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Investment Constraints**

by

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# Oil Futures Prices in a Production Economy with Investment Constraints

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## ABSTRACT

We document a new stylized fact regarding the term-structure of futures volatility. We show that the relationship between the volatility of futures prices and the slope of the term structure of prices is non-monotone and has a “V-shape”. This aspect of the data cannot be generated by basic models that emphasize storage while this fact is consistent with models that emphasize investment constraints or, more generally, time-varying supply-elasticity. We develop an equilibrium model in which futures prices are determined endogenously in a production economy in which investment is both irreversible and is capacity constrained. Investment constraints affect firms’ investment decisions, which in turn determine the dynamic properties of their output and consequently imply that the supply-elasticity of

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the commodity changes over time. Since demand shocks must be absorbed either by changes in prices, or by changes in supply, time-varying supply-elasticity results in time-varying volatility of futures prices. Estimating this model, we show it is quantitatively consistent with the aforementioned “V-shape” relationship between the volatility of futures prices and the slope of the term-structure.

In recent years commodity markets have experienced dramatic growth in trading volume, the variety of contracts, and the range of underlying commodities. There also has been a great demand for derivative instruments utilizing operational contingencies embedded in delivery contracts. For all these reasons there is a widespread interest in models for pricing and hedging commodity-linked contingent claims. Besides practical interest, commodities offer a rich variety of empirical properties, which make them strikingly different from stocks, bonds and other conventional financial assets. Notable properties of futures include, among others: (i) Commodity futures prices are often “backwardated” in that they decline with time-to-delivery (Litzenberger and Rabinowitz (1995)), (ii) Spot and futures prices are mean-reverting for many commodities, (iii) Commodity prices are strongly heteroscedastic (Duffie and Gray (1995)) and price volatility is correlated with the degree of backwardation (Ng and Pirrong (1994) and Litzenberger and Rabinowitz (1995)), and (iv) Unlike financial assets, many commodities have pronounced seasonalities in both price levels and volatilities.

The theory of storage of Kaldor (1939), Working (1948, 1949) and Telser (1958) has been the foundation of the theoretical explorations of futures/forward prices and convenience yields (value of the immediate ownership of the physical commodity). Based on this theory researchers have adopted two approaches to modelling commodity prices. The first approach is mainly statistical in nature and requires an exogenous specification of the ‘convenience yield’ process for a commodity (e.g., Brennan and Schwartz (1985), Brennan (1991), and Schwartz (1997)). The second strand of the literature derives the price processes endogenously in an equilibrium valuation framework with competitive storage (e.g., Williams and Wright (1991), Deaton and Laroque (1992, 1996), Routledge, Seppi, and Spatt (2000)). The appealing aspect of this approach is its ability to link the futures prices to the level of

inventories and hence derive additional testable restrictions on the price processes.

From a theoretical perspective the models based on competitive storage ignore the production side of the economy, and consequently they suffer from an important limitation. Inventory dynamics have little if any impact on the long-run properties of commodity prices, which in such models are driven mostly by the exogenously specified demand process. In particular, prices in such models tend to mean revert too fast relative to what is observed in the data (see Routledge et. al. (2000)), and more importantly these models cannot address the rich dynamics of the term-structure of return volatility.

In this paper we document an important new stylized fact regarding the property of the term structure of volatility of futures prices. We demonstrate that the relation between the volatility of futures prices and the slope of the forward curve (the basis) is non-monotone and convex, i.e., it has a “V-shape”. Specifically, conditional on a negatively sloped term structure, the relation between the volatility of futures prices and the slope of the forward curve is negative. On the other hand, conditional on a positively sloped term structure, the relation between the volatility and the basis is positive. This aspect of the data cannot be generated by basic models that emphasize storage, since such models imply a monotone relation between futures price volatility and the slope of the forward curve (see Routledge et. al. (2000)).

In light of the aforementioned stylized fact, we explore an alternative model characterizing the mechanism of futures price formation. Futures prices are determined endogenously in an equilibrium production economy featuring constraints on investment – irreversibility and a maximum investment rate. These investment constraints lead to investment triggers that affect firms’ investment decisions, which in turn determine the dynamic properties of their

output. More specifically, if the capital stock is much higher than its optimal level, given the current level of demand, firms find it optimal to postpone investment and the irreversibility constraint binds. On the other hand, when the capital stock is much lower than the optimal level, firms invest at the maximum possible rate and the investment rate constraint binds. In either case, the supply of the commodity is relatively inelastic and futures prices are relatively volatile. Since futures prices of longer-maturity contracts are less sensitive to the current value of the capital stock than the spot price, the slope of the forward curve tends to be large in absolute value when the capital stock is far away from its long-run average value. Thus, the absolute value of the slope of the term structure of futures prices is large exactly when the investment constraints are binding. Hence, the model predicts that the volatility of futures prices should exhibit a “V-shape” as a function of the slope of the term structure of futures prices. Stated differently, because of the binding constraints on investment, supply-elasticity of the commodity changes over time. Since demand shocks must be absorbed either by changes in prices, or by changes in supply, time-varying supply-elasticity results in time-varying volatility of futures prices. In our calibration below we show that the model can also generate these patterns in a manner that is quantitatively similar to the data.

There exists very little theoretical work investigating the pricing of futures on commodities using production economy framework. Grenadier (2002) and Novy-Marx (2005) also consider futures prices in a production economy, and discuss how the proximity of the state variable to the investment threshold governs the slope of the forward price curve. Both these papers, which include investment irreversibility, do not include an investment rate bound and thus cannot generate the price volatility predictions in backwardation. Casassus, Collin-Dufresne and Routledge (2004) also analyze spot and futures oil prices in a general

equilibrium production economy but with fixed investment costs and two goods. While also a production economy, the structure and implications of their model are quite different. A recent paper by Carlson, Khoker and Titman (2006) also considers an equilibrium model with production. While we assume that oil reserves are infinite, their model emphasizes exhaustibility of oil reserves. Their model also gives rise to the non-monotone relation between the futures price volatility and the slope of the forward curve. This implication is driven, as in our model, by adjustment costs in the production technology. This provides further evidence of the theoretical robustness of our finding – the exact structure of the model is not particularly important, as long as adjustments of production levels are limited in both directions.

The rest of the paper is organized as follows. In Section 2 we describe our data set and document empirical properties of future prices. Section 3 develops the theoretical model. In Section 4, we study quantitative implications of the model. Section 5 provides conclusions.

## I. Empirical Analysis

We concentrate our empirical study on crude oil. Kogan, Livdan and Yaron (2005) display qualitatively similar findings for heating oil and unleaded gasoline. We choose however to focus on the crude oil contract for several reasons (i) it represents the most basic form of oil where the investment constraints we highlight seem to be the most relevant, (ii) the other contracts clearly use crude oil as an input and thus their analysis may require a more specific ‘downstream’ industry specification. Our data consists of daily futures prices for NYMEX light sweet crude oil contract (CL) for the period from 1982 to 2000. Following previous work by Routledge et. al. (2000), the data is sorted by contract horizon with the ‘one-month’



contract being the contract with the earliest delivery date, the ‘two-month’ contract having next earliest delivery date, etc.<sup>1</sup>. We consider contracts up to 12 months to delivery since liquidity and data availability is good for these horizons.<sup>2</sup> Since we are using daily data, our dataset is sufficiently large and it ranges from 2500 to 3500 data points across different maturities.

Instead of directly using futures prices,  $P(t, T)$ , we use daily percent changes,  $R(t, T) = \frac{P(t, T)}{P(t-1, T)}$ . Percent price changes are not susceptible as much as price levels to seasonalities and trends, and therefore their volatility is more suitable for empirical analysis. We then proceed by constructing the term structure of the unconditional and conditional volatilities of daily percent changes on futures prices. In calculating conditional moments, we condition observations on whether the forward curve was in backwardation or in contango at the end of the previous trading day (based on the third shortest and sixth shortest maturity prices at that time). Figure 1 shows the conditional and unconditional daily volatilities for futures price percent changes. Unconditionally, the volatility of futures price changes declines with maturity, consistent with the Samuelson (1965) hypothesis. The behavior of crude oil (CL) contracts was previously studied by Routledge et. al. (2000). We find, as they did, that the volatility of futures prices is higher when the forward curve is in backwardation. This has been interpreted as evidence in favor of the standard storage theories, emphasizing the effect of inventory stock-outs on price volatility.

[Insert Figure 1 about here]

Next, we study the patterns in volatility of futures prices in more detail. Specifically, we estimate a functional relation between the futures price volatility and the one-day lagged

slope of the forward curve. Following the definition of conditional sample moments, the time series of the slope of the forward curve,  $SL(t)$ , is constructed as a logarithm of the ratio of the futures price of the sixth shortest maturity in months available on any day  $t$ ,  $P(t, 6)$ , to the future price of the third shortest maturity,  $P(t, 3)$ , available on the same day

$$SL(t) \equiv \ln \left[ \frac{P(t-1, 6)}{P(t-1, 3)} \right]. \quad (1)$$

We use demeaned slope

$$\widetilde{SL}(t) \equiv SL(t) - E[SL(t)], \quad (2)$$

in our analysis. We start by using demeaned lagged slope as the only explanatory variable for realized volatility

$$|R(t, T)| = \alpha_T + \beta_T \widetilde{SL}(t-1) + \varepsilon_T(t). \quad (3)$$

Since we are now estimating a different functional form, note that the relation (3) can potentially yield different information than that contained in Figure 1 which was obtained by simply splitting the sample based on the slope of the forward curve. The term structure of  $\beta_T$  as well as the corresponding  $t$ -statistics are shown in Figure 2. We also report these results in Table I for  $T$  equal to 1, 5, and 10 months. The negative sign of  $\beta_T$  for all times to maturity is a notable feature of these regressions. This result seems to be at odds with the relations shown in Figure 1, where volatility conditional on backwardation is for the most part higher than the unconditional volatility.

[Insert Figure 2 about here]

The apparent inconsistency becomes less puzzling in light of the intuition of the model we present below. In particular, our theoretical results motivate one to look for a non-monotone

relation between the volatility of future prices and the slope of their term structure. For that we decompose the lagged demeaned slope into positive and negative parts and use them as separate explanatory variables (i.e., use a piece-wise linear regression on the demeaned slope of the term structure),

$$|R(t, T)| = \alpha_T + \beta_{1,T} \left( \widetilde{SL}(t-1) \right)^+ + \beta_{2,T} \left( \widetilde{SL}(t-1) \right)^- + \varepsilon_T(t), \quad (4)$$

where  $(X)^\pm$  denotes the positive (negative) part of  $X$  respectively. Figure 2 as well as Table I illustrate our results. Both  $\beta_{1,T}$  and  $\beta_{2,T}$  are statistically and economically significant for most maturities. More importantly,  $\beta_{1,T}$  and  $\beta_{2,T}$  differ in sign:  $\beta_{1,T}$  are positive and  $\beta_{2,T}$  are negative. Therefore, the relation between the volatility of futures prices and the slope of the term structure of prices is non-monotone and has a “V-shape” : conditional volatility declines as a function of the slope when the latter is negative, and increases when the latter is positive.<sup>3</sup>

[Insert Table I about here]

One potential concern is that the piecewise linear regression may artificially lead our estimates to highlight the “V-shape” pattern we report. To allow for a more flexible form for the relationship between volatility and the slope we use a nonparametric regression (our specific implementation is based on Atkeson, Moore and Schaal (1997)). Figure 3 shows the results of the nonparametric regression for horizon  $T$  equal to 2, 4, 6, and 8 months. For all maturities it reveals a clear non-monotone “V-shape” relationship between the volatility of futures prices and the slope of the term structure of prices.

