Optimal Operating Strategy for a Storage Facility

by

Ning Zhai

B.Eng., Electrical Engineering
National University of Singapore, 2007

Submitted to the School of Engineering in Partial Fulfillment of the Requirements for the Degree of

Master of Science in Computation for Design and Optimization
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ABSTRACT

In the thesis, I derive the optimal operating strategy to maximize the value of a storage facility by exploiting the properties in the underlying natural gas spot price.

To achieve the objective, I investigate the optimal operating strategy under three different spot price processes: the one-factor mean reversion price process with and without seasonal factors, the one-factor geometric Brownian motion price process with and without seasonal factors, and the two-factor short-term/long-term price process with and without seasonal factors. I prove the existence of the unique optimal trigger prices, and calculate the trigger prices under certain conditions. I also show the optimal trigger prices are the prices where the marginal revenue is equal to the marginal cost. Thus, the marginal analysis argument can be used to determine the optimal operating strategy. Once the optimal operating strategy is determined, I use it to obtain the optimal value of the storage facility in three ways: 1, using directly the net present value method; 2, solving the partial differential equations governing the value of the storage facility; 3, using the Monte Carlo method to simulate the decision making process. Issues about parameter estimations are also considered in the thesis.

Thesis Supervisor: John E. Parsons
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From knowing nothing about natural gas, to writing a thesis on natural gas, I have made it because of my thesis advisor John Parsons. His enthusiasm and inspirations have given me the confidence to tackle the problem, and his timely advice and assistance have provided me the tools to solve the problem. I have learned from him not only the academic knowledge, but the attitude towards work and towards others. I thank my thesis advisor, John Parsons.

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One year in MIT has just passed, but the memory in MIT will last forever!

CONTENTS

LIST OF FIGURES

LIST OF TABLES

LIST OF NOTATIONS

r: the discount rate.

c: the storage cost per unit time in years.

TC: the total cost per unit time (storage cost + financing cost) in dollars.

P_0: the initial price.

P: the current price of the inventory (or the spot price of the natural gas).

P_L: the lower trigger price.

P_H: the upper trigger price.

P_L^*: the optimal lower trigger price.

P_H^*: the optimal upper trigger price.

P_1^*: the lower critical price.

P_2^*: the higher critical price.

P^*: a generic representation of the critical prices, both P_1^* and P_2^*.

t_P: the first hitting time to P, and t_{PH}, t_{PL}, t_{PH^*}, t_{PL^*}, t_{P^*}, t_{P1^*}, t_{P2^*} are similarly defined.

V: the value of a full storage facility.

W: the value of an empty storage facility.

Z: the value of a storage facility satisfies:

$$Z = \begin{cases} V - P & \text{If the current facilityis } \textbf{Full.} \\ W & \text{If the current facilityis } \textbf{Empty.} \end{cases}$$

V^*: the optimal value of a full storage facility.

W^*: the optimal value of a full storage facility.

Z^*: the optimal value of a storage facility when V and W in (2.1.4) is changed to V^* and W^*.

CHAPTER 1
INTRODUCTION

1.1 Chapter Introduction

In the chapter, I introduce the general storage valuation problem, and its relevance to the current market situation, where the link between the optimal operating strategy and the value of a storage facility is discussed. This provides the motivation for the development of the thesis. Next, I conduct a literature review on the storage valuation problem, and compare the different methods used in the literature. Then, the major results of the thesis are summarized. Finally, the organization of the thesis is presented with a brief summary on each of the following chapters.

1.2 Motivation

The US natural gas industry has transformed itself significantly as a result of the deregulation. The storage service in turn has been offered as an independent, distinct service. This change means any qualified company can lease, use and transfer those storage facilities for its own purpose. Therefore, the valuation of a storage facility is becoming a key component for companies to tap into and benefit from the new deregulated market.

In order to value a storage facility, the first step is to understand the functions of a storage facility. Traditionally, there are two key functions offered by a natural gas storage facility:

- Ensuring that the demand in winter months is met by storing excess supply in the summer months.
- Acting as an insurance against unforeseeable accidents or disasters amid other not so severe discontinuities in supply and demand.

Therefore, the storage facility traditionally serves as a cushion for the utility or transportation companies to meet the fluctuating demand.

Because of the deregulation, companies can now benefit from the new structure of the value chain: they can buy and store the natural gas in a storage facility when the price is low, and sell it when the price is high. The change is further reinforced by the liquid financial market, such as the natural gas futures actively traded in New York Mercantile Exchange (NYMEX).

As the new structure of the value chain in the natural gas industry emerges, there is an increasing need for a method to value a storage facility which incorporates this change. And because of the change, the value of the storage is closely related to the timing to operate the storage facility.

One approach to find the timing is to answer the questions of when to inject the natural gas, and when to withdraw the natural gas, based on the spot price. The answers to the above questions are what I call 'the operating strategy'. In the thesis, my aim is to answer the two questions, and this is the basic motivation for this thesis.

Mathematically, the problem can be framed in the following way: let Z be the value of a storage facility, and let P be the current spot price. I want to find the optimal operating strategy P_L^* (lower optimal trigger price) and P_H^* (higher optimal trigger price), such that I can solve the following optimization problem:

$$Z^*\left(P_L^*, P_H^*\right) = \max_{P_L, P_H} Z\left(P_L, P_H\right). \tag{1.1.1}$$

1.3 Literature Review

The value of a storage facility is closely related to the demand fluctuations, as explained by Pilopovic (1998). The fluctuations can be captured by the changes in the underlying natural gas price. Following this logic, researchers in the area use the natural gas price as the first step to tackle the valuation problem. Therefore, the modeling of the natural gas price is a key to accurately value a storage facility.

In the literature, researchers model the spot price of various commodities in different ways: Brennan & Schwartz (1985) uses a geometric Brownian motion process to model the spot price for copper, where the prices grow at a constant rate with the variations grow in proportion to time. However, Dixit & Pindyck (1994), and Smith & McCardle (1999) argue that a mean reversion price model is more appropriate for a general commodity valuation problem. The main argument is: when the price is higher than some fixed level, some higher cost companies will enter the market which leads to more supply, and the price will decrease; when the price is lower than the fixed level, some higher cost companies will exit, which leads to less supply, and the price will increase. For a detailed explanation on the one-factor models, please refer to Baker, Malcolm, Mayfield and Parsons (1998). Further, Gilbson & Schwartz (1990), and Schwartz & Smith (2000) propose two-factor models to better model the commodity price, since more dynamics can be generated by the two sources of uncertainties. Multi-factor models, such as the three-factor model discussed in Schwartz (1997), and models with jump processes, such as Thompson, Davison, & Rasmussen (2003), are also suggested to model the spot price. Other researchers, such as Clewlow & Strickland (2000), argue to use the forward term-structure to find the spot price, which is essentially a multi-factor spot price model.

The complexity of the above price models increases as the accuracy of the price models increases. However, one needs to strike a balance between the complexity of a model, and the manageability of a model. The model that captures the salient features of the natural gas price with the least complexity should be used in the valuation problem.

Once a price process is chosen, one can value a storage facility using one of the two main methods: the contingent claim method and the dynamic programming method. The contingent claim method is inspired by the Black & Scholes (1973) and Merton (1973), which uses a hedging portfolio to eliminate the uncertainties in the price process, and calculates the value from the resulting deterministic portfolio. The dynamic programming method employs a switching concept, where at each time one compares different options, and chooses the option that gives the best value. The contingent claim method requires more stringent assumptions, but it is relatively easy to obtain the parameters; while the dynamic programming method has fewer assumptions, but some parameters are hard to obtain. Sometimes, the two methods are equivalent.

Please refer to Dixit and Pindyck (1994) for a general discussion, or Insley and Wirjanto (2008) for a comparison study.

Using the above two methods, researchers have developed different ways to tackle similar valuation problems: Hodges (2003) and Thompson, Davison, & Rasmussen (2003) use the contingent claim method to derive partial differential equations for the storage valuation problem. Vollert (2003), and Dixit & Pindyck (1994) use the dynamic programming method to solve oil investment related problems. In either case, some differential or partial differential equations are derived, and the solutions can be found by solving the equations using different methods. For examples, Dixit (1989) solves analytically the equations for a simple entry-exit model using two trigger prices – one for entry, and one for exit; Carmona & Ludkovski (2005) uses numerical methods to solve the equations; Chen & Forsyth (2006) uses a semi-Lagrangian method to solve some partial differential equations for the natural gas storage valuation problem.

1.4 Contributions

To answer the questions of when to inject and when to withdraw, I use the concept of dividing the value of the storage facility into the value of an empty storage facility and the value of a full storage facility, as Hodges (2004) has done. By doing so, I can find the optimal operating strategy under which one can get the optimal value of a storage facility. The major contribution of the thesis is as follows:

First, to my best knowledge, this thesis is the first one that discusses the optimal operating strategy problem under both the logarithm mean reversion price process, and the two-factor short-term/long-term price process.

The reason to use the logarithm mean reversion model is that it may be more suited to model the commodity spot price as discussed above, and it is more suitable than the model used in Hodges (2003), because the model he uses has the problem that the spot price may go negative. Under the logarithm mean reversion model, I provide a comprehensive investigation of the problem by considering the deterministic model, the stochastic model, and the related seasonal models. Such a treatment gives readers a good starting point to tackle the optimal operating strategy problem.

However, one-factor price models, which attribute the randomness of the spot price to one source of uncertainty, may not be a good representation of the complex factors that influence the natural gas price, as discussed in Gilbson & Schwartz (1990). Therefore, I use a two-factor short-term/long-term model proposed by Schwartz & Smith (2000). This model is mathematically simple to manipulate while it preserves the important features in the natural gas spot price: the temporary mean reversion to some fixed level in the short term, and the permanent change in the fixed level in the long term. I believe this model is a good example of a model that strikes a balance between complexity and manageability. I show the similarity of this model to the one-factor logarithm mean reversion model using the dynamic programming method to derive the optimal operating strategy. This investigation and comparison can help people better understand the two-factor model so they can make informative judgment on the price process in their research.

Second, I find a general method to tackle the optimal operating strategy problem using the risk-neutral expectation of marginal revenue and marginal cost. This method can be used even when there are transaction costs. Such a method can help the practitioners to find numerically the value of a storage facility when they face a more complex real-life problem.

Lastly, I have obtained the optimal operating strategy for the seasonal models, where I formally consider the effects by adding the seasonal factor in different models, and find the functions and values from the seasonal factor in different models.

1.5 Organization of the Thesis

The thesis consists of six chapters, including this introduction chapter and a conclusion chapter.

In Chapter 2, I find the optimal operating strategy under a one-factor logarithm mean reversion price model. First, I discuss a related deterministic price model, and derive the optimal strategy for the model with the corresponding storage value. Then, I use two differential equations to derive the optimal strategy for the mean reversion model. I also find the values under the optimal strategy by solving the two differential equations. A seasonal factor is added next, and comparisons are made among different models. This chapter shows that the stochastic and seasonal components in the mean reversion model can contribute significantly the value of a storage facility.

In Chapter 3, I find the optimal operation strategy under the geometric Brownian motion (GBM) price model. I discuss both a related deterministic price mode and the GBM model. The key conclusion from this chapter is that this model itself is not adequate to tackle our storage problem, and we need to consider seasonal factor in order to use this model.

In Chapter 4, I present a two-factor short-term/long-term price model, which consists of both a mean reversion component and a geometric Brownian motion component. I derive the optimal operation strategy using the marginal analysis argument, and numerically solve the optimal value of a storage facility using Monte Carlo simulation method. This chapter shows again that both the stochastic and seasonal components are important in the valuation of the storage facility problem.

In Chapter 5, an estimation method for the two-factor model is introduced, where I use the Kalman filter and the forward curve to calibrate the parameters used in the two-factor model. And a parameter estimation example is presented at the last using the Henry Hub data.

In Chapter 6, I conclude the thesis by mentioning the key results in this thesis, and discussing some potential improvements. The appendix and reference are presented at the end of the thesis.

CHAPTER 2

ONE-FACTOR MEAN REVERSION MODELS

2.1 Chapter Introduction

In this chapter, the optimal operating strategy for the valuation of a storage facility with a one-factor mean reversion price process is studied in detail. The spot price is modeled as:

$$\ln(P) = u_t, \text{ where } du_t = k\cdot(m_{real} - u_t)\cdot dt + \sigma dB_{t,\, real}. \tag{2.1.1}$$

u_t follows an Ornstein-Uhlenbeck (OU) process with the reversion rate k, the mean m_{real}, the volatility σ, and the standard Brownian motion B_{real}.

Solving for u_t, I have:

$$u_t = u_0\cdot e^{-k\cdot t} + m_{real}\cdot\left(1 - e^{-k\cdot t}\right) + \int_0^t \sigma\cdot e^{-k\cdot(t-s)}\, dB_s\,. \tag{2.1.2}$$

The same process under the risk-neutral measure can be written as:

$$\ln(P) = u_t, \text{ where } du_t = k\cdot(m - u_t)\cdot dt + \sigma dB_t,\ m = m_{real} - \sigma\cdot\lambda / k\,. \tag{2.1.3}$$

λ is the market price of risk, which I assume is constant and is exogenously determined. Further, B_t is the Brownian motion under the risk-neutral measure. In the following sections, I work with (2.1.3) instead of (2.1.1).

In this chapter, I first investigate the deterministic component of (2.1.3) by setting $\sigma = 0$, and I call this 'the deterministic case'. Then, I proceed to solve the full model of (2.1.3), in which σ is positive, and I call this 'the stochastic case'. I prove and derive the optimal operating prices, with the corresponding value of a storage facility. Next, the seasonal factor is considered, and a discussion on the sources of the storage value is made at last.

In this chapter, I assume V, W, V^*, and W^* are twice differentiable, and there is no storage cost when the facility is empty. Further, the optimal value Z^* must be non-negative in any situation, since I can always choose to stop operation and obtain zero value for a storage facility.

2.2 Deterministic Model without Seasonality

In this section, I assume the volatility parameter $\sigma = 0$ in both (2.1.2) and (2.1.3). Consequently, the price can be expressed as:

$$P = e^{u_t}, \textit{ where } u_t = u_0 \cdot e^{-k \cdot t} + m \cdot \left(1 - e^{-k \cdot t}\right). \tag{2.2.1a}$$

Let the initial value be $P_0 = \ln(u_0)$, and assume $e^m > P \geq P_0 > 0$. Solve equation (2.2.1a) for the first hitting time t_P, which is the first time that reaches the price level P, starting from P_0:

$$t_P = \frac{1}{k} \ln\left(\frac{m - \ln(P_0)}{m - \ln(P)}\right). \tag{2.2.1b}$$

Further in the following sections, I assume the parameters satisfy the relationship:

$$0 > -\frac{c}{k} \cdot e^{-m + \frac{r}{k} - \frac{1}{2 \cdot k} \cdot \sigma^2} > -e^{-1}, \text{ or } 0 > -\frac{c}{k} \cdot e^{-m + \frac{r}{k}} > -e^{-1} \text{ when } \sigma = 0. \tag{2.2.2}$$

2.2.1 Scenario One – Independent Cost Rate

In this section, I make the following assumption: the storage cost is at a constant rate c dollars per year for a full storage facility, and there is neither cost associated with an empty facility nor any cost related to operating activities (i.e. injections and withdrawals).

Assume $0 < P_0 \leq P_L \leq P_H < e^m$, and c, k, r, m are all positive. Let t_{PL}, t_{PH} be the first hitting times for P_L, P_H respectively (the first hitting time passing P in the risk-neutral world is not necessarily the real time passing P, and I use this name here just for convenience). The profit or net present value Z from buying at P_L and selling at P_H is:

$$Z = e^{-r \cdot t_{PH}} \cdot P_H - \int_{t_{PL}}^{t_{PH}} c \cdot e^{-r \cdot t} \, dt - e^{-r \cdot t_{PL}} \cdot P_L, \tag{2.2.3}$$

$$\text{where } t_{PH} = \frac{1}{k} \ln\left(\frac{m - \ln(P_0)}{m - \ln(P_H)}\right), \text{ and } t_{PL} = \frac{1}{k} \ln\left(\frac{m - \ln(P_0)}{m - \ln(P_L)}\right).$$

In the section, I want to find the value of P_L^* and P_H^* such that:

$$Z^*\left(P_L^*, P_H^*\right) = \max_{P_L, P_H} Z\left(P_L, P_H\right). \tag{2.2.4}$$

Therefore, to find the maximum profit, I first differentiate the profit function Z with respect to P_H, and then equate it to 0:

$$\frac{\partial}{\partial P_H} Z = -\frac{r \cdot e^{-r \cdot t_{PH}}}{k(m - \ln(P_H))} + e^{-r \cdot t_{PH}} - \frac{c \cdot e^{-r \cdot t_{PH}}}{k \cdot (m - \ln(P_H)) \cdot P_H}, \tag{2.2.5a}$$

$$-\frac{r \cdot e^{-r \cdot t_{P*}}}{k(m - \ln(P^*))} + e^{-r \cdot t_{P*}} - \frac{c \cdot e^{-r \cdot t_{P*}}}{k \cdot (m - \ln(P^*)) \cdot P^*} = 0. \tag{2.2.5b}$$

To solve equation (2.2.5b) for the critical price P^*, I make the following simplifications:

$$(2.2.5\,b) \Rightarrow -\frac{r}{k(m - \ln(P^*))} + 1 - \frac{c}{k \cdot (m - \ln(P^*)) \cdot P^*} = 0,$$

$$\Rightarrow -r \cdot P^* + k \cdot (m - \ln(P^*)) \cdot P^* - c = 0,$$

$$\Rightarrow \frac{c}{k} = P^* \cdot \left(m - \frac{r}{k} - \ln(P^*)\right).$$

Let $a = \frac{c}{k}$, $b = m - \frac{r}{k}$, $P^* = e^u$, and we have :

$$\Rightarrow a = e^u \cdot (b - u),$$

$$\Rightarrow -a \cdot e^{-b} = e^{u-b} \cdot (u - b).$$

To solve the above equation, I need to introduce the LambertW function, which is the solution to the equation of the form (for some constant *const*): *const* = $e^x \cdot x$. Please refer to Corless, Gonnet, Hare, Jeffrey, & Knuth, D.E. (1996) for further information.

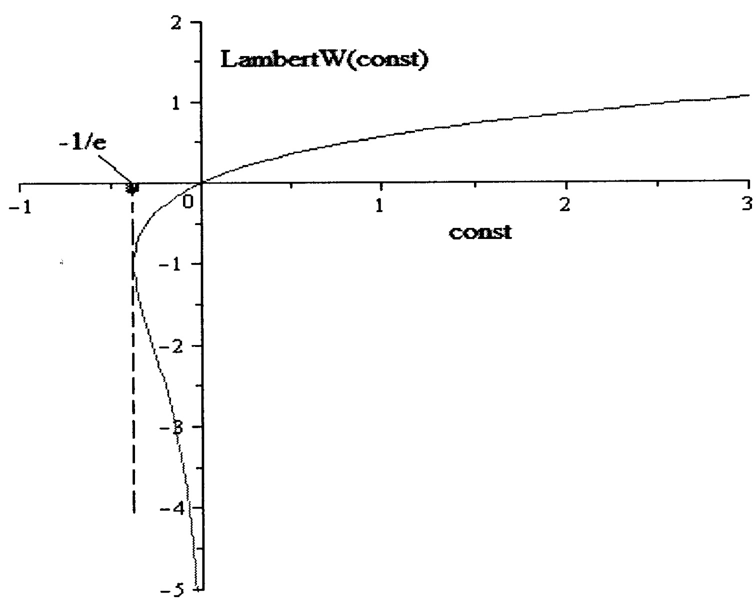

Figure 2.2.1: The LambertW function.

For the storage problem, since all the parameters and solutions are required to be real numbers, we are only interested in those constants and solutions that are real. Therefore, I only plot the LambertW function when its value is real in Figure 2.2.1. With the requirement that all the constants and solutions that are real numbers, the LambertW function has the following properties:

Range of the constant *const*	**Solution Properties of $const = e^x \cdot x$**
(a). When $const < -1/e$:	No Real Solution.
(b). When $-1/e \le const \le 0$, and LambertW($const$) $\ge$ -1:	LambertW($const$) is an increasing function from: LambertW($1/e$) = -1 to LambertW(0) = 0.
(c). When $-1/e \le const \le 0$, and LambertW($const$) $\le$ -1:	LambertW($const$) is a decreasing function from: LambertW($1/e$) = -1 to LambertW(0) = -∞

Table 2.2.1: LambertW function's properties.

Using the LambertW function and the assumption (2.2.2), I can write (2.2.5b) as:

$$\Rightarrow -a \cdot \mathrm{e}^{-b} = \mathrm{e}^{u-b} \cdot (u-b) ,$$

$$\Rightarrow u - b = \mathrm{LambertW}\left(-a \cdot \mathrm{e}^{-b}\right) ,$$

$$\Rightarrow u = m - \frac{r}{k} + \mathrm{LambertW}\left(-\frac{c}{k} \cdot \mathrm{e}^{-m+\frac{r}{k}}\right) .$$

Therefore, the critical price can be written as:

$$\Rightarrow P^* = \mathrm{e}^{m - \frac{r}{k} + \mathrm{LambertW}\left(-\frac{c}{k} \cdot \mathrm{e}^{-m+\frac{r}{k}}\right)} . \tag{2.2.6}$$

With the assumption (2.2.2), the real value requirement, and the property (b) in Table 2.2.1, I have two solutions to (2.2.6), because the following function has two values:

$$\mathrm{LambertW}\left(-\frac{c}{k} \cdot \mathrm{e}^{-m+\frac{r}{k}}\right) .$$

Therefore, I have two critical prices from (2.2.6), denoted as P_1^* and P_2^*, and I assume $P_1^* \le P_2^*$. For example, let $m = 2.3$, $k = 1$, $r = 0.05$, $c = 1$. The LambertW function has two distinct values: −0.1187 and −3.5039. Therefore, the solutions to (2.2.5b) are:

$P_1^* = 0.2854$, and $P_2^* = 8.4260$.

Next, we need to check the sign of the second order derivative of the profit function Z with respect to P_H at the critical prices P_1^* and P_2^*. Let t_{P*} be the first hitting time to P^*. The second derivative of Z with respect to P_H at the critical value P^* (here I use P^* to be a generic representation of both P_1^* and P_2^*) is:

$$\frac{\partial^2}{\partial P_H^2} Z \bigg|_{P_H = P^*} = e^{-r \cdot t_{PH}} \cdot \frac{1}{k \cdot (m - \ln P^*)^2 \cdot (P^*)^2} \cdot \left(-r \cdot P^* + c \cdot (m - \ln P^* - 1)\right)$$

With the assumption $m > P_H$, and the positivity of t_{PH}, r, k, c, to find the sign of the above function is equivalent to find the sign of:

$$\Rightarrow -r \cdot P^* + c \cdot (m - \ln P^* - 1),$$

$$\Rightarrow -r \cdot P^* + c \cdot \left(m - m + \frac{r}{k} - \mathrm{LambertW}\left(-\frac{c}{k} \cdot e^{-m + \frac{r}{k}}\right) - 1\right),$$

$$\Rightarrow -r \cdot P^* + c \cdot \frac{r}{k} - c \cdot \left(\mathit{LambertW}\left(-\frac{c}{k} \cdot e^{-m + \frac{r}{k}}\right) + 1\right),$$

$$\Rightarrow r \cdot \left(-P^* + \frac{c}{k}\right) - c \cdot \left(\mathit{LambertW}\left(-\frac{c}{k} \cdot e^{-m + \frac{r}{k}}\right) + 1\right).$$

Using the assumption (2.2.2) and the LambertW value that is within (-1, 0), I have:

$$\Rightarrow 0 > -\frac{c}{k} \geq -e^{m - \frac{r}{k} - 1} > -e^{m - \frac{r}{k} + \mathit{LambertW}\left(-\frac{c}{k} \cdot e^{-m + \frac{r}{k}}\right)} = -P_2^*,$$

$$\Rightarrow -P_2^* + \frac{c}{k} < 0,$$

$$\Rightarrow r \cdot \left(-P_2^* + \frac{c}{k}\right) < 0,$$

$$\Rightarrow r \cdot \left(-P_2^* + \frac{c}{k}\right) - c \cdot \left(\mathit{LambertW}\left(-\frac{c}{k} \cdot e^{-m + \frac{r}{k}}\right) + 1\right) < 0,$$

$$\Rightarrow \frac{\partial^2}{\partial P_H^2} Z \bigg|_{P_H = P_2^*} < 0.$$

Because the second order derivative at the critical price P_2^* is negative, together with the functional form of Z, I conclude P_2^* is the value that locally maximizes Z for any given P_L^*. Therefore, $P_H^* = P_2^*$.

Next, I argue that P_1^* is the value that minimizes the function Z, for any given P_H^*. For a continuous, differentiable function Z with only two critical values, because the second order derivatives at the two critical values are non-zero, which can be shown using the assumption (2.2.2) and the properties of LambertW function, one can conclude that one critical value must be the local maximum while the other must be the local minimum. Using this argument, since P_2^* is the local maximum, P_1^* must be the local minimum, and I have the following result:

when $P_H \subset [P_1^*, +\infty]$,	$P_H = P_2^*$ is the maximum value for the function Z.
when $P_H \subset [0, P_2^*]$,	$P_H = P_1^*$ is the minimum value for the function Z.

Similar results can be derived for P_L using the same argument. However, observing the function Z, I can find the first and second derivatives of the function Z with respect to P_L are just the opposite of those with respect to P_H.

$$\frac{\partial}{\partial P_H} Z|_{P_H = P} = -\frac{\partial}{\partial P_L} Z|_{P_L = P},$$
$$\frac{\partial^2}{\partial P_H^2} Z|_{P_H = P} = -\frac{\partial^2}{\partial P_L^2} Z|_{P_L = P}. \qquad (2.2.7)$$

Therefore, the critical value for P_L should be the same as P_1^* and P_2^*. However, because the sign difference in the second derivative with respect to P_L comparing to that to P_H, the conclusion for the optimum is different:

when $P_L \subset [P_1^*, +\infty]$,	$P_L = P_2^*$ is the minimum value for the function Z.
when $P_L \subset [0, P_2^*]$,	$P_L = P_1^*$ is the maximum value for the function Z.

I plot the function Z, as a function of P_H while setting $P_0 = P_L = 0.1$ (to satisfy $P_0 = P_L < P_1^* = 0.2854$) in Figure 2.2.2 (left). Then, I plot the function Z, as a function of P_L while setting $P_H = 9$ (to satisfy $P_2^* = 8.4260 < P_H < e^m = 9.9742$) in Figure 2.2.2 (right).

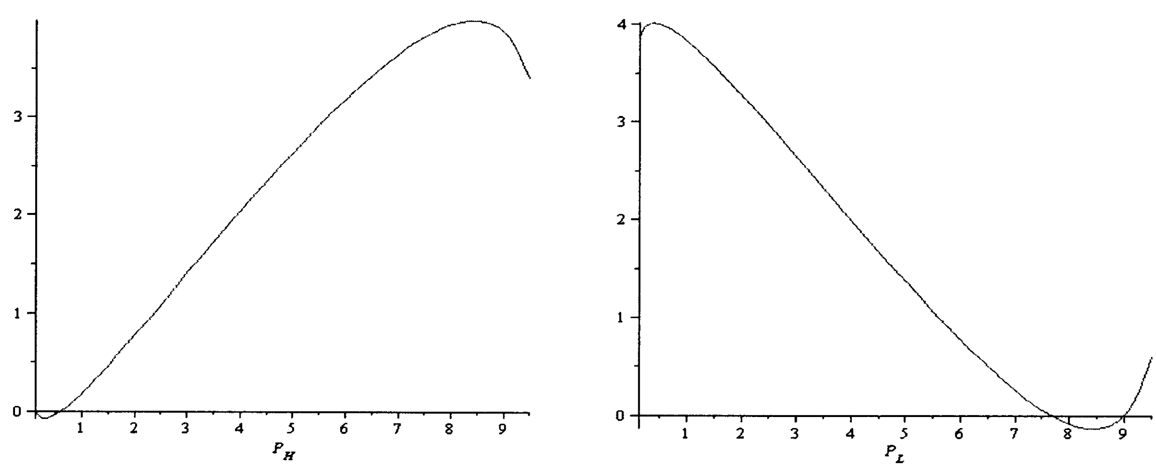

Figure 2.2.2: Z changes as P_H and P_L change.

