>Items</th>
<th>Mass in Slugs</th>
<th>$x^2$ in ft.</th>
<th>$I$ in slug-ft$^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>A + A</td>
<td>0.0523</td>
<td>$(\frac{1.5}{12})^2$</td>
<td>0.005440</td>
</tr>
<tr>
<td>B + B</td>
<td>0.0161</td>
<td>$(\frac{1.1}{12})^2$</td>
<td>0.000135</td>
</tr>
<tr>
<td>C + C</td>
<td>0.0051</td>
<td>$(\frac{6/12}{3})^2$</td>
<td>0.000424</td>
</tr>
<tr>
<td>D + D</td>
<td>0.0204</td>
<td>$(\frac{2}{3})^2$</td>
<td>0.027200</td>
</tr>
<tr>
<td>E + Beam of 3&quot;</td>
<td>0.0062</td>
<td>---</td>
<td>0.00292</td>
</tr>
<tr>
<td>F</td>
<td>0.0340</td>
<td>---</td>
<td>negligibly small</td>
</tr>
<tr>
<td>Model &amp; Fitting</td>
<td>0.0323</td>
<td>---</td>
<td>To be found.</td>
</tr>
</tbody>
</table>

**Total 0.166$^3$ slugs = 5.36 pounds.**

*The moment of inertia of a slender rod about an axis perpendicular to it at its end is $\frac{l^2}{3}$, where $l$ is the length of the rod.*
Let $I$ be the moment of inertia of the apparatus and the model together. Then

$I - (A A + B B + C C)$ gives the moment of inertia of the upright and the model which are fixed in the apparatus. The value of $A A$, $B B$, $C C$, are found in Table V to be

- $A A \ldots \ldots \ldots \ldots \ldots \ldots 0.005440$
- $B B \ldots \ldots \ldots \ldots \ldots \ldots \ldots 0.000135$
- $C C \ldots \ldots \ldots \ldots \ldots \ldots \ldots 0.000424$

Total $0.005999 = 0.006$ say

Hence, in the experiment for moment of inertia, when $A A$, $B B$, $C C$ are removed and longer rods $D D$ are substituted, and when $A A$ are no longer 9" apart but some other distance (say 10" in this experiment), the moment of inertia of the system

\[(I - 0.006) + 0.00544 \times (5/45)^2 + 0.0272\]

Fixed parts New $I$ of $A A$ I of long rods.

(see Table V) $= I + 0.0279$.

When in the second experiment $A A$ are further removed from the axis of rotation (3") apart in this case), the moment of inertia is, by the
same reasoning,

\[
I - 0.006 + 0.005 \frac{1}{24} \times \left( \frac{22}{\frac{1}{4.5}} \right)^2 + 0.0272
\]

I of Fixed Parts. \quad \text{New } I \text{ of } AA \quad I \text{ of long rods.}

\[
= I + 0.1508
\]

These two moment of inertia of the two systems are to each other as the squares of the periods.

Hence

\[
\frac{p^2}{p_1} = \frac{I + 0.1508}{I + 0.0279}
\]

in which \( p_1 > p \). We find that \( p_1 = 1.113 \text{ sec.} \)

\[
\frac{1.113}{0.676} = \frac{I + 0.1508}{I + 0.0279} = 2.72
\]

From this we find \( I = 0.0438 \text{ slug-ft}^2 \).

We will take the average values of \( \frac{\mu_0 + \mu}{2I} \) in Tables II and IV on pages 53 and 56 and multiply them by \( 2I \) to get rid of \( I \).


<table>
<thead>
<tr>
<th>Speed of Wind mi/hr.</th>
<th>( \frac{\mu_0 + \mu}{2 \bar{I}}_1 )</th>
<th>( \frac{\mu_0 + \mu}{2 \bar{I}}_2 )</th>
<th>Ave. ( \frac{\mu_0 + \mu}{2 \bar{I}} )</th>
<th>( \mu_0 + \mu )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.03140</td>
<td>0.0202</td>
<td>0.0271</td>
<td>0.00238</td>
</tr>
<tr>
<td>10</td>
<td>0.0499</td>
<td>...</td>
<td>0.0499</td>
<td>0.00437</td>
</tr>
<tr>
<td>20</td>
<td>0.0674</td>
<td>0.0650</td>
<td>0.0662</td>
<td>0.00580</td>
</tr>
<tr>
<td>25</td>
<td>0.0754</td>
<td>0.0694</td>
<td>0.0724</td>
<td>0.00634</td>
</tr>
<tr>
<td>30</td>
<td>0.0810</td>
<td>0.0920</td>
<td>0.0880</td>
<td>0.00770</td>
</tr>
<tr>
<td>35</td>
<td>0.0946</td>
<td>0.1010</td>
<td>0.0978</td>
<td>0.00856</td>
</tr>
<tr>
<td>40</td>
<td>0.1177</td>
<td>0.1060</td>
<td>0.1180</td>
<td>0.00980</td>
</tr>
</tbody>
</table>

\[ \bar{I} = 0.0438 \quad 2 \bar{I} = 0.0876 \]
It remains for us to get rid of $\omega_0$ in the last column of Table VI. The model was removed and readings for logarithmic decrement both with and without wind were taken, for apparatus damping is itself composed of frictional damping and aerodynamic damping.

