1. Suppose

$$\begin{aligned} X_1, X_2 &\sim \text{i. i. d. } N(\mu, \sigma^2), \\ Y_1, Y_2, Y_3 &\sim \text{i. i. d. } N(\nu, \sigma^2), \\ &\quad \text{and } (X_1, X_2) \text{ is independent of } (Y_1, Y_2, Y_3). \end{aligned}$$

(a) Given the observation that

$$\begin{array}{ll} X_1 = 102 & Y_1 = 110 \\ X_2 = 122 & Y_2 = 130 \\ & Y_3 = 135, \end{array}$$

find a 95% confidence interval for $\mu - \nu$.

**Answer:** Let

$$\begin{aligned} \overline{X} &= (X_1 + X_2)/2, \\ \overline{Y} &= (Y_1 + Y_2 + Y_3)/3, \\ S_X^2 &= \left(X_1 - \overline{X}\right)^2 + \left(X_2 - \overline{X}\right)^2, \\ S_Y^2 &= \left(Y_1 - \overline{Y}\right)^2 + \left(Y_1 - \overline{Y}\right)^2 + \left(Y_1 - \overline{Y}\right)^2. \end{aligned}$$

The four random variables above are mutually independent. Then

$$\overline{X} \sim N\left(\mu, \frac{\sigma^2}{2}\right) \text{ and } \overline{Y} \sim N\left(\nu, \frac{\sigma^2}{3}\right), \text{ and these are independent,}$$

$$\text{so } \left(\overline{X} - \overline{Y}\right) - (\mu - \nu) \sim N\left(0, \sigma^2\left(\frac{1}{2} + \frac{1}{3}\right)\right)$$

$$\text{and therefore } \frac{\left(\overline{X} - \overline{Y}\right) - (\mu - \nu)}{\sigma\sqrt{\dfrac{1}{2} + \dfrac{1}{3}}} \sim N(0, 1).$$

$$\text{Also } S_X^2/\sigma^2 \sim \chi^2_{2-1} \text{ and } S_Y^2/\sigma^2 \sim \chi^2_{3-1}, \text{ and these are independent}$$

$$\text{so } \frac{S_X^2 + S_Y^2}{\sigma^2} \sim \chi^2_{(2-1)+(3-1)} = \chi^2_3,$$

and this $N(0, 1)$ random variable is independent of this $\chi^2_3$ random variable. Consequently

$$\frac{\left(\left(\overline{X} - \overline{Y}\right) - (\mu - \nu)\right) / \left(\sigma\sqrt{(1/2) + (1/3)}\right)}{\sqrt{\left\{(S_X^2 + S_Y^2)/\sigma^2\right\}/3}} \sim t_3.$$

The "$\sigma$" cancels out. According to the table of the $t$ distribution,

$$\mathbf{Pr}\left(-3.182 \leq \frac{\left(\left(\overline{X}-\overline{Y}\right)-(\mu-\nu)\right)/\sqrt{(1/2)+(1/3)}}{\sqrt{(S_X^2+S_Y^2)/3}} \leq 3.182\right) = 0.95.$$

This is equivalent to

$$\mathbf{Pr}\left(\overline{X}-\overline{Y}-3.182\sqrt{\frac{1}{2}+\frac{1}{3}}\sqrt{\frac{S_X^2+S_Y^2}{3}}\right.$$

$$\left.\leq \quad \mu-\nu \quad \leq \quad \overline{X}-\overline{Y}+3.182\sqrt{\frac{1}{2}+\frac{1}{3}}\sqrt{\frac{S_X^2+S_Y^2}{3}}\right) = 0.95.$$

Plugging in the data, we get

$$\begin{aligned}
\overline{X} = (102+122)/2 &= 112,\\
\overline{Y} = (110+130+135)/3 &= 125,\\
S_X^2 = (102-112)^2+(122-112)^2 &= 200,\\
S_Y^2 = (110-125)^2+(130-125)^2+(135-125)^2 &= 350,
\end{aligned}$$

$$\begin{aligned}
\overline{X}-\overline{Y}-3.182\sqrt{\frac{1}{2}+\frac{1}{3}}\sqrt{\frac{S_X^2+S_Y^2}{3}} &= -52.33\ldots\\
\overline{X}-\overline{Y}+3.182\sqrt{\frac{1}{2}+\frac{1}{3}}\sqrt{\frac{S_X^2+S_Y^2}{3}} &= 26.33\ldots
\end{aligned}$$

and so the 95% confidence interval is the interval between the last two of these numbers.

