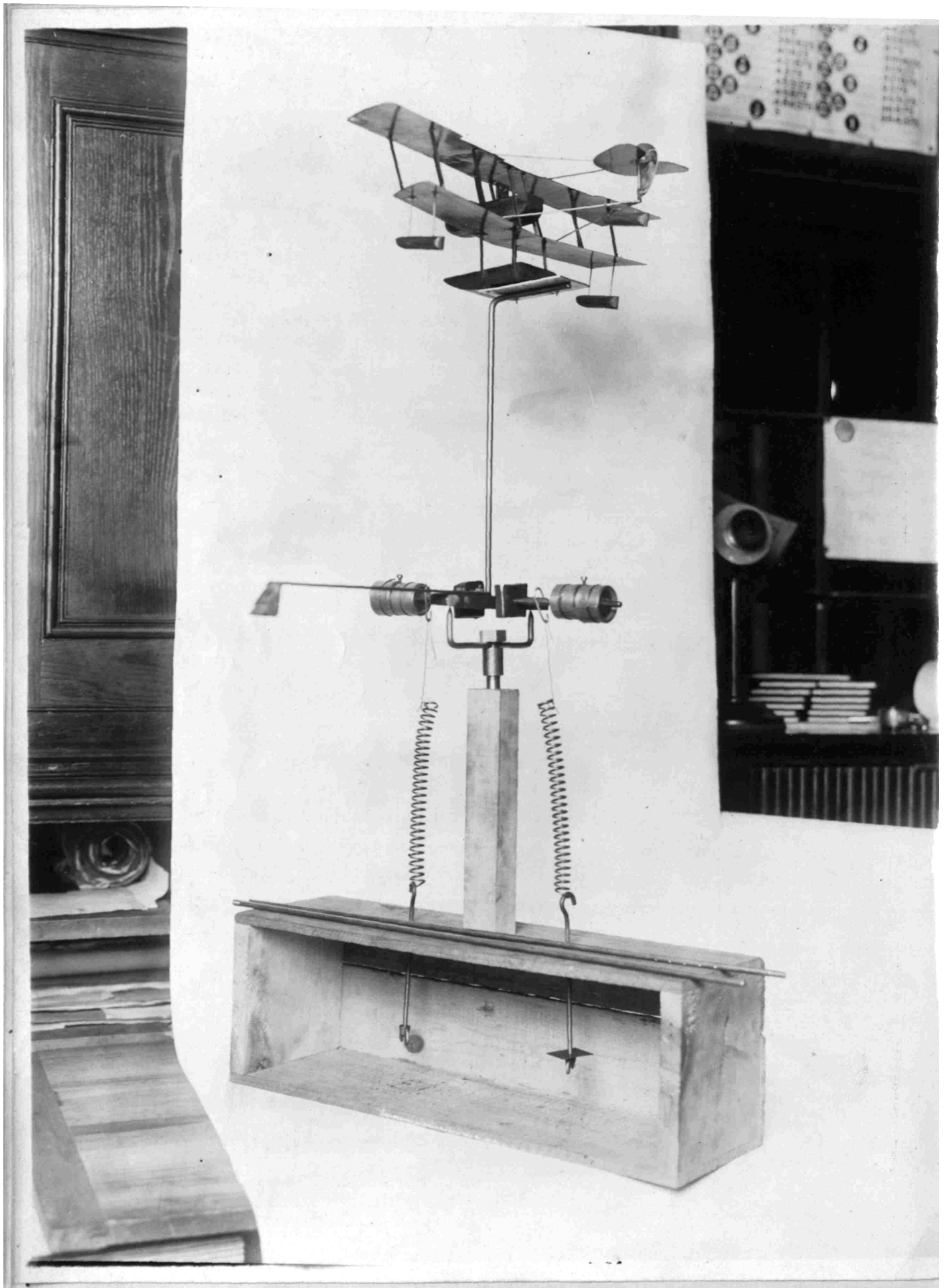Experimental Determination
of
Damping Coefficients
in the Stability of Aeroplanes.

May, 1915.

H. K. Chow

Aeronautical Engineering, XIII.

EXPERIMENTAL DETERMINATION

OF

DAMPING COEFFICIENTS

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CHAPTER I.

THEORY OF SMALL OSCILLATIONS.

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In this study we assume:

(1) that the oscillations are small;

(2) that terms higher than first powers are neglect-
ed;

(3) that there is a force of restitution proportion-
al to the first power of the displacement;

(4) that there is a force of resistance proportion-
al to the first power of velocity;

(5) that the above forces produce an effective
force proportional to the first power of
the acceleration of all the mass of the sys-
tem;

(6) that the state of equilibrium is a function
of the force of restitution, the force of
resistance and the impressed force, and the
final effect is the sum of these three for-
ces, their products being neglected since
the oscillations are small.

In symbols the above assumptions can be con-

tained in the equation

$$\frac{d^2 x}{d t^2} + 2 a \frac{d x}{d t} + b x = 0 \quad \ldots \ldots (1)$$

when x denotes displacement, t, time, a and b are constants. The first term is the effective force; the second term, the force of resistance; the third term, the force of restitution.
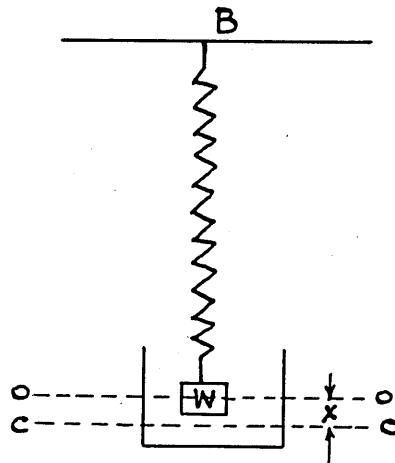
Physical Derivation of Equation (1).

Fig. 1.

Let Fig. 1 represent an oscillating apparatus. B is the support for the coil of spring, to the lower end of which is attached a weight of mass $\frac{W}{g}$

slugs.   When the spring is deflected, the weight
oscillates back and forth in a fluid.   Let us
analyze the motion from the instant when W starts
downward.   x gets greater and greater and $\frac{dx}{dt}$
is positive.   But $\frac{d^2x}{dt^2}$ is negative, the body
getting slower in its motion as x increases.
If the stiffness of spring is such that a force of
one pound elongates it h feet, then for x feet
the upward force will be $\frac{x}{h}$ pounds, which is the
impressed force.   By D'Alembert's Principle the
impressed force should be equal to effective force,

$$- \frac{W}{g} \frac{d^2x}{dt^2} = \frac{x}{h}$$

since $\frac{d^2x}{dt^2}$ is negative.

or $$\frac{W}{g} \frac{d^2x}{dt^2} + \frac{x}{h} = 0 \ \ldots\ldots\ldots (2)$$

At the same time a force of resistance due to the
friction in the fluid is acting in the same direc-
tion as the pull of the spring, being proportion-
al to its velocity, or $k \frac{dx}{dt}$.   Hence we need
to add one more term to equation (2)

$$\frac{W}{g} \frac{d^2 x}{d t^2} + k \frac{dx}{dt} + \frac{x}{h} = 0 \ \ldots \ldots \ (3)$$

Equation (3) can be obtained by the direct application of D'Alembert's Principle. Thus if we put the inertia or effective force on the left hand side of the equation, and equate it to the algebraic sum of the impressed forces, we get

Effective force = Σ impressed forces

= Acting forces – resisting forces.

The acting forces in this case are zero,

$\therefore$ effective force + resisting forces = 0

$$\frac{W}{g} \frac{d^2 x}{d t^2} + k \frac{dx}{dt} + \frac{x}{h} = 0$$

Dividing through by $\dfrac{W}{g}$, we have

$$\frac{d^2 x}{dt^2} + \left(\frac{g k}{W}\right) \frac{d x}{d t} + \left(\frac{g}{W h}\right) x = 0$$

which is identically the same as (1) if we put

$$2 a = \frac{g k}{W}, \qquad b = \frac{g}{W h}.$$

Equation (3) is capable of being further am-

plified by introducing another term in conse-
quence of the motion of support B itself.
Suppose during time t when W has descended x
feet, support B also descends a distance of y
feet.   Then the real displacement is x - y feet,
and the real upward pull is $\dfrac{x - y}{h}$ pounds.
Hence:

$$\frac{W}{g}\frac{d^2x}{dt^2} + k\frac{dx}{dt} + \frac{x - y}{h} = 0$$

or

$$\frac{W}{g}\frac{d^2x}{dt^2} + k\frac{dx}{dt} + \frac{x}{h} = \frac{y}{h} \ \ldots\ldots (4)$$

where y may be zero or f (t).

Equation (3) represents Damped Free Oscilla-
tion, (4) Damped Forced Oscillation.   When y
in (4) is zero, the forced oscillation at once be-
comes Free Oscillation.

Meaning of Terms in Equation (4)

In equation (4),

$$\frac{W}{g}\frac{d^2x}{dt^2} + k\frac{dx}{dt} + \frac{x}{h} = \frac{y}{h},$$

beginning at the left , the first term is the iner-
tia force due to mass; the second, the damping
force due to friction; the third, the restoring
force due to the pull of the spring; the fourth,
the external force due to the forcing of the oscil-
lation.

There are four terms in equation (4).
Let us study what will result from omitting some
of these terms either singly or in groups.

(a) $\frac{dx}{dt} = 0$ $\frac{d^2x}{dt^2} = 0$, motion of $W = 0$.

Spring is merely compressed. $\frac{x}{h} = \frac{y}{h}$

(b) $\frac{x}{h} = 0$, $x = 0$, $\frac{dx}{dt} = 0$, motion of a

rigid body under the influence of external force.
It results in the well known formula

$$\frac{W}{g}\frac{d^2x}{dt^2} = \frac{y}{h} = \text{force.}$$ The spring is

of great stiffness, and it moves with the weight

W as a rigid body by virtue of $\dfrac{x}{h}$.

(c) $\dfrac{dx}{dt} = 0$, $\dfrac{y}{h} = 0$ we have the case of

S.H.M. in frictionless medium: $\dfrac{W}{g}\dfrac{d^2x}{dt^2} + \dfrac{x}{h} = 0$

(d) $\dfrac{dx}{dt} = 0$, we have the case of superposed harmonic motion in a frictionless medium:

$$\frac{W}{g}\frac{d^2x}{dt^2} + \frac{x}{h} = \frac{y}{h}$$

(e) When all the terms are zero, it represents a state of rest of a rigid body.

We are, however, particularly interested in one condition, namely, $\dfrac{y}{h} = 0$. Equation (4) becomes

$$\frac{W}{g}\frac{d^2x}{dt^2} + k\frac{dx}{dt} + \frac{x}{h} = 0 \quad \ldots\ldots (5)$$

which is the case of Free Oscillations.

Equation (5) is exactly the same as Equation (3) and can be put into familiar form like Equation (1) by making suitable substitution of constants.

Thus $\dfrac{d^2x}{dt^2} + 2a\dfrac{dx}{dt} + bx = 0.$

Analogy between Linear Motion and

Angular Motion.

There is a well known analogy between the equations for linear motion and those for angular motion.

