MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Department of Electrical Engineering and Computer Science
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**6.050J/2.110J** **Information and Entropy** **Spring 2003**

# Quiz Solutions

## Solution to Problem 1: Digital Devices (20%)

From the description, we know that $X = AND(A,B)$ and $Y = OR(A,B)$.

### Solution to Problem 1, part a.

The transition diagram of this gate is shown in the diagram below.

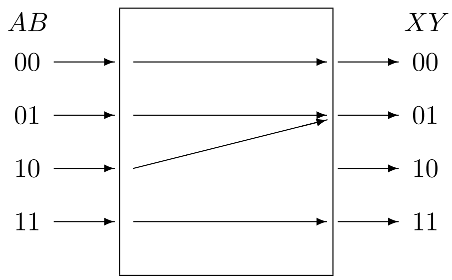


Figure Q–3: The $NEW$ gate, with transition arrows

### Solution to Problem 1, part b.

This gate is lossy because both inputs $AB = 01$ and 10 are transformed into output 01. To make this gate lossless, we can define the input probabilities of 01 or 10 to be 0, or both.

Constraints $p(01)p(10) = 0$

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## Solution to Problem 2: The Price is Right (40%)

### Solution to Problem 2, part a.

The uncertainty is defined as

$$\begin{aligned} U &= \sum_i p_i \log_2\left(\frac{1}{p_i}\right) \\ &= 3\left(0.33\log_2\left(\frac{1}{0.33}\right)\right) \\ &= \log_2\left(\frac{1}{0.33}\right) \\ &= 1.58 \text{ bits} \end{aligned} \tag{Q–1}$$

$U =$ 1.58 bits

## Solution to Problem 2, part b.

We know that the average rate of \$150 is between \$300 and the two lower values, \$50 and \$100. Therefore we know we can make the probabilities of either the group or individual rates zero and still achieve the average. Thus $G_{min} = I_{min} = 0$. If there are no group rates, the average is achieved by setting $I = 3/4$ and $C = 1/4$. Reducing $C$ will reduce the average, and make $I$ larger or $G$ non-zero. But this will only reduce the average further. Therefore this is the minimum value for $C$. The same is true when we try and increase $I$: $C$ will get smaller, lowering the average. Thus we have the maximum value for $I$.

If there are no individual rates, the average is achieved by setting $G = 3/5$ and $C = 2/5$. If we try to increase $C$ this will raise the average. But it will also decrease $G$, raising the average further. Thus this is the maximum value for $C$. Same for $G$: if we try to increase $G$, this will lower the average, and will also lower $C$, which will lower the average further. So we have the maximum value of $G$.

$I_{min}$ 0 $I_{max}$ 3/4 $G_{min}$ 0 $G_{max}$ 3/5 $C_{min}$ 1/4 $C_{max}$ 2/5

## Solution to Problem 2, part c.

Our constraints are as follows:

$$G + I + C = 1 \tag{Q–2}$$

$$\$50G + \$100I + \$300C = \$150 \tag{Q–3}$$

The procedure is to rearrange Equation Q–2

$$G = 1 - I - C \tag{Q–4}$$

$$I = 1 - G - C \tag{Q–5}$$

$$C = 1 - G - I \tag{Q–6}$$

and substitute into Equation Q–3, producing three equations with two variables

$$\$50I + \$250C = \$100 \tag{Q–7}$$

$$\$200C - \$50G = \$50 \tag{Q–8}$$

$$\$250G + \$200I = \$150 \tag{Q–9}$$

which can then be rearranged into six equations casting each variable as a function of the other two

$$G = 4C - 1 \tag{Q–10}$$

$$G = 0.6 - 0.8I \tag{Q–11}$$

$$I = 2 - 5C \tag{Q–12}$$

$$I = 0.75 - 1.25G \tag{Q–13}$$

$$C = 0.4 - 0.2I \tag{Q–14}$$

$$C = 0.25 + 0.25G \tag{Q–15}$$

(Q–16)

these equations can then be substituted into the equation for entropy as appropriate

$$S = G\log_2\left(\frac{1}{G}\right) + I\log_2\left(\frac{1}{I}\right) + C\log_2\left(\frac{1}{C}\right) \tag{Q–17}$$

