6.231 DYNAMIC PROGRAMMING

LECTURE 9

LECTURE OUTLINE

- Deterministic continuous-time optimal control
- Variants of the Pontryagin Minimum Principle
- Fixed terminal state
- Free terminal time
- Examples
- Discrete-Time Minimum Principle

REVIEW

- Continuous-time dynamic system

$$\dot{x}(t) = f\big(x(t), u(t)\big), \quad 0 \le t \le T, \quad x(0) : \text{given}$$

- Cost function

$$h\big(x(T)\big) + \int_0^T g\big(x(t), u(t)\big) dt$$

- $J^*(t, x)$: optimal cost-to-go from x at time t
- HJB equation/Verification theorem: For all (t, x)

$$0 = \min_{u \in U} \big[g(x, u) + \nabla_t J^*(t, x) + \nabla_x J^*(t, x)' f(x, u) \big]$$

with the boundary condition $J^*(T, x) = h(x)$.

- Adjoint equation/vector: To compute an optimal state-control trajectory $\{(u^*(t), x^*(t))\}$ it is enough to know

$$p(t) = \nabla_x J^*\big(t, x^*(t)\big), \qquad t \in [0, T].$$

- Pontryagin theorem gives an equation for $p(t)$.

NEC. CONDITION: PONTRYAGIN MIN. PRINCIPLE

- Define the Hamiltonian function

$$H(x,u,p) = g(x,u) + p'f(x,u).$$

- **Minimum Principle:** Let $\{u^*(t) \mid t \in [0,T]\}$ be an optimal control trajectory and let $\{x^*(t) \mid t \in [0,T]\}$ be the corresponding state trajectory. Let also $p(t)$ be the solution of the adjoint equation

$$\dot{p}(t) = -\nabla_x H\big(x^*(t), u^*(t), p(t)\big),$$

with the boundary condition

$$p(T) = \nabla h\big(x^*(T)\big).$$

Then, for all $t \in [0,T]$,

$$u^*(t) = \arg\min_{u \in U} H\big(x^*(t), u, p(t)\big).$$

Furthermore, there is a constant C such that

$$H\big(x^*(t), u^*(t), p(t)\big) = C, \qquad \text{for all } t \in [0,T].$$

VARIATIONS: FIXED TERMINAL STATE

- Suppose that in addition to the initial state $x(0)$, the final state $x(T)$ is given.
- Then the informal derivation of the adjoint equation still holds, but the terminal condition $J^*(T,x) \equiv h(x)$ of the HJB equation is not true anymore.
- In effect,

$$J^*(T,x) = \begin{cases} 0 & \text{if } x = x(T) \\ \infty & \text{otherwise.} \end{cases}$$

So $J^*(T,x)$ cannot be differentiated with respect to x, and the terminal boundary condition $p(T) = \nabla h\big(x^*(T)\big)$ for the adjoint equation does not hold.

- As compensation, we have the extra condition

$$x(T) : \text{given},$$

thus maintaining the balance between boundary conditions and unknowns.

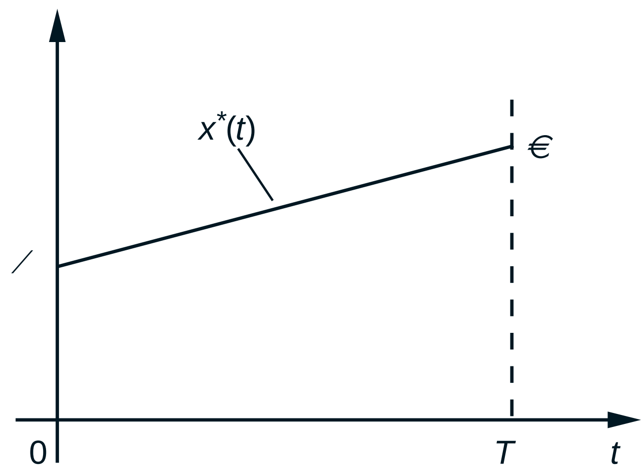
- Generalization: Some components of the terminal state are fixed.