We can clearly see that the two shapes of Z are just the opposite of each other, and the two critical prices are the values that achieve the minimum and the maximum respectively. Therefore, for the problem (2.2.4), with the assumptions of $0 < P_0 \leq P_L \leq P_H < e^m$ and $P_0 \leq P_1^*$, the function Z achieve the maximum Z_{max} when $P_L^* = P_1^*$ and $P_H^* = P_2^*$.

If we allow P_0 to vary when we keep P_L and P_H fixed, the partial derivative with respect to P_0 is:

$$\frac{\partial}{\partial P_0} Z = \frac{r}{k} \cdot \frac{1}{P_0 \cdot (m - \ln(P_0))} \cdot Z. \tag{2.2.8}$$

Using the assumptions that $0 < P_0 \leq P_L \leq P_H < e^m$, and Z is non-negative, (2.2.8) is positive when Z is not equal to 0, therefore, to achieve the maximum value, we should take $P_0 = P_L$. (2.2.8) is equal to 0 only when Z = 0, one such case when $P_L = P_H$.

Combining with the previous result, I have for the function Z, to achieve the maximum value of the function Z_{max} (I assume that it is positive), we must have $P_0 = P_L = P_1^*$, and $P_H = P_2^*$.

If we fix k, r, c, m, σ, P_0 (or u_0), and assume $0 < P_0 \leq P_L \leq P_H < e^m$, I can use the above analysis to find the optimal operating strategy when the initial price P_0 falls in different price ranges. The result is shown in Table 2.2.2.

Condition	Strategy
When $P_0 < P_1^*$,	Wait until the price goes to P_1*. Buy at $P_L^* = P_1^*$, and sell at $P_H^* = P_2^*$.
When $P_1^* \leq P_0 \leq P_2^*$,	Buy immediately at $P_L^* = P_0$, and sell at $P_H^* = P_2^*$.
When $P_0 > P_2^*$,	Do not buy or sell at any time (no operations).

Table 2.2.2: The operating strategy for deterministic case

From the above strategy, I can draw the following conclusions: (1) as long as $0 < P_0 \leq P_2^*$, we always sell at $P_H^* = P_2^*$. (2) When $P_1^* \leq P_0 < P_2^*$, the rate of the price appreciation is bigger than the rate of total cost changes (more will be discussed next), and this is the reason that we need to buy immediately at the current initial price $P = P_0$: the earlier we buy, the more profit we can get. (3) When $P_0 > P_2^*$, the rate of the price appreciation is smaller than the rate of the total cost changes. Therefore, the profit obtained by buying and holding is less than the total cost incurred in the same period, and the best policy is to do nothing. The conclusions (2) and (3) above can be formalized as follows.

The optimal operating strategy can be thought as a tradeoff between the profit gained by the price appreciation from holding the inventory, and the total cost incurred during the holding period. The tradeoff can be examined by the dynamics between the marginal cost (MC), which is the total cost incurred from holding for one additional time period, and the rate of the price appreciation (MR), which is the gain from holding for one additional time period. In our model, the marginal cost consists of the cost of capital and the cost of storage. Let *TC* denote the total cost function. Thus, the marginal cost can be defined as:

$$\frac{dTC}{dt} = r \cdot P + c = r \cdot e^{u} + c \, . \tag{2.2.9}$$

Similarly, the rate of the price appreciation can be written as:

$$\frac{dP}{dt} = k\left(m + \frac{1}{2}\frac{\sigma^2}{k} - u\right) \cdot \exp(u) \tag{2.2.10}$$

In Figure 2.2.3, I plot the dynamics of (2.2.9) and (2.2.10) with the storage cost (not the total cost) set at $c = 1$. The crossing points are given by:

$$\frac{dTC}{dt} = \frac{dP}{dt} \, . \tag{2.2.11}$$

Solve (2.2.11), and it gives the identical solutions for P^* as those in (2.2.6). Equation (2.2.11) is the same equation as (2.2.5b), where we differentiate the profit function *Z* with respect to P_H (or P_L) and equate it to 0. Essentially, the two approaches are the same.

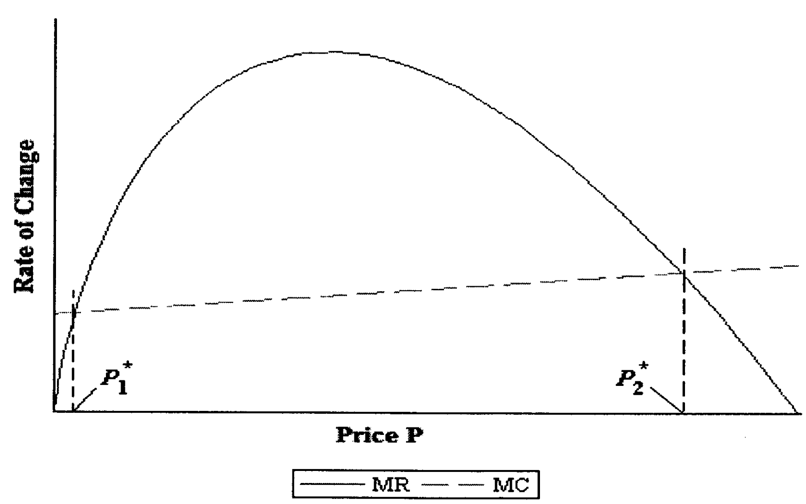

Figure 2.2.3: Plots of (2.2.9) and (2.2.10) as a functions of the price P.

Next, following the strategy in Table 2.2.2, and assuming $0 < P_0 \le P_L \le P_H < e^m$ and setting $P_H^* = P_2^*$, I plot the value of a storage facility when P_0 taking different values in Figure 2.2.4.

By observing Figure 2.2.4, I summarize the key results:

1. When $P_0 > P_2^*$, the value is 0.
2. When $P_0 = P_2^*$, Z = 0. Using (2.2.8), we find the partial derivative $\partial Z/\partial P_0 = 0$ at the point.
3. When $P_1^* < P < P_2^*$, to maximize the profit Z, we buy immediately, and the value is plotted accordingly.
4. When $P_0 = P_1^*$, the maximum possible value Z_{max} of the function Z is achieved only. This has been discussed in the paragraphs following (2.2.8). However, at the point $P_0 = P_1^*$, since Z_{max} is assumed to be positive, the partial derivative $\partial Z/\partial P_0 > 0$ according to (2.2.8).
5. When $P_0 < P_1^*$, the optimal strategy is to wait till the price reaches P_1^*. The increasing trend in Figure 2.2.4 is due to the fact that Z is an increasing function of P_0 from (2.2.8) when Z is positive. When P_0 approaches 0, the first hitting time to P_1^* approaches infinity. Therefore, the value approaches zero as a result. More will be discussed in Section 2.3.

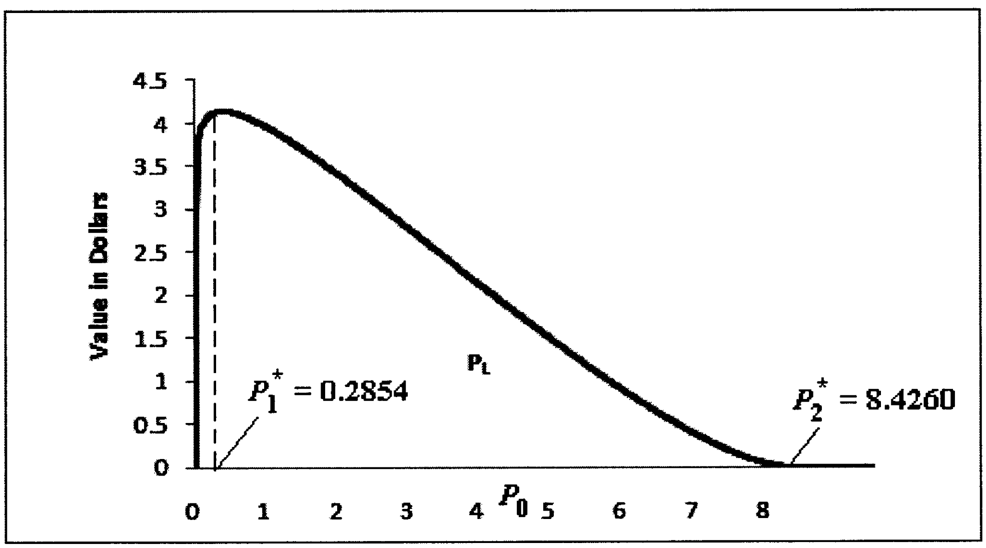

Figure 2.2.4: The optimal value of the facility as a function of initial price P_0.

2.2.2 Scenario Two – Proportionate Cost Rate

In this section, I assume the storage cost is proportional to the current spot price P. That is, at any time, the cost per unit time is $(c \cdot P)$ for a full storage facility, and there are no cost associated with either an empty facility or any transactions (i.e. injections and withdrawals).

Assume $0 < P_0 \le P_L \le P_H < e^m$. The net present profit (Z) by buying at P_L and selling at P_H is:

$$Z = e^{-r \cdot t_{PH}} \cdot P_H - \int_{t_{PL}}^{t_{PH}} c \cdot P_t \cdot e^{m \cdot t} e^{-r \cdot t} \, dt - e^{-r \cdot t_{PL}} \cdot P_L , \tag{2.2.12}$$

Take the derivative of (2.2.12) with respect to P_H, and P_L, and equate it to 0:

$$\frac{\partial}{\partial P_H} Z = \frac{\partial}{\partial P_L} Z = 0 . \tag{2.2.13}$$

I can follow the same argument as that in Section 2.2.1 to derive the critical prices P_H^* and P_L^* that maximize the profit function Z. The followings are an outline of the derivations:

$$\frac{\partial}{\partial P_H} Z = -\frac{r \cdot e^{-r \cdot t_{PH}}}{k\left(m - \ln(P_H)\right)} + e^{-r \cdot t_{PH}} - \frac{c \cdot e^{-r \cdot t_{PH}}}{k \cdot \left(m - \ln(P_H)\right)} ,$$

$$\Rightarrow -\frac{r \cdot e^{-r \cdot t_{P*}}}{k\left(m - \ln(P^*)\right)} + e^{-r \cdot t_{P*}} - \frac{c \cdot e^{-r \cdot t_{P*}}}{k \cdot \left(m - \ln(P^*)\right)} = 0 .$$

$$\Rightarrow k\left(m - \ln(P^*)\right) = r + c ,$$

$$\Rightarrow P^* = e^{m - \frac{r+c}{k}} .$$

Compared to the case in Section 2.2.1, there is only one critical price. The second derivative is:

$$\left.\frac{\partial^2}{\partial P_H^2} Z \right|_{P_H = P^*} = \left. e^{-r \cdot t_H} \cdot (-r - c) \cdot \left(\frac{\partial^2}{\partial P_H^2} t_H \cdot P_H + \frac{\partial}{\partial P_H} t_H \right) \right|_{P_H = P^*}$$

$$= \left. \frac{1}{k} \cdot \frac{1}{\left(m - \ln P_H\right)^2 \cdot P_H} \cdot e^{-r \cdot t_H} \cdot (-r - c) \right|_{P_H = P^*} .$$

Therefore:

$$\left.\frac{\partial^2}{\partial P_H^2} Z \right|_{P_H = P^*} < 0 .$$

I conclude that the $P_H^* = P^*$ is where the function Z achieves maximum for an arbitrary fixed value P_L. Similarly, by the same argument:

$$\left.\frac{\partial^2}{\partial P_L^2} Z\right|_{P_L = P^*} = -\left.\frac{\partial^2}{\partial P_H^2} Z\right|_{P_H = P^*} > 0 .$$

Hence, to maximize the value of Z, we need to buy immediately at $P_L^* = P_0$, and sell at the price $P_H^* = P^*$, assuming $0 < P_0 \leq P_L \leq P_H < e^m$.

Using the marginal argument as in 2.2.1, the rate of price gain is:

$$\frac{dP}{dt} = k\left(m + \frac{1}{2}\frac{\sigma^2}{k} - u\right)\cdot e^u \tag{2.2.14}$$

which is the same as (2.2.10). But the marginal cost now has changed to:

$$\frac{dTC}{dt} = r\cdot P + c\cdot P = (r + c)e^u . \tag{2.2.15}$$

Solve for P^* and it is the same as the solution to (2.2.13). The two functions of (2.2.15) and (2.2.16) are plotted in Figure 2.2.5 (with c=0.1).

Comparing to Figure 2.2.3, we can see that since the MR are the same for both cases, the solid MR line is identical for both of the two cases. But the MC part is different. Especially, there are two non-zero crossing points in Figure 2.2.3 while there is only one in Figure 2.2.5. This difference leads to different operating strategies for the two cases: in the case in Section 2.2.1, we have to wait till the current price reaches or exceeds the critical buying price P_1^*, and buy afterwards, while in this section's case, we buy immediately as long as the current price P exceeds 0. The difference comes from the way we represent the storage cost: in the constant storage cost case, MR can drop below the storage cost rate c since MR is a function of P, but not c, while in the proportional storage cost case, MR decreases as the storage cost rate $c\cdot P$ decreases. That's the reason why there is a difference in the plots in Figure 2.2.3 and Figure 2.2.5.

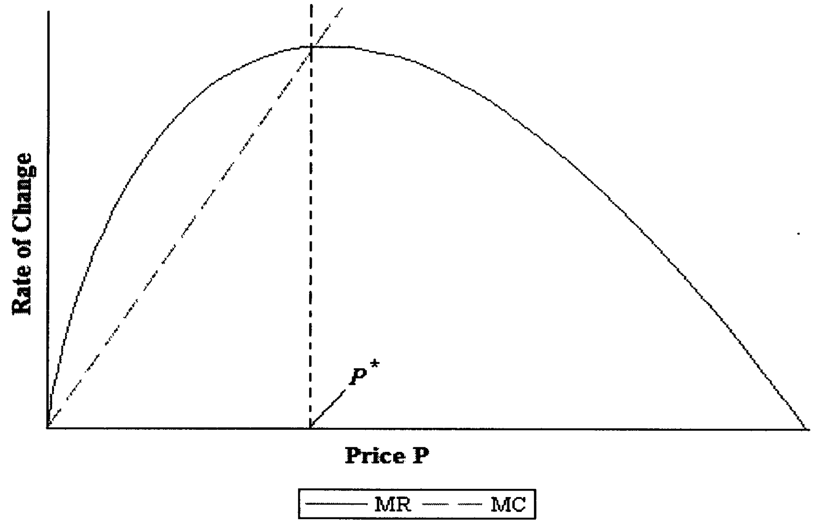

Figure 2.2.5: Plots of marginal cost and rate of the price gains

2.3 Stochastic Model without Seasonality

In this section, I assume the volatility parameter $\sigma > 0$. As a result, we are dealing with a stochastic price process. Assume the storage cost is fixed at a constant rate c dollars per year, and the real price process follows:

$$\ln(P) = u_t, \text{ where } du_t = k\cdot(m_{real} - u_t)\cdot dt + \sigma dB_{t,\,real}. \tag{2.3.1}$$

u_t follows an Ornstein-Uhlenbeck (OU) process with the reversion rate k, the mean m_{real}, the volatility σ, and the standard Brownian motion B_{real}.

The same process under the risk-neutral measure can be written as:

$$\ln(P) = u_t, \text{ where } du_t = k\cdot(m - u_t)\cdot dt + \sigma dB_t\,,\ m = m_{real} - \sigma\cdot\lambda / k\,. \tag{2.3.2}$$

λ is the market price of risk, which I assume is constant and exogenously determine. B_t is the Brownian motion under the risk-neutral measure.

In the following sections, I work with (2.3.2) instead of (2.3.1). Under (2.3.2), with Ito's Lemma, the instantaneous changes of P can be expressed as:

$$dP = k\cdot\left(m + \frac{\sigma^2}{2\cdot k} - \ln(P)\right)\cdot P\cdot dt + \sigma\cdot P\cdot dB_t\,. \tag{2.3.3}$$

I assume there is no trigger cost when changing between the empty and the full states in the storage problem, and I also assume the parameters satisfy the assumption (2.2.2). Let P_H be the higher trigger price, and let P_L be the lower trigger price, such that when $P > P_H$, we sell the inventory; when $P_L < P < P_H$, we buy the inventory; whenever $P < P_L$, we sell the inventory. I first derive the value of a storage facility using arbitrary P_L and P_H. Then, I find the optimal operating strategy P_L^* and P_H^* that maximize the function Z, that is:

$$Z^*\left(P_L^*, P_H^*\right) = \max_{P_L, P_H} Z\left(P_L, P_H\right). \tag{2.3.4}$$

2.3.1 Operating Strategies with Infinite Facility Life

In this section, I first derive the equations that determine the values of V and W under any arbitrary operation strategy P_L and P_H. Then, I derive the optimal operating strategy P_L^* and P_H^*, and find the corresponding values of V^* and W^* when the facility is operated according to the optimal operating strategy P_L^* and P_H^*.

In this section, I assume the storage facility is in the operation condition for infinite time, and the storage facility has a storage capacity of one unit. Since the facility has infinite operating life, the value of the facility is a function of only one parameter P: $V = V(P)$, and the parameter t (time) has no influence on the value of the facility.

2.3.1.1 Value of a Storage Facility with Arbitrary P_L and P_H.

Let Δ represent a small positive price change. I use the following notations:

$W_1(P)$ is the function of an empty facility when $P \leq P_L + \Delta$.

$W_2(P)$ is the function of an empty facility when $P \geq P_H - \Delta$.

$G_1 = V(P_H + \Delta)$.

$G_2 = V(P_L - \Delta)$.

$ProH$ is the probability of hitting $P_H + \Delta$ first, rather than hitting $P_L - \Delta$, starting at $P_H - \Delta$.

$ProL$ is the probability of hitting $P_H + \Delta$ first, rather than hitting $P_L - \Delta$, starting at $P_L + \Delta$.

t_1 is the first hitting time from $P_H + \Delta$ to $P_H - \Delta$.

t_2 is the first hitting time from $P_H - \Delta$ to $P_H + \Delta$.

t_3 is the first hitting time from $P_H - \Delta$ to $P_L - \Delta$.

t_4 is the first hitting time from $P_L - \Delta$ to $P_L + \Delta$.

t_5 is the first hitting time from $P_L + \Delta$ to $P_H + \Delta$.

t_6 is the first hitting time from $P_L + \Delta$ to $P_L - \Delta$.

Assuming we are buying at either $P_H - \Delta$ or $P_L + \Delta$, and selling at either $P_H + \Delta$ or $P_L - \Delta$, I can write the value of G_1 as (in the followings, all the expectations are take with respect to the risk-neutral measure):

$$G_1 = P_H + \Delta + W_2(P_H + \Delta),$$

which means the value of a full facility at $P_H + \Delta$ is equal to the selling price $P_H + \Delta$ plus the empty facility at price $P_H + \Delta$, which can be written as:

$$W_2(P_H + \Delta) = E\left[e^{-r\cdot t_1}\cdot\left(-(P_H - \Delta) + V(P_H - \Delta)\right)\right].$$

The above equation says that the value of the empty facility at price $P_H + \Delta$ is the full facility value at $P_H - \Delta$ minus the buying price of $P_H - \Delta$, discounted by the risk-neutral expected value of $\exp(-r\cdot t_1)$.

Expand the full facility value at $P_H - \Delta$:

$$V(P_H - \Delta) = E\left[e^{-r\cdot t_2}\cdot G_1 - \int_0^{t_2} e^{-r\cdot t}\cdot c\, dt\right]\cdot ProH + E\left[e^{-r\cdot t_3}\cdot G_2 - \int_0^{t_3} e^{-r\cdot t}\cdot c\, dt\right]\cdot (1 - ProH),$$

which says the full storage facility value at $P_H - \Delta$ is divided into two cases: the case it hits $P_H + \Delta$ (where we obtain the value of G_1 but incurs storage costs for the length of t_2); the case it hits $P_L - \Delta$ (where we obtain the value of G_2 but incurs storage costs for the length of t_3). Substitute $V(P_H - \Delta)$ into $W_2(P_H + \Delta)$, and in turn, substitute the resulting equation into G_1, I have:

$$G_1 = P_H + \Delta + E\left\{e^{-r\cdot t_1}\cdot\left(-(P_H - \Delta) + E\left[e^{-r\cdot t_2}\cdot G_1 - \int_0^{t_2} e^{-r\cdot t}\cdot c\, dt\right]\cdot ProH + E\left[e^{-r\cdot t_3}\cdot G_2 - \int_0^{t_3} e^{-r\cdot t}\cdot c\, dt\right]\cdot (1 - ProH)\right)\right\}.$$

Therefore, G_1 can be expressed as a function of G_2, denoted as $G_1 = fun1(G_2)$.

Following the same argument as above, I can write similar equations as:

$$G_2 = P_L - \Delta + W_1(P_L - \Delta),$$

$$W_1(P_L - \Delta) = E\left[e^{-r\cdot t_4}\cdot\left(-(P_L + \Delta) + V(P_L + \Delta)\right)\right]$$

$$V(P_L + \Delta) = E\left[e^{-r\cdot t_5}\cdot G_1 - \int_0^{t_5} e^{-r\cdot t}\cdot c\,dt\right]\cdot ProL + E\left[e^{-r\cdot t_6}\cdot G_2 - \int_0^{t_6} e^{-r\cdot t}\cdot c\,dt\right]\cdot(1 - ProL)\,.$$

Therefore, G_2 can be expressed as:

$$G_2 = P_L - \Delta + E\left\{e^{-r\cdot t_4}\cdot\left(-(P_L + \Delta) + E\left[e^{-r\cdot t_5}\cdot G_1 - \int_0^{t_5} e^{-r\cdot t}\cdot c\,dt\right]\cdot ProL + E\left[e^{-r\cdot t_6}\cdot G_2 - \int_0^{t_6} e^{-r\cdot t}\cdot c\,dt\right]\cdot(1 - ProL)\right)\right\}.$$

Again, G_2 can be written as a function of G_1, denoted as $G_2 = fun2(G_1)$.

Given the parameters k, m, σ, Δ, P_L and P_H, I can use the properties of the mean reversion process (2.3.3) to find the values of *ProL, ProH*, and all the expected values relate to the first hitting time t_i, i = 1…6. Since it is hard to obtain the closed form solutions to derive the above first hitting times and probabilities, one efficient way to find the those values is to use Monte Carlo simulations, which is what I use for the following numerical examples. Therefore, all the parameters in the functions *fun1()* and *fun2()* are known except G_1 and G_2: assuming *fun1(), fun2()* are independent, we have two independent functions *fun1(), fun2(),* and two unknowns. Hence, we can find the value of G_1 and G_2.

Once we find the value of G_1 and G_2, we can find the value of a full facility at P_H and P_L by:

$$V(P_H) = \lim_{\Delta\to 0} G_1\,,\; V(P_L) = \lim_{\Delta\to 0} G_2\,.$$

With the above value at $V(P_H)$ and $V(P_L)$, the value of a storage facility operating according to strategy P_L, P_H are:

1. When the current price $P_L < P_0 < P_H$, let *Prob* be the probability of hitting P_H first rather than hitting P_L first, starting from P_0, the value can be expressed as:

$$V(P_0) = E\left[e^{-r\cdot t_{PH}}\cdot V(P_H) - \int_0^{t_{PH}} e^{-r\cdot t}\cdot c\,dt\right]\cdot Prob + E\left[e^{-r\cdot t_{PL}}\cdot V(P_L) - \int_0^{t_{PL}} e^{-r\cdot t}\cdot c\,dt\right]\cdot(1 - Prob)\,. \tag{2.3.5a}$$

2. When the current price $P_0 \leq P_L$:

$$W_1(P_0) = E\left[e^{-r \cdot t_{PL}} \cdot \left(-P_L + V(P_L)\right)\right]. \tag{2.3.5b}$$

3. When the price $P_0 \geq P_H$:

$$W_2(P_0) = E\left[e^{-r \cdot t_{PH}} \cdot \left(-P_H + V(P_H)\right)\right]. \tag{2.3.5c}$$

(2.3.5a) to (2.3.5c) can be easily calculated using the Monte Carlo simulation method, and examples are given in Section 2.3.3.

2.3.1.2 Optimal Value of a Storage Facility with P_L^* and P_H^*.

Assuming we are in a complete market, I use the contingent claim method pioneered by Black & Scholes (1973) and Merton (1973) to derive the optimal operating strategy together with the corresponding value of the storage facility.

Assume a storage facility is in the operation condition for infinite time, and we are holding one full storage facility. Since the facility has infinite operating life, the value of the facility should be a function of only one parameter P: $V = V(P)$, and the parameter t (time) should have no influence on the value of the facility.

Under the risk-neutral measure, the expected instantaneous gain by holding the full storage facility should satisfy the following relationship:

$$E[\mathrm{d}V(P)] - c \cdot dt = r \cdot dt \cdot V(P).$$

It means in an arbitrage-free market, the expected gain (the gain from holding an asset minus all the cost incurred during the holding period) should be equal to the risk-free return earned on any asset with the value $V(P)$. To expand the expectation term using Ito's Lemma, I have:

$$\begin{aligned} E[dV(P)] &= E\left[V_P \cdot dP + \frac{1}{2} V_{PP} \cdot (dP)^2\right] \\ &= E\left[V_P \cdot \left(k \cdot \left(m + \frac{\sigma^2}{2k} - \ln P\right) \cdot P \cdot dt + \sigma \cdot P \cdot dz\right) + \frac{1}{2} \cdot V_{PP} \cdot \sigma^2 \cdot P^2 \cdot dt\right] \\ &= V_P \cdot k \cdot \left(m + \frac{\sigma^2}{2k} - \ln P\right) \cdot P \cdot dt + \frac{1}{2} \cdot V_{PP} \cdot \sigma^2 \cdot P^2 \cdot dt . \end{aligned}$$

where $V = V(P)$, $V_{PP} = \frac{d^2}{dP^2} V(\mathrm{P})$, $V_P = \frac{d}{dP} V(\mathrm{P})$.

Therefore, combining the above two equations, and simplifying them, I have:

$$V_P \cdot k \cdot \left(m + \frac{\sigma^2}{2 \cdot k} - \ln(P) \right) \cdot P \cdot dt + \frac{1}{2} V_{PP} \cdot \sigma^2 \cdot P^2 \cdot dt - c \cdot dt = r \cdot dt \cdot V$$

Therefore, for a full storage facility, its value should follow:

$$\frac{1}{2} \sigma^2 P^2 V_{PP} + k \left(m + \frac{\sigma^2}{2 \cdot k} - \ln(P) \right) P \cdot V_P - \mathrm{r} \cdot \mathrm{V} = \mathrm{c} . \tag{2.3.6}$$

Similarly, for an empty storage facility, its value should follow:

$$\frac{1}{2} \sigma^2 P^2 W_{PP} + k \left(m + \frac{\sigma^2}{2 \cdot k} - \ln(P) \right) P \cdot W_P - \mathrm{r} \cdot \mathrm{W} = 0 . \tag{2.3.7}$$

Next, I derive the optimal operating prices P_L^* and P_H^* that solve the problem (2.3.4). I prove the optimal trigger prices satisfy $P_H^* = P_2^*$, $P_L^* = P_1^*$, such that whenever $P > P_2^*$, we sell immediately, and whenever $P_1^* < P < P_2^*$, we buy immediately, and when $0 < P < P_1^*$, we sell immediately. First, I prove P_2^* and P_1^* are single values rather than within a certain range.