The following three equations are solutions

$S(G) =$ $G\log_2\left(\frac{1}{G}\right) + (0.75 - 1.25G)\log_2\left(\frac{1}{0.75-1.25G}\right) + (0.25 + 0.25G)\log_2\left(\frac{1}{0.25+0.25G}\right)$

$S(I) =$ $(0.6 - 0.8I)\log_2\left(\frac{1}{0.6-0.8I}\right) + I\log_2\left(\frac{1}{I}\right) + (0.4 - 0.2I)\log_2\left(\frac{1}{0.4-0.2I}\right)$

$S(C) =$ $(4C - 1)\log_2\left(\frac{1}{4C-1}\right) + (2 - 5C)\log_2\left(\frac{1}{2-5C}\right) + C\log_2\left(\frac{1}{C}\right)$

### Solution to Problem 2, part d.

We know that $C_{min} = C_{max} = 0.30$. From Equations Q–13 and Q–15 we know that $G$ and $I$ are fully determined, and their 'max' and 'min' values are equal. Calculating from Equations Q–10 and Q–12 we have

$G_{min}$ 0.2 $G_{max}$ 0.2 $I_{min}$ 0.5 $I_{max}$ 0.5

### Solution to Problem 2, part e.

We can calculate the entropy by substitution directly into the equation for entropy, Equation Q–17. Since we were given $C = 0.30$, let's calculate the entropy using Equation .

$$\begin{aligned} S &= G\log_2\left(\frac{1}{G}\right) + I\log_2\left(\frac{1}{I}\right) + C\log_2\left(\frac{1}{C}\right) \\ S &= 0.2\log_2\left(\frac{1}{0.2}\right) + 0.5\log_2\left(\frac{1}{0.5}\right) + 0.3\log_2\left(\frac{1}{0.3}\right) \\ S &= 0.2 \times 2.32 + 0.5 \times 1 + 0.3 \times 1.58 \\ &= 0.464 + 0.5 + 0.474 \\ &= 1.49 \text{ bits} \end{aligned} \tag{Q–18}$$

$S =$ 1.49 bits

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## Solution to Problem 3: Corporate Pricing Procedure (40%)

### Solution to Problem 3, part a.

Before Betty changed the software the probabilities were $p(R_G) = 0.25$ $p(R_I) = 0.50$, and $p(R_C) = 0.25$. After the change, half of the individual rates get changed to corporate rates. Thus the probability of a group rate is unchanged, and half of the probability for the individual rates gets added to the corporate rate.

$p(R_G) =$ 0.25 $p(R_I) =$ 0.25 $p(R_C) =$ 0.5

### Solution to Problem 3, part b.

The equation for calculating the average is

$$\begin{aligned} R_{avg} &= \sum_i p(R_i)R_i \\ &= p(R_G)R_G + p(R_I)R_I + p(R_C)R_C \end{aligned} \tag{Q–19}$$

Before the change we had the following probabilities:

$$
\begin{aligned}
R_{avg} &= \$50 \times 0.25 + \$100 \times 0.50 + \$300 \times 0.25 \\
&= \$12.50 + \$50.00 + \$75.00 \\
&= \$137.50 \qquad \text{(Q–20)}
\end{aligned}
$$

After the change we had the following probabilities:

$$
\begin{aligned}
R_{avg} &= \$50 \times 0.25 + \$100 \times 0.25 + \$300 \times 0.50 \\
&= \$12.50 + \$25.00 + \$150.00 \\
&= \$187.50 \qquad \text{(Q–21)}
\end{aligned}
$$