**TABLE VII.**

<table>
<thead>
<tr>
<th>$\log_{10} \omega$</th>
<th>$\omega$ Scale</th>
<th>No. of Complete Swings at Wind Speed of $0.30$ mi/hr.</th>
</tr>
</thead>
<tbody>
<tr>
<td>$0.778$</td>
<td>$6$</td>
<td>$0$</td>
</tr>
<tr>
<td>$0.60\frac{1}{4}$</td>
<td>$4$</td>
<td>$1^{1/4}$</td>
</tr>
<tr>
<td>$0.477$</td>
<td>$3$</td>
<td>$17.1$</td>
</tr>
<tr>
<td>$0.301$</td>
<td>$2$</td>
<td>$29.1$</td>
</tr>
<tr>
<td>$0.000$</td>
<td>$0$</td>
<td>$27.5$</td>
</tr>
</tbody>
</table>

**Table Notes:**

On Chart III we find curves similar to those in Chart I and II, from which we obtain the following:
in which $I$ is the moment of inertia of the apparatus alone. The difference of the two gives 0.0218 which must be interpreted as additional damping due to the 30 mi/hr. wind. If we assume, and that assumption is probably correct for the parts exposed to wind stream were very small, that wind damping is a linear function of wind speed, then for other speeds, the wind damping will be

$$\frac{0.0218}{30} \times \text{speed.}$$

Thus
TABLE VIII.

<table>
<thead>
<tr>
<th>Wind Speed (mi/hr.)</th>
<th>Frictional Damping</th>
<th>Aerodynamic Damping</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.0632</td>
<td>0</td>
<td>0.0632</td>
</tr>
<tr>
<td>10</td>
<td>0.0073</td>
<td>0.0705</td>
<td>0.0705</td>
</tr>
<tr>
<td>20</td>
<td>0.0145</td>
<td>0.0777</td>
<td>0.0777</td>
</tr>
<tr>
<td>25</td>
<td>0.0182</td>
<td>0.0814</td>
<td>0.0814</td>
</tr>
<tr>
<td>30</td>
<td>0.0218</td>
<td>0.0850</td>
<td>0.0850</td>
</tr>
<tr>
<td>35</td>
<td>0.0254</td>
<td>0.0886</td>
<td>0.0886</td>
</tr>
<tr>
<td>40</td>
<td>0.0291</td>
<td>0.0923</td>
<td>0.0923</td>
</tr>
</tbody>
</table>

Moment of Inertia

A A 0.0054\text{\textfrac{4}{8}}
B B 0.000135
C C 0.000424
E 0.002920
Total I = 0.008919 = 0.009 slug-ft^2
The Net Damping of the Model.

Combining Tables VI and VIII, we obtain

**TABLE IX.**

<table>
<thead>
<tr>
<th>Wind Speed mi/hr.</th>
<th>Ft./Sec.</th>
<th>$\mu_0$</th>
<th>$\mu$</th>
<th>$\mu_{net}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0.00238</td>
<td>0.00114</td>
<td>0.00124</td>
</tr>
<tr>
<td>10</td>
<td>14.5</td>
<td>0.00137</td>
<td>0.00127</td>
<td>0.00310</td>
</tr>
<tr>
<td>20</td>
<td>29.0</td>
<td>0.00580</td>
<td>0.00140</td>
<td>0.00720</td>
</tr>
<tr>
<td>25</td>
<td>37.0</td>
<td>0.00634</td>
<td>0.00147</td>
<td>0.00787</td>
</tr>
<tr>
<td>30</td>
<td>44.0</td>
<td>0.00770</td>
<td>0.00153</td>
<td>0.00923</td>
</tr>
<tr>
<td>35</td>
<td>51.0</td>
<td>0.00856</td>
<td>0.00160</td>
<td>0.00916</td>
</tr>
<tr>
<td>40</td>
<td>58.5</td>
<td>0.00980</td>
<td>0.00166</td>
<td>0.00846</td>
</tr>
</tbody>
</table>