[Insert Figure 3 about here]

We perform several additional robustness checks. First we estimate conditional variances instead of conditional volatility by using the square of daily price changes instead of their absolute value. We find that the conditional variance leads to very similar conclusions. In most cases, both  $\beta_{1,T}$  and  $\beta_{2,T}$  remain statistically significant. Next, we fit a GARCH(1,1) model to the daily returns percentage price changes of maturity  $T$  to obtain the fitted volatility time series  $\sigma_{GARCH}(t, T)$  and use it as a regressand in (4). The results are reported in Panel 1 of Table II and show that both  $\beta_{1,T}$  and  $\beta_{2,T}$  remain statistically significant. We use the GARCH(1,1) to construct the predicted time series of volatility,  $\hat{\sigma}_{GARCH}(t, T)$ . It is obtained by fitting GARCH(1,1) model to the time series of daily returns percentage price changes of maturity  $T$  up to day  $t - 1$  and then using it to predict the volatility on day  $t$ ,  $\hat{\sigma}_{GARCH}(t, T)$ . The initial window is set to 300 days. We then use it in the following regression,

$$\hat{\sigma}_{GARCH}(t, T) = \alpha_T + \beta_{G,T} \hat{\sigma}_{GARCH}(t-1, T) + \beta_{1,T} \left( \widetilde{SL}(t-1) \right)^+ + \beta_{2,T} \left( \widetilde{SL}(t-1) \right)^- + \varepsilon_T(t). \quad (5)$$

The results are reported in Panel 2 of Table II and show that both  $\beta_{1,T}$  and  $\beta_{2,T}$  still remain statistically significant.

[Insert Table II about here]

As a final robustness check we split our sample into pre- and post-Gulf war sub-samples. We perform the same analysis as in the case of the full sample on the post-Gulf war sub-sample. We find the same “V-shape” in the relationship between the volatility of futures prices and the slope of the term structure of prices.

## I. Model

In this section we present our model for spot prices and derive futures prices.

### A. Setup

We consider a continuous-time infinite-horizon economy. We focus on a competitive industry populated by a large number of identical firms using an identical production technology. Firms produce a non-storable consumption good by means of a production function that exhibits constant returns to scale

$$Q_t = XK_t, \tag{6}$$

where  $K_t$  is capital and  $X$  is the productivity of capital which is assumed to be constant. Without loss of generality, we will assume below that  $X = 1$ . Our results can be easily adjusted to accommodate the case when  $X$  is a stochastic process. We also abstract away from production costs.

Firms can adjust their capital stock according to

$$dK_t = (I_t - \delta K_t)dt, \tag{7}$$

where  $I_t$  is the investment rate and  $\delta$  is the capital depreciation assumed to be a nonnegative constant. We assume the unit cost of capital is equal to one.

We assume that investment is irreversible, i.e.,  $I_t \geq 0$ , and the rate of investment is bounded. Specifically,

$$I_t \in [0, \bar{i}K_t^A], \tag{8}$$

where  $K_t^A$  is the aggregate capital stock in the industry. This constraint implies that the higher the aggregate capital stock in the industry (or the higher the aggregate output rate),

the better the investment opportunities faced by an individual firm. One can think of this as a learning-by-doing technology. These investment frictions give rise to nontrivial dynamic properties of futures prices. It is worth noting that one could derive the same functional form of price dynamics by assuming that investment opportunities depend on the firms' own capital stock. However, the above assumption significantly simplifies formal analysis of the model and leads to fewer restrictions on model parameters.

We do not explicitly model entry and exit in equilibrium. Our assumption of nonnegative investment rates effectively implies that there is no exit. We are assuming that all investment is done by existing firms, so there is no entry either. One could equivalently allow for entry into the industry, as long as the total amount of investment by old and new firms satisfies the constraint (8).<sup>4</sup>

Firms sell their output in the spot market at price  $S_t$ . We assume that financial markets are complete and the firms' objective is to maximize their market value, which in turn is given by

$$V_0 = E_0 \left[ \int_0^\infty e^{-rt} (S_t Q_t - I_t) dt \right]. \quad (9)$$

We assume that the expected value is computed under the risk-neutral measure and the risk-free rate  $r$  is constant.

The consumers in the economy are represented by the demand curve

$$Q_t = Y_t S_t^{-\frac{1}{\gamma}}, \quad Q_t \in (0, \infty), \quad (10)$$

where unexpected changes in  $Y_t$  represent demand shocks. We assume that under the risk-neutral measure  $Y_t$  follows a geometric Brownian motion process

$$\frac{dY_t}{Y_t} = \mu_Y dt + \sigma_Y dW_t. \quad (11)$$

We also assume that  $\gamma > 1$ . Results for the case of  $\gamma \leq 1$  are analogous.

### *B. Equilibrium Investment and Prices*

We adopt a standard definition of competitive equilibrium. Firms must choose an investment policy that maximizes their market value (9), taking the spot price of output and the dynamics of the aggregate capital stock in the industry as exogenous. The spot market must clear, i.e., the aggregate output and the spot price must be related by (10). Finally, the dynamics of the aggregate capital stock in the industry is given by

$$dK_t^A = (I_t^A - \delta K_t^A) dt, \quad (12)$$

where  $I_t^A$  is the aggregate investment rate.

We guess what the equilibrium investment policy and associated price processes should be, and verify formally that firms' optimality conditions are satisfied and markets clear. The details of the solution are provided in the Appendix A.

Intuitively, firms would invest only when the net present value of profits generated by an additional unit of capital is positive. As it turns out, the spot price follows a univariate Markov process in equilibrium, thus firms invest at the maximum possible rate when the spot price is above a certain threshold, and do not invest otherwise. Formally, we prove the following:

PROPOSITION 1: *A competitive equilibrium exists and the equilibrium investment policy is given by*

$$I_t^* = \begin{cases} \bar{i}K_t^A, & S_t \geq S^*, \\ 0, & S_t < S^*, \end{cases} \quad (13)$$

*The investment threshold  $S^*$  is defined in the Appendix A.*

See Appendix A for the proof ■

To make sure that the equilibrium exists and firm value is finite, we need to impose an additional non-trivial restriction on parameter values:

$$\frac{\sigma_Y^2 \gamma^2}{2} - \gamma \mu^+ - (r + \delta) < 0, \quad (14)$$

where we define  $\mu^- = \delta + \mu_Y - \frac{1}{2}\sigma_Y^2$  and  $\mu^+ = \bar{i} - \mu^-$ . When calibrating the model, we impose the above restriction as a weak inequality and verify that it does not bind at the calibrated parameter values.

The risk-neutral dynamics of the spot price is of very simple form:

$$\frac{dS_t}{S_t} = \left( -\gamma(\bar{i}1_{[S_t \geq S^*]} - \mu^-) + \frac{\gamma^2 \sigma_Y^2}{2} \right) dt + \gamma \sigma_Y dW_t. \quad (15)$$

where  $1_{[\cdot]}$  is an indicator function. When the spot price is above the critical value  $S^*$ , it follows a geometric Brownian motion with a drift  $\gamma \mu^- + \frac{\gamma^2 \sigma_Y^2}{2}$ . When it is below  $S^*$ , the drift changes to  $-\gamma(\bar{i} - \mu^-) + \frac{\gamma^2 \sigma_Y^2}{2}$ . As long as  $0 < \mu^- < \bar{i}$ , the spot price process has a stationary long-run distribution with the density function

$$p(S) = \frac{2\mu^-(\bar{i} - \mu^-)}{\gamma \bar{i} \sigma_Y^2 S} \left( \frac{S}{S^*} \right)^{-\frac{2}{\gamma \sigma_Y^2}(\bar{i}1_{[S \geq S^*]} - \mu^-)}. \quad (16)$$

The details of the derivation are provided in the Appendix B.

Before continuing with estimation of the model, it is worth mapping our general investment constraint model into the features of the oil industry. Oil ( $Q$ ) is the output produced using physical capital  $K$  (e.g., oil rigs, pipes, tankers). Implicitly we are assuming there is an infinite supply of underground oil, and production is constrained by the existing capital stock  $K$ . This supply of capital and consequently of oil-output leads to price



fluctuations in response to demand shocks. Futures prices (volatility) depend on anticipated future production which depends on the degree to which investment is constrained.

[Insert Figure 4 about here]

While our model is admittedly stark, it does capture many of the essential features of the investment and supply constraints in the oil industry. Figure 4 displays the relation between oil supply capacity and volatility as given in a Goldman Sachs (1999) publication. The essence of this figure is the “V-shape” relation between the volatility of the spot price and the level of inventories. This pattern is distinct from the one analyzed in our paper: while inventory levels are clearly important for short-run fluctuations in the spot market, their affect on futures prices of longer maturities is much weaker. Thus, a different mechanism must be responsible for the behavior of volatility of longer-maturity futures. However, the logic of time varying supply elasticity applies to the pattern in Figure 4 as well, wherein instead of production constraints one must recognize natural physical constraints on storage levels. Furthermore, market analysts seem to concentrate on two key features of the market: the long term and seasonal demand patterns; and the supply features of this industry. Our model clearly focuses on the second of these two issues. In particular, investments in this industry are concentrated in several key facets of production (i) basic extraction level in the form of finding new fields, constructing, installing, and maintaining rigs, and (ii) the expansion and improvements in the delivery process. Both of these types of investment take time, have capacity constraints, and constrain supply flexibility in this market – the exact channels which our model focuses on.<sup>5</sup>

### *C. Futures Prices*

The futures contract is a claim on the good which is sold on the spot market at prevailing spot price  $S_t$ . The futures price is computed as the conditional expectation of the spot price under the risk-neutral measure:

$$P(t, T) = \mathbf{E}_t[S_{t+T}], \quad \forall T \geq 0. \quad (17)$$

where  $P(t, T)$  denotes the price of a futures contract at time  $t$  with maturity date  $\tau = t + T$ . Since no analytical expression exists for the above expectation, we evaluate it numerically using a Markov chain approximation scheme. Figure 5 illustrates futures prices generated by the model.

[Insert Figure 5 about here]

## II. Estimation and Numerical Simulation

In this section we study how well our model can replicate quantitatively the key features of the behavior of futures prices reported in Section I. Since our model is formulated under the risk-neutral probability measure, while the empirical observations are made under the “physical” probability measure, one has to make an explicit assumption about the relation between these two measures, i.e., about the risk premium associated with the shock process  $dW_t$ . To keep our specification as simple as possible, we assume that the risk premium is constant, i.e., the drift of the demand shock  $Y_t$  under the “physical” probability measure is equal to  $\mu_Y + \lambda$ , where  $\lambda$  is an additional parameter of the model. Clearly, one could achieve greater flexibility and better fit of the data by allowing for a time-varying risk premium process. This, however, is entirely beyond the scope of our paper: our model has implications for the spot price dynamics, but not for the price of risk in the aggregate economy. To have

a meaningful discussion of the price of risk process, one would need a full general equilibrium model. We first estimate the model's parameters using a simulated method of moments and then proceed to analyze and discuss some additional implications of the model.

