

FIRST PASSAGE AND OTHER FAILURE CRITERIA IN
NARROW-BAND RANDOM VIBRATION: A DISCRETE STATE APPROACH

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ABSTRACT

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A closed-form analytical estimate is given for the probability distribution of the time required for a narrow-band random process to cross a specified threshold level for the first time. The results are specialized for the case when the process represents the response of a lightly damped single-degree-of-freedom oscillator to wide-band random excitation.

The solution is based upon a two-state description of the threshold crossings of a stationary narrow-band process and its envelope. The same concept is also used to obtain the mean and variance of the fraction of the peaks which exceed a specified threshold. The stochastic properties of certain measures of the response of simple inelastic systems to random excitation are investigated.

It is shown that the theory of discrete state Markov processes may provide an excellent tool for analyzing the performance of structural systems for which various states of damage may be identified, and where the complete excitation history consists of a sequence of motions isolated in time. The feasibility of performing a dynamic system design optimization by means of this approach is indicated.

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INTRODUCTION

The response of structures to random excitation is of wide engineering interest. The vibration environment may be generated by such diverse sources as atmospheric turbulence, ocean waves in a rough sea, ground motion due to earthquakes or, acoustic pressures caused by jet engines and rocket motors. In each case, the excitation may be characterized by the fact that vibrational energy is generated in a random manner over a broad band of frequencies. A probabilistic outlook on the design of structures or equipment functioning in such an environment seems mandatory.

Those aspects of the theory of random vibration which deal with the first and second order statistics of random processes and with the input-output relations for linear devices has been developed and used for almost 25 years in the field of communication engineering [1-5], and have only fairly recently [6, 7] caught the attention of mechanical and structural engineers. One aspect of the theory is widely recognized [7, 8, 9] not to have been satisfactorily investigated: that of the conversion of random vibration response measures into useful performance or reliability measures. Since the designer's chief aim is to insure satisfactory performance of the structure, it is important that he be given

the (preferably analytical) tools to adequately formulate and then evaluate performance criteria. It is with this aspect of the theory of random vibration that this thesis deals. Most of the work in Chapters II through VII is believed to be original.

One very important failure mechanism is that which postulates failure when the dynamic response quantity first reaches a maximum allowable value. The desired reliability measure is the probability of the time to first passage across a specified barrier. An exact solution to this classical problem has not yet been found [10, 11, 12]. Shinozuka [13] proposed upper and lower bounds, and some widely used approximate results may be obtained if the major assumption is made that the threshold crossings occur independently according to a Poisson process [14, 15]. Actual values of the first passage probability may considerably deviate from this simple estimate, however [16]. From numerical studies [17] and other analytical work [18, 19] useful information has become available about the nature of the dependence of say, the mean first passage time on the barrier configuration, the barrier level and the characteristics of the structure and the excitation. In Chapter III, a new approximate closed-form analytical solution is obtained to the first crossing problem for essentially arbitrary narrow-band random processes, i.e., the solution is not restricted to Gaussian response processes nor to white noise input. The analytical result compares very favorably with the

estimates obtained by simulation techniques and other numerical and analytical work [17, 19]. Also, the proposed solution is in agreement with the available information regarding the behavior of the first passage probability under limiting conditions associated with high and low thresholds [7, 8, 9]. The solution is based on a two-state description of the threshold crossings of a stationary narrow-band random process and its envelope.

The same concept (of a two-state process) is also useful in obtaining other potentially important performance measures, e.g., the fraction of the peaks of the response which exceed a specified threshold, or a closely related measure, the fraction of the time the response envelope exceeds the threshold. These fractions are random variables for which the mean and variance are given in Chapter V. The approach also leads to a new way of deriving the probability distribution of the envelope and of the peaks in a narrow-band random process, and to an uncommon way of viewing the problem of linear cumulative fatigue damage [20, 21].

The above mentioned simple performance criteria do not by any means constitute a final answer as to how random response characteristics should be converted into useful reliability measures. In Section V.4 and also in Chapter VII it is shown (on the basis of some simple three-state models) that the theory of discrete state Markov processes [22, 23]

provides an excellent tool for modeling more complex (but also more realistic) failure mechanisms. All of the Chapters I through VI deal with the response of a structural system during a single excitation (e.g., a single earthquake, a single flight, etc.) of given intensity and duration. In Chapter VII we study the occurrence pattern of a set of single isolated motions and their effect on the structure. One of the most useful properties of the proposed multiple state Markov chain approach is that it provides a framework for quantifying the performance of the structure in terms of losses and rewards. This aspect of the theory of Markov chains has recently received considerable attention in the field of control processes and mathematical optimization [23-26]. The feasibility of performing a dynamic system design optimization by means of this approach is indicated.

Chapters III and IV (see Eqs. III-3.3,4 and IV-4.9-12) contain the principal new results on first passage probabilities. The fundamentals needed to arrive at the first-passage results are developed in Chapter II. Chapters V, VI and VII, each of which is also based on the work in Chapter II, may be studied almost independently. Finally, the reader may wish to bypass Chapter I, which is included mainly for easy reference.

CHAPTER I

SOME BASIC PROPERTIES OF STATIONARY RANDOM PROCESSES

Introduction

No introduction is needed for those probabilistic concepts which are well-known and standard. They will be used freely throughout the thesis. This chapter is devoted mainly to a review of a few basic properties of random processes for which conflicting definitions exist. First, the autocorrelation function and the spectral density function are discussed. Another concept of central importance here for which a formal definition is needed is that of the envelope of a narrow-band random process. Rice's [1] results for the mean stationary threshold crossing rates are given for easy reference. For a more extended treatment on these subjects see, e.g., Rice [1], Crandall [7, 27], Cramer and Leadbetter [9]. In the final section of this chapter the spectral density shape factor of a Gaussian stationary random process is introduced and evaluated for a few well-known spectral density functions.

I.1. Autocorrelation and Spectral Density

Let the zero-mean random process $X(t)$ be stationary in the wide sense (or weakly stationary). Then its autocorrelation function $R_X(\tau)$, defined as the ensemble average of the product of the random variables $X_1 = X(t_1)$ and $X_2 = X(t_2)$, is a function only of the absolute value of the time difference $\tau = t_2 - t_1$. We have

$$R(\tau) = E[X(t_1)X(t_1+\tau)] = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} x_1 x_2 p(x_1, t_1; x_2, t_2) dx_1 dx_2 \quad (\text{I-1.1})$$

where $p(x_1, t_1; x_2, t_2)$ denotes the joint probability density function of X_1 at time t_1 and X_2 at time t_2 . Assuming the process is ergodic, the temporal autocorrelation function of an individual sample function $x(t)$ also equals $R(\tau)$.

$$R(\tau) = \langle x(t_1)x(t_1+\tau) \rangle = \lim_{T \rightarrow \infty} \frac{1}{T} \int_{-T}^{+T} x(t)x(t+\tau) dt \quad (\text{I-1.2})$$

The symbol $\langle \rangle$ denotes the operation of temporal averaging and T is the averaging time.

If the autocorrelation function is absolutely integrable and finite then its Fourier transform defines the power spectral density $G(\omega)$ of $X(t)$. Since $R(\tau)$ is real and even, this can be written as a cosine transform,

$$G(\omega) = \frac{2}{\pi} \int_0^{\infty} R(\tau) \cos \omega \tau \, d\tau \quad (\text{I-1.3})$$

$$R(\tau) = \int_0^{\infty} G(\omega) \cos \omega \tau \, d\omega \quad (\text{I-1.4})$$

The above expressions are known as the Wiener-Kinchine relations [28, 29]. $G(\omega)$ is the one-sided spectral density, which exists only for $\omega > 0$. The two-sided spectral density obtained by taking the exponential transform, takes values for both positive and negative ω ; it is an even function and for $\omega > 0$ is equal to half the value of the one-sided spectral density. The one-sided spectral density will be used exclusively throughout this thesis.

The first few moments of the one-sided spectral density are frequently used in the sequel. We have for the j th moment,

$$\lambda_j = \int_0^{\infty} \omega^j G(\omega) \, d\omega \quad (\text{I-1.5})$$

In particular, for $j=0$, using Eqs. I-1.5, I-1.4 and I-1.1,

$$\lambda_0 = \int_0^{\infty} G(\omega) \, d\omega = R(0) = E[X^2] \quad (\text{I-1.6})$$

Also, for ergodic processes, $\lambda_0 = \langle x^2 \rangle$. Each sample function may loosely be thought of as a superposition of elementary sinusoids each of which is characterized by its frequency ω and has

a random amplitude and phase. $G(\omega)d\omega$ may be interpreted as the ensemble average power (or "long-run" average power in the ergodic case) contained in those elementary sinusoids with frequencies in an infinitesimal band $(\omega, \omega+d\omega)$.

I.2. Formal Spectral Representation of a Random Process

A formal spectral representation of the real stationary random process $X(t)$ is possible. The random time process $X(t)$ may be expressed in terms of the random components $U(\omega)$ and $V(\omega)$ of the spectral process in the following way [2, 30]

$$X(t) = \int_0^{\infty} [U(\omega)\cos\omega t + V(\omega)\sin\omega t]d\omega \quad (\text{I-2.1})$$

Eq. I-2.1 states that the random process $X(t)$ is a superposition (in the frequency domain) of elementary harmonic oscillations [2]

$$[U(\omega)\cos\omega t + V(\omega)\sin\omega t]d\omega = A_{\omega} \cos(\omega t + B_{\omega}) \quad (\text{I-2.2})$$

each of which has an angular frequency ω while the random amplitude A_{ω} and phase B_{ω} are functions of $U(\omega)d\omega$ and $V(\omega)d\omega$. If the random variables $U(\omega)d\omega$ and $V(\omega)d\omega$ have zero mean and variance equal to $G(\omega)d\omega$ and if for all (nonnegative) ω_1 and ω_2 ,

$$E[U(\omega_1)V(\omega_2)] = 0 \quad (\text{I-2.3})$$

then it may be verified that

$$E[\{U(\omega)\cos\omega t + V(\omega)\sin\omega t\}d\omega] = 0 \quad (\text{I-2.4})$$

$$\begin{aligned} E[\{U(\omega)\cos\omega t + V(\omega)\sin\omega t\}d\omega]^2] \\ = E[\{U(\omega)d\omega\}^2\cos^2\omega t + \{V(\omega)d\omega\}^2\sin^2\omega t] \\ = G(\omega)d\omega \end{aligned} \quad (\text{I-2.5})$$

The elementary oscillations, Eq. I-2.2, that contribute to $X(t)$ also have zero mean and variance $G(\omega)d\omega$. Furthermore, if the processes $U(\omega)$ and $V(\omega)$ have orthogonal increments, i.e., $E[U(\omega_1)U(\omega_2)] = E[V(\omega_1)V(\omega_2)] = 0$ for $\omega_1 \neq \omega_2$ then the spectral representation, Eq. I-2.1, of $X(t)$ may be shown to be satisfactory in terms of the first and second order statistics of $X(t)$. Following Crandall [30],

$$E[X(t)] = \int_0^{\infty} E[U(\omega)]\cos\omega t d\omega + \int_0^{\infty} E[V(\omega)]\sin\omega t d\omega = 0 \quad (\text{I-2.6})$$

$$\begin{aligned} E[X(t)X(t+\tau)] &= \int_0^{\infty} \int_0^{\infty} E[\{U(\omega_1)\cos\omega_1 t + V(\omega_1)\sin\omega_1 t\} \\ &\quad \{U(\omega_2)\cos\omega_2(t+\tau) + V(\omega_2)\sin\omega_2(t+\tau)\}d\omega_1 d\omega_2] \\ &= \int_0^{\infty} G(\omega_1)d\omega_1 [\cos\omega_1 t \cos\omega_1(t+\tau) + \sin\omega_1 t \sin\omega_1(t+\tau)] \\ &= \int_0^{\infty} G(\omega_1)\cos\omega_1 \tau d\omega_1 = R(\tau) \end{aligned} \quad (\text{I-2.7})$$

In particular the variance of $X(t)$ becomes

$$E[X^2] = \int_0^{\infty} G(\omega) d\omega \quad (\text{I-2.8})$$

Gaussian random processes are completely characterized by their first and second order statistics. Hence their spectral representation in terms of the processes $U(\omega)$ and $V(\omega)$ with properties given above, will be entirely equivalent to their time representation. The concepts discussed above, which are fundamental to the theory of stationary processes, have been included here mainly to allow a formal definition of the envelope of a process to be introduced in the next section.

I.3. Envelope of a Random Process

Various definitions, formal and informal, of the envelope of a random process are in existence. Those that have direct physical significance are closely related with processes having narrow-band spectral densities. For example, the earliest formal definition due to Rice [1] presumes the existence of a "representative midband frequency". In the case of a narrow-band process the envelope is intuitively thought of as a pair of smoothly varying curves $R(t)$ and $-R(t)$ where $|R(t)|$ is such that $|R(t)| \geq |X(t)|$ for all t , and $|R(t)| = |X(t)|$ at or very nearly at the peaks of $|X(t)|$.

If $X(t)$ represents, say, the displacement response of a simple mechanical system (mass-spring-dashpot) then the following "energy-based" envelope definition due to Crandall [31] may be adopted

$$V[R(t)] = V[X(t)] + \frac{1}{2}\dot{X}^2(t) \quad (\text{I-3.1})$$

$V[X(t)]$ is the potential energy per unit mass and $\frac{1}{2}\dot{X}^2(t)$, where $\dot{X}(t)$ is the time derivative of the process, is the kinetic energy per unit mass. The envelope $R(t)$ is defined as the hypothetical displacement that would result if the total energy were converted into potential energy. A closely related definition also due to Crandall [7] defines $|R(t)|$ as the radius of the point $(\dot{X}/\omega_0, X)$ in the phase-plane representation of the random process

$$R^2(t) = X^2(t) + \dot{X}^2(t)/\omega_0^2 \quad (\text{I-3.2})$$

where ω_0 is the average or expected frequency of the process $X(t)$, to be formally defined in section I.4.

Presume $X(t)$ to be the response of a linear one-degree-of freedom oscillator excited by white noise. Since $X(t)$ is a second-order Markov process, the mathematical difficulties encountered in solving first-crossing problems for $X(t)$ are nearly insurmountable [11, 32]. In the work of Hellstrom [33],

Rosenblueth and Bustamente [17] and Caughey and Gray [34] the envelope (or a closely related quantity in [15]) of $X(t)$ is defined in such a way that the property of polar symmetry may be used to reduce the second-order Markov process $X(t)$ to a first-order Markov process.

Cramer and Leadbetter's [9] definition, although essentially equivalent [9, 30] to that of Rice [1], does not require a "representative midband frequency" to be specified and can strictly be obtained for any stationary random process (whether or not the concept of an envelope is physically meaningful). Also, unlike the envelope definitions mentioned earlier, their definition leads to an envelope process whose mean-square derivative does exist. As will be shown subsequently, the latter property is needed to be able to compute the mean number of times the envelope crosses a level $x=a$, with positive slope, per unit time. According to Cramer and Leadbetter [9], the envelope $R(t)$ of $X(t)$ is defined as the modulus of the envelope vector $\vec{R}(t)$

$$R(t) = |\vec{R}(t)| = [X^2(t) + \hat{X}^2(t)]^{1/2} \quad (\text{I-3.4})$$

in which the random process $\hat{X}(t)$ is defined in terms of the components $U(\omega)$ and $V(\omega)$ of the spectral representation of $X(t)$, Eq. I-2.1,

$$\hat{X}(t) = \int_0^{\infty} [U(\omega) \sin \omega t - V(\omega) \cos \omega t] d\omega \quad (\text{I-3.5})$$

$\hat{X}(t)$ is seen to be composed of elementary oscillations

$$[U(\omega)\sin\omega t - V(\omega)\cos\omega t]d\omega \quad (\text{I-3.6})$$

The vector diagram (figure 1.1) shows how the contribution to the vector $\vec{R}(t)$ of the elementary oscillations with frequencies in the range $(\omega, \omega+d\omega)$, is related to the similar contributions to $X(t)$ and $\hat{X}(t)$ and in turn to $U(\omega)d\omega$ and $V(\omega)d\omega$. This representation is due to Crandall [30].

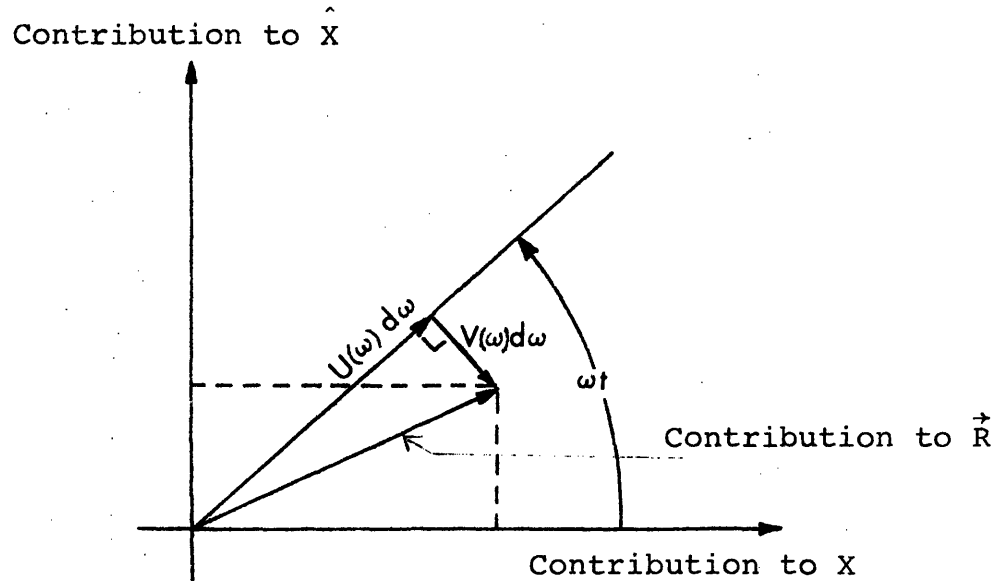


Fig.I.1. Rotating Vector Diagram Showing Contributions in the Frequency Range $(\omega, \omega+d\omega)$ to the Processes $X(t)$, $\hat{X}(t)$ and $\vec{R}(t)$.

It is easy to see that $|X(t)| \leq |R(t)|$, and $|R(t)| = |X(t)|$ for any t such that $\hat{X}(t) = 0$.

The envelope process $R(t)$ is directly related to $X(t)$ and $\hat{X}(t)$ which in turn are functions of $U(\omega)$ and $V(\omega)$. The random variables $X(t)$ and $\hat{X}(t)$ may be shown to be uncorrelated and hence, in the case of Gaussian processes, they will be independent and Gaussian distributed. In that case it is possible to derive the joint distribution of $R(t)$ and its derivative $\dot{R}(t)$ [9, 30]

$$p(r, \dot{r}) = \frac{r}{\lambda_0} e^{-r^2/2\lambda_0} \frac{1}{\sqrt{2\pi} \sqrt{\frac{B}{\lambda_0}}} e^{-\dot{r}^2/(2B/\lambda_0)} \quad (\text{I-3.7})$$

with λ_j given by Eq. I-3.5 and

$$B = \lambda_0 \lambda_2 - \lambda_1^2 \quad (\text{I-3.8})$$

The above result, given by Cramer and Leadbetter [9], reveals that $R(t)$ and $\dot{R}(t)$ are independent random variables; $R(t)$ has a Rayleigh distribution and $\dot{R}(t)$ is normally distributed with zero mean and variance B/λ_0 .

In closing this section on envelope definitions it should be stressed that all available definitions are very similar for narrow-band random processes [9]. Note, for example, that the absolute value of $\hat{X}(t)$ defined by Eq. I-3.6 becomes nearly equal to the absolute value of $\dot{X}(t)/\omega_0$ where $\dot{X}(t)$

is the time derivative of $X(t)$ and ω_0 is the expected circular frequency. We have

$$\begin{aligned}\dot{X}(t) &= \frac{d}{dt} [X(t)] = \frac{d}{dt} \left[\int_0^{\infty} [U(\omega)\cos\omega t + V(\omega)\sin\omega t]d\omega \right] \\ &= - \int_0^{\infty} \omega [U(\omega)\sin\omega t - V(\omega)\cos\omega t]d\omega \quad (\text{I-3.9})\end{aligned}$$

Since, for a narrow band process, all frequencies are closely centered around ω_0 , $\dot{X}(t)$ may be approximated as follows

$$\dot{X}(t) \approx -\omega_0 \hat{X}(t) \quad (\text{I-3.10})$$

I.4. Average Stationary Crossing Rates

The mean rate ν_a , of crossings with positive slope ("upcrossings") of a level $X=a$ by the process $X(t)$ may be obtained from the fundamental result due to Rice [1],

$$\nu_a = \int_0^{\infty} p(x, \dot{x}) \dot{x} dx \Big|_{x=a} \quad (\text{I-4.1})$$

in which $p(x, \dot{x})$ is the joint probability density function of $X(t)$ and its time derivative $\dot{X}(t)$. The mean rate ν_0 of zero crossings with positive slope is

$$\nu_0 = \left[\nu_a \right]_{a=0} \quad (\text{I-4.2})$$

and the apparent circular frequency of the process is $\omega_0 = 2\pi\nu_0$.

Similarly, the mean rate n_a of upcrossings of a threshold $X=a$ by the envelope $R(t)$ may be found from

$$n_a = \int_0^{\infty} p(r, \dot{r}) r dr \Big|_{r=a} \quad (\text{I-4.3})$$

in which $p(r, \dot{r})$ is the joint probability density function of $R(t)$ and $\dot{R}(t)$.

For a stationary Gaussian process $X(t)$ with a (one-sided) spectral density $G(\omega)$ we have

$$p(x, \dot{x}) = \frac{1}{\sqrt{2\pi\lambda_0}} e^{-x^2/2\lambda_0} \frac{1}{\sqrt{2\pi\lambda_2}} e^{-\dot{x}^2/2\lambda_2} \quad (\text{I-4.4})$$

and $p(r, \dot{r})$ is given by Eq. I-3.7. The mean rates v_a and n_a of upcrossings of a level $X=a$ associated with the process $X(t)$ and its envelope $R(t)$, respectively, become

$$v_a = \frac{1}{2\pi} \sqrt{\frac{\lambda_2}{\lambda_0}} e^{-a^2/2\lambda_0} \quad (\text{I-4.5})$$

$$n_a = \sqrt{\frac{\lambda_0\lambda_2 - \lambda_1^2}{2\pi\lambda_0}} \frac{a}{\lambda_0} e^{-a^2/2\lambda_0} \quad (\text{I-4.6})$$

I.5. Spectral Density Shape Factor

Much of our interest will focus on the ratio v_a/n_a , the average number of upcrossings of a level $X=a$ of the ran-

dom process $X(t)$ for each upcrossing (of the same level) of its envelope. For Gaussian processes, using Eqs. I- 4.5 and I- 4.6,

$$\frac{v_a}{n_a} = \sqrt{\frac{\lambda_2}{2\pi(\lambda_0\lambda_2 - \lambda_1^2)}} \frac{\lambda_0}{a} \quad (\text{I-5.1})$$

where λ_j is the j th moment of the spectrum $G(\omega)$, Eq. I-1.5. Let the level $X=a$ be expressed in terms of the r.m.s. value $\sqrt{\lambda_0}$ of the process $X(t)$,

$$r = \frac{a}{\sqrt{\lambda_0}} \quad (\text{I-5.2})$$

and let k be defined as follows

$$k = \left\{ 2\pi \left(1 - \frac{\lambda_1^2}{\lambda_0\lambda_2} \right) \right\}^{1/2} \quad (\text{I-5.3})$$

Then one obtains the simple result,

$$\frac{v_a}{n_a} = \frac{1}{kr} \quad (\text{I-5.4})$$

Full advantage will be taken of the surprising simplicity of the dependence of the ratio v_a/n_a upon the (normalized) barrier level r . The factor k is a characteristic of the shape of the spectral density of the stationary random process and depends only on its 0 th, 1 st and 2 nd moments. It will be

called the spectral density shape factor or simply, the shape factor. From Schwarz' inequality, $0 \leq \frac{\lambda_1^2}{\lambda_0 \lambda_2} \leq 1$, and hence $0 \leq k \leq \sqrt{2\pi}$. If $G(\omega)$ consists of a single spike, then $k=0$; for broad-band systems $k \rightarrow \sqrt{2\pi}$. In the remainder of this section, the shape factor k is computed for a few spectral densities in terms of their descriptive parameters.

i) Band-Limited Filter

The (one-sided) spectral density, $G(\omega)$, of an ideal band-pass filter with Gaussian white noise input is defined by

$$G(\omega) = \begin{cases} G_0, & 0 \leq \omega_a \leq \omega \leq b \\ 0, & \text{otherwise} \end{cases} \quad (\text{I-5.5})$$

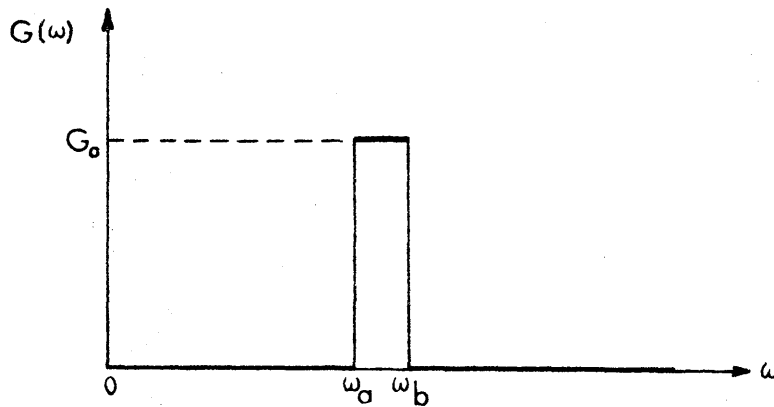


Fig.I.2. Band-Limited Filter

It is depicted in figure I.2. The first few moments λ_j are

$$\begin{aligned}
\lambda_0 &= G_0 (\omega_b - \omega_a) \\
\lambda_1 &= \frac{1}{2} G_0 (\omega_b^2 - \omega_a^2) \\
\lambda_2 &= \frac{1}{3} G_0 (\omega_b^3 - \omega_a^3)
\end{aligned} \tag{I-5.6}$$

Introducing the values of the moments into Eq. I-5.3 the shape factor k becomes

$$k = \sqrt{2\pi} \left[1 - \frac{3\omega_m^2 (\omega_b - \omega_a)}{\omega_b^3 - \omega_a^3} \right]^{1/2} \tag{I-5.7}$$

in which ω_m is the midband frequency, i.e., $\omega_m = \frac{\omega_b + \omega_a}{2}$. Letting $Q = \frac{\omega_m}{\omega_b - \omega_a}$, it may be easily shown that

$$k = \frac{1}{Q} \sqrt{\frac{2\pi}{12 + Q^{-2}}} \tag{I-5.8}$$

For $Q \gg 1$ the following approximation will be sufficiently accurate

$$k \approx \frac{1}{Q} \sqrt{\frac{\pi}{6}} \tag{I-5.9}$$

ii) Gaussian Filter

The (one-sided) spectral density of a Gaussian filter with Gaussian white noise input is defined by [35]

$$G(\omega) = \frac{G_0}{\sqrt{2\pi}\sigma} \exp\left[-\frac{(\omega - \omega_0)^2}{2\sigma^2}\right], \quad \omega \geq 0 \tag{I-5.10}$$

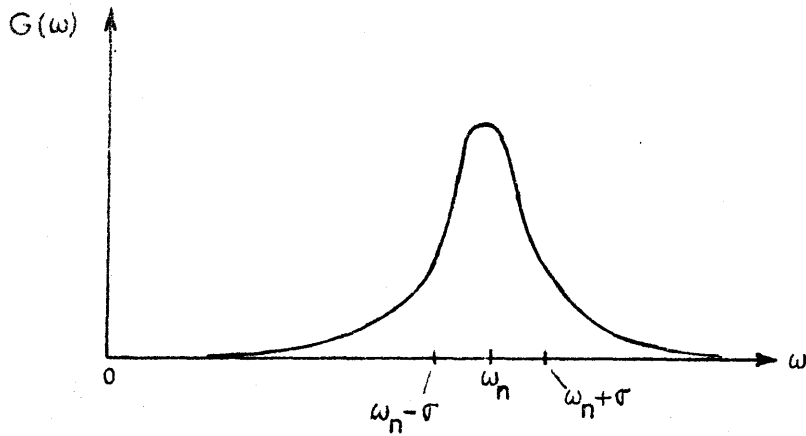


Fig.I.3. Gaussian Filter

as shown in figure I.3. For cases in which σ is substantially smaller than ω_n one easily obtains the following approximate results,

$$\begin{aligned}
 \lambda_0 &\approx G_0 \\
 \lambda_1 &\approx G_0 \omega_n \\
 \lambda_2 &\approx G_0 (\omega_n^2 + \sigma^2)
 \end{aligned}
 \tag{I-5.11}$$

Introducing Eqs. I-5.11 into Eq. I-5.3 and letting $Q' = \omega_n / \sigma$ one obtains

$$k \approx \frac{1}{Q'} \sqrt{\frac{2\pi}{1+Q'^{-2}}} \approx \frac{1}{Q'} \sqrt{2\pi}
 \tag{I-5.12}$$

iii) Response of a Simple Linear Oscillator

The basic input-output relationship of a linear system excited by stationary random excitation is [9]

$$G_X(\omega) = G_I(\omega) |H(\omega)|^2 \quad (\text{I-5.13})$$

where $G_I(\omega)$ is the (one-sided) input spectral density and $G_X(\omega)$ the (one-sided) response spectral density. $H(\omega)$ is the transfer function relating response to input for sinusoidal excitation. For example, the spectral density of the displacement response of a lightly damped linear oscillator when the excitation is a stationary Gaussian ideal white noise with spectral density $G_I(\omega) = G_0$ for $0 \leq \omega < \infty$ is given by

$$G_X(\omega) = \frac{G_0}{(\omega_n^2 - \omega^2)^2 + 4\zeta^2 \omega_n^2 \omega^2}, \quad 0 \leq \omega < \infty \quad (\text{I-5.14})$$

as shown in figure I.4.

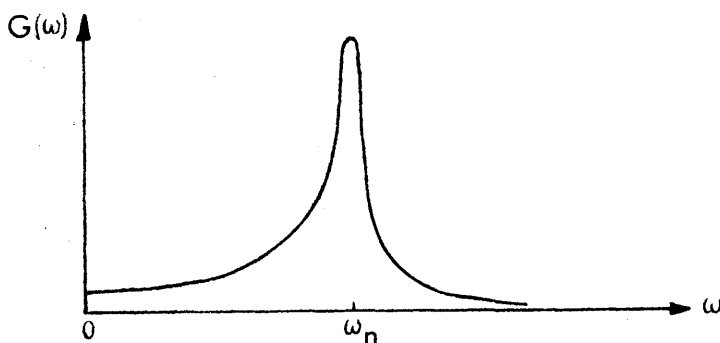


Fig.I.4. Spectral Density of the Response of a Lightly Damped Oscillator to White Noise Excitation.

in which ω_0 is the undamped natural frequency and ζ the ratio of critical damping of the one degree of freedom system. The first few moments of the spectral density are

$$\lambda_0 = \int_0^{\infty} G_X(\omega) d\omega = \frac{\pi G_0}{4\zeta\omega_n^3} \quad (\text{I-5.15})$$

$$\lambda_1 = \frac{\omega_n^2}{\sqrt{\omega_n^2(1-\zeta^2)}} \frac{\pi G_0}{4\zeta\omega_n^3} \left[1 - \frac{1}{\pi} \tan^{-1} \left(\frac{2\zeta\sqrt{1-\zeta^2}}{\sqrt{1-2\zeta^2}} \right) \right] \quad (\text{I-5.16})$$

$$\lambda_2 = \frac{\pi G_0}{4\zeta\omega_n} \quad (\text{I-5.17})$$

Inserting the above expressions into Eq. I-5.3 one may easily obtain the following approximate expression

$$k \approx 2\sqrt{2}\zeta^{1/2}, \quad \text{for small } \zeta \quad (\text{I-5.18})$$

Above expression is "asymptotically" (for $\zeta \rightarrow 0$) exact.

Similar computations may be performed for variety of combinations of excitation spectra (other than white noise) and transfer functions. A paper by Pulgrano and Ablowitz [36] contains a fairly complete list of moments of all possible transfer functions related to one-degree-of-freedom damped oscillators. The paper lists explicit analytical expressions for the incomplete integrals $I(\Omega)$,

$$I(\Omega) = \int_0^{\Omega} \omega^j |H(\omega)|^2 d\omega \quad (\text{I-5.19})$$

thus providing a straightforward method for computing the spectral moments of the response in those cases where the excitation spectrum $G_I(\omega)$ is piece-wise linear.

CHAPTER II

STATIONARY THRESHOLD CROSSING CHARACTERISTICS

II.1. Motivation

Measures of performance of physical systems undergoing random vibration are often quite naturally specified in terms of some fixed threshold value of an oscillatory quantity of the system. This will be the case when failure of the system is due to first excursion up to a certain level. Sometimes failure is related to the number of crossings of a threshold per unit time. For example, in a simple elasto-plastic system subjected to stationary broad-band random excitation, important response measures such as the total permanent deformation or the energy dissipated due to yielding, are closely related to the number of crossings of the yield level of the system.

Let $X(t)$ be the dynamic (response) variable of interest. Also, let $X=a$ and $X=-a$ represent fixed threshold values, in terms of which the performance requirement of the system is specified. Such a two-sided threshold configuration ($X=a$ and $X=-a$) will be called a D type barrier (following Crandall

et al.[17]).Of all the information contained in a complete description of the random process $X(t)$ only a small portion is relevant to the designer attempting to characterize the threshold-related performance of the system. The information contained, say, in the response r.m.s. value seems necessary, but not sufficient to solve first passage problems. The same may be said of the statistic v_a , the expected number of times $X(t)$ crosses the level $X=a$ at positive slope. The information of interest to the designer is believed to be that needed to completely characterize a simple two-state process (a zero-one process) $D_a(n)$ associated with the D type barrier $|X|=a$. The process $D_a(n)$ is observed at the peaks of $|X(t)|$, i.e. when $\dot{X}(t)=0$, and n counts the number of peaks or observations. Figure II.1a shows the relation between the zero-one process $D_a(n)$ and the actual random process $X(t)$. When the i th peak of $|X(t)|$ exceeds the threshold $|X|=a$, then $D_a(i)=1$, otherwise $D_a(i)=0$. It will be of interest to consider the length of sequences of consecutive peaks for which the process $D_a(i)$ has the same value. Let $N_{0,D}$ and $N_{1,D}$ be discrete random variables describing the number of consecutive peaks for which the process $D_a(i)$ takes a value "zero" and "one", respectively. In many cases interest will center on the value $N_{f,D}$ (see figure II.1a) which represents the number of average half-cycles it takes to cross the (two-sided) barrier for the first time.

The preceding paragraph deals with D type barriers.

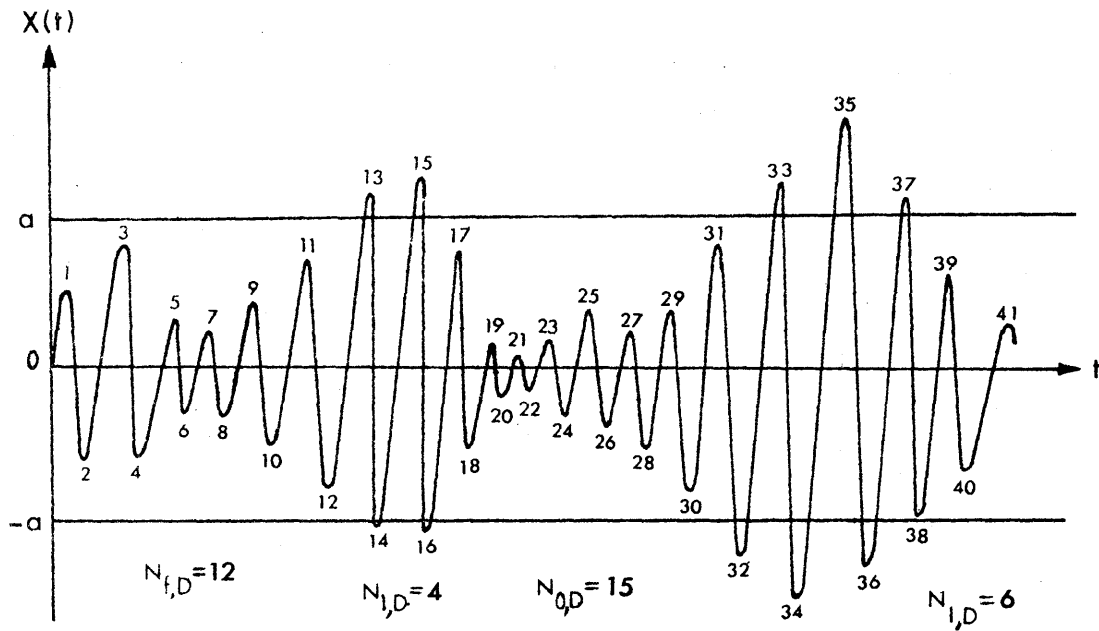


Fig.II.1a. The Discrete Time Two-State Process $D_a(n)$
 (for example, $D_a(1)=0$, $D_a(7)=0$, $D_a(15)=1$, $D_a(16)=1$, etc.)