(b) Given the same data that appear in part (a), test the null hypothesis that $\mu = \nu$ at the 5% level. Explain how you know the outcome of the test.

**Answer:** We reject the hypothesis that $\mu - \nu = 0$ at the 5% level iff 0 lies outside a $(100-5)$% confidence interval for $\mu - \nu$. In this case 0 is well within the 95% confidence interval that we just found, so we do not reject the null hypothesis.

2. Suppose $Y_1, Y_2, Y_3 \sim \text{i.i.d. } N(\mu, \sigma^2)$.

(a) Find two linearly independent vectors $c$ such that the random variable $c'Y$ is an unbiased estimator of zero.

**Answer:**

$$\begin{aligned}
&\mathbf{E}(c_1Y_1+c_2Y_2+c_3Y_3) = c_1\,\mathbf{E}(Y_1)+c_2\,\mathbf{E}(Y_2)+c_3\,\mathbf{E}(Y_3)\\
&= c_1\mu+c_2\mu+c_3\mu = (c_1+c_2+c_3)\mu\\
&= 0 \text{ for } \underline{\textbf{all}} \text{ values of } \mu \text{ if and only if } c_1+c_2+c_3=0.
\end{aligned}$$

So the problem is to find two linearly independent vectors $(c_1, c_2, c_3)$ such that $c_1+c_2+c_3 = 0$. Many answers are possible; here is one: $(1, 1, -2)$ and $(2, -1, -1)$.

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In other words, $Y_1 + Y_2 - 2Y_3$ and $2Y_1 - Y_2 - Y_3$ are two linearly independent unbiased estimators of zero.

(b) Show that every linear unbiased estimator of zero is independent of $\widehat{\mu} = \overline{Y}$.

**Answer:** Since $Y_1, Y_2, Y_3$ are independent and normally distributed, $a_1Y_1+a_2Y_2+a_3Y_3$ must be normally distributed for any constants $a_1, a_2, a_3$. Any linear combination of the two random variables $c'Y$ and $\overline{Y}$ is of the form

$$\begin{aligned}
& b_1(c'Y) + b_2\overline{Y} \\
= \ & b_1c_1Y_1 + b_1c_2Y_2 + b_1c_3Y_3 + b_2(1/3)Y_1 + b_2(1/3)Y_2 + b_2(1/3)Y_3 \\
= \ & (b_1c_1 + b_2(1/3))Y_1 + (b_1c_2 + b_2(1/3))Y_2 + (b_1c_3 + b_2(1/3))Y_3 \\
= \ & a_1Y_1 + a_2Y_2 + a_3Y_3, \text{ and we saw that that is normally distributed.}
\end{aligned}$$

Therefore, any linear combination of $c'Y$ and $\overline{Y}$ is normally distributed. In other words, the two random variables $c'Y$ and $\overline{Y}$ are jointly normally distributed. If they are jointly normally distributed and uncorrelated, then they are independent. That they are uncorrelated means their covariance is zero. Here it is:

$$\begin{aligned}
& \mathbf{cov}(c'Y,\, \overline{Y}\,) = \mathbf{cov}(c'Y,\, [1/3,\, 1/3,\, 1/3]Y) = c'(\mathbf{cov}(Y,\, Y)) \begin{bmatrix} 1/3 \\ 1/3 \\ 1/3 \end{bmatrix} \\
= \ & (1/3)(c_1, c_2, c_3)\left(\sigma^2 I_3\right) \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = (\sigma^2/3)(c_1 + c_2 + c_3) = 0 \text{ because}
\end{aligned}$$

in part (a) we saw that the fact that $c'Y$ is an unbiased estimator of zero is the same as $c_1 + c_2 + c_3 = 0$.

3. Suppose $Y_1, Y_2, Y_3 \sim \text{i.i.d.}\, N(\mu,\, \sigma^2)$. Suppose $\mu$ and $\sigma$ are unknown. If it is observed that $Y_1 = 10.2$, $Y_2 = 9.0$, and $Y_3 = 12.1$, find a 90% confidence interval for $\sigma^2$.