## *A. Simulated Moments Parameter Estimation*

### *A.1 Estimation Procedure*

Our goal is to estimate a vector of structural parameters,  $\theta \equiv \{\gamma, \mu_Y, \sigma_Y, \bar{i}, r, \delta, \lambda\}$ . We do this using a procedure that is similar to those proposed in Lee and Ingram (1991), Duffie and Singleton (1993), Gouriéroux and Monfort (1996), and Gouriéroux et. al. (1993). Let  $x_t$  be the vector-valued process of historical futures prices and output and consider a function of the observed sample  $F_T(x_t)$ , where  $T$  is the sample length. The statistic  $F_T(x_t)$  could represent a collection of sample moments or even a more complicated estimator, such as the slope coefficients in a regression of volatility on the term structure as in (3). Assume that as the sample size  $T$  increases,  $F_T(x_t)$  converges in probability to a limit  $M(\theta)$ , which is a function of the structural parameters. Since many of the useful population moments cannot be computed analytically, we estimate them using Monte Carlo simulation. In particular, let  $m_N(\theta) = \frac{1}{N} \sum_{n=1}^N F_T(x^n; \theta)$  represent the estimate of  $M(\theta)$  based on  $N$  independent model based statistics, where  $x^s$  represents a vector valued process of simulated futures prices and output of length  $T$  based on simulating the model at parameter values,  $\theta$ .<sup>6</sup> Let  $G_N(x, \theta) = m_N(\theta) - F_T(x_t)$ , denote the difference between the estimated theoretical mean of the statistic  $F$  and its observed (empirical) value. Under appropriate regularity conditions, it can be shown that as the size of the sample,  $T$ , and the number of simulations  $N$  increase

to infinity, the GMM estimate of  $\theta$ ,

$$\theta_N = \arg \min_{\theta} J_T = \arg \min_{\theta} G_N(x, \theta)' W_T G_N(x, \theta) \quad (18)$$

will be a consistent estimator of  $\theta$ . The matrix  $W_T$  in the above expression is positive definite and assumed to converge in probability to a deterministic positive definite matrix  $W$ . The SMM approaches in Lee and Ingram (1991) and Duffie and Singleton (1993), focus on one long simulation while Gourieroux and Monfort (1996), and Gourieroux et. al. (1993) also discuss an estimator based on multiple simulations. Our approach simulates samples of equal length to that in the data,  $T$ , and then average across  $N$  such simulations. Given our non-balanced panel data this approach allows for easier mapping from the model to the data and computation of standard errors which are based on the distribution emanating from the cross-section of the simulations.

Assume that  $V$  is the asymptotic variance-covariance matrix of  $F_T(x; \theta)$ . Then, if we use the efficient choice of the weighting matrix,  $W = V^{-1}$ , the estimator  $\theta_N$  is asymptotically normal, with mean  $\theta$  and covariance matrix  $(D'V^{-1}D)^{-1}$ , where  $D = \nabla_{\theta} M(\theta)$ .

We perform estimation in two stages. During the first stage, we use an identity matrix for the weighting matrix  $W$ . During the second stage, the weighting matrix is set equal to the inverse of the estimated covariance matrix:  $W = V_N^{-1}$ , where  $V_N$  is the sample based covariance matrix of  $F_T(x^n; \theta)$ . To compute standard errors, we use as an estimate for  $D$ ,  $D_N = \nabla_{\theta} m_N(\theta)$ .

We estimate the vector of seven model parameters,  $\theta$ , by matching both the unconditional and conditional properties of futures prices. The unconditional properties include mean and volatility of daily percent price changes for futures with maturities equal to 3, 6, and 9

months, as well as the mean, volatility, and the 30-day autoregressive coefficient of the slope of the forward curve of crude oil futures prices. In order to see how far we can push our simple single-factor model, we also fit the relation between the volatility of futures prices and the slope of the term structure. Specifically, the conditional moments include regression coefficients  $\beta_{1,T}$  and  $\beta_{2,T}$  from the equation (4) for  $T$  equal to 3, 6, and 9 months.

### *A.2 Identification*

Not all of the model parameters can be independently identified from the data we are considering. In this subsection we discuss the relations between structural parameters and observable properties of our model economy. These should suggest which of the structural parameters can be identified and what dimensions of the empirical data are likely to be most useful for estimation.

First, we calibrate the risk-free rate. The risk-free rate is determined by many factors outside of the oil industry and consequently it would not be prudent to estimate it solely based on oil-price data. Also, it is clear by inspection that the risk-free rate is not identified by our model. It does not affect any of the moments we consider in our estimation and only appears in the constraint on model parameters in equation (14). Therefore, at best, futures price data can only impose a lower bound on the level of the risk-free rate, as implied by (14). Given all of the above considerations, we set the risk free rate at 2%.<sup>7</sup> Next, consider a simple re-normalization of the structural parameters. Futures prices in our model depend solely on the risk-neutral dynamics of the log of the spot price which evolves according to

$$d \log S_t = - [\gamma \bar{i} 1_{[S \geq S^*]} - \gamma \mu^-] dt + \gamma \sigma_Y dW_t. \quad (19)$$

Since we normalize the productivity parameter in (6) to one, only relative prices are informative, and therefore we can ignore the dependence of  $S^*$  on structural parameters. Thus, the risk-neutral dynamics of futures prices is determined by only three combinations of five structural parameters:  $\gamma\mu^-$ ,  $\gamma\bar{i}$ , and  $\gamma\sigma_Y$ . Therefore, we cannot identify all the model parameters separately from futures data alone.

We obtain an additional identifying condition from the oil consumption data. Cooper (2003) reports that individual growth rates vary for the twenty three countries in his sample, typically falling between  $-3$  to  $3\%$ . For the US, the reported growth rate averaged  $-0.7\%$ . As documented in Cooper (2003), world crude oil consumption increased by 46 per cent per capita from 1971 to 2000, implying an average growth rate of approximately  $1.25\%$  which we attempt to equate with the expected growth rate of oil consumption,  $g_C$ , implied by the model

$$g_C = \bar{i} \Pr(S \geq S^*) - \delta = \lambda + \mu_Y - \frac{1}{2}\sigma_Y^2, \quad (20)$$

where  $\Pr(S \geq S^*) = \int_{-\infty}^{S^*} p^+(S) dS = (\bar{i})^{-1}\mu^-$  is the unconditional probability that  $S$  is below the investment trigger.

Finally, to estimate the risk premium  $\lambda$ , we use average historical daily returns on fully collateralized futures positions (we use three-month contracts). We are thus left with five independent identifying restrictions on six structural parameters. Following Gomes (2001), we fix the depreciation rate of capital at  $\delta = 0.12$  per year and do not infer it from futures prices and thus estimate the remaining five parameters.

[Insert Table III about here]

### *A.3 Parameter Estimates*

Our estimated parameter values and the corresponding standard errors are summarized in Table III. The first parameter value in the table,  $\gamma = 3.15$ , implies that the price elasticity of demand in our model is  $-0.32$ . Cooper (2003) reports estimates of short-run and long-run demand elasticity for a partial adjustment demand equation based on US data of  $-0.06$  and  $-0.45$  respectively. In our model, there is no distinction between short-run and long-run demand, as demand adjustments are assumed to be instantaneous. Our estimate falls half-way between the two numbers reported in Cooper (2003) for the US and is close to the average of the long-run elasticity estimates reported for all 23 countries considered in that study, which is  $-0.2$ .

Our second parameter is  $\bar{i}$ , the maximum investment rate in the model. This variable parameterizes the investment technology used by the firms. In order to make relative empirical comparison we have compared it to the growth rate of the number of operating oil wells between years 1999 and 2000 for several leading crude producers world wide.<sup>8</sup> During this period the number of operating wells has increased by 4.7% in the whole Middle East region, by 9.3% in Russia, by 22.3% in Venezuela, and by 10.3% in Norway.<sup>9</sup> The corresponding growth rate implied by our model is  $\bar{i} - \delta = 0.12$ . These numbers show that the *upper bound* of  $\bar{i} = 24\%$  would allow for a plausible range of realized annual investment rates.

The average growth rate of demand is close to zero. For comparison, the average annualized change in futures prices is approximately 2.8% in the data, which falls within the 95% confidence interval of the model's prediction. The volatility of demand shocks is not directly observable. The estimated value of  $\sigma_Y$ , together with the demand elasticity parameter  $\gamma^{-1}$  imply annualized volatility of the spot price of approximately 33%, which is

close to the observed price volatility of short-maturity futures contracts.

Finally, the market price of risk in our model is equal to  $2.47 \times 10^{-4}$ . This would imply that excess expected returns on a fully collateralized futures strategy should be close to zero. This is consistent with empirical data. While an assumption of constant risk premium is clearly restrictive, it is made for simplicity: nothing in our model prevents one from assuming a time-varying price of oil price risk. However, such an assumption would be exogenous to the model, and hence would not add to our understanding of the underlying economics of the problem.

## *B. Results and Discussion*

### *B.1 Quantitative Results*

We first illustrate the fit of the model by plotting the term structure of unconditional futures price volatility (to facilitate comparison with empirical data, we express our results as daily values, defined as annual values scaled down by  $\sqrt{252}$ ). We chose model parameters, as summarized in Table III, to match the behavior of crude oil futures. Figure 6 compares the volatility of prices implied by our choice of parameters to the empirical estimates. Our model seems capable of reproducing the slow-decaying pattern of futures price volatility. This feature of the data presents a challenge to simple storage models, as discussed in Routledge et. al. (2000). To see why it may not be easy to reproduce the slow-decaying pattern of unconditional volatilities in a simple single-factor model, consider a reduced-form model in which the logarithm of the spot price process follows a continuous-time AR(1) process



(Ornstein-Uhlenbeck process). Specifically, assume that the spot price is given by

$$S_t = e^{y_t}. \quad (21)$$

and under the risk-neutral probability measure  $y_t$  follows

$$dy_t = \theta_y(\bar{y} - y_t)dt + \sigma_y dW_t, \quad (22)$$

where  $\theta_y$  is the mean-reversion coefficient and  $\bar{y}$  is the long-run mean of the state variable.

According to this simple model, the unconditional volatility of futures price changes is an exponential function of maturity  $\tau$ :

$$\sigma^2(\tau) = \sigma_y^2 e^{-2\theta\tau}. \quad (23)$$

To compare the term structure of unconditional volatility implied by this model to the one generate by our model, we calibrate parameters  $\theta_y$  and  $\sigma_y$  so that the simple model exhibits the same volatility of the spot price and the same 30-day autocorrelation of the basis as our model. Figure 6 shows that, as expected, unconditional volatility implied by the simple model above decays too fast relative to our model and data.

[Insert Figure 6 about here]

The main qualitative distinction between the properties of our model and those of basic storage models is in the conditional behavior of futures volatility. As we demonstrate in Section I, the empirical relation between the volatility of futures prices and the slope of the term structure of prices is non-monotone and has a pronounced “V-shape”. Intuitively, we would expect our model to exhibit this pattern. When the spot price  $S_t$  is far away from the investment trigger  $S^*$ , one of the investment constraints is binding and can be expected

to remain binding for some time. If the capital stock  $K_t$  is much higher than its optimal level, given the current level of demand, firms find it optimal to postpone investment and the irreversibility constraint binds. On the other hand, when  $K_t$  is much lower than the optimal level, firms invest at the maximum possible rate and the investment rate constraint binds. In either case, the supply of the commodity is relatively inelastic and futures prices are relatively volatile. The further  $S_t$  travels away from the investment trigger, the larger the effect on volatility of long-maturity futures. At the same time, it is precisely when  $S_t$  is relatively far away from the investment trigger  $S^*$ , when the absolute value of the slope of the term structure of futures prices is large, as illustrated in Figure 5. This is to be expected. All prices in our model are driven by a single mean-reverting stationary spot price, and since futures prices of longer-maturity contracts are less sensitive to the current value of the spot price, the slope of the forward curve tends to be large when the spot price is far away from its long-run average value. The latter, in turn, is not far from  $S^*$ , given that  $S_t$  reverts to  $S^*$ . Thus, our model predicts that the volatility of futures prices should exhibit a “V-shape” as a function of the slope of the term structure of futures prices.

It should be clear from the above discussion that the critical feature of the model is not the precise definition of the production function, but rather the variable-elasticity property of the supply side of the economy. The “V-shape” pattern in volatilities is due to the fact that supply can adjust relatively easily in response to demand shocks when output is close to the optimal level, but supply is relatively inelastic when the output level is far from the optimum.