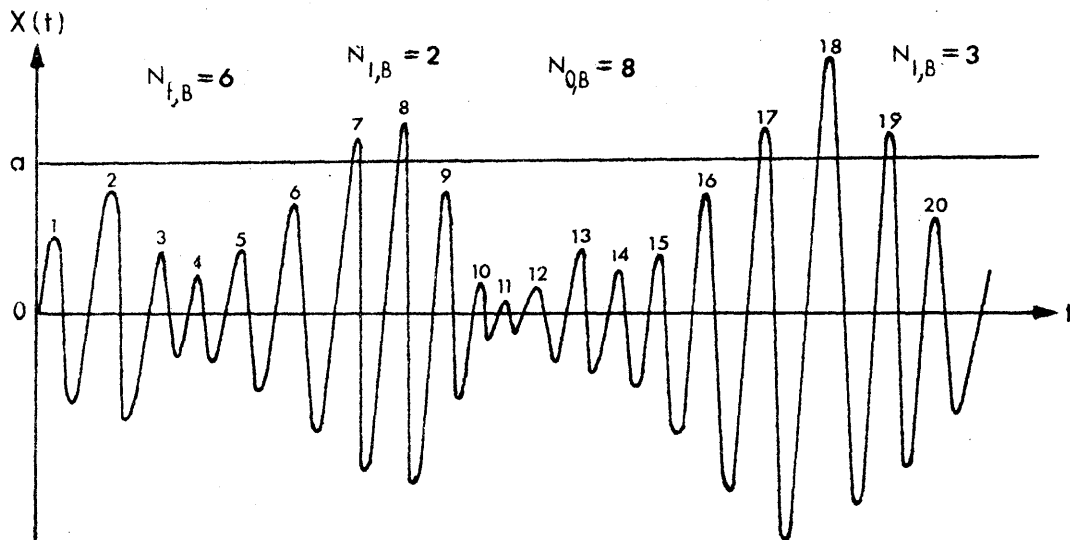


Fig.II.1b. The Discrete Time Two-State Process $B_a(n)$
 (for example, $B_a(7)=1$, $B_a(13)=0$, $B_a(17)=1$, $B_a(20)=0$, etc.)

In many cases of practical importance the barrier configuration is a simple one-sided barrier $X=a$ referred to here as a B type barrier. This quite naturally arises, for example, in processes with nonzero mean, e.g., where the total response may be viewed as the result of a superposition of a random oscillatory component and a slowly varying trend-setting component. For a formal study of performance criteria related to a B type barrier it will again be useful (as in the case of a D type barrier) to introduce a zero-one process. The new process $B_a(n)$ is observed at the peaks of $X(t)$, i.e., every $(v_0)^{-1}$ seconds on the average. See figure II.1b. Let $N_{1,B}$ denote the size of a clump of peaks of $X(t)$ above the threshold and $N_{0,B}$ the number of consecutive peaks of $X(t)$ below the threshold. $N_{f,B}$ denotes the number of average cycles to cross the threshold $X=a$ for the first time.

Important statistical properties of $D_a(n)$ and $B_a(n)$ are derived in sections II.4 and II.5 from the characteristics of still another two-state process related to the envelope $R(t)$ of the process $X(t)$. The latter is discussed in sections II.2 and II.3. The main results derived in this chapter, which will be used extensively throughout the remainder of the thesis, are summarized at the end of the chapter.

II.2. Threshold Crossings Associated with the Envelope
of a Stationary Narrow-Band Random Process

Consider the successive intersections of the envelope $R(t)$ of a stationary narrow-band process $X(t)$ and a fixed two-sided threshold $|X|=a$. See figure II.2. Let the two-state continuous time process $E_a(t)$ be defined as follows

$$E_a(t) = 0 \quad \text{if } R(t) < a$$

$$E_a(t) = 1 \quad \text{if } R(t) \geq a$$

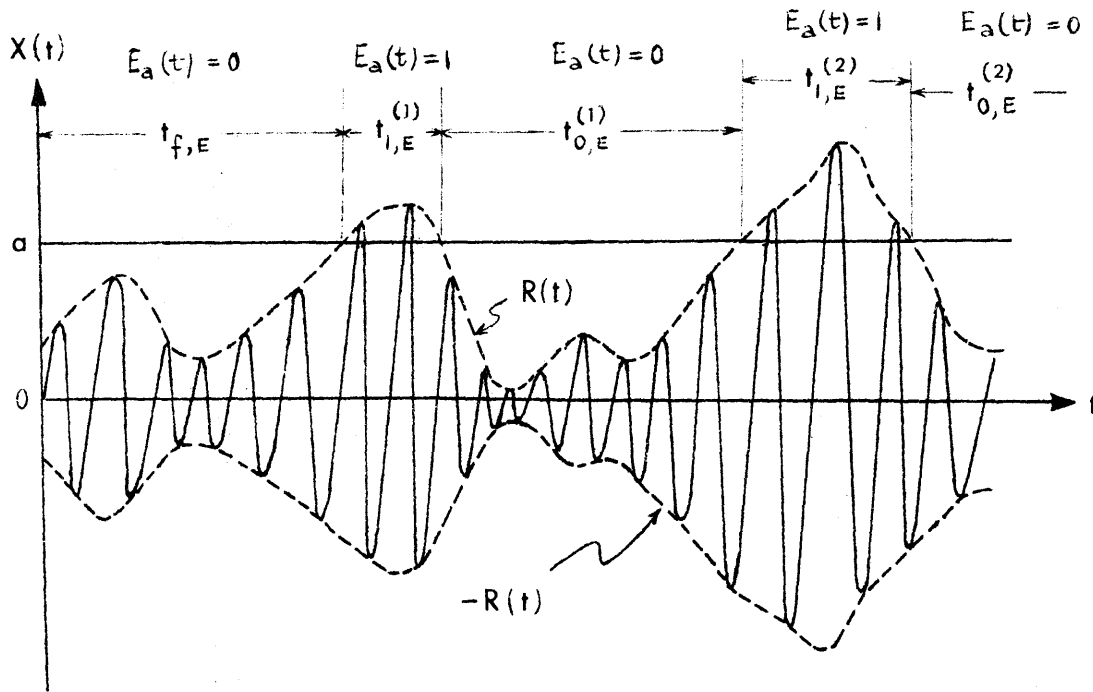


Fig.II.2. The Continuous Time Two-State Process $E_a(t)$

Figure II.2 indicates the sequence of times $t_{f,E}$, $t_{1,E}^{(1)}$, $t_{0,E}^{(1)}$, $t_{1,E}^{(2)}$, $t_{0,E}^{(2)}$, $t_{1,E}^{(3)}$..., i.e., the successive times spent in states 0 or 1. The time $t_{f,E}$ is a sample value of the random time $T_{f,E}$ to first passage of the barrier $R=a$. The times $t_{1,E}^{(1)}$, $t_{1,E}^{(2)}$, ... are sample values of the random variable $T_{1,E}$, the length of time of an excursion above the threshold. Similarly, the times $t_{0,E}^{(1)}$, $t_{0,E}^{(2)}$, ... are sample values of $T_{0,E}$, the length of a "fade" (as it is called in the literature of communications theory [9])

The steady-state behavior of the process $E_a(t)$, i.e., the characteristics of $T_{0,E}$ and $T_{1,E}$, will be examined first. Let $E[T_{0,E}]$ and $E[T_{1,E}]$ be the average holding times in state 0 and state 1, respectively. The average number of envelope crossings of the level $R=a$ with positive slope is n_a as given by Eq. I-4.3. The time lapse between two upcrossings is $T_{0,E}+T_{1,E}$. It is a fundamental result of the theory of recurrent events [37] that

$$E[T_{0,E}+T_{1,E}] = \frac{1}{n_a} \quad (\text{II-2.1})$$

For similar reasons the expected half-period of the narrow-band random process $X(t)$ is related to the average rate $2\nu_0$ of crossings of $X=0$,

$$E[\text{Half-Period}] = \frac{1}{2\nu_0} \quad (\text{II-2.2})$$

Next observe that the process $|X(t)|$ cannot cross the threshold $|X|=a$ without the envelope being above the threshold. All $|X|$ -upcrossings occur in groups or clumps each of which is immediately preceded by an envelope upcrossing and followed by an envelope downcrossing. Let Q denote the number of $|X|$ -upcrossings for every R -upcrossing. If ℓ is the number of $|X|$ -upcrossings during a fixed (very long) period of time s and m the number of R -upcrossings during the same period then it is not without meaning to take the ratio ℓ/m as the average of Q . In fact we have $\lim_{s \rightarrow \infty} \ell/m = E[Q]$. Note, however, that ℓ/m is equivalent to $(\ell/s)/(m/s)$. For the period s (say, in seconds) tending to infinity the ratios ℓ/s and m/s tend to the average rates $2v_a$ and n_a , respectively. Hence we are led to what amounts to a definition [37] of the expected value of Q

$$E[Q] = \frac{2v_a}{n_a} \quad (\text{II-2.3})$$

In narrow-band processes each $|X|$ -upcrossing of the threshold $|X|=a$ corresponds to one crossing of $X=0$. Also the expected time between consecutive $|X|$ -upcrossings of $|X|=a$ will be the expected half-period of the process. Finally, adopting the reasonable assumption that the half-period (time between zero crossings) and the random variable Q are uncorrelated one obtains

$$\begin{aligned}
E[T_{1,E}] &= E[Q]E[\text{Half-Period}] \\
&= \frac{2v_a}{n_a} \frac{1}{2v_0} = \frac{v_a}{n_a} \frac{1}{v_0}
\end{aligned}
\tag{II-2.4}$$

Eq. II-2.4 expresses the expected duration $E[T_{1,E}]$ of an excursion of the envelope $R(t)$ of a stationary narrow-band random process above a fixed threshold, $R=a$, in terms of the basic statistics v_a , v_0 and n_a (defined by Eqs. I-4.1, I-4.2 and I-4.3, respectively.)

From Eqs. II-2.1 and II-2.4 the mean holding time in state "zero" or the mean "length of a fade" is found to be

$$\begin{aligned}
E[T_{0,E}] &= E[T_{0,E} + T_{1,E}] - E[T_{1,E}] \\
&= \frac{v_a}{n_a} \left(\frac{1}{v_a} - \frac{1}{v_0} \right) = \frac{1}{n_a} \left(1 - \frac{v_a}{v_0} \right)
\end{aligned}
\tag{II-2.5}$$

The above expressions hold for arbitrary narrow-band random processes. For Gaussian processes these results become

$$E[T_{1,E}] = \frac{1}{v_0 k r} \tag{II-2.6}$$

$$E[T_{0,E}] = \frac{1}{v_0 k r} \{ \exp(r^2/2) - 1 \} \tag{II-2.7}$$

where $r=a/\sqrt{\lambda_0}$ is the normalized barrier level, v_0 is the average rate of zero crossings at positive level and k is the spectral density shape factor, Eq. I-5.3. The latter expression, Eq. II-2.7 appears as an isolated result in electrical

engineering literature [9]. Its derivation there is based upon some non-trivial theorems due to Volkonski [38].

II.3. Characteristics of Envelope Peaks above a Fixed Threshold

Consider a typical complete excursion of the envelope $R(t)$ of a stationary normal narrow-band process $X(t)$ above a relatively high fixed threshold $R=a$, as shown in figure II.3. Suppose that the prescribed level is sufficiently high that the probability of envelope troughs above the threshold is negligibly small.

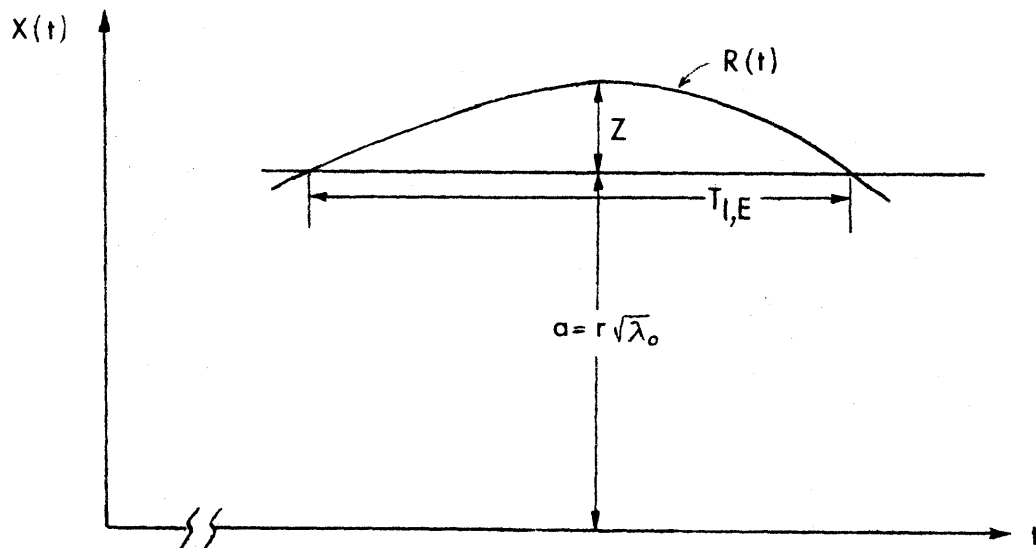


Fig.II.3. An Envelope Peak Above a Fixed Threshold

Two important characteristics of such peaks are the random variable T_1 , the duration of the excursion and Z , the height of the envelope peak above the threshold (see figure II.3) According to Eq. II-2.6, the mean of $T_{1,E}$ equals $(v_0kr)^{-1}$. Here, it will be shown that the asymptotic (for $r \rightarrow \infty$) distribution of Z is exponential with mean $\sqrt{\lambda_0}/r$. Also, that this asymptotic distribution provides a reasonable approximation to the true distribution of Z for threshold values as low as $2.5\sqrt{\lambda_0}$. A proof follows later in this section. First the following implications are noted.

i) For relatively high threshold levels the ratio of the mean values $E[T_{1,E}]$ and $E[Z]$ does not depend on the threshold value (see figure II.4) We have

$$\begin{aligned} \frac{E[Z]}{E[T_{1,E}]} &= \frac{(\sqrt{\lambda_0}/r)}{(v_0kr)^{-1}} \\ &= v_0k\sqrt{\lambda_0} \end{aligned} \quad (\text{II-3.1})$$

ii) Consider the peaks above a fixed relatively high threshold level of the envelopes of Gaussian processes with different power spectral density shapes (but with a common r.m.s. value $\sqrt{\lambda_0}$, i.e., the processes transmit the same average power). The asymptotic distribution of Z is identical in all cases. The means $E[T_{1,E}]$, however, are inverse-

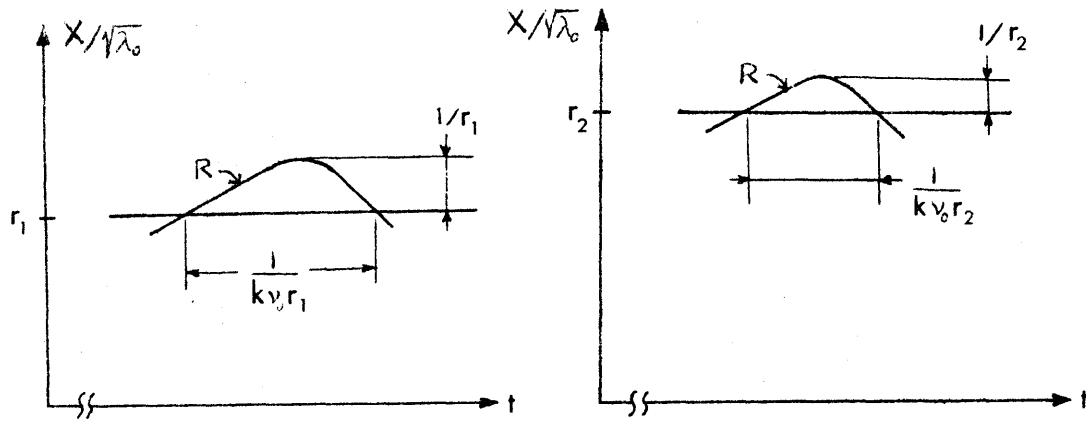


Fig.II.4. "Average" Envelope Peaks for Fixed k -Factor and Different (High) Threshold Levels

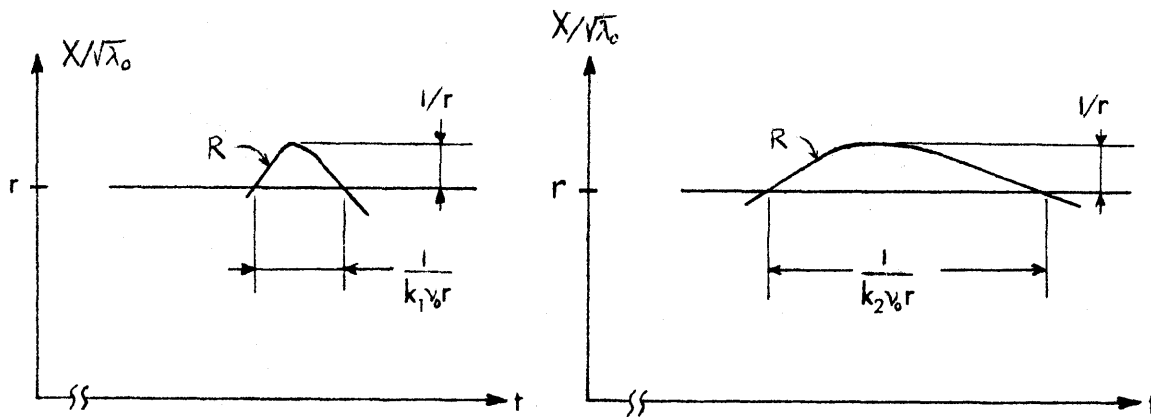


Fig.II.5. "Average" Envelope Peaks for Fixed Threshold and Different k -Factors. ($k_2 < k_1$)

ly proportional to $k\nu_0$, a factor which depends on the first few moments of the spectral density; the average frequency ν_0 equals $(\lambda_2/\lambda_0)^{1/2}$ (see Eqs. I-4.2 and I-4.5) and k is the spectral density shape factor given by Eq. I-2.3 (it is a simple function of $\lambda_0\lambda_2/\lambda_1^2$). Figure II.5 comes close to demonstrating the basic physical significance of the shape factor k . The smaller k , the longer the average duration of an envelope excursion, measured in terms of the average period. Note that, while the horizontal scale in a plot of $X(t)/\sqrt{\lambda_0}$ vs t is affected by a change in $\nu_0 k$, the vertical scale is not.

iii) It is believed that the above remarks suggest the existence of a fairly high degree of correlation between the random variables $T_{1,E}$ and Z associated with envelope peaks above relatively high thresholds. If the variables were indeed completely correlated, i.e., $T_{1,E} = (\nu_0 k \sqrt{\lambda_0})^{-1} Z$ it would follow that $T_{1,E}$ also has an asymptotic (for $r \rightarrow \infty$) exponential distribution, with mean $(\nu_0 k r)^{-1}$. Of course, no proof of that assertion is given here. However the results here do seem to provide some ground for the adoption, in section II-4, of the assumption of exponential holding times $T_{1,E}$.

Proof (that Z is asymptotically exponentially distributed)

The average frequency n_a of intersections at positive slope of the envelope $R(t)$ of a stationary Gaussian random process and the threshold $R=a$ are given by Eq. I-4.6

$$n_a = c \frac{a}{\sqrt{\lambda_0}} e^{-a^2/2\lambda_0} \quad (\text{II-3.2})$$

where

$$c = \left[\frac{\lambda_0 \lambda_2 - \lambda_1^2}{2\pi \lambda_0^2} \right]^{1/2} \quad (\text{II-3.3})$$

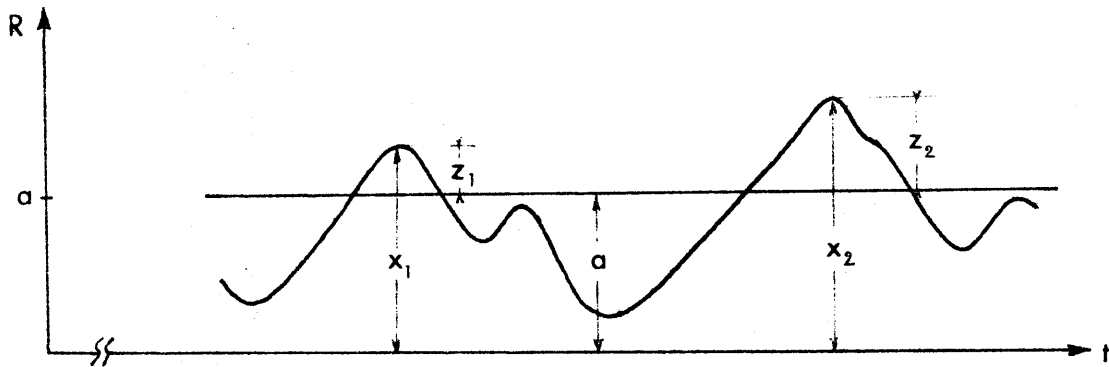


Fig.II.6. Envelope Crossings of a Relatively High Threshold

Let the threshold considered be sufficiently high so that almost every upcrossing is followed by only a single peak of $R(t)$, i.e. the probability of troughs of $R(t)$ above $R=a$ is negligible (see figure II.6). We will follow an approach similar to that first used by Powell [39] to obtain the distribution of the peaks of a narrow-band process. Let Y denote the

random height of those peaks which exceed a specified high threshold $R=a$. The ratio n_y/n_a , where $y \geq a$, may be interpreted as the fraction of envelope peaks above $R=a$ which also exceed the level $R=y$. Hence in accordance with the relative frequency interpretation of probabilities we may write (*)

$$P[Y > y] = \frac{n_y}{n_a} \quad y \geq a \quad (\text{II-3.4})$$

Using Eq. II-3.2, the cumulative probability distribution of Y takes the form (*)

$$\begin{aligned} F_Y(y) = P[Y \leq y] &= 1 - \frac{n_y}{n_a} && y \geq a \\ &= 1 - \frac{y e^{-y^2/2\lambda_0}}{a e^{-a^2/2\lambda_0}} && y \geq a \end{aligned} \quad (\text{II-3.5})$$

It is easy to verify that $F_Y(a)=0$ and $F_Y(\infty)=1$. The random variable of interest is $Z=Y-a$, the height of the envelope peak above the threshold. We have

$$\begin{aligned} F_Z(z) = P[Z \leq z] &= P[Y \leq z+a] \\ &= 1 - \frac{(z+a)}{a} \exp\left\{-\left[\frac{2a+z}{2\lambda_0}\right]z\right\}, \quad z \geq 0 \end{aligned} \quad (\text{II-3.6})$$

The density function of Z is obtained by differentiating $F_Z(z)$

$$f_Z(z) = \frac{dF_Z(z)}{dz} = \frac{1}{a} \left[\frac{(z+a)^2}{\lambda_0} - 1 \right] \exp\left\{-\left[\frac{2a+z}{2\lambda_0}\right]z\right\} \quad z \geq 0 \quad (\text{II-3.7})$$

(*) This result is also given by Crandall [32, 41].

Consider the following approximations to Eqs. II-3.6 and II-3.7

$$F_Z^*(z) = 1 - \exp\left\{-\left(\frac{a}{\lambda_0}\right)z\right\} \quad (\text{II-3.8})$$

$$f_Z^*(z) = \left(\frac{a}{\lambda_0}\right) \exp\left\{-\left(\frac{a}{\lambda_0}\right)z\right\} \quad (\text{II-3.9})$$

$F_Z^*(z)$ and $f_Z^*(z)$ represent respectively the cumulative distribution and density function of an exponential random variable with mean $\lambda_0/a = \sqrt{\lambda_0}/r$.

Consider the ratio $f/f^* = f_Z(z)/f_Z^*(z)$ of the true and the approximate density functions of Z

$$\frac{f}{f^*} = \frac{\lambda_0}{a^2} \left[\frac{(z+a)^2}{\lambda_0} - 1 \right] e^{-z^2/2\lambda_0} \quad (\text{II-3.10})$$

Expressing z in terms of its approximate mean value λ_0/a as follows

$$z = u(\lambda_0/a) \quad (\text{II-3.11})$$

one obtains

$$\begin{aligned} \frac{f}{f^*} &= \frac{\lambda_0}{a} \left[\left(\frac{u\sqrt{\lambda_0}}{a} + \frac{a}{\sqrt{\lambda_0}} \right)^2 - 1 \right] e^{-u^2\lambda_0/2a^2} \\ &= \frac{1}{r^2} \left[\left(\frac{u}{r} + r \right)^2 - 1 \right] e^{-u^2/2r^2} \end{aligned} \quad (\text{II-3.12})$$

It is easy to see that for any fixed value of u

$$\lim_{r \rightarrow \infty} \frac{f}{f^*} = 1 \quad (\text{II-3.13})$$

The ratio f/f^* is plotted in figure II.7 as a function of r for several values of u . Figure II.8 shows that for low threshold levels ($r=2, 3$) the approximate exponential density function $f_z^*(z)$ overestimates the likelihood of large values of z (say, several times the mean value).

Remark

It is of interest to note that the functional form of $f_z(z)$ as given by Eq. II-3.7, if not restricted by the condition $z > 0$, exhibits the following behavior. It has a peak at $z=z^*$ where z^* is determined by the condition

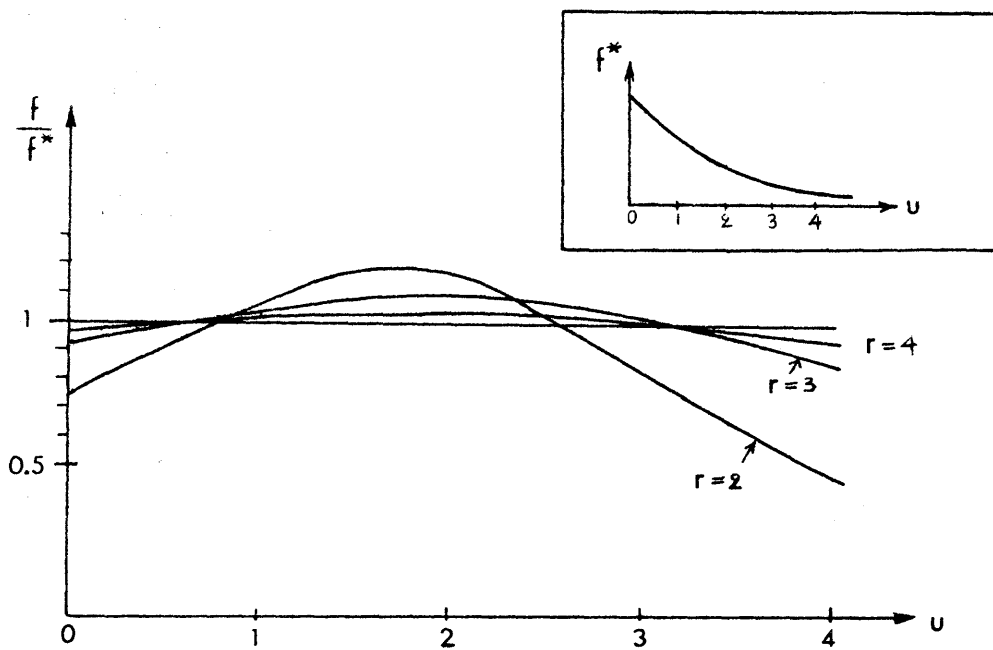
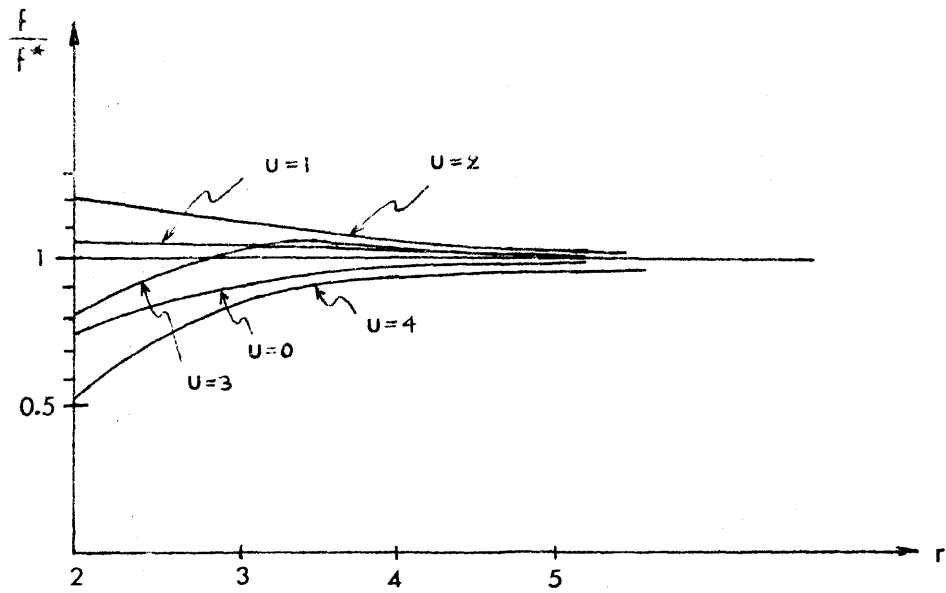
$$\frac{df_z(z)}{dz} = 0 \quad \rightarrow \quad z+a=\sqrt{3\lambda_0} \quad (\text{II-3.14})$$

Its maximum value, corresponding to $z=\sqrt{3\lambda_0}-a$, is

$$\text{Max}_{\text{all } z} [f_z(z)] = \frac{2}{a} e^{(r^2-3)/2} \quad (\text{II-3.15})$$

Its value at $z=0$ is

$$f_z(0) = \frac{1}{a} [r^2-1] \quad (\text{II-3.16})$$



Figs.II.7 and 8. The ratio of the density function $f_z(z)$ and $f_z^*(z)$, where $z=u(\lambda_0/a)$

For a threshold level $a = \sqrt{3\lambda_0}$ or $r = \sqrt{3}$ the above equations yield identical results. The above information is displayed in figure II.9. The theorem proved in this section essentially states that the probability density of Z which corresponds to the tail portion (labeled $f_Z(z)$, $z \geq 0$) of the dotted curve has the exponential form, Eq. II-3.9, for large $r = a/\sqrt{\lambda_0}$.

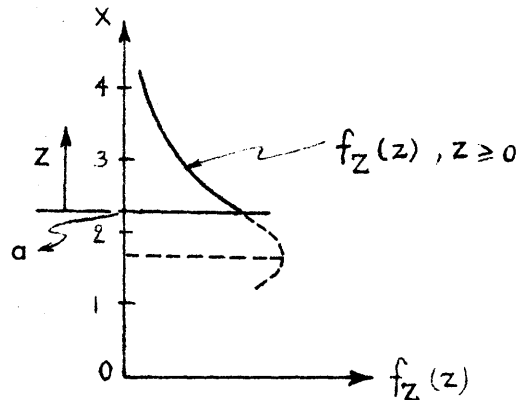


Fig.II.9. The Density of Envelope Peaks

II.4. Properties of Crossings of a D Type Barrier

Most crossing problems of practical significance are directly related to the dynamic variable $X(t)$ of interest rather than to its envelope $R(t)$. Fortunately, the threshold crossing characteristics of the two processes are quite similar. Note for example, that

- i) peaks of $|X(t)|$ exceed a given threshold only at times when the envelope is above that threshold.
- ii) peaks of $|X(t)|$ above the threshold tend to occur in

clumps, the size of which is closely related to $T_{1,E}$, the duration of an envelope excursion above the threshold. However some significant differences do exist. Note that not all envelope upcrossings are immediately followed by a clump of $|X|$ -peaks exceeding the threshold. For any fixed threshold value, the frequency of envelope upcrossings always exceeds the frequency of occurrence of clumps. One of the main problems here is to find a way to identify and eliminate ("search and destroy") those envelope upcrossings which are not immediately followed by at least one $|X|$ -upcrossing. Another more obvious difference is the essentially discrete nature of the occurrence of threshold upcrossings (or peaks) of the random process $X(t)$. It is also worth noting here that the time scale of interest is presumed to be at least an order of magnitude larger than the average period of $X(t)$. Hence, no consideration is given to the actual durations of single excursions (each corresponding to a single peak) of $|X(t)|$ above the threshold.

Recall the nature of the zero-one process $D_a(n)$ defined in the introduction to this chapter (see figure II.1a). The subscript a refers to the absolute value of the (two-sided) threshold and the counter n records the total number of peaks of $|X(t)|$ observed. Such peaks occur at time intervals of average length $(2\nu_0)^{-1}$. When, say, the i th peak exceeds the threshold $X=a$, then $D_a(i)=1$, otherwise $D_a(i)=0$. We

are mainly interested in the length or the size of sequences of consecutive peaks for which the corresponding value of $D_a(i)$ is a constant, zero or one. Let $N_{0,D}$ and $N_{1,D}$ be (discrete) random variables describing the size of such clumps of "zeros" and "ones". The mean values $E[N_{0,D}]$ and $E[N_{1,D}]$ may be obtained in terms of the characteristics of the holding times $T_{0,E}$ and $T_{1,E}$ of the envelope-related two-state process $E_a(t)$. To see this, recall that n_a represents the frequency of envelope excursion above the level $X=a$ and that $T_{1,E}$ is the random duration of each such excursion. Whenever the duration $T_{1,E}$ is larger than $(2\nu_0)^{-1}$, an average half-period of $X(t)$, it is almost certain that $|X(t)|$ will cross the threshold, at positive slope, at least once during the envelope excursion. However, if $T_{1,E}$ is smaller than $(2\nu_0)^{-1}$, one has to account for the possibility that no peaks of $X(t)$ occur during the envelope excursion. The fraction ρ_D of envelope excursions above a threshold that thus has to be eliminated may be computed as follows

$$\rho_D = P[N_{1,D}=0] = \int_0^{(2\nu_0)^{-1}} P[N_{1,D}=0 | T_{1,E}=t_1] f_{T_{1,E}}(t_1) dt_1 \quad (\text{II-4.1})$$

where $f_{T_{1,E}}(t_1)$ is the probability density function of $T_{1,E}$. To arrive at a value for the conditional probability in the integrand of Eq. II-4.1, visualize a segment of length t_1 to have fixed position on a time scale. Then let a pattern of

points which are regularly spaced at intervals of length $(2\nu_0)^{-1}$ be randomly placed on the time scale (see figure II.10) What is the probability that none of the points will cover the segment?

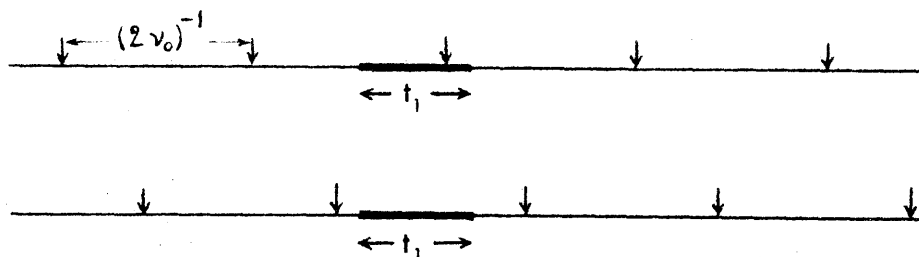


Fig.II.10

It seems reasonable to take, for $t_1 < (2\nu_0)^{-1}$,

$$P[N_{1,D}=0 | T_{1,E}=t_1] = \frac{(2\nu_0)^{-1} - t_1}{(2\nu_0)^{-1}} = 1 - 2\nu_0 t_1 \quad (\text{II-4.2})$$

Combining Eqs. II-4.1 and II-4.2, ρ_D becomes

$$\rho_D = \int_0^{(2\nu_0)^{-1}} (1 - 2\nu_0 t_1) f_{T_{1,E}}(t_1) dt_1 \quad (\text{II-4.3})$$

In section II.2 only the mean value $E[T_{1,E}]$ is computed. Here the density function is needed. The results of section II.3, although by no means a proof, do provide an argument in favor of the adoption of the assumption that $T_{1,E}$ is exponentially distributed. A major advantage is also that only a single parameter is needed to characterize the density function. Thus

we take

$$f_{T_{1,E}}(t_1) = \beta_E e^{-\beta_E t_1} \quad t_1 \geq 0 \quad (\text{II-4.4})$$

where β_E , the decay rate of the exponential function, is related to $E[T_{1,E}]$ in the following way,

$$E[T_{1,E}] = \int_0^{\infty} t_1 \beta_E e^{-\beta_E t_1} dt_1 = \frac{1}{\beta_E} \quad (\text{II-4.5})$$

Hence, from Eqs. II-2.4 and II-4.5,

$$\beta_E = \frac{n_a v_0}{v_a} \quad (\text{II-4.6})$$

where v_a , v_0 and n_a are defined in section I.4.

