9th problem set,

1. It is desirable to make widgets produced on an assembly line as nearly identical as possible. If $X_1, \ldots, X_n$ are the logarithms of the masses n randomly chosen widgets, and $X_1, \ldots, X_n \sim$ i. i. d. $N(\mu, \sigma^2)$, then the above desideratum means making σ as small as possible.

   Recall that the normal density is

$$\varphi_{\mu,\sigma^2}(x) = \text{constant} \cdot \frac{1}{\sigma} \exp\left(\frac{-1}{2} \left(\frac{x-\mu}{\sigma} \right)^2 \right)$$

   and that

$$\sum_{i=1}^{n} (x_i - \mu)^2 = n(\overline{x} - \mu)^2 + \underbrace{\sum_{i=1}^{n} (x_i - \overline{x})^2 = n\left[(\overline{x} - \mu)^2 + s^2 \right]}_{\uparrow\ \text{This defines } s^2.} .$$

   (a) Suppose $0 < \sigma_0 < \sigma_1$.

   In order to test the null hypothesis $H_0 : \sigma = \sigma_0$
   against the alternative hypothesis $H_1 : \sigma = \sigma_1$

   we have used the likelihood-ratio statistic

$$\Lambda(x_1, \ldots, x_n) = \frac{f_{X_1,\ldots,X_n}(x_1, \ldots, x_n \mid \sigma = \sigma_1)}{f_{X_1,\ldots,X_n}(x_1, \ldots, x_n \mid \sigma = \sigma_0)}.$$

   Show that $\Lambda(x_1, \ldots, x_n)$ is an increasing function of s^2.

   (b) Let H_0 and H_1 be as in part (a). Let capital S = the same function of $X_1, \ldots, X_n$ that lower-case s is of $x_1, \ldots, x_n$. Let $K = K(X_1, \ldots, X_n)$ be some other statistic. Suppose $\mathbf{Pr}(K > \kappa \mid H_0) = 0.03$ and $\mathbf{Pr}(\chi^2_{n-1} > \ell) = 0.03$. How do you know that the test that rejects H_0 if and only if $nS^2 > \ell$ is at least as powerful as the one that rejects H_0 if and only if $K > \kappa$?

   (c) Suppose $0 < \sigma_0 < \sigma_1$. Suppose $K = K(X_1, \ldots, X_n)$ is some statistic that is worthless for testing hypotheses about the value of σ because the probability distribution of K in no way depends on σ. Suppose $\mathbf{Pr}(K > \kappa) \leq 0.03$. We indicate the lack of dependence on σ by saying

   "$\alpha = \mathbf{Pr}_\sigma(K > \kappa)$ is the same regardless of the value of σ."

   Choose m so that $\mathbf{Pr}(\chi^2_{n-1} > m) = \alpha$. Pretend you are completely ignorant of the nature of the chi-square distribution, you know nothing about the distribution of S^2 except that, as it says in parts (a) and (b) above, S^2 is an increasing function of Λ, and you otherwise know the results of parts (a) and (b) above, and of #12 on the 8th problem set. Explain how, in this state of ignorance, you would justify each step labelled with a "?" below.

$$P_{\sigma_0}(nS^2 > \ell) \overset{\boxed{?}\atop\downarrow}{=} P_{\sigma_0}(K > \kappa) \overset{\boxed{?}\atop\downarrow}{=} P_{\sigma_1}(K > \kappa) \overset{\boxed{?}\atop\downarrow}{\leq} P_{\sigma_1}(nS^2 > \ell).$$

(Summary: $P_\sigma(nS^2 > \ell)$ is an increasing function of σ.)

(d) Suppose the null and alternative hypotheses are:

$$\begin{aligned} H_0 : \sigma &\leq 1, \\ H_1 : \sigma &> 1. \end{aligned}$$

Observe that these hypotheses make sense in our assembly-line scenario. Let K be some statistic such that $\mathbf{Pr}_\sigma(K > \kappa) \leq 0.03$ whenever $\sigma \leq 1$. Show that for <u>any</u> $\sigma_1 > 1$ we

↑ "≤", not "=".

have $\mathbf{Pr}_{\sigma_1}(nS^2 > \ell) \geq \mathbf{Pr}_{\sigma_1}(K > \kappa)$, i.e., the test based on S^2 is at least as powerful as the test based on K.

2. DeGroot & Schervish, p. 541, #4.

3. DeGroot & Schervish, p. 548, #2.

4. DeGroot & Schervish, p. 672, #5.

5. Recall that for $0 < p < 1$ we have $\text{logit}(p) = \log \dfrac{p}{1-p}$. Suppose $X \mid [\mu = \mu_i] \sim N(\mu_i, 1^2)$ for $i = 1, 2$. A prior probability distribution is assigned to μ, so that $\mathbf{Pr}(\mu = \mu_1) + \mathbf{Pr}(\mu = \mu_2) = 1$. Show that for some A, B,

$$\text{logit}\,\mathbf{Pr}(\mu = \mu_1 \mid X = x) = Ax + B + \text{logit}\,\mathbf{Pr}(\mu = \mu_1),$$

i.e., the logit of the posterior probability is some function of x whose graph is a straight line plus the logit of the prior probability. Find the values of A and B.