[Insert Table IV about here]

We now report the quantitative properties of the model. All data moments used to estimate the model are reported in the first column of Table 5. The expected growth rate of oil consumption implied by the model is equal to 0.9% and is close to an average world-wide growth rate of 1.25%. The long-run average of the slope of the forward curve,  $\ln \left[ \frac{P(t-1,6)}{P(t-1,3)} \right]$ , is 0.0065 in the model, compared to the empirical value of  $-0.0125$ . Both values are statistically indistinguishable from zero. The long-run standard deviation of the slope in the model, which equals 0.0285, is almost identical to the empirical value of 0.0287. The 30-day autocorrelation coefficient of the slope implied by the model is equal to 0.83, as compared to the value of 0.77 in the data. Overall, our model fits the basic behavior of the slope of the forward curve quite well. The Table IV shows the estimates of linear and piece-wise linear specifications of conditional variance of futures price changes (4) implied by the model for one-, five-, and ten-month futures. The coefficients of the linear regressions are negative and close in magnitude to their empirical counterparts. Such a negative relation between conditional volatility of futures prices and the basis would typically be interpreted as supportive of simple storage models. Note, however, that our model without storage can reproduce the same kind of relation. Our model, however, has a further important implication: the linear model is badly misspecified, since the theoretically predicted relation is non-monotone. Our piece-wise linear specification produces coefficients  $\beta_{1,T}$  and  $\beta_{2,T}$  that agree well their empirical counterparts for longer maturities (3 to 12 months), but the fit worsens for shorter maturities (1 and 2 months). Given the extremely streamlined nature of our model (e.g., the slope of the forward curve is a sufficient statistic for conditional volatility), this should not be surprising. In order to capture the properties of the short end of the term structure, one must take into account storage, which we do not allow in our model. The entire distribution of regression

coefficients across maturities of the futures contracts is shown in Figure 7. Finally, Figure 3 helps visualize the “V-shape” pattern.

[Insert Figure 7 about here]

### *B.2 Sensitivity Analysis*

In order to understand the sensitivity of our results to the baseline parameters summarized in Table III, we compute elasticities of basic statistics of the model output with respect to these parameters. Each elasticity is calculated by simulating the model twice: with a value of the parameter of interest ten percent of one standard deviation below (above) its baseline value. Next, the change in the moment is calculated as the difference between the results from the two simulations. This difference is then divided by the change in the underlying structural parameter between the two simulations. Finally, the result is then multiplied by the ratio of the baseline structural parameter to the baseline moment. The elasticities are reported in Table V.

An increase in the demand volatility,  $\sigma_Y$ , or in the elasticity of the inverse demand curve,  $\gamma$ , leads to an increase in the volatility of the spot price, which equals  $\gamma^2\sigma_Y^2$ . As one would expect, volatility of futures prices of various maturities increases as well. Qualitatively, both of the parameters  $\sigma_Y$  and  $\gamma$  affect the level of the unconditional volatility curve plotted in Figure 6. However, the demand volatility has strong positive effect on the expected growth rate of oil consumption since it increases the long-run growth rate of the level of the demand curve,  $Y_t$ .  $\gamma$  has no such effect.

The constraint on the investment rate  $\bar{i}$  has no effect on the volatility of the spot price. However, it affects volatility of futures prices. A higher value of  $\bar{i}$  allows capital stock to

adjust more rapidly in response to positive demand shocks, thus reducing the impact of demand shocks on the future value of the spot price and therefore lowering the volatility of futures prices. We thus see that  $\bar{i}$  effectively controls the slope of the term structure of volatility, higher values of  $\bar{i}$  imply a steeper term structure.  $\bar{i}$  has no effect on the expected growth rate of oil consumption, in agreement with equation (20).

[Insert Table V about here]

An increase in the unconditional mean of the demand shock,  $\mu_Y$ , has little affect on the level of futures price volatility. This is not surprising given the role  $\mu_Y$  plays in the evolution of the spot price  $S_t$ . An increase in  $\mu_Y$  raises the drift of  $S_t$  uniformly. The impact of this on the volatility of futures price is ambiguous and depends on the relative magnitude of the drift of  $S_t$  above,  $\mu^-$ , and below,  $\mu^+ \equiv \bar{i} - \mu^-$ , the investment threshold  $S^*$ . By symmetry considerations, if  $\mu^+ = \mu^-$ , an infinitesimal change in  $\mu_Y$  has no impact on the volatility of futures prices. Under the calibrated parameter values,  $\mu^- = 0.1289$  and  $\mu^+ = 0.1083$  and futures volatility is not very sensitive to  $\mu_Y$ . The same is true for the risk premium,  $\lambda$ . Both  $\mu_Y$  and  $\lambda$  have strong positive effect on  $g_C$  in agreement with equation (20).

In general, the affect of model parameters on the slope of the forward curve is difficult to interpret intuitively and depends on the chosen parameter values. However, the fact that the moments of the slope have different sensitivities to various model parameters makes them useful in estimating these parameters.

### III. Conclusions

This paper contributes along two dimensions. First, we show that volatility of future prices has a “V-shape” relationship with respect to the slope of the term structure of

futures prices. Second, we show that such volatility patterns arise naturally in models that emphasize investment constraints and, consequently, time-varying supply-elasticity as a key mechanism for price dynamics. Our empirical findings seem beyond the scope of simple storage models, which are currently the main focus of the literature, and point towards investigating alternative economic mechanisms, such as the one analyzed in this paper. Adding finite storage capacity to this economy would likely affect the very short end of the forward curve. It is likely then that the volatility of spot prices would be related to the level of inventories. However, adding storage is not likely to have material effect on the long end of the forward curve, since in practice the total storage capacity is rather small relative to annual consumption. What is difficult to predict, however, is how storage would interact with production constraints at intermediate maturities. This requires formal modeling and future work will entail a model that nests both storage and investment in an attempt to isolate their quantitative effects.

## Appendix A. Proof of Proposition 1

We conjecture that the equilibrium investment policy  $I_t^*$  is given by (13). Then, market clearing in the spot market implies that the spot price process  $S_t$  satisfies (15).

A competitive firm chooses an investment policy  $I_t$  to maximize the firm value, i.e., the present value of future output net of investment costs:

$$\max_{I_t} E_0 \left[ \int_0^\infty e^{-rt} (K_t S_t - I_t) dt \right], \quad (\text{A1})$$

subject to the capital accumulation rule

$$dK_t = (I_t - \delta K_t) dt, \quad (\text{A2})$$

$$I_t \geq 0, \quad (\text{A3})$$

$$I_t \leq \bar{i}K_t^A. \quad (\text{A4})$$

From (A2), we obtain

$$K_t = \int_0^t e^{-\delta(t-s)} I_s ds + K_0 e^{-\delta t}. \quad (\text{A5})$$

Using this expression for the capital stock, and relaxing the constraint (A4), we re-write the above optimization problems as

$$\max E_0 \left[ \int_0^\infty e^{-rt} (V_t - 1) I_t dt \right] + K_0 V_0, \quad (\text{A6})$$

subject to (A3, A4), where

$$V_t = E_t \left[ \int_t^\infty e^{-(r+\delta)(u-t)} S_u du \right]. \quad (\text{A7})$$

Assuming that the process  $V_t$  is well defined (we prove that next), the optimal solution of the firm's problem is

$$I_t^* = \begin{cases} \bar{i}K_t^A, & V_t - 1 > 0, \\ [0, \bar{i}K_t^A], & V_t - 1 = 0, \\ 0, & V_t - 1 < 0. \end{cases} \quad (\text{A8})$$

The above policy will coincide with (13), as long as  $V_t = 1$  whenever  $S_t = S^*$ . Let  $V(S, S^*)$  denote the value of  $V_t$  when  $S_t = S$  and the optimal investment threshold is  $S^*$ . Note that the conjectured form of the equilibrium spot price process implies that

$$V_t = V(S_t, S^*) = S^* V \left( \frac{S_t}{S^*}, 1 \right) \quad (\text{A9})$$

Thus, the equilibrium value of  $S^*$  can be found as  $S^* = V(1, 1)^{-1}$ .

We now characterize  $V(S; 1)$ , and show that  $V_t$  is finite. Let  $S^* = 1$  and define  $\omega_t = -(1/\gamma) \ln S_t$ . Then

$$d\omega_t = (\bar{i}1_{[\omega_t \leq 0]} - \mu^-) dt - \sigma_Y dW_t. \quad (\text{A10})$$

Let  $B$  be an arbitrary positive number and define a stopping time  $\tau_B = \inf\{t : \omega_t \leq -B\}$ .

Let

$$F_t^B = \mathbb{E}_t \left[ \int_t^\infty 1_{[s \leq \tau_B]} e^{-(r+\delta)(s-t) - \gamma \omega_s} ds \right]. \quad (\text{A11})$$

We look for  $F_t^B = F^B(\omega_t)$ . We start by heuristically characterizing  $F^B(\omega)$  as a unique solution of the Feynman-Kac equation

$$\frac{\sigma_Y^2}{2} \frac{d^2 F^B(\omega)}{d\omega^2} + [\bar{i} 1_{[\omega \leq 0]} - \mu^-] \frac{dF^B(\omega)}{d\omega} - (r + \delta) F^B(\omega) + e^{-\gamma \omega} = 0 \quad (\text{A12})$$

with the boundary condition

$$F^B(-B) = 0. \quad (\text{A13})$$

We look for a solution of (A12) of the form

$$F^B(\omega) = \begin{cases} A e^{\varkappa^- \omega} + \frac{e^{-\gamma \omega}}{r - \gamma \mu^- - \frac{\gamma^2}{2} \sigma_Y^2}, & \omega \geq 0, \\ C_1 e^{\bar{\varkappa}^+ \omega} + C_2 e^{\bar{\varkappa}^- \omega} + \frac{e^{-\gamma \omega}}{r + \gamma \mu^+ - \frac{\gamma^2}{2} \sigma_Y^2}, & \omega < 0. \end{cases} \quad (\text{A14})$$

Substituting these solutions into ODE (A12) yields quadratic equations on  $\varkappa_\pm$  and  $\bar{\varkappa}_\pm$

$$\frac{\sigma_Y^2}{2} \varkappa_\pm^2 - \mu^- \varkappa_\pm - (r + \delta) = 0, \quad (\text{A15})$$

$$\frac{\sigma_Y^2}{2} \bar{\varkappa}_\pm^2 + \mu^+ \bar{\varkappa}_\pm - (r + \delta) = 0. \quad (\text{A16})$$

$\varkappa_\pm$  and  $\bar{\varkappa}_\pm$  denote, respectively, positive and negative roots of the above equations. Define

$$M = \frac{\gamma \bar{i}}{\left( r + \delta - \gamma \mu^- - \frac{\gamma^2}{2} \sigma_Y^2 \right) \left( r + \delta + \gamma \mu^+ - \frac{\gamma^2}{2} \sigma_Y^2 \right)}. \quad (\text{A17})$$

Using the boundary condition (A13) and imposing continuity of the function  $F^B$  and its first derivative across 0 (to verify that the solution of the differential equation (A12) characterizes the expected value  $F_t^B$ , we only need the first derivative to be continuous at 0), we obtain

the following system of equations on coefficients  $C_1$ ,  $C_2$ , and  $A$ :

$$\begin{cases} C_1 + C_2 = A + M, \\ \bar{\varkappa}_+ C_1 + \bar{\varkappa}_- C_2 = \left( \frac{2\bar{i}}{\sigma_Y} + \varkappa_- \right) A + 2 \frac{r + \delta + \gamma \mu^+ - \gamma^2 \sigma_Y^2}{\gamma \sigma_Y^2} M, \\ C_1 e^{-\bar{\varkappa}_+ B} + C_2 e^{-\bar{\varkappa}_- B} = \frac{r + \delta - \gamma \mu^- - \frac{\gamma^2}{2} \sigma_Y^2}{\gamma \bar{i}} M e^{\gamma B}. \end{cases} \quad (\text{A18})$$



Solving this system yields the following expression for  $C_2$ :

$$C_2 = M \frac{\frac{r+\delta-\gamma\mu^- - \frac{\gamma^2}{2}\sigma_Y^2}{\gamma^2} \left( \bar{\varkappa}_+ - \frac{2\bar{\nu}}{\sigma_Y^2} - \varkappa_- \right) e^{(\bar{\varkappa}_- + \gamma)B} - \left( 2\frac{r+\delta+\gamma\mu^+ - \gamma^2\sigma_Y^2}{\gamma\sigma_Y^2} - \varkappa_- \right) e^{(\bar{\varkappa}_- - \bar{\varkappa}_+)B}}{\bar{\varkappa}_+ - \frac{2\bar{\nu}}{\sigma_Y^2} - \varkappa_- - \left( \bar{\varkappa}_- - \frac{2\bar{\nu}}{\sigma_Y^2} - \varkappa_- \right) e^{(\bar{\varkappa}_- - \bar{\varkappa}_+)B}}. \quad (\text{A19})$$