Finally inserting Eqs. II-4.4 and II-4.6 into Eq. II-4.3 and integrating one obtains

$$\rho_D = 1 - \frac{2v_a}{n_a} (1 - \exp\{-n_a/2v_a\}) \quad (\text{II-4.7})$$

$1 - \rho_D$ is the fraction of envelope excursions above the threshold $X=a$ during which at least one peak of $|X(t)|$ occurs above the same threshold. Let it be called the fraction of qualified envelope excursions or the fraction of qualified envelope upcrossings. In figure II.11, ρ_D is plotted as a function of n_a/v_a (or kr for Gaussian processes). Recall that

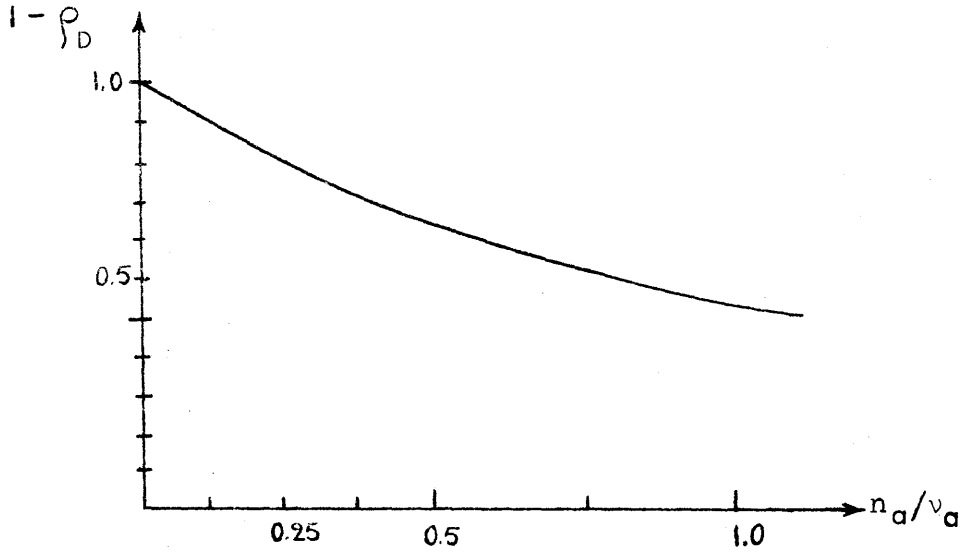


Fig.II.11. The Fraction of Qualified Envelope Excursions Above a Threshold.

average time between envelope upcrossing of a threshold is $E[T_{1,E} + T_{0,E}]$. The average time between qualified envelope upcrossings becomes

$$\frac{1}{1-\rho_D} E[T_{1,E} + T_{0,E}] \quad (\text{II-4.8})$$

It also equals $\frac{1}{2v_0} E[N_{1,D} + N_{0,D}]$. Thus we have

$$E[N_{1,D} + N_{0,D}] = \frac{2v_0}{1-\rho_D} E[T_{1,E} + T_{0,E}] \quad (\text{II-4.9})$$

An analogous relationship may be assumed to exist between $E[N_{1,D}]$ and $E[T_{1,E}]$

$$E[N_{1,D}] = \frac{2v_0}{1-\rho_D} E[T_{1,E}] \quad (\text{II-4.10})$$

Clearly, if almost all envelope upcrossings qualify ($E[T_{1,E}]$ large compared to $(2\nu_0)^{-1}$) we should have $E[N_{1,D}] \approx 2\nu_0 E[T_{1,E}]$. On the other hand if $E[T_{1,E}]$ is very small compared to $(2\nu_0)^{-1}$, very few envelope upcrossings will qualify and $E[N_{1,D}]$ should be very close to one. To verify that Eq. II-4.10 properly reflects these properties it may be rewritten, by inserting Eqs. II-2.4 and II-4.7, into the following form:

$$E[N_{1,D}] = \frac{1}{1 - \exp\{-1/2\nu_0 E[T_{1,E}]\}} \quad (\text{II-4.11})$$

In figure II.12 the mean $E[N_{1,D}]$ is shown as a function of $2\nu_0 E[T_{1,E}]$. This statistic has been called [41] the "mean clump size". It is discussed in some more detail at the end of this section. We also have, using Eq. II-2.4,

$$E[N_{1,D}] = \frac{1}{1 - \exp\{-n_a/2\nu_a\}} \quad (\text{II-4.12})$$

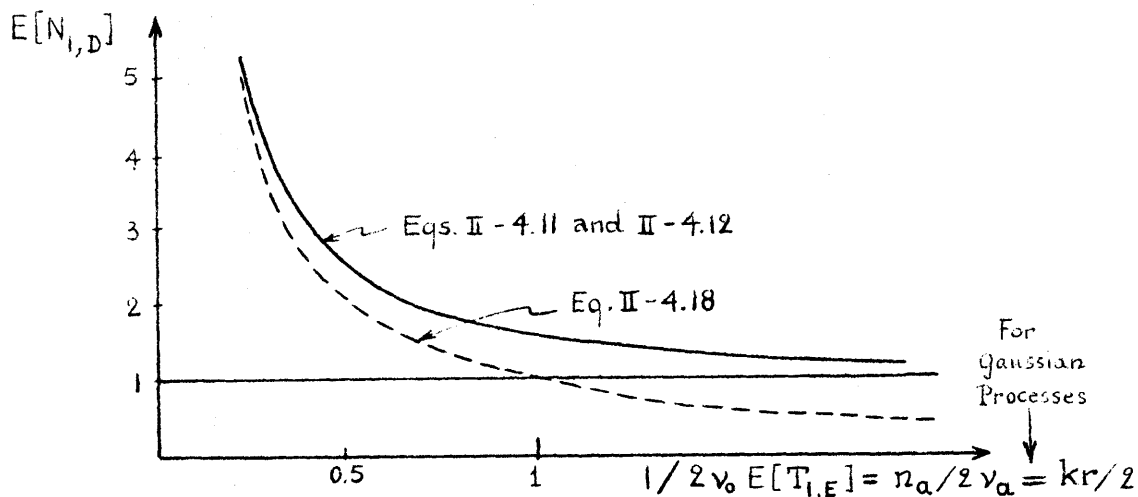


Fig.II.12. The Mean Clump Size

Subtracting Eq. II-4.10 from Eq. II-4.9 one obtains

$$E[N_{0,D}] = \frac{2\nu_0}{1-\rho_D} E[T_{0,E}] \quad (\text{II-4.13})$$

$$= \frac{\nu_0}{\nu_a} \frac{1-\nu_a/\nu_0}{1-\exp\{-n_a/2\nu_a\}} \quad (\text{II-4.14})$$

All the results in this section apply for arbitrary narrow-band random processes for which the needed statistics ν_a and n_a exist. For Gaussian processes all results may be specialized in terms of r , the normalized threshold value, $r=a/\sqrt{\lambda_0}$, and the spectral density shape factor k (Eq. I-2.3). We have

$$\rho_D = 1 - \frac{2}{kr} (1 - \exp\{-kr/2\}) \quad (\text{II-4.15})$$

$$E[N_{1,D}] = \frac{1}{1 - \exp\{-kr/2\}} \quad (\text{II-4.16})$$

$$E[N_{0,D}] = \exp\{r^2/2\} \frac{1 - \exp\{-r^2/2\}}{1 - \exp\{-kr/2\}} \quad (\text{II-4.17})$$

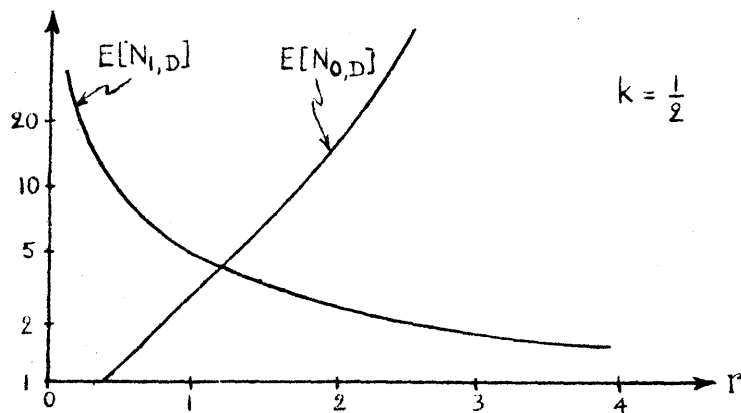


Fig.II.13. Average Number of Consecutive Peaks Above and Below a Fixed Double Threshold.

In figure II.13 the means $E[N_{1,D}]$ and $E[N_{0,D}]$ are shown as functions of $r=a/\sqrt{\lambda}$ for a fixed value of the shape factor k .

Clump Size

The random number $N_{1,D}$ has been called the "clump size" by Racicot [40]. He proposed a numerical integration procedure for estimating its mean value $E[N_{1,D}]$ for the case in which $X(t)$ is the response of a linear oscillator to Gaussian wide-band excitation. The concept and terminology, however, have been introduced by Lyon [35], who essentially argued that the quotient $2v_a/n_a$ may be interpreted as the mean clump size. His argument fails to account for the fact that a fraction of envelope crossings are not followed by a clump of $|X|$ -crossing. In fact, the quotient $2v_a/n_a$ may easily become smaller than one, in which case it loses much of its meaning as an estimate of $E[N_{1,D}]$ (since we must have $E[N_{1,D}] > 1$). Note that Lyon's [35] estimate of $E[N_{1,D}]$ is closely related to the expected value of the holding time $T_{1,E}$ in the envelope-based zero-one process $E_a(t)$.

$$E[N_{1,D}] \approx 2v_0 E[T_{1,E}] = 2v_a/n_a \quad (\text{II-4.18})$$

To compare the above approximation to the estimate proposed here, Eq. II-4.11 or II-4.12, see figure II.12 where Eq. II-4.18 is represented by a dotted line. As may be expected, the two estimates converge when r approaches zero, i.e., when the clump size becomes much larger than one.

II.5 Properties of Crossings of a B Type Barrier

The two-state process $B_a(n)$ is very similar to the process $D_a(n)$. Recall from section II.1 (see figure II-1b) that the index n of $B_a(n)$ counts the number of peaks of $X(t)$ (rather than those of $|X(t)|$). Peaks of $X(t)$ occur every $(v_0)^{-1}$ seconds on the average. If the i th peak of $X(t)$ exceeds the threshold $X=a$, then $B_a(i)=1$. Otherwise, $B_a(i)=0$. Also, $N_{1,B}$ denotes the size of a clump of peaks of $X(t)$ above the threshold and $N_{0,B}$ the number of consecutive peaks of $X(t)$ below the threshold. To derive expressions for the means $E[N_{0,B}]$ and $E[N_{1,B}]$ from the characteristics of the envelope-related zero-one process $E_a(t)$ we need the fraction $(1-\rho_B)$ of envelope crossings that are followed by at least one crossing of the one-sided barrier $X=a$. Following the arguments of section II.4 one obtains

$$\begin{aligned} \rho_B &= \int_0^{(v_0)^{-1}} (1-v_0 t_1) \beta_Z e^{-\beta_Z t_1} dt_1 \\ &= 1 - \frac{v_a}{n_a} (1 - \exp\{-n_a/v_a\}) \end{aligned} \quad (\text{II-5.1})$$

and also

$$E[N_{1,B}] = \frac{1}{1 - \exp\{-n_a/v_a\}} \quad (\text{II-5.2})$$

$$E[N_{0,B}] = \frac{v_0}{v_a} \frac{1 - v_a/v_0}{1 - \exp\{-n_a/v_a\}} \quad (\text{II-5.3})$$

Specializing the above results for Gaussian processes,

$$1 - \rho_B = \frac{1 - \exp\{-rk\}}{rk} \quad (\text{II-5.4})$$

$$E[N_{1,B}] = \frac{1}{1 - \exp\{-rk\}} \quad (\text{II-5.5})$$

$$E[N_{0,B}] = \frac{\exp\{rk/2\} - 1}{1 - \exp\{-rk\}} \quad (\text{II-5.6})$$

From Eqs. II-4.16 and II-5.5 note that, for small rk (and hence, for large clump sizes) one finds, not unexpectedly, $E[N_{1,D}] \approx 2E[N_{1,B}]$.

Summary of Results

A schematic summary of the important results to be carried to the next chapters, is presented in table II.1.

Stationary Two-State Processes
Related to a Narrow-Band Random Process X(t)

State Process:	E _a (t) Continuous Time	D _a (n) Discrete Time	B _a (n) Discrete Time
Characteristics	T _{0,E} : holding time in state 0 T _{1,E} : holding time in state 1	N _{0,D} : no. of consecutive peaks of X below a N _{1,D} : no. of consecutive peaks of X with values ≥ a	N _{0,B} : no. of consecutive peaks of X below a N _{1,B} : no. of consecutive peaks of X with values ≥ a
For an Arbitrary Narrow-Band Process	$E[T_{0,E}] = \frac{1}{n_a} (1 - \frac{v_a}{v_0})$ $E[T_{1,E}] = \frac{v_a}{n_a} \frac{1}{v_0}$	$E[N_{0,D}] = \frac{v_0/v_a - 1}{1 - \exp\{-n_a/2v_a\}}$ $E[N_{1,D}] = \frac{1}{1 - \exp\{-n_a/2v_a\}}$	$E[N_{0,B}] = \frac{v_0/v_a - 1}{1 - \exp\{-n_a/v_a\}}$ $E[N_{1,B}] = \frac{1}{1 - \exp\{-n_a/v_a\}}$
For a Gaussian Narrow-Band Process	$E[T_{0,E}] = \frac{1}{v_0 k r} [\exp\{r^2/2\} - 1]$ $E[T_{1,E}] = \frac{1}{v_0 k r}$	$E[N_{0,D}] = \frac{\exp\{r^2/2\} - 1}{1 - \exp\{-rk/2\}}$ $E[N_{1,D}] = \frac{1}{1 - \exp\{-kr/2\}}$	$E[N_{0,B}] = \frac{\exp\{r^2/2\} - 1}{1 - \exp\{-rk\}}$ $E[N_{1,B}] = \frac{1}{1 - \exp\{-kr\}}$
$r = a/\sqrt{\lambda_0}$ $v_0 = \frac{1}{2\pi} \sqrt{\lambda_2/\lambda_0}$ $k = \{2\pi(1 - \lambda_1^2/\lambda_0\lambda_2)\}^{1/2}$ $\lambda_j = \int_0^\infty \omega^j G(\omega) d\omega$			

CHAPTER III

FIRST CROSSING PROBABILITIES

III.1. Scope

In the safety analysis and design of civil, mechanical and aerospace structures undergoing random vibration it is of considerable practical importance to be able to evaluate certain measures of quality of performance or reliability. One quantity of central interest is the probability that the dynamic response variable $X(t)$ will be kept within specified bounds during the system operating time, or, the probability distribution of the time to first passage across the bounds.

In this chapter a relatively simple analytical solution is proposed to the first passage problem for essentially arbitrary stationary narrow-band random processes, i.e., the solution is not restricted to Gaussian response processes nor to white noise input. The starting condition considered throughout this chapter is also random or stationary [17]. The effect of nonstationarity is investigated in the next

chapter. The nonstationarity may be due to the transients of the response motion at its start or to the nonstationarity of the input or both. Also in chapter IV, the results obtained here are specialized for the case in which the random process represents the response of a lightly damped linear oscillator to wide-band Gaussian excitation. The results are seen to compare favorably with the first passage time estimates obtained by simulation techniques and other numerical methods [17, 42]. Also, the proposed solution is shown to be in agreement with the available information [7, 43, 14, 32] regarding the behavior of the first passage probability under limiting conditions associated with high and low threshold levels.

Three types of failure conditions are considered each of which corresponds to a particular barrier configuration. In section III.3, the distribution of first passage times is sought for the failure condition $R(t) \geq a$. Using Cramer and Leadbetter's definition of the envelope $R(t)$ of a stationary random process $X(t)$ (Eq. I-3.4), the failure condition may restated as follows

$$R^2(t) = X^2(t) + \hat{X}^2(t) \geq a^2 \quad (\text{III-1.1})$$

where $\hat{X}(t)$, formally defined by Eq. I-3.5, is a (real, stationary) random process associated with $X(t)$. Recall that, for narrow-band processes, $\hat{X}^2(t) \approx \dot{X}^2(t) / \omega_0^2$ (Eq. I-3.10), where

$\dot{X}(t)$ is the time derivative of $X(t)$ and ω_0 is the average circular frequency (Eq. I-4.2). The barrier associated with this failure condition is a circle of radius $R=a$ in the $X-\hat{X}$ plane, as shown in Figure III.1a. Very similar "circular" barrier

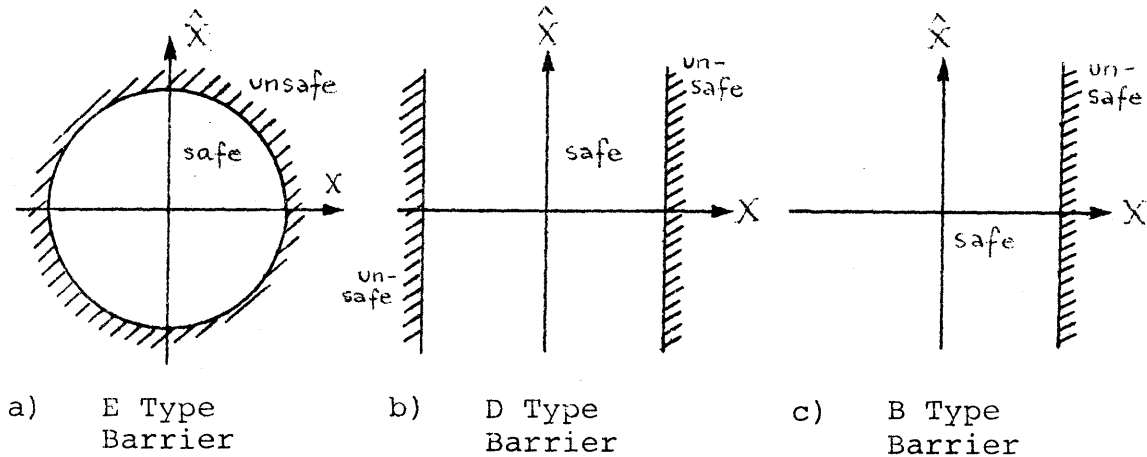


Fig.III.1. Barrier Configurations (adapted from Ref. 17)

configurations have been considered by Rosenblueth and Bustamente [15], Caughey and Gray [34] and Hellstrom [33] to obtain an approximate (and conservative) estimate of the distribution of the time to first passage of the double barrier $|X(t)|=a$, shown in Figure III.1b. The latter consists of a pair of lines $X(t)=a$ and $X(t)=-a$. In those cases where negative values of $X(t)$ do not cause failure, the single barrier $X(t)=a$, corresponding to the failure criterion $X(t) \geq a$, will be an appropriate model. Following Crandall et al. [17], and consistent with the notation in the preceding chapter, the above

barrier configurations are called, respectively, E type barrier, D type barrier and B type barrier. The E type barrier is discussed in section III.3; the two linear barrier models will be discussed in section III.4. The treatment is almost entirely similar for all three types. The basic concepts leading to the new first-crossing results are discussed in the next section.

III.2. Derivation of the Reliability Function in Case of Random Start

The probability distribution of the time to first passage of a barrier is directly related to the characteristics of a corresponding zero-one process. To keep the discussion fairly general, presume that a complete stochastic description of the particular zero-one process of interest, say $Z(t)$, is available. A sample function of the process $Z(t)$ is shown in figure III.2. It may be seen as a sequence of times $t_0^{(1)}$, $t_1^{(1)}$, $t_0^{(2)}$, $t_2^{(2)}$, etc., where $t_0^{(i)}$ is a sample value of the random time T_0 the process $Z(t)$ holds the value zero and $t_1^{(i)}$ is a sample value of T_1 , the random "holding time" in state one.

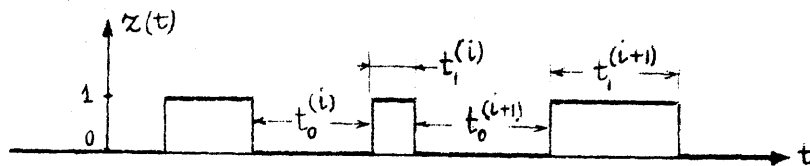


Fig. III.2. A Sample Function of the Zero-One Process $Z(t)$.

Assume the times T_0 and T_1 to be independent random variables with density functions $f_{T_0}(t_0)$ and $f_{T_1}(t_1)$, respectively. Also, consecutive pairs of holding times are assumed to be mutually independent.

Let the event $Z(t)=1$ be labeled "failure". Interest focuses on the time T_f to first occurrence of $Z(t)=1$. Consider a few sample functions of the two-state process as shown in figure III.3. Corresponding to each sample function is a sample value of T_f .

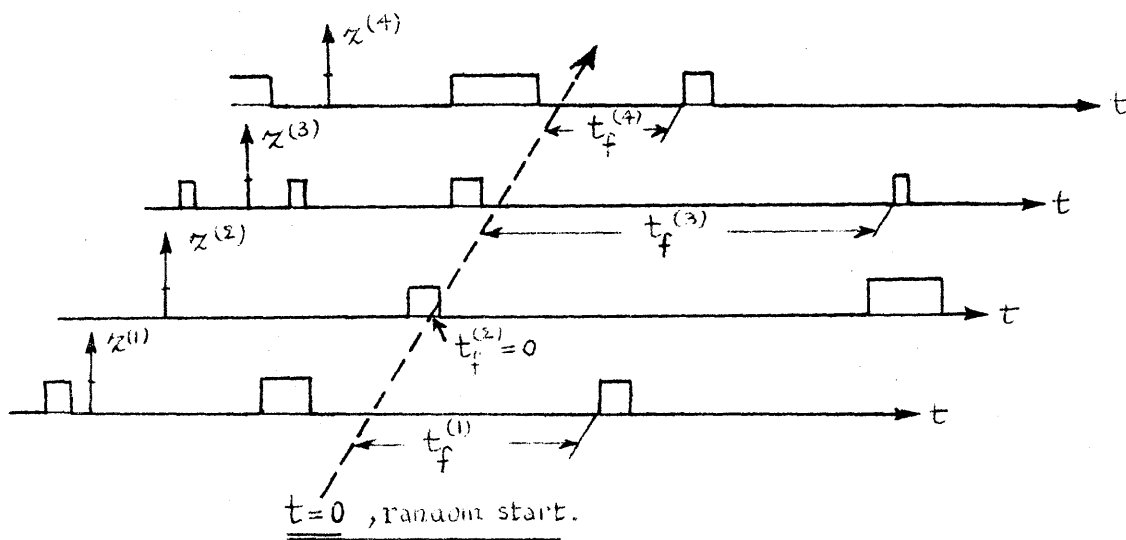


Fig.III.3. Some Sample Functions of the Ensemble of Possible Histories of the Zero-One Process $Z(t)$.

When the start is random or stationary there is a finite probability that failure will be instantaneous. That probability is equal to the fraction of the ensembles for which $Z(0)$, the

value of the process at $t=0$, is one. It may be shown that^(*)

$$P\{T_f=0\} = P\{Z(0)=1\} = \frac{E\{T_1\}}{E\{T_0\}+E\{T_1\}} \quad (\text{III-2.1})$$

The above result, which seems plausible, does in fact follow from a basic theorem of renewal theory [44]^(*). The fraction of the ensemble of two-state processes for which the time to failure is positive, i.e., non-zero equals

$$\begin{aligned} P\{T_f > 0\} &= P\{Z(0)=0\} = 1 - P\{T_f=0\} \\ &= \frac{E\{T_0\}}{E\{T_0\}+E\{T_1\}} \end{aligned} \quad (\text{III-2.3})$$

Let the reliability function^(**), $L_{T_f}(t)$, be defined as the fraction of samples of the ensemble of the two-state processes that still survive at time t , i.e., for which $Z(t_0)=0$, $0 \leq t_0 \leq t$,

$$L_{T_f}(t) = P\{T_f > t\} = 1 - F_{T_f}(t) \quad (\text{III-2.3})$$

where $F_{T_f}(t)$ is the cumulative distribution function of the time to failure. The density function $f_{T_f}(t)$ of T_f is related to the reliability function $L_{T_f}(t)$ in the following way^(**)

(*) The theorem requires that i) T_0 and T_1 be independent random variables ii) (T_0+T_1) have a continuous distribution iii) $E\{T_0\}$ and $E\{T_1\}$ be finite.

(**) This representation is quite common in reliability literature [45, 46]

$$f_{T_f}(t) = \frac{d}{dt} F_{T_f}(t) = -\frac{d}{dt} L_{T_f}(t) \quad (\text{III-2.4})$$

$f_{T_f}(t)dt$ is the probability that T_f takes a value in the interval $(t, t+dt)$. The density function contains a spike at the origin, with enclosed area equal $P[T_f=0]$. See figure III.4

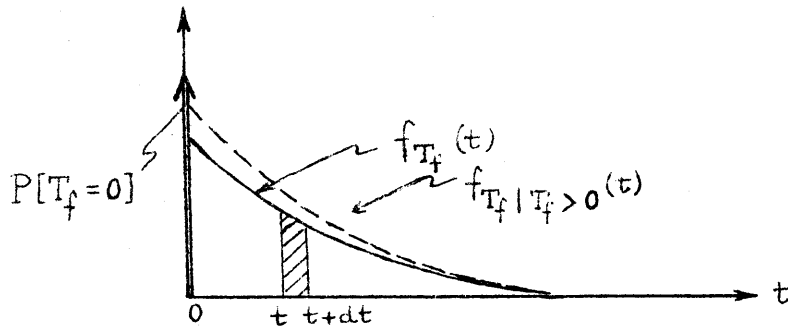


Fig.III.4

It is useful to introduce a conditional density function $f_{T_f|T_f>0}(t)$ where $f_{T_f|T_f>0}(t)dt$ is the probability that T_f takes a value in the interval $(t, t+dt)$, given no instantaneous failure occurred. In other words, of the original ensemble, only those sample functions qualify for which $Z(0)=0$. The condition eliminates, for example, sample function $z^{(2)}$ in figure III-3. It is easy to verify that

$$f_{T_f}(t) = P[T_f>0] f_{T_f|T_f>0}(t) \quad t>0 \quad (\text{III-2.5})$$

There is a close relationship between the density function $f_{T_f|T_f>0}(t)$ and $f_{T_0}(t)$. Following an argument very similar to

that given by Rice and Beer [47] for the case in which $P[T_f > 0]$ is assumed to be equal to one, (i.e., for very high threshold levels), it may be shown that (*)

$$f_{T_f | T_f > 0}(t) = \frac{1}{E[T_0]} \int_t^{\infty} f_{T_0}(t_0) dt_0 \quad (\text{III-2.6})$$

In particular, if the holding time T_0 is exponentially distributed then the time to failure T_f , $T_f > 0$, also has an exponential distribution. Quite apart from the analytical simplicity that the assumption of exponential T_0 carries, its adoption also seems indicated by the following facts:

i) The results of simulation studies [17] suggest $f_{T_f}(t)$ to have the form

$$f_{T_f}(t) = A\alpha e^{-\alpha t} \quad (\text{III-2.7})$$

ii) For high threshold levels, when crossings become rare events, the assumption of Poisson occurrences (which has been proved to be asymptotically correct [9, 43]) also leads to exponential first-passage times.

(*) Their argument [47] further leads to a simple relationship which has been used [17, 46] to determine the expected first excursion time from statistically measured recurrence times. Its extension to the case of relatively low thresholds (for which the number of T_0 -observations during a given observation time will be much larger) has the form

$$E[T_f | T_f > 0] = E[T_0^2] / 2E[T_0]$$

iii) If $E[T_0]$ is relatively large compared to the "correlation time" of the response process (as may be expected in most cases of practical interest) then the exponential assumption may be expected to be quite satisfactory. (A unique property of the exponential distribution is its being "memory-less" [46]). The actual distribution probably exhibits some peculiar behavior in a range close to zero but this should not substantially influence the results.

Now it will be shown how the first-passage solution proposed here may be derived from the basic results of the preceding chapter. An exponential distribution is adopted for the holding time T_0 . i.e.,

$$f_{T_0}(t) = \alpha e^{-\alpha t} \quad t \geq 0 \quad (\text{III-2.8})$$

The decay rate α is related to the mean value of T_0 ,

$$\alpha = \frac{1}{E[T_0]} \quad (\text{III-2.9})$$

where $E[T_0]$ has been derived in the preceding chapter as a function of the barrier configuration, of the magnitude of the threshold and of certain properties of the response process. Since $f_{T_f | T_f > 0}(t) = f_{T_0}(t)$, we have from Eqs. III-2.5, III-2.4 and III-2.3,

$$f_{T_f}(t) = A\alpha e^{-\alpha t} \quad t > 0 \quad (\text{III-2.7})$$

$$L_{T_f}(t) = P[T_f > t] = Ae^{-\alpha t} \quad t \geq 0 \quad (\text{III-2.10})$$

where A, the parameter in the simulation [17] solutions (Eq. III-2.6), is equal to the probability, $P[T_f > 0]$, of no instantaneous failure. From Eq. III-2.2,

$$A = P[T_f > 0] = \frac{E[T_0]}{E[T_0] + E[T_1]} \quad (\text{III-2.11})$$

Like α in Eq. III-2.8, it is obtainable from the means of T_0 and T_1 given in the preceding chapter. In the next two sections, the above results are specialized for the three barrier configurations (E, D and B type barriers) discussed in section III.1

III.3. First Passage Time for E Type Barriers

The two-state process corresponding to the "circular" or E type barrier configuration is the process $E_a(t)$ discussed in section II.2. Let $T_{f,E}$ denote the first passage time. The decay rate in the reliability function $L_{T_{f,E}}(t)$ will be denoted by α_E . It is related to $E[T_{0,E}]$, the expected length of a fade. From Eq. II-2.5,

$$\alpha_E = \frac{1}{E[T_{0,E}]} = n_a (1 - v_a/v_0)^{-1} \quad (\text{III-3.1})$$

Also let the probability $P[T_{f,E} > 0]$ be denoted by A_E . From Eqs. III-2.11, II-2.5 and II-2.4

$$A_E = P[T_{f,E} > 0] = \frac{E[T_{0,E}]}{E[T_{0,E}] + E[T_{1,E}]} = 1 - v_a/v_0 \quad (\text{III-3.2})$$

Finally, inserting the above equations into the "standard" form, Eq. III-2.10, we find for the reliability function associated with an E type barrier, $R(t)=a$, under stationary starting conditions

$$\begin{aligned} L_{T_{f,E}}(t) &= P[T_{f,E} > t] = A_E \exp\{-\alpha_E t\} \\ &= \left(1 - \frac{v_a}{v_0}\right) \exp\left\{-n_a \left(1 - \frac{v_a}{v_0}\right)^{-1} t\right\} \quad t \geq 0 \quad (\text{III-3.3}) \end{aligned}$$

The above expression applies for arbitrary stationary narrow-band processes for which the basic statistics v_a and n_a exist. An extension to nonstationary processes will be discussed in the next chapter. The narrow-band restriction entered into the analysis in section II.2, where it is assumed that the average of the time interval between peaks of $|X(t)|$ is equal to $(2v_0)^{-1}$ or one - half the average period. For Gaussian processes, by inserting Eqs. I-4.2, I-4.5 and I-5.4 into III-3.3, one obtains

$$\begin{aligned}
L_{T_{f,E}}(t) &= A_E \exp\{-\alpha_E t\} \\
&= (1 - \exp\{-r^2/2\}) \exp\{-v_0 t \frac{kr}{\exp\{r^2/2\} - 1}\} \quad t \geq 0 \quad (\text{III-3.4})
\end{aligned}$$

where $r = a/\sqrt{\lambda_0}$, a is the threshold level and k is the spectral density shape factor given by Eq. I-5.3. These last two equations are basic results of this thesis.

III.4. First Passage Time for D Type and B Type Barriers

The two-state process $E_a(t)$ which corresponds to an E type barrier is defined in the preceding chapter as a continuous time process. On the contrary, the two-state processes $D_a(n)$ and $B_a(n)$ are defined only at discrete points in time. The process $D_a(n)$ associated with a D type barrier, $|X|=a$, takes a value at times when a peak of $|X(t)|$ occurs, i.e., about every $(2v_0)^{-1}$ seconds. $B_a(n)$ is observed at the peaks of $X(t)$, at intervals of approximate length $(v_0)^{-1}$. The simplest approach here is to modify the discrete-time two-state processes $D_a(n)$ and $B_a(n)$ essentially with the aim of restoring the time continuity. This approach will be shown to give good results. An alternate approach, discussed in some detail in the Appendix, is to adopt a geometric distribution for (say, in the case of D type barriers) $N_{0,D}$, the number of consecutive $|X|$ -peaks below the threshold $|X|=a$. The results

of the two approaches are shown there to be almost identical except in the range of low threshold levels when the average number of half-cycles to failure is, say, less than five.

Let the modified continuous time two-state processes be denoted by $D'_a(t)$ and $B'_a(t)$. They are characterized by the holding times $T_{0,D}$, $T_{1,D}$ and $T_{0,B}$, $T_{1,B}$, the means of which are related to the corresponding mean clump sizes of the discrete time processes $D_a(n)$ and $B_a(n)$.

$$\begin{aligned} E[T_{0,D}] &= E[N_{0,D}]/2v_0 \\ E[T_{1,D}] &= E[N_{1,D}]/2v_0 \end{aligned} \quad (\text{III-4.1})$$

$$\begin{aligned} E[T_{0,B}] &= E[N_{0,B}]/v_0 \\ E[T_{1,B}] &= E[N_{1,B}]/v_0 \end{aligned} \quad (\text{III-4.2})$$

Also, let $T_{f,D}$ and $T_{f,B}$ be the times to first passage of the threshold configuration the second subscript refers to. Assume $T_{0,D}$ and $T_{0,B}$ to be exponentially distributed with decay rates α_D and α_B , respectively. The reliability functions corresponding to D and B type barriers then take the form

$$L_{T_{f,D}}(t) = A_D e^{-\alpha_D t}, \quad t \geq 0 \quad (\text{III-4.3})$$

$$L_{T_{f,B}}(t) = A_B e^{-\alpha_B t}, \quad t \geq 0 \quad (\text{III-4.4})$$

The decay rates α_D and α_B may be obtained from Eqs. III-4.1, III-4.2, II-4.14 and II-5.3,

$$\alpha_D = \frac{1}{E[T_{0,D}]} = 2v_a \frac{1 - \exp\{-n_a/2v_a\}}{1 - v_a/v_o} \quad (\text{III-4.5})$$

$$\alpha_B = \frac{1}{E[T_{0,B}]} = v_a \frac{1 - \exp\{-n_a/v_a\}}{1 - v_a/v_o} \quad (\text{III-4.6})$$

The probabilities $A_D = P[T_{f,D} > 0]$ and $A_B = P[T_{f,B} > 0]$ may be evaluated as follows

$$A_D = P[T_{f,D} > 0] = \frac{E[T_{0,D}]}{E[T_{0,D}] + E[T_{1,D}]} = \frac{E[N_{0,D}]}{E[N_{0,D}] + E[N_{1,D}]} = 1 - \frac{v_a}{v_o} \quad (\text{III-4.7})$$

$$A_B = P[T_{f,B} > 0] = \frac{E[T_{0,B}]}{E[T_{0,B}] + E[T_{1,B}]} = 1 - \frac{v_a}{v_o} \quad (\text{III-4.8})$$

It should be noted that the above estimate of A_D is strictly correct only if the start ($t=0$) is given to coincide with the occurrence of a peak of the process $|X(t)|$. The probability of instantaneous failure, $P[T_f=0]$, may then be redefined to be the probability that the first peak exceeds the threshold $|X|=a$. Note furthermore that the assumption that a peak $|X(t)|$ occurs at the start, i.e., $\dot{X}(0)=0$, leads to a conservative estimate of the probability of instantaneous failure. Thus the true value of A_D should be somewhat larger than that predicted by Eq. III-4.7. This is precisely what is observed in the simulation results [17] as is shown in figure III.8.

The same remarks apply a fortiori to B type barriers. Some

further aspects of this problem are discussed in the Appendix.

Inserting Eqs. III-4.5 and III-4.7 into Eq. III-4.3, we find for the reliability function associated with a D type barrier, under "almost" stationary starting conditions, (that is, strictly, $t=0$ corresponds to a randomly selected peak of $|X|$)

$$\begin{aligned} L_{T_{f,D}}(t) &= P[T_{f,D} > t] = A_D \exp\{-\alpha_D t\} \\ &= \left(1 - \frac{v_a}{v_0}\right) \exp\{-2v_a t \frac{1 - \exp\{-n_a/2v_a\}}{1 - v_a/v_0}\} \quad t \geq 0 \quad (\text{III-4.9}) \end{aligned}$$

For Gaussian random processes, Eq. III-4.9 becomes

$$L_{T_{f,D}}(t) = (1 - \exp\{-r^2/2\}) \exp\{-2v_0 t \frac{1 - \exp\{-kr/2\}}{\exp\{r^2/2\} - 1}\} \quad t \geq 0 \quad (\text{III-4.10})$$

where $r = a/\sqrt{\lambda_0}$ and k is the spectral density shape factor (Eq. I-5.3)

Similarly, the reliability function associated with a B type barrier, again under "almost" stationary starting conditions (in the sense that $t=0$ is assumed to correspond to randomly selected peak of X), takes the form

$$\begin{aligned} L_{T_{f,B}}(t) &= P[T_{f,B} > t] = A_B \exp\{-\alpha_B t\} \\ &= \left(1 - \frac{v_a}{v_0}\right) \exp\{-v_a t \frac{1 - \exp\{-n_a/v_a\}}{1 - v_a/v_0}\} \quad t \geq 0 \quad (\text{III-4.11}) \end{aligned}$$

For Gaussian random processes, Eq. III-4.11 becomes

$$L_{T_{f,B}}(t) = (1 - \exp\{-r^2/2\} \exp\{-v_0 t \frac{1 - \exp\{-kr\}}{\exp\{r^2/2\} - 1}\}) \quad t \geq 0 \quad (\text{III-4.12})$$

where $r = a/\sqrt{\lambda_0}$ and k is given by Eq. I-5.3. These last four equations are basic results of this thesis. Eqs. III-4.9 and III-4.11 apply for any stationary narrow-band process for which the basic statistics v_a and n_a exist.

III.5. Relationship to Asymptotic First Passage Distribution. Comparison of Estimates

D type barriers and E type barriers

A well-known approximate first passage time distribution [7, 14, 16] for D type barriers is based on the assumption that $|X|$ -upcrossings of a (high) threshold $|X|=a$ occur independently according a Poisson process with average occurrence rate $2v_a$. It has been formally proved [9, 14] that this assumption is asymptotically correct, that is, when the threshold value tends to infinity. According to this simple model, the reliability function has the form of Eq. III-4.3, but with $A_D = P[T_{f,D} > 0] = 1$ and with $2v_a$ substituting α_D .

To compare the decay rates α_D , α_E and the approximate (but asymptotically correct) mean rate $2v_a$, it will be convenient to let $2v_a$ act as a reference quantity. This procedure has also been used by Crandall, Chandiramani and Cook [17].

Thus consider the quotients

$$\frac{\alpha_E}{2\nu_a} = \frac{n_a/2\nu_a}{1-\nu_a/\nu_0} \quad (\text{III-5.1})$$

$$\frac{\alpha_D}{2\nu_a} = \frac{1-\exp\{-n_a/2\nu_a\}}{1-\nu_a/\nu_0} \quad (\text{III-5.2})$$

For Gaussian random processes, the above rates become, with $2\nu_a = 2\nu_0 e^{-r^2/2}$,

$$\frac{\alpha_E}{2\nu_a} = \frac{kr/2}{1-\exp\{-r^2/2\}} \quad (\text{III-5.3})$$

$$\frac{\alpha_D}{2\nu_a} = \frac{1-\exp\{-kr/2\}}{1-\exp\{-r^2/2\}} \quad (\text{III-5.4})$$

These ratios are plotted in figure III.5 as a function of the normalized barrier level $r=a/\sqrt{\lambda_0}$ for various values of the spectral density shape factor k , given by Eq. I-5.3. Note that $\alpha_D/2\nu_0$ tends to one for $r \rightarrow \infty$. For low values of k the ratio $\alpha_D/2\nu_a$ may be considerably smaller than one in the range of threshold levels $a=2\sqrt{\lambda_0}$ to $5\sqrt{\lambda_0}$. To use the approximate mean failure rate $2\nu_a$ to estimate first crossing probabilities in such a case is very likely to be too conservative.

Figure III.6 shows the probability of no instantaneous failure $A_E=A_D=P[T_f>0]=1-e^{-r^2/2}$ as a function of r . Its value increases monotonically to an asymptotic value one.