In Chart IV, we plot $\mu_{net}$ against speed in miles per hour. With due allowance for experimental errors, a straight line can be drawn through these points. Then the tangent $= \frac{\mu}{U}$ will be constants for all speeds, where $U$ is the relative velocity of the wind to the model or to an aeroplane. Taking $\mu$ at 30 miles per hour,
We find the constant rate to be
\[
\frac{0.00617}{30} = 0.000206 \quad \text{if } U \text{ is in mi/hr.}
\]
or
\[
\frac{0.00617}{4.4} = 0.00140 \quad \text{if } U \text{ is in ft./sec.}
\]

Hence, using ft. sec. units,
\[
= 0.00140 U \quad \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots (46)
\]

Since \( \mu = -(m M_q + m X_u) \), page 18
\[
m M_q + m X_u = -0.0014 U \quad \ldots \ldots (47)
\]

From (47) when \( X_u \) is known, \( m M_q \) may be found. In general \( X_u \) is very small compared with \( M_q \) and may be neglected.
\[
:\quad m M_q = -0.0014 U \quad \ldots \ldots \ldots \ldots (48)
\]
Stepping up of the Result for Model to Full Scale Aeroplane.

$m H_q$ in (48) applies to model only. It is necessary to consider the dimension of $m H_q$.

In the original equation

$$\frac{d^2 \theta}{dt^2} + (\mu_o + \mu) \frac{d \theta}{dt} + (K_2 \varepsilon - \omega) \theta = 0,$$

Each term expresses a moment and has the dimension

$$\left( \frac{\mu_o + \mu}{T} \right) \frac{d \theta}{dt} = \text{moment}$$

$$= \text{force} \times \text{distance}$$

$$= m \frac{1}{T^2} X 1 = m \frac{l^2}{T^2}$$

$$\frac{\mu_o + \mu}{T} = m \frac{l^2}{T^2}$$

$$\frac{\mu_o + \mu}{T} = m \frac{l^2}{T} = m \frac{1}{T}$$

$$= \frac{l^3}{1^3} \frac{m}{T} \frac{1}{T} = \frac{m}{l^3} l^4 v$$

$$= \rho l^4 v \ldots \ldots (49)$$
where \( \rho \) is the density of the fluid. This is correct provided a certain function

\[
\frac{v^1 l \dot{\rho}}{\nu}
\]

is kept approximately constant, in which \( \nu \) is the kinetic viscosity of the fluid.

From (49) it is evident that \( \mu \) is proportional to \( l^{ \frac{1}{2} } \). If the scale of the model is \( 1/20 \) of the full size machine then

\[
mH_q = - (20)^{ \frac{1}{2} } \times 0.0001^{ \frac{1}{2} } U
\]

\[
= - 22.4 U
\]

If \( U = 80 \text{ ft./sec.} \) (55 mi p.h.),

\( m = \frac{40}{40} \) slugs, then

\[
H_q = - \frac{22.4 \times 80}{40} = - 44.8
\]
Discussion of the Result.

In Report No. 78 of "Technical Report of the Advisory Committee on Aeronautics, 1912-1'13," an experimental investigation of a similar character was undertaken upon a model of a monoplane of the Blériot Type. The drawing of the model is found in Fig. 4 Report No. 75. The logarithmic decrement was photographed instead of being observed with the eye, as in the case of the present investigation. Among other things, the Report gives for \( m M_q \)

\[
m M_q = -\omega = -0.00069 \text{ U for model}
\]

on top of page 177, which is about five times larger than the value obtained in this experiment.

\[
-0.00069\div -0.00014 = 5
\]

This discrepancy can be explained by the fact that the two models differ a great deal both in the disposition of tail surfaces, and in the
character of the body. The British model has a large tail far removed from the c.g. and has an enclosed body with its knife-edge horizontal. Not only the damping surface is large but also the moment arm of the damping moment is much longer. On the other hand, our model has a small tail comparatively close to the c.g. of the model, and has not anything like a body to speak of. (See Frontispiece). We naturally expect less damping in our model than that in the British model.

If we take into consideration the scale of linear dimensions of the moment arms of the damping moments of the two models, we shall find the results agree very well. The damping coefficients are to each other as the fourth power of the linear dimensions.

\[
\left( \frac{l_1 \text{ (British Model)}}{l_2 \text{ (our model)}} \right)^4 = 5
\]

\[
\frac{l_1}{l_2} = \sqrt[4]{5} = 1.5 \text{ approximately}
\]

\[
l_1 = 1.5 \times l_2
\]
\[ l_1 = 1\frac{1}{2} \text{ inches} \quad l_2 = \frac{1\frac{1}{4}}{1.5} = 9.3 \text{ ins.} \]

\[ l_2 \text{ actually measured} = 8 \text{ inches}. \]
Conclusion.

In the absence of other coefficients which are necessary for the calculation of stability, it is difficult to say whether the machine will be longitudinally stable or unstable. One thing is certain, however, that the damping coefficient for pitching for this particular machine is only one-fifth that of the Blériot Monoplane as reported in the British Report.

The End.