Claim 2.3.1:

Let V^* and W^* be the optimal values for the full and empty facility respectively, and assume:

$$0 > -\frac{c}{k} \cdot e^{-m + \frac{r}{k} - \frac{1}{2 \cdot k} \cdot \sigma^2} > -e^{-1}. \tag{2.3.8}$$

Then P_H^*, P_L^* are single values rather than within a certain range for the problem (2.3.4).

Proof:

I prove the uniqueness of the upper trigger price (P_H^*), and the same argument can be made for the uniqueness of the lower trigger price (P_L^*).

Assume the upper trigger prices fall in the range (P_{H1}, P_{H2}), then for any value P (P_{H1}, P_{H2}), it satisfies (2.3.6) and (2.3.7) simultaneously. Therefore, when P in (P_{H1}, P_{H2}), we must have:

$$P + W^*(P) = V^*(P) \tag{i}$$

If not, let's say $P + W^*(P) > V^*(P)$, then it means $V^*(P)$ is not optimal which contradicts to the optimality assumptions of V^*.

Because of (2.3.8) and P in (P_{H1}, P_{H2}), together with the assumptions of twice differentiability of V^* and W^*, I have:

$$\frac{d}{dP} W^*(P) + 1 = \frac{d}{dP} V^*(P) . \tag{ii}$$

$$\frac{d^2}{dP^2} W^*(P) = \frac{d^2}{dP^2} V^*(P) . \tag{iii}$$

Therefore, subtracting (2.3.6) from (2.3.7) using (i), (ii) and (ii), I have:

$$k\left(m + \frac{\sigma^2}{2 \cdot k} - \ln(P)\right) P - r \cdot P = c . \tag{vi}$$

Clearly, if $P_{H1} \neq P_{H2}$, then any P in (P_{H1}, P_{H2}), satisfy (vi). This is not possible given the positivity of the constants of k, r, c.

The same is true for the lower trigger price.

Therefore, P_H^*, P_L^* are single values.□

Next, I claim the following.

Claim 2.3.2:

Assume (2.3.8). Let V^* and W^*(W_1^* *and* W_2^*) represent the optimal values for the full and empty facility respectively. I can expand the definition of V^* and W^* by defining:

$$V_{all}^* = \begin{cases} W_1^* + P & 0 < P \leq P_1^* \\ W_2^* + P & P \geq P_2^* \\ V^* & P_1^* \leq P \leq P_2^* \end{cases}, \textbf{and } W_{all}^* = \begin{cases} W_1^* & 0 < P \leq P_1^* \\ W_2^* & P \geq P_2^* \\ V^* - P & P_1^* \leq P \leq P_2^* \end{cases} . \tag{i}$$

Then for any value of P, $V_{all}^*(P)$ *and* $W_{all}^*(P)$ are continuously twice differentiable at P_2^*. (This is an alternative proof of higher order smooth pasting property.)

Proof:

I will consider only V_{all}^* in the proof, and W_{all}^* can be proved similarly. Since I have assumed V and W are twice differentiable, to prove the claim, we only need to prove that V_{all}^* is once and twice differentiable at P = P_2^*.

If the first derivative of $V_{all}^*(P)$ at $P = P_2^*$ is not differentiable, then it is not unique (it cannot explode due to the assumptions I have made on V and W). Using the definition in (i), I ***assume*** *the following relationship:*

$$V_P^*(P_2^*) > W_P^*(P_2^*) + 1 \ (V_P^*(P_2^*) < W_P^*(P_2^*) + 1 \text{ can be proved similarly}). \tag{ii}$$

If there is a small positive price change ΔP, I can multiply ΔP to both sides of (ii):

$$V_P^*(P_2^*)\cdot\Delta P > W_P^*(P_2^*)\cdot\Delta P + \Delta P. \tag{iii}$$

It indicates that the changes in the full facility values are bigger than the changes in the empty facility value plus the changes in price. Adding both sides by $V^*(P_2^*) = W^*(P_2^*) + P_2^*$, I have:

$$V_P^*(P_2^*)\cdot\Delta P + V^*(P_2^*) > W_P^*(P_2^*)\cdot\Delta P + W^*(P_2^*) + P_2^* + \Delta P, \tag{iv}$$

As ΔP goes to zero, (iv) which can be approximated by:

$$V^*(P_2^*+\Delta P) \geq W^*(P_2^*+\Delta P) + (P_2^* + \Delta P), \tag{v}$$

$$\text{or} \quad V^*(P_2^*+\Delta P) - (P_2^* + \Delta P) \geq W^*(P_2^*+\Delta P).$$

This means that we can switch at the price $P_2^* + \Delta P$, and in that case, we have a larger value $V^*(P_2^*+\Delta P) - (P_2^* + \Delta P)$ for the value of the empty facility, and this contradicts to the optimality of $W^*(P_2^*+\Delta P)$.

Therefore, we must have $V_P^*(P_2^*) = W_P^*(P_2^*) + 1$. Since V_P^* is twice differentiable, second order derivative can be proved similarly using Taylor expansion as (iv) and (v), which gives $V_{PP}^*(P_2^*) = W_{PP}^*(P_2^*)$.

Hence, $V_{all}^*(P)$ are continuously twice differentiable at P_2^*. *($W_{all}^*(P)$ can be proved similarly).* □

Similarly, I can also prove it is true for P_1^*, the lower trigger price. To find the functional form of P_2^*, as proved in Claim 2.3.2, at the optimal switch point P_2^*, I have: $V_P^*(P_2^*) = W_P^*(P_2^*) + 1$, $V_{PP}^*(P_2^*) = W_{PP}^*(P_2^*)$, and together with the definition that $V^*(P_2^*) = W^*(P_2^*) + P_2^*$. Using the three equations, I subtract (2.3.7) from (2.3.6), and get:

$$k\left(m + \frac{\sigma^2}{2\cdot k} - \ln\left(P_2^*\right)\right)P_2^* - r\cdot P_2^* = c \tag{2.3.9a}$$

$$\Rightarrow k\left(m + \frac{\sigma^2}{2\cdot k} - \ln\left(P_2^*\right)\right)P_2^* = r\cdot P_2^* + c\,. \tag{2.3.9b}$$

(2.3.9b) is exactly the equation of the risk-neutral expected rate of the price appreciation (MR) = the risk-neutral expected rate of the total cost changes (MC). Solving (2.3.9b), I have:

$$P^* = e^{m - \frac{r}{k} + \frac{1}{2\cdot k}\cdot\sigma^2 + \mathrm{LambertW}\left(-\frac{c}{k}\cdot e^{-m + \frac{r}{k} - \frac{1}{2\cdot k}\cdot\sigma^2}\right)}. \tag{2.3.10}$$

Under the assumption (2.3.8), there are two values for (2.3.10), which I denote them a P_1^* and P_2^*, and I assume $P_1^* \leq P_2^*$. Hence, using the same analysis as what I have made for the deterministic case, I can conclude that the optimal upper trigger price is where $P_H^* = P_2^*$, and the optimal lower trigger price is $P_L^* = P_1^*$. The detailed optimal strategy is shown in Table 2.3.1.

Case	Sign of MR – MC	Operation Strategy:
$P \geq P_2^*$:	Non-positive.	Sell the inventory immediately.
$P_1^* < P < P_2^*$:	Positive.	Buy the inventory immediately, and hold the inventory.
$P \leq P_1^*$:	Non-positive.	Sell the inventory immediately.

Table 2.3.1: The optimal operating strategies for the stochastic mean reversion model.

The above results can be better understood by the plot of MR and MC in Figure 2.3.1.

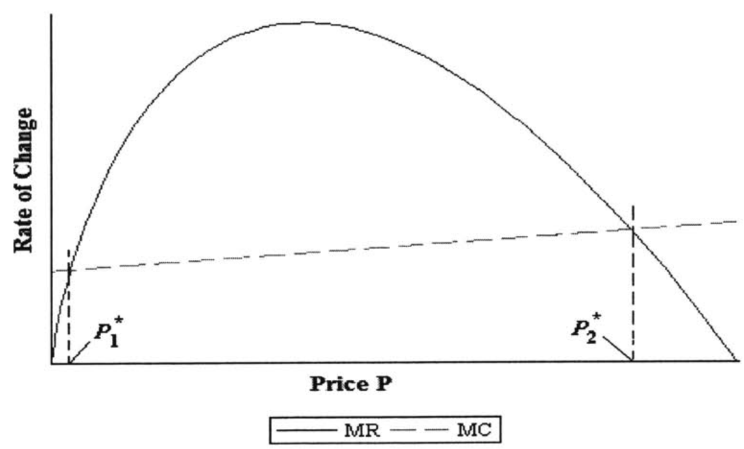

Figure 2.3.1: The value of the functions MR and MC.

Let's compare the optimal trigger prices for both the deterministic and stochastic cases. In deterministic model from Section 2.2, I have:

$$P^*_{deterministic} = e^{m - \frac{r}{k} + \mathrm{LambertW}\left(-\frac{c}{k}\cdot e^{-m + \frac{r}{k}}\right)}. \tag{2.3.11}$$

In the stochastic case in this section, I have:

$$P^*_{stochastic} = e^{m - \frac{r}{k} + \frac{1}{2\cdot k}\cdot\sigma^2 + \mathrm{LambertW}\left(-\frac{c}{k}\cdot e^{-m + \frac{r}{k} - \frac{1}{2\cdot k}\cdot\sigma^2}\right)} \qquad (2.3.12)$$

It is clear that the difference is caused by the volatility term in the underlying stochastic process. In (2.3.12), when $\sigma = 0$, the two equations coincides which proves the validity of both approaches (the contingent claim method and the net present value method). Using the results about LambertW function in Section 2.2 and a little algebra, we can show $P^*_{1,stochastic} < P^*_{1,deterministic}$, and $P^*_{2,stochastic} > P^*_{2,deterministic}$. That is to say, the holding price range of the stochastic case is larger than the deterministic case. The differences can be explained as follows: in the deterministic model, the value of P_H^* is price at which the rate of price gain is equal to marginal cost. The expected value of the price in the deterministic case is:

$$\overline{P} = e^{\overline{u}}.$$

In the stochastic case, the marginal cost is identical to the deterministic case in each state, but the rate of the price gain is volatile with the expected value being a function of the volatility:

$$\overline{P} = e^{\overline{u} + \frac{1}{2}\frac{\sigma^2}{k}}.$$

In the stochastic case, as shown in (2.3.10), where if we use the risk-neutral expected value to calculate the rate of the price appreciation, and equate it to the risk-neutral expected marginal cost, I can get exactly (2.3.12). One conclusion from this section is that we can use the risk-neutral expected price appreciation and the marginal cost to find the optimal trigger prices easily, and this method gives us a general formula to derive the optimal trigger price: that is, **for a general storage problem, I can always equate the risk-neutral expected rate of the price appreciation to the risk-neutral expected rate of the overall cost changes, and the resulting prices are the optimal trigger prices.**

Using the optimal trigger prices P_1^* and P_2^*, next, I present an analytic solution to the simultaneous differential equations (2.3.6) and (2.3.7) to find the optimal value of the storage facility. Because both (2.3.6) and (2.3.7) have the same structure, I solve the homogeneous part first. I first make the following substitutions:

$$\ln(P) = u. \qquad (2.3.13)$$

I have:

$$\frac{1}{2}\sigma^2 \cdot V_{uu,\,Homo} + k\left(m + \frac{\sigma^2}{2\cdot k} - \mathrm{u}\right)\mathrm{V}_{u,\,Homo} - r\,V_{Homo} = 0\,, \tag{2.3.14}$$

where $V_{Homo} = V_{Homo}(u)$, $V_{u,\,Homo} = \frac{\mathrm{d}}{\mathrm{d}u} V_{Homo}(u)$, $V_{uu,\,Homo} = \frac{\mathrm{d}^2}{\mathrm{d}u^2} V_{Homo}(u)$.

Next, I make the following substitution:

$$x = \frac{\sqrt{k}\left(u - m - \frac{\sigma^2}{2\cdot k}\right)}{\sigma} \quad \text{and}\ y(x) = V_{Homo}(u(x))\,. \tag{2.3.15}$$

Hence, the homogeneous equation (2.3.14) can be written as:

$$y_{xx} - 2\cdot x\cdot y_x - \frac{2\cdot r}{k}\cdot y = 0\,, \tag{2.3.16}$$

where $y = y(x)$, $y_x = \frac{\mathrm{d}}{\mathrm{d}x} y(x)$, and $y_{xx} = \frac{\mathrm{d}^2}{\mathrm{d}x^2} y(x)$.

Equation (2.3.16) is in the form of the Hermite equation structure, with solutions can be expressed as a linear combination of the following two components:

$$y_1(x) = H\left(\frac{r}{2\cdot k}, \frac{1}{2}, x^2\right),$$

$$y_2(x) = x\cdot H\left(\frac{r}{2\cdot k} + \frac{1}{2}, \frac{3}{2}, x^2\right),$$

where $H(a, b, q) = 1 + \sum_{n=1}^{\infty}\left(\prod_{i=0}^{n-1}\frac{(\mathrm{a}+i)}{(\mathrm{b}+i)}\right)\cdot\frac{q^n}{n!}$.

The general solution to the Hermite equation is:

$$y(x) = A\cdot y_1(x) + B\cdot y_2(x)\,. \tag{2.3.17}$$

Substitute x and $y(x)$ using (2.3.13) and (2.3.15), and the solution for (2.3.14) is:

$$V_{Homo}(P) = A\cdot Y_1(P) + B\cdot Y_2(P)\,, \tag{2.3.18}$$

where $Y_1(P) = y_1\left(\frac{\sqrt{k}\left(\ln(P) - m - \frac{\sigma^2}{2\cdot k}\right)}{\sigma}\right)$, $Y_2(P) = y_2\left(\frac{\sqrt{k}\left(\ln(P) - m - \frac{\sigma^2}{2\cdot k}\right)}{\sigma}\right)$

and A, B are constants to be determined.

Given the optimal trigger price P_2^* and P_1^*, I have, for the optimal value of the full storage facility:

$$V^*(P) = v_1\cdot Y_1(P) + v_2\cdot Y_2(P) - \frac{c}{r}\,, \text{when } P_1^* \le P \le P_2^*, \tag{2.3.19}$$

where v_1 and v_2 are constants to be determined.

The optimal value for the empty storage facility is:

$$W^*(P) = \begin{cases} W_1^*(P) = n_{11} \cdot Y_1(P) + n_{12} \cdot Y_2(P) & \text{when } 0 < P \le P_1^*, \\ W_2^*(P) = n_{21} \cdot Y_1(P) + n_{22} \cdot Y_2(P) & \text{when } P_2^* \le P. \end{cases} \tag{2.3.20}$$

where $n_{11}, n_{12}, n_{21}, n_{22}$ are constants to be determined.

To find the constants in (2.3.19) and (2.3.20), I use the following boundary conditions:

$$\lim_{P \to 0} W_1^*(P) = 0, \tag{2.3.21}$$

$$\lim_{P \to +\infty} W_2^*(P) = 0, \tag{2.3.22}$$

$$V^*\left(P_1^*\right) = P_1^* + W_1^*\left(P_1^*\right), \tag{2.3.23}$$

$$V^*\left(P_2^*\right) = P_2^* + W_2^*\left(P_2^*\right), \tag{2.3.24}$$

$$V_P^*\left(P_1^*\right) = 1 + W_{1,P}^*\left(P_1^*\right), \tag{2.3.25}$$

$$V_P^*\left(P_2^*\right) = 1 + W_{2,P}^*\left(P_2^*\right). \tag{2.3.26}$$

The validity of the boundary conditions (2.3.21) and (2.3.22) have been proved in Appendix A and B respectively; (2.3.23) and (2.3.24) are the properties for the value of a storage facility at any price P; (2.3.25) and (2.3.26) are the properties by the definition of the optimal trigger prices (see Claim 2.3.2). Giving P_1^* and P_2^*, and assuming (2.3.21) to (2.3.26) are independent, we can determine the six unknowns, $v_1, v_2, n_{11}, n_{12}, n_{21}, n_{22}$. Therefore, the solution for $V(P)$, and $W(P)$ can be found accordingly. Next, I will solve the system of equations (2.3.21) to (2.3.26). From (2.3.21), it is identical to require:

$$\lim_{P \to 0} n_{11} \cdot Y_1(P) + n_{12} \cdot Y_2(P) = 0,$$

$$\Rightarrow \lim_{P \to 0} n_{11} \cdot y_1\left(\frac{\sqrt{k}\left(\ln(P) - m - \frac{\sigma^2}{2 \cdot k}\right)}{\sigma}\right) + n_{12} \cdot y_2\left(\frac{\sqrt{k}\left(\ln(P) - m - \frac{\sigma^2}{2 \cdot k}\right)}{\sigma}\right) = 0,$$

$$\Rightarrow \lim_{P \to 0} n_{11} \cdot H\left(\frac{r}{2 \cdot k}, \frac{1}{2}, \frac{k\left(\ln(P) - m - \frac{\sigma^2}{2 \cdot k}\right)^2}{\sigma^2}\right) + n_{12} \cdot \frac{\sqrt{k}\left(\ln(P) - m - \frac{\sigma^2}{2 \cdot k}\right)}{\sigma}$$
$$\cdot H\left(\frac{r}{2 \cdot k} + \frac{1}{2}, \frac{3}{2}, \frac{k\left(\ln(P) - m - \frac{\sigma^2}{2 \cdot k}\right)^2}{\sigma^2}\right) = 0.$$

I can use the asymptotic property of $y_1(x)$ and $y_2(x)$:

$$\lim_{x \to \infty} y_1(x) \cdot x^{1-\eta} \cdot e^{-x^2} = \frac{\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{\eta}{2}\right)},$$

$$\lim_{x\to\infty} y_2(x)\cdot x^{1-\eta}\cdot e^{-x^2} = \frac{\Gamma\left(\frac{3}{2}\right)}{\Gamma\left(\frac{1}{2}+\frac{\eta}{2}\right)},$$

This leads to:

$$n_{12} = n_{11}\cdot\frac{\Gamma\left(\frac{1}{2}\right)\cdot\Gamma\left(\frac{r}{2k}+\frac{1}{2}\right)}{\Gamma\left(\frac{r}{2k}\right)\cdot\Gamma\left(\frac{3}{2}\right)}.$$

Similarly, for (2.3.22), I have:

$$\lim_{P\to+\infty} n_{21}\cdot H\left(\frac{r}{2\cdot k},\frac{1}{2},\frac{k\left(\ln(P)-m-\frac{\sigma^2}{2\cdot k}\right)^2}{\sigma^2}\right) + n_{22}\cdot\frac{\sqrt{k}\left(\ln(P)-m-\frac{\sigma^2}{2\cdot k}\right)}{\sigma}\cdot H\left(\frac{r}{2\cdot k}+\frac{1}{2},\frac{3}{2},\frac{k\left(\ln(P)-m-\frac{\sigma^2}{2\cdot k}\right)^2}{\sigma^2}\right) = 0.$$

With similar relationship from the asymptotic property:

$$n_{22} = -n_{21}\cdot\frac{\Gamma\left(\frac{1}{2}\right)\cdot\Gamma\left(\frac{r}{2k}+\frac{1}{2}\right)}{\Gamma\left(\frac{r}{2k}\right)\cdot\Gamma\left(\frac{3}{2}\right)}.$$

Therefore, the system of equations can be re-written as:

$$(2.3.21)\Rightarrow n_{12} = n_{11}\cdot\frac{\Gamma\left(\frac{1}{2}\right)\cdot\Gamma\left(\frac{r}{2k}+\frac{1}{2}\right)}{\Gamma\left(\frac{r}{2k}\right)\cdot\Gamma\left(\frac{3}{2}\right)} \tag{1}$$

$$(2.3.22)\Rightarrow n_{22} = -n_{21}\cdot\frac{\Gamma\left(\frac{1}{2}\right)\cdot\Gamma\left(\frac{r}{2k}+\frac{1}{2}\right)}{\Gamma\left(\frac{r}{2k}\right)\cdot\Gamma\left(\frac{3}{2}\right)} \tag{2}$$

$$(2.3.23)\Rightarrow v_1\cdot Y_1\left(P_1^*\right) + v_2\cdot Y_2\left(P_1^*\right) - \frac{c}{r} = P_1^* + n_{11}\cdot Y_1\left(P_1^*\right) + n_{12}\cdot Y_2\left(P_1^*\right) \tag{3}$$

$$(2.3.24)\Rightarrow v_1\cdot Y_1\left(P_2^*\right) + v_2\cdot Y_2\left(P_2^*\right) - \frac{c}{r} = P_2^* + n_{21}\cdot Y_1\left(P_2^*\right) + n_{22}\cdot Y_2\left(P_2^*\right) \tag{4}$$

$$(2.3.25)\Rightarrow v_1\cdot\frac{d}{dP}Y_1\left(P_1^*\right) + v_2\cdot\frac{d}{dP}Y_2\left(P_1^*\right) = 1 + n_{11}\cdot\frac{d}{dP}Y_1\left(P_1^*\right) + n_{12}\cdot\frac{d}{dP}Y_2\left(P_1^*\right) \tag{5}$$

$$(2.3.26)\Rightarrow v_1\cdot\frac{d}{dP}Y_1\left(P_2^*\right) + v_2\cdot\frac{d}{dP}Y_2\left(P_2^*\right) = 1 + n_{21}\cdot\frac{d}{dP}Y_1\left(P_2^*\right) + n_{22}\cdot\frac{d}{dP}Y_2\left(P_2^*\right) \tag{6}$$

To solve (1) to (6), I use (1) and (2) to eliminate n_{12} and n_{22}. Then, the equations (3) to (6) are becoming a linear system of equations:

$$M \cdot v = b,$$

where

$$M = \begin{bmatrix} Y_1(P_1^*) & Y_2(P_2^*) & -Y_1(P_1^*) - Y_2(P_1^*) \cdot Par & 0 \\ Y_1(P_2^*) & Y_2(P_2^*) & 0 & -Y_1(P_2^*) + Y_2(P_2^*) \cdot Par \\ \frac{d}{dP} Y_1(P_1^*) & -\frac{d}{dP} Y_2(P_1^*) & -\frac{d}{dP} Y_1(P_1^*) - \frac{d}{dP} Y_2(P_1^*) \cdot Par & 0 \\ -\frac{d}{dP} Y_1(P_2^*) & -\frac{d}{dP} Y_2(P_2^*) & 0 & -\frac{d}{dP} Y_1(P_2^*) + \frac{d}{dP} Y_2(P_2^*) \cdot Par \end{bmatrix},$$

$$b = \begin{bmatrix} v_1 & v_2 & n_{11} & n_{21} \end{bmatrix}^T, \text{ and } Par = \frac{\Gamma\left(\frac{1}{2}\right) \cdot \Gamma\left(\frac{r}{2k} + \frac{1}{2}\right)}{\Gamma\left(\frac{r}{2k}\right) \cdot \Gamma\left(\frac{3}{2}\right)}.$$

Solving the above system of equations, we can obtain the six unknown parameters, v_1, v_2, n_{11}, n_{12}, n_{21}, n_{22}. Once the parameters are found, V^* and W^* are obtained. (This is one practical problem in solving the above system of equations. If P_1^* is small, the absolute values of $Y_1(P_1^*)$ and $Y_2(P_1^*)$ will be in the order of more than the computer can accurately process. Most numerical simulation software will truncate such a large number, which make the resulting solutions incorrect. One way to resolve this problem is to introduce long-bits representation, which is out of the scope of the thesis.)

2.3.2 Optimal Operating Strategies with Finite Facility Life

For a finite time horizon, the value for the full and empty storage facilities are functions of both the current spot price P and the current time t, denoted as $V = V(P, t)$ and $W = W(P, t)$ respectively. Further, I assume both V and W are continuously twice differentiable with respect to P, and once differentiable with respect to t.

By the contingent claim method, the value functions have to satisfy the following two equations for both the full and the empty storage facilities:

$$\frac{1}{2}\sigma^2 P^2 V_{PP} + k\left(m + \frac{\sigma^2}{2 \cdot k} - \ln(P)\right) P \cdot V_P - r \cdot V + V_t = c, \tag{2.3.27}$$

$$where\ V = V(P,t),\ V_{PP} = \frac{d^2}{dP^2} V(P,t),\ V_P = \frac{d}{dP} V(P,t),\ V_t = \frac{d}{dt} V(P,t).$$

$$\frac{1}{2} \sigma^2 P^2 W_{PP} + k \left(m + \frac{\sigma^2}{2 \cdot k} - \ln(P) \right) P \cdot W_P - r \cdot W + W_t = 0. \qquad (2.3.28)$$

$$where\ W = W(P,t),\ W_{PP} = \frac{d^2}{dP^2} W(P,t),\ W_P = \frac{d}{dP} W(P,t),\ W_t = \frac{d}{dt} W(P,t).$$

(The above equations are derived using the same argument as in Section 2.3.1.2, The only difference is that now the V and W are functions of both the current time t and current price P. Thus, by Ito's lemma, an extra term $V_t \cdot dt$ is added. All the other steps follow exactly the same as those in Section 2.3.1.)

In order to solve for the optimal trigger prices, I first prove the following claim.

Claim 2.3.3:

Under the assumption (2.3.8), assume that the life of the storage facility is T. Then, in the finite facility life case, P_L^* and P_H^* are single values rather than within a certain range for our valuation problem (2.3.4).

Proof:

I prove the uniqueness of the upper trigger price (P_H^*), and the same argument can be made for the uniqueness of the lower trigger price (P_L^*).

Let's say I have two ***arbitrary*** *time instants t_1 and t_2, satisfying:*

$t_2 = t_1 + \delta$, *where* $T - t_1 > \delta > 0$.

Assume the upper trigger prices fall in the range (P_{H1}, P_{H2}), then for any value P (P_{H1}, P_{H2}), it satisfies (2.3.27) and (2.3.8) simultaneously. Therefore, when P in (P_{H1}, P_{H2}), we must have:

$$P + W^*(P) = V^*(P) \qquad \text{(i)}$$

If not, let's say $P + W^*(P) > V^*(P)$, then it means $V^*(P)$ is not optimal which contradicts to the optimality assumptions of V^*.