It is of considerable interest to note the striking similarity between the curves shown in figures III.5 and III.6

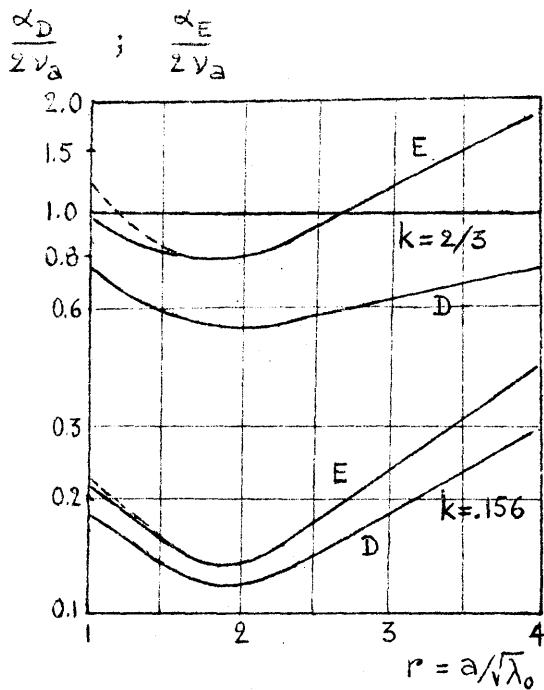


Fig.III.5. First Passage Probability Decay Rates. Eqs.III-5.3 and III-5.4. (Dotted lines: $\alpha'_D/2\gamma_a$; See Appendix)

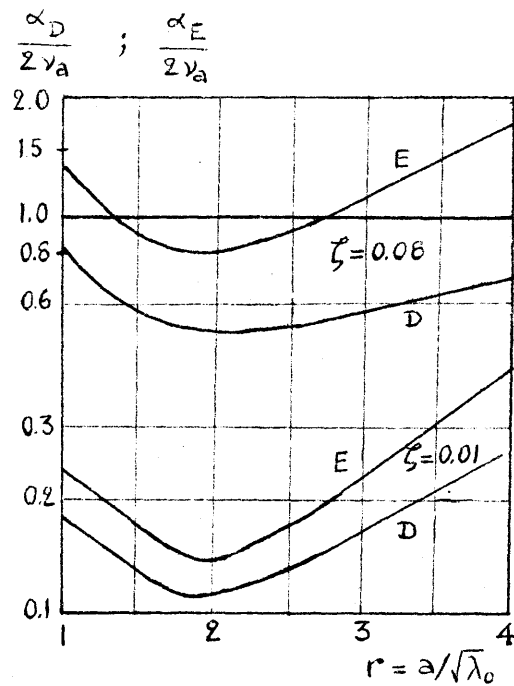


Fig.III.7. First Passage Probability Decay Rates. Simulation Results (Crandall et al.[17]).

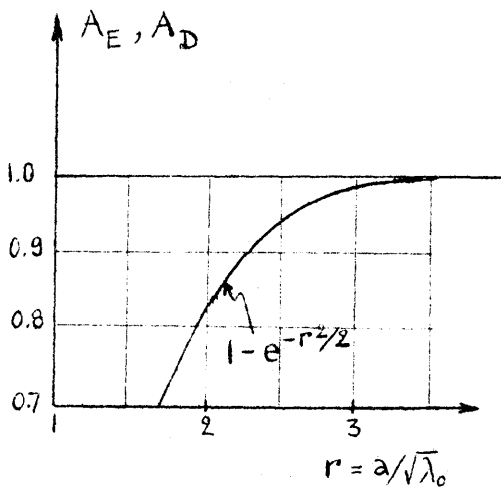


Fig.III.6. Probability of Instantaneous Failure.

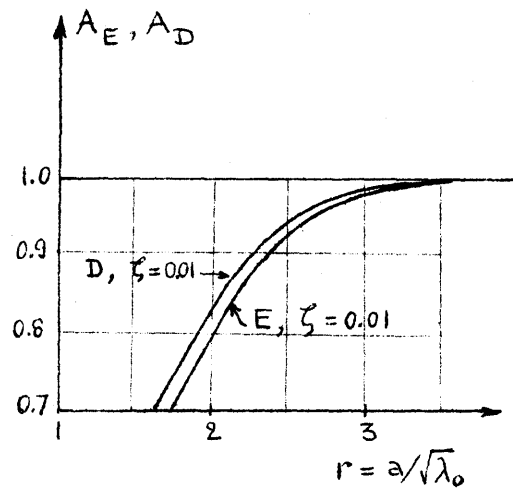


Fig.III.8. Factors A_E and A_D in Simulation Results (Crandall et al.[17]).

and similar ones shown in figures III.7 and III.8. The latter have been obtained by numerically estimating the distribution function of the first passage time of the response of lightly damped linear oscillators to wide-band Gaussian excitation [17]. While there is an unmistakable qualitative agreement between the analytical estimates proposed here and the experimental results, the quantitative aspects of this relationship need further investigation. This is done in chapter IV.

B type barriers

The assumption that X-upcrossings of a positive threshold $X=a$ occur independently according to a Poisson process leads to a reliability function of the same form as Eq. III-44 but with ν_a substituting α_B and with $A_D = P[T_{f,D} > 0] = 1$. Again consider the ratios α_B / ν_a

$$\frac{\alpha_B}{\nu_a} = \frac{1 - \exp\{-n_a / \nu_a\}}{1 - \nu_a / \nu_0} \quad (\text{III-5.5})$$

which for Gaussian processes takes the form

$$\frac{\alpha_B}{\nu_a} = \frac{\alpha_B}{\nu_0 \exp\{-r^2/2\}} = \frac{1 - \exp\{-kr\}}{1 - \exp\{-r^2/2\}} \quad (\text{III-5.6})$$

This ratio is plotted in figure III.9 as a function of r for a few values of the shape factor k . Again the ratio α_B / ν_a tends asymptotically (for $r \rightarrow \infty$) to one, in accordance with Cramer's [43, 9] limit theorem.

Remark: Two-State Markov Process Assumption

A number of assumptions have been adopted regarding the holding times of the two-state processes studied in this and the preceding chapter. These assumptions almost add up to one major assumption, i.e., that the processes $E_a(t)$, $D'_a(t)$ and $B'_a(t)$ are continuous time two-state Markov processes [44]. Note, however, that none of the particular results derived earlier, strictly requires the Markov assumption. For example, to derive the first passage distributions, no assumption needed to be made regarding the form of the density function of $T_{1,E}$. Nonetheless, it will be convenient in further work (in chapters V and VII) to "cover" all earlier (somewhat less restrictive) assumptions by taking the two-state processes to be Markovian.

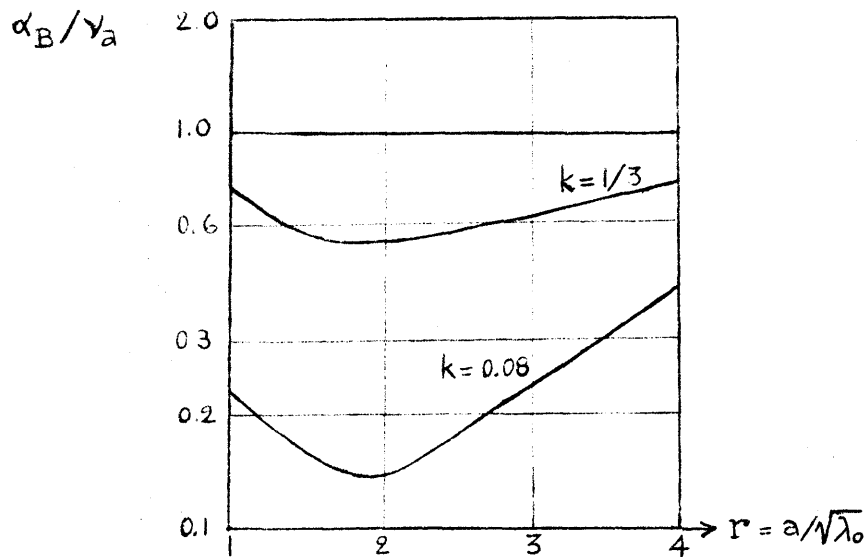


Fig. III.9. First Passage Probability Decay Rate for B Type Barriers.

CHAPTER IV

FIRST CROSSING PROBABILITY AND MAXIMUM RESPONSE OF RANDOMLY EXCITED LIGHTLY DAMPED OSCILLATORS

IV.1. Scope

In the preceding chapter an approximate analytical solution is proposed to the classical problem of determining the probability that the maximum value of a random vibration $X(t)$, remains below a specified threshold during a given time interval. Recall that the solution is based on the concept of a two-state process, whose properties are explored in Ch. II. Here, the results are applied to the case where $X(t)$ represents the response of a lightly damped linear or nonlinear oscillator to random wide-band excitation.

Note that all the results in chapters II and III are first obtained in a general (i.e., non-Gaussian) form. They are stated in terms of a few basic statistical properties, i.e., v_a , n_a and v_0 , which may be computed by means of Rice's [1] fundamental results, Eqs. I-4.1 and I-4.3. The basic information needed consists of the joint probability distributions, $p(x, \dot{x})$ and $p(r, \dot{r})$, respectively related to the process $X(t)$ and its envelope $R(t)$, and their respective time derivatives.

Gaussian Random Processes

For a stationary Gaussian random process $X(t)$, the needed joint distributions are available (see Eqs. I-3.7 and I-4.4). It has been shown that all general expressions (including the first-passage probability estimates) may be conveniently re-stated in terms of the first few moments λ_j of the one-sided spectral density $G(\omega)$ of the process $X(t)$

$$\lambda_j = \int_0^{\infty} \omega^j G(\omega) d\omega \quad (\text{I-1.5})$$

In fact, the results have been found to depend upon the normalized (i.e. unit area) power spectral density, $G(\omega)/\lambda_0$, only through ν_0 , the average frequency, and k , the "spectral density shape factor",

$$\nu_0 = \frac{1}{2\pi} \sqrt{\lambda_2/\lambda_0} \quad (\text{IV-1.1})$$

$$k = \sqrt{2\pi(1-\lambda_1^2/\lambda_0\lambda_2)} \quad (\text{I-5.3})$$

The properties of the shape factor k are briefly discussed in section I.5 and the value of k is derived there for some well-known spectral densities. Furthermore, the threshold value may be conveniently normalized with respect to the r.m.s. value $\sqrt{\lambda_0} = \sqrt{E[X^2]}$. The normalized threshold level is designated by r ,

$$r = a/\sqrt{\lambda_0} \quad (\text{I-5.2})$$

In this form, the "first-crossing" results are directly applicable to the case where $X(t)$ represents the response of a lightly damped linear oscillator to broad-band Gaussian excitation. This very important class of structures is examined in sections V.2 and V.3. The effect of nonstationarity of the response is considered in section V.3.

To determine the shape factor k analytically by means of Eq. I-5.3, the spectral density function $G(\omega)$ has to be available. Often, in practice, the following experimental method of determining k (and the other needed parameters, ν_0 and λ_0) will be indicated. First, note that, to estimate the shape factor k of a process experimentally (from a sample history of the, presumably ergodic, process), one does not need to estimate the whole spectral density function first, and then, compute its first few moments. A much simpler method is to obtain a sample estimate of the average clump size, $E[N_{1,D}]$, associated with some fixed threshold a . Recall that $E[N_{1,D}]$ is the average number of consecutive peaks of $|X(t)|$ above the double threshold, $|X|=a$. Presume that $\bar{N}_{1,D}$ is the sample estimate of $E[N_{1,D}]$. Then from Eq. II-4.16, one may obtain the following estimate, \bar{k} , of the shape factor,

$$\bar{k} = -\frac{2}{\bar{r}} \ln\left(1 - \frac{1}{\bar{N}_{1,D}}\right) \quad (\text{IV-1.2})$$

where $\bar{r} = a/(\overline{x^2})^{1/2}$, with $(\overline{x^2})$ denoting a sample estimate of the mean square of $X(t)$. Note that a single sample history suf-

fices to obtain many point estimates, $\bar{k}^{(1)}$, $\bar{k}^{(2)}$, etc., each of which will correspond to a different threshold level setting and hence, may be based on a different sample size. An interesting study could be done on the reliability of the estimator, Eq. IV-1.2. This will, however, not be pursued in this thesis.

Non-Gaussian Processes

The results of chapters II and III, in their general form, i.e., in terms of v_a , n_a and v_0 , will be directly useful in obtaining pertinent response statistics, e.g., first-crossing probabilities, for nonlinear narrow-band systems subjected to wide-band excitation or to linear narrow-band systems subjected to non-Gaussian input. Basically, what is needed are expressions for v_a and n_a , in terms of the mechanical properties of the nonlinear oscillator and of the characteristics of the random input. Some relevant results are available for certain nonlinear vibratory systems [31, 35, 48, 49, 50]. This will be further discussed in section IV.4., where the usefulness of the proposed approach is demonstrated for the case of a hard-spring oscillator subjected to Gaussian white noise excitation.

IV.2. First-Crossing Probability for the Damped
Linear One-Degree-of-Freedom Oscillator Sub-
jected to Gaussian White Noise Excitation

Preliminaries

A viscously damped single-degree-of-freedom oscillator is shown in figure IV.1. The system may be excited either by base motion $\ddot{u}_0(t)$, or by an applied force $F(t)$.

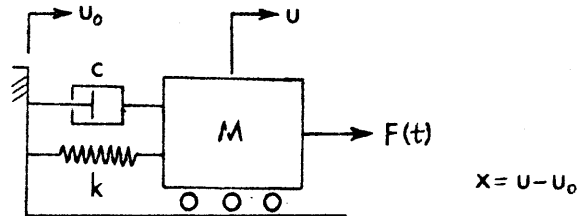


Fig.IV.1. A Viscously Damped One-Degree-of-Freedom Oscillator.

The general equation of motion for the system is

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2x = f(t) - \ddot{u}_0(t) \quad (\text{IV-2.1})$$

where

$$\begin{aligned} \omega_n &= K/M \\ \zeta &= c/2M\omega_n \\ f(t) &= F(t)/M \end{aligned} \quad (\text{IV-2.2})$$

in which ω_n is the natural frequency and ζ is the damping ratio. The transfer function associated with various input-

output combinations may be obtained. Presume that the structure is subjected to a support motion, characterized by its acceleration history $\ddot{u}_0(t)$, and that one is interested in the relative displacement x between the mass and the support. The transfer function (or complex frequency function) $H(\omega)$ may be obtained by substituting $f(t)=0$, $\ddot{u}_0(t)=e^{i\omega t}$ and $x=H(\omega)e^{i\omega t}$ in Eq. IV-2.1. Then

$$H(\omega) = \frac{-1}{\omega_n^2 - \omega^2 + 2i\zeta\omega_n\omega} \quad (\text{IV-2.3})$$

Let the excitation be a stationary Gaussian white noise with (one-sided) spectral density G_0 (for all frequencies from $\omega=0$ to $\omega=+\infty$). The stationary response spectral density is [9]

$$\begin{aligned} G_X(\omega) &= |H(\omega)|^2 G_0 \\ &= \frac{G_0}{(\omega_n^2 - \omega^2)^2 + 4\zeta^2 \omega_n^2 \omega^2} \end{aligned} \quad (\text{IV-2.4})$$

The first few moments, λ_0 , λ_1 and λ_2 , of the above spectrum are given in section I.5 by Eqs. I-1.15, I-1.16 and I-1.17, respectively. The average frequency ν_0 of the response process is

$$\nu_0 = \frac{1}{2\pi} \sqrt{\lambda_2/\lambda_0} = \omega_n/2\pi \quad (\text{IV-2.5})$$

The spectral density shape factor k has been shown in section

I.5, Eq. I-5.18 to be approximately equal to

$$k = 2\sqrt{2}\zeta^{1/2} \quad \text{for small } \zeta \quad (\text{IV-2.6})$$

The above approximation closes in on the true value of k as $\zeta \rightarrow 0$, (i.e., higher order terms in ζ vanish.)

First-Crossing Probability

We will concentrate here on the absolute value, $|X(t)|$, of the displacement response of the linear oscillator and on its position w.r.t. a specified (double) threshold level $|X|=d$. Failure is postulated to occur when $|X(t)|$ first exceeds the level d . Recall that $L_{T_{f,D}}(t)$, the reliability function for D type barriers, designates the probability that $|X(t)|$ remains below the threshold d during the time interval 0 to t_0 . For stationary random processes and under random starting conditions, we have, from Eq. III-4.9

$$L_{T_{f,D}}(t_0) = P[T_{f,D} > t_0] = A_D \exp\{-\alpha_D t\}, \quad t \geq 0 \quad (\text{IV-2.7})$$

where A_D equals the probability that failure is not instantaneous, (i.e., $A_D = P[T_{f,D} > 0]$) and α_D is the decay rate or the failure rate of the first-crossing probability. For Gaussian processes, A_D and α_D take the form,

$$A_D = 1 - \exp\{-r^2/2\} \quad (\text{IV-2.8})$$

$$\alpha_D = 2v_0 \exp\{-r^2/2\} \frac{1 - \exp\{-kr/2\}}{1 - \exp\{-r^2/2\}} \quad (\text{IV-2.9})$$

where $r=d/\sqrt{\lambda_0}$. Replacing v_0 and k by Eqs. IV-2.5 and IV-2.6, respectively, one obtains

$$\alpha_D = \frac{\omega_n}{\pi} \exp\{-r^2/2\} \frac{1 - \exp\{-r\sqrt{2}\zeta\}}{1 - \exp\{-r^2/2\}} \quad (\text{IV-2.10})$$

Our estimate of the first crossing probability will now be compared to other approximations [8,9,51] and to the results obtained by simulation [17, 42] and numerical methods [17, 19]. Theoretical and experimental evidence will prompt a revision of the estimate, Eq. IV-2.6, of the shape k .

Comparison with Available Simple Approximations

a) The approximation most often used by investigators (e.g., Refs. 7, 9, 14, 15, 52, 53, 54) has the form

$$L_{T_{f,D}}(t_0) = P[T_{f,D} > t_0] \approx \exp\{-\alpha t\} \quad (\text{IV-2.11})$$

where

$$\alpha = 2v_0 \exp\{-r^2/2\} = \frac{\omega_n}{\pi} \exp\{-r^2/2\} \quad (\text{IV-2.12})$$

The above approximation is based on the assumption that successive upcrossings of the threshold d by the stationary absolute displacement $|X(t)|$, occur independently according to a Poisson process [7, 15]. This assumption has been shown [9, 43] to be asymptotically exact, i.e., when $r \rightarrow \infty$.

The two expressions, Eqs. IV-2.10 and IV-2.12 will be nearly equal under the following conditions: i) if the numerator of the fraction in Eq. IV-2.10 tends to one, i.e., if kr becomes large, and ii) if r is sufficiently large so that $\exp\{r^2/2\} \gg 1$.

b) Another approximate expression for the failure rate, proposed by Lin [8] and Lyon [35], presumes that envelope-upcrossings of the threshold d occur independently according to a Poisson process. Again, $L_{T_{f,D}}(t)$ has a form as given by Eq. IV-2.11, but α is now substituted by a new decay rate $\alpha' = n_d$, where n_d is the average rate of envelope-upcrossings of the level d . Using our notation (see Eq. I-5.4 and I-4.5),

$$\begin{aligned} \alpha' &= n_d = 2v_d \frac{kr}{2} = 2v_0 \frac{kr}{2} \exp\{-r^2/2\} \\ &= 2\sqrt{2}v_0 r \zeta^{1/2} \exp\{-r^2/2\} \end{aligned} \quad (\text{IV-2.13})$$

The above estimate will be nearly equivalent to Eq. IV-2.10 under the following conditions:

- i) if $kr \ll 1$, such that $1 - \exp\{-kr/2\} \approx kr/2$ and
- ii) if r is sufficiently large that $\exp\{r^2/2\} \gg 1$.

The two requirements are somewhat contradictory. The latter requires the relative threshold level, $r = d/\sqrt{\lambda_0}$, to be high. But recall that, at those high levels, a substantial fraction of envelope upcrossings is not (immediately) followed by an $|X|$ -upcrossing. It is easy to see that the estimate based on

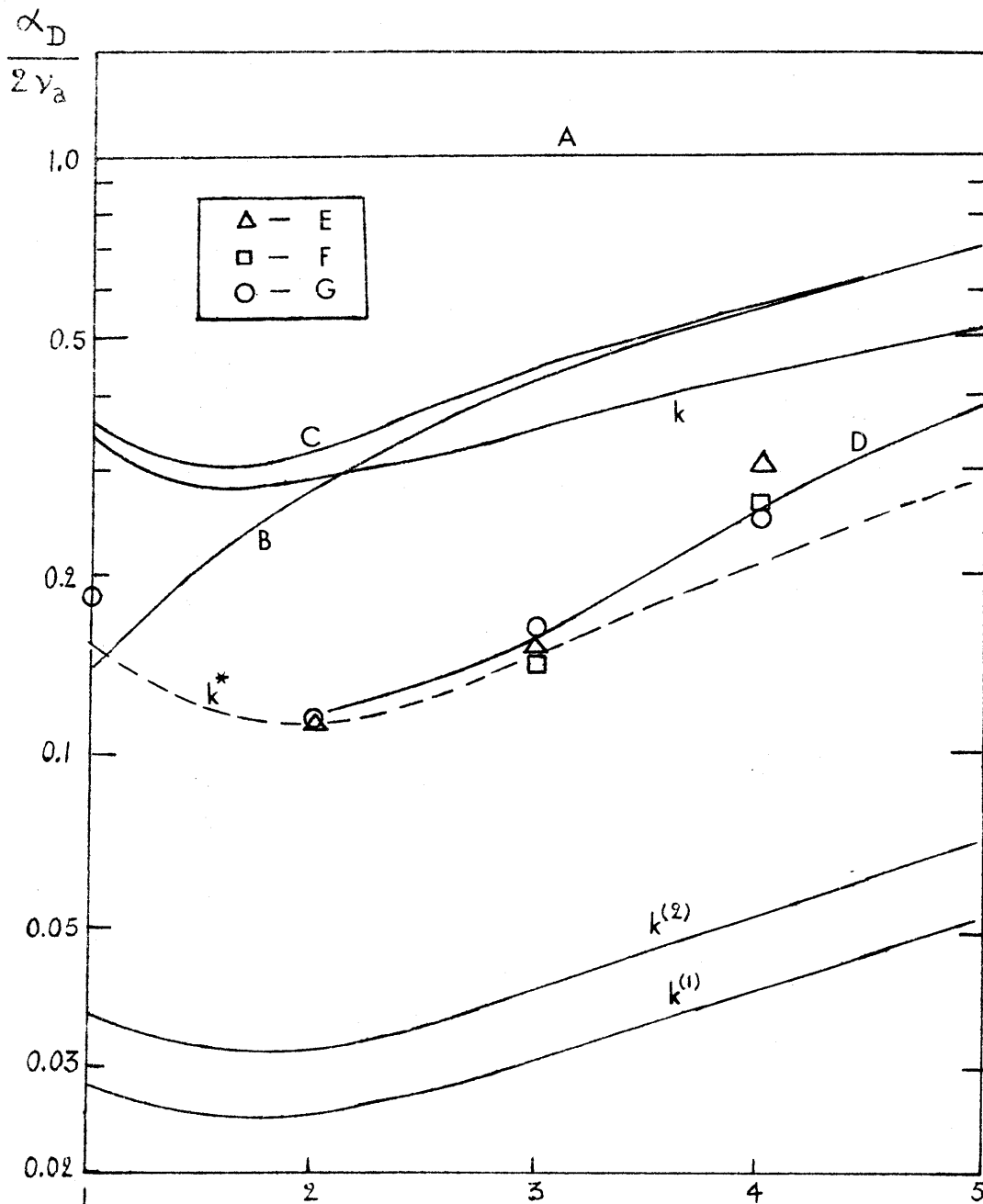


Fig.IV.2. Comparison of Estimates for the Decay Rate α_D , $\zeta=0.01$
 A: Independent Crossings; B: Independent Envelope Crossings[8];
 C: Decay Rate α_D ; D: Mark's Analytical Work [19].
 E: Digital Simulation [17]; F: Analog Simulation [42]; G: Numerical Diffusion of Probability [17].
 k , $k^{(1)}$, $k^{(2)}$: Various Analytical Estimates of the Spectral Shape Factor; k^* : "Experimentally" determined k-factor.

the assumption of independent envelope upcrossings, does not exhibit the proper limiting behavior for high thresholds.

c) Two other estimates, quite similar to the above, may be given. The first one is based on the assumption that envelope peaks are independent [51,32]. The second one amounts to using α_E , the decay rate of the reliability function (Eq.III-3.4) corresponding to type E barriers. The difference between the three envelope-based approximations is notable only for low threshold levels. Figure IV.2 (which is adapted from a recent review paper by Crandall[32]) shows several of the above approximations.

Sensitivity of k-Factor Estimate

Before comparing our estimate of the failure rate α_D , given by Eq.IV-2.10, to results obtained by simulation [17,42] and other analytical work [19], it will be of interest to illustrate the fact that the k-factor is highly sensitive to the shape of the power spectral density function. Recall that k is a simple function of $\lambda_1^2/\lambda_0\lambda_2$ (see Eq. I-5.3) where the λ_j are moments of the power spectrum. For the response spectral density $G_X(\omega)$, given by Eq. IV-2.4 and shown in figure IV.3a, we found $k \approx 2\sqrt{2}\zeta^{1/2}$ for low values of the damping ratio ζ . Consider, however, the following two approximations to the actual power spectral density function $G_X(\omega)$.

- i) a band-limited filter (Fig. IV.3b)
- ii) a Gaussian filter (Fig. IV.3c)

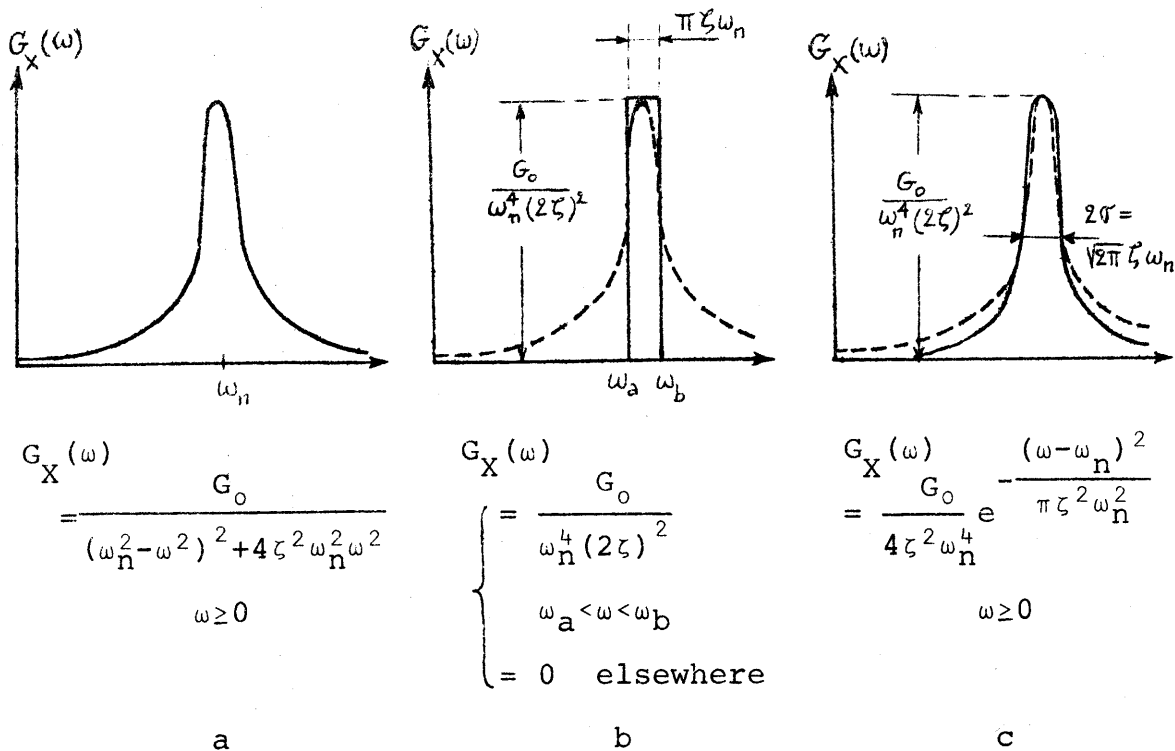


Fig.IV.3. Approximate Spectra.

An expression for the shape factor k in terms of the parameters of each of the above filters is given in section I.5, by Eqs. I-5.9 and I-5.12, respectively. The parameters of the approximate spectra are so chosen, that the following two conditions are satisfied

- i) they transmit the same total power, i.e., λ_0 is given by Eq. I-5.15
- ii) their peak value equals $G_X(\omega_n)$ in Eq. IV-2.4.

The approximate spectra are depicted in figures IV.3b and c, respectively. It is easy to show that the following estimates of k result,

"Band-Limited" Approximation $k^{(1)} \approx \pi \sqrt{\frac{\pi}{6}} \zeta$ (IV-2.14)

(Fig. IV.3b)

"Gaussian" Approximation $k^{(2)} \approx \pi \zeta$ (IV-2.15)

(Fig. IV.3c)

The above estimates show the shape factor to be linear in ζ . Our earlier estimate, Eq. IV-2.6, indicates that k is proportional to $\zeta^{1/2}$. The ideal white noise input upon which the latter estimate is based, is, of course, physically unrealizable. It is also interesting to note that the "band-limited" approximation for k has been used by Lin [8], to evaluate the approximate decay rate α' , given by Eq. IV-2.13. For a damping ratio $\zeta=0.02$, the estimates k and $k^{(1)}$ differ by a ratio of 1 to 9. This discrepancy between the various approximations to the shape factor, strongly suggests its rather pronounced sensitivity to the extremes and/or skewness of the power spectral density function. Note that all three estimates of the spectral shape factor have the form

$$k = c_1 \zeta^{c_2} \quad (\text{IV-2.16})$$

where the constant c_1 has the value 2.83, 2.22 and 3.14 in Eqs. IV-2.6, IV-2.14 and IV-2.15, respectively, and c_2 takes the respective values 0.5, 1.0 and 1.0.

Comparison with Numerical Solutions

Crandall [32] recently compared the author's estimate,

Eq. IV-2.10, of the decay rate α_D of the first-crossing probability, to the results obtained by simulation [17, 42] and numerical diffusion [17]. Also included in the comparison are the analytical results due to Mark [19]. He obtained the eigenvalues of a first order "equivalent" of the integral equation characterizing a second-order Markov process. The dominant eigenvalue is taken as an approximation for the decay rate α_D . His method also requires a considerable amount of computation.

Figure IV.2 shows the various estimates of α_D , normalized w.r.t. α , for a damping ratio $\zeta=0.01$. The curve, labeled "k", represents the case where the shape factor is estimated by Eq. IV-2.6. If the estimates $k^{(1)}$ or $k^{(2)}$, given by Eqs. IV-2.14 and IV-2.15, respectively, are used then a considerably smaller value of the ratio α_D/α results, as shown in figure IV.2. In the face of the uncertainty as to the validity of the available analytical estimates of k, an "experimental" determination of k seems indicated. The form, given by Eq. IV-2.16, of the relationship between k and ζ is adopted on theoretical grounds. It is found that excellent quantitative agreement may be achieved between the numerical results and the proposed analytical solution for α_D , if we take $c_1=2.5$ and $c_2=0.65$,

$$k = 2.5\zeta^{0.65} \quad (\text{IV-2.17})$$

It should be emphasized that the agreement is achieved over the whole range of threshold levels r and damping ratio's, $0.01 < \zeta < 0.08$.

This "semi-empirical" estimate of the functional relationship between the shape factor k and the damping ratio ζ , may now be back-substituted into the expressions, derived earlier, in terms of k , for a number of important response statistics. For example, the expression for the mean clump size, given by Eq. II-4.16, now becomes

$$E[N_{1,D}] = (1 - \exp\{-\frac{c_1}{2} r \zeta^{c_2}\})^{-1} \quad (\text{IV-2.18})$$

$$= (1 - \exp\{-1.25 r \zeta^{0.65}\})^{-1} \quad (\text{IV-2.19})$$

where r designates the normalized (w.r.t. the response r.m.s. value) threshold level. The above equation is plotted in Fig. IV.4 as a function of r for several values of the damping ratio ζ .

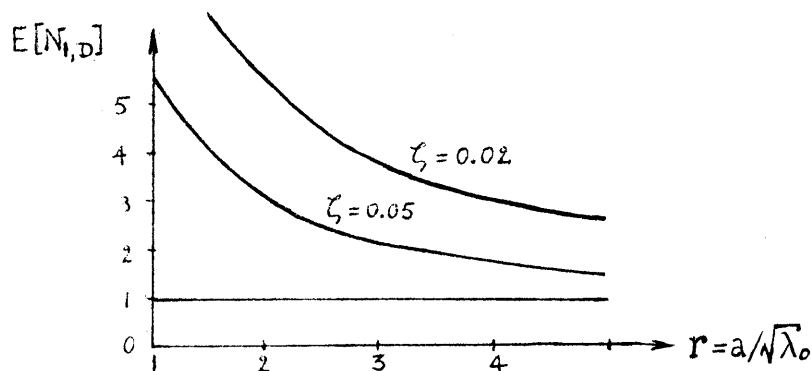


Fig. IV.4. Mean Clump Size for Stationary Response of a Lightly Damped Oscillator to White Noise. (Based on Eq. IV-2.19)

IV.3. Characteristics of the Maximum Response

Some General Results

a) Stationary Narrow-Band Gaussian processes

Consider an arbitrary zero-mean stationary narrow-band random process $X(t)$. The distribution function of the time of first passage of a D type barrier, $|X|=a$, is directly related to the distribution function of X_s , the maximum amplitude (in absolute value) of $X(t)$ during the time interval 0 to s . We have

$$F_{X_s}(a) = P[X_s \leq a] = P[T_{f,D;a} > s] = L_{T_{f,D;a}}(s) \quad (\text{IV-3.1})$$

where the symbols have the following meaning

- $F_{X_s}(a)$: the cumulative distribution function of X_s
- X_s : the maximum relative displacement (in absolute value) during the time interval 0 to s
- s : the motion duration
- a : the threshold value
- $T_{f,D;a}$: the time to first passage of a D type barrier with threshold value a
- $L_{T_{f,D;a}}(s)$: the value at time s , of the reliability function for a D type barrier, $|X|=a$.

Inserting Eq. III-3.3 into Eq. IV-3.1, one obtains

$$F_{X_s}(a) = L_{T_{f,D;a}}(s) = A_D \exp\{-\alpha_D s\} \quad s \geq 0 \quad (\text{IV-3.2})$$

where α_D is the decay rate of the first passage density and A_D is a factor which depends on the starting conditions. In the case of a stationary start it equals the probability of no instantaneous failure. For Gaussian processes, using Eq. III-3.4, Eq. IV-3.2 becomes

$$F_{X_S}(a) = (1 - \exp\{-r^2/2\}) \exp\{2\nu_0 s \frac{1 - \exp\{-kr/2\}}{\exp\{r^2/2\} - 1}\} \quad (\text{IV-3.3})$$

where $r = a/\sigma_X$, $\sigma_X = \sqrt{\lambda_0}$ being the standard deviation of the stationary process. Also, k is the spectral density shape factor (Eq. I-5.3) and ν_0 denotes the average number of zero upcrossings per unit time. Let R_S be defined as the reduced maximum amplitude

$$R_S = X_S/\sigma_X \quad (\text{IV-3.4})$$

It is easy to see that $F_{X_S}(a) = F_{R_S}(r)$.

