6.231 DYNAMIC PROGRAMMING

LECTURE 12

LECTURE OUTLINE

- DP for imperfect state info
- Sufficient statistics
- Conditional state distribution as a sufficient statistic
- Finite-state systems
- Examples

REVIEW: PROBLEM WITH IMPERFECT STATE INFO

- Instead of knowing x_k, we receive observations

$$z_0 = h_0(x_0, v_0), \quad z_k = h_k(x_k, u_{k-1}, v_k), \ \ k \geq 0$$

- I_k: information vector available at time k:

$$I_0 = z_0, \ I_k = (z_0, z_1, \ldots, z_k, u_0, u_1, \ldots, u_{k-1}), \ k \geq 1$$

- Optimization over policies $\pi = \{\mu_0, \mu_1, \ldots, \mu_{N-1}\}$, where $\mu_k(I_k) \in U_k$, for all I_k and k.
- Find a policy π that minimizes

$$J_\pi = \mathop{E}_{\substack{x_0, w_k, v_k \\ k=0,\ldots,N-1}} \left\{ g_N(x_N) + \sum_{k=0}^{N-1} g_k\big(x_k, \mu_k(I_k), w_k\big) \right\}$$

subject to the equations

$$x_{k+1} = f_k\big(x_k, \mu_k(I_k), w_k\big), \qquad k \geq 0,$$

$$z_0 = h_0(x_0, v_0), \ z_k = h_k\big(x_k, \mu_{k-1}(I_{k-1}), v_k\big), \ k \geq 1$$

DP ALGORITHM

- DP algorithm:

$$J_k(I_k) = \min_{u_k \in U_k} \Big[\mathop{E}_{x_k,\, w_k,\, z_{k+1}} \Big\{ g_k(x_k, u_k, w_k) + J_{k+1}(I_k, z_{k+1}, u_k) \mid I_k, u_k \Big\} \Big]$$

for $k = 0, 1, \ldots, N-2$, and for $k = N-1$,

$$J_{N-1}(I_{N-1}) = \min_{u_{N-1} \in U_{N-1}} \left[\mathop{E}_{x_{N-1},\, w_{N-1}} \Big\{ g_N\big(f_{N-1}(x_{N-1}, u_{N-1}, w_{N-1})\big) + g_{N-1}(x_{N-1}, u_{N-1}, w_{N-1}) \mid I_{N-1}, u_{N-1} \Big\} \right]$$

- The optimal cost J^* is given by

$$J^* = \mathop{E}_{z_0}\Big\{ J_0(z_0) \Big\}.$$

SUFFICIENT STATISTICS

- Suppose that we can find a function $S_k(I_k)$ such that the right-hand side of the DP algorithm can be written in terms of some function H_k as

$$\min_{u_k \in U_k} H_k\bigl(S_k(I_k), u_k\bigr).$$

- Such a function S_k is called a *sufficient statistic*.
- An optimal policy obtained by the preceding minimization can be written as

$$\mu_k^*(I_k) = \overline{\mu}_k\bigl(S_k(I_k)\bigr),$$

where $\overline{\mu}_k$ is an appropriate function.

- Example of a sufficient statistic: $S_k(I_k) = I_k$
- Another important sufficient statistic