Next, we show that indeed  $F_t^B = F^B(\omega_t)$ . To see this, note that the process  $X_t = e^{-(r+\delta)t} F^B(\omega_t) + \int_0^t e^{-(r+\delta)s - \gamma\omega_s} ds$  is a local martingale. This follows from the fact that, by Ito's lemma, the drift of the process is equal to zero (due to (A12)). Next, since the diffusion coefficient of  $X_t$ ,  $\sigma dF^B(\omega)/d\omega$ , is bounded on the domain  $\omega \geq -B$ , the stopped process  $X_{t \wedge \tau_B}$  is a martingale. Thus,

$$X_0 = F^B(\omega_0) = \mathbb{E}_0[X_T] = \mathbb{E}_0 \left[ \mathbb{1}_{[T \leq \tau_B]} e^{-(r+\delta)T} F^B(\omega_T) + \int_0^T \mathbb{1}_{[s \leq \tau_B]} e^{-(r+\delta)s - \gamma\omega_s} ds \right].$$

Since the function  $F^B(\omega)$  is bounded on the domain  $\{\omega_t \geq -B\}$ , we know that  $\lim_{T \rightarrow \infty} \mathbb{E}_0[\mathbb{1}_{[T \leq \tau_B]} e^{-(r+\delta)T} F^B(\omega_T)] = 0$ . Thus, as we take  $T \rightarrow \infty$ , by monotone convergence theorem,  $F_B(\omega_0) = F_0^B$ .

Having found  $F_t^B$ , we now take a limit of  $B \rightarrow \infty$ . The  $\lim_{B \rightarrow \infty} F_t^B$  is well defined. Because  $\frac{\sigma_Y^2 \gamma^2}{2} - \gamma\mu^+ - (r + \delta) < 0$ , it follows that  $\bar{\varkappa}_- + \gamma < 0$  and, therefore,  $\lim_{B \rightarrow \infty} C_2 = 0$ . Also, constants  $C_1$  and  $A$  converge to finite limits as  $B \rightarrow \infty$ .

To show that  $\lim_{B \rightarrow \infty} F_t^B = V_t$ , we use the monotone convergence theorem, combined with an observation that  $\lim_{B \rightarrow \infty} \tau_B = \infty$ . The latter follows from the fact that  $\omega_t \leq \omega_0 + \mu^- t + \sigma(W_t - W_0)$ , which is an arithmetic Brownian motion and for which the corresponding stopping time converges to infinity as  $B \rightarrow \infty$ .

Thus,  $V(S, 1)$  is well defined, it is a function of the state variable  $\omega$ ,  $V(e^{-\gamma\omega}, 1)V(e^{-\gamma\omega}, 1) = F(\omega)$ , which satisfies the equation

$$\frac{\sigma_Y^2}{2} \frac{d^2 F(\omega)}{d\omega^2} + [\bar{i}\mathbb{1}_{[\omega \leq \omega^*]} - \mu^-] \frac{dF(\omega)}{d\omega} - (r + \delta)F(\omega) + e^{-\gamma\omega} = 0 \quad (\text{A20})$$

and is given explicitly by

$$F(\omega) = \begin{cases} Ae^{\varkappa_-\omega} + \frac{e^{-\gamma\omega}}{r-\gamma\mu^- - \frac{\gamma^2}{2}\sigma_Y^2}, & \omega \geq 0, \\ Ce^{\varkappa_+\omega} + \frac{e^{-\gamma\omega}}{r+\gamma\mu^+ - \frac{\gamma^2}{2}\sigma_Y^2}, & \omega < 0. \end{cases} \quad (\text{A21})$$

where

$$A = \frac{2^{\frac{r+\delta+\gamma\mu^+ - \gamma^2\sigma_Y^2}{\gamma\sigma_Y^2} - \varkappa_+}}{\varkappa_+ - \frac{2\bar{i}}{\sigma_Y^2} - \varkappa_-} M, \\ C = \left( 1 + \frac{2^{\frac{r+\delta+\gamma\mu^+ - \gamma^2\sigma_Y^2}{\gamma\sigma_Y^2} - \varkappa_+}}{\varkappa_+ - \frac{2\bar{i}}{\sigma_Y^2} - \varkappa_-} \right) M.$$

This completes our proof.

## Appendix B. Stationary long-run distribution of $S_t$

Define  $\omega = -(1/\gamma) \ln S_t$ . Then

$$d\omega_t = (\bar{i}1_{[\omega_t \leq \omega^*]} - \mu^-) dt - \sigma_Y dW_t, \quad \omega^* \equiv -\frac{1}{\gamma} \ln S^*. \quad (\text{B1})$$

It is enough to calculate the stationary long-run distribution of the  $\omega$ ,

$$p(\omega) = \begin{cases} p^+(\omega) & \omega \leq \omega^* \\ p^-(\omega) & \omega > \omega^* \end{cases}. \quad (\text{B2})$$

It exists if  $0 < \mu^- < \bar{i}$  and it satisfies the forward Kolmogorov ODE

$$\frac{d^2 p(\omega)}{d\omega^2} - \frac{2[\mu^+ 1_{[\omega \leq \omega^*]} - \mu^- 1_{[\omega > \omega^*]}]}{\sigma_Y^2} \frac{dp(\omega)}{d\omega} = 0. \quad (\text{B3})$$

$p(\omega)$  also satisfies the normalization condition

$$\int_{-\infty}^{\omega^*} p^+(\omega) d\omega + \int_{\omega^*}^{\infty} p^-(\omega) d\omega = 1. \quad (\text{B4})$$

Condition (B4) eliminates a constant as a trivial solution of the ODE (B3). We solve the

ODE (B3) separately for  $p^+(\omega)$  and  $p^-(\omega)$ :

$$p^+(\omega) = A^+ e^{\frac{2\mu^+}{\sigma_Y^2} \omega}, \quad (\text{B5}) \\ p^-(\omega) = A^- e^{-\frac{2\mu^-}{\sigma_Y^2} \omega},$$

and paste the solutions together so that  $p(\omega)$  is continuous at  $\omega^*$ . Thus,  $A^\pm$  can be found from the boundary condition at  $\omega^*$  and the normalization condition:

$$A^+ e^{\frac{2\mu^+}{\sigma_Y^2} \omega^*} = A^- e^{-\frac{2\mu^-}{\sigma_Y^2} \omega^*}, \quad (\text{B6})$$

$$\frac{\sigma_Y^2}{2\mu^+} A^+ e^{\frac{2\mu^+}{\sigma_Y^2} \omega^*} + \frac{\sigma_Y^2}{2\mu^-} A^- e^{-\frac{2\mu^-}{\sigma_Y^2} \omega^*} = 1. \quad (\text{B7})$$

Solving equations (B6), (B7) for  $A^+$  and  $A^-$ , and changing variables according to  $\omega = -\frac{1}{\gamma} \ln S$ , we obtain (16).

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# Notes

<sup>1</sup>In our data set on any given calendar day there are several contracts available with different time to delivery measured in days. The difference in delivery times between these contracts is at least 32 days or more. We utilize the following procedure for converting delivery times to the monthly scale. For each contract we divide the number of days it has left to maturity by 30 (the average number of days in a month), and then round off the resultant. For days when a contract with time to delivery of less than 15 days is traded, we add one “month” to the contract horizon obtained using the above procedure for all contracts traded on such days. The data is then sorted into bins based on the contract horizon measured in months.

<sup>2</sup>We refer to this time to delivery as time to maturity throughout the paper.

<sup>3</sup>An alternative parametric form would be quadratic. While the quadratic specification does not allow us to test for the conditional sign of the relationship between the slope of the forward curve and conditional volatility of futures returns, such specification provides a test for convexity of that relationship. The estimated relationship (not reported here) shows that the second-order term is highly statistically significant.

<sup>4</sup>For example, other recent applications in finance using competitive industry equilibrium include Fries, Miller, and Perraudin (1997) and Miao (2005).

<sup>5</sup>Analysts often describe the ‘cushion’ in this market as spare productive capacity. For example, according to estimates in Goldman Sachs (1999) publication the spare capacity was about 7-12% around fourth quarter of 2000. The report continues to say that “These fields will require significant investment and drilling to not only increase production but offset the decline rates. This will take time., ....., even if rig counts rebound substantially, during the fourth quarter, new supplies would not be available until late second quarter at the earliest”. While undoubtedly inventories and stock-outs affect capacity constraints they seem to be measured in days (around 20 days for full coverage of which only a few days are full working inventory as the rest represents minimum operating requirements). This is consistent with our view that the final inventories affect the short end of the forward curve but not likely to affect it along the several month horizon we investigate.

<sup>6</sup>Specifically, for any given value of  $\theta$ , we draw  $N$  realizations of the spot price  $S_t$  from its long-run steady-state distribution (which itself depends on model parameters and is given by (16)). Then, for each set of initial conditions, we simulate a path of the state variable of the same length as the historical sample and evaluate the function  $F(x, \theta)$  for each simulated path of the economy.

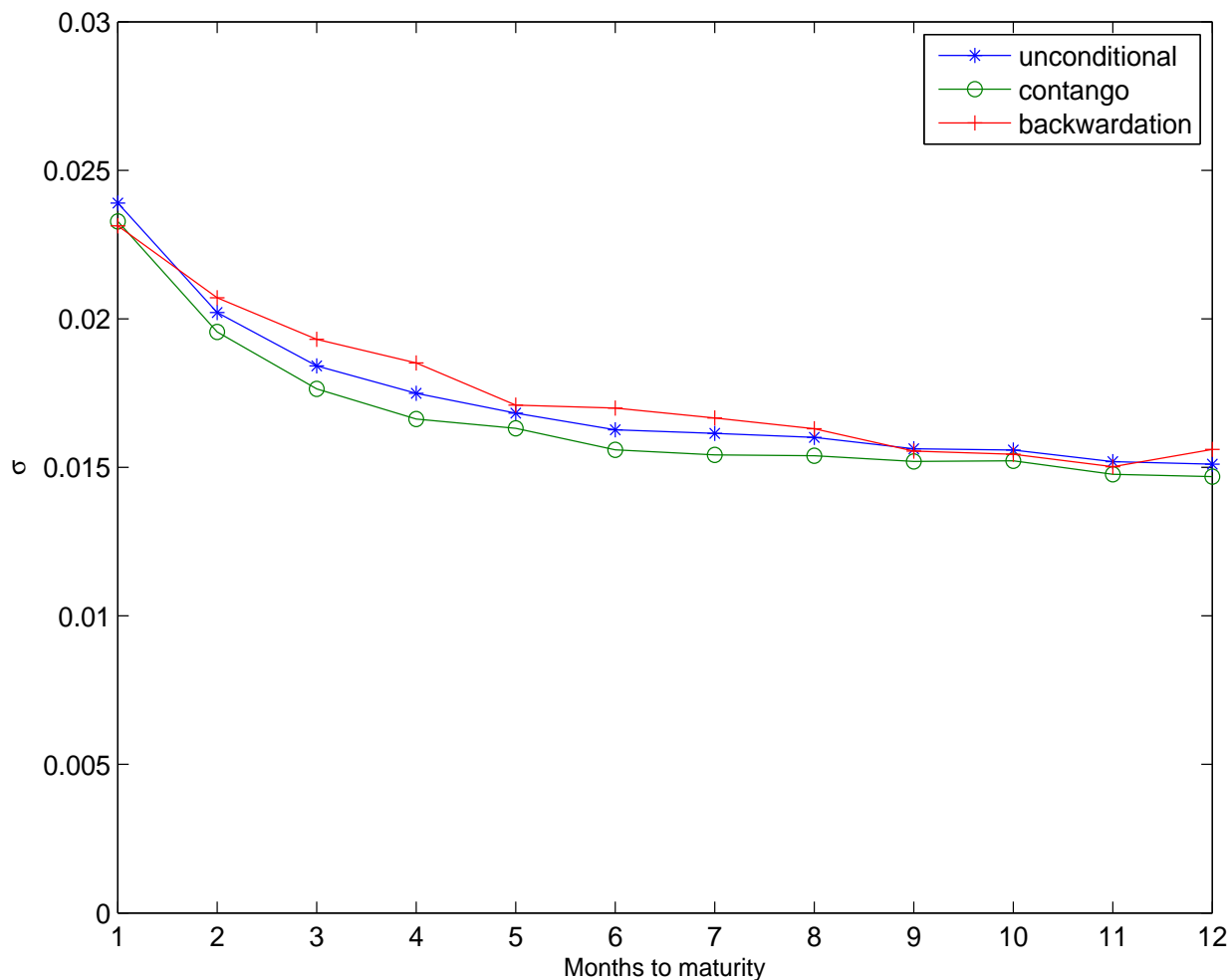
<sup>7</sup>Our results regarding the ‘V’ shape response in prices are not affected by this choice.