For any value P in (P_{H1}, P_{H2}), with the assumptions of twice differentiability of V^* and W^*, I have:

$$V^*(P,\, t_1) = W^*(P,\, t_1) + P, \tag{ii}$$

$$V^*(P,\, t_2) = W^*(P,\, t_2) + P. \tag{iii}$$

(iii) – (ii) gives:

$$V^*(P,\, t_1) - V^*(P,\, t_2) = W^*(P,\, t_1) - W^*(P,\, t_2). \tag{ii}$$

Divide both side of (iii) by $t_2 - t_1$, I have:

$$[V^*(P,\, t_1) - V^*(P,\, t_2)\,] / (t_2 - t_1) = [W^*(P,\, t_1) - W^*(P,\, t_2)] / (t_2 - t_1). \tag{iv}$$

With the assumption that V^* and W^* are continuous functions of t and P, I have:

$$\frac{d}{dt} V^*(P,t) = \lim_{\delta \to 0} \frac{V^*\left(P, t_1 + \delta\right) - V^*\left(P, t_1\right)}{\delta} \tag{v}$$

$$= \lim_{\delta \to 0} \frac{W^*\left(P, t_1 + \delta\right) - W^*\left(P, t_1\right)}{\delta} = \frac{d}{dt} W^*(P,t)\,.$$

Since t_1 is chosen arbitrarily, I have for any t:

$$\frac{d}{dt} V^*(P,t) = \frac{d}{dt} W^*(P,t)\,. \tag{vi}$$

Sum up the results:

$$\frac{d}{dP} W^*(P,t) = 1 + \frac{d}{dP} V^*(P,t)\,. \tag{vii}$$

$$\frac{d^2}{dP^2} W^*(P,t) = \frac{d^2}{dP^2} V^*(P,t)\,. \tag{viii}$$

$$\frac{d}{dt} W^*(P,t) = \frac{d}{dt} V^*(P,t)\,. \tag{ix}$$

Therefore, subtracting (2.3.27) from (2.3.28) and using (vii), (viii) and (xi), I have:

$$k\left(m + \frac{\sigma^2}{2 \cdot k} - \ln(P)\right) P - r \cdot P = c\,. \tag{x}$$

Clearly, if $P_{H1} \neq P_{H2}$, then any P in (P_{H1}, P_{H2}), satisfy (x). This is not possible given the positivity of the constants k, r, c.

The same is true for the lower trigger price.

Therefore, P_1^* and P_2^* are single values. □

Next, I need to prove a result similar to Claim 2.3.2.

Claim 2.3.4:

In the finite facility life case, assume (2.3.8), and let V^* and W^* represent the optimal values for the full and empty facility. Again define:

$$V_{all}^* = \begin{cases} V^* & P_1^* \le P \le P_2^* \\ W_2^* + P & P \ge P_2^* \\ W_1^* + P & 0 < P \le P_1^* \end{cases}, \textbf{ and } W_{all}^* = \begin{cases} V^* - P & P_1^* \le P \le P_2^* \\ W_2^* & P \ge P_2^* \\ W_1^* & 0 < P \le P_1^* \end{cases}. \qquad \text{(i)}$$

Then for any value of P, $V_{all}^*(P)$ and $W_{all}^*(P)$ are continuously twice differentiable with respect to P, and once differentiable with respect to t

Proof:

I will consider only V_{all}^* in the proof, and W_{all}^* can be proved similarly.

Because I have assumed V and W are continuously twice differentiable with respect to P, and once differentiable with respect to t, to prove the claim, we only need to prove that V_{all}^* is once and twice differentiable with respect to P at $P = P_2^*$ for all t, and it is once differentiable with respect to t for all values of P.

Fix a fixed value of t, as proved in Claim 2.3.2, we must have $V^*(P_2^*) = W^*(P_2^*) + P_2^*$, $V_P^*(P_2^*) = W_P^*(P_2^*) + 1$, and $V_{PP}^*(P_2^*) = W_{PP}^*(P_2^*)$, because of the optimality condition.

Now, for a fixed $P = P_2^*$, using the argument in the proof of Claim 2.3.3 (v), I have:

$$\begin{aligned} \frac{d}{dt} V^*\left(P_2^*, t\right) &= \lim_{\delta \to 0} \frac{V^*\left(P_2^*, t_1 + \delta\right) - V^*\left(P_2^*, t_1\right)}{\delta} \\ &= \lim_{\delta \to 0} \frac{W^*\left(P_2^*, t_1 + \delta\right) - W^*\left(P_2^*, t_1\right)}{\delta} = \frac{d}{dt} W^*\left(P_2^*, t\right). \end{aligned} \qquad \text{(ii)}$$

Therefore,

$$\frac{d}{dt} V^*(P, t) = \frac{d}{dt} W^*(P, t). \qquad \text{(iii)}$$

Using the definition of V_{all}^* and W_{all}^*, because of (iii), I conclude V_{all}^* and W_{all}^* are twice differentiable with respect to P, and once differentiable with respect to t. □

To find the optimal trigger prices, using the results from Claim 2.3.3 and Claim 2.3.4, I have:

$W^*(P, t) = P + V^*(P, t)$.

$$\frac{d}{dP}W^*(P,t) = 1 + \frac{d}{dP}V^*(P,t)\ .$$

$$\frac{d^2}{dP^2}W^*(P,t) = \frac{d^2}{dP^2}V^*(P,t)\ .$$

$$\frac{d}{dt}W^*(P,t) = \frac{d}{dt}V^*(P,t)\ .$$

Therefore, we can subtract (2.3.27) from (2.3.28) using the above equations. I have:

$$P^* = e^{m - \frac{r}{k} + \frac{1}{2\cdot k}\cdot\sigma^2 + \mathrm{LambertW}\left(-\frac{c}{k}\cdot e^{-m + \frac{r}{k} - \frac{1}{2\cdot k}\cdot\sigma^2}\right)}. \qquad \square$$

The result shows that the optimal trigger price for the finite facility life is the same as that for the infinite facility life. This can also be derived by the marginal analysis argument, since the rate of the price appreciation and the rate of the total cost changes do not change as we change from the infinite case to the finite case. This conclusion leads to an important result: **the life of storage facility does not change the optimal operating strategy we would make if we maximize the value of a storage facility.**

2.3.3 A Numerical Example to Find the Value of a Storage Facility

Once we know the trigger prices, different methods can be employed to solve the storage valuation problem: we can solve the problem using the method outlined in (2.3.1.1); we can solve the differential equations analytically (i.e., the analytic solutions from solving the Hermite equation in Section 2.3.1); we can use the numerical methods such as the finite difference method, and Monte Carlo simulation to solve the equation (2.3.6), (2.3.7), (2.3.27) and (2.3.28).

For simplicity and flexibility, I use the Monte Carlo Method together with the optimal trigger prices to calculate the optimal value of a storage facility Z^*. I first simulate the risk-neutral price price paths according to the following price dynamics:

$$P_{n+1} = \exp\left\{\ln(P_n)\cdot e^{-k\cdot\Delta t} + m\cdot\left(1 - e^{-k\cdot\Delta t}\right) + \sigma\cdot\sqrt{\frac{1 - e^{-2k\cdot\Delta t}}{2\cdot k}}\cdot z_{n+1}\right\},$$

where Δt is the time step, z_{n+1} is the standard Gaussian random variable with the mean zero, and the variable one.

Use the values: $m = 2.3$, $k = 1$, $r = 0.05$, $c = 1$, $\sigma = 0.3$, $\Delta t = 1/24$. I solve (2.3.10) numerically using Matlab, and obtain the optimal trigger prices $P_1^* = 0.2804$ and $P_2^* = 8.8659$ in the simulation. I generate 0.1 million price paths to value the facility Z^* (with 20 years life) using the optimal operating strategy shown in Table 2.3.1. The optimal value as a function of the current price P is plotted as the dotted line in Figure 2.3.2. To compare the optimal operating strategy to the non-optimal strategies, I change P_2* to different values, and find the corresponding values of the facility Z. The results are also plotted in Figure 2.3.2.

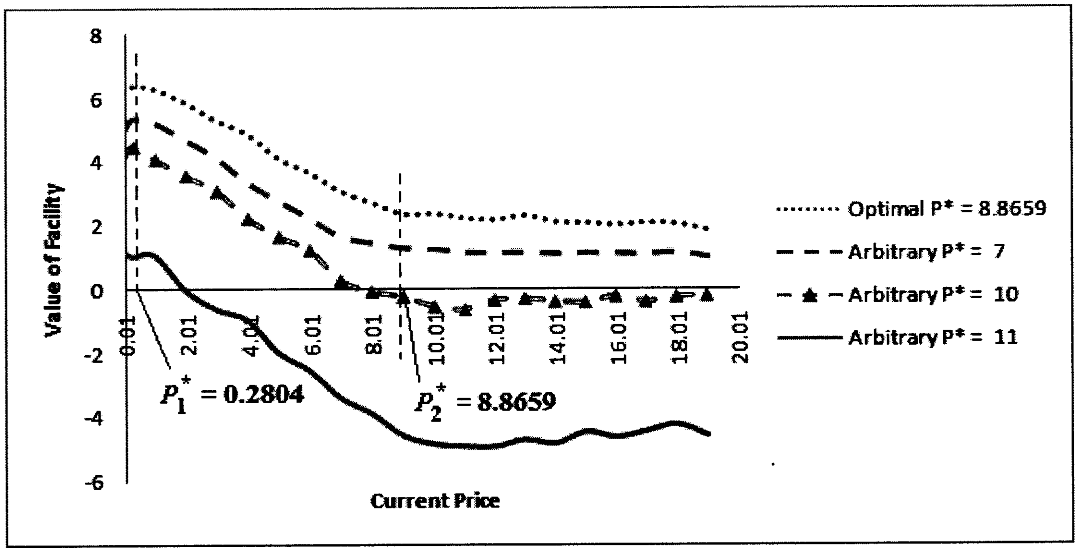

Figure 2.3.2: The value of the storage facility Z for different trigger prices.

Figure 2.3.2 also indicates that for a fixed upper trigger price P_2^*, the value of the facility increases when the current price P goes towards P_1^* from left. It reaches the maximum at the point P_1^*, then it decreases monotonically.

Comparing the optimal strategy (the dotted line in Figure 2.3.2) to Figure 2.2.4 plotted in Section 2.2.1 (the deterministic case), we can find several differences:

1. The overall value of the storage facility increases in the stochastic case, comparing to the deterministic case, when all the parameters take the same values in the two cases, except the volatility term σ. It means the volatility can add value to the value of a storage facility (here I assume the same value of m for the two cases. However, we need to adjust m for the increasing risk as σ increases, and the explicit adjustment is discussed later in Section 2.7).

2. When the current price $P > P_2^*$, the value is positive in the stochastic case, while in the deterministic case, it is zero. This again indicates the price uncertainty does provide values even the deterministic counterpart shows zero value.

In the above, I have not discussed the situations where the current price goes to some extreme values. Now I explore what happens when the current price is extremely low (P approaches 0) or extremely high (P approaches infinity). As explained and proved in Appendix A, when the current price approaches 0, the value of a full facility approaches 0. Hence, at the left end (where the current price approaches 0), the pure facility value is 0. For the right end (where the current price approaches positive infinity), I prove in Appendix B that the value is also 0. Combining the two results, I plot the value of the facility by the following steps:

- First solve for the optimal trigger price P^* (both P_1^* and P_2^*);
- Use either analytic solution or numerical methods (i.e., Monte Carlo Simulation) to get the value of the facility when the current price is at either P_1^* or P_2^*.
- Find the first hitting time t_1 and t_2, and the corresponding discounted factor, and *Prob* value in (2.3.5a) in Section 2.3.1.1, by either analytic solution or numerical methods (i.e., Monte Carlo Simulation), given the current price P;
- Use equation (2.3.5a), (2.3.5b) and (2.3.5c) to numerically find the value of the storage facility as a function of the current price P.

Following the above procedure, I plot the value of the facility as a function of P in Figure 2.3.3.

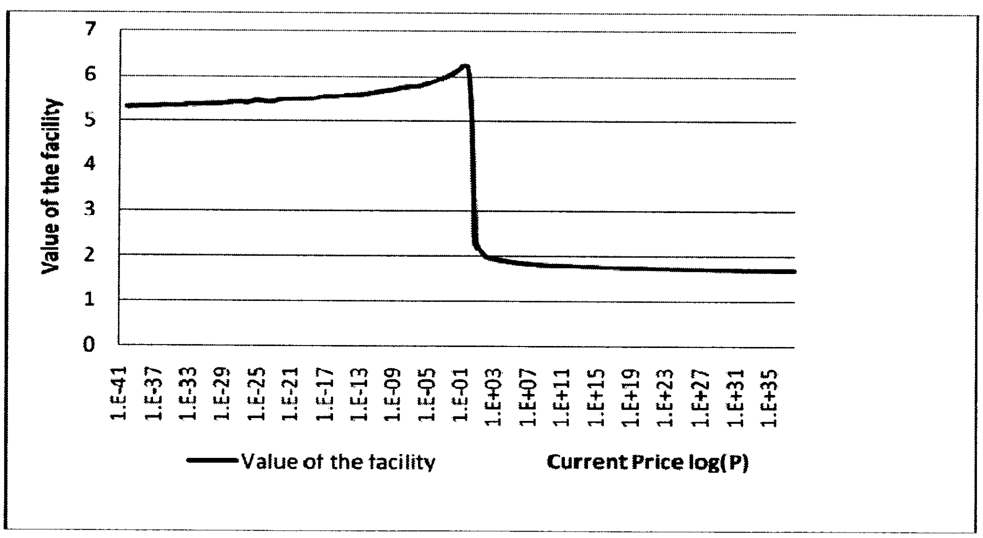

Figure 2.3.3: Value of a storage facility as a function of current price in log scale.

Some observations can be made from Figure 2.3.3.

1. In the initial phase, as the current price goes from 0 to P_1^*, the hitting time to P_2^* becomes shorter, and the discount factor $e^{-r \cdot t}$ is becoming bigger. The increase in the discount factor leads to the increase in the value of a storage facility.
2. From $P = P_1^*$ to the point $P = P_2^*$, the effect from the increase in the discount factor becomes less prominent, and now it is the difference between the current price P and P_2^* dominates the value of a storage facility. As the difference decreases, the profit from buying low and selling high reduces. Therefore, it leads to decreasing value of a storage facility.
3. After the current price goes beyond P_2^*, we are in the range of an empty facility. Now the only dominant factor is the discount factor $e^{-r \cdot t}$. Therefore, as the current price grows bigger, the first hitting time from current price to P_2^* becomes longer, and thus, the discount factor becomes smaller. Using (2.3.5c), it is clearly that both the value of the facility and the discount factor go to 0, as the first hitting time to P_2^* goes to infinity (the detailed proof is given in Appendix B). However, the rate of the value of the storage facility approaching to zero is very slow, which can be seen from the log plot in the Figure 2.3.3.

Further, the optimal value Z^* obtained from (2.3.5a), (2.3.5b) and (2.3.5c) should be the same as those values obtained by solving the differential equations (2.3.6) and (2.3.7) for V^* and W^*. But the two approaches tackle the problem from different perspectives: in the differential equations case, we stipulate how V^* and W^* should move individually, and then find the connections between the two by different boundary conditions. While in Section 2.3.1.1, we obtain V^* and W^* by looking at how the two interact with each others, and by using the probability properties of the underlying price process directly.

2.4 Seasonal Factors

To account for the seasonal factor in the mean reversion model, I choose to add a function $f(t)$ into the price process in the following way:

$$\ln(P) = u_t + f(t), \tag{2.4.1}$$

where $du_t = k\cdot(m - u_t)\cdot dt + \sigma\cdot dB$, and $f(t)$ is a periodic, once differentiable function.

By Ito's Lemma, the instantaneous change dP can be expressed as:

$$dP = \left\{k\cdot(m - \ln(P) + f(t)) + \frac{1}{2}\sigma^2 + \frac{df(t)}{dt}\right\}\cdot P\cdot dt + P\cdot\sigma\cdot dB_t . \tag{2.4.2}$$

Using the marginal analysis argument, I equate the risk-neutral expected rate of price appreciation (MR) to the risk-neutral expected total cost (MC):

$$E[MR] = E[MC] \tag{2.4.3}$$

$$\Rightarrow E\left[\frac{dP}{dt}\right] = E\left[\frac{dTC}{dt}\right]$$

$$\Rightarrow \left\{k\cdot(m - \ln(P) + f(t)) + \frac{1}{2}\sigma^2 + \frac{df(t)}{dt}\right\}\cdot P = r\cdot P + c .$$

Solve (2.4.3) for the critical price which makes the $MR = MC$ for a fixed t:

$$P^* = e^{m - \frac{r}{k} + \frac{1}{2\cdot k}\cdot\sigma^2 + f(t) + \frac{df(t)}{k\cdot dt} + \mathrm{LambertW}\left(-\frac{c}{k}\cdot e^{-m + \frac{r}{k} - \frac{1}{2\cdot k}\cdot\sigma^2 - f(t) - \frac{df(t)}{k\cdot dt}}\right)} . \tag{2.4.4}$$

Let $F(P,t)$ be the function:

$$F(P,t) = E[MR] - E[MC] = \left\{k\cdot(m - \ln P + f(t)) + \frac{1}{2}\sigma^2 + \frac{df(t)}{dt}\right\}\cdot P - (r\cdot P + c). \tag{2.4.5}$$

One property of (2.4.5) is that when $f(t)$ satisfies certain conditions, we have $F < 0$ regardless the value of P, and I prove this conclusion formally in the Claim 2.4.1.

Claim 2.4.1:

Under the assumptions: $f(t)$ is a periodic, once differentiable function; k, m, σ, c are positive constants; $F(P)$ is the function defined by (2.4.5).

If $f(t)$ satisfies:

$$\min_t k\cdot f(t) + \frac{df(t)}{dt} < \beta + \alpha, \textit{ where } \alpha = r - k\cdot m - 0.5\cdot\sigma^2, \beta = k\cdot\left(\ln\left(\frac{c}{k}\right) + 1\right). \tag{i}$$

Then, there exists a set Q of values of t such that for every t in set Q, the function $F < 0$.

Proof:

To prove $F < 0$, we need to prove:

$$k \cdot f(t) + \frac{df(t)}{dt} < \left(k \cdot \ln(P) + \frac{c}{P}\right) + \alpha\,,\ \textit{where}\ \alpha = r - k \cdot m - 0.5 \cdot \sigma^2. \qquad \text{(ii)}$$

Now let us examine the function:

$$Y(P) = k \cdot \ln(P) + \frac{c}{P}\,.$$

To find the minimum value, I use the first order condition:

$$\frac{dY(P)}{dP} = 0 \Rightarrow \frac{k}{P} - \frac{c}{P^2} = 0 \Rightarrow P = \frac{c}{k}\,.$$

Take the second derivative of Y(P), I have:

$$\frac{d^2Y(P)}{dP^2} = -\frac{k}{P^2} + \frac{2 \cdot c}{P^3},\ \text{substitute}\, P = \frac{c}{k}, \Rightarrow \frac{d^2Y(P)}{dP^2} = \frac{k^3}{c^2} > 0\,.$$

Since Y(P) is a continuous function, with the positive second order derivative, I have:

$$\min_P\ Y(P) = k \cdot \left(\ln\left(\frac{c}{k}\right) + 1\right),\ \textit{where equality is taken when}\ P = \frac{c}{k}\,.$$

Therefore, using the given condition (i), I have:

$$\min_t\ k \cdot f(t) + \frac{df(t)}{dt} < \beta + \alpha = \min_P\ Y(P) + \alpha \le k \cdot \ln(P) + \frac{c}{P} + \alpha\ \ \text{for any}\ P > 0\,.$$

Therefore, we prove the (ii), which in turn proves the claim: under (i), there exists some t, such that F < 0 for all P>0. □

Claim 2.4.1 essentially expresses the idea that for certain time periods of some periodic function *f(t)*, even though the current price *P* is very low, the rate of the price appreciation is always smaller than the total cost.

We can also derive the Claim 2.4.1 from the LambertW function, which has real solution only when c ≥ -1/e. Using this criteria in (2.4.4), we can obtain the same result as that in Claim 2.4.1. From (2.4.4) and Claim 2.4.1., it is clear that in the seasonal case, the trigger prices are functions of the calendar time *t*.

Let *X(t)* denote the term inside the LambertW function in (2.4.4), and *X(t)* is negative since we require *k, m, σ, c* are positive. I use the following strategy to maximize the value of the facility:

Calendar Time t	Optimal Strategy
(a). When *X(t)* < *-1/e*:	There is no solution to (2.4.4). Use the result from Claim

	2.4.1, $F(P,t) < 0$. We will empty the inventory immediately.
(b). When $-1/e \leq X(t) \leq 0$	There are two solutions P_1^*, P_2^* to (2.4.4). Assuming $P_1^* \leq P_2^*$, I can compare the current price P_0 with P_1^* and P_2^* and use the strategies outlined in Table 2.2.2.

Table 2.4.1: Optimal strategy for the seasonal models.

Because in the seasonal case the trigger prices are functions of time, in the following sections, unless indicated otherwise, I will not directly use the trigger prices as the decision variables. Instead, I will use the sign of the function $F(P,t)$ to make the buying or selling decision. There should be no confusion about the changes, since the trigger prices are derived by setting $F(P,t) = 0$, and there are just the two ways to express the same information.

2.5 Deterministic Model with Seasonality

In this section, I assume $f(t) = b\cdot\sin(2\pi\cdot w\cdot t + \varphi)$, and the price process is:

$$\ln(P) = u_t + b\cdot\sin(2\pi\cdot w\cdot t + \varphi), \tag{2.5.1}$$

where $du_t = k\cdot(m - u_t)\cdot dt$.

In (2.5.1), the seasonality is expressed as a sine function, where b is the amplitude, w is the frequency, and φ is the variable to account for the time shift. By default, I assume that when $t = 0$, the sine function achieves the minimum – 1, and hence $\varphi = -0.5\pi$.

In Figure 2.5.1, I plot the spot price process together with reversion value m, the pure u_t term, and pure sine term (assuming initial value is smaller than the reversion value m). Figure 2.5.1 shows that although the pure sinusoidal term (seasonal term, denoted as '—··—' in Figure 2.5.1) varies mildly, the effect on the spot price is very prominent. This is due to the way I add the seasonal factor into the price process.

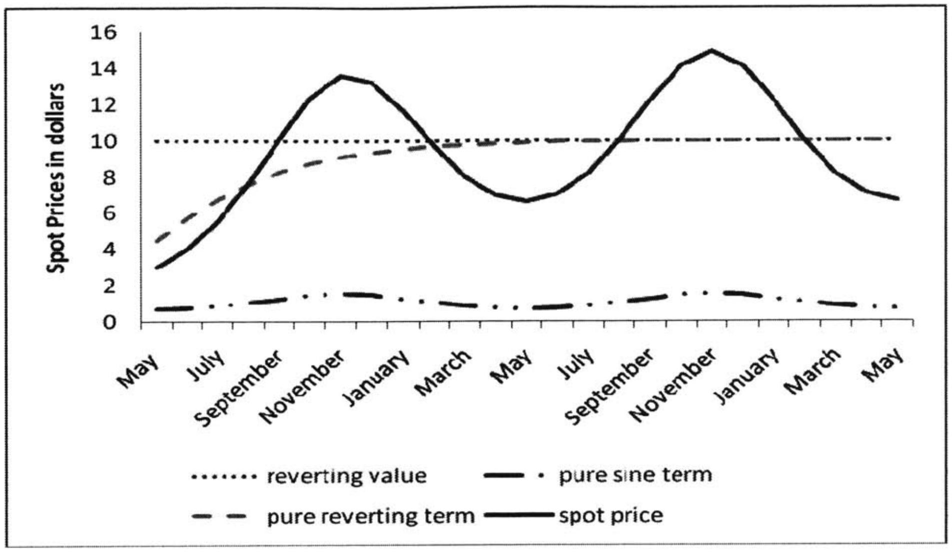

Figure 2.5.1: One realization of a seasonal price process

Next, I derive the optimal operating strategies for the deterministic seasonal price model. Using the concept of the marginal analysis (let MR be the rate of the price appreciation, and let MC be the rate of the total cost changes), and the *F(P,t)* function defined as (2.4.5), I have:

$$F(P,t) = \left\{k\left(m - \ln P + b\cdot\sin(2\pi wt + \varphi)\right) + \frac{1}{2}\sigma^2 + b\cdot 2\pi w\cdot\cos(2\pi wt + \varphi)\right\}P - (rP + c) \quad (2.5.2)$$

To characterize function *F(P,t)* graphically, I substitute *P* using relation (2.5.1) and change it into a function of time *t*, written as *F(t)*. I plot it in Figure 2.5.2.

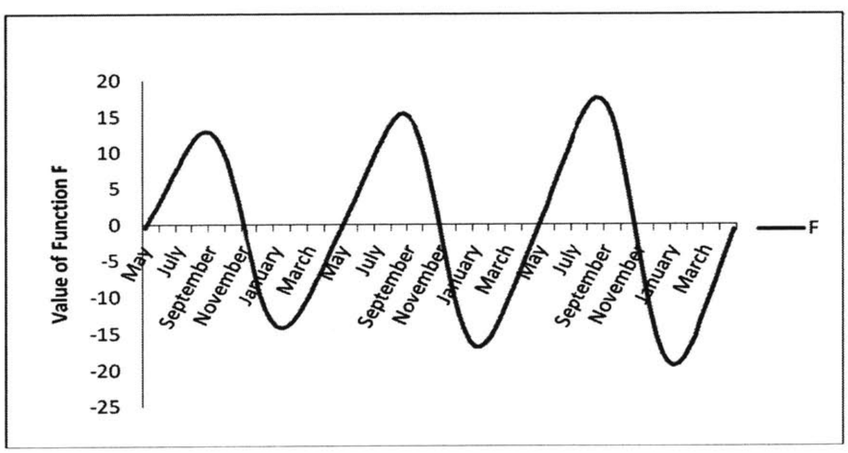

Figure 2.5.2: The value of *F* as a function of time t.

In Figure 2.5.2, the points where the *F* crosses the zero axis are where MC = MR. All those points are the trigger points, and we use the following operating strategy to maximize our profit:

- The regions where $F(P,t) > 0$ is where MR > MC, and we will buy and hold the inventory because we will gain by holding for an extra period.
- The regions where $F(P,t) < 0$ indicate MR < MC, and we will sell any inventory because the cost of holding exceed the revenue in the regions.

In Figure 2.5.1 and 2.5.2, I use the following set of parameters:

k	m	u_0	b	w	φ
0.3	2.3	1.5	0.4	1	– 0.5π.

2.6 Stochastic Model with Seasonality

In this section, I explore the stochastic model under the seasonal mean reversion model developed in Section 2.4. The risk-adjusted spot price with seasonality is modeled as:

$$\ln(P) = u_t + f(t), \tag{2.6.1}$$

where $du_t = k\cdot(m - u_t)\cdot dt + \sigma\cdot dB$, and $f(t)$ is a periodic, once differentiable function.

And I list again the equations in 2.4 for convenience:

$$dP = \left\{k\cdot(m - \ln(P) + f(t)) + \frac{1}{2}\sigma^2 + \frac{df(t)}{dt}\right\}\cdot P\cdot dt + P\cdot\sigma\cdot dB_t . \tag{2.6.2}$$

$$E[MR] = E[MC] \tag{2.6.3}$$

$$\Rightarrow E\left[\frac{dP}{dt}\right] = E\left[\frac{dTC}{dt}\right]$$

$$\Rightarrow \left\{k\cdot(m - \ln(P) + f(t)) + \frac{1}{2}\sigma^2 + \frac{df(t)}{dt}\right\}\cdot P = r\cdot P + c .$$

$$P^* = e^{m - \frac{r}{k} + \frac{1}{2\cdot k}\cdot\sigma^2 + f(t) + \frac{df(t)}{k\cdot dt} + \mathrm{LambertW}\left(-\frac{c}{k}\cdot e^{-m + \frac{r}{k} - \frac{1}{2\cdot k}\cdot\sigma^2 - f(t) - \frac{df(t)}{k\cdot dt}}\right)} . \tag{2.6.4}$$

$$F(P,t) = E[MR] - E[MC] = \left\{k\cdot(m - \ln P + f(t)) + \frac{1}{2}\sigma^2 + \frac{df(t)}{dt}\right\}\cdot P - (r\cdot P + c). \tag{2.6.5}$$

First, I illustrate the decision variable $F(P,t)$ using an example. Let $f(t) = b\cdot\sin(2\cdot\pi\cdot\omega\cdot t + \varphi)$, and I plot (2.6.5) in Figure 2.6.1.