	Linear Motion	Angular Motion.
Displacement	x	θ
Velocity	$\dfrac{dx}{dt} = v$	$\dfrac{d\theta}{dt} = \omega$
Acceleration	$\dfrac{d^2x}{dt^2} = a$	$\dfrac{d^2\theta}{dt^2} = \alpha$
Mass	m	I
Time	t	t
Displacement	$x = 1/2\,a\,t^2$	$\theta = 1/2\,\alpha\,t^2$
K.E.	$1/2\,M\,v^2$	$1/2\,I\,\omega^2$

Hence we infer that what $\dfrac{d^2x}{dt^2} + 2a\dfrac{dx}{dt} + bx = 0$ is for linear motion will be $\dfrac{d^2\theta}{dt^2} + 2a\dfrac{d\theta}{dt} + b\theta = 0$ ---- (6) for angular oscillations. The constants a and b contain m which will be substituted by I accordingly. That Equation (5) is correct can be proved by the direct application of D'Alembert's Principle, the procedure being the same as for the linear

oscillations.

We shall in future make use of Equation (6)
in place of Equation (1)
since the oscillations in this special investigation are of an angular nature.

Analogy between Mechanical and

Electrical Oscillations.

Equation (4) gives for mechanical oscillations

$$\frac{W}{g}\frac{d^2x}{dt^2} + b\frac{dx}{dt} + \frac{x}{h} = \frac{y}{h}$$

But it is well known in electricity that

$$L\frac{d^2v}{dt^2} + R\frac{dv}{dt} + \frac{v}{K} = \frac{e}{K}.$$

Comparing the two equations, we find

The mass $\dfrac{W}{g}$ corresponds with self-induction L.

The friction per foot per second b, corresponds with the resistance R.

The displacement x corresponds with voltage v, v being a direct function of Q which is the real electrical displacement.

The want of stiffness of spring h corre-

sponds with the capacity of condenser K.

The forced displacement y corresponds with the forced E.M.F. of an alternator.

Thus the theory of small Oscillation is applicable to all forms of oscillation, be it mechanical, electrical, etc.

Solution of Equation (6).

Equation (6) is linear, and can be solved with the symbolic method. Let D denote $\frac{d}{dt}$, $D^2 = \frac{d^2}{dt^2}$, then Equation (6) becomes

$$(D^2\theta + 2 a D + b)\,\theta = 0 \quad \ldots \ldots (7)$$

The roots of (7) are

$$\lambda_1 \text{ and } \lambda_2 = -a \pm \sqrt{a^2 - b}.$$

The nature of the roots depends upon the character of the constants under the radical sign. There are four cases to be considered.

I. $a^2 > b$, roots all real, all negative

$$-\alpha, \quad -\beta$$

II. $a^2 = b$, roots all real, negative, equal

$$-a, \quad -a$$

III. $a^2 <$ b   roots complex imaginary

$\qquad - g + h\,i, - g - h\,i$

IV. $a^2 = 0$ roots pure imaginary

$\qquad + \sqrt{b}\,i, - \sqrt{b}\,i$

In case I, our answer is:

$$\theta = A\,e^{-\alpha t} + B\,e^{-\beta t}, \quad \dots (8)$$

In case II,

$$\theta = (A + B\,t)\,e^{-at}; \quad \dots\dots (9)$$

In case III,

$$\theta = e^{-gt}\,(A \sin h\,t + B \cos h\,t) \dots (10)$$

In case IV,

$$\theta = A \sin \sqrt{b}\,t + B \cos \sqrt{b}\,t, \quad \dots\dots (11)$$

Where A and B in all cases are arbitrary con-
stants. They can be determined by the conditions
that $\theta = 0$ when $t = 0$ and $\dfrac{d\theta}{dt} = \omega_0$ when
$t = 0$.

The graphs of equations (8), (9), (10), (11)
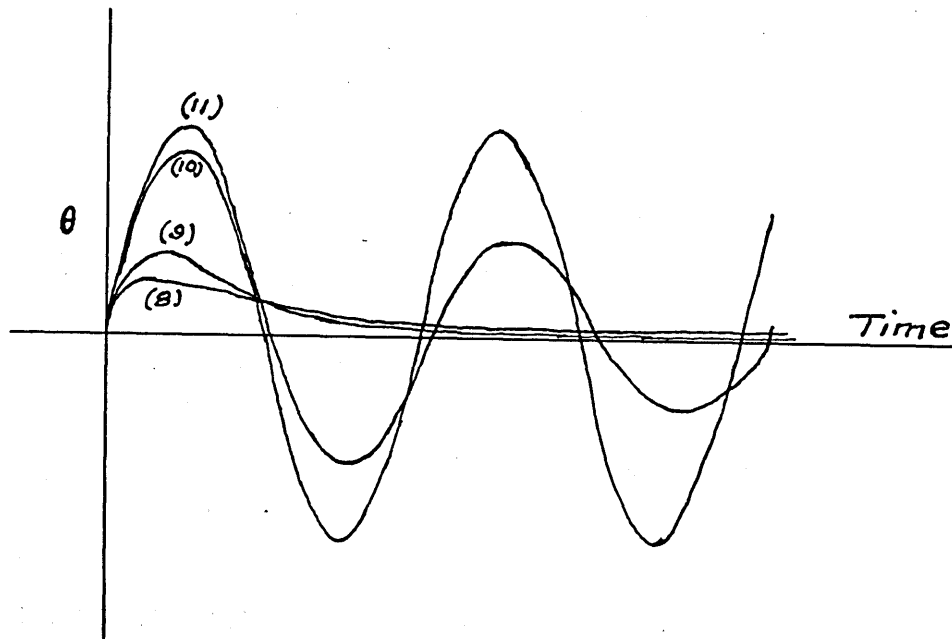are shown in Fig. 2.

Fig. 2.

It is evident both from the equations and from their graphs that in order that the motion may be oscillatory, the roots of (7) must be imaginary , complex or pure. That is, the quantity under the radical sign should be negative

$$\sqrt{a^2 - b} = \sqrt{-c}$$

Conditions of Instability of Motion.

In the four cases considered, the motions are all stable. That this is so may be shown by an inspection of the graphs in Fig. 2. None of the graphs exhibits any tendency to increased amplitude with increase in time which is the cause of instability.

This stable condition is obtained by the assumption that b in

$$\frac{d^2\theta}{dt^2} + 2a\frac{d\theta}{dt} + b\theta = 0$$

is a positive quantity, which renders the roots of

$$D^2 + 2aD + b = 0$$

either negative or imaginary which are criteria for stability.   When we assume,- and it does happen physically,- that b is a negative quantity, then the quantity under the radical will be positive, and the roots are no longer imaginary.   A real root means non-oscillatory motion.   The roots may be both positive or one positive and the other negative.   In the former case the solution of the equation becomes:

$$\theta = A e^{\alpha_1 t} + B e^{\alpha_2 t} \quad\dots\dots\ (12)$$

where α_1 and α_2 are positive roots of the equation.   In the latter case the solution is

$$\theta = A e^{\beta_1 t} + B e^{-\beta_2 t} \quad\dots\dots\ (13)$$

where β_1 and $- \beta_2$ are roots of the equation. A and B are arbitrary constants to be determined.

Equation (12) shows that the displacement in-

creases with time.   Hence it represents un-
stable motion.

Equation (13) may be written

$$\theta = A\, e^{\beta_1 t} + B\, \frac{1}{e^{\beta_2 t}}.$$

when t is large, the second term may be neglect-
ed.   When t is small, the first term may be
neglected.   The former corresponds to unstable
motion, and the latter stable motion.   It is
therefore a case of doubtful stability.

Dependence of Dynamic Stability upon
Static Stability.

We have seen that when b is positive a
negative quantity under the radical sign,

$$\sqrt{a^2 - b},$$

means oscillatory and stable motion; and that
when b is negative, the quantity under the rad-
ical sign will always be positive which means non-
oscillatory and unstable motion.   It is clear
then that the dynamical stability depends upon
whether b is positive or negative.

Let us examine the meaning of b.   Referring

to the original equation

$$\frac{d^2\theta}{dt^2} + 2a\frac{d\theta}{dt} + b\theta = 0$$

we find, paralleling the interpretation of the terms

on page 6, $\frac{d^2\theta}{dt^2}$ represents ~moment of~ angular momentum;

$2a\frac{d\theta}{dt}$ damping moment; $b\theta$, static restoring

moment.   As a matter of fact, $b\theta$ does not

represent static restoring moment alone; it repre-

sents the algebraic sum of all static moments

which has θ as a multiplier.   The final numer-

ical result may be positive or negative.   Usually

we call a static restoring moment positive.   Then

all static upsetting moments will be negative.

It is evident that when $b\theta$ is negative, the sum

of static upsetting moment is greater than the sum

of static restoring moment, which is a condition

of static instability.   Hence when $b\theta$ is posi-

tive, i.e., b is positive, the machine is stati-

cally stable.   When b is negative, the machine

is statically unstable.   But a positive b and

a negative b also account for dynamical stability

and instability.   Hence the dependance of dy-
namical stability upon static stability.

The preceding discussion only applies to the
case of free oscillation.   For forced oscillation
where the forcing and natural oscillations may syn-
chronize, there may be dynamic instability without
static instability.

Summary.

The foregoing conclusions may be summarized
in tabular form for reference.

Differential Equation for Small Natural Oscillations $\dfrac{d^2\theta}{dt^2} + 2a\dfrac{d\theta}{dt} + b\theta = 0$

Signs of b	Conditions	Roots	Solution of the Equation	Graph	Type of Motion	Type of Dynamic Stability	Remarks
Statically Stable (b = +)	$a^2 > b$	Both real & negative $-\alpha, -\beta$	$\theta = Ae^{-\alpha t} + Be^{-\beta t}$		non-periodic. Sub-sidence	Stable	
	$a^2 = b$	Both equal real & negative $-a, -a$	$\theta = (A+Bt)e^{-at}$		″	″	
	$a^2 < b$	Both complex imaginary $-g+hi$ $-g-hi$	$\theta = e^{-gt}(A\sin ht + B\cos ht)$ or $\theta = Ae^{-gt}\cos(ht+r)$		Damped oscilla-tion	″	Period $\dfrac{2\pi}{h}$
	$a^2 = 0$	Both pure imaginary $+\sqrt{b}i, -\sqrt{b}i$	$\theta = A\sin\sqrt{b}t + B\cos\sqrt{b}t$		undamp-ed osci-llation	Indifferent	Period $\dfrac{2\pi}{\sqrt{b}}$
b = 0 statically indifferent		one zero one real & negative $0, -2a$	$\theta = Ae^{-2at}$			″	
b = − statically unstable		Both real one positive one negative $-\omega_1, +\omega_2$	$\theta = Ae^{-\omega_1 t} + Be^{\omega_2 t}$			Doubtful	

CHAPTER II.

APPLICATION OF THE THEORY TO THE

STABILITY OF AEROPLANES.

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APPLICATION OF THE THEORY TO THE
STABILITY OF AEROPLANES

The Object of the Investigation.

The present investigation is undertaken with a view to determining experimentally certain of the rotary coefficients necessary for the solution of differential equations as set forth in Professor G. H. Bryan's book "Stability of Aeroplanes". In particular, attention was directed towards obtaining the damping coefficient due to pure pitching. However, the methods developed and the apparatus designed are equally applicable to the determination of other rotary coefficients where the oscillations are not of a forced nature.

System of Co-ordinates and Notation.

The system of co-ordinates and the notations used are the same as that used in British Blue Book 1912 - 1913. Report No. 77, page 142*.