EXAMPLE WITH FIXED TERMINAL STATE

- Consider finding the curve of minimum length connecting two points $(0, \alpha)$ and (T, β). We have

$$\dot{x}(t) = u(t), \qquad x(0) = \alpha, \qquad x(T) = \beta,$$

and the cost is $\int_0^T \sqrt{1 + \big(u(t)\big)^2}\, dt$.



- The adjoint equation is $\dot{p}(t) = 0$, implying that

$$p(t) = \text{constant}, \qquad \text{for all } t \in [0, T].$$

- Minimizing the Hamiltonian $\sqrt{1 + u^2} + p(t)u$:

$$u^*(t) = \text{constant}, \qquad \text{for all } t \in [0, T].$$

So optimal $\{x^*(t) \mid t \in [0, T]\}$ is a straight line.

VARIATIONS: FREE TERMINAL TIME

- Initial state and/or the terminal state are given, but the terminal time T is subject to optimization.

- Let $\big\{\big(x^*(t), u^*(t)\big) \mid t \in [0, T]\big\}$ be an optimal state-control trajectory pair and let T^* be the optimal terminal time. Then $x^*(t), u^*(t)$ would still be optimal if T were fixed at T^*, so

$$u^*(t) = \arg\min_{u \in U} H\big(x^*(t), u, p(t)\big), \quad \text{for all } t \in [0, T^*]$$

where $p(t)$ is given by the adjoint equation.

- In addition: $H(x^*(t), u^*(t), p(t)) = 0$ for all t [instead of $H(x^*(t), u^*(t), p(t)) \equiv$ constant].

- Justification: We have

$$\nabla_t J^*\big(t, x^*(t)\big)\Big|_{t=0} = 0$$

Along the optimal, the HJB equation is

$$\nabla_t J^*\big(t, x^*(t)\big) = -H\big(x^*(t), u^*(t), p(t)\big), \quad \text{for all } t$$

so $H\big(x^*(0), u^*(0), p(0)\big) = 0$.

MINIMUM-TIME EXAMPLE I

- Unit mass moves horizontally: $\ddot{y}(t) = u(t)$, where $y(t)$: position, $u(t)$: force, $u(t) \in [-1, 1]$.
- Given the initial position-velocity $(y(0), \dot{y}(0))$, bring the object to $(y(T), \dot{y}(T)) = (0, 0)$ so that the time of transfer is minimum. Thus, we want to

$$\text{minimize } T = \int_0^T 1 dt.$$

- Let the state variables be

$$x_1(t) = y(t), \qquad x_2(t) = \dot{y}(t),$$

so the system equation is

$$\dot{x}_1(t) = x_2(t), \qquad \dot{x}_2(t) = u(t).$$

- Initial state $\big(x_1(0), x_2(0)\big)$: given and

$$x_1(T) = 0, \qquad x_2(T) = 0.$$

MINIMUM-TIME EXAMPLE II

- If $\{u^*(t) \mid t \in [0,T]\}$ is optimal, $u^*(t)$ must minimize the Hamiltonian for each t, i.e.,

$$u^*(t) = \arg \min_{-1 \le u \le 1} \big[1 + p_1(t)x_2^*(t) + p_2(t)u\big].$$

Therefore

$$u^*(t) = \begin{cases} 1 & \text{if } p_2(t) < 0, \\ -1 & \text{if } p_2(t) \ge 0. \end{cases}$$