Now, let $r_{S;p}$ denote the value of the reduced maximum amplitude which has a given probability p of being exceeded during s seconds of stationary motion. It may be obtained by solving the equation

$$p = F_{R_S}(r_{S;p}) \quad (\text{IV-3.5})$$

where the right side of the above equation may be substituted by the right side of Eq. IV-3.3, with $r_{S;p}$ replacing r . It

is clear that $r_{s;p}$ will, in general, be a function of $p, \nu_0 s$ and k . Many earlier studies [14, 43, 52, 53, 54] of the distribution of the maximum value of a random process are essentially based on the following "asymptotic" expression for $F_{R_s}(r)$

$$F_{R_s}(r) = \exp\{-2\nu_0 s \exp\{-r^2/2\}\} \quad (\text{IV-3.6})$$

For the case in which kr becomes large and $\exp\{r^2/2\} \gg 1$, the two expressions, Eq. IV-3.3 and IV-3.6, will be equal. As may be seen from Fig. IV.5, however, for very small k (or, in case

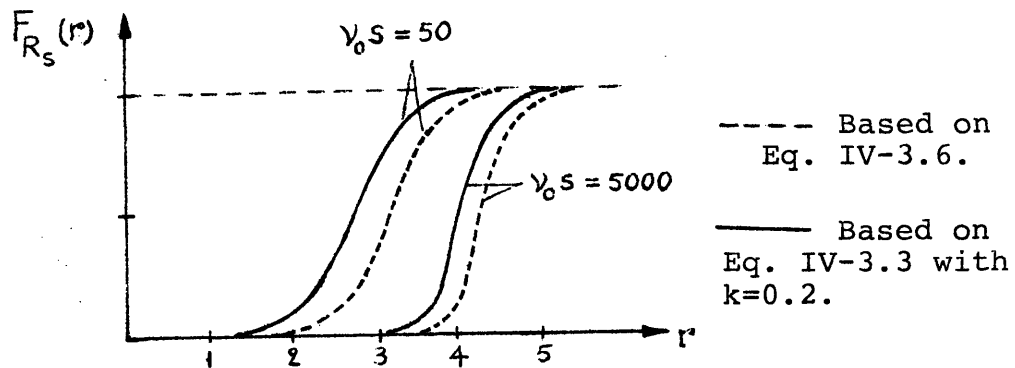


Fig.IV.5. Distribution Function of the Reduced Maximum Amplitude

$X(t)$ represents the response of a linear oscillator, for a very low damping ratio) and for low threshold levels, a substantial difference may result in the estimates of $r_{s;p}$ computed by means of Eqs. IV-3.3 and IV-3.6, respectively. The "clumping" of peaks (which becomes more pronounced for

low values of k) has the effect of increasing the mean first passage time and hence, of decreasing the expected maximum value.

b) Nonstationary Narrow-Band Gaussian Processes

Visualize a zero-mean narrow-band Gaussian random process $X(t)$ with a time-dependent mean square value, but whose unit-area power spectrum does not change in time. The spectral density shape factor k and the average frequency ν_0 will then also be independent of time. To estimate the probability distribution of the time to first passage of a double threshold level $|X|=a$ for such a nonstationary narrow-band random process, it will be useful to introduce a time-dependent failure rate $\alpha_D(t)$, which may be interpreted as follows: $\alpha_D(t)dt$ is the probability that failure occurs in the time interval t to $t+dt$ given that it did not occur in the interval $(0,t)$. In fact, a complete similarity between the stationary and the nonstationary case may be achieved by defining a time-dependent two-state Markov process, with time-dependent rates $\alpha_D(t)$ and $\beta_D(t)$ (*). The first-passage time distribution takes the form

(*) This representation is an extension of a non-homogeneous Poisson model for the occurrence of crossings of a high level. Such a model has recently been proposed by Ang and Amin [55] to study the characteristics of the response of structures to strong-motion earthquakes. Of course, the two-state Markov process will reduce to a Poisson process when the average clump size tends to one.

$$L_{T_{f,D;a}}(s) = P[T_{f,D;a} > 0] \exp\left\{-\int_0^s \alpha_D(t) dt\right\} \quad (IV-3.7)$$

If the system starts at rest, then the probability of instantaneous failure is zero, i.e., $P[T_{f,D;a} > 0] = 1$. Since only the standard deviation of the process depends on time, $\alpha(t)$ may be written as follows

$$\alpha_D(t) = 2\nu_0 \frac{1 - \exp\{-ka/2\sigma_X(t)\}}{\exp\{a^2/2\sigma_X^2(t)\} - 1} \quad (IV-3.8)$$

Inserting Eq. IV-3.8 into Eq. IV-3.7 and using the basic relation, $F_{X_S}(a) = L_{T_{f,D;a}}(s)$, one obtains the following expression for the distribution function of X_S , the maximum amplitude (in absolute value), during the time interval 0 to s, of a Gaussian narrow-band random process with a time-dependent mean square value $\sigma_X(t)$, which starts at zero level ($\sigma_X(0) = 0$),

$$F_{X_S}(a) = \exp\{-2\nu_0 \int_0^s \frac{1 - \exp\{-ka/2\sigma_X(t)\}}{\exp\{a^2/2\sigma_X^2(t)\} - 1} dt\} \quad (IV-3.9)$$

In Eqs. IV-3.8 and IV-9 the value of the shape factor k is assumed to be independent of time. Further work is needed to verify this assumption.

Statistical Properties of Lightly Damped Response Spectra

An important notion in practical and theoretical studies of the response of structures to dynamic loads is the response spectrum. It is a plot indicating the maximum

response of linear single-degree-of-freedom system to a given (deterministic) excitation versus the natural period of the (spectrum of) system(s) for various fractions of critical damping. Corresponding to each sample function of an ensemble of excitations is a "sample" set of response spectra. The aim here is to obtain a complete probabilistic description of the response spectra in terms of the properties of the ensemble of input motions, i.e., its power spectral density and duration. Our study will be limited to a "quasi-stationary" Gaussian excitation of duration s and with smoothly varying spectral density function $G(\omega)$. Its intensity function has a step-function-like form, as shown in Fig. IV.6. The nonstationary behavior of the response of a spectrum of lightly damped oscillators to such an excitation has been studied by Caughey and Stumpf [56]. They found that, for small values of the damping ratio ζ , the time-dependent mean square value of the response, $\sigma_x^2(t) = E[X^2(t)]$, has the following form

$$\sigma_x^2(t) = E[X^2(t)] \approx \frac{\pi G(\omega_n)}{4\zeta\omega_n^3} [1 - \exp\{-2\zeta\omega_n t\}] \quad 0 \leq t \leq s \quad (\text{IV-3.10})$$

When t becomes large compared to the "correlation time", $1/(2\zeta\omega_n)$, of the response process, then $\sigma_x^2(t)$ will become equal to its "stationary" value σ_x^2 , which equals

$$\sigma_x^2 \approx \frac{\pi G(\omega_n)}{4\zeta\omega_n^3} \quad (\text{IV-3.11})$$

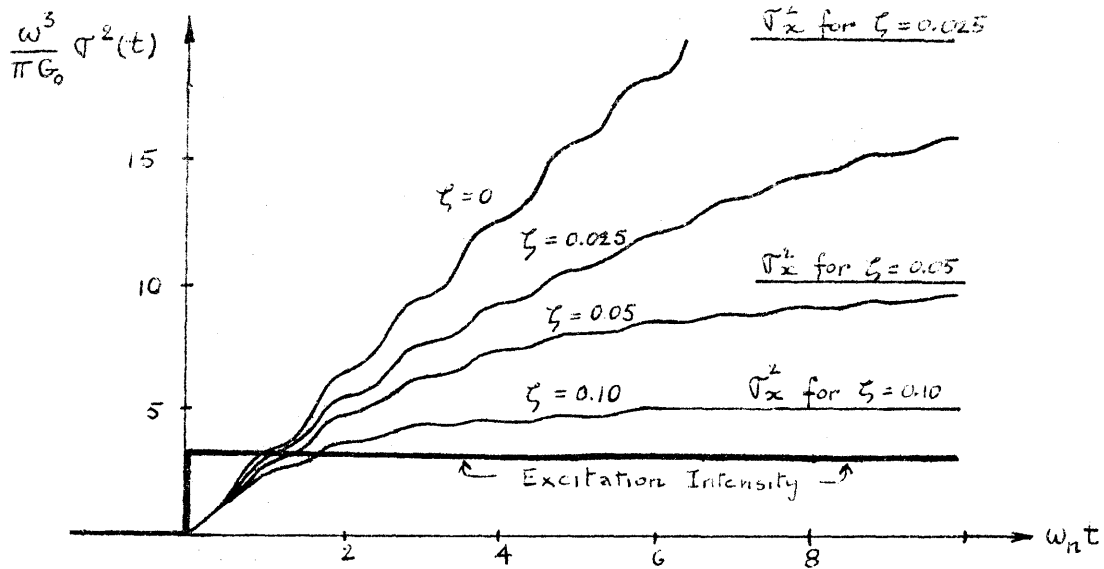


Fig.IV.6. Mean Square Value of the Transient Response of a Single-Degree-of-Freedom System under White Noise Excitation (taken from Caughey and Stumpf [56])

The transient nature of the response becomes important when the total time s during which the system is exposed to the excitation, is small compared to $1/(2\zeta\omega_n)$. When the damping ratio approaches zero, $\sigma_x^2(t)$ in Eq. IV-3.7 becomes [56]

$$\sigma_x^2(t) \Big|_{\zeta=0} = \lim_{\zeta \rightarrow 0} E[X^2(t)] \approx \frac{\pi G(\omega_n)}{2\omega_n^2} t \quad (\text{IV-3.12})$$

In Fig. IV.6, the mean square value $\sigma_x^2(t)$ is plotted as a function of t for various values of the damping ratio.

The probability distribution function of X_s , the maximum displacement (in absolute value) of the oscillator, is found by inserting Eq. IV-3.10 into Eq. IV-3.9,

$$F_{X_S}(a) = \exp\{-2v_0 \int_0^s \frac{1 - \exp\{-\frac{ka}{2\sigma_x} [1 - \exp(-2\zeta\omega_n t)]^{-1/2}\}}{\exp\{\frac{a^2}{2\sigma_x^2} [1 - \exp(-2\zeta\omega_n t)]^{-1}\} - 1} dt\} \quad (\text{IV-3.13})$$

For oscillators with a "correlation time", $1/(2\omega_n \zeta)$, which is much smaller than the motion duration s , the response standard deviation $\sigma_x(t)$ will rapidly reach its stationary level, and hence, the integrand in Eq. IV-3.13 will be nearly constant throughout most of the interval $(0, s)$. It will then be convenient to approximate Eq. IV-3.13 as follows,

$$\begin{aligned} F_{X_S}(a) &= \exp\{-2v_0 s^* \frac{1 - \exp\{-ka/2\sigma_x\}}{\exp\{a^2/2\sigma_x^2\} - 1}\} \\ &= \exp\{-\alpha_D s^*\} \end{aligned} \quad (\text{IV-3.14})$$

where s^* may be interpreted as the "equivalent motion duration". It is always smaller than s because the actual failure rate at the start of the response motion is smaller than the "stationary" failure rate. The larger the "correlation time", the more s^* and s will differ.

Another useful form of presenting $F_{X_S}(a)$ is that in which Crandall's simulation results [17] are reported, for the case where the oscillator starts from rest.

$$F_{X_S}(a) = A'_D \exp\{-\alpha_D s\} \quad s \gg 1/2\omega_n \zeta \quad (\text{IV-3.15})$$

The factor A'_D may be computed from Eqs. IV-3.14 and IV-3.15

$$A'_D = \exp\{\alpha_D (s-s^*)\} \quad (\text{IV-3.16})$$

$s-s^*$ may be interpreted as the "rise time" of the response r.m.s. value. It depends only on the oscillator properties, ω_n and ζ . The failure rate α_D (sharply) decreases when the threshold value increases. In fact, α_D is inversely proportional to the (stationary) mean time between threshold crossings. The product $\alpha_D (s-s^*)$ tends to zero, and hence $A'_D \rightarrow 1$, when the threshold level increases. The predicted behavior corresponds well with the simulation results [17], as shown in figure IV.7.

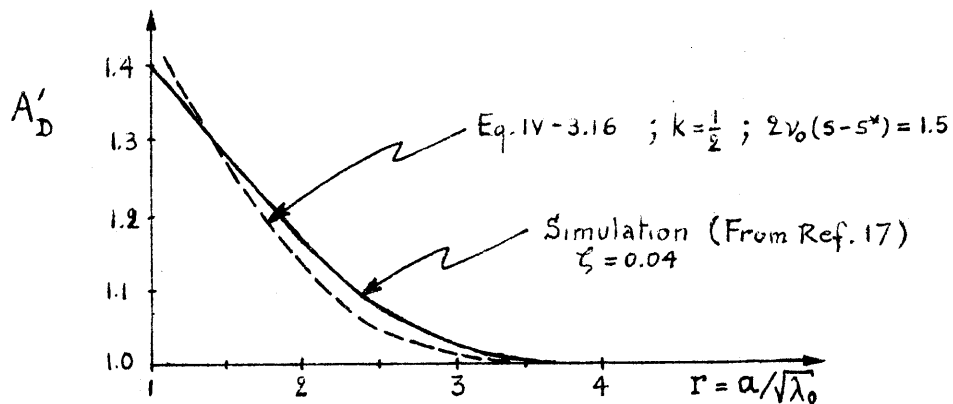


Fig.IV.7. The Factor A'_D in the Reliability Function (Eq.IV-3.15) Corresponding to a "Zero Start".

Let us now concentrate attention on $\sigma_x(s)$, the r.m.s. value of the response at the end of the "exposure". In arriving at Eqs. IV-3.14 and IV-3.15, it has been assumed that

$\sigma_x(s)$ is very nearly equal to the stationary r.m.s. value σ_x . For an oscillator with a very low damping ratio or a very long natural period, however, $\sigma_x(s)$ may be much smaller than σ_x , as shown in Fig. IV. 8.

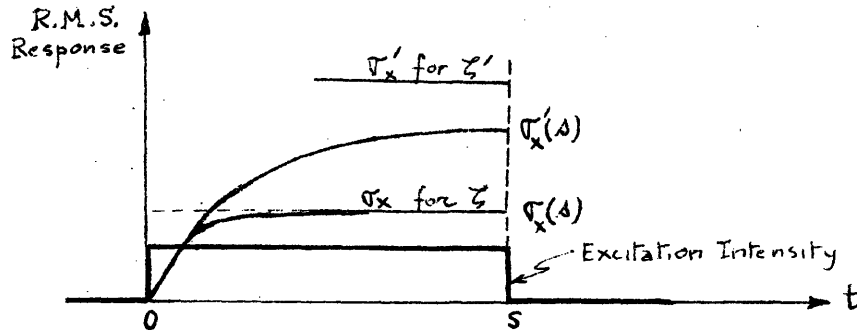


Fig.IV.8.

In fact, an undamped oscillator will not (ever) reach a stationary state under stationary excitation. It is not meaningful, in such cases, to express the reliability function (or the distribution of the maximum response) in terms of the stationary failure rate α_D . The following approach may be adopted, however. The stationary properties σ_x and α_D may be replaced by $\sigma_x(s)$ and $\alpha_D(s)$, respectively, in their role to act as reference quantities for the time-dependent statistics $\sigma_x(t)$ and $\alpha_D(t)$, $0 < t < s$. Let us define

$$\begin{aligned} r' &= a/\sigma_x(s) \\ R'_s &= X_s/\sigma_x(s) \end{aligned} \tag{IV-3.17}$$

Then, from Eqs. IV-3.10 and IV-3.13, one may write

$$F_{R'_S}(r') = \exp\{-2v_0 \int_0^s \frac{1 - \exp\{-kr'g(s,t)/2\}}{\exp\{r'^2g^2(s,t)/2\} - 1} dt\} \quad (\text{IV-3.18})$$

where

$$g(s,t) = \left[\frac{1 - \exp\{-2\zeta\omega_n s\}}{1 - \exp\{-2\zeta\omega_n t\}} \right]^{1/2} \quad (\text{IV-3.19})$$

It may be useful in practice to write the above result in the following form

$$F_{R'_S}(r') = \exp\{-\alpha_D(s)s^{**}\} \quad (\text{IV-3.20})$$

where s^{**} will be some fraction of the exposure time s , and $\alpha_D(s)$ is given by

$$\alpha_D(s) = 2v_0 \frac{1 - \exp\{-kr'\}}{\exp\{r'^2/2\} - 1} \quad (\text{IV-3.21})$$

Let $x_{s;p}$ designate the ordinate of the response spectrum corresponding to a quasi-stationary Gaussian random excitation of duration s and with a smooth power spectrum $G(\omega)$, which has a 100 % likelihood of not being exceeded. The quantity, $x_{s;p}$ may be expressed in terms of $\sigma_x(s)$,

$$x_{s;p} = r'_{s;p} \sigma_x(s) \quad (\text{IV-3.22})$$

where

$$\sigma_x(s) = \frac{\sqrt{\pi G(\omega_n)}}{2\zeta^{1/2} \omega_n^{3/2}} [1 - \exp\{-2\zeta\omega_n s\}]^{1/2} \quad (\text{IV-3.23})$$

and $r'_{s;p}$ is implicitly given by the relationship

$$p = F_{R'_S}(r'_{s;p}) \quad (\text{IV-3.24})$$

where $F_{R'_S}(\cdot)$ is given by Eq. IV-3.18.

The results given in the previous few pages are based on the assumption that the shape factor k remains a constant. There are some indications that this may not be a reasonable assumption. Further investigation is indicated.

IV.4. Approximate Characteristics of the Response of a Class of Oscillators with Nonlinear Restoring Forces

The response to stationary Gaussian white noise excitation of a nonlinear one-degree-of-freedom oscillator is of course, non-Gaussian. It has been pointed out in the introduction to this chapter that the theory developed in Chapters II and III does, in principle at least, accommodate this class of structures. The basic information needed to render the results useful, essentially consists of the average crossing rates, v_d and n_d corresponding to a specified threshold \underline{d} . The latter, in turn, may be found from the joint probability density functions $p(x, \dot{x})$ and $p(r, \dot{r})$, respectively, by means of Rice's fundamental relations, Eqs. I-4.1 and I-4.3. The joint density, $p(x, \dot{x})$, of displacement and velocity of the response of certain classes of nonlinear oscillators, excited by purely random Gaussian noise, may be obtained from the solution to the Fokker-Planck equation (for example, see Ref. 8, 48, 49, 50). To compute, for a nonlinear response quantity, the joint density function $p(r, \dot{r})$, of the envelope and its time derivative, (provided the latter exists!), is a task of formidable mathematical complexity. No exact solution has yet been obtained. Lyon [35] reported an approximate solution for the case of a randomly excited hard-spring oscillator. His aim was to compare the values of the "mean clump size", (which,

recall from section II.5, he defines to be equal to the ratio v_a/n_a), corresponding to oscillators with different spring characteristics, i.e., linear vs. hardening spring. It should be noted that Lyon's [35] analysis holds only for the case where the nonlinearity effects are "small" (where "small" is to be clarified later). His estimates of the basic statistics v_0 , v_a and n_a are accepted here without much further questioning, as the (limited) purpose of this section is, essentially, to demonstrate the potential and the generality of the proposed solution to first-crossing problems and other reliability measures.

Consider the differential equation governing the displacement response of a one-degree-of-freedom system with a nonlinear restoring force

$$\ddot{x} + 2\zeta\omega_n\dot{x} + g(x) = f(t) \quad (\text{IV-4.1})$$

In particular, taking

$$g(x) = \omega_n^2 x(1 + \beta x^2) \quad (\text{IV-4.2})$$

one obtains the classical nonlinear equation of a (linearly) damped Duffing system [57]. The function $g(x)$ is shown in figure IV. 9. The Duffing equation has been shown, through experimental and theoretical studies, to describe in an ap-

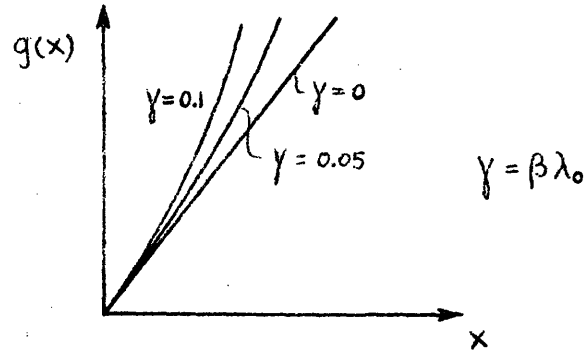


Fig.IV. 9. Hardening Spring Characteristics.

proximate way the periodic vibrations of beams and plates (see for example, Ref. 8). The constant β introduces the effect of the nonlinearity, and the other symbols in Eqs. IV-4.1 and IV-4.2 have the same meaning as in Eq. IV-2.2. It will be useful to express the equations of motion, Eqs. IV-4.1 and IV-4.2 in terms of $y = x/\sqrt{\lambda_0}$, in which $\sqrt{\lambda_0}$ is the r.m.s. (relative) displacement corresponding to the linear response, i.e., with $\beta = 0$. The equation of motion becomes

$$\ddot{y} + 2\zeta\omega_n\dot{y} + \omega_n^2 y(1 + \gamma y^2) = \frac{1}{\sqrt{\lambda_0}} f(t) \quad (\text{IV-4.3})$$

where $\gamma = \beta\lambda_0$.

By solving the Fokker-Planck equations [11], Chuang and Kazda [58] obtained $p(y, \dot{y})$, the joint density of the normalized nonlinear response $Y(t)$ and its time derivative $\dot{Y}(t)$, for a stationary white noise Gaussian input. Lyon and Smith [50, 35] found approximate expressions for other pertinent statistics needed here. To evaluate the first-crossing proba-

bility corresponding to a D type barrier, under random starting conditions, we need the decay rate or failure rate α_D .

From Eq. III-4.5,

$$\alpha_D = 2v_0 \frac{1 - \exp\{-n_d/2v_d\}}{(v_0/v_d) - 1} \quad (\text{IV-4.4})$$

where d is the specified threshold level. The failure criterion is, $|X| \geq d$, or written in terms of the normalized threshold value, $r = d/\sqrt{\lambda_0}$, it becomes, $|Y| \geq r$.

We now summarize the pertinent results due to Lyon and Smith [50, 35]. The rate of zero up-crossing has the form

$$v_0 = \frac{\omega_n}{2\pi} \frac{1}{K(\gamma)} \quad (\text{IV-4.5})$$

where the function $K(\gamma)$ is shown in figure IV.10. Its precise (and fairly complex) mathematical form [35] will be of no interest here. Not unexpectedly, the nonlinearity has the effect of increasing the rate of zero crossings, or the average

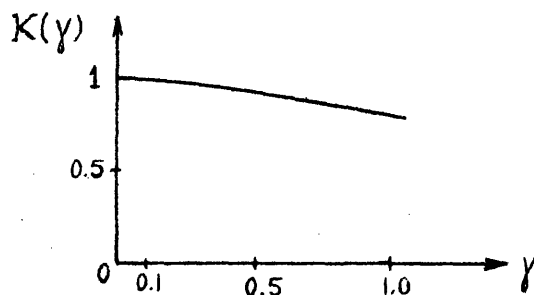


Fig. IV.10

frequency, of the hard-spring oscillator. Note, however, that, for small values of γ , say, of the order of 0.2, there will only be a few percent increase in the average frequency. It has also been established [50, 35] that

$$\frac{\nu_d}{\nu_0} = \exp\left\{-\frac{r^2}{2}\left(1+\frac{1}{2}\gamma r^2\right)\right\} \quad (\text{IV-4.6})$$

where r is the normalized threshold value, $r=d/\sqrt{\lambda_0}$. When $\gamma=0$, Eq. IV-4.6 reduces to the well-known "Gaussian" result. Lyon [35] also found the following approximate relationship between the ratios n_d/ν_d and $(n_d/\nu_d)_{LO}$, respectively corresponding to the hard-spring oscillator and to the linear oscillator

$$\frac{n_d}{\nu_d} = \left(\frac{n_d}{\nu_d}\right)_{LO} K(\gamma) \left[1+\frac{1}{2}\gamma r^2\right]^{1/2} \quad (\text{IV-4.7})$$

This (approximate) expression is valid only for values of γ less than 0.2. The ratio $(n_d/\nu_d)_{LO}$ is given by Eq. I-5.4,

$$\left(\frac{n_d}{\nu_d}\right)_{LO} = kr \quad (\text{IV-4.8})$$

where k is the spectral density shape factor, associated with the response of a linear ($\gamma=0$) oscillator to Gaussian white noise excitation. Its dependence on the damping ratio ζ is studied in section IV-2. From Eqs. IV-4.7 and IV-4.8, it follows that

$$\frac{n_d}{v_d} = K(\gamma) kr [1 + \frac{1}{2}\gamma r^2]^{1/2} \quad (\text{IV-4.9})$$

Inserting IV-4.5, IV-4.6 and IV-4.9 into the expression for the decay rate α_D , given by Eq. IV-4.4, one obtains

$$\alpha_D = \frac{\omega_0}{\pi K(\gamma)} \frac{1 - \exp\{-K(\gamma) \frac{kr}{2} [1 + \frac{1}{2}\gamma r^2]^{1/2}\}}{\exp\{\frac{r^2}{2} (1 + \frac{1}{2}\gamma r^2)\} - 1} \quad (\text{IV-4.10})$$

Finally, by making the following substitution,

$$r_* = r (1 + \frac{1}{2}\gamma r^2)^{1/2} \quad (\text{IV-4.11})$$

the decay rate α_D of the first-passage density of the hard-spring oscillator becomes,

$$\alpha_D = \frac{\omega_0}{\pi K(\gamma)} \frac{1 - \exp\{-K(\gamma) kr_*/2\}}{\exp\{r_*^2/2\} - 1} \quad (\text{IV-4.12})$$

We already noted that, for $\gamma < 0.2$, $K(\gamma)$ will only be slightly different from 1. Observe that, if we take $K(\gamma) = 1$, then Eq. IV-4.12 has the form of Eq. IV-2.9, which expresses the decay rate α_D , of the first-passage density of the corresponding linear oscillator (Eq. IV-4.3, but with $\gamma = 0$). The essential difference is that r is substituted by r_* .

In fact, it may further be shown, using Eq. III-4.7, that A_D , the probability that $|Y(0)| < r$, (i.e., the probability that failure is not instantaneous), for a stationary start, has the form

$$\begin{aligned}
A_D = 1 - v_d/v_o &= 1 - \exp\left\{-\frac{r^2}{2}\left(1 + \frac{1}{2}\gamma r^2\right)\right\} \\
&= 1 - \exp\{-r_*^2/2\}
\end{aligned}
\tag{IV-4.13}$$

where r_* is given by Eq. IV-4.11.

The net result of the above analysis is surprisingly simple. Presume that we seek the probability of first passage across a D type barrier, $|X(t)|=d$, where $X(t)$ represents the response of a hard-spring oscillator to stationary white noise Gaussian excitation. It suffices to compute the first-passage probability for the associated linear oscillator (,i.e., $\beta=\gamma=0$,) but with an elevated threshold level d_* . The ratio d_*/d may be found from equation IV-4.11,

$$\frac{d_*}{d} = \frac{r_*}{r} = \left(1 + \frac{1}{2}\gamma r^2\right)^{1/2}
\tag{IV-4.14}$$

γ is the non-dimensional "nonlinearity-factor", restricted here to values less than 0.2; r is the normalized threshold level, i.e., $r=d/\sqrt{\lambda_0}$, where $\sqrt{\lambda_0}$ is the r.m.s. value of the associated linear response process. As noted before, the small increase in the average frequency, which is given in Eq. IV-4.5, is neglected. The ratio d_*/d is plotted in figure IV.11 as a function of r for several values of γ . Consider the following example. Let γ be equal to 0.1 and let the specified threshold

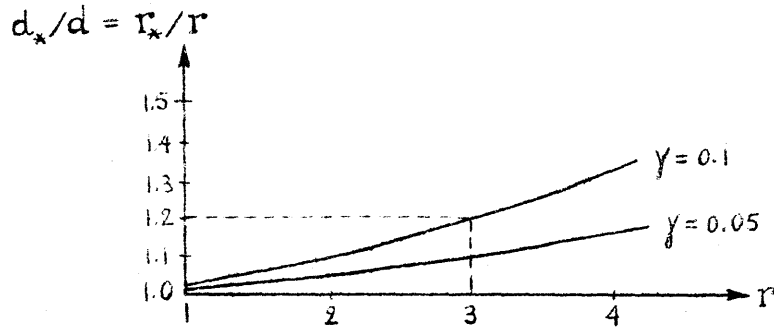


Fig.IV.11. "Threshold Amplification Factor".

value d be three times larger than the linear response r.m.s. value, i.e., $r=3$. The new "fictitious" threshold level d_* may be computed from Eq. IV-4.14,

$$\begin{aligned} d_* &= 3\sqrt{\lambda_0} \left(1 + \frac{1}{2} \times 0.1 \times 3^2\right)^{1/2} \\ &= 3.6\sqrt{\lambda_0} \end{aligned} \quad (\text{IV-4.15})$$

The probability that the hard-spring oscillator response will cross the threshold $3\sqrt{\lambda_0}$, during a given time interval, is approximately the same as the probability that the response of the associated linear oscillator will cross the threshold $3.6\sqrt{\lambda_0}$, during the same time interval.

The above described method for solving first-crossing problems for "lightly" nonlinear narrow-band oscillators seems very promising. Briefly, the idea is to express the basic statistics, v_0 , v_d and n_d , and, subsequently, the first passage probability, in terms of the mechanical properties of the non-

linear oscillator. It may then be possible, (as in the case treated,) to reduce the problem to a linear oscillator first-crossing problem, essentially by defining an "equivalent" threshold level and possibly also an "equivalent" average frequency. It is not claimed that the particular results, for the hard-spring oscillator, necessarily represent a final answer. The proposed methodology, however, should lead to results that are quite similar.

CHAPTER V

OTHER FAILURE CRITERIA AND MEASURES OF RELIABILITY

V.1. Introduction

The preceding chapters primarily deal with one very important mechanism of failure, i.e., that which postulates failure when the dynamic response quantity $X(t)$ first reaches a maximum allowable value $X=a$. Recall that our solution to the problem of determining the statistical properties of the time to first passage of a threshold $X=a$ is essentially based on the concept of a two-state Markov process. Take, for example, the case of a D type-barrier. As long as the peaks of $|X(t)|$ remain below the double barrier $|X|=a$, the two-state process $D_a(n)$ will have a value zero. However, as soon as the response $X(t)$ crosses the threshold it is no longer of interest as failure is postulated to have occurred, i.e. state 1 is an absorbing state. In a flow graph representation, (see, for example, reference [23]) this situation may be depicted by a single directed arc joining the nodes labeled zero and one, which represent the two states. See figure V.1.

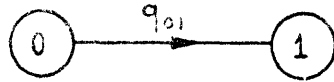


Fig.V.1. Flow Graph Representation: First Crossing Failure.

The quantity q_{01} , called the intensity of transition from state 0 to state 1, is equal to the decay rate in the reliability function corresponding to the particular threshold configuration of interest. For example, for a D type barrier using the modified (continuous time) two-state process $D_a'(t)$, we have $q_{01} = \alpha_D$ where α_D is given by Eq. III-4.5. It is clear that the two-state processes, introduced in chapter II, contain more probabilistic information than that needed to solve first crossing problems. They provide the groundwork for analyzing other potentially useful mechanisms of failure.

Often the response continues to be of interest after the first crossing of a predetermined threshold has occurred. In particular a designer might be more interested in the question: "How often do the peaks cross the threshold during a time interval 0 to T?" In that case we need to specify, not only the characteristics of the time to make a transition from state 0 to 1, but also of the time it takes to return from state 1 to state 0. The two-state Markov model is depicted in

the flow-graph, figure V.2

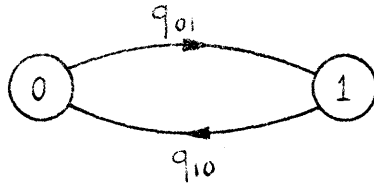


Fig.V.2. Flow Graph Representation: Fractional Occupation Time.

Again the transition intensities q_{01} and q_{10} have values which depend on the particular threshold configuration and threshold value considered. For example, for a D type barrier, $|X|=a$, we have $q_{01}=\alpha_D$ where α_D is given by Eq.III-4.5 and $q_{10}=\beta_D=\frac{1}{E[T_{1,E}]}$, where $E[T_{1,E}]$ is given by Eq. II-2.4.

A measure of the quality of performance that is dealt with in some detail in the first part of this chapter, is the fraction of peaks of the response process that exceed a specified threshold level in a given time. Failure is then defined to occur when the fraction of peaks becomes greater than a prescribed fraction. This failure criterion is essentially the same as that proposed by Spence and Luhrs [59]. They presume the performance to be unacceptable when the average number of crossings of the threshold per unit time exceeds a prescribed rate. Another closely related measure is the fraction of total time for which the response envelope process $R(t)$ exceeds the threshold. These fractions are random variables whose ensemble mean and variance are reported in section V-2.

In the second part of this chapter the information gathered in chapter II on the properties of the two-state processes $E_a(t)$ and $D'_a(t)$ is used, to derive and, in a sense, re-interpret the probability distribution of the envelope and of the peaks in a narrow-band stationary process. It is also seen to lead to an uncommon way of viewing the problem of fatigue damage accumulation. All results are found to agree with those available. This material is included here mainly to demonstrate one of the major features of the two-state representation, proposed in chapter II, namely, that it leads to a consistent and conceptually unified set of explicit analytical expressions for all common reliability measures related to narrow-band vibration.

In the final part of the chapter the concept of a two-state process is expanded. It is shown that a simple three state Markov process may be a satisfactory mathematical model for analyzing certain important failure mechanisms which have hitherto not received the attention they deserve, mainly due to a lack of analytical capability. In general, a Markov model may be expected to be satisfactory in cases where it is reasonable to assume that the system dependence upon the response history is completely characterized by the present state of the system, i.e., future behavior depends only on the present state. This is seldom strictly true. The response of a simple linear oscillator, for example, even to memory-less (white

noise) input, is a vector-valued process. However, the usefulness of two-state Markov processes to model failure criteria related to linear oscillator response, has been demonstrated quite convincingly in the preceding chapters. It is therefore believed that multiple state Markov processes may prove to be equally useful in modeling more complex (but also more realistic) failure mechanisms.

V.2. Failure Criteria Based on Fractional Occupation Time

A. Mean and Variance of the Fraction of the Time the Envelope Exceeds a Fixed Threshold

Recall that the continuous time two-state process $E_a(t)$ is based on the envelope $R(t)$ of a stationary random process $X(t)$. The envelope $R(t)$ is below the threshold $R=a$ for a random time $T_{0,E}$, and makes excursions of random length $T_{1,E}$ above the threshold. See figure II.3. The means $E[T_{0,E}]$ and $E[T_{1,E}]$ are derived in section II.2.

Let $\gamma_T(a)$ be the fraction of time the envelope process $R(t)$ exceeds the threshold, $R=a$, during the interval 0 to T , i.e., the fraction of time for which the two-state process $E_a(t)$ has the value "one". The fraction $\gamma_T(a)$ may be represented as a temporal average [44]

$$\gamma_T(a) = \frac{1}{T} \int_0^T E_a(t) dt \quad (V-2.1)$$

There is no contribution to $\int_0^T E_a(t) dt$ during time intervals for which $E_a(t)=0$, i.e., when the envelope is below the threshold. The times during which the envelope exceeds the threshold are simply added. Some results may be stated that do not require an assumption regarding the form of the density functions of $T_{0,E}$ and $T_{1,E}$. In particular, the following limit theorem of probability, due to Rényi [60], may be called upon to obtain the mean and variance and asymptotic distribution of the fraction $\gamma_T(a)$. The following assumptions need to be made

- i) the times $T_{0,E}$ and $T_{1,E}$ are independent random variables
- ii) $T_{0,E}$ and $T_{1,E}$ have finite variances

Note that these assumptions are no more restrictive than those made in chapters II and III. The theorem asserts that the fraction $\gamma_T(a)$ is asymptotically, for $T \rightarrow \infty$, normally distributed with mean and variance given below,

$$\lim_{T \rightarrow \infty} E[\gamma_T(a)] = \frac{E[T_{1,E}]}{E[T_{1,E}] + E[T_{0,E}]} \quad (V-2.2)$$

$$\lim_{T \rightarrow \infty} \text{Var}[\gamma_T(a)] = \frac{1}{T} \frac{E^2[T_{1,E}] \text{Var}[T_{0,E}] + E^2[T_{0,E}] \text{Var}[T_{1,E}]}{(E[T_{1,E}] + E[T_{0,E}])^3} \quad (V-2.3)$$

The above expressions for mean and variance are also valid for small and moderate values of T , when the starting condi-

tions are random or stationary. Substituting $E[T_{0,E}]$ and $E[T_{1,E}]$ by their values given in Eqs. II-2.4 and II-2.5, we have, in the case of a stationary start,

$$E[\gamma_T(a)] = v_a/v_0 \quad (V-2.4)$$

To evaluate the variance of $\gamma_T(a)$, estimates of $\text{Var}[T_{0,E}]$ and $\text{Var}[T_{1,E}]$ are needed. Let $T_{0,E}$ and $T_{1,E}$ be assumed to be exponentially distributed, with means $E[T_{0,E}] = 1/\alpha_E$ and $E[T_{1,E}] = 1/\beta_E$, respectively. (In other words, the two-state process $E_a(t)$ becomes a Markov process.) Then, using $\text{Var}[T_{0,E}] = 1/\alpha_E^2$ and $\text{Var}[T_{1,E}] = 1/\beta_E^2$, Eq. V-2.3 becomes

$$\text{Var}[\gamma_T(a)] = \frac{1}{T} \frac{2\alpha_E\beta_E}{(\alpha_E + \beta_E)^3} \quad (V-2.5)$$

Again inserting Eqs. II-2.4 and II-2.5, one obtains,

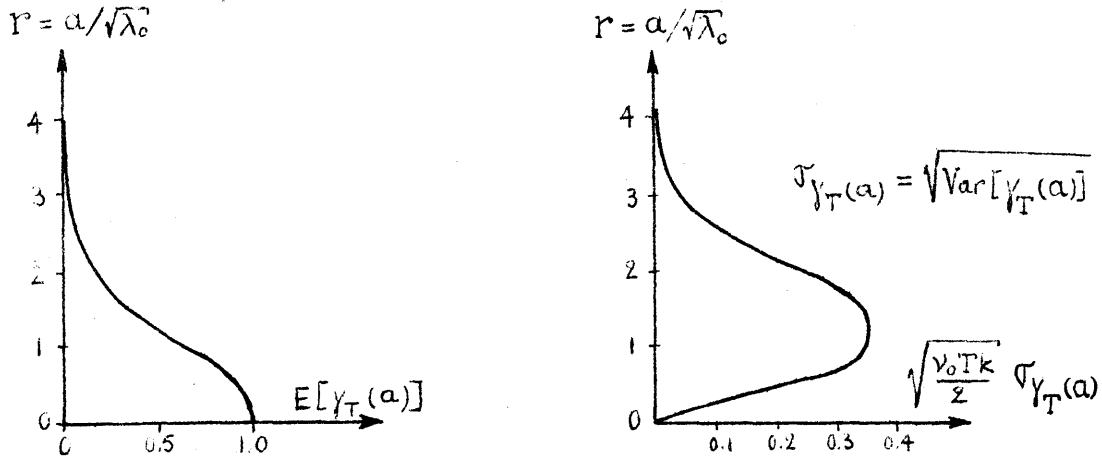
$$\text{Var}[\gamma_T(a)] = \frac{1}{v_0 T} \frac{2v_a}{n_a v_0} (1 - v_a/v_0)^2 \quad (V-2.6)$$

For Gaussian processes, and under stationary starting conditions, the mean and variance of the fraction of time the envelope process exceeds the threshold $X=a$ during the interval 0 to T , respectively take the form

$$E[\gamma_T(a)] = \exp\{-a^2/2\lambda_0\} = \exp\{-r^2/2\} \quad (V-2.7)$$

$$\text{Var}[\gamma_T(a)] = \frac{1}{v_0 T} \frac{2 \exp\{-r^2/2\}}{kr} (1 - \exp\{-r^2/2\})^2 \quad (\text{V-2.8})$$

where $r = a/\sqrt{\lambda_0}$ and k is the spectral density shape factor. The above expressions are plotted in figures V.3 and V.4. Note



Figs.V.3 and 4. Mean and Standard Deviation of the Fraction of Time the Envelope Exceeds the Threshold during a Time Interval 0 to T.

that $\text{Var}[\gamma_T(a)] \rightarrow 0$ when $r = a/\sqrt{\lambda_0} \rightarrow 0$ or when $r \rightarrow \infty$. There is little uncertainty about the value of $\gamma_T(r)$ corresponding to very high or to very low threshold levels. Of course, for $r \rightarrow 0$, $E[\gamma_T(a)] \rightarrow 1$ and for $r \rightarrow \infty$, $E[\gamma_T(a)] \rightarrow 0$. It is also of interest to note that the statistics of $1 - \gamma_T(a)$, the fraction of time the envelope process spends below the threshold $R = a$ during the interval 0 to T, is simply related to the characteristics of $\gamma_T(a)$, as follows

$$\begin{aligned}
 E[1-\gamma_T(a)] &= 1-E[\gamma_T(a)] \\
 \text{Var}[1-\gamma_T(a)] &= \text{Var}[\gamma_T(a)]
 \end{aligned}
 \tag{V-2.9}$$

The coefficient of variation (c.o.v.) of $\gamma_T(a)$ has the form

$$\frac{\sqrt{\text{Var}[\gamma_T(a)]}}{E[\gamma_T(a)]} = \frac{1}{\sqrt{\nu_0 T}} \frac{\sqrt{2}}{\sqrt{k}} \frac{\exp\{r^2/4\}}{r^{1/2}} (1-\exp\{-r^2/2\}) \tag{V-2.10}$$

It may be seen as a product of three factors:

i) the c.o.v. decreases in proportion to $1/\sqrt{\nu_0 T}$, where $\nu_0 T$ is the expected total number of cycles.

ii) the c.o.v. decreases in proportion to $1/\sqrt{k}$, where k is the spectral density shape factor, defined by Eq. I-5.3. If the stationary narrow-band process represents the response of a lightly damped one-degree-of-freedom system to wide-band Gaussian excitation, then we may take $k=2\sqrt{2}\zeta^{1/2}$ where ζ is the damping ratio of the oscillator. The c.o.v. is seen to increase for decreasing ζ , but the dependence is rather weak.

iii) the effect of changing the value of the threshold level on the c.o.v. of the fraction $\gamma_T(a)$, may be seen from figure V.5.