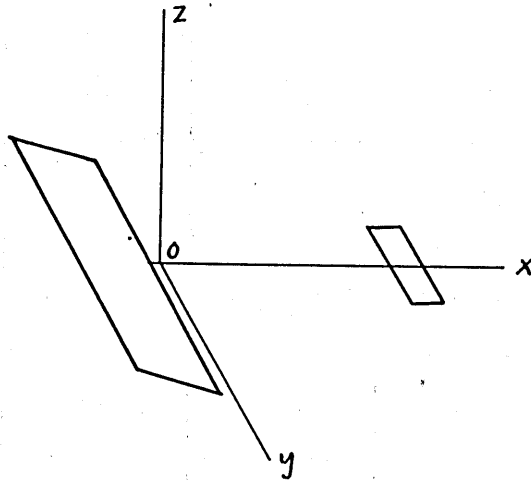*Technical Report of the Advisory Committee for Aeronautics 1912 - 1913.

Fig. 3.

The usual mathematical conventions are observ-
ed as to signs.   Forces are positive when acting
along the positive directions of the axes, angles
and moments are positive when turning occurs

Table of Notations

Axis	Name of Axis	Name of Force	Symbol for Force	Velocity Finite	Velocity very small	Name of Angle of	Symbol for Angle Finite	Symbol for Angle very small	Name of Moment	Symbol for Moment	Angular Vel. Finite	Angular Vel. very small
x	Longitudinal	Longitudinal	X	U	u	Roll	Φ	ϕ	Rolling	L	P	p
y	Lateral	Lateral	Y	V	v	Pitch	Θ	θ	Pitching	M	Q	q
z	Normal	Normal	Z	W	w	Yaw	Ψ	ψ	Yawing	N	R	r

*Forces of X, Y, Z, are forces of unit mass.
Moments L, M, N, are moments of forces of unit mass.
For actual forces and moments, multiply these with m, the mass.

or tends to occur from oy to oz, oz to ox,

and ox to oy.

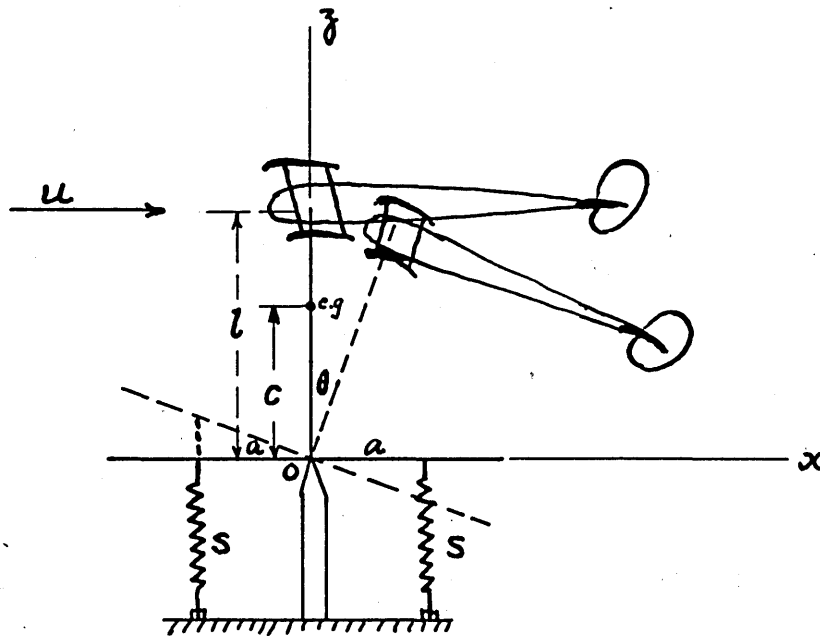### Preliminary Discussion of the Apparatus.

Fig. 4.

(Angle θ is exaggerated for the sake of clear-

ness).

Let Fig. 4 represent an apparatus for produc-

ing small oscillations.   The model of an aero-

plane is mounted on the top of an upright piece
which is rigidly attached to the horizontal rod.
The whole rests on a knife edge, held down by two
powerful helical springs.

Let the horizontal and vertical pieces be
axes of x and z, and the angular displacement be
called θ; C be the distance of c.g. of the
whole apparatus from the axis of rotation;
a, a, the moment arm of springs s, s, l the
distance of force X from axis of rotation.

Let the apparatus be deflected one way by
giving it a slight push.   A strong restoring mo-
ment s a will bring it back to its original po-
sition.   The inertia of the apparatus, however,
carries it further than the original position.
It is brought back again by s a, and carried
farther by the inertia and so on.   Thus we ob-
tain a motion of oscillation.

It has been noted that a powerful spring
was used.   This is necessary in order that an
oscillatory motion can be produced.   For consid-
er again the original equation

$$\frac{d^2 \theta}{d t^2} + 2 a \frac{d \theta}{d t} + b \theta = 0$$

or $$D^2 + 2 a D + b = 0$$

the roots of which are $- a \pm \sqrt{a^2 - b}$.

We must make b large so that the roots will be imaginary and the motion oscillatory. A powerful spring gives a large b. With this arrangement we expect the solution of Equation (6) to come under Case III, page 11, viz.:

$$\theta = e^{-gt} (A \sin ht + B \cos ht)$$

or what is the same thing

$$\theta = C e^{-gt} \cos (ht + \gamma)$$

where C and γ are arbitrary constants.

Derivation of Equation Similar to Equation (6).

The motion of an aeroplane in space is three-dimensional. Whatever effect a small variation of one quantity in one direction has will also set up variations of different amount in other directions.

Hence X, Y, Z, L, M, N, are functions of
u, v, w, p, q, r, Θ, ϕ and ψ, which, however,
can not be expressed in any simple form.
For the study of small oscillation such as the
problem in hand, the general expression can be
reduced to the form:

$$X = g \sin \Theta + X_0 + uX_u + vX_v + wX_w$$
$$+ pX_p + qX_q + rX_r \ldots (14)$$

$$Y = -g \cos \Theta \sin \phi + Y_0 + uY_u + vY_v + wY_w$$
$$+ pY_p + qY_q + rY_r \ldots (15)$$

$$Z = -g \cos \Theta \cos \phi + Z_0 + uZ_u + vZ_v + wZ_w$$
$$+ pZ_p + qZ_q + rZ_r \ldots (16)$$

$$L = L_0 + uL_u + vL_v + wL_w$$
$$+ pL_p + qL_q + rL_r \ldots (17)$$

$$M = M_0 + uM_u + vM_v + wM_w$$
$$+ pM_p + qM_q + rM_r \ldots (18)$$

$$N = N_0 + uN_u + vN_v + wN_w$$
$$+ pN_p + qN_q + rN_r \ldots (19)$$

Equations (14), (15), (16), (17), (18), and (19)
are obtained by expansion into Maclaurin series to
terms of the first order, all terms higher than
first power and all products of first powers are

neglected.   This assumption is fundamental
in Differential Calculus.

In the present investigation, $Y = 0$,
$L = 0$, $N = 0$, $g = 0$.   The aerodynamic force
and moment acting on the model are respectively
X and M.   Again with this special arrangement
of the apparatus, $p = 0$, $r = 0$, $v = 0$, $W = 0$.
Hence

$$X = X_0 + uX_u + qX_q \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (20)$$

$$M = M_0 + uM_u + qM_q \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (21)$$

Since   drift $= mX$

$$\therefore \text{ Drift} = mX_0 + m\,uX_u + m\,qX_q \quad \ldots\ldots\ldots\ldots \quad (22)$$

Again since pitching couple $= m\,M$

$$\therefore \text{ Pitching couple} = m\,M_0 + m\,uM_u + m\,qM_q \quad (23)$$

Having simplified so far, let us analyze
the forces and couples acting on the model in a
new position of equilibrium after being displaced
θ to the right, Fig. 4.   The force of the wind
can be resolved into a force of drift and a cou-
ple of pitching.   They are moments acting about
the pivot.   To this must be added the moment
due to the down pull of gravity at c.g.   The

resisting moment is the moment of spring.   By
D'Alembert's Principle,

Effective Moment =   Acting Moments

             −   Resisting Moments

or in symbols

$$I\frac{d^2\,\theta}{d\,t^2} = D\;l + G + c\;W\sin\theta - S\;a \;\ldots\ldots (24)$$

In (24) I stands for Moment of Inertia in
slug-ft^2, D, for drift; l, the distance from
c.g. of model to pivot; G, the pitching couple;
c, the distance of C.G. of whole apparatus to
the pivot; W, the mass of the whole apparatus;
S, the pull in the spring; a, the moment arm of
the spring.   Making substitution of (22) for D,
(23) for G, θ for $\sin\theta$ since θ is very small,
and $S = S_0 + K\,a\,\theta$ where K is the stiffness of
the spring, we obtain

$$I\frac{d^2\theta}{dt^2} = (m\;X_0 + m\;uX_u + m\;qX_q)\;l + (m\;M_0 + mM_u + mqM_q)$$

$$+ c\;W\,\theta - S_0 a - K\;a^2\theta \;\ldots\ldots (25)$$

For initial condition

$$mX_0\;l + mM_0 = S_0\;a$$

or $\quad(mX_0\;l + mM_0) - S_0\;a = 0 \;\ldots\ldots\ldots\ldots (26)$

Subtracting (26) from (25), there remains

$$I\frac{d^2\theta}{dt^2} = (m\,uX_u + m\,qX_q)l + m\,uX_u + m\,qM_q$$
$$+ C\,W\,\theta - K\,a^2\,\theta \dots (27)$$

Now we know, using $\theta^{\cdot} = \frac{d\theta}{dt}$,

$$\dot{u} = \frac{\dot{\theta}}{dt}, \quad u = \dot{\theta}\,l, \quad q = \dot{\theta}. \quad \text{Making the sub-}$$

stitution, we have

$$I\theta'' = (m\,\dot{\theta}\,l\,X_u + m\,\dot{\theta}\,X_q)l + m\,\dot{\theta}\,l\,M_u + m\,\dot{\theta}\,M_q$$
$$+ C\,W\,\theta - K\,a^2\,\theta \dots (28)$$

Collecting terms of $\dot{\theta}$ and θ, we obtain

$$I\theta'' = (m\,l^2X_u + m\,l\,X_q + m\,l\,M_u + m\,M_q)\,\dot{\theta}$$
$$+ (C\,W - K\,a^2)\,\theta \dots (29)$$

Previous experiment* shows that the value
for X_q is negligible when compared with M_q.
Also from the consideration of horizontal steady
flight, $M_u = 0$. Hence (29) may be simplified
further to

$$I\ddot{\theta} = (m\,M_q + m\,l^2X_u)\dot{\theta} + (C\,W - K\,a^2)\theta \quad (30)$$

or $$\ddot{\theta} - \left(\frac{m\,M_q + ml^2X_u}{I}\right)\dot{\theta} + \left(\frac{K\,a^2 - CW}{I}\right)\theta = 0 \dots (31)$$

*British Report 1912-1913, page 159.
$X_u = -0.14; \quad X_q = \pm 0.05; \quad M_q = -210.$

Let $(\mu_0 + \mu) \, \dot{\theta}' = -\left(m M_1 + m \, l \, X_u \right) \dot{\theta}$, where

$\mu_0 \dot{\theta}'$ and $\mu_i \dot{\theta}'$ stand for damping moments of the apparatus without the model and of the model alone respectively. Therefore (31) becomes, putting $\dfrac{d^2 \theta}{d\,t^2}$ for $\ddot{\theta}''$, $\dfrac{d\theta}{dt}$ for $\dot{\theta}'$,

$$\frac{d^2 \theta}{d\,t^2} + \frac{(\mu_0 + \mu)}{I} \frac{d\theta}{dt} + \left(\frac{K\,a^2 - CW}{I}\right)\theta = 0 \ldots (32)$$

which is exactly similar to equation (6) page 8. The solution of (32) falls under Case III, page 11. It is to be remembered that the apparatus was designed to make the solution of the differential equation representing the motion of the system, come under Case III. Hence the solution of (32) is

$$\theta = C\, e^{-\frac{\mu_0 + \mu}{2\,I}\,t} \; \cos\left\{\sqrt{\frac{K a^2 - CW}{I} - \left(\frac{\mu_0 + \mu}{2\,I}\right)^2}\,t + \gamma\right\}$$

$$(33)$$

where C and γ are arbitrary constants, the first depending upon initial displacement the second depending upon initial velocity. At the moment when θ becomes θ_0, the amplitude of the swing

velocity is zero.   Rewriting (33), we obtain

$$\theta = \theta_0 c^{-\frac{\mu_0 + \mu}{2\,I}t} \cos\left\{\left(\sqrt{\frac{Ka^2 - CW}{I} - \left(\frac{\mu_0 + \mu}{2\,I}\right)^2}\,t + O\right)\right\}$$

(34)

Meaning of Terms.