**Answer:** We use the fact that

$$\frac{(Y_1 - \overline{Y}\,)^2 + (Y_2 - \overline{Y}\,)^2 + (Y_3 - \overline{Y}\,)^2}{\sigma^2} \sim \chi^2_{3-1}.$$

Therefore

$$\mathbf{Pr}\left(0.1026 \le \frac{(Y_1 - \overline{Y}\,)^2 + (Y_2 - \overline{Y}\,)^2 + (Y_3 - \overline{Y}\,)^2}{\sigma^2} \le 5.991\right) = 0.90$$

and each tail separately has probability 0.05.

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Taking reciprocals term-by-term requires inversion of each occurrence of "≤" (Here we are relying that the numbers on each side of "≤" are both positive. Both negative would also work. One positive and one negative would invalidate this step.) yields

$$\textbf{Pr}\left(9.7466 \geq \frac{\sigma^2}{(Y_1-\overline{Y})^2+(Y_2-\overline{Y})^2+(Y_3-\overline{Y})^2} \geq 0.1669\right) = 0.90$$

and then

$$\textbf{Pr}\left(9.7466\sum_{i=1}^{3}(Y_i-\overline{Y})^2 \geq \sigma^2 \geq 0.1669\sum_{i=1}^{3}(Y_i-\overline{Y})^2\right) = 0.90.$$

The confidence interval is from $0.1669\sum_{i=1}^{3}(Y_i-\overline{Y})^2 = (0.1669)(4.886666\ldots) \cong 0.8156$ to $9.7466\sum_{i=1}^{3}(Y_i-\overline{Y})^2 = (9.7466)(4.886666\ldots) \cong 47.628$.

You got less than full credit if you asserted without qualification anything like this:

$$\textbf{Pr}\left(0.8156 \leq \sigma^2 \leq 47.628\right),$$

i.e., that the probability statement remains intact after a particular non-random value has been substituted for the random variable $\sum_{i=1}^{2}(Y_i-\overline{Y})^2$. Confidence intervals are an inherently frequentist method.

4. Suppose $X = \begin{bmatrix} 1 & x_1 \\ \vdots & \vdots \\ 1 & x_n \end{bmatrix}$, the columns of $X$ are linearly independent, and $H = X(X'X)^{-1}X'$.

(a) Show that if $u$ is in the column space of $X$, then $Hu = u$.

**Answer:** A vector $u$ is in the column space of $X$ if and only if for some vector $a$ we have $u = Xa$. Then

$$Hu = \left(X(X'X)^{-1}X'\right)u = \left(X(X'X)^{-1}X'\right)Xa = X\left((X'X)^{-1}X'X\right)a = Xa = u.$$

(b) Show that if $u$ is orthogonal to the column space of $X$, then $Hu = 0$.

**Answer:** A vector $u$ is orthogonal to the column space of $X$ iff $u$ is orthogonal to each column of $X$. If $X_i$ is the $i$th column of $X$, then $u$ is orthogonal to $X_i$ iff $X_i'u = 0$. That holds for both values of $i$ iff $X'u = 0$. If $X'u = 0$ then

$$Hu = \left(X(X'X)^{-1}X'\right)u = X(X'X)^{-1}\left(X'u\right) = X(X'X)^{-1}0 = 0.$$

(c) Suppose $H = A\Lambda A^{-1}$ where $A$ and $\Lambda$ are $n \times n$ matrices, $A$ is invertible, and $\Lambda$ is diagonal. What can you say about $A$ and $\Lambda$?

**Answer:** A real symmetric matrix can be diagonalized by an orthogonal matrix, but can a real symmetric matrix be diagonalized only by an orthogonal matrix? Not quite: If $H = A\Lambda A'$, where $\Lambda$ is a diagonal matrix and $A$ is an orthogonal matrix, then $H = \frac{A}{3}\Lambda(3A)$, and $\frac{A}{3}$ and $3A$ are not orthogonal matrices.

But all you had to say to get all 10 points on this one was:

- $A$ is an orthogonal matrix, and
- Two columns of $A$ span the column space of $X$, and
- Two of the diagonal entries in $\Lambda$ are 1 and all of the others are 0.