Clarification of part of the 4th problem set

Redo the problem with this in mind, and turn it in on Monday, Session #18.

7. Suppose $X_1, X_2, X_3 \sim \text{i. i. d. Uniform}(0, \theta)$. By using a Fisher factorization, we showed in class that $\max\{X_1, X_2, X_3\}$ is a sufficient statistic for θ. I have decided to hand you the following fact free of charge rather than have you work out the details: The conditional distribution of X_1, X_2 given that $X_3 = \max\{X_1, X_2, X_3\}$ is given by

$$X_1, X_2 \mid [X_3 = \max\{X_1, X_2, X_3\}] \sim \underbrace{\text{i. i. d. Uniform}(0, X_3)}_{\uparrow \ \boxed{\text{No “}\theta\text{” appears here.}}} .$$

CLARIFICATION: I originally intended the conditional distribution of X_1, X_2 given, not just the event that $X_3 = \max$, but also the actual value of X_3. Therefore I should have said:

$$\begin{aligned} X_1, X_2 \mid [X_3 = \max,\ \max] &\sim \overbrace{\text{i. i. d. Uniform}(0, \max)}^{\boxed{\text{No “}\theta\text{” appears here.}} \downarrow} \\ &= \text{i. i. d. Uniform}(0, \max\{X_1, X_2, X_3\}) \\ &= \text{i. i. d. Uniform}(0, X_3). \end{aligned}$$

(The absence of “θ” from this distribution is what sufficiency is.) We also observed that

$$2\overline{X} = 2 \cdot \frac{X_1 + X_2 + X_3}{3}$$

is an unbiased estimator of θ.

(a) Use the case $X_1 = 2, X_2 = 1, X_3 = 12$ to explain why $2\overline{X}$ is a flawed estimator of θ. (This is no different from what we did in class on March 4th, so don't work too hard on this part.)

(b) Find $\mathbf{Pr}(X_3 = \max\{X_1, X_2, X_3\})$.

(c) Use what is given above to find $\mathbf{E}(2\overline{X} \mid X_3 = \max\{X_1, X_2, X_3\})$

(d) Finally, find the Rao-Blackwell estimator $\mathbf{E}(2\overline{X} \mid \max\{X_1, X_2, X_3\})$.