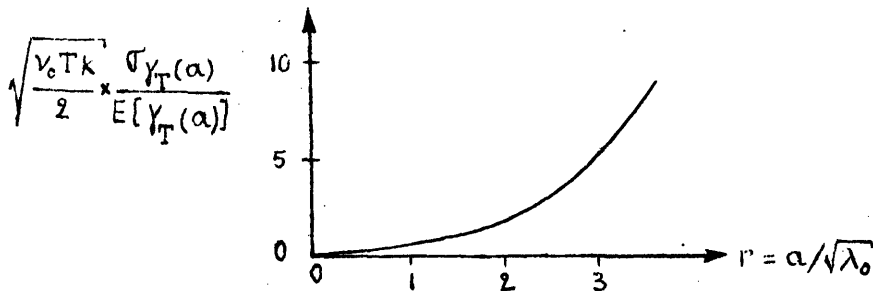


Fig.V.5. The Coefficient of Variation of the Fraction $\gamma_T(a)$.

Starting Conditions Other Than Random

Let q be the probability of finding the envelope below the threshold at the start of the observation period, i.e. the probability that the two-state process has a value zero at time $t=0$.

$$q = \text{Prob}[E_a(0)=0] \quad (\text{V-2.11})$$

Again assume the holding times $T_{0,E}$ and $T_{1,E}$ to be exponentially distributed with mean values $1/\alpha_E$ and $1/\beta_E$, respectively. The value of the two-state process at time t , $t \geq 0$, is of course a random variable. Its expected value (ensemble average) may be expressed in terms of the parameters α_E and β_E , the starting characteristic q and the time t . Parzen [44] shows that

$$E[E_a(t)] = \frac{\alpha_E}{\alpha_E + \beta_E} - \left(q - \frac{\beta_E}{\alpha_E + \beta_E} \right) e^{-(\alpha_E + \beta_E)t} \quad (\text{V-2.12})$$

Note that the effect of the starting condition fades away for $t \rightarrow \infty$. The expected value of $\gamma_T(a)$, the fraction of time the envelope exceeds the threshold $X=a$ in the interval 0 to T , may be obtained from Eqs. V-2.1 and V-2.12,

$$\begin{aligned} E[\gamma_T(a)] &= \frac{1}{T} \int_0^T E[E_a(t)] dt \\ &= \frac{\alpha_E}{\alpha_E + \beta_E} - \frac{1}{T} \left(q - \frac{\beta_E}{\alpha_E + \beta_E} \right) \frac{1}{\alpha_E + \beta_E} \left[1 - e^{-(\alpha_E + \beta_E)T} \right] \end{aligned} \quad (\text{V-2.13})$$

The value of q , which corresponds to stationary or random starting conditions, is that which eliminates the dependence on T of the ensemble averages $E[\gamma_T(a)]$. Let that value be $q_{\text{stat.}}$. We have

$$q_{\text{stat.}} = \frac{\beta_E}{\alpha_E + \beta_E} \quad (\text{V-2.14})$$

Replacing α_E and β_E , Eq. V-2.14 becomes

$$q_{\text{stat.}} = 1 - v_a/v_0 \quad (\text{V-2.15})$$

Taking $q=q_{\text{stat.}}$ it is easy to check that the expression for $E[\gamma_T(a)]$ in Eq. V-2.13 does reduce to the estimate obtained earlier (Eq. V-2.4).

B. Mean and Variance of the Fraction of Peaks Exceeding a Fixed Threshold

A similar analysis may be performed for the modified (continuous time) two-state process $D'_a(t)$. Recall that it is derived from the discrete time two-state process $D_a(n)$ by taking

$$\begin{aligned} E[T_{0,D}] &= 1/\alpha_D = E[N_{0,D}]/2v_0 \\ E[T_{1,D}] &= 1/\beta_D = E[N_{1,D}]/2v_0 \end{aligned} \quad (\text{V-2.16})$$

The quantity of interest here is $\zeta_T(a)$, the fraction of the peaks of $|X(t)|$, in the interval 0 to T, which exceed the threshold $|X|=a$.

$$\zeta_T(a) = \frac{1}{T} \int_0^T D'_a(t) dt \quad (V-2.17)$$

As the occurrence of the peaks is, in fact, a discrete time process, the right side of Eq. V-2.17 is really meant to be a good approximation for

$$\frac{1}{N} \sum_{n=1}^N D_a(n) \quad (V-2.18)$$

where N is the random number of peaks occurring in the interval 0 to T. When properly interpreted, all the results of the preceding section still hold here, if the subscript D is substituted for E and $\gamma_T(a)$ for $\zeta_T(a)$. In particular, when the rates α_D and β_D are replaced by their actual values (see Eqs. III-4.5, V-2.16 and III-5.4), the following estimates are obtained for the mean and variance of the fraction $\zeta_T(a)$, in the case of a stationary start,

$$E[\zeta_T(a)] = v_a/v_0 \quad (V-2.19)$$

$$\text{Var}[\zeta_T(a)] = \frac{1}{v_0 T} \frac{v_a/v_0}{(1 - \exp\{-n_a/2v_a\})} (1 - v_a/v_0)^2 \quad (V-2.20)$$

It is easy to establish the following relations between the statistics of the fractions $\gamma_T(a)$ and $\zeta_T(a)$,

$$E[\zeta_T(a)] = E[\gamma_T(a)] \quad (V-2.21)$$

$$\begin{aligned} \text{Var}[\zeta_T(a)] &= \frac{n_a/2\nu_a}{1-\exp\{-n_a/2\nu_a\}} \text{Var}[\gamma_T(a)] \\ &= \frac{1}{(1-\rho_D)} \text{Var}[\gamma_T(a)] \end{aligned} \quad (V-2.22)$$

where ρ_D is given by Eq. II-4.7, and for Gaussian random processes, by Eq. II-4.15. It may be seen from figure V.6 that

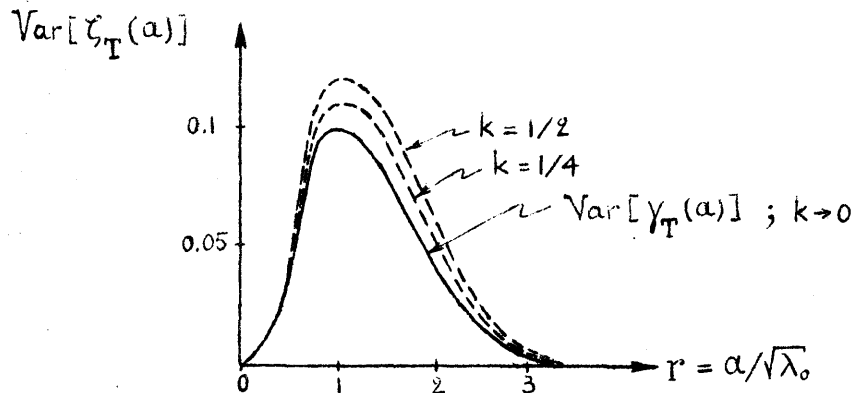


Fig.V.6. Comparison of the Variances of the Fractions $\gamma_T(a)$ and $\zeta_T(a)$.

the variance of the fraction of the peaks of $|X(t)|$ exceeding a fixed threshold, is somewhat higher than that of the time the envelope exceeds the same threshold during the same time interval. The corresponding mean values are identical.

V.3. Distribution of the Envelope and of the Peaks in a Stationary Narrow-Band Process. Fatigue Failure

Let $g_T(a)da$ be the fraction of the time interval 0 to T, during which the envelope $R(t)$ of a stationary random process $X(t)$ is between the threshold levels $R=a$ and $R=a+da$. The fractions $g_T(a)da$ and $\gamma_T(a)$ are related in the following way

$$g_T(a) = -\frac{d}{da}[\gamma_T(a)] \quad (V-3.1)$$

$$\gamma_T(a) = \int_a^{\infty} g_T(a)da \quad (V-3.2)$$

Note that Eqs. V-3.1 and V-3.2 essentially state that the random variables on both sides of the equality sign are equivalent. Recall that the fraction $\gamma_T(a)$ takes a value zero for sure when $a \rightarrow \infty$, and one for sure when $a \rightarrow 0$. For intermediate values of a it has a mean given by Eq. V-2.4 and variance given by Eq. V-2.6. $g_T(a)$ may be interpreted as being proportional to the (random) fraction of time the envelope spends in the neighborhood of the level $R=a$. Its ensemble average is found by introducing an expected value operator into Eq. V-3.1.

$$E[g_T(a)] = -\frac{d}{da} E[\gamma_T(a)] \quad (V-3.3)$$

If $X(t)$ is a Gaussian random process and assuming the starting

conditions to be random, i.e., $q=q_{\text{stat}}$. (Eq. V-2.14), we have $E[\gamma_T(a)] = e^{-a^2/2\lambda_0}$ (Eq. V-2.7). Hence

$$\begin{aligned} E[g_T(a)] &= -\frac{d}{da} [\exp\{-a^2/2\lambda_0\}] \\ &= \frac{a}{\lambda_0} [\exp\{-a^2/2\lambda_0\}], \quad a \geq 0 \end{aligned} \quad (\text{V-3.4})$$

One recognizes the right side of Eq. V-3.4 to represent the Rayleigh probability density function. It has been shown [1,7] to be the density function for the envelope of a narrow-band normal process.

A similar argument may be used to obtain the expected likelihood of the height of the peaks of $|X(t)|$. Of all the peaks of $|X(t)|$ occurring in the interval $(0, T)$, let $g'_T(a)da$ be the fraction for which the height of the peak lies between a and $a+da$. We have

$$g'_T(a) = -\frac{d}{da} [\zeta_T(a)] \quad (\text{V-3.5})$$

$$\zeta_T(a) = \int_a^{\infty} g'_T(a) da \quad (\text{V-3.6})$$

Again making use of Eqs. V-3.5, V-2.2 and V-2.7, one obtains, for Gaussian random processes, and assuming stationary starting conditions

$$\begin{aligned}
E[g'_T(a)] &= -\frac{d}{da} E[\zeta_T(a)] \\
&= -\frac{d}{da} \exp\{-a^2/2\lambda_0\} \\
&= \frac{a}{\lambda_0} \exp\{-a^2/2\lambda_0\} \quad a>0 \quad (V-3.7)
\end{aligned}$$

Again the Rayleigh distribution is obtained, which has been shown [39] to represent the probability distribution for the peaks of a narrow-band normal process.

To analyze the cumulative fatigue failure mechanism a parametric representation of the two-state model, in terms of all the possible threshold levels $X=a$, $0<a<\infty$, is required. Note that $Tg'_T(a)da$ equals the total number (during the interval 0 to T) of peaks whose height lies between the threshold levels a and $a+ada$. Adopting the Palmgren-Miner hypothesis [61, 62] of incremental damage, one may compute the value of the damage, $D(T)$, accumulated during the time period 0 to T, in the following way

$$D(T) = T \int_0^{\infty} \frac{g'_T(a)}{N(a)} da \quad (V-3.8)$$

where $N(a)$ is the number of cycles to failure in a constant-amplitude fatigue test with stress amplitude a . It is easy to show that, under random starting conditions, the mean value $E[D(T)]$ equals that obtained by Mark [7, 63]. Also, the vari-

ance, $\text{Var}[D(T)]$, may be shown, according to this approach, to be proportional to $v_0 T/k$.

V.4. Discrete State Markov Models of Failure Mechanisms Caused by a Random Vibration Environment

The two types of structural failure most commonly considered are first-excursion failures and fatigue failures. Another type, where failure is postulated to occur when the fractional occupation time exceeds a prescribed value, has also been proposed [7, 59]. It has been shown in the preceding sections that reasonable estimates of the relevant properties of all three types may be obtained by considering a simple two-state model of the type discussed in chapter II. In fact the approach has led to new and useful results for the first crossing and fractional occupation time failure modes.

Although the above mentioned performance criteria are widely accepted they do not by any means constitute a final answer to the question as to how the random response characteristics should be converted into useful reliability measures. On the basis of a few examples the author hopes to demonstrate in this section that the (well-developed) theory of continuous time discrete state Markov processes provides an excellent tool for describing and analyzing certain failure mechanisms which are essentially due to the random vibration environment.

A somewhat similar method of treating complex failure criteria in earthquake response reliability has recently been proposed by Zsuttty [64]. Here, attention focuses on some simple continuous time Markov models. The usefulness of the two-state model has already been established. Several potentially useful three-state models will be discussed next.

Model A. If the absolute response process $|X(t)|$ crosses the barrier $|X|=a_2$, the system fails irreversibly (and the response process ceases to exist). If $|X(t)|$ makes a first excursion into the range (a_1, a_2) where $a_1 < a_2$, a change occurs in the system's properties (say, stiffness decreases due to crack formation or perhaps, the ultimate capacity, barrier a_2 , is lowered). A flow graph representation of the process is shown in figure V.7.

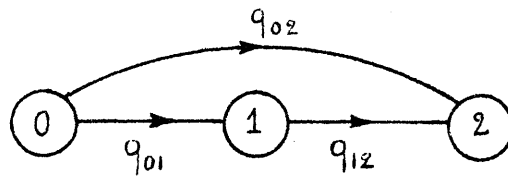


Fig.V.7. Flow Graph Representation of a Single Deterioration Process.

The three states are:

- 0 $\text{Max}_{0 \leq \tau \leq t} |X(\tau)| < a_1$
- 1 $a_1 \leq \text{Max}_{0 \leq \tau \leq t} |X(\tau)| < a_2$
- 2 $\text{Max}_{0 \leq \tau \leq t} |X(\tau)| \geq a_2$

A transition from state 0 to state 2 is said to occur when the first crossing of the lower threshold is immediately followed by a crossing of the higher threshold. The labels q_{jk} on the branches are called the transition rates. They are defined as follows: $q_{jk}\Delta t$ equals the probability that a process which presently occupies state j , will make a transition to k in the short time interval Δt . The probability of two or more state transitions in the same time interval is assumed to be zero if Δt is sufficiently small. The assumptions imply that the times between transitions are assumed to be exponentially distributed with mean rate q_{jk} .

Let us define the "failure rates":

$\alpha_{D;a_1}$: the decay rate of the first passage density corresponding to a D type barrier with a threshold value a_1 , for the undamaged structure

$\alpha_{D;a_2}$: the decay rate of the first passage density corresponding to a D type barrier with a threshold value a_2 , for the undamaged structure

$\bar{\alpha}_{D;a_2}$: the decay rate of the first passage density corresponding to a D type barrier with a threshold value a_2 , for the damaged structure.

Each of the above decay rates may be evaluated in terms of the threshold value, the structural properties and the input characteristics. For example, $\alpha_{D;a_2}$ has the form

$$\alpha_{D;a_2} = 2\nu_0 \frac{1 - \exp\{-ka_2/2\sigma_x\}}{\exp\{a_2^2/2\sigma_x^2\} - 1} \quad (V-4.1)$$

where ν_0 is the average (response) frequency, σ_x is the r.m.s. value of the response and k is the (response) spectral density shape factor. All quantities are related to the initial properties of the structure. To compute $\bar{\alpha}_{D;a_2}$, Eq. V-4.1 may still be used, but with ν_0 , k and σ_x referring to the damaged structure. The transition rates shown on the flow-graph (Fig.V.7) are related to the above defined first passage probability decay rates in the following way

$$\begin{aligned} q_{01} &= \alpha_{D;a_1} - \alpha_{D;a_2} \\ q_{02} &= \alpha_{D;a_2} \\ q_{12} &= \bar{\alpha}_{D;a_2} \end{aligned} \quad (V-4.2)$$

The quantities sought here are the probabilities $p_{jk}(t)$, that a structure which starts in state j (e.g., $j=0$, the structure is undamaged at the start) will be in state k after t seconds of stationary random response. The probabilities $p_{jk}(t)$ may be found by solving a set of differential equations [23]. The use of Laplace transforms (see, for ex. Ref.23) considerably simplifies this task. Only some of the principal results are stated here:

$$p_{00}(t) = \exp\{-(q_{01} + q_{02})t\} = \exp\{-\alpha_{D;a_1}t\} \quad (V-4.3)$$

$$\begin{aligned}
p_{01}(t) &= \frac{q_{01}}{q_{01}+q_{02}-q_{12}} [\exp\{-q_{12}t\}-\exp\{-(q_{01}+q_{02})t\}] \\
&= \frac{\alpha_{D;a_1}^{-\alpha_{D;a_2}}}{\alpha_{D;a_1}^{-\alpha_{D;a_2}}} [\exp\{-\bar{\alpha}_{D;a_2}\}-\exp\{-\alpha_{D;a_1}\}] \quad (V-4.4)
\end{aligned}$$

$$p_{02}(t) = 1-p_{00}(t)-p_{01}(t) \quad (V-4.5)$$

$$\begin{aligned}
p_{12}(t) &= \exp\{-q_{12}t\} \\
&= \exp\{-\bar{\alpha}_{D;a_2}t\} \quad (V-4.6)
\end{aligned}$$

The possibility of instantaneous damage or instantaneous failure may also be incorporated in the analysis. This will not be further pursued here, however.

Model B. If the absolute response process $|X(t)|$ crosses the barrier $|X|=a_2$ the system fails irreversibly. If it ever makes an excursion into the range (a_1, a_2) a loss or damage of non-structural nature is suffered. The structural properties (and hence, the response characteristics) are not affected. The loss may be due to sensitive equipment becoming non-operational, nuisance to people, cracking of finishings, etc. The state definition is identical to that given above for model A. The main difference between Model A and Model B, apart from the economic aspects (which will be discussed at some length in Chapter VII), is that, in the present case

$$q_{12} = q_{02} = \alpha_{D;a_2} \quad (V-4.7)$$

This is so because the structural properties are not affected by the damage. The flow graph is shown in figure V.8.

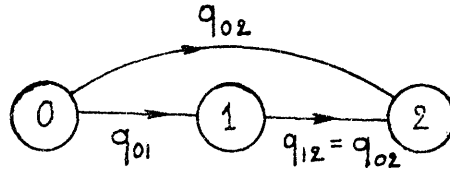


Fig.V.8. Flow Graph Representation for the Case in which the Performance Criterion Involves Both Structural and Non-Structural Damage.

Using Eqs. V-4.3 to V-4.6 and Eq. V-4.7, it is easy to show that the probabilities $p_{jk}(t)$, defined earlier, reduce to

$$\begin{aligned} p_{00}(t) &= p_{11}(t) = \exp\{-\alpha_{D;a_1} t\} \\ p_{02}(t) &= p_{12}(t) = 1 - \exp\{-\alpha_{D;a_2} t\} \\ p_{01}(t) &= 1 - p_{00}(t) - p_{02}(t) \\ p_{12}(t) &= 1 - p_{11}(t) \end{aligned} \quad (V-4.8)$$

where $\alpha_{D;a_2}$ and $\alpha_{D;a_1}$ have a form as in Eq. V-4.1.

The probabilities $p_{jk}(t)$ essentially characterize the behavior (w.r.t. the multiple-damage performance criterion) of the

structural system during a single "quasi-stationary" excitation (e.g., a single earthquake, a single flight, etc.) of given intensity and given duration t . Both three-state models that have been introduced in this section, will be further discussed in Chapter VII, where we study the occurrence pattern of a set of single isolated motions and their effect on structures.

CHAPTER VI

SOME APPLICATIONS OF THE TWO-STATE PROCESS CONCEPT IN THE ANALYSIS OF THE RESPONSE OF SIMPLE NONLINEAR HYSTERETIC SYSTEMS TO RANDOM EXCITATION

VI.1 Introduction

In chapter IV we studied some important statistical properties of the random response of elastic single-degree-of-freedom systems. Recall that this was done by specializing the more general results on first passage times obtained in chapters II and III.

For many structural and mechanical systems it is permissible to allow for plastic deformations during severe but infrequent random vibratory motions. Taking advantage of their plastic capacity often provides an efficient means of absorbing energy and for damping the response of structures or equipment subjected to dynamic loading. The growing interest in the analysis of the response of nonlinear hysteretic structures to random excitation is indicated by the large number of numerical and simulation studies that recently have been reported (e.g., Refs.65,66,67,68). Other work, (e.g.,

Refs. 69, 70] attempting at rigourously analyzing the (random) response of such systems attests to the fact that the mathematical complexity is formidable.

The stationary two-state process concept, discussed at length in chapter II, will be shown here to lead to a potentially very useful set of approximate analytical results for the statistical properties of certain important response measures of hysteretic systems. Among those: the maximum absolute displacement or the ductility factor, the plastic work dissipated, the time required for yielding to progress to the point of collapse. Several well-known (see, for example, Refs. 66, 71, 72) types of force-deformation relationships are considered. They are shown in figure VI.1.

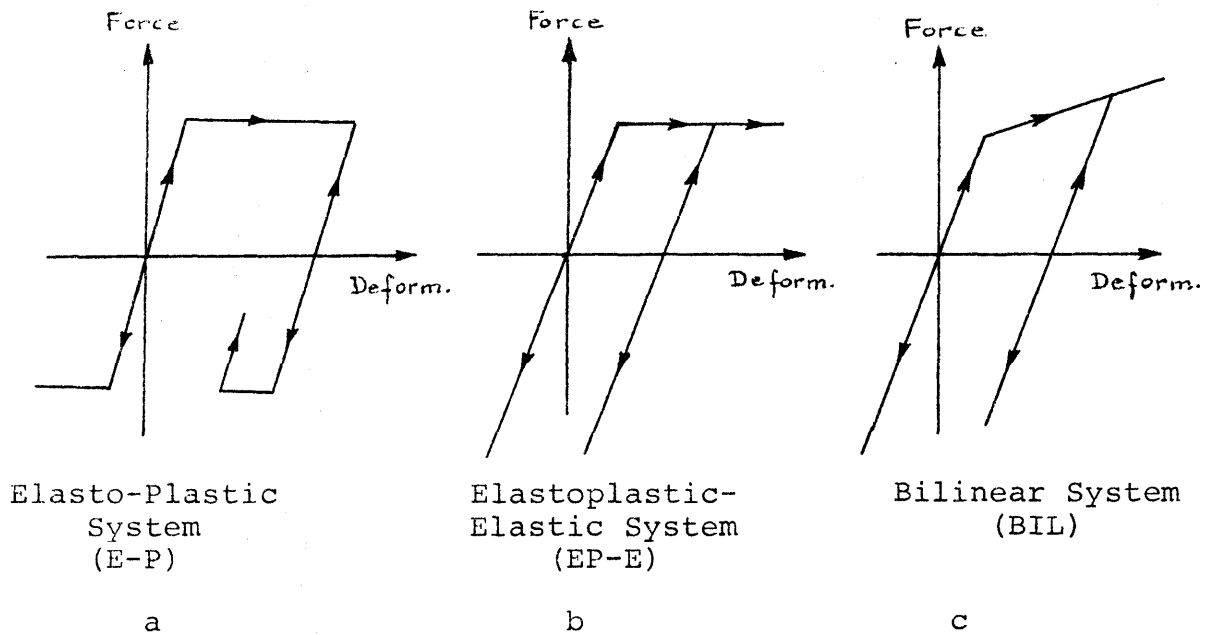


Fig.VI.1. Simple Idealized Hysteretic Systems.

The elasto-plastic (E-P) system, shown in figure VI.1a, is an idealized inelastic system whose strains are reversible. The systems, labeled b and c, have irreversible plastic strains. Following Veletsos [66], we call system b an elastoplastic-elastic (EP-E) system. Fig. VI.1c pictures a bilinear (BIL) system.

A widely accepted way of representing the response of a hysteretic system is by the ductility factor, which is defined as the ratio of the peak displacement to the displacement at yielding [66, 71, 72]. For E-P systems, however, where strain reversal is likely to occur, the ductility factor may not properly represent the state of damage of the structure [73]. For such systems, a more appropriate response measure may be the amount of energy dissipated in yielding [73, 74]. In fact, Coffin [75, 76] established, on the basis of extensive experimental evidence, that a close correlation exists between the latter measure and the fatigue life of some materials.

It will be useful, in our study of the statistical properties of these inelastic response measures, to distinguish two basically different types:

- i) measures that monotonically increase with time, e.g., the total plastic deformation in EP-E and BIL systems, the amount of energy dissipated in yielding (say, in E-P systems).
- ii) measures that may fluctuate in time, e.g. the perma-

ment set or plastic deformation in E-P systems. Measures of the first type are studied in section VI.3, those of the second type in section VI.4. The idea of approximating the inelastic response by an artificial two-regime linear process has first been suggested by Karnopp and Scharon [74]. They obtained an estimate of the average rate of energy dissipation due to yielding, for elasto-plastic systems, excited by stationary Gaussian white noise excitation. A number of new results are obtained in this chapter. They are believed to be very useful in helping one gain physical insight into the behavior, under random vibration, of important inelastic response measures, e.g., the ductility factor. Further study is needed, however, in verifying some of the results, through comparison with available data and possibly by simulation.

VI.2 The Basic Model

Let $Y(t)$ represent the stationary displacement response to broad-band Gaussian random excitation, of an EP-E (elastoplastic-elastic) structure with a yield level $Y=a$, as shown in Fig. VI.2a.

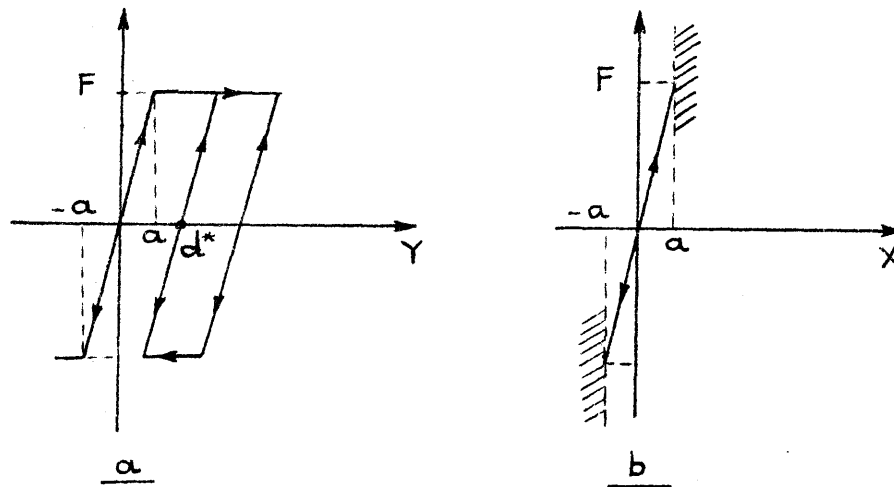


Fig.VI.2. Elasto-Plastic System and its Associated Linear System (and D type barrier).

At the start of the motion, the response of the EP-E system is identical to that of an associated linear system, shown in Fig. VI.2b. This is so until $Y(t)$ crosses the yield level for the first time. It is clear then, that problems surrounding the onset of plastic deformation are equivalent to a D type barrier first-crossing problem for the associated linear oscillator. Hence, before plastic deformations occur, the inelastic response process, $Y(t)$, is described by the linear differential equation

$$\ddot{Y} + 2\zeta\omega_n\dot{Y} + \omega_n^2 Y = F(t) \quad (\text{VI-2.1})$$

where ω_n and ζ are the natural (circular) frequency and damping ratio, respectively, of the associated linear system. In between yield level crossings, the EP-E system also behaves like a linear oscillator. In fact, presume that the most recent yield level crossing brought the total plastic deformation up to the value d^* . The total displacement will then consist of a permanent set d^* and an (linear elastic) oscillatory component $X(t)$, i.e.,

$$Y(t) = d^* + X(t) \quad (\text{VI-2.2})$$

For $d^*=0$, i.e., before any plastic yield occurred, we have $Y(t)=X(t)$. The differential equation describing the oscillatory part, $x(t)$, of the total displacement of the EP-E oscillator takes the form

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = F(t) \quad (\text{VI-2.3})$$

The process $X(t)$ may be viewed as the response of the "artificial linear system" shown in figure VI.2b. It is directly related to the inelastic response $Y(t)$ at times when d^* is fixed, i.e., in between plastic excursions. It will be useful to concentrate attention to the peaks of $X(t)$.

Recall that the response $X(t)$ of a lightly damped linear oscillator to random broad-band excitation may be described by means of a discrete two-state process $D_a(n)$, whose characteristics were discussed in chapter II. They are briefly repeated here. The process $D_a(n)$ is observed at times t_1, t_2 , etc., at which peaks of $|X(t)|$ occur, i.e., $\dot{X}(t)=0$ for $t=t_1, t_2$, etc. If $|X(t_j)|$, the absolute value of the process at the j th peak, equals or exceeds a specified level a , then the two-state process assumes the value one, i.e., $D_a(j)=1$. If the magnitude of the n th peak is below the threshold a , then $D_a(j)=0$. The proposed scheme of approximating the inelastic response by an "artificial linear process" X , will be restricted to cases where plastic deformations occur relatively infrequently, i.e., when a large fraction of the peaks of the inelastic response process occur below the yield level. It is important to note that this restriction does not imply that yield level impacts need to be isolated events, occurring approximately according to a Poisson process. To see this, recall that, in linear systems, the peaks above a threshold $X=a$, i.e., with $D_a(j)=1$, tend to occur in clumps. Let $\mu_{a,D}$ be defined as the average rate at which such clumps occur. For Gaussian processes, $\mu_{a,D}$ may be written as follows

$$\mu_{a,D} = n_a (1 - \rho_D) = 2v_a (1 - \exp\{-kr/2\}) \quad (\text{VI-2.4})$$

where n_a is the average number of envelope crossings (at posi-

tive slope) of the threshold \underline{a} , and $(1-\rho_D)$, given by Eq. II-4.7, is a reduction factor which accounts for the fact that some envelope crossings of the threshold \underline{a} are not immediately followed by peaks of $|X|$ above \underline{a} . Also, k is the spectral density shape factor and $r=a/\sigma_X$, σ_X being the (linear) response r.m.s. value ($\sigma_X=\sqrt{\lambda_0}$). The characteristics of the random number $N_{1,D}$, where $N_{1,D}=1,2,\dots$, of peaks in a clump have also been studied in chapter II. The expected clump size is given in Eq. II-4.16

$$E[N_{1,D}] = (1-\exp\{-rk/2\})^{-1} \quad (\text{VI-2.5})$$

From Eqs. VI-2.4 and VI-2.5 it follows that

$$\mu_{a,D} E[N_{1,D}] = 2\nu_a = 2\nu_0 \exp\{-r^2/2\} \quad (\text{VI-2.6})$$

The larger the mean clump size corresponding to a given (normalized) threshold level, the smaller the average rate of occurrence of clumps. Hence, if the inelastic system is characterized by a (viscous) damping ratio of, say, 1%, then the average time between clumps of plastic excursions will be (much) larger than it would be for a system with, say, $\zeta=5\%$, the normalized threshold level being constant. In fact, the average time between clumps is smallest when the average clump size tends to one. The isolated clumps will be treated here as points in time at which the permanent set d^* changes its value. Furthermore, these points which represent the times

at which d^* -jumps occur, will be assumed to arrive in a Poisson-like manner, with mean rate $\mu_{a,D}^{(*)}$. The relationship between the size of clumps in the "artificial" linear process $X(t)$ and the size of d^* -jumps in the "real" elasto-plastic process $Y(t)$ poses some yet unsolved problems. Note, however, that, if $E[N_{1,D}] \rightarrow 1$ then $\mu_{a,D} \rightarrow 2v_a$. All results in the following sections are in fact derived for this limiting case. A more exact analysis is needed to investigate the dependence of the inelastic response measures on the damping ratio, particularly for very low values of ζ , when $E[N_{1,D}]$ may be substantially larger than one. This is further discussed at the end of Section VI.3.

An EP-E (elastoplastic-elastic system) corresponds to a B type barrier (see Fig. VI.3) in the same way as an E-P system corresponds to a D type barrier. The mean rate of d^* -jumps is designated by $\mu_{a,B}$, a being the yield level. The following relations may be shown to hold

$$\mu_{a,B} = v_a (1 - \exp\{-rk\}) \quad (\text{VI-2.7})$$

$$E[N_{1,B}] = (1 - \exp\{-rk\})^{-1} \quad (\text{VI-2.8})$$

$$\mu_{a,B} E[N_{1,D}] = v_a = v_0 \exp\{-r^2/2\} \quad (\text{VI-2.9})$$

(*) This is in line with our earlier assumption that the time $T'_{0,D}$ is exponentially distributed. Since we restrict ourselves to cases where $E[T'_{1,D}] \ll E[T'_{0,D}]$, the sum $T'_{1,D} + T'_{0,D}$ may also be expected to be approximately exponentially distributed.

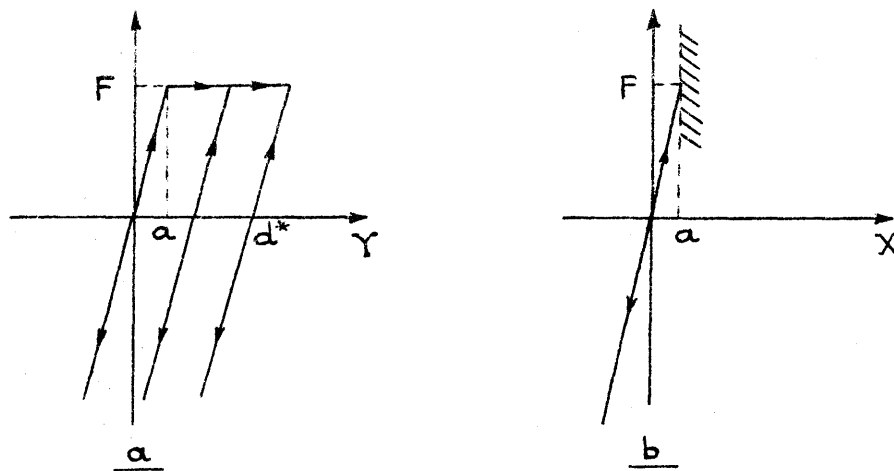


Fig.VI.3. Elastoplastic-Elastic System and its Associated Linear System (and B type barrier).

All earlier remarks about the E-P response also apply in this case. In particular, the limiting value of $\mu_{a,B}$, the mean occurrence rate of d^* -jumps, becomes ν_a .

Amount of Plastic Deformation Due to a First Excursion into the Plastic Domain

The results outlined here are due to Karnopp and Scharon [74]. Suppose that the elasto-plastic response process $Y(t)$ has been below the yield level for some time. Let the current value of the permanent set be d^* , as shown in Fig. VI.4. During that time the oscillating part of the process is governed by the linear differential equation, Eq. VI-2.3. Let M and K respectively denote the mass and the (initial) spring stiffness of the inelastic system. When the pro-

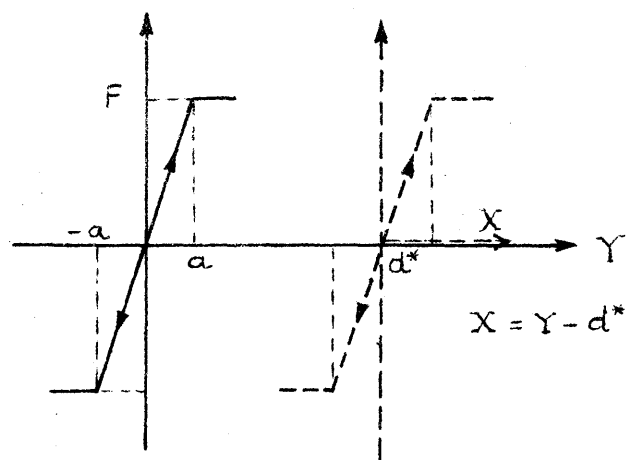


Fig.VI.4

cess $X(t)$ impacts a barrier a corresponding to the yield level, it is known to have a potential energy $\frac{1}{2}Ka^2$. The kinetic energy $\frac{1}{2}M\dot{X}^2$ is a random variable which depends on the impact velocity \dot{X} . If the excitation is Gaussian white noise then the stationary response $X(t)$ and its derivative $\dot{X}(t)$ are independent Gaussian random variables [7]. Therefore the conditional distribution of \dot{X} , given $X=a$, is simply Gaussian and does not depend on a . Also, since \dot{X} has a zero mean, it follows that $E[\dot{X}^2] = \sigma_{\dot{X}}^2$, $\sigma_{\dot{X}}$ being the standard deviation of the linear response velocity. Hence, the expected kinetic energy at impact is given by

$$E[\text{K.E}] = E\left[\frac{1}{2}M\dot{X}^2\right] = \frac{1}{2}M\sigma_{\dot{X}}^2 \quad (\text{VI-2.10})$$

Karnopp and Scharon argue that all the kinetic energy at impact will be released by yielding action. (Note that the

influence of the forcing function during the excursion is ignored. The higher the normalized yield level the more accurate the result is likely to be.) The elasto-plastic response process $Y(t)$ will only then reverse itself (and re-enter the "linear" domain) when $\dot{Y}(t)$ becomes zero. The approximate absolute plastic deformation $|\Delta|$, corresponding to an impact velocity \dot{X} may be obtained from the energy equation

$$|\Delta|_F = \frac{M\dot{X}^2}{2} \quad (\text{VI-2.11})$$

Using the relations $\omega_n^2 = K/M$ and $F = Ka$, one obtains

$$|\Delta| = \frac{1}{2\omega_n^2 a} \dot{X}^2 \quad (\text{VI-2.12})$$

If the excitation is a stationary white noise, then we have $\sigma_{\dot{X}}^2 = \omega_n^2 \sigma_X^2$, where σ_X and $\sigma_{\dot{X}}$ denote the r.m.s. values of the linear response displacement and velocity respectively. The expected absolute plastic deformation is [74]

$$E[|\Delta|] = \frac{1}{2\omega_n^2 a} E[\dot{X}^2] = \frac{\sigma_X^2}{2a} \quad (\text{VI-2.13})$$

where \underline{a} designates the yield level.