The expression $\dfrac{\mu_0 + \mu}{2\,I}$ is called loga-

rithmic decrement.   The expression

$$\sqrt{\frac{Ka^2 - CW}{I} - \left(\frac{\mu_0 + \mu}{2\,I}\right)^2}$$ divided into 2π

gives period, i.e.,

$$p = \frac{2\pi}{\sqrt{\dfrac{Ka^2 - CW}{I} - \left(\dfrac{\mu_0 + \mu}{2\,I}\right)^2}}$$ . . . . (35)

Which one to Measure, Logarithmic Decrement or Period.

It is very easy to find p and it looks
as though this is the way to find the coeffi-
cients X_u and M_q which are connected with
μ_0 and μ by the relation $\mu_0 + \mu = mM_q + ml^2 X_u$,

assuming that the moment of inertia of the whole apparatus is known.   But the expression involves quantities like K, the stiffness of the spring, and C, the distance of c.g. of whole apparatus from the pivot, both of which are difficult to get, especially K.

Turning now to the expression for logarithmic decrement, we find that it contains coefficient of $\dfrac{d\theta}{dt}$ only which does not involve K and C.   It contains I which must be determined just as I in the expression for period must be determined.   We see, therefore, there is every advantage in making use of the expression for logarithmic decrement for the determination of mX_u and mM_q.   It is to be noted that X_u can be determined by a static test of a model, and therefore the only unknown quantities in the expression are mM_q and I.

Referring to equation (33), if we draw a curve through the points of oscillation corresponding with

$$\cos \sqrt{\frac{Ka^2 - CW}{I} - \left(\frac{\mu_0 + \mu}{2I}\right)^2}\; t = 1 \quad \ldots\ldots (36)$$

i.e., through the highest points, its equation is

$$\theta = \theta_0\, e^{-\frac{\mu_0 + \mu}{2I} t} \quad \ldots\ldots\ldots (37)$$

or

$$\log_e \frac{\theta}{\theta_0} = -\frac{\mu_0 + \mu}{2I}\, t \quad \ldots\ldots (38)$$

If we plot a curve of $\log_e \dfrac{\theta}{\theta_0}$ on the

time t as base, the slope of the curve which is

a straight line will give $\dfrac{\mu_0 + \mu}{2I}$, from which,

when I is known, $m M_q$ can be obtained.

This decision of observing logarithmic decrement instead of period suggests the design of an apparatus such as above outlined with the additional auxiliary apparatus to make such observation possible. In the next chapter, we shall describe the apparatus and the manner in which it was handled to perform an experiment.

CHAPTER III

DESCRIPTION OF THE APPARATUS

AND METHODS OF

PERFORMING THE EXPERIMENTS.

CHAPTER III.

DESCRIPTION OF APPARATUS
AND METHOD OF EXPERIMENT.

Requisites of a Good Apparatus.

The value of $(m \ M_q + m \ 1^2 \ X_u)$ obtained
from the logarithmic decrement gives the total
damping effect of the apparatus and the model,
i.e., $\mu_0 + \mu$. In order to find the damp-
ing of the model, we have to subtract μ_0 from
the observed values of the total damping. It
is clear then, that the smaller the apparatus
damping the greater the accuracy of the result
for the model damping.

Apparatus damping is itself composed of
two items, viz.: friction damping and wind damp-
ing. To reduce the former, all pivots were point
contact pivots; to reduce the latter, parts of the
apparatus were shielded.

The apparatus must have means for adding a
known moment of inertia to the system without dis-
turbing the position of c.g. either horizontally
or vertically. This adding or subtracting of

moment of inertia from the system permits the ad-

justment of long or short period which is very nec-

essary in view of the fact the oscillations are to

be observed with eyes instead of a photographic

plate.

The apparatus must have means to indicate

the oscillation in a magnified degree.   A mechan-

ical magnifier, such as a long stick will offer too

much apparatus damping by the air, and it is also

impossible to obtain a stick of small mass rigid

enough.   We are to use a beam of light in place

of a stick to do the magnification.

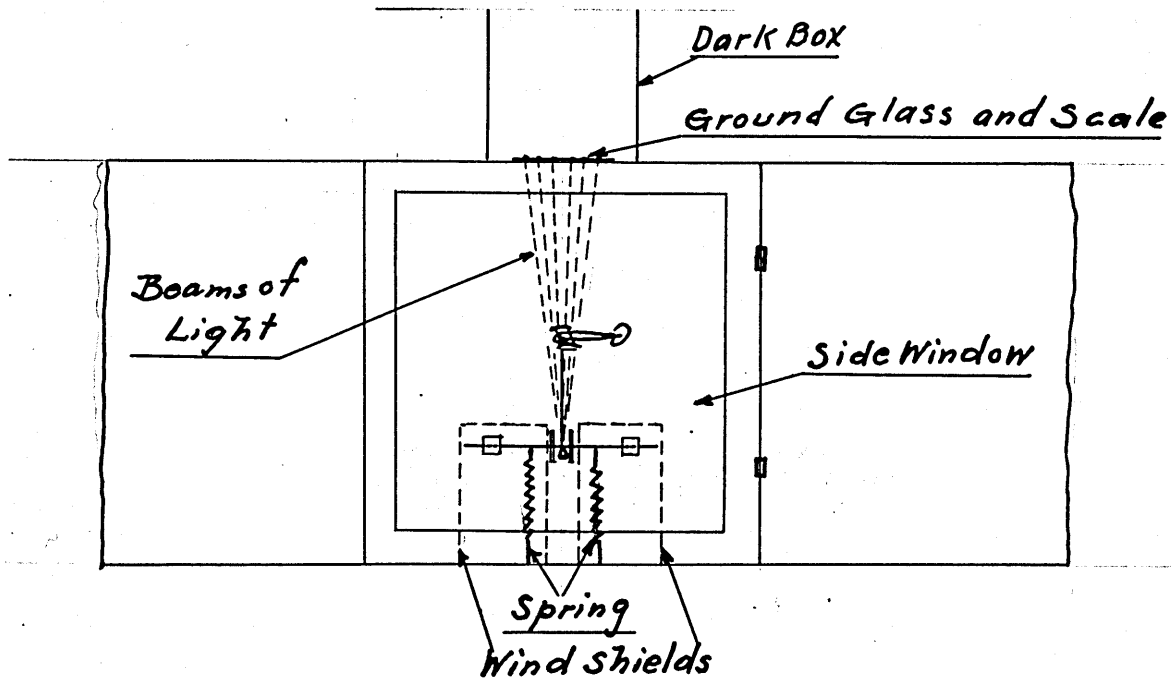The apparatus must have means to produce
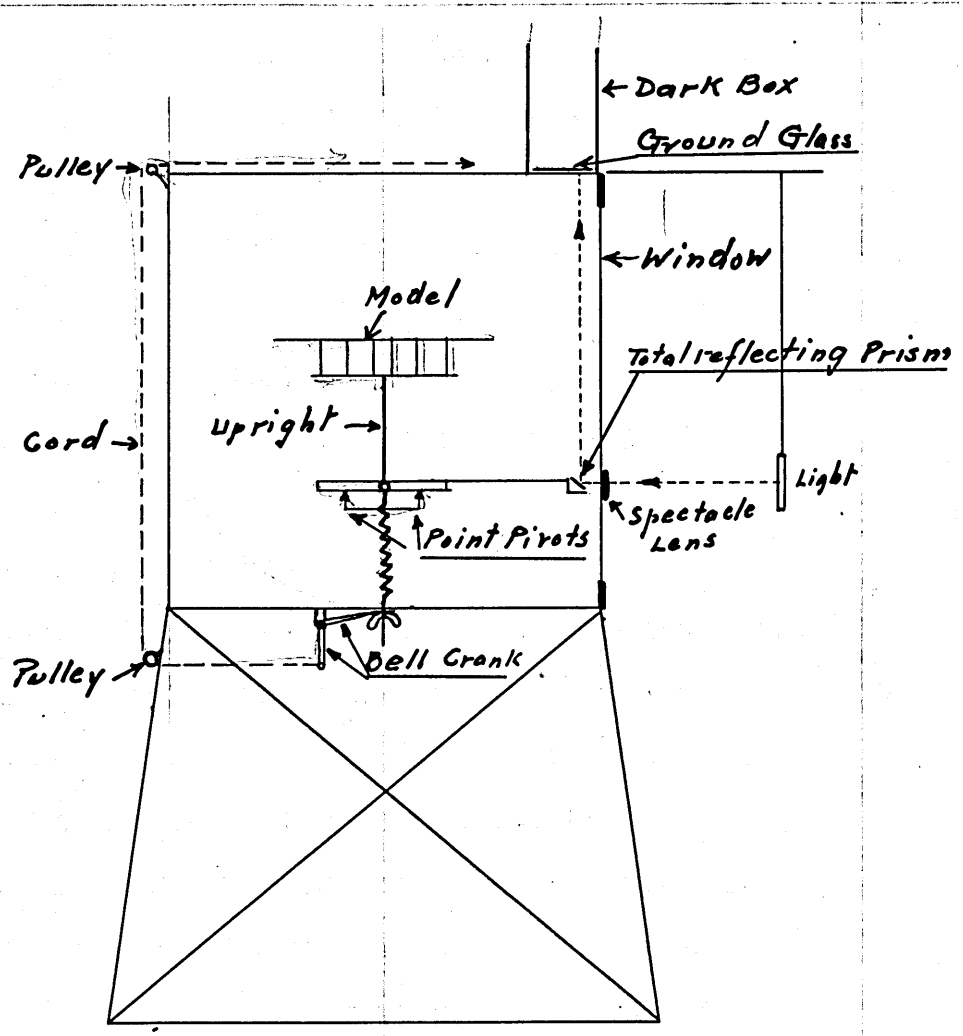
the oscillation.

Fig. 5a   Side View

Fig. 5b   End View

The sketch shows in ~~two~~ views the ar-
rangement of the apparatus for pitching experi-
ments in the middle portion of the 4-foot wind
tunnel in the Aerodynamic Laboratory of the Massa-
chusetts Institute of Technology.