- The adjoint equation is

$$\dot{p}_1(t) = 0, \qquad \dot{p}_2(t) = -p_1(t),$$

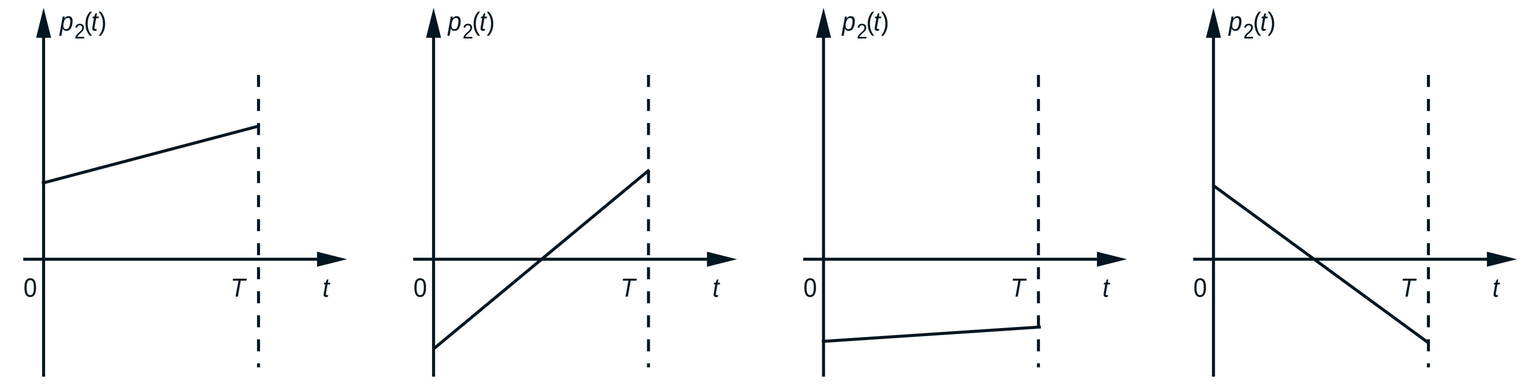
so

$$p_1(t) = c_1, \qquad p_2(t) = c_2 - c_1 t,$$

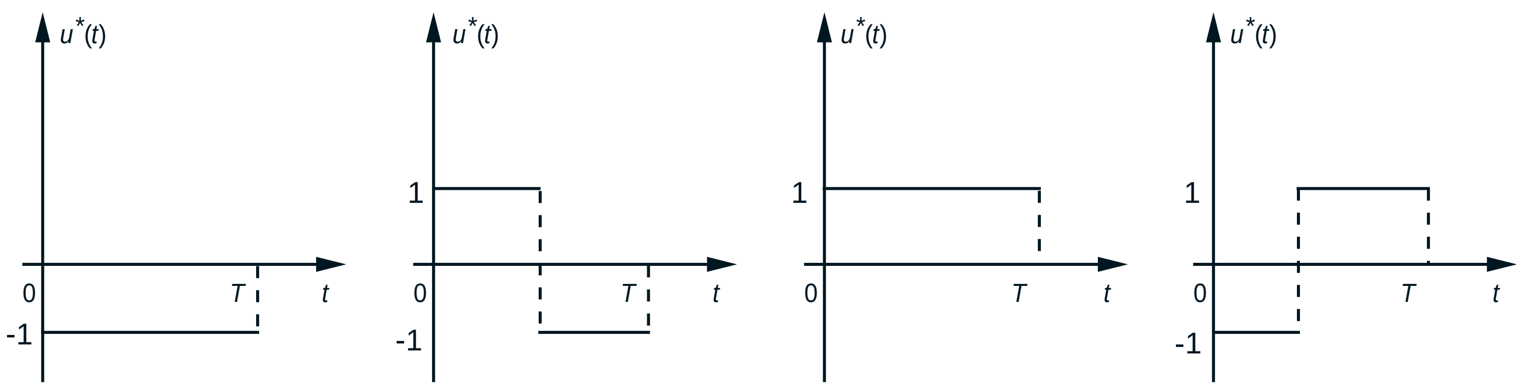
where c_1 and c_2 are constants.

- So $\{p_2(t) \mid t \in [0,T]\}$ switches at most once in going from negative to positive or reversely.

MINIMUM-TIME EXAMPLE III



(a)



(b)

- For $u(t) \equiv \zeta$, where $\zeta = \pm 1$, the system evolves according to

$$x_1(t) = x_1(0) + x_2(0)t + \frac{\zeta}{2}t^2, \qquad x_2(t) = x_2(0) + \zeta t.$$

Eliminating the time t, we see that for all t

$$x_1(t) - \frac{1}{2\zeta}\big(x_2(t)\big)^2 = x_1(0) - \frac{1}{2\zeta}\big(x_2(0)\big)^2.$$