<sup>8</sup>This data is publicly available from [www.WorldOil.com](http://www.WorldOil.com).

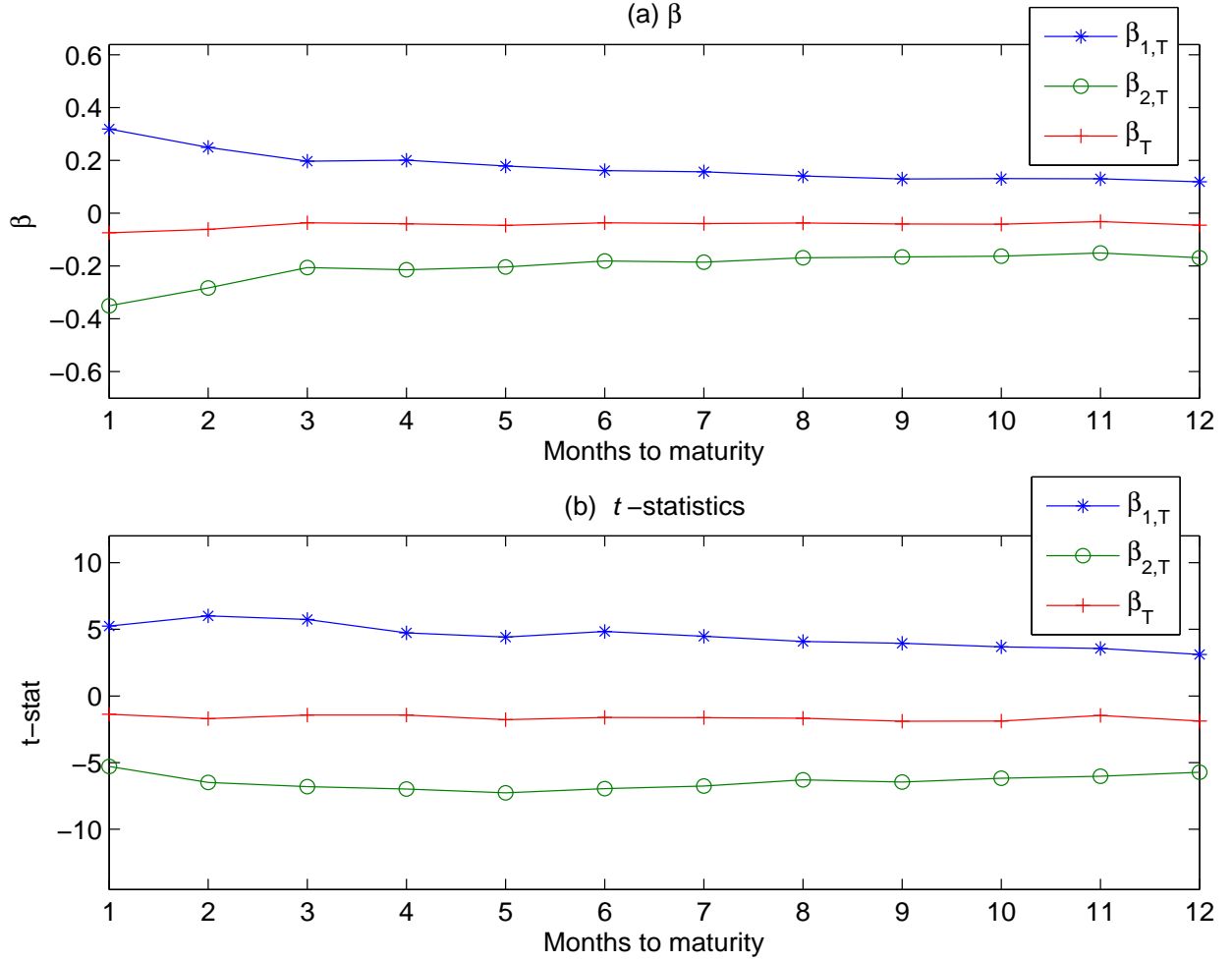
<sup>9</sup>The total worldwide growth in the number of operating oil wells has been around 0.6% for the same



period. This number is largely driven by the negative two per cent growth in the U.S., where most of these wells produce relatively small volumes of oil, often on an intermittent and marginally economic basis (commonly called “stripper” wells). The number of producing “stripper” wells changes depending on how many wells enter the ranks (by declining in production) and leave the ranks of stripper wells (by increasing production or being plugged and abandoned) each year. The United States’ stripper oil well population has been gradually declining over the past decade.



**Figure 1. Term structure of volatility, crude oil futures.** The data are daily percentage price changes on NYMEX crude oil (CL) futures from 1983 to 2000.  $\sigma$  denotes the standard deviation of daily percentage price changes. The time to maturity is defined as number of months left until the delivery month. The unconditional standard deviation is constructed using sample's first and second moments, while standard deviations conditional on backwardation and contango are conditioned on the shape of the forward curve one day prior.



**Figure 2. Conditional volatility, crude oil futures** The data are daily percentage price changes on NYMEX crude oil (CL) futures from 1983 to 2000, denoted by  $R(t, T)$ . Two different specifications are used to relate the instantaneous volatility of futures prices to the beginning-of-period slope of the forward curve defined as

$$SL(t) = \ln \left[ \frac{P(t, 6)}{P(t, 3)} \right].$$

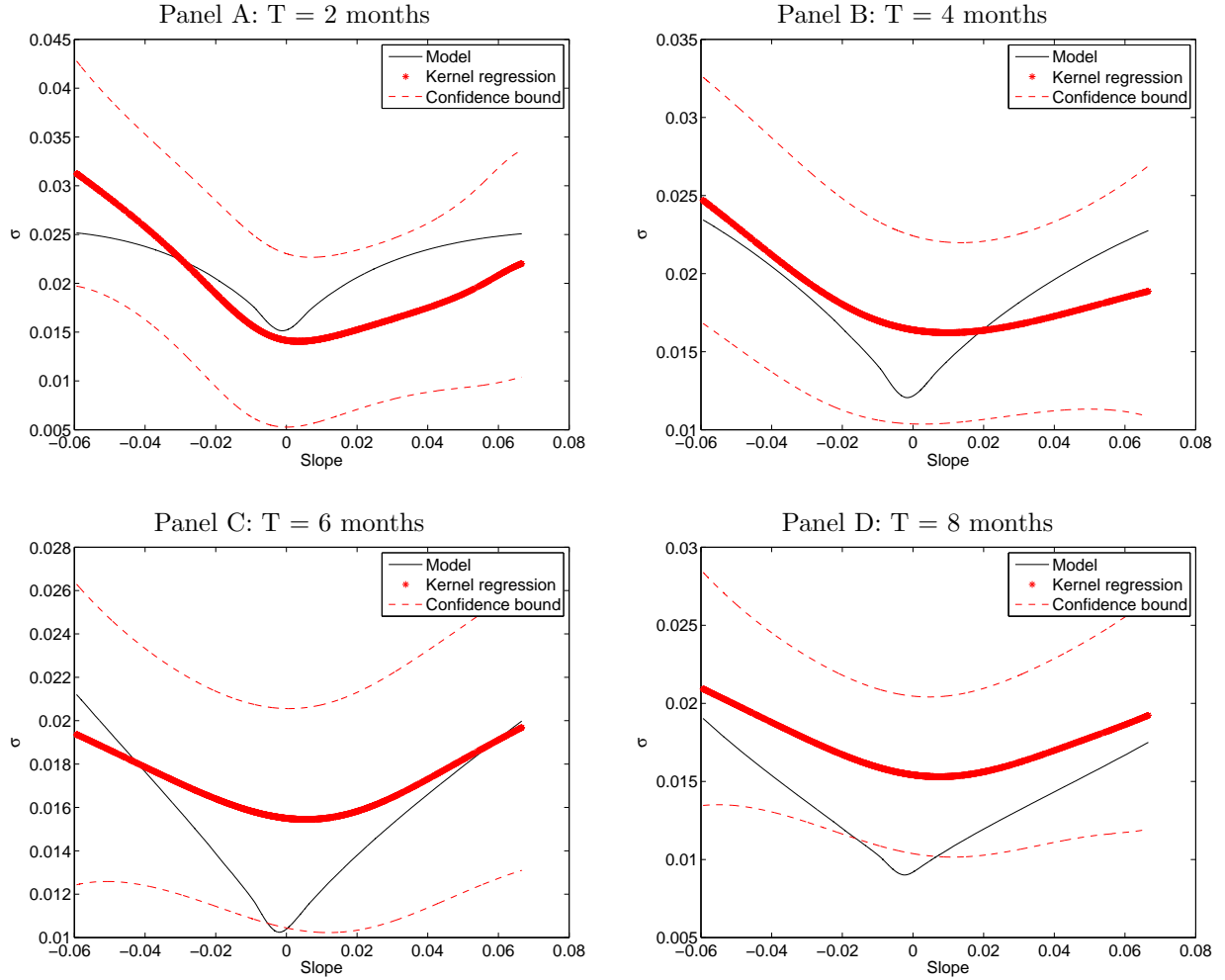
The first specification is

$$|R(t, T)| = \alpha_T + \beta_T \widetilde{SL}(t-1) + \varepsilon_T(t).$$

The second specification decomposes the slope into positive and negative parts

$$|R(t, T)| = \alpha_T + \beta_{1,T} \left( \widetilde{SL}(t-1) \right)^+ + \beta_{2,T} \left( \widetilde{SL}(t-1) \right)^- + \varepsilon_T(t),$$

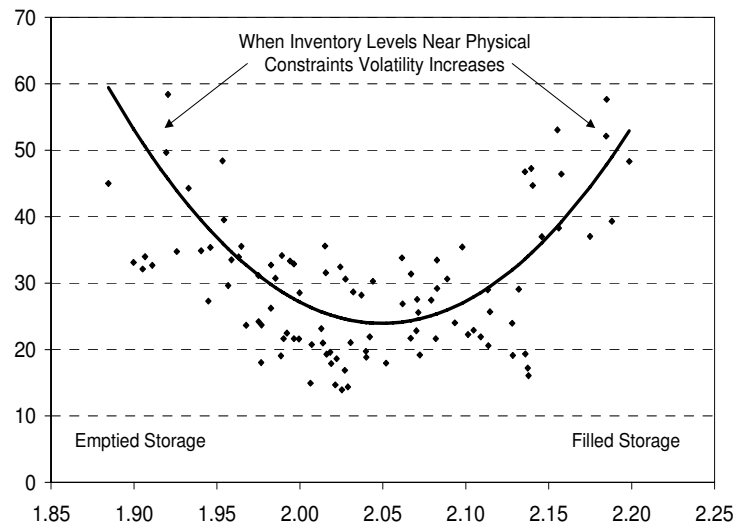
where  $(X)^\pm$  denotes positive (negative) part of  $X$  respectively. The figure shows the estimates of all three coefficients (exhibit A) and their respective  $t$ -statistics (exhibit B) for different times to maturity. All  $t$ -statistics are White-adjusted.