VI.3. Properties of Irreversible Inelastic Response Measures

The Amount of Energy Dissipated Due to Yielding in E-P Systems

Karnopp and Scharon [74] combined the result of the preceding analysis with an approximate measure of the frequency of excursions into the plastic domain, to obtain the expected value, $E[D_t]$, of the sum of the absolute deformations, $|\Delta_i|$, accumulated during the time interval 0 to t . Assuming that single plastic excursions occur independently, according to a Poisson process, with mean rate $2\nu_a$, they estimate

$$E[D_t] = 2\nu_a t E[|\Delta_i|] \quad (\text{VI-3.1})$$

Inserting Eq. VI-2.13 into Eq. VI-3.1, and taking $r = a/\sigma_x$, a being the yield level, and σ_x the ("linear") r.m.s. displacement, they obtain

$$E[D_t] = \nu_o t \sigma_x \frac{1}{r} e^{-r^2/2} \quad (\text{VI-3.2})$$

Using the same general model, a number of additional results will now be obtained, again for the case where the mean rate of occurrence of yield level crossings may be estimated by $2\nu_a$. First it is observed that, since the impact velocity \dot{X} is a zero-mean Gaussian random variable, $\dot{X}/\sigma_{\dot{X}}$ will be a standard Gaussian random variable, and therefore, $\dot{X}^2/\sigma_{\dot{X}}^2$

will have a χ^2 -distribution with one-degree-of-freedom (or, a χ_1^2 -distribution). The total of the absolute plastic deformations, accumulated in the time interval 0 to t, is

$$D_t = |\Delta_1| + |\Delta_2| + \dots + |\Delta_{N_t}| \quad (\text{VI-3.3})$$

It is a sum of a random number, N_t , of identically distributed contributions. Assuming the sizes of these (infrequent) contributions to D_t to be mutually independent^(*) and (also) independent^(*) of the number of yield level impacts, one may write [44]

$$E[D_t] = E[N_t]E[|\Delta_i|] \quad (\text{VI-3.4})$$

$$\text{Var}[D_t] = E[N_t]\text{Var}[|\Delta_i|] + \text{Var}[N_t]E^2[|\Delta_i|] \quad (\text{VI-3.5})$$

Eq. VI-3.4 leads to Karnopp and Scharon's [74] estimate of the average of D_t (see Eq. VI-3.1). We will now evaluate the above expression for the variance of D_t . If the yield level crossings are Poisson distributed, with approximate mean rate $2v_a$, then

$$E[N_t] = \text{Var}[N_t] = 2v_a t \quad (\text{VI-3.6})$$

(*) Actually, one only needs N_t and $|\Delta_i|$ to be uncorrelated in Eq. VI-3.4. In Eq. VI-3.5, one needs N_t and $|\Delta_i|$, and the $|\Delta_i|$'s uncorrelated.

It may be seen from Eq. VI-2.12, that $|\Delta_i|$ is proportional to \dot{X}^2/σ_x^2 . The latter is a χ_1^2 -distributed random variable. Hence, its expected value equals 1 and its variance is 2. Using Eq. VI-2.12, it follows that

$$E[|\Delta_i|] = \left(\frac{\sigma_x}{2a}\right) E\left[\frac{\dot{X}^2}{\sigma_x^2}\right] = \left(\frac{\sigma_x}{2a}\right) \quad (\text{VI-3.7})$$

$$\text{Var}[|\Delta_i|] = \left(\frac{\sigma_x}{2a}\right)^2 \text{Var}\left[\frac{\dot{X}^2}{\sigma_x^2}\right] = 2\left(\frac{\sigma_x}{2a}\right)^2 \quad (\text{VI-3.8})$$

Inserting Eqs. VI-3.6, VI-3.7 and VI-3.8 into Eq. VI-3.5 one obtains the following estimate of the variance of D_t .

$$\begin{aligned} \text{Var}[D_t] &= 6v_a t (\sigma_x^2/2a)^2 \\ &= \frac{3}{2} v_o t \sigma_x^2 \frac{1}{r^2} e^{-r^2/2} \end{aligned} \quad (\text{VI-3.9})$$

where $r=a/\sigma_x$, a being the yield level and σ_x the ("linear") r.m.s. displacement. From Eqs. VI-3.2 and VI-3.9, the coefficient of variation of D_t is found:

$$\frac{\sqrt{\text{Var}[D_t]}}{E[D_t]} = \sqrt{\frac{3}{2v_o t}} e^{r^2/4} \quad (\text{VI-3.10})$$

The c.o.f. of D_t is plotted in figure VI.5. It decreases in proportion to \sqrt{t} . It increases for increasing values of the ratio $r=a/\sigma_x$, of the yield level to the ("linear") r.m.s. response.

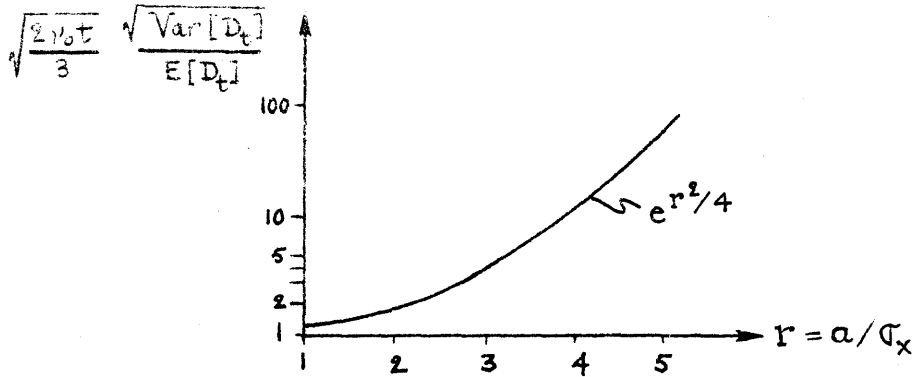


Fig.VI.5. The Coefficient of Variation of the Total Absolute Plastic Deformation in E-P System Undergoing Stationary Gaussian Random Vibration.

For the sake of completeness, it should be mentioned that it is possible to obtain the characteristic function, $\phi_{D_t}(u) = E[e^{iuD_t}]$, of D_t . The latter is well-known to contain the same information as the probability density function [44]. It is particularly useful to generate higher order moments of Δ_t . The characteristic function of the sum D_t , of a random number, N_t , of independent identically distributed random variables $|\Delta_i|$, is obtained from the characteristic functions $\phi_{N_t}(u)$ and $\phi_{|\Delta_i|}(u)$ by the following relationship [44, 77]

$$\phi_{D_t}(u) = \phi_{N_t}[\phi_{|\Delta_i|}(u)] \quad (\text{VI-3.11})$$

It is somewhat more convenient to work with the normalized deformations $|\Delta_i^*| = 2a|\Delta_i|/\sigma_x^2$ (that are χ_1^2 -distributed) and the normalized total of the absolute deformations, $D_t^* = 2aD_t/\sigma_x^2$. Then

$$\phi_{D_t^*}(u) = \phi_{N_t}[\phi_{|\Delta_i^*|}(u)] \quad (\text{VI-3.12})$$

where $\phi_{N_t}(u)$ is the Poisson characteristic function [44],

$$\phi_{N_t}(u) = e^{2\nu_a t(u-1)} \quad (\text{VI-3.13})$$

and $\phi_{|\Delta_i^*|}$ is the χ_1^2 characteristic function [44],

$$\phi_{|\Delta_i^*|}(u) = (1-2iu)^{-1/2} \quad (\text{VI-3.14})$$

Hence,

$$\phi_{D_t^*}(u) = \exp\{2\nu_a t[(1-2iu)^{-1/2} - 1]\} \quad (\text{VI-3.15})$$

If the number of excursions into the plastic range were a given number n_t , then the normalized absolute deformation would be a χ^2 -distribution with n_t degrees of freedom. To acquire information as to the shape of the density function of D_t^* one might compute $E[N_t] = 2\nu_a t$. The density function of D_t^* will not be unlike the density function of a χ^2 variable with $E[N_t]$ degrees of freedom. Thus, figure VI.6 shows that, for $E[N_t]$ small, say, less than 5, the density function of D_t^* may be expected to be significantly skewed. For large values of $E[N_t]$, the density function actually approaches that of the normal distribution.

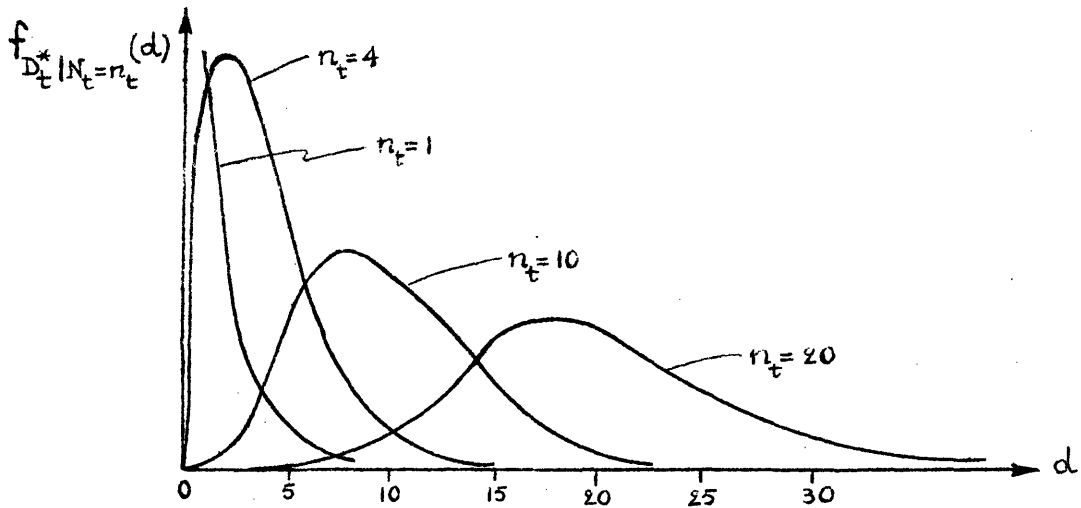


Fig.VI.6. The Density Function of D_t^* for a Given Number n_t of Yield Impacts.

Total Plastic Deformation in EP-E Systems

According to an analysis completely similar to that given in the preceding pages, it is possible to obtain the statistical properties of the total plastic deformation Δ_t' , developed, during a time interval 0 to t, in an elastoplastic-elastic system (shown in Fig. VI.1b) subjected to stationary Gaussian random excitation. The "asymptotic" mean rate of yield level impacts is now v_a , rather than $2v_a$, because we deal with a single barrier. The results are given here for the expected value and for the coefficient of variation of Δ_t'

$$E[\Delta_t'] = \frac{1}{2} v_a t \sigma_x r \frac{1}{r} e^{-r^2/2} \tag{VI-3.16}$$

$$\frac{\sqrt{\text{Var}[\Delta_t']}}{E[\Delta_t']} = \sqrt{\frac{3}{v_a t}} e^{r^2/4} \tag{VI-3.17}$$

Effect of Clump Size.

Threshold crossings are likely to occur in clumps when the damping ratio of a linear oscillator becomes small. It has been argued in section VI.2 that the rate at which such clumps occur equals $\mu_{a,D}$, given by Eq. VI-2.4. To be able to extend the analysis given in the preceding pages, an estimate is needed of the characteristics of the amount of plastic deformation due to all yield level crossings in a single clump. The size of a clump of yield level impacts is likely to be much smaller than that of a clump of threshold crossings of the associated linear oscillator. This is so because the initial plastic deformation is likely to destroy much of the energy that would otherwise cause a (long) envelope excursion above the threshold (and hence a large clump size) in the linear system. This problem is studied in some detail by Karnopp and Scharon [74]. It will not be further considered here.

VI.4. Properties of Reversible Inelastic Response Measures

Total Plastic Deformation in E-P Systems

The total plastic deformation in an elasto-plastic system, responding to a zero-mean stationary random excitation, tends to fluctuate about a zero mean value. This is so, because a "randomly selected" single plastic excursion is equally likely to produce a negative contribution to Δ_t , the total plastic deformation, as a positive one (see figure VI.2). Again, we consider the case where the mean rate of occurrence of jumps in Δ_t may be approximated by $2\nu_a$. This implies that the number of $|x|$ -peaks which contribute to the size of single jump in Δ_t , is nearly always equal to one.

Recall that the normalized absolute value, $|\Delta_i|$, of the inelastic deformation, corresponding to a single plastic excursion, has been found to be approximately χ_1^2 distributed. It is clear that the density of Δ_i is symmetrical and defined by,

$$f_{\Delta_i}(\Delta_i) = \frac{1}{2} f_{|\Delta_i|}(\Delta_i) \quad , \Delta_i > 0 \quad (VI-4.1)$$

$$f_{\Delta_i}(\Delta_i) = \frac{1}{2} f_{|\Delta_i|}(-\Delta_i) \quad , \Delta_i < 0$$

The density function of Δ_i is shown in figure VI.7.

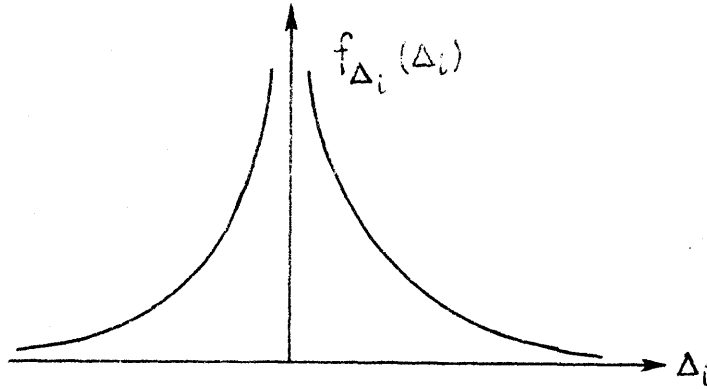


Fig.VI.7. The Density Function of the Amount of Deformation due to a Single Yield Level Impact in Elasto-Plastic Systems.

It will again be found useful to work with the normalized incremental deformation, $\Delta_i^* = 2a\Delta_i/\sigma_x^2$, and with the normalized value, $\Delta_t^* = 2a\Delta_t/\sigma_x^2$, of the total plastic deformation accumulated in the time interval 0 to t. The first two moments of Δ_i^* are

$$E[\Delta_i^*] = (-1)\frac{1}{2} + (+1)\frac{1}{2} = 0 \quad (\text{VI-4.2})$$

$$E[\Delta_i^{*2}] = \text{Var}[\Delta_i^*] = (3)\frac{1}{2} + (3)\frac{1}{2} = 3 \quad (\text{VI-4.3})$$

The total plastic deformation or the total permanent set Δ_t , developed in the time interval 0 to t, equals

$$\Delta_t = \Delta_1 + \Delta_2 + \dots + \Delta_{N_t} \quad (\text{VI-4.4})$$

The random number N_t , of contributions to Δ_t , is again assumed to have a Poisson distribution with average rate

$\mu_{a,D} = 2v_a$. The contributions Δ_i have a distribution given by

Eq. VI-4.1 and are assumed to be mutually independent and uncorrelated with N_t . Then

$$E[\Delta_t] = E[N_t]E[\Delta_i] = 0 \quad (\text{VI-4.5})$$

$$\begin{aligned} \text{Var}[\Delta_t] &= E[N_t]\text{Var}[\Delta_i] + \text{Var}[N_t]E^2[\Delta_i] \\ &= (2\nu_a t) 3(\sigma_x^2/2a)^2 \\ &= \frac{3\nu_0 t}{2} \sigma_x^2 \frac{1}{r^2} e^{-r^2/2} \end{aligned} \quad (\text{VI-4.6})$$

in which $r=a/\sigma_x$, a being the yield displacement, and σ_x the ("linear") r.m.s. displacement. The r.m.s. value of the total plastic deformation, expressed in terms of the ("linear") r.m.s. displacement σ_x , has the form

$$\text{R.m.s. of } \left(\frac{\Delta t}{\sigma_x}\right) = \frac{\sqrt{\text{Var}\Delta_t}}{\sigma_x} = \sqrt{\frac{3\nu_0 t}{2}} \frac{1}{r} e^{-r^2/4} \quad (\text{VI-4.7})$$

It is seen to increase as \sqrt{t} , and to decrease for increasing relative yield levels, $r=a/\sigma_x$, as shown in figure VI.8.

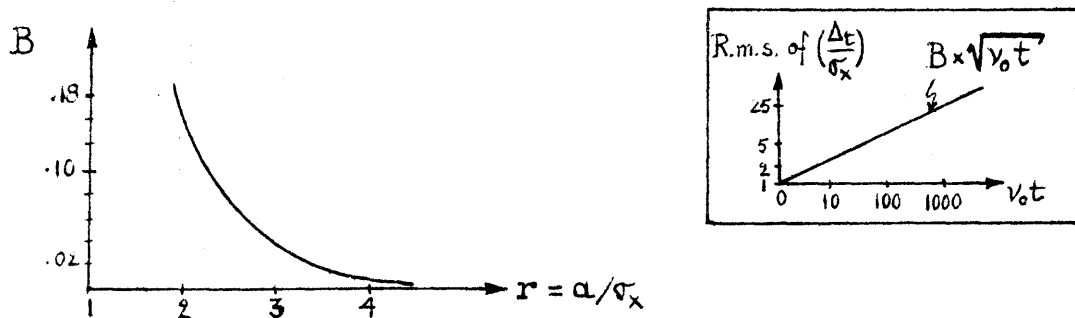


Fig.VI.8. The Standard Deviation of the Total Plastic Deformation Developed in an Elasto-Plastic System During t Seconds of Stationary Random Vibration. 159

For the sake of completeness, it should be mentioned that it is again possible to obtain the characteristic function, $\phi_{\Delta_t^*}(u)$, of the normalized total plastic deformation Δ_t^* . Recall that Δ_t and Δ_t^* are related in the following way,

$$\Delta_t = \frac{\sigma_x^2}{2a} \Delta_t^* = \frac{\sigma_x}{2r} \Delta_t^* \quad (\text{VI-4.8})$$

The characteristic function of $|\Delta_i^*|$ is given by Eq. VI-3.14. Using Eq. VI-4.1, we may write

$$\begin{aligned} \phi_{\Delta_i^*}(u) &= E[e^{iu\Delta_i^*}] = \frac{1}{2}E[e^{iu|\Delta_i^*|}] + \frac{1}{2}E[e^{-iu|\Delta_i^*|}] \\ &= \frac{1}{2}\phi_{|\Delta_i^*|}(u) + \frac{1}{2}\phi_{|\Delta_i^*|}(-u) \\ &= \frac{1}{2}(1-2iu)^{-1/2} + \frac{1}{2}(1+2iu)^{-1/2} \end{aligned} \quad (\text{VI-4.9})$$

Inserting Eqs. VI-4.9 and VI-3.13 into the relationship

$$\phi_{\Delta_t^*}(u) = \phi_{N_t}[\phi_{\Delta_i^*}(u)] \quad (\text{VI-4.10})$$

one obtains the characteristic function of Δ_t^* , the normalized total plastic deformation in E-P systems responding to stationary zero-mean random excitation

$$\phi_{\Delta_t^*}(u) = \exp\{2v_a t [\frac{1}{2}(1-2iu)^{-1/2} + \frac{1}{2}(1+2iu)^{-1/2} - 1]\} \quad (\text{VI-4.11})$$

VI.5. Average Time to Collapse in Bilinear Systems

Consider a structure characterized by a bilinear force-displacement relationship, as shown in figure VI.9.

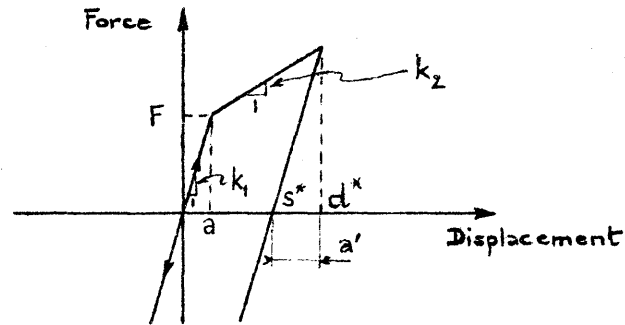


Fig.VI.9

A problem of some practical interest is the following:

"What is the average time for the maximum displacement D , of a randomly excited bilinear oscillator to reach some predetermined allowable level d^* ?" The problem is, of course, equivalent to determining the average time required for the permanent set S to reach a level $s^* = d^* - a'$, where a' is depicted in Fig. VI.9.

We first obtain an expression for the average time, $E[T_{s, s+\Delta s}]$, it takes the system to "travel" from s to $s+\Delta s$. Let f be the force level corresponding to the displacement s . See figure VI.10. Note that jumps in s occur at (unknown) discrete points $s=s_0, s=s_1, \dots, s=s_i$. If none of the points $s_i, i=0,1, \dots$ lie in the interval $(s, s+\Delta s)$, it means that the interval is travelled during a single plastic excursion. The time $T_{s, s+\Delta s}$ will then be negligibly small compared to its value which

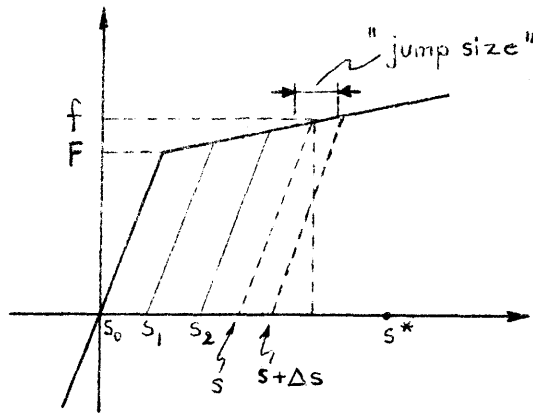


Fig.VI.10

corresponds to the case, $s < s_i < s + \Delta s$, for at least one i . In fact, the characteristics of the time $T_{s, s+\Delta s}$ clearly depend on just how many jumps occur in the interval $(s, s+\Delta s)$. We have

- i) $T_{s, s+\Delta s} \approx 0$, if no jump occurs in $(s, s+\Delta s)$
- ii) $T_{s, s+\Delta s}$ has an exponential distribution, (with a mean rate which depends on s , or f), if one jump occurs in $(s, s+\Delta s)$
- iii) $T_{s, s+\Delta s}$ is approximately Gamma distributed if two jumps occur in $(s, s+\Delta s)$

etc.

If two jumps occur in $(s, s+\Delta s)$, then, in accordance with earlier assumptions, $T_{s, s+\Delta s}$ will be the sum of two independent nearly identical (for Δs very small) exponentially distributed random variables. Its distribution may therefore be expected to be nearly equal to the second member of the Gamma family [77]. The expected value of $T_{s, s+\Delta s}$ may be computed as follows,

$$\begin{aligned}
E[T_{s,s+\Delta s}] &= 0 \times P[\text{no jumps in } s,s+\Delta s] \\
&+ E[T_{s,s+\Delta s} | 1 \text{ jump in } s,s+\Delta s] P[1 \text{ jump in } s,s+\Delta s] \\
&+ E[T_{s,s+\Delta s} | 2 \text{ jumps in } s,s+\Delta s] P[2 \text{ jumps in } s,s+\Delta s] \\
&+ \dots
\end{aligned} \tag{VI-5.1}$$

When Δs becomes infinitesimally small, the probability $P[i \text{ jumps in } s,s+\Delta s]$, for $i > 1$, will vanish. Eq. VI-5.1 becomes

$$\begin{aligned}
E[T_{s,s+ds}] &= E[T_{s,s+ds} | 1 \text{ jump in } s,s+ds] P[1 \text{ jump in } s,s+ds] \\
&+ 0[(ds)^2]
\end{aligned} \tag{VI-5.2}$$

The time between jumps is exponentially distributed with a mean rate which depends on the threshold level. Let the mean rate, corresponding to the force level f , be denoted by $\alpha_B(f)$, where the subscript B refers to the fact that we deal with a B type barrier. The mean of $T_{s,s+ds}$, given the occurrence of one jump "at s ", equals,

$$E[T_{s,s+ds} | 1 \text{ jump in } s,s+ds] = \frac{1}{\alpha_B(f)} \tag{VI-5.3}$$

Also, it seems reasonable to take^(*)

$$P[1 \text{ jump in } s,s+ds] \approx \frac{ds}{E[\text{jump size}]} \tag{VI-5.4}$$

(*) A more detailed analysis may be made. It involves conditioning on the jump size and requires an assumption of equal likelihood.

where $E[\text{jump size}]$ is taken to be that which corresponds to a fixed yield force level f . From Eq. VI-2.13

$$E[\text{jump size}] = \frac{M\omega_n^2 \sigma_x^2}{2f} \quad (\text{VI-5.5})$$

Inserting Eq. VI-5.5 into Eq. VI-5.4, one obtains

$$P[1 \text{ jump in } s, s+ds] = \frac{2f}{M\omega_n^2 \sigma_x^2} ds \quad (\text{VI-5.6})$$

From Eqs. VI-5.2, VI-5.5 and VI-5.6,

$$E[T_{s, s+ds}] \approx \frac{2}{M\omega_n^2 \sigma_x^2} \frac{f}{\alpha_B(f)} ds \quad (\text{VI-5.7})$$

k_1 and k_2 , where $0 < k_2 < k_1$, are the stiffnesses which characterize the bilinear force-displacement relationship. Note that, for positive values of s , $df = k_2 ds$. Therefore

$$E[T_{s, s+ds}] \approx \frac{2}{k_2 M\omega_n^2 \sigma_x^2} \frac{f}{\alpha_B(f)} df \quad (\text{VI-5.8})$$

or

$$E[T_{s, s+ds}] \approx \frac{2}{k_1 k_2 \sigma_x^2} \frac{f}{\alpha_B(f)} df \quad (\text{VI-5.9})$$

Finally, the average time needed for the bilinear system to develop a permanent set s^* , or to reach a force level $F + (d^* - a)k_2$ (see Fig. VI.9 or 11) is found by integrating Eq. VI-5.9

$$E[T_{d^*}] = \int_0^{F+(d^*-a)k_2} \frac{2}{F} \frac{f}{k_1 k_2 \sigma_x^2} \alpha_B(f) df \quad (\text{VI-5.13})$$

For simplicity, let us again take the case where excursions into the plastic domain are independent events, i.e.,

$$\alpha_B(f) = v_0 \exp\{-f^2/2k_1^2 \sigma_x^2\} \quad (\text{VI-5.11})$$

Using the notation, $r=F/k_1 \sigma_x$, $r^*=r + \frac{(d^*-a)k_2}{k_1 \sigma_x}$ and $r'=f/k_1 \sigma_x$, the meaning of which is clarified in Fig. VI.11, Eq. VI-5.10 becomes

$$\begin{aligned} E[T_{d^*}] &= \frac{2}{v_0} \left(\frac{k_1}{k_2}\right) \int_r^{r^*} r' \exp\{r'^2/2\} dr' \\ &= \frac{2}{v_0} \left(\frac{k_1}{k_2}\right) \left[\exp\left\{\frac{1}{2}\left[r + \frac{(d^*-a)k_2}{\sigma_x k_1}\right]^2\right\} - \exp\left\{\frac{1}{2}r^2\right\} \right] \quad (\text{VI-5.12}) \end{aligned}$$

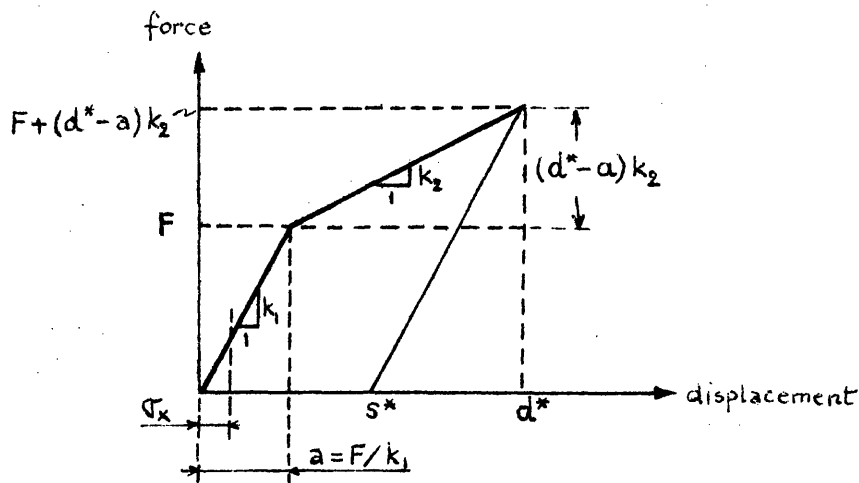


Fig.VI.11.

When k_2 becomes zero, the bilinear system reduces to an elasto-plastic-elastic (EP-E) system. In that case, Eq. VI-5.12 leads to an indefinite expression. Using de L'Hospital's rule, one finds

$$E[T_{s^*}] \Big|_{k_2=0} = \frac{1}{v_0} \frac{s^*}{\sigma_x} 2re^{+r^2/2} \quad (\text{VI-5.13})$$

The above expression may be seen as the product of the following two factors

- i) the average time between yield level crossings, $\frac{1}{v_0} e^{r^2/2}$
- ii) the average number of yield level crossings, $s^*/E[\Delta_i]$,
 where, $E[\Delta_i] = \sigma_x/2r$, is the average jump size.

CHAPTER VII

STRUCTURAL SYSTEM PERFORMANCE ANALYSIS FOR MULTIPLE OCCURRENCES OF RANDOM EXCITATION

VII.1. Occurrence Patterns of Random Vibration

Phenomena such as atmospheric turbulence, ground motion or ocean waves, giving rise to random excitations of interest in structural engineering are basically intermittent in nature (see, for example, ref. [78]). Relatively short active periods of varying intensities are separated by long quiescent periods. In fact, to be distinguishable, the active periods have to be examined on a more detailed time-scale of minutes or seconds compared to the total operational lifetime of the system, which may be measured in months or years. This intermittency phenomenon makes it impractical to perform a harmonic analysis of a complete sample history, including quiescent periods. Clearly only the active periods are of any real interest to the designer of engineering systems. Another fact, first drawn attention to by Mandelbrot [79, 80] is the hierarchical nature of these clusters of activity. The energy of wind is concentrated in a hierarchy of puffs, gusts and storms. Earthquake energy release occurs in a hierarchy of

foreshocks and aftershocks, respectively preceding and following the main shock. Observing a single shock on a finer time scale, the intensity builds up to a peak value, perhaps levels off for a few seconds, and finally gradually drops to a low level. Mandelbrot [80] calls this property by which phenomena appear similar whichever the scale at which they are observed the property of self-similarity. Particularly in reference to the behavior of error clusters in communication systems he proposes to order clusters of activity according to their intensity. Thus, for example, what he calls "clusters of the highest order", are rare events, essentially by definition. Rare events tend to occur independently according to a Poisson process. Fortunately, for several important types of random excitation of structural engineering interest, the lowest intensity level at which system integrity may become in jeopardy is often sufficiently high that the events may reasonably be assumed to have a Poisson occurrence pattern. For example, in earthquake risk analysis studies [81, 82, 83, 84], the use of a Poisson model for the occurrence of high intensity earthquakes at a site is widely accepted.

Other types of random excitation causing high levels of dynamic response have an almost perfectly predictable occurrence pattern: the acoustic excitation induced by jet noise during take-off of missiles, the vibrations due to runway unevenness, experienced by airplanes during take-off and landing, blast induced vibrations, etc. These operational loads may be

assumed to occur at known discrete points in time. Their intensity level, duration and non-stationarity characteristics may often be predicted much more accurately than the corresponding properties of random excitations due to natural phenomena.

In conclusion, most types of disturbing forces that may give rise to high levels of random vibratory response have occurrence patterns that may be modeled in one of the following two ways (*):

- i) Events, with an intensity exceeding a fixed level i_0 , occur according to a Poisson process with average rate λ_{i_0} .
- ii) Events (with an intensity exceeding a fixed level i) occur according to some fixed pattern, i.e., the number of events occurring during a given time interval is perfectly predictable or, alternatively, time or "life" may be counted in number of loads, e.g., number of flights.

(*) More general inter-arrival distributions may in some cases have to be considered. Much of the work to be presented in this chapter may then be extended by making use of the theory of semi-Markov processes [83]. A semi-Markov process is one that makes transitions according to the transition probability matrix of a Markov process, but whose times between transition can be an arbitrary random variable that depends on the transition. The discrete- and continuous-time Markov processes are special cases of the semi-Markov process, which is, on the other hand, much less tractable in its general form.

VII.2. Representation of a Sequence of Excitations

Consider a structure to be subjected to a sequence of events (say, strong earthquakes or wind storms) each of which causes a random vibratory response of the structure. Let the times at which these events occur be denoted by τ_1, τ_2, \dots , etc. A sample function of the complete excitation process $V(t)$, $0 \leq t \leq T$, where T is the structure's operational life, is shown in figure VII.1

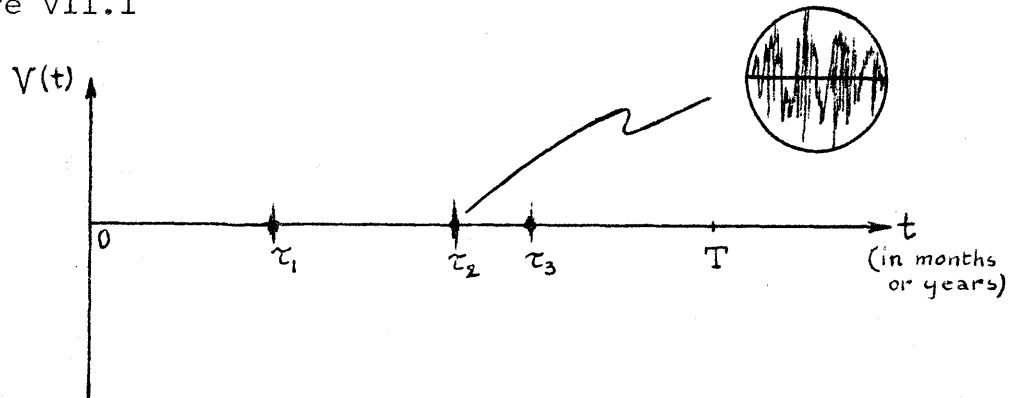


Fig.VII.1. A Sample History of the Excitation Applied to a Structure.

The characteristics of the excitation corresponding to each event in the sequence are assumed to be, in a probabilistic sense, of the same type. Specifically, let the following ground rules be adopted:

- i) The intensity I and the duration S randomly vary from event to event in the sequence. Each pair of values (i, s) constitutes a sample from the joint distribution $p(i, s)$, which characterizes the sequence.

ii) The sequence is also characterized by its normalized (i.e., unit area) input power spectrum $G_V(\omega)$. A refinement, which will otherwise not be further pursued here, is possible by assuming $G_V(\omega)$ to have a few random parameters, i.e., $G_V(\omega) = g(\omega, A, B, \dots)$, where A and B are random variables, which may either vary randomly from event to event, or take a common (but random) value for all events in the sequence. For example, in earthquake spectra, the former (randomly varying components) may be due to uncertainty regarding the regional geologic environment, and the latter (non-changing components) to inadequate knowledge of the local soil conditions [85].

iii) Nonstationarity properties of the excitation corresponding to each event, may also be dealt with through some parametric representation similar to that proposed for $G_V(\omega)$. Often, however, explicit consideration of nonstationarity of the input may be avoided by using the self-explanatory concepts of a quasi-stationary motion and an equivalent duration. These concepts have been rather successfully applied to earthquake loading (Barnstein [86], Housner and Jennings [87] and others) and wind loading (Davenport [88]).

In conclusion, we assume the sequence of excitations to be characterized by its occurrence characteristics, (i.e., lower bound intensity i_0 and, in the case of Poisson occurrences, the arrival rate λ_i), its unit-area power spectrum $G_V(\omega)$ and the joint distribution $p(i, s)$ of intensity I and

duration S . The excitation corresponding to a single event, say the j th event in the sequence, may be described by its occurrence time τ_j , the unit power spectrum $G_V(\omega)$, and the sample intensity i_j and sample duration s_j (where the pair (i_j, s_j) is statistically independent of (i_k, s_k) for $j \neq k$).

VII.3. Conditional and Marginal Transition Probability Matrices

Consider a simple structural system subjected to a sequence of excitations with properties as described in the preceding section. Let $V(t)$, $0 \leq t \leq T$ represent the complete excitation process and let $X(t)$, $0 \leq t \leq T$ be the corresponding random response process. See figure VII.2.

