3rd problem set,

1. Suppose $M \sim N(\mu, \sigma^2)$ and $X_1, X_2, X_3, \ldots\ldots \mid M \sim \text{i.i.d.}\, N(M, \tau^2)$.
   Let $\overline{X}_n = (X_1 + \cdots + X_n)/n$.

   (a) Find the conditional distribution of $\overline{X}_n$ given M.

   (b) Find the conditional distribution of M given $\overline{X}_n$.

   (c) Show that the conditional expectation $\mathbf{E}(M \mid \overline{X}_n)$ is a weighted average of $\overline{X}_n$ and the prior expected value of M, and that the weights are inversely proportional to certain variances — say what they are variances of.

   (d) Show that the conditional distribution of M given $X_1, \ldots, X_n$ is the same as the conditional distribution of M given $\overline{X}_n$.

   (e) Find a 90% posterior probability interval for M.

2. DeGroot & Schervish, p. 346, #17. (The model is reasonable if θ is the time between the arrival of each bus and the next, and it is known that the time between arrivals is never less than four minutes. This prior distribution is a Pareto distribution. Concerning the family of distributions that go by that name, see also #18 on the same page.)

3. Suppose $X_1, \ldots, X_n \sim \text{i.i.d.}\, \text{Poisson}(\lambda)$.

   (a) Find the likelihood function induced by observation of the values of $X_1, \ldots, X_n$. Then find the maximum likelihood estimator of λ based on that observation.

   (b) Find the likelihood function induced by observation of the value of $X_1 + \cdots + X_n$. Then find the maximum likelihood estimator of λ based on that observation. Comment on similarities and differences between the answers to (a) and (b).

4. DeGroot & Schervish, page 363, #7.

5. DeGroot & Schervish, page 369, #7.

6. DeGroot & Schervish, page 376, #3.

7. DeGroot & Schervish, page 382, #1.

8. Suppose $\theta \in \{2, 3\}$, i.e., it is known that either $\theta = 2$ or $\theta = 3$, but it is not known which. Suppose $f_X(x) = [\text{constant}] \cdot e^{-|x-\theta|}$. Let

$$T(X) = \begin{cases} 2 & \text{if } X \le 2, \\ X & \text{if } 2 < X < 3, \\ 3 & \text{if } X \ge 3. \end{cases}$$

   Show that $T(X)$ is a sufficient statistic for θ.