$$S_k(I_k) = P_{x_k \mid I_k}$$

DP ALGORITHM IN TERMS OF $P_{X_K|I_K}$

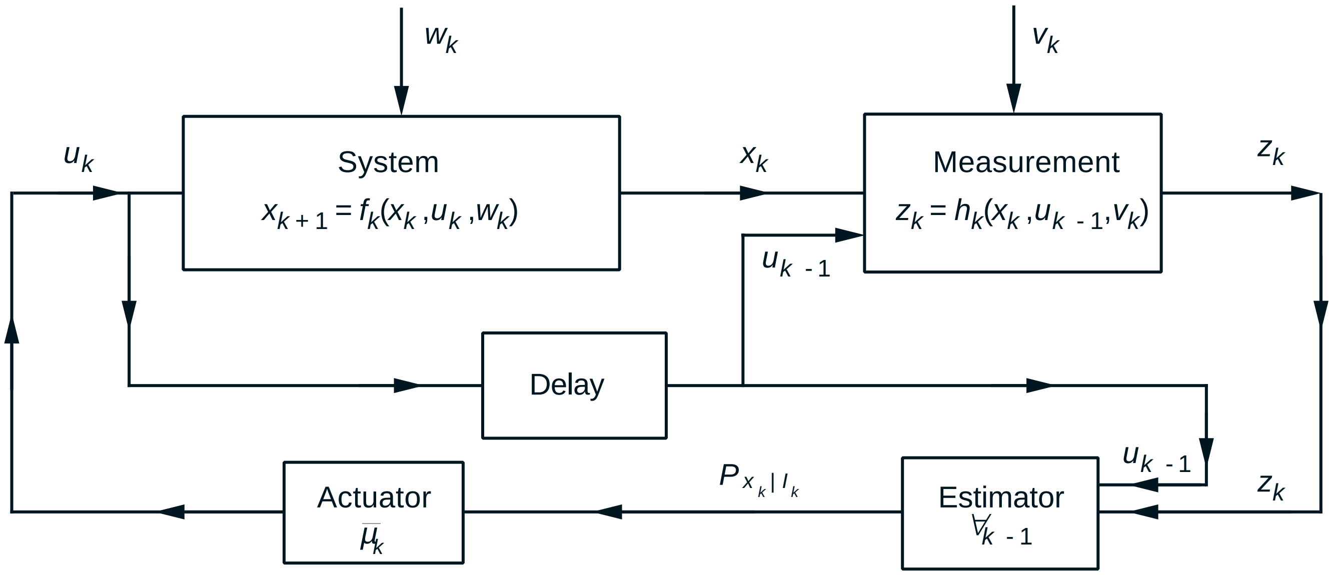
- It turns out that $P_{x_k|I_k}$ is generated recursively by a dynamic system (estimator) of the form

$$P_{x_{k+1}|I_{k+1}} = \Phi_k\big(P_{x_k|I_k}, u_k, z_{k+1}\big)$$

for a suitable function Φ_k

- DP algorithm can be written as

$$\overline{J}_k(P_{x_k|I_k}) = \min_{u_k \in U_k} \Big[\underset{x_k, w_k, z_{k+1}}{E} \big\{ g_k(x_k, u_k, w_k) + \overline{J}_{k+1}\big(\Phi_k(P_{x_k|I_k}, u_k, z_{k+1})\big) \mid I_k, u_k \big\} \Big]$$



EXAMPLE: A SEARCH PROBLEM

- At each period, decide to search or not search a site that may contain a treasure.
- If we search and a treasure is present, we find it with prob. β and remove it from the site.
- Treasure's worth: V. Cost of search: C
- States: treasure present & treasure not present
- Each search can be viewed as an observation of the state
- Denote

p_k : prob. of treasure present at the start of time k

with p_0 given.

- p_k evolves at time k according to the equation

$$p_{k+1} = \begin{cases} p_k & \text{if not search,} \\ 0 & \text{if search and find treasure,} \\ \frac{p_k(1-\beta)}{p_k(1-\beta)+1-p_k} & \text{if search and no treasure.} \end{cases}$$

SEARCH PROBLEM (CONTINUED)

- DP algorithm

$$\overline{J}_k(p_k) = \max\Bigg[0,\ -C + p_k\beta V + (1 - p_k\beta)\overline{J}_{k+1}\left(\frac{p_k(1-\beta)}{p_k(1-\beta) + 1 - p_k}\right)\Bigg],$$

with $\overline{J}_N(p_N) = 0$.

- Can be shown by induction that the functions $\overline{J}_k$ satisfy

$$\overline{J}_k(p_k) = 0, \qquad \text{for all } p_k \leq \frac{C}{\beta V}$$

- Furthermore, it is optimal to search at period k if and only if

$$p_k\beta V \geq C$$

(expected reward from the next search $\geq$ the cost of the search)

FINITE-STATE SYSTEMS

- Suppose the system is a finite-state Markov chain, with states $1, \ldots, n$.
- Then the conditional probability distribution $P_{x_k \mid I_k}$ is a vector

$$\bigl(P(x_k = 1 \mid I_k), \ldots, P(x_k = n \mid I_k)\bigr)$$

- The DP algorithm can be executed over the n-dimensional simplex (state space is not expanding with increasing k)
- When the control and observation spaces are also finite sets, it turns out that the cost-to-go functions $\overline{J}_k$ in the DP algorithm are piecewise linear and concave (Exercise 5.7).
- This is conceptually important and also (moderately) useful in practice.

INSTRUCTION EXAMPLE

- Teaching a student some item. Possible states are L: Item learned, or $\overline{L}$: Item not learned.

- Possible decisions: T: Terminate the instruction, or $\overline{T}$: Continue the instruction for one period and then conduct a test that indicates whether the student has learned the item.

- The test has two possible outcomes: R: Student gives a correct answer, or $\overline{R}$: Student gives an incorrect answer.