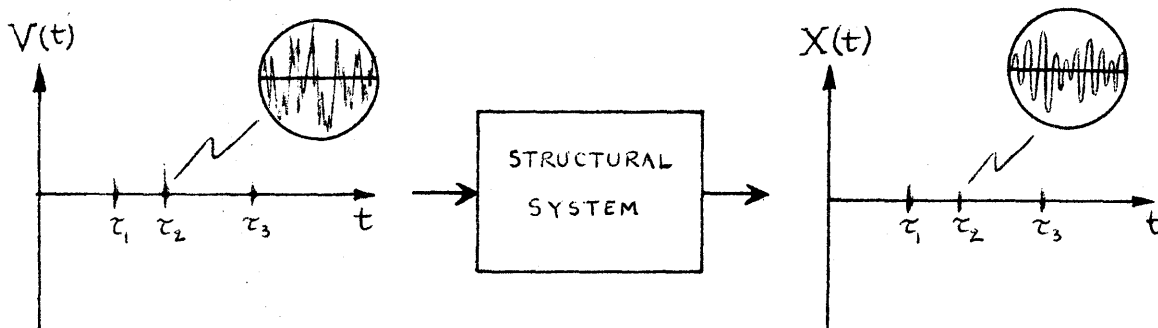


Fig.VII.2. Excitation and Response Sample Functions.

One visualizes the structure, under the influence of the excitation $V(t)$, to make a series of transitions from one state

to another. The initial state may be labeled "survival"; the structure may then go through several states of "damage" (e.g., failure of sensitive equipment stored in the structure, or say, a decrease in structural stiffness properties) or may perhaps, through corrective action, be made to return to a state of less severe damage. Finally, it may reach a state of permanent un-serviceability, called "failure". Jumps from one state to another may occur only at times τ_1, τ_2, \dots which may be either fixed or random, as discussed in section VII.1.

Conditional Transition Probability Matrices

In section V.4 it is shown that, for an m-state model, the uncertain effect of a single random excitation with given intensity and given finite duration may be summarized in terms of an $m \times m$ matrix $P = [p_{jk}]$, the "one-step" transition probability matrix. The element p_{jk} denotes the probability that the event will cause the system to make a jump to state k , given that it occupied state j just before the event. The exact structure of the matrix P depends on the definition of the states and the nature of the failure mechanism the model is meant to represent. For example, consider the two three-state models studied in section V.4. Both models have in common the fact that system performance relates to two threshold levels, a_1 and a_2 , $a_2 > a_1$.

Model A represents a simple deterioration process. Failure (state 2) occurs when the response exceeds the higher

threshold (say, its ultimate strength). When it first exceeds the lower threshold (i.e., reaches state 1), a basic change is postulated to occur in the mechanical properties of the system, and hence, in the nature of the response process. The flow graph representation, in figure VII.3, depicts what may happen during a single event.

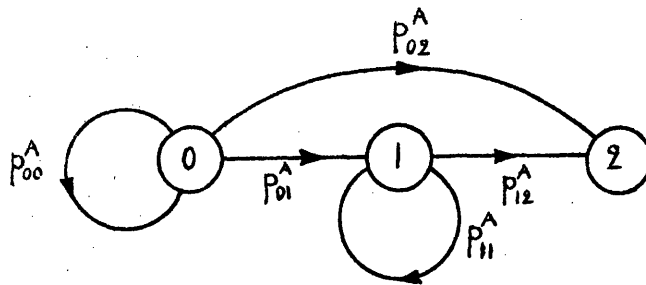


Fig.VII.3. Markov Model of a Simple Deterioration Process.

Just before an event takes place the system may occupy either state 0 or state 1. If in state 0 just before an event, the system may be found in either of the three states immediately after the event. If in state 1 just before the event, it may either stay there or move to state 2. The excitation ceases to be of interest once state 2 is reached. The one-step transition probability matrix P^A takes the form

$$P^A = \begin{bmatrix} P_{00}^A & P_{01}^A & P_{02}^A \\ 0 & P_{11}^A & P_{12}^A \\ 0 & 0 & 1 \end{bmatrix} \quad (\text{VII-3.1})$$

where the probabilities p_{jk}^A , the elements of P^A , are given in section V.4 in terms of the structural properties, the excitation characteristics (including input intensity and duration) and the threshold levels. Since each row in Eq. VII-3.1 must sum to one, the one-step transition matrix P^A is seen to be specified by 3 (independent) probabilities (say, p_{00}^A , p_{02}^A and p_{12}^A).

According to model B a loss of non-structural nature is sustained when the response process, corresponding to a single event, first crosses the lower of the two fixed threshold levels. Structural failure is postulated to occur when the higher threshold is first crossed. Since the mechanical properties of the structural system are not altered when state 1 is reached, the transition probabilities are not influenced by the state of the system (0 or 1) at the start of each motion. The system is defined to be in state 1 if, during the most recent event, its response crossed the lower, but not the higher threshold level. The system is in state 0 if, during the most recent event, its response did not exceed any of the barriers. State 2, the absorbing state, is reached when the response process crosses the higher barrier for the first time. Again, the response process then ceases to exist, or say, ceases to be of interest. The flow graph in figure VII.4 shows what transitions are possible during a single event. For example, visualize a structure, in which sensitive equipment has been installed, to be subjected to a number of strong-

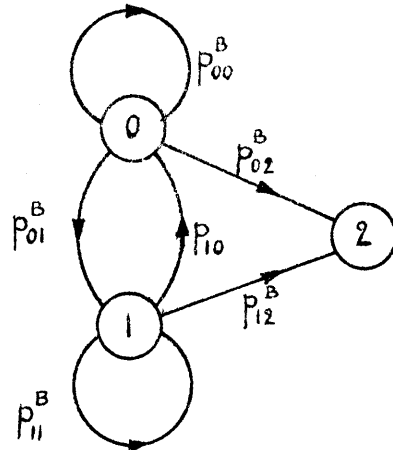


Fig.VII.4. Markov Model of a Performance Criterion Involving Both Repairable Non-Structural Damage and Structural Failure.

motion earthquakes. A sequence, e.g., 001012, means that the structure is initially in state 0 (i.e., satisfactory performance); it is still there immediately after the first earthquake; it moves to state 1 (say, equipment fails) as a result of the 2nd event; the equipment is repaired or replaced and survives the 3rd earthquake; the equipment again becomes unoperational during the 4th event; finally, the structure itself becomes unserviceable as a result of the 5th ground motion.

The one-step transition probability matrix has the form

$$P^B = \begin{bmatrix} P_{00}^B & P_{01}^B & P_{02}^B \\ P_{10}^B & P_{11}^B & P_{12}^B \\ 0 & 0 & 1 \end{bmatrix} \quad (\text{VII-3.2})$$

P^B is completely specified by only two probabilities, say, p_{00}^B and p_{02}^B . These are given in section V.4 again in terms of the structural properties, the excitation characteristics, and the threshold levels.

Before discussing the implications of the fact that the duration and intensity corresponding to each event are not given (as the preceding analysis presumes), but rather random and different from event to event, some elementary properties of discrete state Markov processes are noted (mainly for the benefit of the reader not familiar with discrete state Markov processes). They will be entirely relevant in the discussion to follow.

Presume that events in a sequence occur at known discrete points in time. Let $P=[p_{jk}]$ be the $m \times m$ one-step transition probability matrix. Also, let $P(v)=[p_{jk}(v)]$ be the v -step transition probability matrix, where $p_{jk}(v)$ denotes the probability that the system will be in state k immediately after the v th event, given that it is in state j immediately before the first event. Of course, we have $P(1)=P$. The Chapman-Kolmogorov equations [44] have the following matrix form,

$$P(v) = P \times P(v-1) \quad v=1,2,\dots \quad (\text{VII-3.3})$$

From VII-3.3, it follows by induction that

$$P(v) = P \times P \dots \times P = P^v \quad (\text{VII-3.4})$$

The above relationship holds for so-called homogeneous Markov processes, that is, processes where P remains unchanged.

Eq. VII-3.3 constitutes the basic relationship from which all important statistics of discrete time discrete state Markov processes may be obtained. Fast flow-graph techniques, based on z-transformations [23], may be used to carry out the actual computations (at least in those cases where the number of states is fairly small, say, $m < 5$)

Non-homogeneous discrete Markov processes are somewhat less tractable. The one-step transition probability matrices $P^{(1)}, P^{(2)}, \dots, P^{(v)}$, differ from event to event. The Chapman-Kolmogorov equations take the form,

$$P(v) = P^{(v)} \times P^{(v-1)} \quad , v=1,2,\dots \quad (\text{VII-3.5})$$

and, hence, by induction,

$$P(v) = P^{(1)} \times P^{(2)} \dots \times P^{(v)} \quad (\text{VII-3.6})$$

Marginal Transition Probability Matrices

The analysis in the preceding section and throughout the earlier part of this work presumes the values of important excitation parameters such as motion intensity and duration to be given. Therefore, the one-step transition probability

matrix, $P=[p_{jk}]$, referred to in the last few pages, may be thought of as being conditioned, i.e., the elements of P are conditional probabilities, given $I=i$ and $S=s$. In the present context, intensity and duration corresponding to a single event are random variables. What is worse, they take different values from event to event, thus rendering the resulting Markov chain non-homogeneous, i.e., each event in the sequence, say the v th, is characterized by a one-step transition probability matrix $P^{(v)}$ which depends upon the sample intensity i_v and sample duration s_v . For any given sequence $(i_1, s_1), (i_2, s_2), \dots, (i_v, s_v)$, a non-homogeneous Markov chain results. The pairs of sample values of intensity and duration are mutually independent and identically distributed. Let $p(i_v, s_v)$ be the joint density function of I_v and S_v . The marginal one-step transition probability p_{jk}^* may be found by integrating $p_{jk}^{(v)}$ over all possible values of intensity and duration

$$p_{jk}^* = \int_i \int_s p_{jk} p(i_v, s_v) di_v ds_v \quad (\text{VII-3.7})$$

or

$$p_{jk}^* = E[p_{jk}^{(v)}] \quad (\text{VII-3.8})$$

where it is understood that the expectation is with respect to I_v and S_v only. Also, let the marginal (with respect to I and S) one-step transition probability matrix be denoted by $P^*=[p_{jk}^*]$. We may write

$$P^* = E[P^{(v)}] \quad \text{(VII-3.9)}$$

Eq. VII-3.6 shows that the v -step transition probability matrix $P^{(v)} = [p_{jk}^{(v)}]$ of a non-homogeneous Markov chain depends upon all the one-step matrices $P^{(1)}, P^{(2)}, \dots, P^{(v)}$, which in turn depend on $i_1, s_1, \dots, i_v, s_v$. Hence, the marginal v -step transition probability $p_{jk}^*(v)$ becomes

$$p_{jk}^*(v) = E[p_{jk}^{(v)}] = \int_{i_1} \int_{s_1} \dots \int_{i_v} \int_{s_v} p_{jk}^{(v)} p(i_1, s_1, \dots, i_v, s_v) di_1 ds_1 \dots di_v ds_v \quad \text{(VII-3.10)}$$

The expectation here is with respect to all the random variables that $p_{jk}^{(v)}$ depends on, i.e., $i_1, s_1, \dots, i_v, s_v$.

The crucial question upon which the tractability (and hence feasibility) of our approach rests is, in mathematical terms, the following,

$$P^*(v) \stackrel{?}{=} P^* \times P^*(v-1) \quad \text{(VII-3.11)}$$

or: $E[P^{(v)}] \stackrel{?}{=} E[P^{(v)}] E[P^{(v-1)}]$

In words, is it possible to reproduce the homogeneous form of the Chapman-Kolmogorov equations (Eq. VII-3.3), with P^* replacing P ? The answer is yes.

Proof:

First, take $r=2$. Let the pairs (i_1, s_1) and (i_2, s_2) be given. Let the one-step transition probability matrix corresponding

to (i_1, s_1) be denoted by $P^{(1)}$ and that corresponding to (i_2, s_2) by $P^{(2)}$. The two-step transition matrix $P(2)$ will depend on i_1, s_1, i_2, s_2 . We have

$$P(2) = P^{(1)} P^{(2)} \quad (\text{VII-3.12})$$

An element of $P(2)$, say $p_{jk}(2)$, has the form,

$$p_{jk}(2) = p_{j0}^{(1)} p_{0k}^{(2)} + p_{j1}^{(1)} p_{1k}^{(2)} + \dots + p_{jm}^{(1)} p_{mk}^{(2)} = \sum_{t=0}^m p_{jt}^{(1)} p_{tk}^{(2)} \quad (\text{VII-3.13})$$

Its expected value, with respect to all random variables involved is found by integrating over i_1, s_1, i_2 and s_2 ,

$$\begin{aligned} p_{jk}^*(2) &= E[p_{jk}(2)] = \int_{i_1} \int_{s_1} \int_{i_2} \int_{s_2} p_{jk} p(i_1, s_1, i_2, s_2) di_1 ds_1 di_2 ds_2 \\ &= \sum_{t=0}^m \left\{ \int_{i_1} \int_{s_1} p_{jt}^{(1)} p(i_1, s_1) di_1 ds_1 \right\} \left\{ \int_{i_2} \int_{s_2} p_{tk}^{(2)} p(i_2, s_2) di_2 ds_2 \right\} \end{aligned} \quad (\text{VII-3.14})$$

Note that this derivation critically depends on the assumption that the pairs (i_1, s_1) and (i_2, s_2) are statistically independent. Eq. VII-3.14 may further be written

$$\begin{aligned} p_{jk}^*(2) &= \sum_{t=0}^m E[p_{jt}^{(1)}] E[p_{tk}^{(2)}] \\ &= \sum_{t=0}^m p_{jt}^* p_{tk}^* \end{aligned} \quad (\text{VII-3.15})$$

And therefore,

$$P^*(2) = [P^*] \quad (\text{VII-3.16})$$

An simple induction argument may be used to complete the proof.

It has been shown in this section that the theory of homogeneous discrete state Markov processes may be applied to problems of the type encountered here, i.e., where each of the one-step transition probability matrices depend on a (set of) random variable(s), whose sample value(s) varies (vary) randomly from event to event. It suffices to compute the expected values of the one-step transition probabilities. The expected value P^* , of the random one-step transition probability matrix, may then be used as a constant and many homogeneous Markov chain results remain valid (as they are essentially based on the Chapman-Kolmogorov equations.) In particular, let $\underline{\pi}^*(r) = \{\pi_0^*(r); \pi_1^*(r); \dots; \pi_{m-1}^*(r)\}$ be the vector of (unconditioned) state probabilities. $\pi_j^*(r)$ is the (marginal w.r.t. $i_1, s_1, \dots, i_r, s_r$) probability of being in state j immediately after the r th event. The initial conditions are expressed in terms of the vector $\underline{\pi}(0)$ of initial state probabilities. For example, if the system is certain to start in state 0, then $\pi_0(0)=1$ and $\pi_j(0)=0$ for $j \neq 0$. It is easy to establish that

$$\underline{\pi}^*(r) = \underline{\pi}(0)P^*(r) = \underline{\pi}(0)[P^*]^r \quad (\text{VII-3.17})$$

The results may be shown to hold also for the case in which

events occur according to a Poisson process with average rate, say, λ . Let N be defined as the number of events in a given time interval 0 to T . It will be a Poisson distributed random variable. Also, let $\pi_j^*(T)$ designate the probability that the system will be in state j at time T . It is easy to show that

$$\pi_j^*(T) = \sum_{r=0}^{\infty} \pi_j^*(r) P[N=r] = \sum_{r=0}^{\infty} \pi_j^*(r) e^{-\lambda T} \frac{(\lambda T)^r}{r!} \quad (\text{VII-3.18})$$

The continuous time discrete state chain (i.e., when inter-arrival times of events are exponential) may be analyzed more efficiently using the notion of transition intensities discussed in section V.4. Important statistics, such as $\pi_j^*(T)$, may then be obtained by using fast Laplace Transform Techniques [23].

VII.4. Decision Making in the Design of Structural Systems to Resist Random Vibration

One of the most useful properties of the proposed discrete state Markov process approach to random vibration reliability analysis, is that it provides a framework for quantifying system performance in terms of losses or rewards. This aspect of the theory of discrete state Markov processes has recently received considerable attention in the field of control processes and mathematical optimization [23, 25, 26].

Basically, the structural system is visualized to generate a sequence of rewards (or losses) as it travels from one state (of damage) to another. In the analysis of a given system, our interest focuses on the total value that system is likely to realize during its "life". In design, we seek the optimal values of the design parameters, i.e., those that maximize the total expected value of the system.

First, the notions of the "reward matrix" and the "present value" are discussed, and some important relations, due to Howard [23], are quoted. Next, the theory is applied to several simple two- and three-state Markov models of failure mechanisms.

Some Basic Relations

a. Events Occur at Discrete Times

Let us define an $m \times m$ reward matrix $R = [r_{jk}]$, whose element r_{jk} represents the "reward" received (or loss sustained) if the system makes a transition from state j to state k during a single event. Let $v_j(v)$ be the expected total reward earned during v transitions if the system starts in j . Often, in decision making involving engineering structures, it will be very important to discount future returns. Assume for simplicity, that events occur at unit time intervals. Let the factor β denote the present value of a unit reward to be received at the end of a unit time interval.

Presume that the system makes a transition from j to k during the first event (with $v-1$ events remaining), then the expected accumulated reward $v_j(v)$ will be equal to $r_{jk} + \beta v_k(v-1)$. Considering all possible values of k , one is led to the following simple iterative relationship [23]

$$v_j(v) = \sum_{k=1}^m p_{jk} [r_{jk} + \beta v_k(v-1)] \quad v=1,2,\dots \quad (\text{VII-4.1})$$

It is useful to define a quantity q_j as the expected immediate reward for state j as follows

$$q_j = \sum_{k=1}^m p_{jk} r_{jk} \quad (\text{VII-4.2})$$

It is the reward expected to be received during the next event, if the present state is j . Inserting Eq. VII-4.2 into Eq. VII-4.1, we may write

$$v_j(v) = q_j + \beta \sum_{k=1}^m p_{jk} v_k(v-1) \quad v=1,2,\dots \quad (\text{VII-4.3})$$

A closed form solution may be obtained for $v_j(v)$ by means of z -transformations [23]. For large values of v , however, the difference between $v_j(v)$ and $v_j(v-1)$ will become negligible. The limit $v_j = \lim_{v \rightarrow \infty} v_j(v)$ is called the present value of the system if it starts in state j . The present values v_j may be obtained by solving the system of m equations [23]

$$v_j = q_j + \beta \sum_{k=1}^m p_{jk} v_k \quad j=1,2,\dots,m \quad (\text{VII-4.4})$$

b. Events Occur According to a Poisson Process

Very similar results may be obtained for continuous time discrete state Markov processes. Some quantities have to be re-interpreted, however. The quantity r_{jj} becomes the reward earned by the system per unit time it occupies the state j . If the system makes a transition from state j to state k , it earns a reward r_{jk} , $j \neq k$. (Note that r_{jj} and r_{jk} , $j \neq k$, do not have the same dimensions.) A quantity q'_j is defined as the "earning rate" of the system [23]

$$q'_j = r_{jj} + \sum_{j \neq k} \lambda p_{jk} r_{jk} \quad (\text{VII-4.5})$$

where λ is the mean rate of arrival of events. Let $v_j(t)$ be the expected (discounted) total reward that the system will earn in a time t if it starts in j . The following differential equation, which is the continuous time equivalent of the difference equation in Eq. VII-4.4, may be obtained [23]

$$\frac{dv_j(t)}{dt} + (\delta + \lambda)v_j(t) = q'_j + \lambda \sum_{j=1}^m p_{jk} v_k(t) \quad (\text{VII-4.6})$$

where δ is the (continuous) discount rate (a unit quantity of money received after a very short interval dt is now worth $1 - \delta dt$). Laplace Transformations [23] may be used to obtain

closed form solutions for $v_j(t)$, as shown in Ref. 23. For large t , however, $v_j(t)$ will remain almost constant, i.e., $\frac{dv_j(t)}{dt} = 0$. Hence to obtain the present values, $v_j = \lim_{t \rightarrow \infty} v_j(t)$, one only has to solve the following system of m equations,

$$(\delta + \lambda)v_j = q_j' + \lambda \sum_{k=1}^m p_{jk} v_k \quad (\text{VII-4.7})$$

Two-and Three State Models of Failure Mechanisms

a) Models Involving a Single State of Damage

The simple two-state models, corresponding to a first-passage failure criterion, may be described by a 2×2 matrix P^* of "expected" (w.r.t. motion duration and intensity) transition probabilities, and a 2×2 reward matrix R . For example, presume that failure is postulated to occur when the response process $X(t)$, (which essentially consists of a sequence of isolated motions,) first surpasses a D type barrier, $|X=a|$, where a denotes the predetermined threshold value. At least two different "replacement policies" are possible [see, for example, Esteva [81]] leading to somewhat different state definitions and different sets of matrices, P^* and R .

Policy 1: The structure is not replaced after failure has occurred. State 0 corresponds to "survival" and state 1 to "failure". The flow-graph of this 2-state model is shown in figure VII.5.

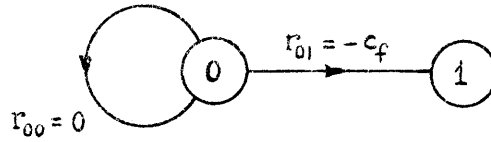


Fig.VII.5. Reward Structure With Policy 1.

Let c_f designate the "cost of failure". The matrices P^* and R take the form

$$P^* = \begin{bmatrix} p_{00}^* & p_{01}^* \\ 0 & 1 \end{bmatrix} \quad R = \begin{bmatrix} 0 & -c_f \\ 0 & 0 \end{bmatrix} \quad (\text{VII-4.8})$$

The probability $p_{01}^* = 1 - p_{00}^*$, that the structure will fail due to a single "typical event" will be computed subsequently. Let event occurrences be Poisson-like with average rate λ . The earning rates q_0' and q_1' may be evaluated using Eq. VII-4.5. We have $q_0' = -\lambda c_f p_{01}^*$ and $q_1' = 0$. To compute v_0 , the present value of the system if it starts in state 0, Eqs.VII-4.7 need to be solved.

$$\begin{aligned} (\delta + \lambda)v_0 &= -\lambda c_f p_{01}^* + \lambda (p_{00}^* v_0 + p_{01}^* v_1) \\ (\delta + \lambda)v_1 &= 0 + \lambda (v_1) \end{aligned} \quad (\text{VII-4.9})$$

The solution is

$$\begin{aligned}
 v_0 &= -\lambda c_f p_{01}^* / (\delta + \lambda p_{01}^*) \\
 v_1 &= 0
 \end{aligned}
 \tag{VII-4.10}$$

where δ designates the (continuous) discount rate. This result has been derived, in a very different way, by Esteva [81].

Policy 1: After structural failure has occurred, the structure is "instantaneously" replaced by an identical structure. As shown in the flow graph (Fig. VII.6), a loss c_f

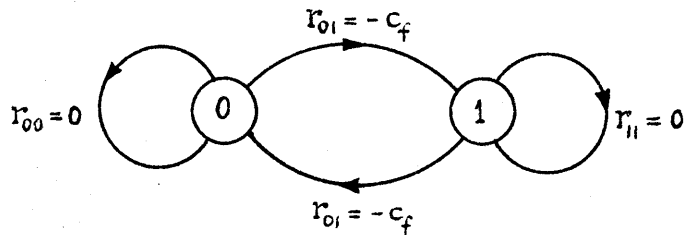


Fig.VII.6. Reward Structure With Policy 2.

is now sustained each time a transition $0 \rightarrow 1$ or $1 \rightarrow 0$ occurs. The emphasis here is not on whether or not state 1 is ever reached, but rather, on how many times the structure changes its state. The matrices P^* and R may be written as follows

$$P^* = \begin{bmatrix} p_{00}^* & p_{01}^* \\ p_{10}^* & p_{11}^* \end{bmatrix} \quad R = \begin{bmatrix} 0 & -c_f \\ -c_f & 0 \end{bmatrix}
 \tag{VII-4.11}$$

The Eqs. VII-4.7 now yield the following set of present values

$$v_0 = v_1 = \frac{-c_f p_0^* \lambda}{\delta} \quad (\text{VII-4.12})$$

which again corresponds to Esteva's result [81]. Note that, if δ tends to zero, v_0 and v_1 tend to infinity. As rewards keep accumulating in time at a constant average rate, the concept of "present value" loses its meaning. It may be substituted by the "gain" of the process, i.e., the average reward received per unit time when time becomes very large. The gain is directly related to the so-called "steady state probabilities" which may be obtained by simply solving a system of linear equations. For further details, see Ref.23.

Let c_0 designate the cost of acquisition of the structure, (which is presumably a non-decreasing function of the value a of the threshold level.) In design the aim is essentially to minimize the total present cost or to maximize the total present value of the structural system

$$\text{Maximize } (v_0 - c_0) \quad (\text{VII-4.13})$$

where v_0 is the present value of the "built" structure (which is likely to increase as the threshold value a increases). Note that v_0 in Eq. VII-4.10 or in Eq. VII-4.12 depends upon

the structural properties and the input statistics only through p_{01}^* , which is the probability that failure occurs as a result of a "typical" event in the sequence of motions that characterize the response process $X(t)$. If failure is postulated to occur when $X(t)$ first surpasses a D type barrier, $|X|=a$, where a denotes the predetermined (double) threshold value, then we have

$$p_{01}^* = \int_i \int_s P[T_{f,D} \leq s] p(i,s) di ds \quad (\text{VII-4.14})$$

where $P[T_{f,D} \leq s] = 1 - L_{T_{f,D}}(s)$, denotes the probability that $T_{f,D}$, the time to first passage across the D type barrier, will be less than or equal to the motion duration. Using Eq. III-3, one may write

$$p_{01}^* = 1 - \int_i \int_s A_D e^{-\alpha_D s} p(i,s) di ds \quad (\text{VII-4.15})$$

In many practical problems, where failure is very unlikely to be due to any given "typical" event, it will be reasonable to take $A_D \approx 1$ and to approximate $\exp\{-\alpha_D s\}$ by $1 - \alpha_D s$. Eq. VII-4.14 then reduces to

$$p_{01}^* \approx E[s] \int_i \alpha_D p(i) di \quad (\text{VII-4.16})$$

where $E[s]$ is the expected motion duration. By substituting

α_D by its value given by Eq. IV-3.9, one finally obtains an expression for p_{01}^* in terms of several potentially important design parameters, e.g., the barrier level a .

$$p_{01}^* \approx 2v_0 E[s] \int \frac{1 - \exp\{-ka/2\sigma_x\}}{i \exp\{a^2/2\sigma_x^2\} - 1} \rho(i) di \quad (\text{VII-4.17})$$

where σ_x^2 , the mean square response, is proportional to the (input motion) intensity i . Furthermore, k , v_0 and σ_x all depend on the first few moments of the output spectral density, which in turn depends on the structural properties and characteristics of the excitation.

b) Models Involving Two Damage States

The Markov approach becomes particularly attractive when more than a single damage state needs to be defined to adequately describe a failure mechanism. Consider, for example, the three-state processes, models A and B, discussed in the preceding section.

Presume that, in model A, which represents a simple deterioration process, actual loss is suffered only when the response exceeds the higher of the two prescribed threshold levels, i.e., when the system enters state 2. This occurs as a result of the transition $0 \rightarrow 2$, or $1 \rightarrow 2$. The corresponding "rewards" are denoted by r_{02} and r_{12} . Furthermore, let $r_{02} = r_{12} = -c_f$, where c_f denotes the "cost of failure". No penalty

or reward is received as a result of other possible transitions. Figure VII.7 shows the flow-graph for model A with the

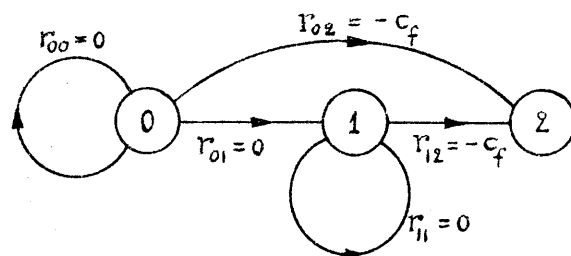


Fig.VII.7. Model A Reward Structure

values of all nonzero "rewards" appearing as labels on the corresponding branches. Dropping the superscript A in Eq. VII-3.1, the matrix P^* of "expected" (w.r.t. intensity and duration) transition probabilities takes the form

$$P^* = \begin{bmatrix} p_{00}^* & p_{01}^* & p_{02}^* \\ 0 & p_{11}^* & p_{12}^* \\ 0 & 0 & 1 \end{bmatrix} \quad (\text{VII-4.18})$$

The vector $\underline{q}' = \{q_1'; q_2'; q_3'\}$ of "earning rates" may be evaluated using Eqs. VII-4.5

$$\underline{q}' = \{ -\lambda c_f p_{02}^* ; -\lambda c_f p_{12}^* ; 0 \} \quad (\text{VII-4.19})$$

The present values v_j may be obtained by solving the system of equations, Eqs. VII-4.7

$$\begin{aligned}
(\delta + \lambda)v_0 &= -\lambda c_f p_{02}^* + \lambda (p_{00}^* v_0 + p_{01}^* v_1 + p_{02}^* v_2) \\
(\delta + \lambda)v_1 &= -\lambda c_f p_{12}^* + \lambda (p_{11}^* v_1 + p_{12}^* v_2) \\
(\delta + \lambda)v_2 &= 0 + \lambda v_2
\end{aligned}
\tag{VII-4.20}$$

One easily finds:

$$\begin{aligned}
v_0 &= -\lambda c_f \left[p_{02}^* + \frac{\lambda p_{01}^* p_{12}^*}{\delta + \lambda p_{12}^*} \right] / [\delta + \lambda (p_{01}^* + p_{02}^*)] \\
v_1 &= -\lambda c_f p_{12}^* / (\delta + \lambda p_{12}^*) \\
v_2 &= 0
\end{aligned}
\tag{VII-4.21}$$

Note that, if the discount rate δ equals zero, then one obtains, not unexpectedly, $v_0 = v_1 = -c_f$, $v_2 = 0$. The probabilities p_{02}^* , p_{12}^* and p_{01}^* , may be expressed in terms of the pertinent design parameters (e.g., the threshold levels a_1 and a_2) by taking the expectation with respect to motion duration S and intensity I of $p_{02}(s)$, $p_{12}(s)$ and $p_{01}(s)$, respectively. The latter are given by Eqs.V-4.4 to V-4.6.

In model B, a loss of non-structural nature is sustained when the response first crosses the lower of the two fixed thresholds. An additional (and presumably larger) loss is suffered if the higher barrier is also exceeded. Let those

losses be represented, respectively, by c_d , the cost of non-structural damage, and c_f , the cost of structural failure. The flow graph for model B, again with all nonzero rewards shown as labels on the corresponding branches, is shown in figure VII.8.

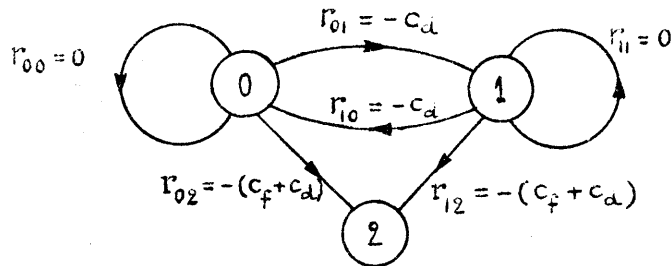


Fig.VII.8. Model B Reward Structure.

Again dropping the superscript B in Eq. VII-3.2, the "expected" (w.r.t. intensity and duration) one-step transition matrix, P^* , becomes

$$P^* = \begin{bmatrix} p_{00}^* & p_{01}^* & p_{02}^* \\ p_{10}^* & p_{11}^* & p_{12}^* \\ 0 & 0 & 1 \end{bmatrix} \quad (\text{VII-4.22})$$

From Eqs. VII-4.5 the vector of earning rates becomes

$$\underline{q}' = \{-\lambda[p_{01}^* c_d + p_{02}^* (c_d + c_f)]; -\lambda[p_{10}^* c_d + p_{12}^* (c_d + c_f)]; 0\} \quad (\text{VII-4.23})$$

Eqs. VII-4.7, with the p_{jk}^* and q'_j substituted by the above values, yield the following set of "present values":

$$v_0 = v_1 = - \frac{\lambda [p_{01}^* c_d + p_{02}^* (c_d + c_f)]}{\delta + \lambda p_{02}^*} \quad (\text{VII-4.24})$$

$$v_2 = 0$$

For $\delta = 0$, we obtain

$$v_0 = v_1 = c_f + \left(1 + \frac{p_{01}^*}{p_{02}^*}\right) c_d \quad (\text{VII-4.25})$$

The probabilities p_{01}^* and p_{02}^* are the expected values, w.r.t. the motion duration S and intensity I , of $p_{01}(s)$ and $p_{02}(s)$, respectively. The latter quantities have been derived in Ch. V, and are given by Eqs. V-4.8. For example,

$$\begin{aligned} p_{01}^* &= 1 - \int_i \int_s [p_{02}(s) + p_{00}(s)] p(i, s) di ds \\ &= \int_i \int_s [\exp\{-\alpha_{D;a_2} s\} - \exp\{-\alpha_{D;a_1} s\}] p(i, s) di ds \end{aligned} \quad (\text{VII-4.26})$$

where $\alpha_{D;a_2}$ and $\alpha_{D;a_1}$ are the decay rates in the D type barrier first passage densities corresponding to the threshold values a_1 and a_2 , respectively. When a crossing of the lower threshold is unlikely to be due to any given "typical" event in the sequence, then it may be reasonable to approximate Eq.VII-4.26 by

$$p_{01}^* \approx 2v_0 E[s] \int_i [\alpha_{D;a_1}^{-\alpha_{D;a_2}}] p(i) di \quad (\text{VII-4.27})$$

$$p_{01}^* \sim 2v_0 E[s] \int_i \left[\frac{1 - \exp\{-ka_1/2\sigma_x\}}{\exp\{a_1^2/2\sigma_x^2\} - 1} - \frac{1 - \exp\{ka_2/2\sigma_x\}}{\exp\{a_2^2/2\sigma_x^2\} - 1} \right] p(i) di \quad (\text{VII-4.28})$$

where σ_x^2 , the mean square response, is proportional to the (excitation) intensity. The integrand depends (through the spectral shape factor k , the average frequency v_0 and the r.m.s. value of the response, σ_x) on the first few moments of the response spectral density, which in turn depends on the structural properties and on the characteristics of the excitation. It also depends on the threshold levels a_1 and a_2 . The total present value v_0 , given by Eq. VII-4.25, will depend (explicitly) on all above quantities, and also on the (relative) value of the losses c_d and c_f , on the discount rate δ and finally, on the event arrival rate λ .

The designer's aim is to balance cost and (expected) performance [89]. For the particular damage criterion considered (i.e., Model B) the objective is to maximize the total present value ($v_0 - c_0$) of the structural system, where c_0 denotes the acquisition cost, and v_0 is the present value of the "built" structure. The factors which influence ($v_0 - c_0$) are related to the excitation, the structural system, and the characteristics of the damage criterion.

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BIOGRAPHY

The author was born in Belgium on August 6, 1941.

He entered the Catholic University of Louvain, Belgium in 1960 and received the degree of Burgerlijk Bouwkundig Ingenieur (similar to B.S. in Civil Engineering) in June 1965.

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APPENDIX

To derive the distribution of the time to first passage of a D type or a B type barrier in the same way as that of an E type barrier, it was proposed (in Section III.4) to substitute the discrete two-state processes $D_a(n)$ and $B_a(n)$ by the continuous time two-state processes $D'_a(t)$ and $B'_a(t)$, respectively. The respective holding times in state zero, $T_{0,D}$ and $T_{0,B}$ were further assumed to be exponential random variables. This led to the estimates, Eqs. III-4.9 and III-4.11, for the first passage time distributions.

An alternate approach is to adopt a geometric distribution for (say, in the case of D type barriers) $N_{0,D}$, the number of consecutive $|X|$ -peaks below the threshold $|X|=a$. The two approaches lead to first passage distributions that are nearly identical except when the average of $N_{0,D}$ is very small, say, less than five. Note that the time to first passage is really a continuous random variable. It will be estimated here by $(2v_0)^{-1}N_{f,D}$, where $N_{f,D}$ is the total number of peaks of $|X|$ below the threshold value before "failure" occurs (see Fig.II.1). This estimate is expected to deviate from the true first passage time in several ways. First, the time until the arrival of the first peak of $|X|$ will not be correctly accounted for. Also, the actual time between peaks of $|X|$ is not a constant, but a random variable with mean $(2v_0)^{-1}$.

Let a geometric distribution be adopted for the discrete random variable $N_{0,D}$. The probability mass function of

$N_{0,D}$ has the form

$$p_{N_{0,D}}(n) = \Pr[N_{0,D}=n] = (1-p)^{n-1} \quad (A1)$$

where p denotes the probability that the process $D_a(n)$ will make a transition to state 1 when it is next observed, given that it is presently in state 0. To satisfy Eq. II-4.14, we must have,

$$p = \frac{1}{E[N_{0,D}]} = \left(\frac{v_a}{v_0}\right) \frac{1 - \exp\{-n_a/2v_a\}}{1 - v_a/v_0} \quad (A2)$$

The following reliability statement involving $N_{f,D}$ may be made in the case of a random start,

$$P[N_{f,D} > n] = A_D (1-p)^n \quad n=0,1,\dots \quad (A3)$$

where p is defined by Eq. A2, and A_D equals the probability that failure does not occur before the first arrival of the first peak of $|X|$. Note that, if the first peak happens to be above the threshold, the actual first crossing occurs some time before the arrival of that peak. Hence, A_D will be nearly equal to the probability that a randomly selected peak of $|X|$ has a value exceeding the threshold value. We have

$$A_D \approx \frac{E[N_{1,D}]}{E[N_{0,D}] + E[N_{1,D}]} = 1 - \frac{v_a}{v_0} \quad (A4)$$

Equation A3 may be cast into the form of Eq.III-4.3 by defining a new decay rate estimate α'_D , in the following way

$$(1-p)^{2v_0 t} = e^{-\alpha'_D t} \quad (A5)$$

Therefore,

$$\alpha'_D = -2v_0 \log_e (1-p) \quad (A6)$$

$$\alpha'_D = -2v_0 \log_e \left(1 - \frac{v_a}{v_0} \frac{1 - e^{-n_a/2v_a}}{1 - v_a/v_0} \right) \quad (A7)$$

Using Eqs. I-4.5 and I-5.4 the above result may be easily specialized for Gaussian processes. The new estimate of the decay rate is shown in figure III.5 (by a dotted line). An entirely similar analysis may be performed for B type barriers.