On the trumpet of the balance which forms
the vertical axis is screwed a special fitting
which gives the support to two pivot points.
Above the points and across the tunnel rests a
horizontal bar in well shaped shallow cavities.
The line joining the two points is the axis of
rotation for the system.   A horizontal rod
at right angles to the horizontal bar is secured
to the latter, on which are made a number of cav-
ities for the placing of the pointed ends of
hooks attached to the vertical springs which
maintain the oscillation.   The tension in
springs can be adjusted by turning the butter-
fly nuts at the lower ends of the spring.

To the two horizontal pieces is rigidly
secured a vertical piece with such attachment
at its top as to permit the mounting of an aero-
plane model.

The transverse horizontal bar has a total reflecting mirror at its end.   A beam of light coming from outside the window through a double convex spectacle lens from a long filament incandescent lamp is reflected upwards to the glass ceiling of the tunnel and is received on a piece of fine ground glass.   When experiments are performed during day time, it is necessary to cover the glass window and ceiling with black cloth, leaving openings for the artificial light to pass through.   The ground glass screen is also enclosed in a dark box. · When the apparatus rocks about the lateral axis, i.e., o y , the reflected light is made to deviate from its normal or zero position.   The oscillations are thus exactly reproduced and indicated on the ground glass screen.   The angular motion is transformed to linear motion on the screen which can be observed and recorded. For this purpose a scale is devised whereby the lin ear displacements can be measured.

We have referred to the fact that the apparatus should include a device for producing the

motion.   This is accomplished by the bell
crank lever and cord underneath the floor of
the tunnel, whereby the spring is given a slight
jerk through the chord by the observer on the
top of the tunnel.   The spring reacts upon
the apparatus and, upon releasing, sets the sys-
tem to oscillation, which is a free or natural
one since the supports for the springs are fix-
ed during the motion.

Method of Experiment

How to Obtain Logarithmic Decrement.

The model used in this experiment is the
model of a biplane hydroplane supplied by a com-
pany.   It has a single pontoon, and the engine
and passengers are enclosed in a short, blunt,
sudden-terminating body, to the rear of which is
placed a propeller which is, however, absent on
the model.   The tail and elevator are carried
at the end of four sticks attached to the rear
struts, sufficiently separated to allow the turn-
ing of the propeller.   The ends of the sticks
are collected two and two vertically, which, when
covered, form a horizontal knife edge, but which
is not covered in this case.   The distance of
the tail from c.g. of the model is about 6.5
inches.   The machine has a small tail and a
short lever arm for the tail.

The model is first mounted on the top of
the upright piece with the nose towards the wind
and with correct altitude.   This arrangement
is good for experiments in pitching only.   For

yawing, with the nose heading the wind as before,
the wings will be placed vertical.   For rolling,
the presentation and altitude of the model being
the same as for pitching;   the whole apparatus
has to be turned 90° in the horizontal plane, so
that the rolling will take place about an axis
in the direction of the wind.   It is to be not-
ed that this shifting of apparatus necessitates
a corresponding change in the arrangement of
spring attachments and in the optical part of
the apparatus.

The model having been mounted in the
manner above described, the springs are then hook-
ed on to the horizontal rods.   If the period
is too rapid, we can do one of the following
things or the combination of them.

    (a) Slacken up the spring.

    (b) Put the hooks closer to the axis
        of rotation.

    (c) Move out the weights on the horizon-
        tal rods.

(a) and (b) diminish the restoring moment of the
spring, i.e., make $K a^2$ small in Equation (35).

$$p = \frac{2\pi}{\sqrt{\dfrac{Ka^2 - CW}{I} - \left(\dfrac{\mu_0 + \mu)}{2I}\right)^2}}$$

resulting in a larger p.   (c) increases the mo-
ment of inertia I in the same equation which af-
fects the period directly.

The zero is seen to have changed when the
wind is on.   This is due to the drift of the
wind upon the model.   ~~He~~ *The experimenter* should not shift the
box to adjust the new zero.   He should signal
to the other person to tighten or slacken the
spring by turning the butterfly nuts.   While
his assistant is doing that he should watch the
movement of the light till it comes back to the
original zero, and then signal to the assistant
to stop.   This insures a true zero angle of
incidence of the planes with the wind.

The damping effect of the tail is to be
investigated at different speeds of the wind,
and the wider the range of speeds the more re-
liable will be the result.   This tunnel gives
a wind of forty miles per hour as the highest
speed.   Let us say the speeds are to be 40,

35, 30, 25, 20 and 10 miles per hour.   It is preferable that we use the high speed first and· then slow it down.

The observer now signals to his assistant to give him a wind of forty miles per hour to begin with.   The assistant makes the necessary adjustment for the new zero.   The observer then gives a pull on the cord to set the apparatus to oscillate, counts the beats, and jots them down. He repeats the same process for at least a dozen times while his assistant is keeping the wind constant, and gets a dozen sets of numbers.   If they come out pretty uniform, he need not repeat any further.   He now takes up a stop watch and gets the period of the oscillation at that speed of the wind.   (See Equation (35).)

He now signals to his assistant to slow down the wind to a speed previously agreed upon; tells him to make the adjustment of the new zero since the drift is changed with change of wind speed.   Then he does the pulling of the cord, counting of the beats, and the taking of the period as described above.   The whole thing is *repeated for other wind speeds.*

The Determination of Moment of Inertia

of the Whole Apparatus.

Thus far we have dealt with the problem of finding the logarithmic decrement. In order that μ_θ and μ can be determined it is necessary to eliminate I by direct experiment or by calculation. The latter method is impracticable inasmuch as the aeroplane model is a complicated structure and the work is laborious as well as inaccurate.

There is a well known experimental method of finding the moment of inertia of a body that does not admit of geometrical treatment. The body is mounted on some oscillating apparatus and the period of oscillation is observed. The quantities, period and moment of inertia, are connected according to Equation (35)

$$p = \frac{2\pi}{\sqrt{\dfrac{Ka^2 - CW}{I} - \left(\dfrac{\mu_\theta + \mu}{2I}\right)^2}}$$

The correction damping term $\left(\dfrac{\mu_\theta + \mu}{2I}\right)^2$ can be

neglected since the experiment is always perform-
ed in still air and apparatus friction is either
zero or very small.   Hence,

$$p = \frac{2\pi}{\sqrt{\dfrac{Ka^2 - CW}{I}}} \dots\dots\dots\dots (39)$$

By adding a known moment of inertia I_1, we get
a new period,

$$p_1 = \frac{2\pi}{\sqrt{\dfrac{Ka^2 - CW}{I + I_1}}} \dots\dots\dots\dots (40)$$

From (39) and (40) we can find I provided
$Ka^2 - CW$ does not change in the two experi-
ments, and is equal to some unknown constant K
which can be eliminated since we have two equa-
tions for two unknowns K and I.   From (39),

$$p^2 = \frac{4\pi^2 I}{K} \dots\dots\dots\dots\dots (41)$$

From (40)

$$p_1^2 = \frac{4\pi^2 (I + I_1)}{K} \dots\dots\dots\dots (42)$$

Dividing (42) by (41), we have

$$(\frac{p_1}{p})^2 = \frac{I + I_1}{I}, \quad \ldots\ldots\ldots\ldots \quad (43)$$

From which we obtain

$$I = I_1 \left\{(\frac{p_1}{p})^2 - 1\right\} \quad \ldots\ldots \quad (44)$$

The condition that $Ka^2 - CW$ must not
change during the two experiments implies the
fact that the setting of the spring must not
change both as to its tension and moment arm;
and further that the adding of the known moment
of inertia I_1 must not change the mass of the ap-
paratus nor the vertical position of the c.g.
The first condition is easily met.   The second
condition suggests that we must not introduce any
extra mass from outside into the system, and that
in order not to change the vertical c.g. the
shifting weights must lie in the horizontal
plane containing the axis of rotation.   Both
are satisfied by shifting the weights on the hor-
izontal rods outwards preferably in equal amounts
although not necessary.

The precaution taken for the constant c.g.
is not absolutely necessary in view of the fact

that the influence of gravitational moment is small compared with spring and aerodynamic moments for small oscillations.

From Equations (41) and (42) we see that the period is affected by the square root of moment of inertia. It is evident, therefore, that in order to obtain an appreciable difference in period in the second experiment, we need to add a large known moment of inertia. This is done by substituting long arms for the short ones used in the experiments for logarithmic decrement, taking care that in doing so we have not introduced any extra mass into the system. By shifting the original weights on the new long rods to a good distance outward, we add a large moment of inertia. We must remember that $I \propto m\, x^2$, $p \propto \sqrt{I} \propto \sqrt{m}\, x$, and a change in x is much more powerful than a change in m as far as its influence on the period is concerned.*

---

*The statement is only general. m is not permitted to change in this case in accordance with the restriction stated on page 43.

Elimination of μ_o

We have on page 28 separated the apparatus
damping and the model damping by calling the for-
mer μ_o and the latter μ.   It remains for us
to determine the numerical value of μ_o with a
view to determining the numerical value of μ
which may be used for stability calculations after
proper correction for a scale of model and speed
of wind.   To do so, we must take away the model
and repeat the experiments of logarithmic decre-
ment and those of moment of inertia, the parts ex-
posed to the wind as well as the mass of the appa-
ratus being different from before.   The equation
expressing the motion is similar to Equation (32),
page 28.

$$\frac{d^2\theta}{dt^2} + \frac{\mu_o}{I'}\frac{d\theta}{dt} + \left(\frac{K\,a'^2 - c'w'}{I'}\right)\theta = 0 \ldots (45)$$

the primes denoting new values of the quantities
when the model is off the apparatus.   The dif-
ference of $(\mu_o + \mu)$, obtained from the first set
of experiments with the model, and μ_o obtained

from the second set of experiments without the
model, gives   net for the model.

C H A P T E R   I V.

METHOD OF COMPUTATION FROM

OBSERVED DATA AND ANALYSIS OF RESULTS.

CHAPTER IV.

METHOD OF COMPUTATION FROM

OBSERVED DATA AND ANALYSIS OF RESULTS.

Data for Logarithmic Decrement

when the Model is on.

TABLE I

Common log θ	θ † Scale $\theta_0=7$	No. of Complete Swings†† at Wind Speed of						
		0 mi/hr	10 mi/hr	20 mi/hr	25 mi/hr	30 mi/hr	35 mi/hr	40 mi/hr
0.846	7—	8.0	6.0	6.0	6.0	5.0	4.4	4.0
0.699	5—	17.9	11.0	9.0	7.7	7.4	5.6	4.3
0.477	3—	18.5	10.6	8.0	7.9	6.6?	6.6	4.8
0.301	2—	27.1	17.4	14.2	15.0	14.4?	8.5	7.4
0.000	1—	....	....	....	....	....	...	...
	0—							
Period of Complete Swing in Seconds.		0.834	0.820*	0.794	0.774	0.747	0.734	0.727

*Interpolated.
† θ in Column two are half amplitudes.
†† Complete Swing =   A ⇌ B

Note:— The values of number of swings are the average values of a number of observations ~~as shown under each wind speed.~~
Common logs are used for convenience.
They should be multiplied by 2.30 to get $\log_e$.
Period in seconds was obtained with a stopwatch.