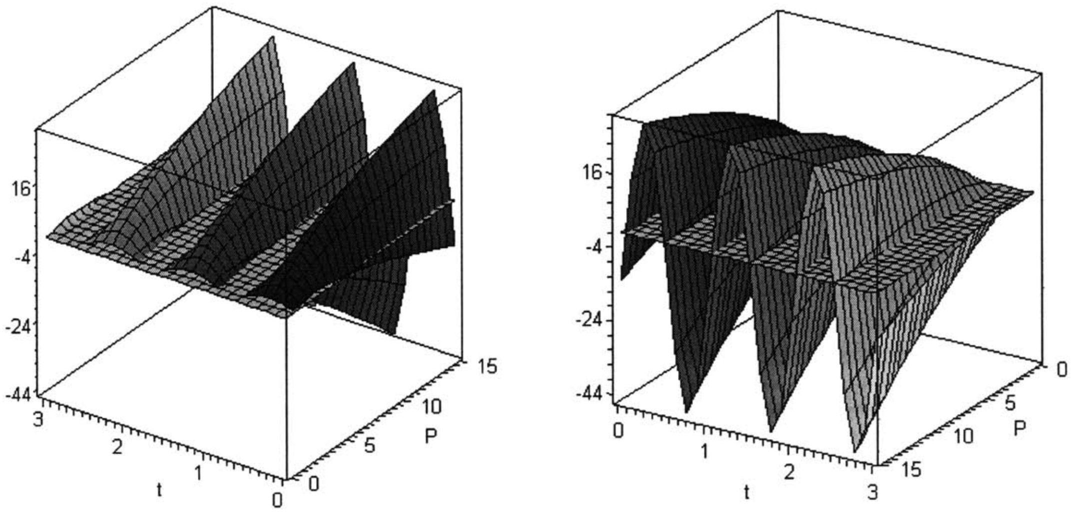

Figure 2.6.1: Plot of *F* as a function of t and P.

Figure 2.6.1 shows, for certain value of time *t*, *F(P,t)* is always negative regardless of the value of *P*. Hence, no matter what the current price is, we will not buy or hold any inventory during the time. This is the result because of the assumptions made on the structure of the model (2.6.1), and I have proved the result formally in the Claim 2.4.1, which expresses the idea for certain time period, even though the current price *P* is very low, the rate of price gain by holding is always smaller than loss due to both the holding cost and the cost of money (at a risk-free rate).

Let P_1^* and P_2^* be the solutions to $F(P, t) = 0$, We can find the optimal trigger prices at time t with the optimal strategy listed in Table 2.6.1.

1. If $F(P, t) = 0$ has solutions.	**MR – MC**	**Operation Strategy:**
$P_t \geq P_2^*$:	Non-positive.	Sell the inventory immediately.
$P_1^* < P_t < P_2^*$:	Positive.	Buy the inventory immediately, and hold the inventory.
$P_t \leq P_1^*$:	Non-positive.	Sell the inventory immediately.
2. If $F(P, t) = 0$ does not has solutions.	**MR – MC**	**Operation Strategy:**
Any P_t	Non-positive.	Sell the inventory immediately.

Table 2.6.1: Optimal trading strategy for the seasonal case.

The above strategy can be seen from the Figure 2.6.2, where the regions between the dotted line and the dashed line are the places we hold the inventory; otherwise, we sell it immediately.

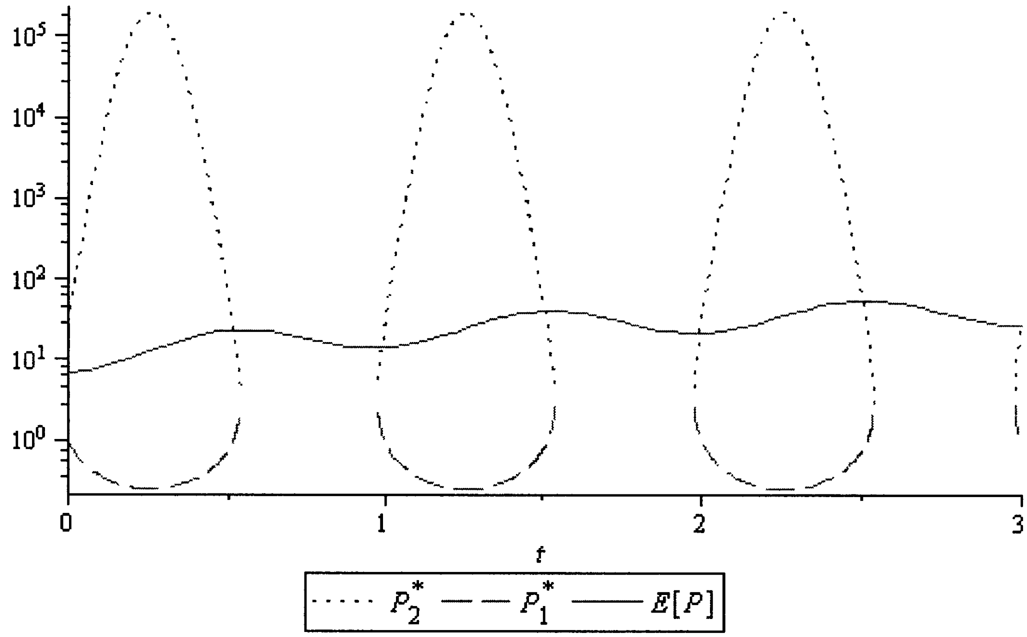

Figure 2.6.2: The optimal trigger prices.

2.7 Value Decomposition

The value of the storage facility can be divided into three components: the value from the reversion components shown as the drift term in (2.3.2), the value from the volatility term, and the value from the seasonality term. To examine each of the components, I do the followings:

A. Set the volatility σ in (2.6.1) to 0 in order to examine the value from the drift term.

B. Set σ some positive value in (2.6.1) to examine the volatility term.

C. Let the price follow $P_t = exp(f(t)+m)$, where $f(t) = b \cdot sin(2 \cdot \pi \cdot \omega \cdot t + \varphi)$ to examine the value from the pure seasonal factors.

D. Let the price follow the model (2.3.1) – the stochastic mean reversion model. This is the case when we remove the seasonal factor $f(t)$ in case B.

In Case B, because the volatility term σ changes, we need to just the drift term under the risk-neutral price process. Since under the risk-neutral measure $m = m_{real} - \lambda \cdot \sigma / k$, I adjust the above term m as σ changes in the followings: in Case A and Case C, let $m_A = m_C = m_{real}$, and in Case B and D, I let $m = m_{real} - \lambda \cdot \sigma / k$. Using Monte Carlo simulation, I generate the price paths in the following way:

$$u_{n+1} = u_n + k \cdot (m - u_n) \cdot \Delta t + \sigma \cdot \sqrt{\Delta t} \cdot z_{n+1}, \tag{2.7.1}$$

$$P_{n+1} = e^{b \cdot \sin(2 \cdot \pi \cdot w \cdot (n+1) \cdot \Delta t) + u_{n+1}},$$

where Δt is the time step and z_{n+1} is the standard Gaussian random variable.

I follow the strategy outlined in Table 2.6.1 to find the optimal values of the storage facility in Case A, B, and C, and use the strategies in Section 2.3.2 to find the value for Case D. The values are plotted in Figure 2.7.1.

The observation of Figure 2.7.1 can lead to the conclusion: both the seasonal factor and the stochastic factor in the mean reversion model play an important role in the valuation of a storage facility. By considering both of the factors, we can significantly increase the value of a storage facility, and substantial reduction in the storage value can happen if either of the two are ignored.

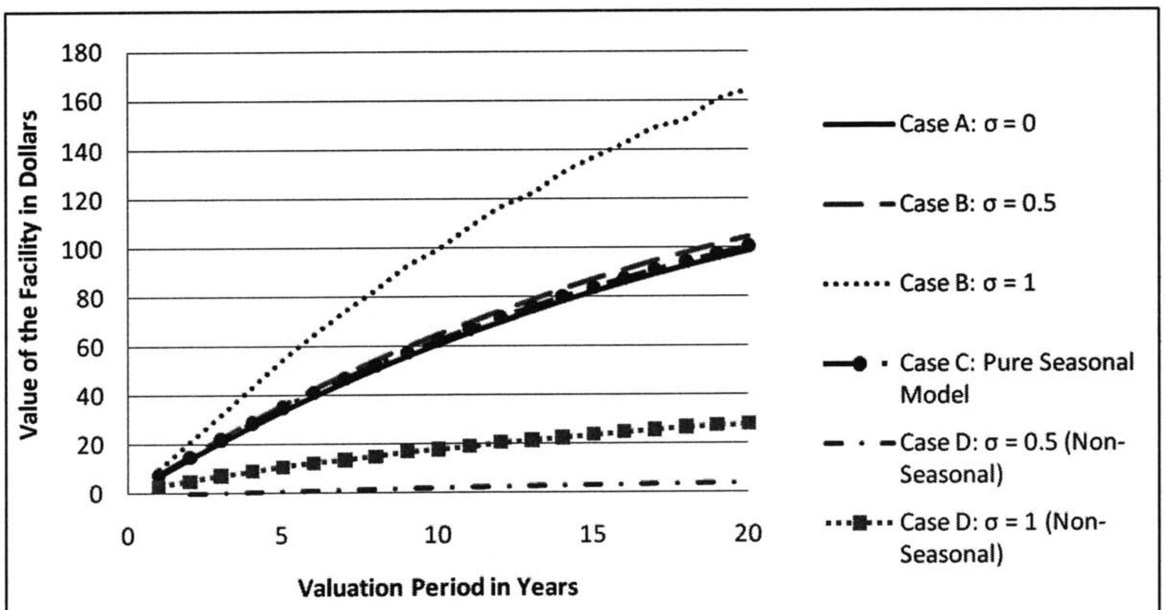

Figure 2.7.1: Comparisons of the effects from the seasonal and stochastic components.

In the simulation, the values of the parameters are:

k	r	m_{real}	u_0	b	w	Φ	σ	λ
0.3	0.05	2.3	2.2	0.4	1	0	1	0.1

CHAPTER 3

ONE-FACTOR GEOMETRIC BROWNIAN MOTION MODELS

3.1 Chapter Introduction

I derive the optimal operating strategy for the valuation of a storage facility with the one-factor geometric Brownian motion (GBM) price process where the spot price is modeled as:

$$\frac{dP}{P} = m_{real} \cdot dt + \sigma \cdot dB_{t,\, real}, \tag{3.1.1}$$

where m_{real} is a real constant, σ is a non-negative real constant, and $B_{t,\, real}$ represents the standard Brownian Motion.

The stochastic differential equation (3.1.1) has the solution:

$$P = P_0 \cdot e^{\left(m_{real} - \frac{1}{2} \cdot \sigma^2\right) \cdot t + \sigma \cdot B_t}. \tag{3.1.2}$$

In this chapter, as what I have done in Chapter 2, I investigate separately the deterministic case by setting $\sigma = 0$ in (3.1.1), and the stochastic case by considering the whole model of (3.1.1). This separation can better illustrate how the different components contribute to the overall value of a storage facility. Then, a seasonal factor is added into the model, and the models in Chapter 2 and in Chapter 3 are compared and their differences are discussed at the end of the chapter.

In this chapter, I make the following assumptions:

- $0 < P_0 \leq P_L \leq P_H < \infty$, where P_0, P_L, P_H are the initial spot price, the buying price, and the selling price respectively.
- The risk-neutral expected appreciation rate is: $m = m_{real} - \lambda \cdot \sigma / k$, where λ is the market price of risk, and hence the risk-neutral price process can be written as:

$$P = P_0 \cdot e^{\left(m - \frac{1}{2} \cdot \sigma^2\right) \cdot t + \sigma \cdot B_t}. \tag{3.1.3}$$

where B_t is the standard Brownian motion under the risk-neutral measure.

- The fixed storage cost $c > 0$, and it is only incurred when the facility is full.
- We buy at time t_{PL} with price P_L, and sell at time t_{PH} ($t_{PH} \geq t_L$) at price P_H.

With the above assumption, the objective of this chapter is to solve to the optimal valuation problem:

$$Z^*\left(P_L^*, P_H^*\right) = \max_{P_L, P_H} Z\left(P_L, P_H\right). \tag{3.1.4}$$

The key conclusion from the chapter is that the pure GBM model is not adequate to the natural gas storage valuation problem. Only when we add the seasonal factor into the GBM model, we can have a realistic model that tackles the storage valuation problem effectively.

3.2 Deterministic Model without Seasonality

In this section, assuming $\sigma = 0$ in (3.1.3), I have:

$$P = P_0 \cdot e^{m \cdot t}. \tag{3.2.1}$$

The net present value by buying at price P_L and selling at P_H, starting at P_0, is:

$$Z = e^{-r \cdot t_{PH}} \cdot P_H - e^{-r \cdot t_{PL}} P_L - \int_{t_{PL}}^{t_{PH}} c \cdot e^{-r \cdot t}\, dt = e^{-r \cdot t_{PH}} \cdot \left(P_H + \frac{c}{r}\right) - e^{-r \cdot t_{PL}} \cdot \left(P_L + \frac{c}{r}\right). \tag{3.2.2}$$

To maximize the net present value Z, I first differentiate Z with respect to P_H:

$$\frac{\partial}{\partial P_H} Z = -\frac{r\, e^{-r \cdot t_{PH}}}{m} + e^{-r \cdot t_{PH}} - \frac{c\, e^{-r \cdot t_{PH}}}{m\, P_H}. \tag{3.2.3}$$

Equate (3.2.3) to 0, and solve for the critical price. I have:

$$P^* = \frac{c}{m - r}. \tag{3.2.4}$$

Together with the second derivative:

$$\frac{\partial^2}{\partial P_H^2} Z\left(P^*\right) = \frac{c}{m \cdot \left(P^*\right)^2} \cdot e^{-r \cdot t_{PH}} > 0. \tag{3.2.5}$$

Therefore, $P_H^* = P^*$ is the price that minimizes the profit function Z, given $0 < P_0 \leq P_L \leq P_H < \infty$.

Similarly, the critical price for P_L can be derived by first finding:

$$\frac{\partial}{\partial P_L} Z = \frac{r\,e^{-r \cdot t_{PL}}}{m} - e^{-r \cdot t_{PL}} + \frac{c\,e^{-r \cdot t_{PL}}}{m\,P_L}. \qquad (3.2.6)$$

Then, equate (3.2.6) to 0, and solve for the critical price. I have:

$$P^* = \frac{c}{m-r}. \qquad (3.2.7)$$

The second derivative with respect to P_L is:

$$\frac{\partial^2}{\partial P_L^2} Z(P^*) = -\frac{c}{m \cdot (P^*)^2} \cdot e^{-r \cdot t_{PL}} < 0. \qquad (3.2.8)$$

Therefore, $P_L^* = P^*$ is the price that maximizes the profit function Z, given $0 < P_0 \leq P_L \leq P_H < \infty$.

With the above information, in order to maximize Z, P_H^* must take the boundary values given the form of Z, which is either P_L or $+\infty$. If $P_L = P_L$, the value Z = 0. Therefore, we need to compare the value of Z at $P_H = +\infty$ to 0, so to decide the optimal value of P_H^*.

Thus, the optimal operating strategy is:

1. When $m - r > 0$:	*If* $0 < P_0 \leq P^*$:	The optimal strategy is to buy at $P_L = c\,/\,(m - r)$, and sell at $P_H = +\infty$. This strategy violates the assumption $0 < P_0 \leq P_L \leq P_H < \infty$. The $m - r > 0$ case may also be problematic for a deterministic case since it assumes that the convenient yield is negative.
	If $P_0 > P^*$:	The optimal strategy is to buy at $P_L = P_0$, and sell at $P_H = +\infty$. This strategy violates the assumption $0 < P_0 \leq P_L \leq P_H < \infty$.
2. When $m - r = 0$	P^* does not exist:	The optimal strategy is to buy at $P_L = +\infty$ and sell it immediately, with $Z^* = 0$. This is because the first derivative of Z with respect to P_L is positive by observing (3.2.6), which indicates Z is an increasing function. The first derivative of Z with respect to

		P_H is negative by observing (3.2.3), which indicates Z is a decreasing function. Therefore, we should buy as late as possible, and sell as early as possible. That is $P_L = +\infty$ and sell it immediately. This strategy violates our assumptions.
3. When $m - r < 0$	P^* is negative.	The optimal strategy is either to buy at $P_L = +\infty$, and sell it immediately, or to do nothing. The optimal value is $Z^* = 0$. This is because when $m - r < 0$, (3.2.6) increases as P_L increases, while (3.2.3) decreases as P_H increases. This strategy violates our assumptions.

Table 3.2.1: Optimal strategy to maximize the value of a storage facility.

Table 3.2.1 shows, using this price model, the optimal operating strategy either violate our assumption $0 < P_0 \leq P_L \leq P_H < \infty$, or it gives a storage value of 0. Hence, I conclude this particular deterministic model is not adequate for the storage valuation problem.

3.3 Stochastic Model without Seasonality

Assume both $\sigma > 0$ and the facility can operate for infinite life. I follow the contingent claim method introduced in Section 2.3, and obtain the following two differential equations for a full and an empty storage facility respectively:

$$\frac{1}{2}\sigma^2 P^2 \cdot V_{PP} + m \cdot P \cdot V_P - r \cdot V = c\,, \tag{3.3.1}$$

$$\frac{1}{2}\sigma^2 P^2 \cdot W_{PP} + m \cdot P \cdot W_P - r \cdot W = 0\,. \tag{3.3.2}$$

Further, assuming $0 < P_L \leq P \leq P_H < +\infty$, in this section, I am going to find the optimal operating strategies P_L^* and P_H^* that solves the optimal valuation problem (3.1.4). The optimal values V^* and W^* must satisfy (refer to Claim 2.3.1 for a proof):

$W^*(P^*) = P^* + V^*(P^*)$, where P^* is the optimal trigger price. (3.3.4)

Next, I show that as in the deterministic case, this stochastic model (3.1.3) is not adequate so model the optimal valuation problem.

Claim 3.3:

Assume that V^* and W^* are twice differentiable functions, and $0 < P_L \leq P \leq P_H < +\infty$, then the optimal pair of trigger prices that satisfy (3.3.1), (3.3.2) and (3.3.3) does not exist.

Proof:

Assume there exists at least one solution P^* *satisfying* $0 < P_L \leq P_L^* < P^* < P_H^* \leq P_H < +\infty$, *to the optimal operating problem (3.1.4). I can follow the argument in Claim 2.3.1 and Claim 2.3.2 to find the following relationships:*

$$W^*(P^*) = P^* + V^*(P^*) . \tag{i}$$

$$\frac{\mathrm{d}}{\mathrm{d}P} W^*(P^*) = 1 + \frac{\mathrm{d}}{\mathrm{d}P} V^*(P^*) . \tag{ii}$$

$$\frac{\mathrm{d}^2}{\mathrm{d}P^2} W^*(P^*) = \frac{\mathrm{d}^2}{\mathrm{d}P^2} V^*(P^*) . \tag{iii}$$

Using (i), (ii), (iii), and subtracting (3.3.2) from (3.3.1), I have:

$$m \cdot P^* = r \cdot P^* + c . \tag{iv}$$

(iv) is again, exactly the equation of E[MC] = E[MR] discussed in the Chapter 2. I can use a similar argument as those in Table 3.2.1 to argue that in order to maximize the profit Z, when $m - r > 0$, *one has to sell at* $P_H = +\infty$; *when* $m - r = 0$, *one has to buy at* $P_L = +\infty$; *when* $m - r < 0$, *one has to buy at* $P_L = +\infty$.

The above strategy that maximizes the value of the facility violate the assumption $0 < P_L \leq P \leq P_H < +\infty$.

Therefore, under our assumption, there does not exist the optimal trigger prices that maximize the value of a storage facility □

From Claim 3.3.1, I can conclude that (3.1.3) is not a good model to solve our optimal valuation problem.

3.4 Deterministic Model with Seasonality

Once we add the seasonal factor into the model (3.1.3), there are some new properties. Let the seasonal deterministic price process follows:

$$P = P_0 \cdot \exp(f(t) + m \cdot t)\,, \tag{3.4.1}$$

where $f(t)$ is a periodic, once-differentiable function.

Using the marginal analysis argument, I define the function $F(P,t)$ as:

$$F(P,t) = MR - MC = \frac{dP}{dt} - \frac{dTC}{dt} = P \cdot \left(\frac{df(t)}{dt} + m \right) - r \cdot P - c. \tag{3.4.2}$$

To find the optimal operating strategies, similar to Table 2.4.1, one can operate the storage facility according to the rules: buying and holding the inventory when $F(P_t\,,\, t) > 0$; sell the inventory immediately when $F(P_t\,,\, t) \; < 0$.

3.5 Stochastic Model with Seasonality

In this section, I use both the contingent claim argument and the marginal analysis argument to find the optimal trigger prices with the corresponding optimal value of a storage facility.

In this section, the stochastic risk-neutral price process with seasonality is modeled as:

$$P = P_0 \cdot \mathrm{e}^{f(t) + \left(m - \frac{1}{2}\sigma^2\right) t + \sigma B_t}, \tag{3.5.1}$$

where $f(t)$ is periodic, once differentiable function.

The stochastic equation for the instantaneous return is:

$$\frac{dP}{P} = \left(\frac{df(t)}{dt} + m \right) \cdot dt + \sigma \cdot dB_t\,. \tag{3.5.2}$$

Using the contingent claim argument, I have:

$$\frac{1}{2}\sigma^2 P^2 \cdot V_{PP} + \left(\frac{df(t)}{dt} + m \right) P \cdot V_P + V_t - r\, V(P,t) = c\,, \tag{3.5.3}$$

where $V_P = \frac{\mathrm{d}}{\mathrm{d}P} V(P,t),\; V_{PP} = \frac{\mathrm{d}^2}{\mathrm{d}P^2} V(P,t),\; V_t = \frac{\mathrm{d}}{\mathrm{d}t} V(P,t)$.

$$\frac{1}{2}\sigma^2 P^2 \cdot W_{PP} + \left(\frac{df(t)}{dt} + m \right) P \cdot W_P + W_t - r\, W(P,t) = 0\,, \tag{3.5.4}$$

where $W_P = \frac{\mathrm{d}}{\mathrm{d}P} W(P,t)$, $W_{PP} = \frac{\mathrm{d}^2}{\mathrm{d}P^2} W(P,t)$, $W_t = \frac{\mathrm{d}}{\mathrm{d}t} W(P,t)$.

The trigger price P^*, have to satisfy the following conditions (the proof is similar to the proof of Claim 2.3.3 and Claim 2.3.4):

$$W^*(P,t) = P + V^*(P,t) . \tag{3.5.5}$$

$$\frac{\mathrm{d}}{\mathrm{d}P} W^*(P,t) = 1 + \frac{\mathrm{d}}{\mathrm{d}P} V^*(P,t) .$$

$$\frac{\mathrm{d}^2}{\mathrm{d}P^2} W^*(P,t) = \frac{\mathrm{d}^2}{\mathrm{d}P^2} V^*(P,t) .$$

$$\frac{\mathrm{d}}{\mathrm{d}t} W^*(P,t) = \frac{\mathrm{d}}{\mathrm{d}t} V^*(P,t) .$$

Subtracting (3.5.4) from (3.5.3), and using the conditions in (3.5.5), I have:

$$P^* = P_H^* = P_L^* = \frac{c}{df(t)/dt + m - r} . \tag{3.5.6}$$

From the above equation, I have two scenarios:

$\frac{df(t)}{dt} + m > r$	At those time t, we have an optimal P^* that is positive and finite, and thus it is the optimal value that satisfies (3.5.3) and (3.5.4).
$\frac{df(t)}{dt} + m \leq r$	At those time t, there does not exist positive and finite P^*, and thus, no optimal operating strategy exists. (A similar conclusion is also made in Section 2.6).

For example, let $f(t) = b \cdot sin(2 \cdot \pi \cdot w \cdot t + \varphi)$. I plot the trigger price P^* in Figure 3.5.1. The solid line indicates the regions where $m + 2 \cdot b \cdot \pi \cdot w \cdot cos(2 \cdot \pi \cdot w \cdot t) - r > 0$, and where the trigger price exists.

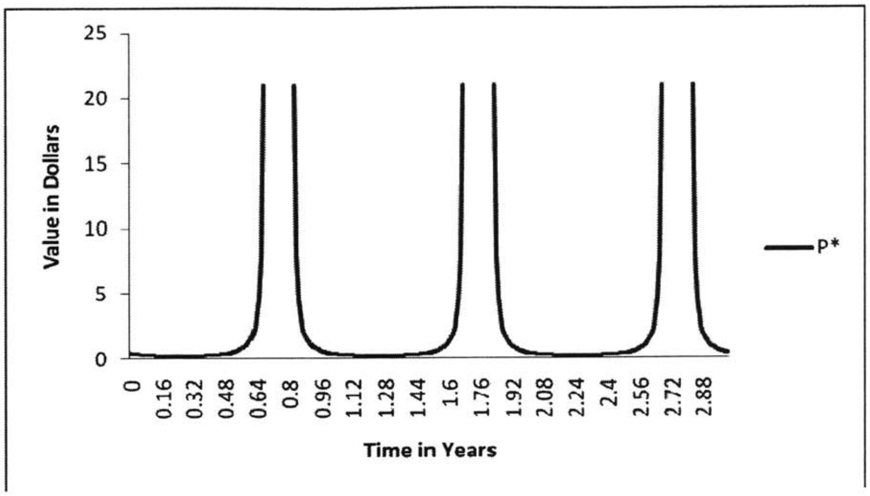

Figure 3.5.1: The optimal trigger price P* at each time t.

Therefore, the optimal operating strategy can be formulated as (*P* is the current price):

Condition	Strategy
When *(3.5.6)* exits, and $P > P_t^*$,	Buy and hold the inventory.
Otherwise,	Sell the inventory immediately.

Table 3.5.1: Optimal operating strategy

The above strategy can also be understood by plotting the function $F(P, t)$, which is defined in Section 3.4) in Figure 3.5.2. The correspondence between $F(P, t)$ and P_t^* is:

- The regions where $F(P, t) > 0$ correspond to the solid line regions in Figure 3.5.1.
- The regions where $F(P, t) \leq 0$ correspond to the empty regions in Figure 3.5.1, where the optimal trigger price does not exist.

With the correspondence between $F(P, t)$ and P_t^* the optimal strategy can also be expressed in terms of $F(P_t, t)$ which is identical to Table 2.4.1.

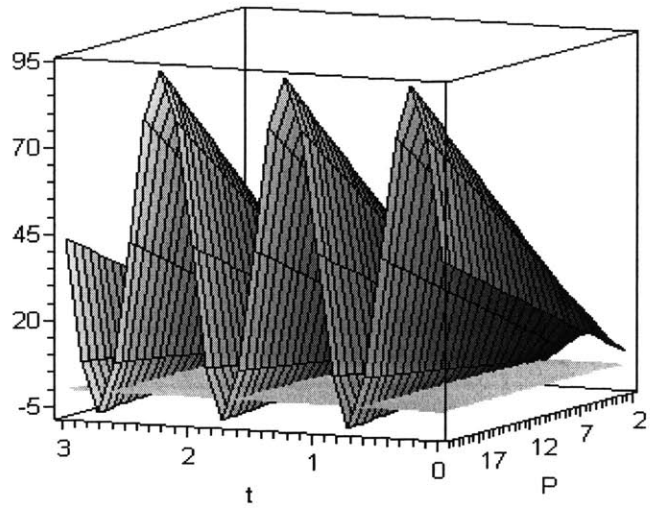

Figure 3.5.2: Function $F = MR - MC$ as a function of both t and price P.