Before $137.50 After $187.5

## Solution to Problem 3, part c.

The transition probabilities are as follows

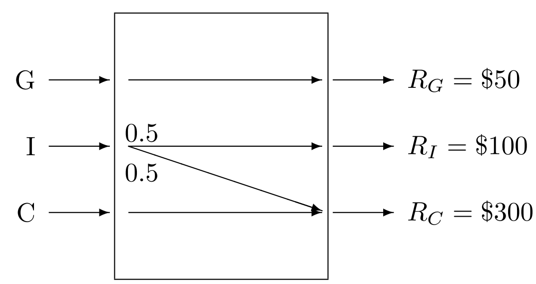


Figure Q–4: Fly-By-Night Transition Diagram

## Solution to Problem 3, part d.

We know if you were charged $50 there was a 100% chance that you asked for a group rate. Thus the first column of the table is 1 0 0. If you were charged a $100 there was also a 100% chance that you asked for an individual rate. So the second column of the table is 0 1 0. The only uncertainty comes when you are charged $350. In this case, half of the people (0.25 out of 0.5) asked for corporate rates, and the other half asked for individual rates. Thus the column reads 0.5 0 0.5

| $R =$ | $50 | $100 | $300 |
|---|---|---|---|
| $p(G \mid R)$ | 1 | 0 | 0 |
| $p(I \mid R)$ | 0 | 1 | 0.5 |
| $p(C \mid R)$ | 0 | 0 | 0.5 |

Table Q–3: Fly-By-Night Customer to Price Transition Table Solution

## Solution to Problem 3, part e.

The input information is given by

$$\begin{aligned} I_{in} &= G\log_2\left(\frac{1}{G}\right) + I\log_2\left(\frac{1}{I}\right) + C\log_2\left(\frac{1}{C}\right) \\ &= 0.25\log_2\left(\frac{1}{0.25}\right) + 0.5\log_2\left(\frac{1}{0.5}\right) + 0.25\log_2\left(\frac{1}{0.25}\right) \\ &= 0.25\times 2 + 0.5\times 1 + 0.25\times 2 \\ &= 1.5 \text{ bits} \end{aligned} \qquad \text{(Q–22)}$$

The output information is the same

$$\begin{aligned} I_{out} &= p(R_G)\log_2\left(\frac{1}{p(R_G)}\right) + p(R_I)\log_2\left(\frac{1}{p(R_I)}\right) + p(R_C)\log_2\left(\frac{1}{p(R_C)}\right) \\ &= 0.25\log_2\left(\frac{1}{0.25}\right) + 0.25\log_2\left(\frac{1}{0.25}\right) + 0.5\log_2\left(\frac{1}{0.5}\right) \\ &= 0.25\times 2 + 0.25\times 2 + 0.5\times 1 \\ &= 1.5 \text{ bits} \end{aligned} \qquad \text{(Q–23)}$$

Since the input and output information are equal, the noise and the loss must be the same, so we need only calculate one of them. Also, since we already have the forward joint probabilities in Table Q-3 we should calculate the loss. All entries except two are 0 or 1, meaning those terms will drop out.

$$\begin{aligned} L &= \sum_j p(B_j)\sum_i p(A_i \mid B_j)\log_2\left(\frac{1}{p(A_i \mid B_j)}\right) \\ &= p(R_C)\left(p(I \mid R_C)\log_2\left(\frac{1}{p(I \mid R_C)}\right) + p(C \mid R_C)\log_2\left(\frac{1}{p(C \mid R_C)}\right)\right) \\ &= 0.5\left(0.5\log_2\left(\frac{1}{0.5}\right) + 0.5\log_2\left(\frac{1}{0.5}\right)\right) \\ &= 0.5\times(0.5+0.5) \\ &= 0.5 \text{ bits} \end{aligned} \qquad \text{(Q–24)}$$

Finally, the mutual information is the input information minus the loss (or the output information minus the noise).

$I_{in} =$ 1.5 bits $I_{out} =$ 1.5 bits $N =$ 0.5 bits $L =$ 0.5 bits $M =$ 1