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On page 29 we find the solution of the differential equation to be

$$\theta = \theta_0 e^{-\frac{\mu_0 + \mu}{2\,I}\,t} \cos \sqrt{\frac{Ka^2 - CW}{I} - \left(\frac{\mu_0 + \mu}{2\,I}\right)^2}\,t$$

If we draw a curve through the points of oscillation corresponding with

$$\cos \sqrt{\frac{Ka^2 - CW}{I} - \left(\frac{\mu_0 + \mu}{2\,I}\right)^2}\,t = 1,$$

i.e., through practically the highest points its equation is

$$\theta = \theta_0\, e^{-\frac{\mu_0 + \mu}{2\,I}\,t}$$

Taking logarithms on both sides, we have

$$\log_e \theta = -\frac{\mu_0 + \mu}{2\,I}t + \log_e \theta_0$$

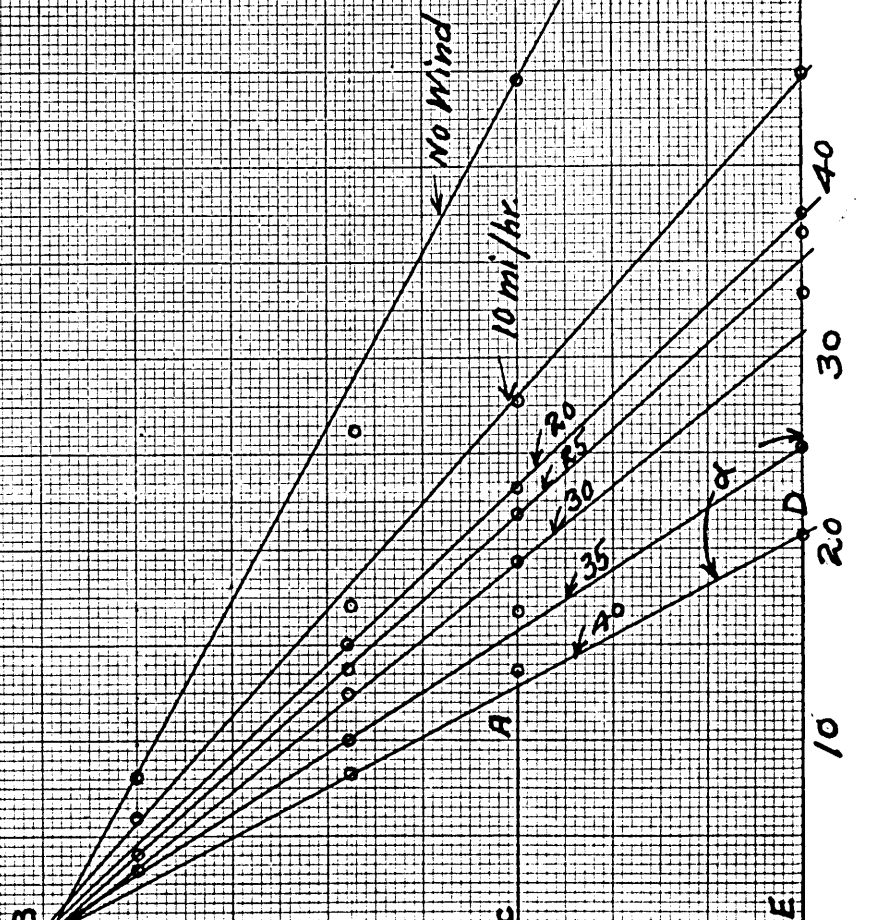If we plot $\log_e \theta$ on a base of t, the graph will be a straight line and $\frac{\mu_0 + \mu}{2\,I}$ will be the slope of the line. Chart I shows the general disposition of the curves. The ordinates represent the common log of the half amplitude, and the abscissae the number of complete swings which is a

Chart I

Log. of Half Amplitude

No. of Complete Swings

No Wind

10 mi/hr.

20
25
30
35
40

B

A

C

D

E

α

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function of time.   A point A on the chart has
the meaning that it takes little over 13 complete
swings to reduce the half amplitude 7 on the scale
to the half amplitude 2 on the same scale.   Re-
ferring to Table I, we find the period of oscil-
lation for 40 miles per hour to be .727 second.
Therefore, it takes 13 X .727 = 9.45 seconds for
the wind to damp the motion to 5/7 of what it was.
The rate of damping is:

$$- \frac{\log_e 5/7}{9.45} = \frac{\log_e 5 - \log_e 7}{9.45} = \frac{2.3(\log_{10} 5 - \log_{10} 7)}{9.45}$$

$$= - \frac{2.3(0.7-0.845)}{9.45} = - \frac{-.334}{9.45}$$

$$= \quad 3.40 \text{ % per second.}$$

The following table shows the rate of damp-
ing in % per second at different wind speeds, which
is evidently the tangent of the angles made by the
straight lines with the axis of abscissae in Chart
I.   In calculating the tangent, the sides of the
largest triangle made by the inclined line with the

axes have been used.   For instance, for $\tan \alpha$,
(see Chart I) $-\dfrac{B\ C}{C\ A}$ is just as good as $-\dfrac{B\ E}{E\ D}$.

But the latter is preferred for the reason that
we can measure B E, E D much more accurately
than B C, C A.

TABLE II. (See Chart I).

Speed of Wind Miles per Hr.	Tan α	$\dfrac{\mu_0 + \mu}{2 I}$
0	$\dfrac{2.30 \times .8}{72 \times .834} =$	.... 0.0340
10	$\dfrac{2.30 \times .8}{45 \times .820} =$	.... 0.0499
20	$\dfrac{2.30 \times .85}{36.5 \times .794} =$	.... 0.0674
25	$\dfrac{2.30 \times .85}{33.5 \times .774} =$	.... 0.0754
30	$\dfrac{2.30 \times .82}{30 \times .747} =$	.... 0.0840
35	$\dfrac{2.30 \times .80}{26.5 \times .734} =$	.... 0.0946
40	$\dfrac{2.30 \times .80}{21 \times .727} =$	.... 0.1177

~~Chart II shows the relation between the rate of damping and the speed of wind.~~

In saying that tan α is the rate of damping expressed in per cent per second, we are as-

signing a unit to $\dfrac{\mu_o + \mu}{2\ I}$, which is equal to

tan α, and which is hitherto undefined.   That

the unit is correct may be seen from the expres-

sion $\dfrac{\log_e 5/7}{9.45}$ on page 51, which is similar to

those under column of tan α in Table II.   The

numerator expresses the ratio of the half ampli-

tudes and the denominator expresses the time in

seconds during which this change takes place.

From this it is obvious that we are interested

only in the ratio of the amplitudes and not in the

absolute value of the individual amplitudes, and

that it is immaterial as to the unit of the scale,-

inches, feet, radians, degrees, etc.   This fact

permits changing the scale at will.   For example,

the scale for the experiment without model was dif-

ferent from the one with the model.

Reference has been made to the fact that we

were taking the half amplitudes.   It is possible

that the two halves are not symmetrical.   In or-

der to *eliminate* this error, the above experiment

was repeated and the other half amplitude was taken.

The following tables are identical with

Table I, page 49, and Table II on page 53.

TABLE III.

Common log θ	θ Scale $\theta_0=7$	No. of Complete Swings at Wind of Speed						
		0	10 mi/hr	20 mi/hr	25 mi/hr	30 mi/hr	35 mi/hr	40 mi/hr
	0							
		---	---	---	---	---	---	---
0.000	**1**							
		47	----	12.8	14.0	9.8	9.9	10.0
0.301	**2**							
		28	----	8.5	7.8	6.5	6.1	6.7
0.477	**3**							
		31	----	9.5	8.8	6.7	6.7	7.0
0.699	**5**							
		21	----	7.5	7.0	5.7	5.0	5.0
0.846	**7**							
		0.830	----	0.775	0.755*	0.740	0.700	0.678

Period of
Complete
Swing in
Seconds.

*Interpolated.

TABLE IV.   (See Chart II).

Speed of Wind Miles per Hr.	Tan	$\dfrac{\mu_0 + \mu}{2\ I}$
0	$\dfrac{2.3 \times .845}{116 \times .83}$	0.0202
20	$\dfrac{2.3 \times .845}{38.5 \times .775}$	0.0650
25	$\dfrac{2.3 \times .845}{37 \times .755}$	0.0694
30	$\dfrac{2.3 \times .845}{28.5 \times .74}$	0.0920
35	$\dfrac{2.3 \times .845}{27.5 \times .70}$	0.1010
40	$\dfrac{2.3 \times .845}{27 \times .678}$	0.1060

It is necessary to calculate the moment of inertia of the apparatus and the model together as a unit to eliminate I in $\dfrac{\mu_0 + \mu}{2\ I}$ shown in Table II and IV.

Chart II

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No. of Complete Swings

No Wind abscissae 1″=20 swings

20 mi/hr.
25
30
35
40

Log. of Half Amplit.