**Figure 3. “V-shape” of volatility of futures prices, data and model output.** Futures volatility is plotted as a function of the slope of the forward curve for different values of time to maturity. We use Receptive Field Weighted Regression (RFWR) to construct the functional form of volatility implied by the data. In both case, data and model, volatility exhibits a “V-shape” pattern as a function of the slope. The slope is demeaned. Volatility is expressed in annual terms. Model parameters are given in Table III.

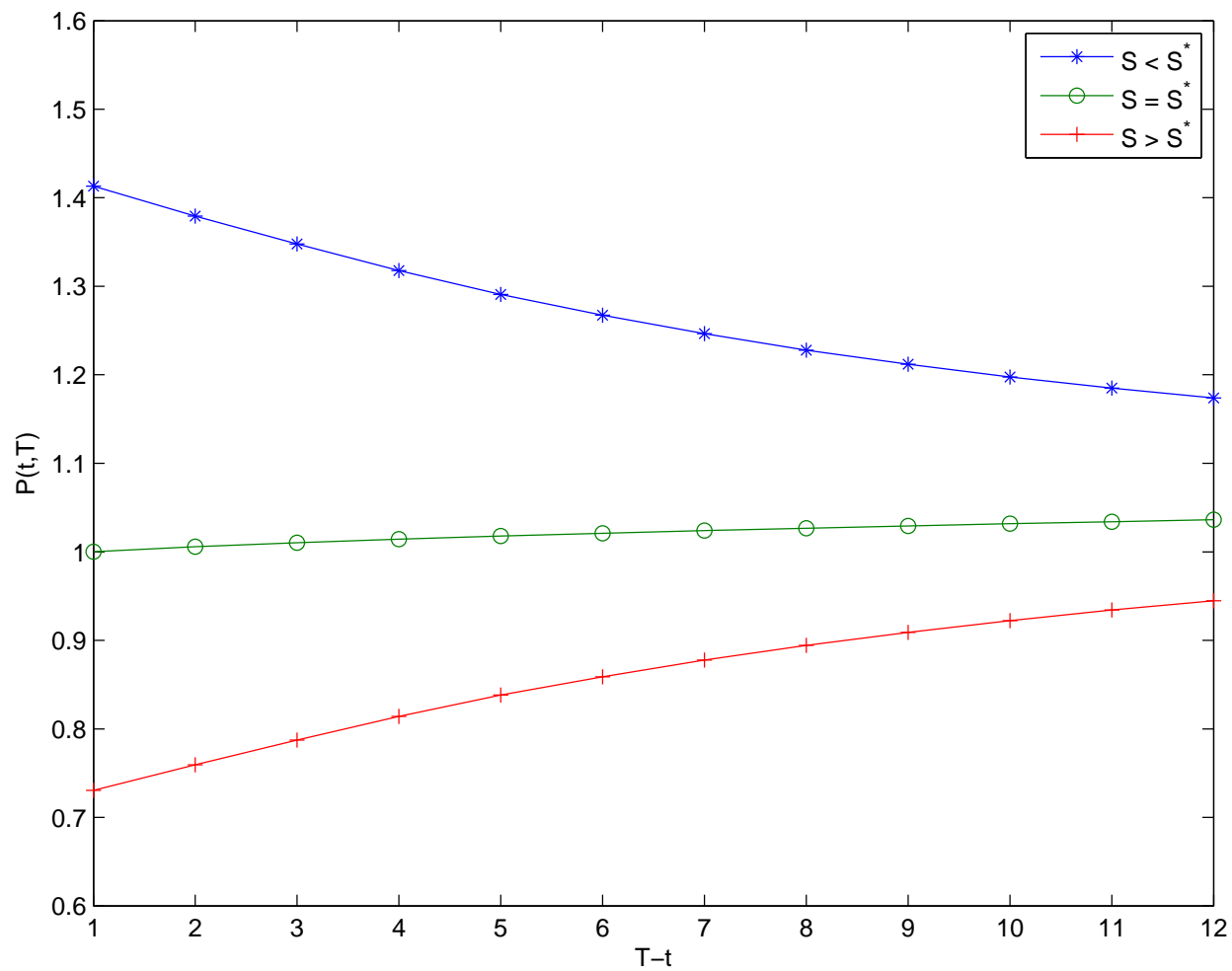
## Actual WTI Volatility Versus Expected End-of-March Atlantic Basin Inventories

billions of barrels, horizontal axis; percent volatility, vertical axis

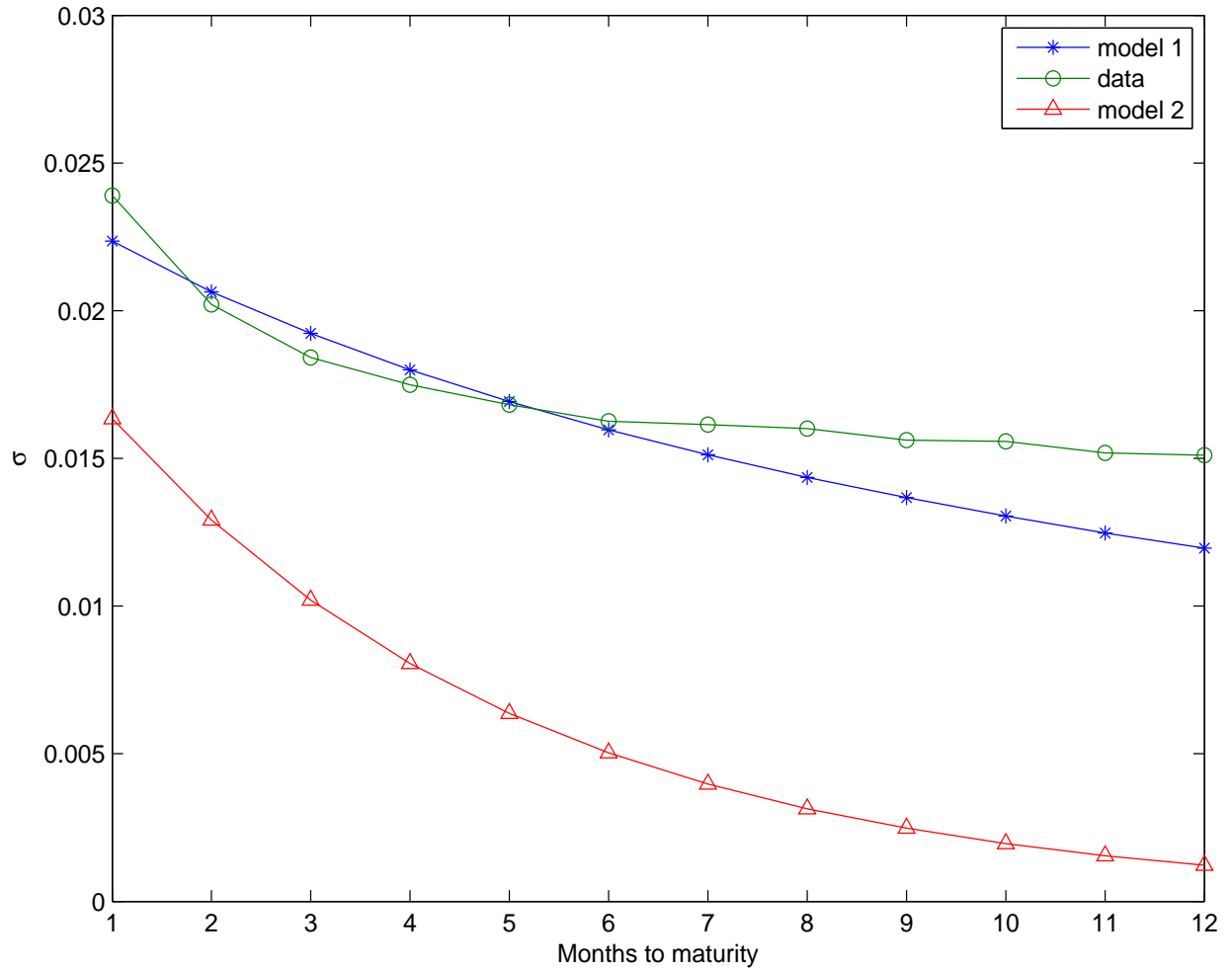


Source: Goldman, Sachs & Co.

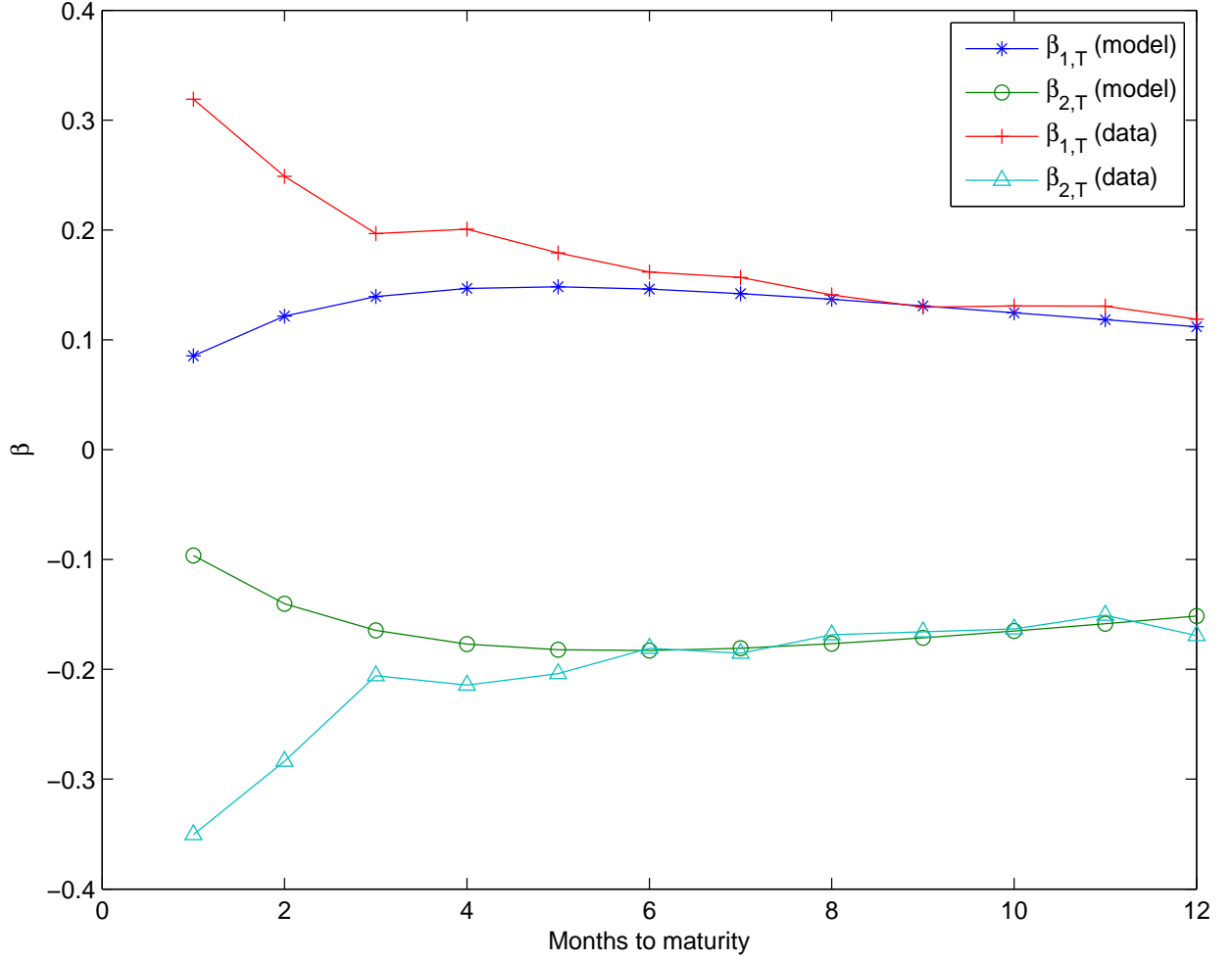
Figure 4. Volatility and Supply Capacity Constraints.