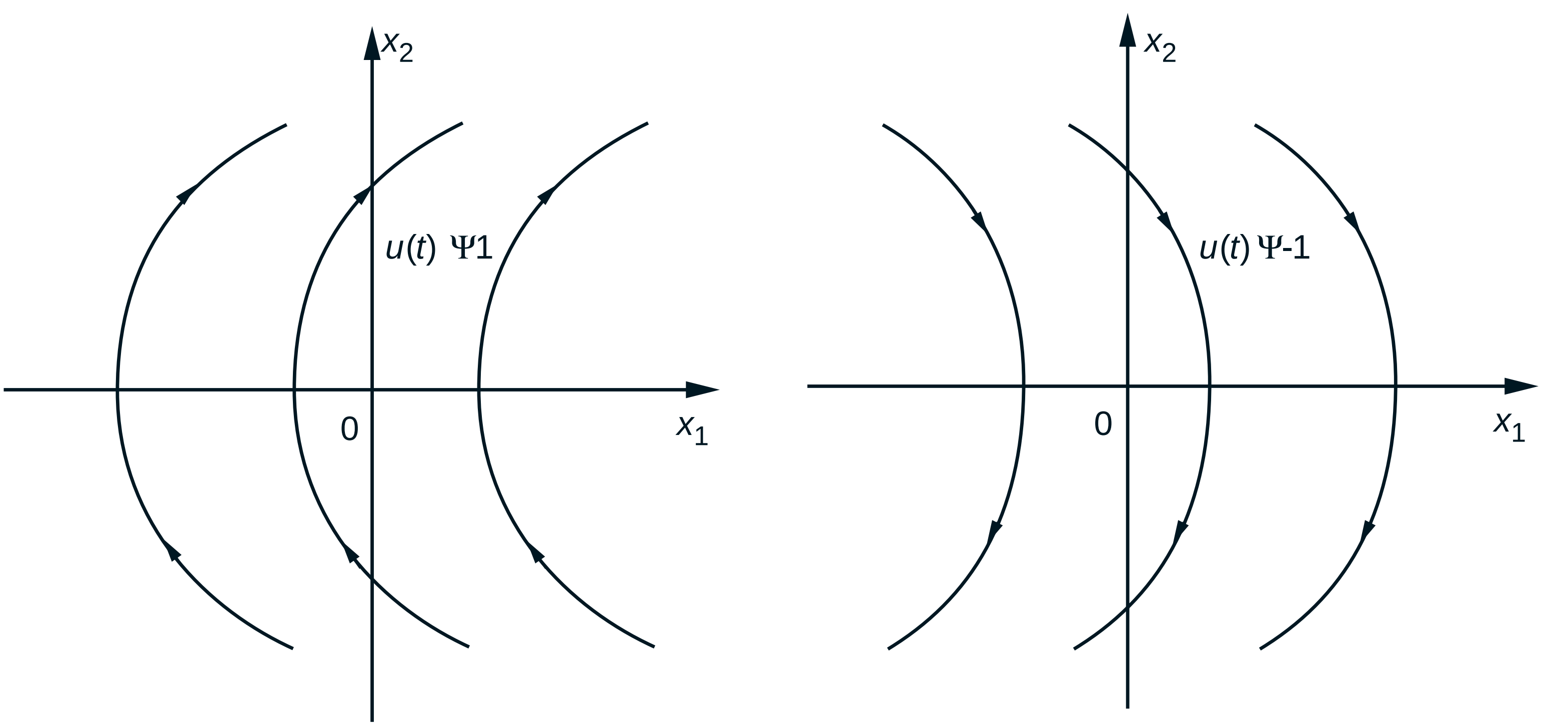
MINIMUM-TIME EXAMPLE IV

- For intervals where $u(t) \equiv 1$, the system moves along the curves

$$x_1(t) - \frac{1}{2}\big(x_2(t)\big)^2 : \text{ constant.}$$

- For intervals where $u(t) \equiv -1$, the system moves along the curves

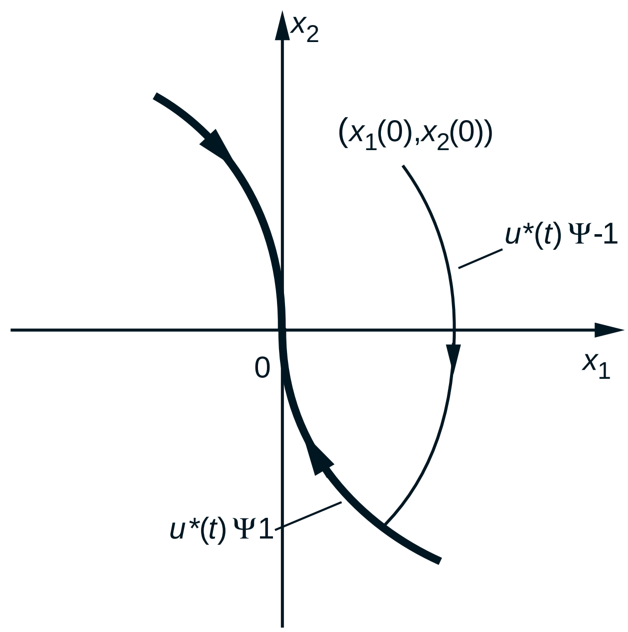
$$x_1(t) + \frac{1}{2}\big(x_2(t)\big)^2 : \text{ constant.}$$



(a) (b)

MINIMUM-TIME EXAMPLE V

- To bring the system from the initial state $x(0)$ to the origin with at most one switch, we use the following switching curve.



(a) If the initial state lies *above* the switching curve, use $u^*(t) \equiv -1$ until the state hits the switching curve; then use $u^*(t) \equiv 1$.

(b) If the initial state lies *below* the switching curve, use $u^*(t) \equiv 1$ until the state hits the switching curve; then use $u^*(t) \equiv -1$.

(c) If the initial state lies on the top (bottom) part of the switching curve, use $u^*(t) \equiv -1$ [$u^*(t) \equiv 1$, respectively].

DISCRETE-TIME MINIMUM PRINCIPLE

- Minimize $J(u) = g_N(x_N) + \sum_{k=0}^{N-1} g_k(x_k, u_k)$, subject to $u_k \in U_k \subset \Re^m$, with U_k: convex, and

$$x_{k+1} = f_k(x_k, u_k), \quad k = 0, \ldots, N-1, \quad x_0 : \text{ given.}$$

- Introduce Hamiltonian function

$$H_k(x_k, u_k, p_{k+1}) = g_k(x_k, u_k) + p'_{k+1} f_k(x_k, u_k)$$