.8 .7 .6 .5 .4 .3 .2 .1

0 10 20 30 40 50 60 70 80 90

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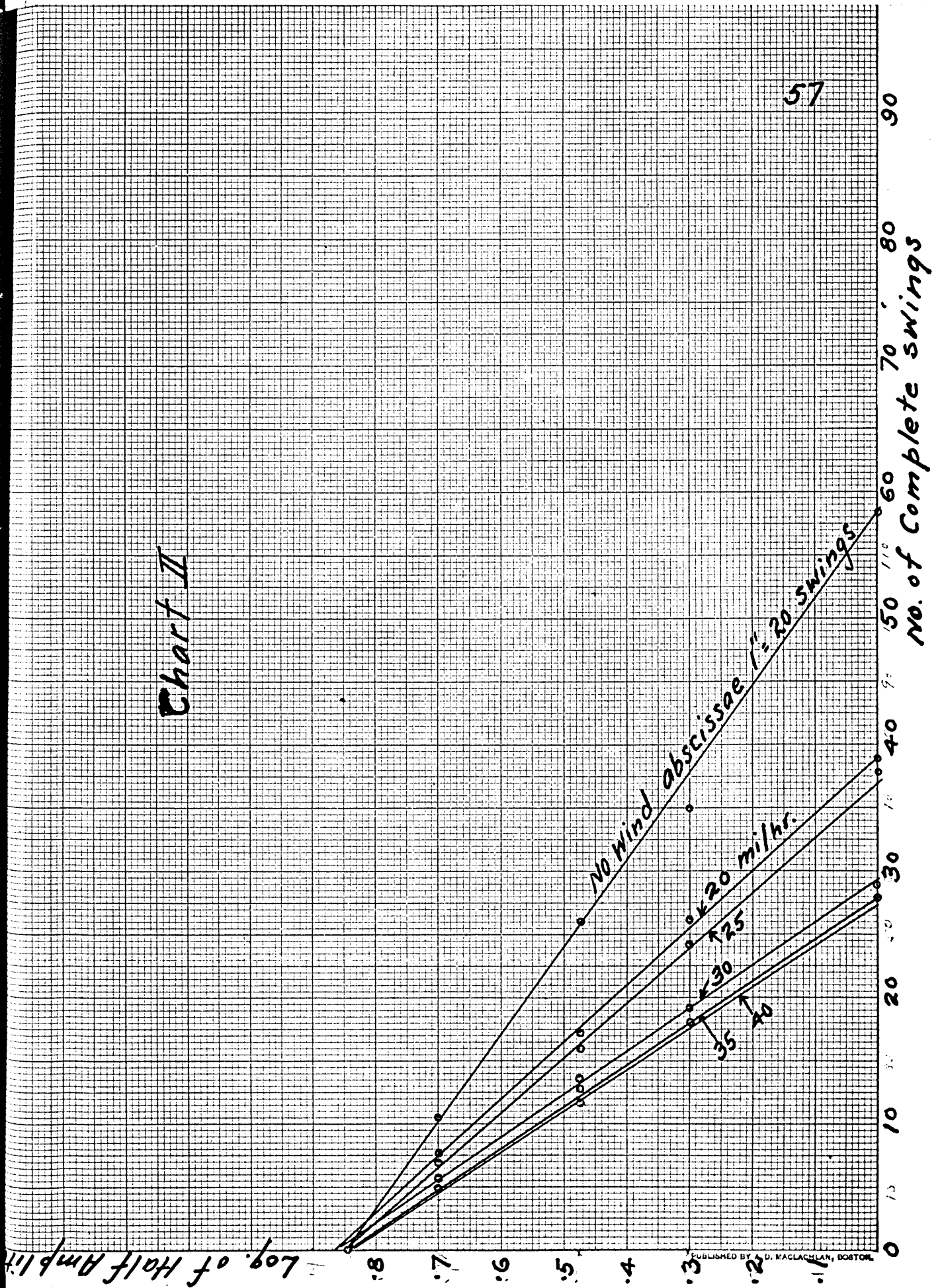
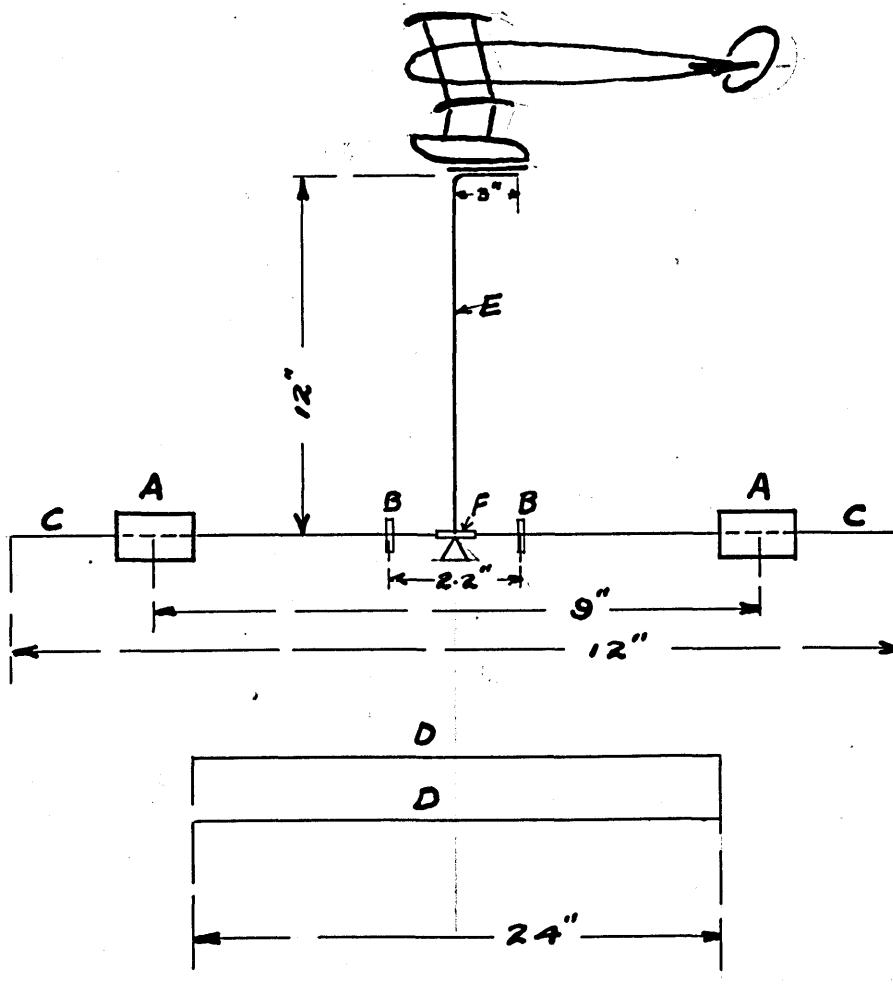The distribution of mass is shown in Fig.6, and the values of known moment of inertia are given in Table V.

Fig. 6

TABLE V.

Items	Mass in Slugs	x^2 in ft.	I in slug-ft^2
A + A	0.0523	$(\frac{4.5}{12})^2$	0.005440
B + B	0.0161	$(\frac{1.1}{12})^2$	0.000135
C + C	0.0051	$\frac{*(6/12)^2}{3}$	0.000424
D + D	0.0204	$\frac{*(2)^2}{3}$	0.027200
E + Bend of 3"	0.0062	------	0.00292
F	0.0340	------	negligibly small.
Model & Fitting	0.0323	------	To be found.
Total	0.1664 slugs = 5.36 pounds.		

*The moment of inertia of a slender rod about
an axis perpendicular to it at its end is

$m\frac{l^2}{3}$, where l is the length of the rod.

Let I be the moment of inertia of the apparatus and the model together.   Then
I - (A A + B B + C C) gives the moment of inertia of the upright and the model which are fixed in the apparatus.   The value of AA, BB, CC, are found in Table V to be

A+A 0.005440

B B 0.000135

C C 0.000424

Total   0.005999 = 0.006 say

Hence, in the experiment for moment of inertia, when A A, B B, C C are removed and longer rods D D are substituted, and when A A are no longer 9" apart but some other distance (say 10" in this experiment), the moment of inertia of the system

$$(I - 0.006) \quad + \quad 0.00544 \times (5/45)^2 \quad + \quad 0.0272$$

Fixed parts   New I of A A   I of long rods.

(see Table V) = I + 0.0279.

When in the second experiment A A are further removed from the axis of rotation (44" apart in this case), the moment of inertia is, by the

same reasoning,

$$\underbrace{I - 0.006}_{\text{I of Fixed Parts.}} + \underbrace{0.00544 \times (\frac{22}{4.5})^2}_{\text{New I of A A}} + \underbrace{0.0272}_{\text{I of long rods.}}$$

I of Fixed Parts.   New I of A A   I of long rods.

$$= I + 0.1508$$

These two moment of inertia of the two systems
are to each other as the squares of the periods.
Hence

$$(\frac{p_1}{p})^2 = \frac{I + 0.1508}{I + 0.0279}$$

in which $p_1 > p$. We find that $p_1 = 1.113$ *Sec.*

$p = 0.676$ *sec.*

$$(\frac{1.113}{0.676})^2 = \frac{I + 0.1508}{I + 0.0279} = 2.72$$

From this we find $I = 0.0438$ slug-ft.^{2}

We will take the average values of

$$\frac{\mu_0 + \mu}{2 I}$$ in Tables II and IV on pages 53 and 56

and multiply them by $2 I$ to get rid of I.

TABLE VI.

Speed of Wind mi/hr.	$\left(\dfrac{\mu_0+\mu}{2I}\right)_1$	$\left(\dfrac{\mu_0+\mu}{2I}\right)_2$	Ave. $\left(\dfrac{\mu_0+\mu}{2I}\right)$	$\mu_0+\mu$
0	0.0340	0.0202	0.0271	0.00238
10	0.0499	. . . .	0.0499	0.00437
20	0.0674	0.0650	0.0662	0.00580
25	0.0754	0.0694	0.0724	0.00634
30	0.0840	0.0920	0.0880	0.00770
35	0.0946	0.1010	0.0978	0.00856
40	0.1177	0.1060	0.118	0.00980

$$I = 0.0438 \quad 2I = 0.0876$$
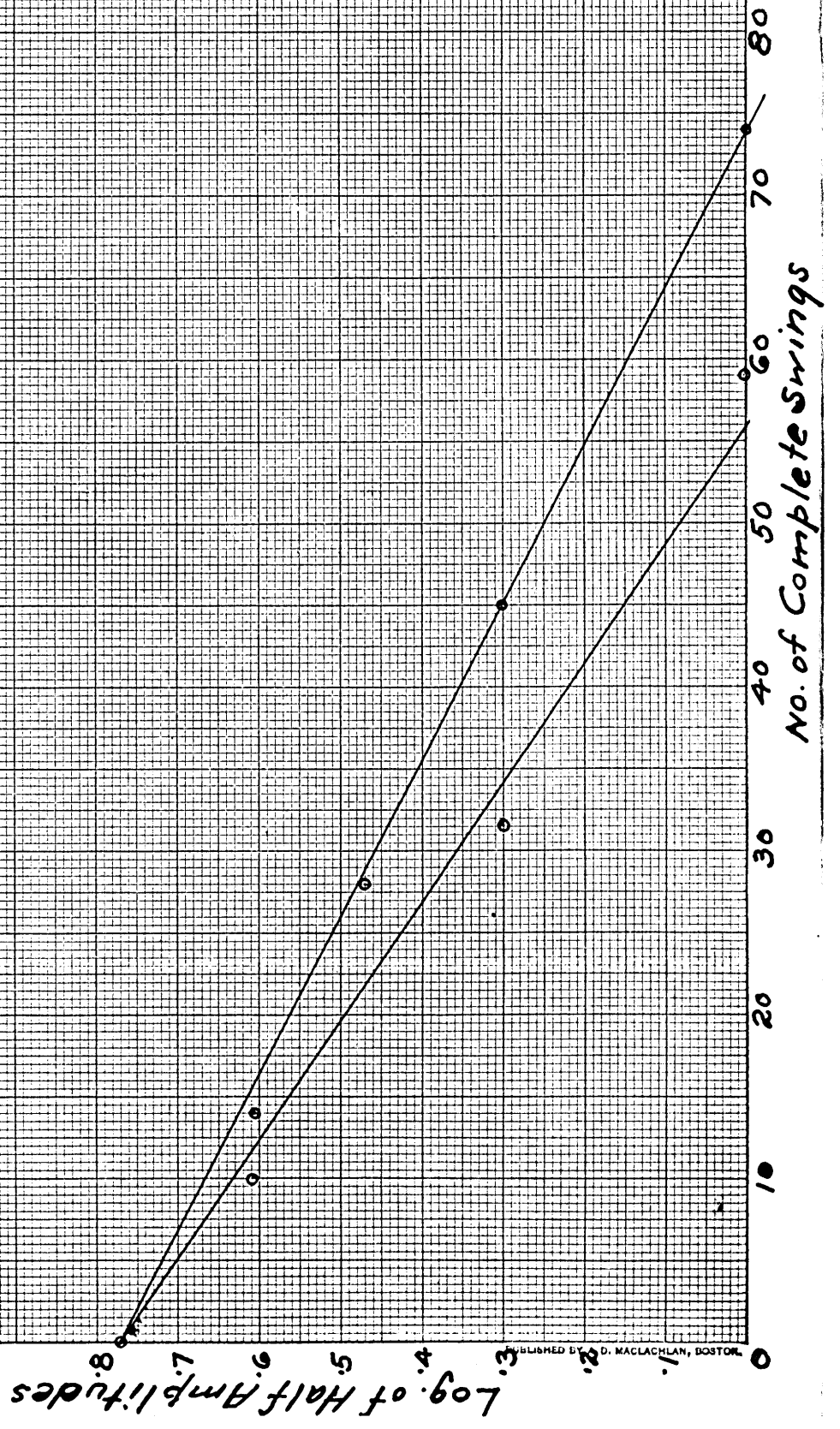
Apparatus Damping.

We have *eliminated* I. It remains for us to get rid of μ_0 in the last column of Table VI. The model was removed and readings for logarithmic decrement both with and without wind were taken, for apparatus damping is itself composed of frictional damping and aerodynamic damping.

TABLE VII.