**Figure 5. Futures prices, model output.** Futures prices are generated by the model using parameter values reported in Table III. The investment threshold,  $S^*$ , is normalized to one.



**Figure 6. Unconditional volatility of futures prices, model output.** The unconditional standard deviation of daily changes in futures prices is constructed based on the output from the model (model 1). The model is fitted to the data on crude oil futures (CL) using a two-step simulated method of moments described in Section . Parameter values are reported in Table III. Model 2 corresponds to the unconditional standard deviation of changes in futures prices implied by Eq. (23). We report average across 2000 simulations.



**Figure 7. Conditional volatility of futures prices, model output.** For each time to maturity,  $T$ , we simulate a time series of daily futures prices using parameter values reported in Table III. We then compute daily percentage changes in futures prices, denoted by  $R(t, T)$ . The instantaneous volatility of futures prices is related to the beginning-of-period slope of the forward curve, defined as

$$SL(t) = \ln \left[ \frac{P(t, 6)}{P(t, 3)} \right],$$

according to the following specification

$$|R(t, T)| = \alpha_T + \beta_{1,T} \left( \widetilde{SL}(t-1) \right)^+ + \beta_{2,T} \left( \widetilde{SL}(t-1) \right)^- + \varepsilon_T(t),$$

where  $(X)^\pm$  denotes positive (negative) part of  $X$  respectively. This procedure is repeated 2000 times and we report the average across simulations. The figure shows  $\beta_{1,T}$  and  $\beta_{2,T}$  for different times to maturity. See the caption to Figure ?? for further details.



**Table I**

**Conditional volatility, crude oil futures**

This Table reports results for three different regressions. The data are daily percentage price changes on NYMEX crude oil (CL) futures from 1983 to 2000. The specification in the both panels is the same as in Figure ???. All results are reported for times to maturity equal to one, five and ten months. All  $t$ -statistics are White-adjusted for conditional heteroscedasticity.

|  | 1 Month            | 5 Months           | 10 Months          |
|--|--------------------|--------------------|--------------------|
| 1. $ R(t, T)  = \alpha_T + \beta_T \widetilde{SL}(t-1) + \varepsilon_T(t)$   |                    |                    |                    |
| $\beta_T$<br>( $t$ -stat)  | -0.0743<br>(-1.37) | -0.0462<br>(-1.77) | -0.0419<br>(-1.89) |
| $R^2$  | 0.0122             | 0.0111             | 0.0117             |
| 2. $ R(t, T)  = \alpha_T + \beta_{1,T} \left(\widetilde{SL}(t-1)\right)^+ + \beta_{2,T} \left(\widetilde{SL}(t-1)\right)^- + \varepsilon_T(t)$ |                    |                    |                    |
| $\beta_{1,T}$<br>( $t$ -stat)  | 0.3191<br>(5.25)   | 0.1791<br>(4.42)   | 0.1308<br>(3.68)   |
| $\beta_{2,T}$<br>( $t$ -stat)  | -0.3505<br>(-5.29) | -0.2039<br>(-7.26) | -0.1632<br>(-6.17) |
| $R^2$  | 0.1320             | 0.1058             | 0.0790             |

Table II

GARCH regressions, crude oil futures

This Table reports results for two different regressions. The data are daily returns percentage price changes on NYMEX crude oil (CL) futures from 1985 to 2000. See the caption to Table I. The volatility time series  $\sigma_{GARCH}(t, T)$  is obtained by fitting GARCH(1,1) model to the daily returns percentage price changes of maturity  $T$ . The volatility time series  $\hat{\sigma}_{GARCH}$  is obtained by fitting GARCH(1,1) model to the time series of daily returns percentage price changes of maturity  $T$  up to day  $t - 1$  and then using it to predict the volatility on day  $t$ ,  $\hat{\sigma}_{GARCH}(t, T)$ . The initial window is set to 300 days.

|   | 1 Month            | 5 Months           | 10 Months          |
|---|--------------------|--------------------|--------------------|
| 1. $ R(t, T)  = \alpha_T + \beta_{G,T}\sigma_{GARCH}(t, T) + \beta_{1,T}(\widetilde{SL}(t-1))^+ + \beta_{2,T}(\widetilde{SL}(t-1))^- + \varepsilon_T(t)$                          |                    |                    |                    |
| $\beta_{G,T}$<br>( $t$ -stat)   | 0.5323<br>(8.85)   | 0.6145<br>(9.48)   | 0.6612<br>(12.47)  |
| $\beta_{1,T}$<br>( $t$ -stat)   | 0.1610<br>(4.32)   | 0.0823<br>(4.01)   | 0.0595<br>(3.81)   |
| $\beta_{2,T}$<br>( $t$ -stat)   | -0.1605<br>(-2.84) | -0.0796<br>(-2.92) | -0.0599<br>(-2.91) |
| $R^2$   | 0.2601             | 0.2605             | 0.2495             |
| 2. $\hat{\sigma}_{GARCH}(t, T) = \alpha_T + \beta_{G,T}\hat{\sigma}_{GARCH}(t-1, T) + \beta_{1,T}(\widetilde{SL}(t-1))^+ + \beta_{2,T}(\widetilde{SL}(t-1))^- + \varepsilon_T(t)$ |                    |                    |                    |
| $\beta_{G,T}$<br>( $t$ -stat)   | 0.9403<br>(68.82)  | 0.9748<br>(155.06) | 0.9739<br>(161.50) |
| $\beta_{1,T}$<br>( $t$ -stat)   | 0.0269<br>(3.06)   | 0.0079<br>(3.34)   | 0.0064<br>(2.90)   |
| $\beta_{2,T}$<br>( $t$ -stat)   | -0.0228<br>(-3.04) | -0.0060<br>(-2.23) | -0.0053<br>(-1.83) |
| $R^2$   | 0.9108             | 0.9627             | 0.9586             |

**Table III****Parameter estimates, crude oil futures**

This table reports our parameter values. We use a two step SMM procedure to estimate a vector of seven structural parameters  $\theta \equiv \{\gamma, \mu_Y, \sigma_Y, \bar{i}, r, \delta, \lambda\}$ . Since only five model parameters can be independently identified from the data (see Section ), we fix  $\delta$  and  $r$  and estimate the remaining five parameters. We match the unconditional properties of crude oil futures prices, specifically the historic daily return on fully collateralized three-month futures position, the unconditional volatility of daily percent price changes for futures of various maturities as well as the mean, volatility, and the 30-day autoregressive coefficient of the slope of the forward curve. In addition, we match the expected growth rate of crude oil consumption. The standard errors are reported where applicable.

| Parameter  | Value                 | Std. Error           |
|------------|-----------------------|----------------------|
| $\gamma$   | 3.1484                | 0.2396               |
| $\bar{i}$  | 0.2362                | 0.0531               |
| $r$        | 0.0200                | NA                   |
| $\mu_Y$    | 0.01438               | 0.0048               |
| $\sigma_Y$ | 0.1043                | 0.0216               |
| $\delta$   | 0.1200                | NA                   |
| $\lambda$  | $2.47 \times 10^{-4}$ | $1.4 \times 10^{-4}$ |

**Table IV**

**Conditional volatility, model output**

Conditional variance of crude oil futures prices are compared with the output of the model under two different econometric specifications. See the caption to Table I.

|  | 1 Month |         | 5 Months |         | 10 Months |         |
|--|---------|---------|----------|---------|-----------|---------|
|  | Data    | Model   | Data     | Model   | Data      | Model   |
| 1. $ R(t, T)  = \alpha_T + \beta_T \widetilde{SL}(t-1) + \varepsilon_T(t)$   |         |         |          |         |           |         |
| $\beta_T$  | -0.0743 | -0.0030 | -0.0462  | -0.0100 | -0.0419   | -0.012  |
| 2. $ R(t, T)  = \alpha_T + \beta_{1,T} (\widetilde{SL}(t-1))^+ + \beta_{2,T} (\widetilde{SL}(t-1))^- + \varepsilon_T(t)$ |         |         |          |         |           |         |
| $\beta_{1,T}$  | 0.3191  | 0.0854  | 0.1791   | 0.1482  | 0.1308    | 0.1246  |
| $\beta_{2,T}$  | -0.3505 | -0.0964 | -0.2039  | -0.1822 | -0.1632   | -0.1651 |

**Table V**  
**Sensitivity of model moments to parameters**

This table presents elasticities of model moments with respect to the model parameters. The baseline parameters are given in Table III. Each elasticity is calculated by simulating the model twice: once with a value of the parameter of interest ten percent of one standard deviation below its baseline value, and once with a value ten percent of one standard deviation above its baseline value. Then the change in the moment is calculated as the difference between the results from the two simulations. This difference is then divided by the change in the underlying structural parameter between the two simulations. The result is then multiplied by the absolute value of the ratio of the baseline structural parameter to the baseline moment.

|  | Data    | Baseline Moments | $\gamma$ | $\bar{i}$ | $\mu_Y$ | $\sigma_Y$ | $\lambda (\times 10^{-2})$ |
|--|---------|------------------|----------|-----------|---------|------------|----------------------------|
| Average Slope of the Forward Curve                                   | -0.0124 | 0.0065           | 1.8401   | -1.2862   | 0.1814  | 2.3471     | -0.5004                    |
| Volatility of the Slope of the Forward Curve                         | 0.0287  | 0.0285           | 0.9704   | 0.5516    | -0.0377 | 0.7005     | 0.4189                     |
| 30-days Autoregressive Coefficient of the Slope of the Forward Curve | 0.7653  | 0.8337           | -0.0041  | -0.1221   | -0.0038 | 0.2112     | 0.1177                     |
| Long Term Crude Oil Consumption Growth                               | 0.0092  | 0.0092           | 0.0001   | -0.0003   | 1.5652  | -1.1843    | 2.6939                     |
| Unconditional Variance 3-month CL Futures                            | 0.0184  | 0.0186           | 0.7444   | -0.5535   | 0.0036  | 1.2367     | 0.0646                     |
| Unconditional Variance 6-month CL Futures                            | 0.0163  | 0.0149           | 0.6807   | -0.9276   | 0.0104  | 1.4487     | 0.0625                     |
| Unconditional Variance 9-month CL Futures                            | 0.0156  | 0.0124           | 0.6176   | -1.2162   | 0.0173  | 1.5690     | 0.0706                     |
| $\beta_{1,T}$ 3-month CL Futures                                     | 0.1968  | 0.1393           | -0.0367  | 1.2496    | -0.1222 | -0.1146    | -0.0488                    |
| $\beta_{1,T}$ 6-month CL Futures                                     | 0.1616  | 0.1462           | -0.0634  | 1.0368    | -0.1475 | 0.4632     | -0.1799                    |
| $\beta_{1,T}$ 9-month CL Futures                                     | 0.1298  | 0.1309           | -0.0795  | 0.8254    | -0.1634 | 0.9137     | -0.0662                    |
| $\beta_{2,T}$ 3-month CL Futures                                     | -0.2059 | -0.1646          | 0.0745   | -1.7212   | 0.1753  | -0.0084    | 0.3828                     |
| $\beta_{2,T}$ 6-month CL Futures                                     | -0.1808 | -0.1829          | 0.0897   | -2.6308   | 0.2155  | 0.5637     | 0.4722                     |
| $\beta_{2,T}$ 9-month CL Futures                                     | -0.1659 | -0.1713          | 0.0983   | -3.5200   | 0.2551  | 1.0333     | 0.5976                     |