Comparing the result from this section to that from Section 2.6, although the two operating strategies are the same, there are some fundamental differences about the sources of the storage values between the two price process: in the mean reversion price model, once the price goes far from the reversion level (m), the price has a strong tendency to revert to the mean level regardless of the seasonal factor; in the geometric Brownian motion model, since there is no price reversion tendency, any reversion is due to the seasonal factor. Therefore, in the two cases, the underlying driving factors of 'reversion' are fundamentally different.

This difference can be shown more clearly by studying the value from the different components in the GBM with the seasonal factor model. (A numerical example with this price process is given in the next section.)

3.6 Value Decomposition

The value of a storage facility can be investigated by comparing the following values:

1. The value from the pure drift component (denoted by Z1), which is the $m \cdot dt$ in (3.1.1) or the price follows $P = P_0 \cdot exp(m \cdot t)$.
2. The value from the pure volatility component (denoted by Z2), which is the $\sigma \cdot dB$ in (3.1.1) or the price follows $P = P_0 \cdot \exp(f(t)+\sigma \cdot B)$.

3. The value from the pure seasonal factor $f(t)$ (denoted by Z3), which is $P = P_0 \cdot \exp(f(t))$. And for simplicity, assume $f(t) = b \cdot sin(2 \cdot \pi \cdot w \cdot t)$ in this section, and other forms of $f(t)$ can be discussed in a similar manner.
4. The value from the full model (3.5.1) by setting σ to different values (denoted by Z4).

To facilitate the above comparisons, I define $F(P, t)$ as:

$$F(P,t) = E[MR] - E[MC] = E\left[\frac{dP}{dt}\right] - E\left[\frac{dTC}{dt}\right] = P \cdot \left(\frac{df(t)}{dt} + m\right) - r \cdot P - c. \tag{3.6.1}$$

For a numerical example, I use the following values to find the values in the above cases:

k	r	m	u_0	b	w	Φ	σ
0.3	0.05	0.04	2.2	0.4	1	0	1

In the case 1, since $m < r$, Z1 = 0. For Case 2 to 4, I use the strategies from Table 3.5.1 to find the optimal value of the storage facility. I plot Z1 to Z4 in Figure 3.6.1, where I simulation 0.5 million price paths, according to the following equation:

$$P_{n+1} = P_n \cdot e^{\left(m - \frac{1}{2} \cdot \sigma^2\right) \cdot \Delta t + \sigma \cdot \sqrt{\Delta t} \cdot z_{n+1}}, \tag{3.6.2}$$

where Δt is the step size, and z_{n+1} is the standard Gaussian random variable.

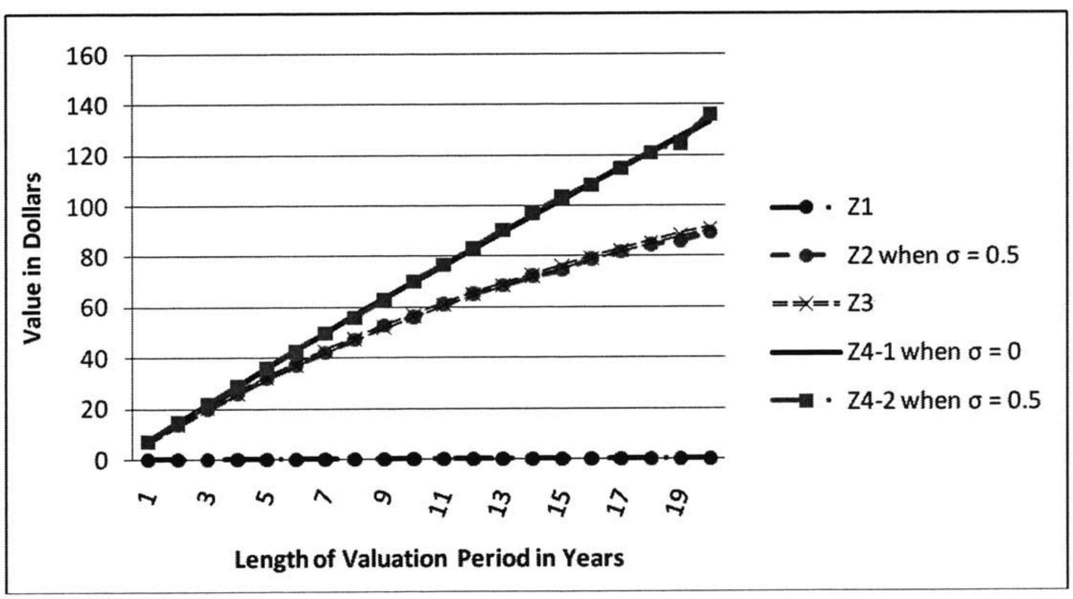

Figure 3.6.1: Comparison of seasonal and stochastic components.

From Figure 3.6.1, we can infer the following results:

1. There is a seasonality value in the storage facility for a GBM price process. Comparing Z1 and Z4, where the only difference is the addition of the seasonal factor *f(t)*, we can see a significant increase in the value of the storage facility.
2. There is no volatility value in the storage for a GBM price process. Comparing either Z4-1 to Z4-2, or Z2 to Z3, we cannot find any increase in the value of the storage facility due to the addition of the stochastic factor.

The first point is easy to understand, while the second point needs some calculations. Let $\sigma > 0$ to incorporate the stochastic component. Then the risk-neutral expectation of the price for Z4 is:

$$E[P_t] = P_0 \cdot e^{b \cdot \sin(2\pi \cdot w \cdot t) + m \cdot t}, \tag{3.6.3}$$

which is not a function of volatility σ.

The risk-neutral price process follows the stochastic equation:

$$\frac{dP}{P} = m \cdot dt + \sigma \cdot dB_t \tag{3.6.4}$$

Which is also not a function of volatility σ.

Because the value of a storage facility is derived under the risk-neutral measure, which will use only (3.6.3) and (3.6.4), the value for different volatility should give the same result. And that explains the second observation.

In conclusion, for the price process modeled in (3.5.1), the value of a geometric price process is derived only from the seasonal effect while the volatility does not contribute to the value of storage facilities. This result is distinctly different from the mean reversion price process where the value of the storage is a combination of the two sources – the volatility factor and the seasonal factor.

CHAPTER 4
TWO-FACTOR PRICE MODELS

4.1 Chapter Introduction

I derive the optimal operating strategy for the valuation of a storage facility when the underlying spot price follows a two-factor stochastic process.

In the following sections, I first introduce the two-factor short-term/long-term spot price model and its mathematical properties. Then, I derive the optimal operating strategy using the dynamics programming method, where again, two differential equations are derived. The value of a storage facility is calculated based on the optimal operating strategy by running Monte Carlo simulation. Next, I add the seasonal factor to the model, and compare this seasonal form to the non-seasonal model. The sources of the value of a storage facility in the seasonal two-factor model are discussed at the end of the chapter.

4.2 Introduction to the Two-Factor Price Model

The one-factor price models in the previous chapters, which attribute the randomness of the spot price to one source of uncertainty, may not be a good representation of the complex underlying factors that could affect the natural gas spot price in the market. This limitation has been discussed in Gilbson & Schwartz (1990), and Baker et al. (1998). Thus, a multi-factor model is better suited to capture the dynamics of the natural gas price, which is influenced by different market factors.

Schwarts & Smith (2000) proposes a two-factor short-term/long-term model that captures both the short-term and long-term properties observed from the natural gas spot price. This model strikes a good balance between complexity and manageability: it provides richer price dynamics than the one-factor models, while the problem is still mathematically tractable for practical purposes.

In this model, the spot price consists of two components: a short-term factor x_t which follows an Ornstein-Uhlenbeck (OU, or mean reversion) process, and a long-term factor ε_t which follows a geometric Brownian motion (GBM) process. Mathematically, the model can be represented as:

$$\ln(P_t) = x_t + \epsilon_t \tag{4.2.1a}$$

where $dx_t = -k\cdot x_t\cdot dt + \sigma_x\cdot dz_{x,\,real}$, $d\epsilon_t = v_{real}\cdot dt + \sigma_\epsilon\cdot dz_{\epsilon,\,real}$, and $\rho = dz_\epsilon\cdot dz_x$.

The price process P_t under the risk-neural measure can be expressed as:

$$\ln(P_t) = x_t + \epsilon_t \tag{4.2.1b}$$

where $dx_t = -k\cdot(x_t - \lambda)\cdot dt + \sigma_x\cdot dz_x$, $d\epsilon_t = v_\epsilon\cdot dt + \sigma_\epsilon\cdot dz_\epsilon$, and $\rho = dz_\epsilon\cdot dz_x$.

In (4.2.1b), λ and v_ε are the risk adjustment parameters.

It is proved in Schwarts & Smith (2000) that x_t and ε_t are bivariate normally distributed under the risk-neutral measure with the mean and covariance matrix:

$$E[(x_t, \epsilon_t)] = \left[\, e^{-k\cdot t}\cdot x_0 + \lambda\cdot(1 - e^{-k\cdot t}),\ \epsilon_0 + v_\epsilon\cdot t \right], \tag{4.2.2}$$

$$Cov[(x_t, \epsilon_t)] = \begin{bmatrix} (1 - e^{-2k\cdot t})\cdot\dfrac{\sigma_x^2}{2\cdot k} & (1 - e^{-k\cdot t})\cdot\dfrac{\rho_{x,\epsilon}\sigma_\epsilon\sigma_x}{k} \\ (1 - e^{-k\cdot t})\cdot\dfrac{\rho_{x,\epsilon}\sigma_\epsilon\sigma_x}{k} & \sigma_\epsilon^2\cdot t \end{bmatrix}.$$

Because x_t and ε_t are bivariate normally distributed, P_t is log-normally distributed with the mean:

$$E[P_t] = \exp\left(E[\log(P_t)] + \frac{1}{2}V[\log(P_t)]\right) \tag{4.2.3}$$

where $E[\log(P_t)] = e^{-k\cdot t}\cdot x_0 + \lambda\cdot(1 - e^{-k\cdot t}) + \epsilon_0 + v_\epsilon\cdot t$,

$$Var[\log(P_t)] = (1 - e^{-2k\cdot t})\cdot\frac{\sigma_x^2}{2\cdot k} + \sigma_\epsilon^2\cdot t + 2\cdot(1 - e^{-k\cdot t})\cdot\frac{\rho_{x,\epsilon}\sigma_\epsilon\sigma_x}{k}.$$

The model shown in (4.2.1a, b) reflects two difference natural gas price properties: first, the natural gas spot price is likely to revert to some fixed level because the production can be adjusted to bring the price back to the level once the price is relatively low or high; second, the fixed level may change due to some permanent changes, such as the decrease in the natural gas reserve, or the new breakthrough in the related technology.

The variables x_t and ε_t in this model capture the above two properties: the short-term deviations are captured by x_t which reverts to zero and follows an OU process, while the long-term equilibrium level is captured by ε_t which is assumed to evolve according to the geometric Brownian motion. This short-term/long-term division offers an intuitive decomposition of the complex factors that affect the price, and it provides a more realistic representation of the underlying spot price process than the one-factor price models.

4.3 Valuation Using the Two-Factor Price Model

In this section, I use the dynamic programming method to derive the optimal operating strategy in order to find the optimal value of the storage facility.

The reason for the change from the contingent claim method to the dynamic programming method is to introduce the use of the dynamic programming in the storage valuation problem. The dynamic programming method can accommodate more complex price models with less required assumptions than the contingent claim method. However, sometimes, it is hard to find the values of some parameters using this method. Please refer to Dixit & Pindyck (1994) for a detailed explanation.

Using the dynamic programming method, I first derive two governing equations: one for the empty storage facility W, and another for the full storage facility V. As demonstrated in the previous chapters, the value of the storage facility can be thought as the net present value of the expected future cash flows (both inflows and outflows) generated by the operations of the storage facility (i.e., buying, holding, and selling), discounted at an appropriate rate (i.e., under the risk-neutral measure, the discount rate is the risk-free rate available in the market). By this argument, the two governing equations can be readily derived using the dynamic programming method.

After finding the governing equations, I follow the procedures used in the previous chapters to find the optimal operating strategy. Then I use the optimal operating strategy to find the value of the storage facility. Again, the optimal operating strategy is defined as the optimal trigger prices P_L^* and P_H^* that maximizes the value Z of a storage facility, expressed as:

$$Z^*\left(P_L^*, P_H^*\right) = \max_{P_L, P_H} Z(P_L, P_H) .$$

4.3.1 Problem Formulation

Assume the life of the storage facility is infinite. Let V, W represent the values of a full and an empty storage facility respectively, which are functions of the price P only (since I assume the facility has infinite life, and thus the time dimension is irrelevant here). The value of a storage facility in the full state and the empty state can be model as:

$$V(P) = \max\left\{P + W(P),\ -c \cdot dt + \frac{1}{1 + r \cdot dt} \cdot E_P[V(P + dP)]\right\}, \tag{4.3.1}$$

$$W(P) = \max\left\{-P + V(P),\ \frac{1}{1 + r \cdot dt} \cdot E_P[W(P + dP)]\right\}, \tag{4.3.2}$$

where r is the discount rate, c is the storage cost per time period, and $E_P[\cdot]$ represents the risk-neutral expectation conditioned on the current spot price P.

We can interpret (4.3.1) as follows: for the value of the full facility $V(P)$, the first term inside the bracket on the right hand side of (4.3.1) represents the full facility value if we sell inventory immediately, and the value of the full facility is equal to the proceeds obtained from the selling price P plus the value of the empty facility $W(P)$; the second term on the right hand side of (4.3.1) means if we hold the inventory instead of selling, we incur the storage cost $c \cdot dt$, and get the discounted expected value of the full storage in the next time period where the price changes from P to $P + dP$. One can compare the two options, and choose the option that offers the larger value. Similar interpretation can be made for (4.3.2).

Using the Ito's Lemma, I have the following relationship derived from (4.2.1b):

$$\frac{dP}{P} = \left[-k \cdot \left(x_t - \lambda\right) + v_\epsilon + \rho \cdot \sigma_x \cdot \sigma_\epsilon + \frac{1}{2} \cdot \sigma_x^2 + \frac{1}{2} \cdot \sigma_\epsilon^2\right] \cdot dt + \sigma_x \cdot dz_x + \sigma_\epsilon \cdot dz_\epsilon .$$

Use Ito's Lemma to expand the expectation terms in (4.3.1). With the above relationship, I have:

$$E_P[V(P+dP)] \tag{4.3.3}$$
$$= E_P\left[V(P) + V_p \cdot dP + \frac{1}{2} \cdot V_{pp} \cdot (dP)^2\right]$$
$$= E_P\left[V(P) + V_p \cdot dP + \frac{1}{2} \cdot V_{pp} \cdot P^2\left(\sigma_x^2 \cdot dt + \sigma_\epsilon^2 \cdot dt + 2 \cdot \rho \sigma_x \sigma_\epsilon dt\right)\right]$$
$$= V(P) + V_p \cdot P \cdot \left(-kx_t^* + k \cdot \lambda + v_\epsilon + \frac{1}{2} \cdot \left(\sigma_x^2 + \sigma_\epsilon^2\right) + \rho \sigma_x \sigma_\epsilon\right) \cdot dt$$
$$+ \frac{1}{2} \cdot V_{pp} \cdot P^2 \cdot \left(\sigma_x^2 + \sigma_\epsilon^2 + 2 \cdot \rho\, \sigma_x \sigma_\epsilon\right)\Big) \cdot dt .$$

where $x^* = E_P[x_t]$, it is the conditional risk-neutral expectation given the current price P.

Substituting (4.3.3) into (4.3.1), and assuming we are in the continuation region (the right hand side of (4.3.1) is the larger of the two), I have the following equation for a full storage facility:

$$-r \cdot V(P) + V_p \cdot P \cdot \left(-k \cdot \left(x_t^* - \lambda\right) + v_\epsilon + \frac{1}{2}\sigma_x^2 + \frac{1}{2}\sigma_\epsilon^2 + \rho \cdot \sigma_x \cdot \sigma_\epsilon\right) \tag{4.3.4}$$
$$+ \frac{1}{2} \cdot V_{pp} \cdot P^2 \cdot \left(\sigma_x^2 + \sigma_\epsilon^2 + 2 \cdot \rho\, \sigma_x \sigma_\epsilon\right) = c .$$

Similarly, for (4.3.2), in the continuation region, the equation for an empty facility is:

$$-r \cdot W(P) + W_p \cdot P \cdot \left(-k \cdot \left(x_t^* - \lambda\right) + v_\epsilon + \frac{1}{2}\sigma_x^2 + \frac{1}{2}\sigma_\epsilon^2 + \rho \cdot \sigma_x \cdot \sigma_\epsilon\right) \tag{4.3.5}$$
$$+ \frac{1}{2} \cdot W_{pp} \cdot P^2 \cdot \left(\sigma_x^2 + \sigma_\epsilon^2 + 2 \cdot \rho\, \sigma_x \sigma_\epsilon\right) = 0 .$$

4.3.2 Derivations of the Optimal Operating Strategies

To find the optimal operating strategy, I follow the same logic as that in Section 2.3, and argue that for the optimal value V^* and W^*, they must satisfy $V^*(P) = W^*(P) + P$ for all positive P (the proof can be derived similar to the proof of Claim 2.3.1). Similar to the proof of Claim 2.3.2, the optimal trigger price P^* must satisfy:

$$V_P^*(P) = 1 + W_P^*(P) , \tag{4.3.6}$$
$$V_{PP}^*(P) = W_{PP}^*(P) .$$

Using the relationships, and subtracting (4.3.4) from (4.3.5), I have:

$$P^* \cdot \left(-k \cdot \left(x_t^* - \lambda\right) + v_\epsilon + \frac{1}{2}\sigma_x^2 + \frac{1}{2}\sigma_\epsilon^2 + \rho \cdot \sigma_x \cdot \sigma_\epsilon\right) = r \cdot P^* + c , \tag{4.3.7}$$

Where the left hand side of equation (4.3.7) is exactly the risk-neutral expected rate of price appreciation while the right hand side is the risk-neutral expected marginal cost.

Thus, marginal analysis argument still applies.

Unlike in the previous chapters, x_t^* in (4.3.7) is not given even though we know the current price P (in the one-factor models, x_t can be deterministically decided once P is given: i.e. in the one-factor mean reversion model, $\ln P = x_t$, and one can find x_t deterministically once P is given). Here, x_t^* is the expected value given the current price P. Using the fact that x_t and ε_t are bivariate normally distributed, I have:

$$x_t^* = E_P[x_t] = E[x_t \mid P] = E[x_t \mid x_t + \epsilon_t = \ln(P)] = E[x_t \mid \epsilon_t = \ln(P) - x_t]. \tag{4.3.8}$$

In probability, if the random variables $y \sim \mathrm{N}(u_y, \sigma_y)$ and $z \sim \mathrm{N}(u_z, \sigma_z)$ are bivariate-normally distributed with correlation coefficient ρ, they have the properties that the conditional probability density of z given $y = y_0$ is also normally distributed:

$$f(z \mid y = y_0) \sim N(u, \sigma) \text{, where } u = u_z + \rho \cdot \sigma_z \cdot \frac{(y_0 - u_y)}{\sigma_y} \text{, and } \sigma = \sigma_z \cdot \sqrt{1 - \rho^2}. \tag{4.3.9}$$

Before continuing, I assume that we are now at time $t = 0$, and we know accurately the values of x_0 and ε_0 (hence we know the current price P_0). We want to find that with this information, what are the optimal switching prices P_t^* for $t > 0$. In the following derivations, I use P_t to replace P in order to show the time dependence explicitly.

With (4.3.9), I can write the conditional distribution of x_t given ε_t as:

$$f(x_t \mid \epsilon_t) \sim N(u, \sigma) \text{, where } u = u_{x'} + \rho' \cdot \sigma_{x'} \cdot \frac{(\varepsilon_0 - u_{\epsilon'})}{\sigma_{\epsilon'}}, \quad \sigma = \sigma_{x'} \cdot \sqrt{1 - (\rho')^2}, \tag{4.3.10}$$

where

$$u_{x'} = e^{-k \cdot t} \cdot x_0 + \lambda \cdot (1 - e^{-k \cdot t}),$$

$$\sigma_{x'} = \sqrt{(1 - e^{-2 \cdot k \cdot t}) \cdot \frac{\sigma_x^2}{2 \cdot k}},$$

$$u_{\epsilon'} = \epsilon_0 + v_\epsilon \cdot t,$$

$$\sigma_{\epsilon'} = \sqrt{\sigma_\epsilon^2 \cdot t},$$

$$\rho' = \rho \cdot \sqrt{\frac{2}{k \cdot t} \cdot \frac{1 - \exp(-k \cdot t)}{1 + \exp(-k \cdot t)}}.$$

With the above properties, (4.3.8) can be written as:

$$x_t^* = \int_{-\infty}^{+\infty} x_t \cdot f(x_t \mid \epsilon_t = \ln(P_t) - x_t) \mathrm{d}x \tag{4.3.11}$$

$$= \int_{-\infty}^{+\infty} x_t \cdot \frac{1}{\sqrt{2 \cdot \pi} \cdot \sigma} \cdot e^{-\frac{1}{2\sigma^2} \cdot \left(x_t - u_{x'} - \rho' \cdot \sigma_{x'} \cdot \frac{\left(\ln(P_t) - x_t - u_{\epsilon'} \right)}{\sigma_{\epsilon'}} \right)^2} dx$$

Manipulating the term in the integral, I have:

$$= \int_{-\infty}^{+\infty} x_t \cdot \frac{1}{\sqrt{2 \cdot \pi} \cdot \sigma} \cdot e^{-\frac{1}{2\sigma^2} \cdot \left(1 + \frac{\rho' \cdot \sigma_{x'}}{\sigma_{\epsilon'}} \right)^2 \cdot \left(x_t - \frac{u_{x'} \cdot \sigma_{\epsilon'} + \rho' \cdot \sigma_{x'} \cdot \left(\ln(P_t) - u_{\epsilon'} \right)}{\sigma_{\epsilon'} + \rho' \cdot \sigma_{x'}} \right)^2} dx$$

$$= \left(\int_{-\infty}^{+\infty} x_t \cdot \frac{1}{\sqrt{2 \cdot \pi} \cdot \sigma} \cdot \left(1 + \frac{\rho' \cdot \sigma_{x'}}{\sigma_{\epsilon'}} \right) \cdot e^{-\frac{1}{2\sigma^2} \cdot \left(1 + \frac{\rho' \cdot \sigma_{x'}}{\sigma_{\epsilon'}} \right)^2 \cdot \left(x_t - \frac{u_{x'} \cdot \sigma_{\epsilon'} + \rho' \cdot \sigma_{x'} \cdot \left(\ln(P) - u_{\epsilon'} \right)}{\sigma_{\epsilon'} + \rho' \cdot \sigma_{x'}} \right)^2} dx \right) \cdot \left(1 + \frac{\rho' \cdot \sigma_{x'}}{\sigma_{\epsilon'}} \right)^{-1} .$$

Now it is in the form of an expectation of another random variable multiplying some constant. Therefore, with the parameters defined in (4.3.10), I have the following result:

$$x_t^* = \left(1 + \frac{\rho' \cdot \sigma_{x'}}{\sigma_{\epsilon'}} \right)^{-2} \cdot \left(u_{x'} + \frac{\rho' \cdot \sigma_{x'}}{\sigma_{\epsilon'}} \cdot \left(\ln(P_t) - u_{\epsilon'} \right) \right) \tag{4.3.12}$$

$$= \frac{e^{-kt} x_0 + \lambda \left(1 - e^{-kt} \right) + \frac{\rho \cdot \sigma_x \cdot \left(1 - e^{-kt} \right)}{\sigma_\epsilon \cdot k \cdot t} \left(\ln(P_t) - \epsilon_0 - \nu_\epsilon \cdot t \right)}{\left(1 + \frac{\rho \cdot \sigma_x \cdot \left(1 - e^{-kt} \right)}{\sigma_\epsilon \cdot k \cdot t} \right)^2} .$$

From (4.3.12), we have several observations:

- When the correlation coefficient $\rho = 0$, $x_t^* = E[x_t]$. This means that the prediction of x_t given P_t is the same as that only given P_0, and x_t and ε_t are acting independently (property of the bivariate normal distribution).
- When $\rho > 0$, x_t^* is an increasing function of P_t. This means the best prediction of x_t becomes larger as P_t becomes larger, since x_t and ε_t tend to move in the same direction.
- When $\rho < 0$, x_t^* is an decreasing function of P_t. This means the best prediction of x_t becomes smaller as P_t becomes larger.

- When t goes to infinity, x_t^* goes to $\lambda - v_\varepsilon \cdot \rho \cdot \sigma_x / (\sigma_\varepsilon \cdot k)$, which is a constant (not a function of P), and it is smaller than the reversion value m, which is the result of both the mean reversion tendency of x_t and the interplay between x_t and ε_t . Figure 4.3.1 presents a graphic illustration the value $x_t^* = E_P[x_t]$ as a function of t and P_t.

To obtain x_t^*, we need to know the latest accurate value of x_t and ε_t , and set the time at that point be t = 0. Then we have x_0, ε_0. Then using (4.3.12), we can obtain the value of $x_t^* = x_t^*(P_t)$ – as a function of the given price P_t at each time t > 0.

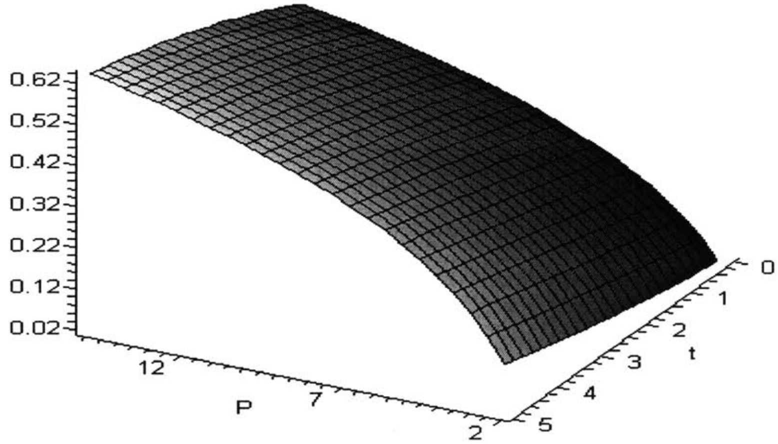

Figure 4.3.1: Expected x_t conditioned on the price P.

Next, substitute (4.3.12) into (4.3.7):

$$P_t^* \cdot \left(-k \cdot \left(x_t^* - \lambda \right) + v_\epsilon + \frac{1}{2}\sigma_x^2 + \frac{1}{2}\sigma_\epsilon^2 + \rho \cdot \sigma_x \cdot \sigma_\epsilon \right) = r \cdot P_t^* + c\,.$$

Comparing to the mean reversion model where $x_t^* = x^*$ does not change as time changes, we find in the two-factor model, x_t^* changes as time changes. However, for each fixed time t, the above equation is in the same form of as (2.3.9b) of the mean reversion stochastic model. The similarities lead to two optimal trigger prices $P_{1,t}^*$ and $P_{2,t}^*$, which act the same role as those derived in Chapter 2, under the condition that the above equation is solvable. The difference now is that instead of being constant, they are now a function of both time t, and the current price P.