- Suppose $\{(u_k^*, x_{k+1}^*) \mid k = 0, \ldots, N-1\}$ are optimal. Then for all k,

$$\nabla_{u_k} H_k\big(x_k^*, u_k^*, p_{k+1}\big)'(u_k - u_k^*) \geq 0, \quad \text{for all } u_k \in U_k,$$

where $p_1, \ldots, p_N$ are obtained from

$$p_k = \nabla_{x_k} f_k \cdot p_{k+1} + \nabla_{x_k} g_k,$$

with the terminal condition $p_N = \nabla g_N(x_N^*)$.

- If, in addition, the Hamiltonian H_k is a convex function of u_k for any fixed x_k and p_{k+1}, we have

$$u_k^* = \arg \min_{u_k \in U_k} H_k\big(x_k^*, u_k, p_{k+1}\big), \qquad \text{for all } k.$$

DERIVATION

- We develop an expression for the gradient $\nabla J(u)$. We have, using the chain rule,

$$\nabla_{u_k} J(u) = \nabla_{u_k} f_k \cdot \nabla_{x_{k+1}} f_{k+1} \cdots \nabla_{x_{N-1}} f_{N-1} \cdot \nabla g_N$$
$$+ \nabla_{u_k} f_k \cdot \nabla_{x_{k+1}} f_{k+1} \cdots \nabla_{x_{N-2}} f_{N-2} \cdot \nabla_{x_{N-1}} g_{N-1}$$
$$\cdots$$
$$+ \nabla_{u_k} f_k \cdot \nabla_{x_{k+1}} g_{k+1}$$
$$+ \nabla_{u_k} g_k,$$

where all gradients are evaluated along u and the corresponding state trajectory.

- Iintroduce the discrete-time adjoint equation

$$p_k = \nabla_{x_k} f_k \cdot p_{k+1} + \nabla_{x_k} g_k, \qquad k = 1, \ldots, N-1,$$

with terminal condition $p_N = \nabla g_N$.

- Verify that, for all k,

$$\nabla_{u_k} J(u_0, \ldots, u_{N-1}) = \nabla_{u_k} H_k(x_k, u_k, p_{k+1})$$