$\log_{10} \theta$	θ Scale $\theta_0 = 6$	No. of Complete Swings at Wind Speed of	
		0	30 mi/hr.
0.778	6		
		14.0	10
0.604	4		
		14.0	
0.477	3		21.6
		17.1	
0.301	2		
		29.1	27.5
0.000	1		
	0		
Period in Sec.		0.384	0.384

On Chart III we find curves similar to those in Chart I and II, from which we obtain the following:

Chart III

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Log. of Half Amplitudes

No. of Complete Swings

No wind $\quad \dfrac{2.30 \times .788}{55 \times 0.384} = \dfrac{2\,I}{0.0632}$

30 mi/hr. $\quad \dfrac{2.30 \times .788}{55 \times 0.384} = 0.0850$

in which I is the moment of inertia of the apparatus alone.   The difference of the two gives 0.0218 which must be interpreted as additional damping due to the 30 mi/hr. wind.   If we assume,- and that assumption is probably correct for the parts exposed to wind stream were very small,- that wind damping is a linear function of wind speed, then for other speeds, the wind damping will be

$\dfrac{0.0218}{30}$ X speed.   Thus

TABLE VIII.

Wind Speed mi/hr.	Frictonal Damping.	Aerodynamic Damping $\dfrac{0.0218}{30} \times \text{Speed}$	Total $\dfrac{\mu_0}{2 I}$	μ_0 $(2 I = 0.018)$
0	0.0632	0	0.0632	0.00114
10	"	0.0073	0.0705	0.00127
20	"	0.0145	0.0777	0.00140
25	"	0.0182	0.0814	0.00147
30	"	0.0218	0.0850	0.00153
35	"	0.0254	0.0886	0.00160
40	"	0.0291	0.0923	0.00166

Moment of Inertia

A A	0.005440
B B	0.000135
C C	0.000424
E	0.002920
Total I =	0.008919 = 0.009 slug-ft^2

The Net Damping of the Model.

Combining Tables VI and VIII, we obtain

TABLE IX.

Wind Speed mi/hr.	Ft./Sec.	$\mu_0 + \mu$	μ_0	μ_{net}
0	0	0.00238 ..	0.00114 ..	0.00124
10	14.5 ...	0.00437 ..	0.00127 ..	0.00310
20	29.0 ...	0.00580 ..	0.00140 ..	0.00440
25	37.0 ...	0.00634 ..	0.00147 ..	0.00487
30	44.0 ...	0.00770 ..	0.00153 ..	0.00617
35	51.0 ...	0.00856 ..	0.00160 ..	0.00696
40	58.5 ...	0.00980 ..	0.0.166 ..	0.00814

In Chart IV, we plot μ_{net} against speed in miles per hour. With due allowance for experimental errors, a straight line can be drawn through these points. Then the tangent $= \dfrac{\mu}{U}$ will be constants for all speeds, where U is the relative velocity of the wind to the model or to an aeroplane. Taking μ at 30 miles per hour,

We find the constant rate to be

$$\frac{0.00617}{30} = 0.000206 \quad \text{if} \quad U \text{ is in mi/hr.}$$

or $\dfrac{0.00617}{44} = 0.000140 \quad \text{if} \quad U \text{ is in ft./sec.}$

Hence, using ft.sec. units,

$$= 0.000140 \, U \quad \ldots\ldots\ldots\ldots\ldots \quad (46)$$

Since $\mu = - (m \, M_q + m \, X_u)$, page 28

$$m \, M_q + m \, X_u = - \, 0.00014 \, U \quad \ldots\ldots \quad (47)$$

From (47) when X_u is known, $m \, M_q$ may be found. In general X_u is very small compared with M_q and may be neglected.

$$\therefore \quad m \, M_q = - \, 0.00014 \, U \quad \ldots\ldots\ldots\ldots \quad (48)$$

0.009

0.008

0.007

0.006

μ NST

0.005

0.004

Chart IV

0.003

0.002

0.001

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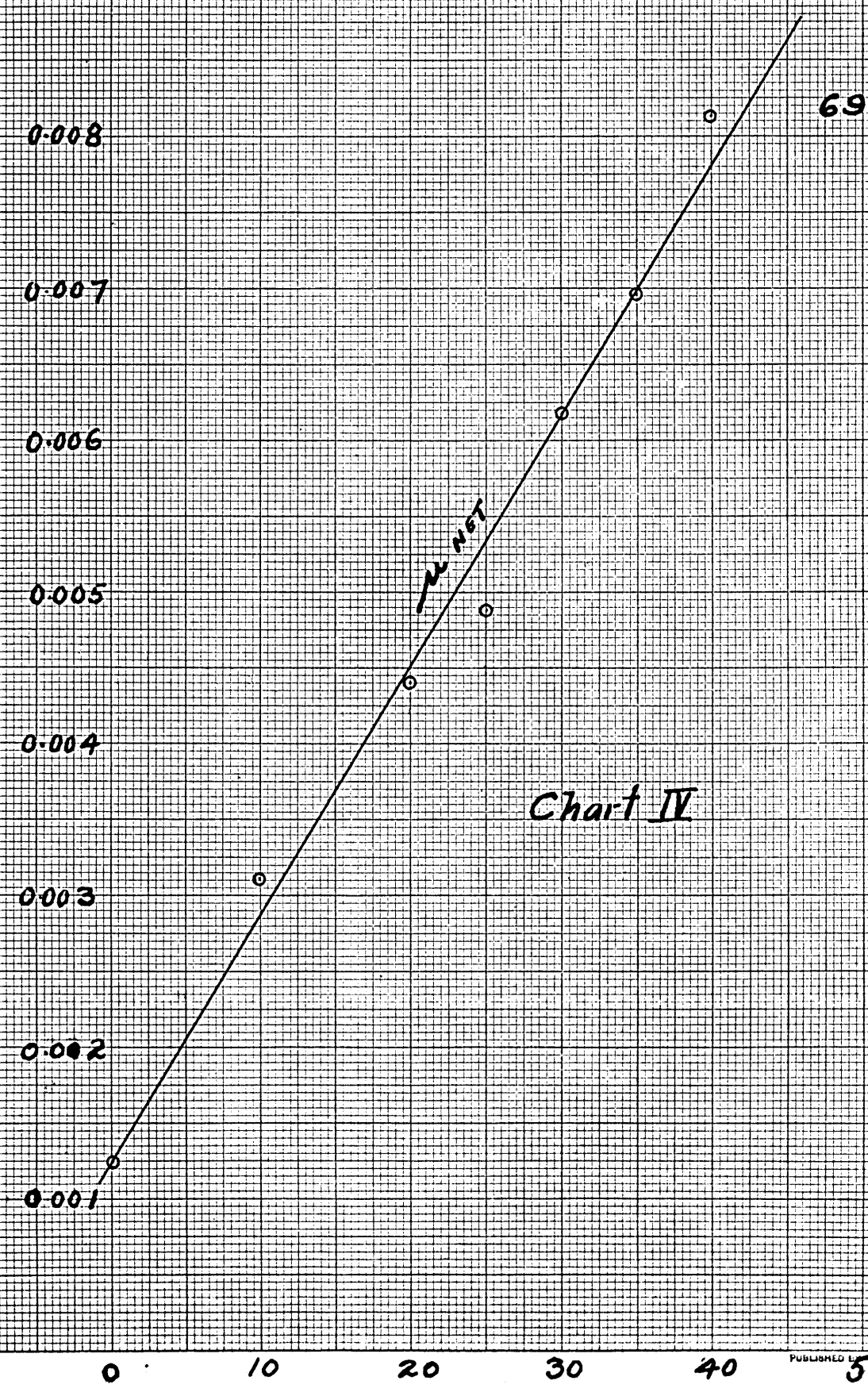0   10   20   30   40   50

Stepping up of the Result for Model

to Full Scale Aeroplane.

$m M_q$ in (**48**) applies to model only.
It is necessary to consider the dimension of

or $m M_q$.

In the original equation

$$I \frac{d^2\theta}{dt^2} + (\mu_0 + \mu) \frac{d\theta}{dt} + (Ka^2 - cW) \theta = 0,$$

Each term expresses a moment and has the dimension

$$(\mu_0 + \mu) \frac{d\theta}{dt} = \text{moment}$$
$$= \text{force X distance}$$
$$= m \frac{1}{T^2} \times 1 = m \frac{1^2}{T^2}$$

$$\frac{\mu_0 + \mu}{T} = m \frac{1^2}{T^2}$$

$$\mu_0 + \mu = m \frac{1^2}{T} = m 1 \left(\frac{1}{T}\right)$$

$$= \frac{1^3}{1^3} m 1 \left(\frac{1}{T}\right) = \frac{m}{1^3} 1^4 v$$

$$= \rho 1^4 v \quad \ldots\ldots\ldots \quad (49)$$

where ρ is the density of the fluid.   This
is correct provided a certain function

$$f \left(\frac{v\ 1}{\nu} ,\ \frac{1\ \dot{\theta}}{v} \right)$$

is kept approximately constant, in which ν is
the kinetic viscosity of the fluid.

From (**49**) it is evident that μ is pro-
portional to 1^4.   If the scale of the model
is $1/20$ of the full size machine then

$$m\ M_q = - (20)^4 \times 0.00014\ U$$
$$= -\ 22.4\ U$$

If $U\ =\ 80$ ft./sec. (55 mi p.h.),

m = 40 slugs, then

$$M_q = -\ \frac{22.4 \times 80}{40}\ =\ -\ 44.8$$

Discussion of the Result.

In Report No. 78 of "Technical Report of the Advisory Committee ~~of~~ for Aeronautics, 1912-1'13" an experimental investigation ~~of~~ of a similar character was undertaken upon a model of a monoplane of the Blériot Type.   The drawing of the model is found in Fig. 4 Report No. 75.   The logarithmic decrement was photographed instead of being observed with the ~~human~~ eye, as in the case of the present investigation.   Among other things, the Report gives for m M_q

$$m \ M_q = - \mu = - \ 0.00069 \ U \ for \ model$$

on top of page 177, which is about five times larger than the value obtained in this experiment.

$$\frac{- \ 0.00069}{- \ 0.00014} = 5$$

This discrepancy can be explained by the fact that the two models differ a great deal both in the disposition of tail surfaces, and in the

character of the body.   The British model has a large tail far removed from the c.g. and has an enclosed body with its knife-edge horizontal. Not only the damping surface is large but also the moment arm of the damping moment is much longer."   On the other hand, our model has a small tail comparatively close to the c.g. of the model, and has not anything like a body to speak of. . (See Frontispiece).   We naturally expect less damping in our model than that in the British model.

If we take into consideration the scale of linear dimensions of the moment arms of the damping moments of the two models, we shall find the results agree very well.   The damping coefficients are to each other as the fourth power of the linear dimensions.

$$\left\{ \frac{l_1 \text{ (British Model)}}{l_2 \text{ (our model)}} \right\}^4 = 5$$

$$\frac{l_1}{l_2} = \sqrt[4]{5} = 1.5 \text{ approximately}$$

$$l_1 = 1.5\, l_2$$

$l_1 = 14$ inches $l_2 = \dfrac{14}{1.5} = 9.3$ ins.

l_2 actually measured $= 8$ inches.

Conclusion..

In the absence of other coefficients which are necessary for the calculation of stability, it is difficult to say whether the machine will be longitudinally stable or unstable.   One thing is certain, however, that the damping coefficient for pitching for this particular machine is only one-fifth that of the Blériot Monoplane as reported in the British Report.

The End.