I can use the $F(P, t)$ function to find the optimal operating strategy, which is defined as the risk-neutral expected rate of price appreciation minus the corresponding marginal cost:

$$F(P,t) = P \cdot \left(-k \cdot \left(x_t^* - \lambda \right) + v_\epsilon + \frac{1}{2}\sigma_x^2 + \frac{1}{2}\sigma_\epsilon^2 + \rho \cdot \sigma_x \cdot \sigma_\epsilon \right) - (r \cdot P + c)\,. \tag{4.3.13}$$

F(P, t) indicates that for the price *P* in certain ranges (depending on the current time *t*), one can generate a positive expected return by buying and holding the inventory for an extra period; while in some other time, one have to sell the inventory immediately to avoid the expected loss by holding for an extra period. To visualize *F(P, t)*, I plot it in Figure 4.3. The optimal operating prices at each time are the points where *F* intersects with the zero plane (blue plane shown in the graph).

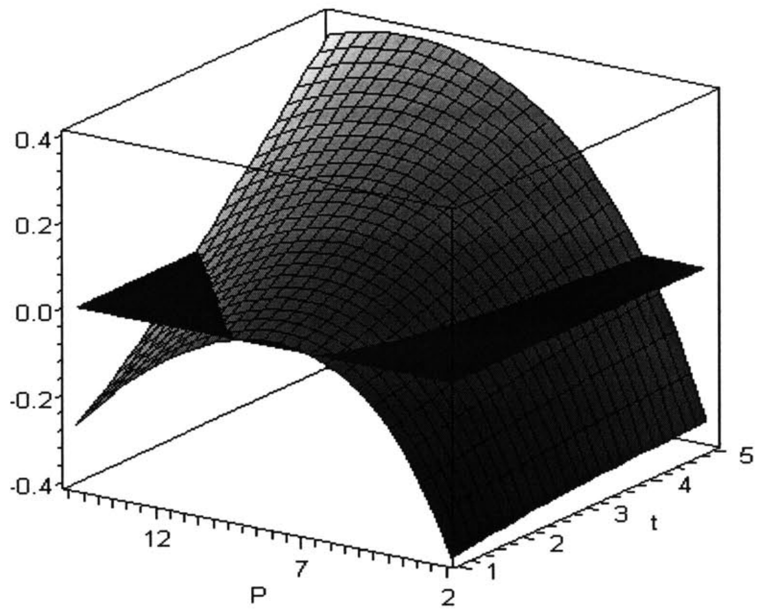

Figure 4.3.2: F as a function of *t* and *P*.

4.3.3 Valuations from the Optimal Operating Strategies

Once we find the function *F(P, t)* at each time t (or the corresponding optimal trigger price P^* obtained from either (4.3.7) or *F(P, t)* = 0), the optimal operation strategy can be derive by comparing the value of *F(P, t)* to zero: if *F(P, t) > 0*, we buy and hold; otherwise, we sell. Following the above optimal trading strategy, I use the Monte Carlo simulation to estimate the value of the storage facility for the two-factor model. The price process is simulated as follows:

$$x_{n+1} = -k\cdot(x_n - \lambda)\cdot \Delta t + z_{n+1} + x_n, \qquad (4.3.14)$$

$$\epsilon_{n+1} = v_\epsilon \cdot \Delta t + w_{n+1} + \epsilon_n,$$

$$P_{n+1} = e^{x_{n+1} + \epsilon_{n+1}}.$$

where Δt is the step size, and z_{n+1} and w_{n+1} are correlated Gaussian random variables with means zero, and the covarance matrix given by:

$$\Sigma = \begin{bmatrix} \sigma_x^2 \cdot \Delta t & \rho_{x\epsilon} \cdot \sigma_x \cdot \sigma_\epsilon \cdot \Delta t \\ \rho_{x\epsilon} \cdot \sigma_x \cdot \sigma_\epsilon \cdot \Delta t & \sigma_\epsilon^2 \cdot \Delta t \end{bmatrix}.$$

To generate z_{n+1} and w_{n+1}, I use the Cholesky decomposition. Let J be equal to the result from the Cholesky decomposition of Σ. The z_{n+1} and w_{n+1} can be found by:

$$\begin{bmatrix} z_{n+1} \\ \epsilon_{n+1} \end{bmatrix} = J' \cdot \begin{bmatrix} i_{1,n+1} \\ i_{2,n+1} \end{bmatrix}, \tag{4.3.15}$$

Where $i_{1,\,n+1}$ and $i_{2,\,n+1}$ are independent standard Gaussian random variables.

Using the above strategy, and simulating 0.5 million price paths for each facility life, I find the values of the storage facility Z^* for different facility life, which is plotted in Figure 4.3.3.

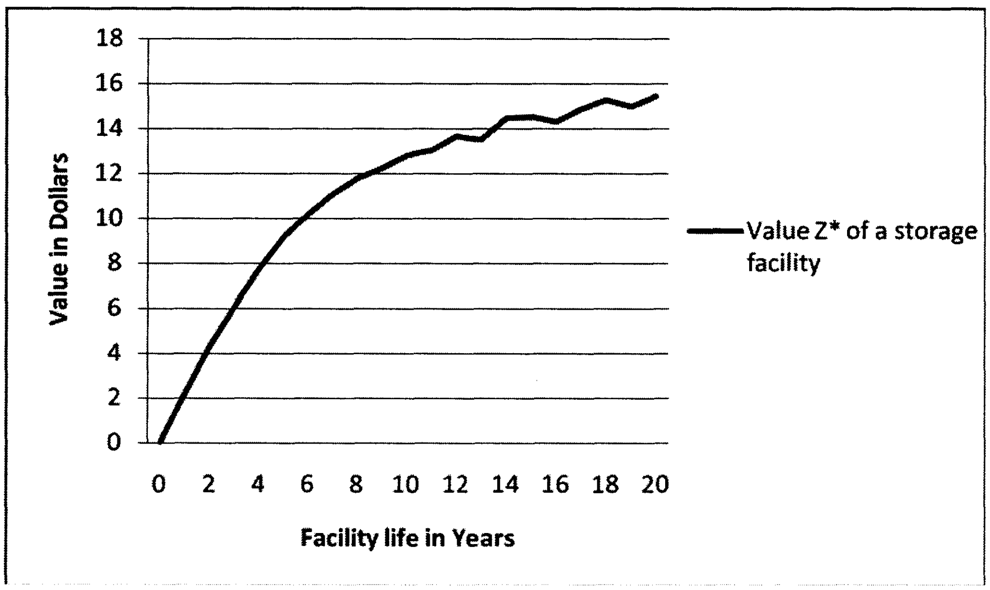

Figure 4.3.3: The value of storage facility Z^* when using different strategies.

We can see the value of the storage facility increases monotonically with the length of the facility life. The shape of this value function is similar to the plot of the value under the pure mean reversion price process, which is shown as – · – in Figure 2.7.1. The resemblance shows the important influence the mean-reversion factor x_t has on the two factor model.

In plotting Figure 4.3.3, I use the following parameters:

k	λ	v_ε	ε_0	x_0	ρ	σ_x	σ_ε	Δt	r	c

1	2.3	0.04	0	1.5	0.5	0.6	0.2	0.01	0.05	1

4.4 Valuation with Seasonality

As argued in the previous chapters, the seasonal variations of spot prices are observed prominently in the natural gas market, and the seasonal effect could increase the value of a storage facility if it is managed correctly. To account for such a seasonal effect, in this section, I modify the two-factor price model by adding an extra deterministic price component. The new price process is thus written as:

$$\ln(P_t) = x_t + \epsilon_t + f(t) \tag{4.4.1}$$

where x_t and ε_t are defined in (4.2.1b), and $f(t)$ is a periodic, once-differentiable function.

Using the dynamic programming method (similar to the derivations in Section 4.3), I can find two partial differential equations for the seasonal model (4.4.1):

$$\begin{aligned} -r\cdot V(P) + V_t + V_p\cdot P\cdot\left(f'(t) - k\cdot\left(x_t^* - \lambda\right) + v_\epsilon + \frac{1}{2}\sigma_x^2 + \frac{1}{2}\sigma_\epsilon^2 + \rho\cdot\sigma_x\cdot\sigma_\epsilon\right) \\ + \frac{1}{2}\cdot V_{pp}\cdot P^2\cdot\left(\sigma_x^2 + \sigma_\epsilon^2 + \sigma_x\cdot\sigma_\epsilon\cdot\rho\right) = c\,. \end{aligned} \tag{4.4.2}$$

$$\begin{aligned} -r\cdot W(P) + W_t + W_p\cdot P\cdot\left(f'(t) - k\cdot\left(x_t^* - \lambda\right) + v_\epsilon + \frac{1}{2}\sigma_x^2 + \frac{1}{2}\sigma_\epsilon^2 + \rho\cdot\sigma_x\cdot\sigma_\epsilon\right) \\ + \frac{1}{2}\cdot W_{pp}\cdot P^2\cdot\left(\sigma_x^2 + \sigma_\epsilon^2 + \sigma_x\cdot\sigma_\epsilon\cdot\rho\right) = 0\,. \end{aligned} \tag{4.4.3}$$

To find the optimal trigger prices, I make use of the fact that $V^*(P, t) = W^*(P, t) + P$ as well as the relations in (4.3.6) for the optimal trigger price. Subtracting (4.4.3) from (4.4.2), I get:

$$P\cdot\left(f'(t) - k\cdot\left(x_t^* - \lambda\right) + v_t + \frac{1}{2}\sigma_x^2 + \frac{1}{2}\sigma_\epsilon^2 + \rho\cdot\sigma_x\cdot\sigma_\epsilon\right) = \left(r\cdot P + c\right)\,. \tag{4.4.4}$$

In (4.4.4), $x_t^* = E_P[x_t]$, and it can be derived in a similar manner as what I have done in Section 4.3.2. By replacing ln(P) with ln(P) – $f(t)$ in (4.3.12), the result is:

$$x^* = \frac{e^{-k\,t}x_0 + \lambda\left(1 - e^{-k\,t}\right) + \dfrac{\rho\cdot\sigma_x\cdot\left(1 - e^{-k\,t}\right)}{\sigma_\epsilon\cdot k\cdot t}\left(\ln(P) - f(t) - \epsilon_0 - v_\epsilon\cdot t\right)}{\left(1 + \dfrac{\rho\cdot\sigma_x\cdot\left(1 - e^{-k\,t}\right)}{\sigma_\epsilon\cdot k\cdot t}\right)^2}\,. \tag{4.4.5}$$

Similarly, I can define function $F(P, t)$ as:

$$F(P,t) = P\cdot\left(f'(t) - k\cdot\left(x_t^* - \lambda\right) + v_t + \frac{1}{2}\sigma_x^2 + \frac{1}{2}\sigma_\epsilon^2 + \rho\cdot\sigma_x\cdot\sigma_\epsilon\right) - (r\cdot P + c)\,. \tag{4.4.6}$$

Given the functional form of *f(t),* I can plot *F(P, t)* as what I have done in Section (4.3.3), where the intersections with the zero plane represent the optimal trigger prices at each time *t*. The optimal trading strategy can also be derived by determining the sign of *F(P, t)*: when *F(P, t) >0,* we buy and hold the inventory; otherwise, we sell immediately.

4.5 Value Decomposition

In this section, I compare the values from the different components in the seasonal two-factor model. Specifically, I examine the following values:

1. The pure seasonal case: Z1 = $e^{f(t)}$;
2. Z2, the value by the model (4.2.1);
3. Z3, the value by the model (4.4.1) with $\sigma_x = 0$, $\sigma_\varepsilon = 0$;
4. Z4, the value by the model (4.4.1) with $\sigma_x = 0.6$, $\sigma_\varepsilon = 0.2$;

Z1 has been derived in Chapter 3, and Z2 has been derived in Section 4.3. The value for Z3 and Z4 can be similar determined using the strategy discussed in (4.4.6). I use Monte Carlo simulation described in Section 4.3.3 with the price path given by:

$$x_{n+1} = -k\cdot(x_n - m)\cdot\Delta t + z_{n+1} + x_n\,, \tag{4.3.14}$$

$$\epsilon_{n+1} = u_\epsilon\cdot\Delta t + w_{n+1} + \epsilon_n\,,$$

$$P_{n+1} = e^{x_{n+1} + \epsilon_{n+1} + f((n+1)\cdot\Delta t)}\,.$$

where Δt is the step size, and z_{n+1} and w_{n+1} are correlated Gaussian random variables with means zero, and the covarance matrix given by:

$$\Sigma = \begin{bmatrix} \sigma_x^2\cdot\Delta t & \rho_{x\epsilon}\cdot\sigma_x\cdot\sigma_\epsilon\cdot\Delta t \\ \rho_{x\epsilon}\cdot\sigma_x\cdot\sigma_\epsilon\cdot\Delta t & \sigma_\epsilon^2\cdot\Delta t \end{bmatrix}.$$

I use the following parameters:

k	λ	v_ε	ε_0	x_0	ρ	σ_x	σ_ε	Δt	r	c
0.5	0.2	0.1	1	0.2	0.5	0.6	0.2	0.01	0.05	1

The plot for different value is plot in Figure 4.5.1.

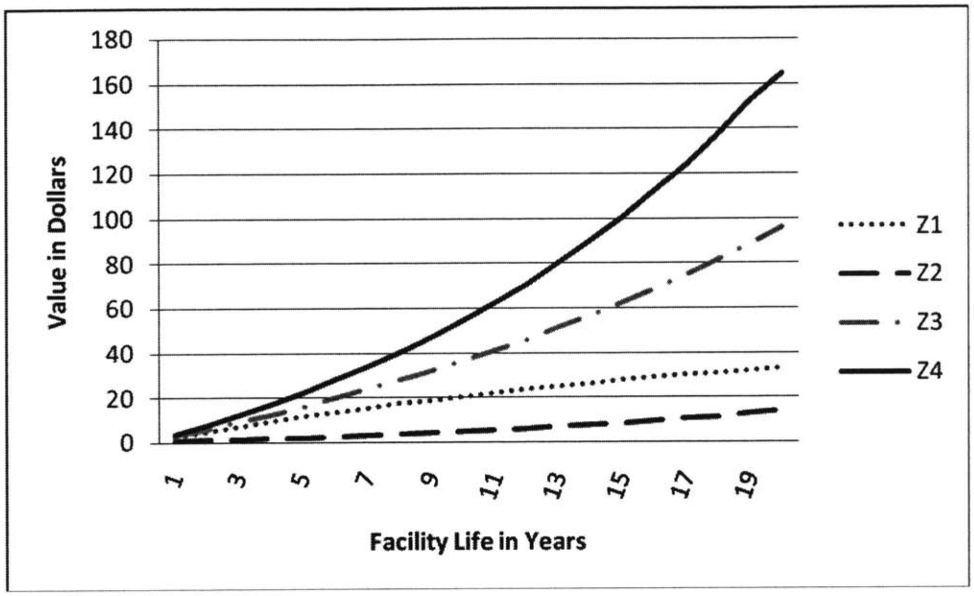

Figure 4.4.1: Comparisons of the values of a storage facility due to different processes.

From Figure 4.4.1, I have the following observations:

- Comparing Z2 to Z1, Z3, and Z4, we can see that without considering the seasonal factor, there are substantial losses on the value of the storage facility. This observation further demonstrates the assertion that the seasonal factor should be considered in the storage valuation problem.
- Comparing Z3 and Z4, we can infer that the volatility factors do contribute to the overall storage value, and this contribution can be substantial.

The observations are similar to those for the one-factor mean reversion models, where both the seasonal variations and the price uncertainties are crucial to the value of a storage facility.

CHAPTER 5

ESTIMATIONS FOR TWO-FACTOR MODELS

5.1 Chapter Introduction

Given the seasonal two-factor models discussed in Chapter 4, I have not yet known how to derive the state variables x_t and ε_t from the natural gas price data. Although both of the variables are not directly observable, the information from the prices of the short-term and long-term futures contracts can provide information on the true values of the two state variables. In this chapter, I discuss the techniques to obtain the values of x_t and ε_t from different futures contracts.

In the natural gas market, both the spot prices and the futures prices exhibit some observable seasonal property. For example, using the NYMEX Henry Hub data for May 31, April 30, March 31, 2008, I plot the futures prices maturing from 1 month to 60 months in Figure 5.1.1. It is very clear that the seasonal variations are present at a period of 12 months.

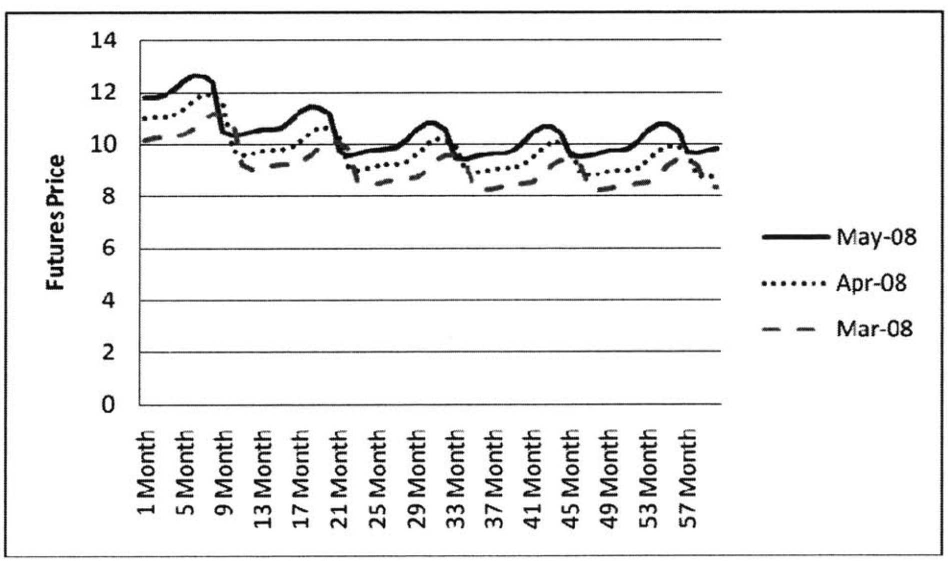

Figure 5.1.1: Futures prices with different maturities.

Therefore, it is important to estimate the model with seasonal factor included. Hence, I choose to estimate the seasonal model (4.4.1). To estimate the state variables from the model (4.4.1), which is a highly non-linear stochastic model, I use the Kalman filter from the control theory. The Kalman filter updates the estimated data continuously as new information comes. The updates are conducted in a way such that the new estimated data at time t are the best prediction (in terms of the maximum probability conditioned upon all the information up to time t). Please refer to Harvey (1989) for an introduction to the general Kalman filter techniques.

To use the Kalman filter, two steps are needed:

1. One needs to find the relationship between the estimated variables (i.e., x_t and ε_t) and the observable variables (i.e., the futures prices), and use the following form of equations to relate them:

$$\begin{cases} \mathbf{X}_t = A \cdot \mathbf{X}_{t-1} + C + w_{1,t} \\ Y_t = H \cdot \mathbf{X}_t + \mathrm{D} + w_{2,t} \end{cases}, \tag{5.1.1}$$

   where $\boldsymbol{X_t}$ is a vector to be estimated, $\boldsymbol{Y_t}$ is a vector containing the observable variables, and $w_{1,t}$, $w_{2,t}$ are independent normal vectors.

2. One needs to run the Kalman filter using an updating mechanism which feeds the filter with the observable vector $\boldsymbol{Y_t}$ to estimate $\boldsymbol{X_t}$ at each time t.

In Section 5.2, I outline the steps to find the relationship in the form of (5.1.1) between the state variables x_t, ε_t, and the observable futures prices in the market. However, in order to find the state variables x_t and ε_t at each time t, there is an obstacle to use the relationship (5.1.1) directly: in (5.1.1), we do not know the parameters A, H, d, c (they are functions of the parameters $f(t)$, k, m, u, σ_x, σ_ε and ρ). Section 5.3 is devoted to explain how to estimate these parameters.

Equipped with the above methods, in Section 5.4, I use the NYMEX Henry Hub futures data to estimate both the parameters and the state variables.

In the following sections, S_t represents the spot price at time t (I do not use P_t for reasons that will be clear later).

5.2 Setup of Kalman Filter

To use the Kalman filter, the first step one needs to take is to relate the state variables to the observable variables in the form of (5.1.1). Therefore, I derive the relationship in the section.

By the fundamental theorem of finance, in a complete financial market, the price of any asset can be expressed as the risk-neutral expectation of all the future cash flows discounted at the risk-free rate. By this theorem, and together with the assumptions that all the cash settlements of futures contracts happen at their maturity dates, I can express the futures prices as the risk-neutral expected spot prices at their respective maturity dates. Please refer to Duffie (1992) for a mathematical proof. Formally, the price of the futures contract matures at time T starting at time 0 can be written as:

$$\begin{aligned} F_{T,0} &= E[S_T] \qquad (5.2.1)\\ &= \exp\left(E[log(S_T)] + \frac{1}{2}\cdot Var[\log(S_T)]\right)\\ &= \exp\left(e^{-k\cdot t}\cdot x_0 + \epsilon_0 + A(T)\right), \end{aligned}$$

$$\text{where } A(T) = f(T) + v_\epsilon T + \left(1 - e^{-kT}\right)\lambda + \frac{1}{2}\left(\left(1 - e^{-2kT}\right)\frac{\sigma_x^2}{2k} + \sigma_\epsilon^2 T + 2\left(1 - e^{-kT}\right)\frac{\rho_{x,\epsilon}\sigma_\epsilon\sigma_x}{k}\right).$$

In (5.2.1), I use the properties (4.2.2) and (4.2.3) for the seasonal model (4.4.1) to derive the price for the futures contract maturing at time T.

The first equation in (5.1.1) can be derived by the following time series relationships between the state variables x_t and ε_t: from (4.4.1), I solve the two stochastic differential equations for state variables x_t and ε_t:

$$x_t = x_0\cdot e^{-k\cdot t} + \lambda\cdot\left(1 - e^{-k\cdot t}\right) + \int_0^t \sigma_x\cdot e^{k\cdot(s-t)}\, d\left(z_{x,s}\right),\ \epsilon_t = \epsilon_0 + v_\epsilon\cdot t + \sigma_\epsilon\cdot z_{\epsilon,t}. \qquad (5.2.2)$$

(5.2.2) is the process under the risk-neutral measure. However, for the real price process, I have:

$$x_t = x_0\cdot e^{-k\cdot t} + \int_0^t \sigma_x\cdot e^{k\cdot(s-t)}\, d\left(z_{real\,x,s}\right),\ \epsilon_t = \epsilon_0 + v_{real}\cdot t + \sigma_\epsilon\cdot z_{real\,\epsilon,t}. \qquad (5.2.3)$$

Using the real price process, and letting the time interval to be Δt, I can discretize (5.2.3) and write the state variables in the following matrix form:

$$\mathbf{X}_t = A \cdot \mathbf{X}_{t-1} + C + w_{1,t}, \tag{5.2.4}$$

where $\mathbf{X}_t = [x_t, \epsilon_t]$, and $A = \begin{bmatrix} e^{-k\cdot\Delta t} & 0 \\ 0 & 1 \end{bmatrix}$, $C = \begin{bmatrix} 0 \\ v_{real}\cdot\Delta t \end{bmatrix}$.

$w_{1,t}$ is a bivariate normally distributed vector with mean of zero, and:

$$var(w_{1,t}) = W_1 = cov(x_{\Delta t}, \epsilon_{\Delta t}) \tag{5.2.5}$$

$$= \begin{bmatrix} \left(1 - e^{-2k\cdot\Delta t}\right)\cdot\dfrac{\sigma_x^2}{2\cdot k} & \left(1 - e^{-k\cdot\Delta t}\right)\cdot\dfrac{\rho_{x,\epsilon}\sigma_\epsilon\sigma_x}{k} \\ \left(1 - e^{-k\cdot\Delta t}\right)\cdot\dfrac{\rho_{x,\epsilon}\sigma_\epsilon\sigma_x}{k} & \sigma_\epsilon^2\cdot\Delta t \end{bmatrix}.$$

Equation (5.2.4) is also called the **state equation** in the Kalman filter literature.

The second equation in (5.1.1) can be derived by relating the state variables and the futures prices, maturing at time *T1, T2 ...Tn*. Using (5.2.1), it can be written as:

$$Y_t = H\cdot\mathbf{X}_t + D + w_{2,t}, \tag{5.2.6}$$

where $\mathbf{Y}_t = \begin{bmatrix} \log(F_{t,T1}) \\ \log(F_{t,T2}) \\ \vdots \\ \log(F_{t,Tn}) \end{bmatrix}$, $H = \begin{bmatrix} e^{-kT1} & 1 \\ e^{-kT2} & 1 \\ \vdots & \vdots \\ e^{-kTn} & 1 \end{bmatrix}$, and $D = \begin{bmatrix} A(T1) \\ A(T2) \\ \vdots \\ A(Tn) \end{bmatrix}$.

$w_{2,t}$ is a multivariate normally distributed vector with mean of zero, and $var(w_{2,t}) = W_2$. (5.2.6) is called the measurement equation in the literature.

After finding both the state equation and the measurement equation, I can use the following updating equations at each time t to incorporate the new information given by $\boldsymbol{Y}_t$:

$$\bar{\mathbf{X}}_t = A\cdot\mathbf{X}_{t-1} + C, \tag{5.2.7}$$

$$\bar{P}_t = A\cdot P_{t-1}\cdot A^T + W_1,$$

$$K = \bar{P}_t\cdot H^T\cdot\left(H\cdot\bar{P}_t\cdot H^T + W_2\right)^{-1},$$

$$\mathbf{X}_t = \bar{\mathbf{X}}_t + K\cdot\left(Y_t - H\cdot\bar{\mathbf{X}}_t - D\right),$$

$$P_t = \bar{P}_t - K\cdot H\cdot\bar{P}_t,$$

where I define $P_{t-1} = E\left[(\mathbf{X}_{t-1} - \mathbf{X}_{t-1,real})\cdot(\mathbf{X}_{t-1} - \mathbf{X}_{t-1,real})^T\right]$.

The Kalman filter can be initialized by supplying an initial estimate of $\boldsymbol{X_0}$ and P_0. As time changes, I use (5.2.7) to estimate the value of $\boldsymbol{X_t}$ for time T > t > 0, where T is the terminal time of the estimation. (To make the estimation more accurate, generally a Kalman smoother can be used after we have reached time T. Given the last estimated value $\boldsymbol{X_T}$, the smoother will update all the previous estimated $\boldsymbol{X_t}$ when 0 < t < T, using information up to time T. However, the detailed updating equations are omitted here and interested readers can refer to Shumway & Stoffer (2006) for reference).

5.3 Parameter Estimations

To estimate the parameters A, H, D, C (they are functions of the underlying the parameters *f(t)*, *k*, *m*, *u*, σ_x, σ_ε and ρ), I use the maximum likelihood method, which is briefly discussed by Shumway & Stoffer (2006). The maximum likelihood estimation method of the parameter set $\Theta = \{f(t), k, m, u, \sigma_x, \sigma_\varepsilon, \rho\}$ follows the likelihood function:

$$\max_{\Theta} P\left(inov_t, inov_{t-1}, inov_{t-2} \ldots, inov_1 \,\middle|\, \Theta\right) \tag{5.3.1}$$

where $inov_t = Y_t - H \cdot \bar{X}_t - \mathrm{D}$, and $cov\left(inov_t\right) = H \cdot \bar{P}_t \cdot H^T + W_2$.

I want to find the Θ such that at that point, (5.3.1) is maximized. Under the assumptions of i.i.d normal variables of both $\{w_t\}$ and $\{v_t\}$, the innovation terms $\{inov_t\}$ are independent Gaussian random variables (please see Shumway & Stoffer (2006) for a proof). Therefore, I can express the logarithm version of the (5.3.1) as (I omit the constant term for simplicity):

$$\mathrm{L} = \max_{\Theta} -\frac{1}{2} \cdot \sum_{t=1}^{n \cdot \Delta t} \log\left|cov\left(inov_t\right)\right| - \frac{1}{2} \cdot \sum_{t=1}^{n \cdot \Delta t} inov_t^T \cdot \left(cov\left(inov_t\right)\right)^{-1} \cdot inov_t \tag{5.3.2}$$

Therefore, the goal is to find Θ that solves (5.3.2). This can be accomplished using the following procedures:

Step 1: Choose an initial estimation, $\Theta_{(\mathrm{old})}$.

Step 2: Use equations (5.2.7) to run the Kalman filter from time 0 to time T = $n \cdot \Delta t$ to estimate the value $\boldsymbol{X_T}$. Record all the innovation terms $\{inov_t\}$, and calculate all the covariance matrix $\{\mathrm{cov}(inov_t)\}$ associated with the innovation terms.

Step 3: Solve the optimized problem (5.3.2) for $\Theta_{(new)}$, given the inputs $\{inov_t\}$ and $\{cov(inov_t)\}$. Then use the result as a new estimate $\Theta_{(new)}$ and record the likelihood value L. Let $\Theta_{(old)} = \Theta_{(new)}$.

Step4: Repeat Step 1 to 3, until the value of L is stabilized (or it satisfies some predetermined value).

Following the above procedure, the optimal values of the parameters can be numerically determined. However, it is noted that the numerically solutions may be unstable due to Step 3, which solves the highly non-linear optimization problem. Gaussian related methods could be employed; however, more stable algorithms are needed to tackle this problem.

To reduce the number of the parameters to be estimated, I can estimate the seasonal factor *f(t)* independently from the forward curve, and I discuss one approach I adopt next.

From (5.2.1), the futures price maturing at time T can be written in the following form:

$$F_{T,0} = e^{f(T) + G(T)}, \text{ where } G(T) \text{ can be derived from (5.2.1)}. \tag{5.3.3}$$

Therefore, the seasonal factor *f(t)* is added to the futures price in a multiplicative form. For an estimate of *f(t)*, I assume that *G(t)* does not change much in any successive 12 months, *f(t)* is a periodic function with period = 1 (year), and further, for any consecutive 12 months, I have:

$$\sum_{i=0}^{11} f\left(\frac{1}{12}\cdot i + t\right) = 0, \text{ for any } t > 0. \tag{5.3.4}$$

With the above assumptions, I can remove *G(t)* from (5.3.3) by finding the average value of *G(t)*: for any non-negative integer n:

$$mean(n) \approx \frac{1}{12}\left(\sum_{i=0}^{11} \ln\left(F_{\frac{1}{12}\cdot i + n,0}\right) - \sum_{i=0}^{11} f\left(\frac{1}{12}\cdot i + n\right)\right) = \frac{1}{12}\sum_{i=0}^{11} \ln\left(F_{\frac{1}{12}\cdot i + n,0}\right), \tag{5.3.5}$$

$G(t) = mean(n)$, when t is in the interval $[n, n+1)$ for any non-negative integer n.

Then, I subtract the mean from the logarithm of the futures price. Thus, when t is in the interval $[n, n+1)$ for any non-negative integer n, I can express *f(t)* as:

$$f(t) = \ln(F_{t,0}) - mean(n). \tag{5.3.6}$$

Using (5.3.5) and (5.3.6), for each $i = 0, 1, \ldots 11$, I sum $f(t = i/12 + n)$, for all the non-negative integer n, and use the average of the summation to represent $f(t = i/12)$, for each $i = 0, 1, \ldots 11$.

5.4 An Empirical Example

In this section, I use the natural gas futures data to estimate both the parameters and the state variables x_t and ε_t using the approaches discussed in Section 5.3.

The data I use in this section are the natural gas futures prices traded on the New York Mercantile Exchange (NYMEX). The price is based on the delivery price at the Henry Hub in Louisiana, and it is denominated in \$/mmBtu (dollars per millions of British Thermal Units). It is generally considered to be the benchmark of the natural gas futures prices in the United States. Since some of the long-term futures are only available after 2001, I collect the monthly data for 1-month to 60-month maturity futures prices, from November 2001 to May 2008. Because the longer time period is preferred for more accurate statistical estimations, the short time span I use (7 years) may have some stability problems. However, for an illustration of the estimation techniques, it is sufficient.

The basic statistics for the raw futures contract from November 2001 to May 2008 is:

	1 Month	**10 Month**	**30 Month**	**60 Month**	**Overall**
Mean	6.733595	6.987519	6.477734	5.978886	6.484417
Variance	6.116126	5.636126	4.700934	3.031152	4.639233
STD	2.47308	2.374053	2.168164	1.74102	2.153888

The correlations among different maturities futures are:

	Jan	**Feb**	**Mar**	**Apr**	**May**	**Jun**	**Jul**	**Aug**	**Sep**	**Oct**	**Nov**	**Dec**
Jan	1	1	0.979	0.985	0.984	0.982	0.982	0.981	0.98	0.977	0.973	0.971
Feb		1	0.982	0.983	0.981	0.98	0.979	0.978	0.977	0.974	0.969	0.966
Mar			1	0.991	0.987	0.986	0.985	0.985	0.983	0.973	0.967	0.966
Apr				1	1	0.999	0.998	0.998	0.997	0.992	0.985	0.979
May					1	1	1	0.999	0.998	0.992	0.984	0.978
Jun						1	1	0.999	0.998	0.99	0.977	0.969
Jul							1	1	0.999	0.989	0.975	0.966
Aug								1	1	0.989	0.972	0.962
Sep									1	0.992	0.975	0.965
Oct										1	0.993	0.985
Nov											1	0.998
Dec												1

I first estimate the seasonal parameters $f(t)$. For each month, I use (5.3.5) and (5.3.6) to estimate $f(t)$ with the data of 1-month to 60-month maturity futures prices traded at that month. I do the same for all the remaining month from December 2001 to May 2008. After adjusting for the starting month of $f(t)$, I obtain the average of all the $f(t)$ with the values plotted on Figure 5.4.1.

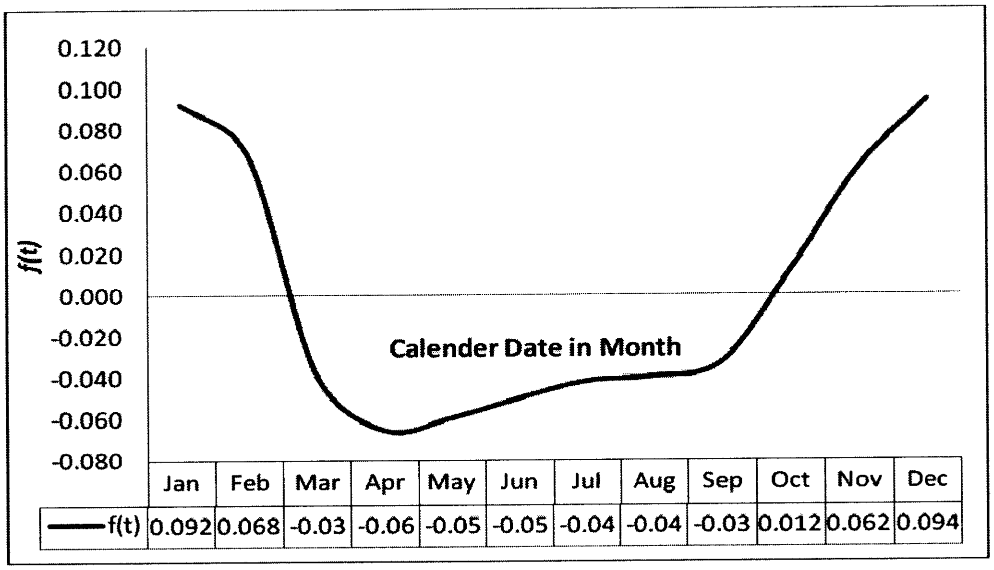

Figure 5.4.1: Estimated seasonal values from Henry Hub data.

Once I know values of the function $f(t)$ at each month, I can make seasonal adjustments to the original futures data by subtracting $f(t)$ from the logarithm of the futures prices. For example, the deseasonalized forward curve at May 2008 using the $f(t)$ obtained from the May 2008 forward curve together with the original forward curve is plotted in Figure 5.4.2, where the seasonal effect (peaks and troughs) has been reduced considerably.

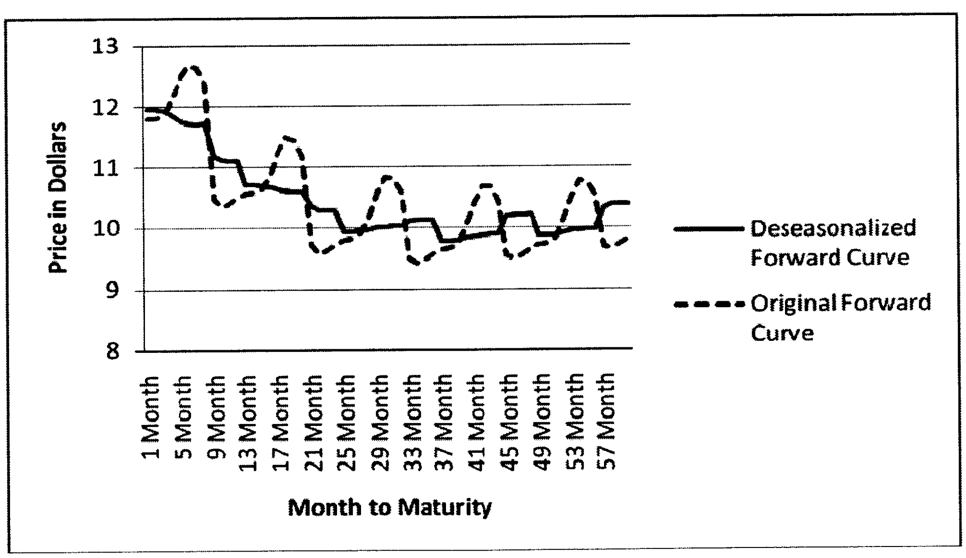

Figure 5.4.2: Deseasonalized forward curve in May 2008.

Using the above method, I can deseasonalize all the forward curves for each month from November 2001 to May 2008.

Before I continue, there is one remaining problem: how many, and what kinds of futures contracts one should choose to estimate the values of the parameters. Based on the futures trading volumes obtained from the NYMEX, only those futures contracts with large trading volumes are selected. For example, the trading volumes on August 6th, 2008 are plotted in Figure 5.4.3, from which it is clear that some of the long-term maturity contracts are not actively traded. Based on the trading volumes, I select the following 20 contracts for the parameter estimations: 1-month to 16-month futures contracts, 24-month futures contract, 36-month futures contract, 48-month futures contract, 60-month futures contract.

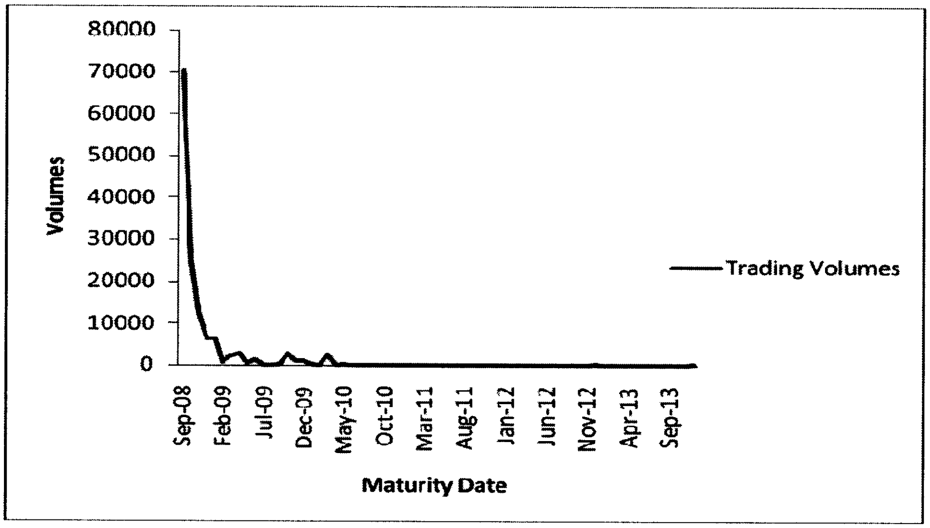

Figure 5.4.3: Futures contracts trading volumes on August 7th, 2008

With the above information, I use Step 1 to 4 outlined in Section 5.3 to estimate the remaining parameters. For simplicity, assuming for all $t \geq 0$, in (5.2.6), I have:

$$W_{2,t} = \begin{bmatrix} 0.1 & & & & \\ & 0.1 & & & \\ & & \ddots & & \\ & & & \ddots & \\ & & & & \ddots & \\ & & & & & 0.1 \end{bmatrix},$$

which is a fixed matrix. (This is for simplicity of the estimation. Ideally, one needs to estimate the matrix as parameters of the model as well).

I use the following initial values, and follow Step 1 to 4 to get the values of parameters k, m, u, σ_x, σ_ε and ρ with the corresponding state variables x_t and ε_t:

$$x_0 = 0.5, \qquad \varepsilon_0 = 0.6, \qquad P_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

(I use the Matlab function fminunc() to solve the non-linear optimization problem in Step 3)

I estimated the parameters using the maximum likelihood function (5.3.2):

k=1.215 λ=0.049 v_{real}=0.193 v_ε= −0.054 σ_x=0.123 σ_ε=0.119 ρ= 1

With the parameters, I re-run the Kalman filter from time 0 to $T = n \cdot \Delta t$ to estimate the values of the state variables x_t and ε_t (One can also use the Kalman Smoother after finding X_T, starting from T to 0, and update the estimated $\boldsymbol{X_t}$ for $0 \leq t < T$. The step of Kalman smoother is omitted here, and interested readers can refer to Shumway & Stoffer (2006) for details).

Once the parameters and the state variables have been estimated, I plot the deseasonalized spot price $\exp(x_t + \varepsilon_t)$, with the 5 years predicted price in Figure 5.4.4. From Figure 5.4.4, it is clear to see that the predicted price in the two-factor model grows almost exponentially as time passes. This is due to the exponential long term factor ε_t. However, the speed of growth is different than the pure exponential model because of the mean reversion factor x_t.

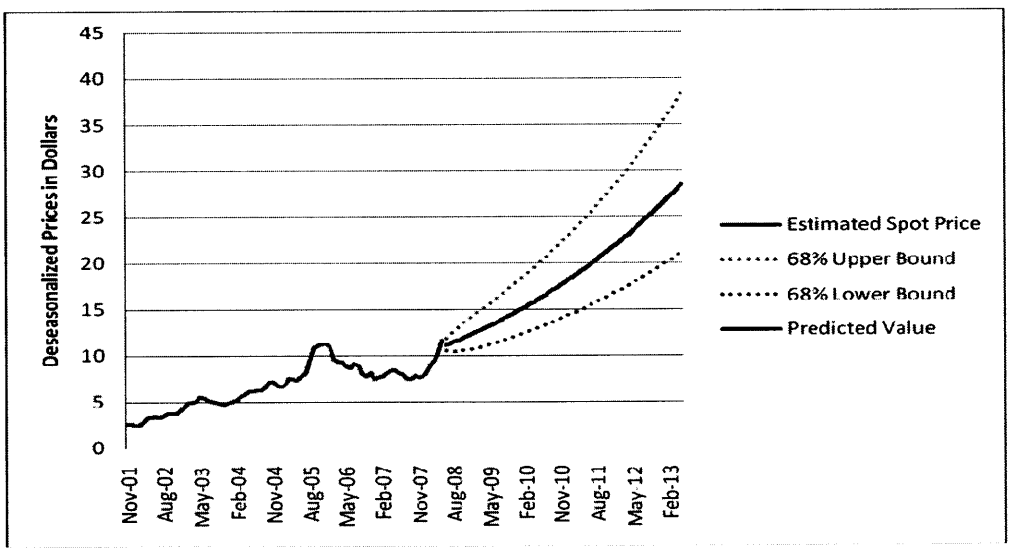

Figure 5.4.4: Estimated natural gas Price Using Kalman Filter

CHAPTER 6
CONCLUSION

6.1 Thesis Conclusion

In this thesis, I have answered the question of when to inject and when to withdraw natural gas optimally, when operating a gas storage facility. I have found the optimal operating strategy in terms of solving the problem (1.1.1) under which one gets the optimal value of a facility.

In the thesis, I have investigated the problem under both the logarithm mean reversion price process, the geometric Brownian motion price process and the two-factor short-term/long-term price process. The reason to use the logarithm mean reversion model is that it may be more suited to model the commodity spot price, and it is more suitable than the model used in Hodges (2003), because the model he uses has the problem that the spot price may go negative. Under the logarithm mean reversion model, I have provided a comprehensive investigation of the problem by considering the deterministic model, the stochastic model, and the related seasonal model. The same logic is applied to all the other sections of the thesis.

However, one-factor price models may not be adequate to model the real world price process. Therefore, I use a two-factor short-term/long-term model proposed by Schwartz & Smith (2000). This model is mathematically simple to manipulate while it preserves the important features in the natural gas price. I have used the dynamic programming method to derive the optimal operating strategy.

In the thesis, I have found a general method to tackle the optimal operating strategy problem by the risk-neutral expectation of marginal revenue and marginal cost. This method can be used even when there are transaction costs. The method can help practitioners find numerically the value of a storage facility when they face a more complex real-life problem.

For all the price processes, I have also obtained the optimal operating strategy for the seasonal models, which are more realistic for natural gas.

In the end, I have conducted the parameter estimations for the two-factor seasonal model for the completeness of the thesis.

6.2 Limitations and Future Directions

In the thesis, I have assumed that there are no transaction costs, no operating costs, no operating limitations, and the price follows a continuous stochastic process. All of the above assumptions can be violated when one faces a real world problem. Especially, all the parameters in the models can become stochastic, and the stability assumptions for the models can be relaxed. Future research can be conducted in any of the above areas. Further, although I formulate the optimal operating problem in terms of the spot price, there are values to exploit directly in the forward curve, which is another interesting problem.

APPENDIX

A. Proof of W_1 goes to 0 as P goes to 0 in Section 2.3

Assume the value of a storage facility is finite. To prove that the value W_1 of an empty facility goes to zero when the price P goes to 0, I follow a three-step procedure:

Step 1: Given the optimal trigger price $P_L^* = P_1^*$, W_1 is a function of t_L – the first hitting time from the current price P to P_L^*. It can be written as:

$$W_1(P) = E\left[\mathrm{e}^{-r\cdot t_H}\cdot\left(V\left(P_L^*\right) - P_L^*\right)\right]. \tag{A1}$$

Step 2: t_L goes to infinite as P goes to 0.

Step 3: Because V is finite, V goes to 0 as P goes to 0.

Before the formal proof, I make the following assumptions:

- The current price $P \geq 0$;
- There exists a pair of the optimal trigger prices P_L^* and P_H^* (proved in Claim 2.3.1 and Claim 2.3.2), and I assume they are positive.
- The value of a storage facility is non-negative and finite.
- Assume at time $t = 0$, with the current price $0 \leq P < P_L^* \leq P_H^* < +\infty$, we are in an empty facility status.

With the above assumptions, the following is the formal proof.

Proof of Appendix A:

Step 1:

Since the value W_1 of an empty storage facility is the net present value of all the risk-neutral expected future cash flows discounted at the risk free rate r, the first time we have such cash flows starting from $P < P_L^*$ is when the price first hits P_L^*. Immediately at that time, we have the cash outflow P_L^*, and we obtain all the future cash flows starting from P_L^* represented by $V(P_L^*)$.

Step 2:

Use the definition of P, where $\ln(P) = u$, and u follows $du = k\cdot(m-u)\cdot dt + \sigma\cdot dB$. To prove $t_L \to \infty$ as lim P $\to$ 0 is equivalently to prove $t_L \to \infty$ when lim u $\to -\infty$, and t_L is the same as the first hitting time going from $u = \ln(P)$ to $u^* = \ln(P_L^*)$. I use the following Lemma 1 to prove this assertion.

Lemma 1: For a OU process with the form $du = k\cdot(m-u)\cdot dt + \sigma\cdot dB$, starting from an initial value $u = |x| = \infty$, the expected first hitting time to a finite value u^* is infinite with probability 1.

Proof of Lemma 1:

For simplicity, let $m = 0$. The OU process is $du = -k\cdot u\cdot dt + \sigma\cdot dB$, with the first hitting time to some value 'a' defined as $T_a = inf\{t: u_t = a\}$ starting from some initial value $|x|$.

In the following proof, let $u_t = a = 0$, and denote the corresponding expected first hitting time as t, with the initial value $|x|$.

According to Göing-Jaeschke and Yor (2003), Alili and Pedersen (2005), the probability density function of the first hitting time t, starting from $|x|$ to $u_t = 0$ is:

$$f_x = \sqrt{\frac{2}{\pi}}\cdot\frac{|x|\cdot e^{-t}}{\left(1-e^{-2\cdot t}\right)^{1.5}}\cdot\exp\left(-\frac{x^2\cdot e^{-2t}}{2\cdot\left(1-e^{-2t}\right)}\right) \tag{L1}$$

From (L1), I want to calculate the probability of the first hitting time t when t is in the range of $[1/|x|, t_0]$. First assumed $1/|x| \le t_0$, the probability equation can be expressed as:

$$P\left(\frac{1}{|x|}\le t\le t_0\right)=\int_{\frac{1}{|x|}}^{t_0}\sqrt{\frac{2}{\pi}}\cdot\frac{|x|\cdot e^{-t}}{\left(1-e^{-2\cdot t}\right)^{1.5}}\cdot\exp\left(-\frac{x^2\cdot e^{-2t}}{2\cdot\left(1-e^{-2t}\right)}\right)dt \tag{L2}$$

Next I prove the equation (L2) approach to 0 for any finite t_0 as x goes to infinity. When $t \ge 0$, I have increasing functions (since e^{-2t} is increasing):

$$y = \frac{1}{1 - e^{-2t}}, \quad y = \frac{e^{-2t}}{1 - e^{-2t}}$$

Hence, (L2) can be bounded by:

$$\le \sqrt{\frac{2}{\pi}} \cdot \frac{|x| \cdot e^{-\frac{1}{|x|}}}{\left(1 - e^{-\frac{2}{|x|}}\right)^{1.5}} \cdot \exp\left(- \frac{x^2 \cdot e^{-2t_0}}{2 \cdot \left(1 - e^{-2t_0}\right)} \right) \cdot \left(t_0 - \frac{1}{|x|} \right)$$

$$\le \sqrt{\frac{2}{\pi}} \cdot \frac{|x| \cdot 1}{\left(1 - e^{-\frac{2}{|x|}}\right)^{1.5}} \cdot \exp\left(- \frac{x^2 \cdot e^{-2t_0}}{2 \cdot \left(1 - e^{-2t_0}\right)} \right) \cdot \left(t_0 - \frac{1}{|x|} \right)$$

To continue, I make the following claim, when $|x| > 20$, we have

$$\frac{1}{\left(1 - e^{-\frac{2}{|x|}}\right)} < \frac{|x|^2}{4} \qquad \text{(L3)}$$

$$\frac{1}{\left(1 - e^{-\frac{2}{|x|}}\right)^{1.5}} < \frac{|x|^3}{8}$$

This can be derived using the fact that:

$$h(a) = a^2 + e^{-a} \textbf{ and } h'(a) = 2a - e^{-a} \le 0 \textit{ when } 0 \le a < 0.1 \qquad \text{(L4)}$$

Therefore, $h(a)$ is a non-increasing function for $0 \le a < 0.1$, hence in this range:

$$h(a) \le h(0) = 1 \qquad \text{(L5)}$$

$$\Rightarrow a^2 + e^{-a} \le 1 \Rightarrow \frac{1}{1 - e^{-a}} < \frac{1}{a^2}$$

Let $a = 1/|x|$, I have, when $|x| > 20$,

$$\frac{1}{\left(1 - e^{-\frac{2}{|x|}}\right)} < \frac{|x|^2}{4} \qquad \text{(L6)}$$

Thus I have (L3). Using (L3), I can further bound (L2) as:

$$\le \sqrt{\frac{2}{\pi}} \cdot \frac{|x| \cdot |x|^3}{8} \cdot \exp\left(- \frac{x^2 \cdot e^{-2t_0}}{2 \cdot \left(1 - e^{-2t_0}\right)} \right) \cdot \left(t_0 - \frac{1}{|x|} \right) \qquad \text{(L7)}$$

$$\le \sqrt{\frac{2}{\pi}} \cdot \frac{|x|^4}{8} \cdot \exp\left(- \frac{x^2 \cdot e^{-2t_0}}{2 \cdot \left(1 - e^{-2t_0}\right)} \right) \cdot \left(t_0 - \frac{1}{|x|} \right)$$

Now let $|x| \to \infty$, I have:

$$\frac{|x|^4}{8}\cdot\exp\left(-\frac{x^2\cdot e^{-2t_0}}{2\cdot\left(1-e^{-2t_0}\right)}\right)\rightarrow 0 \text{ and } \left(t_0-\frac{1}{|x|}\right)\rightarrow t_0$$

Hence, for any finite t_0, I have:

$$\sqrt{\frac{2}{\pi}}\cdot\frac{|x|^4}{8}\cdot\exp\left(-\frac{x^2\cdot e^{-2t_0}}{2\cdot\left(1-e^{-2t_0}\right)}\right)\cdot\left(t_0-\frac{1}{|x|}\right)\rightarrow 0$$

This leads to:

$$0\le\lim_{|x|\rightarrow\infty}P\left(\frac{1}{|x|}\le t\le t_0\right)\le 0$$

$$\Rightarrow P\left(0<t\le t_0\right)=0$$

=> The probability of finite hitting time to zero is 0 if the starting point $|x|\rightarrow\infty$.

=> The probability of infinite hitting time to zero is 1 if the starting point $|x|\rightarrow\infty$.

In addition, the first hitting time from 0 to any finite value is finite with positive probability (which can be derived using (L1)). Therefore, the expected first hitting time from $|x|\rightarrow\infty$, to any finite value is infinite with probability 1. End of Lemma 1 □

Step 3:

Using (A1), and use the result from Step 2, I have:

$$\lim_{P\rightarrow 0}W_1(P)=\lim_{x\rightarrow-\infty}e^{-r\cdot t_L}\cdot\left(V\left(P_L^*\right)-P_L^*\right)\cdot Prob\left(t_L \text{ is finite}\right)$$
$$+\lim_{x\rightarrow-\infty}e^{-r\cdot t_L}\cdot\left(V\left(P_L^*\right)-P_L^*\right)\cdot Prob\left(t_L \text{ is infinite}\right)=0.$$

End of Proof. □

B. Proof of W_2 goes to 0 as P goes to positive infinity in Section 2.3

Given the proof in Appendix A, I can prove this claim by a similar infinite hitting time argument.

First, the value of an empty facility can be expressed as:

$$W_2(P) = E\left[\mathrm{e}^{-r \cdot t_H} \cdot \left(V\left(P_H^*\right) - P_H^*\right)\right], \tag{B.1}$$

where P_H^* is the optimal upper trigger price, and t_H is the first hitting time to the optimal upper trigger price starting from current price $P > P_H^*$. The expectation is taken under the risk-neutral measure. This equation can be argued in the same way as the proof of Step 1 in Appendix A.

Under the assumptions in Appendix A, I can use Lemma 1 in Appendix A to show that t_H goes to infinity as P goes to positive infinity. Therefore, we have:

$$\lim_{P \to \infty} W(P) = \lim_{x \to +\infty} E\left[\mathrm{e}^{-r \cdot t_H} \cdot \left(V\left(P_H^*\right) - P_H^*\right)\right]. \tag{B.2}$$

Therefore, we finish the proof. □

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