- Probabilistic structure

$$\begin{tikzcd}
L \arrow[r, "1"] & L \arrow[r, "1"] & R \\
\overline{L} \arrow[ur, "t"'] \arrow[r, "1-t"'] & \overline{L} \arrow[ur, "r"'] \arrow[r, "1-r"'] & \overline{R}
\end{tikzcd}$$

- Cost of instruction is I per period

- Cost of terminating instruction; 0 if student has learned the item, and $C > 0$ if not.

INSTRUCTION EXAMPLE II

- Let p_k: prob. student has learned the item given the test results so far

$$p_k = P(x_k|I_k) = P(x_k = L \,|\, z_0, z_1, \ldots, z_k).$$

- Using Bayes' rule we can obtain

$$\begin{aligned} p_{k+1} &= \Phi(p_k, z_{k+1}) \\ &= \begin{cases} \frac{1-(1-t)(1-p_k)}{1-(1-t)(1-r)(1-p_k)} & \text{if } z_{k+1} = R, \\ 0 & \text{if } z_{k+1} = \overline{R}. \end{cases} \end{aligned}$$

- DP algorithm:

$$\overline{J}_k(p_k) = \min \left[(1-p_k)C,\ I + \mathop{E}_{z_{k+1}} \left\{ \overline{J}_{k+1}\Big(\Phi(p_k, z_{k+1})\Big) \right\} \right].$$

starting with

$$\overline{J}_{N-1}(p_{N-1}) = \min\big[(1-p_{N-1})C,\ I+(1-t)(1-p_{N-1})C\big].$$

INSTRUCTION EXAMPLE III

- Write the DP algorithm as

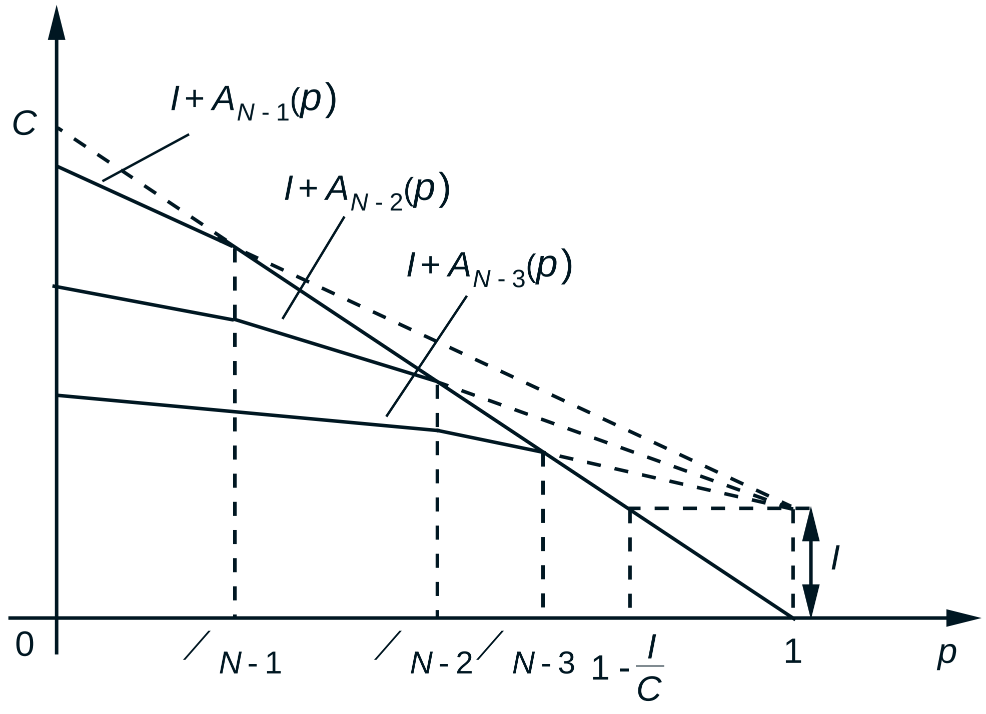
$$\overline{J}_k(p_k) = \min\big[(1-p_k)C,\, I + A_k(p_k)\big],$$

where

$$A_k(p_k) = P(z_{k+1} = R \mid I_k)\overline{J}_{k+1}\big(\Phi(p_k, R)\big) + P(z_{k+1} = \overline{R} \mid I_k)\overline{J}_{k+1}\big(\Phi(p_k, \overline{R})\big)$$

- Can show by induction that $A_k(p)$ are piecewise linear, concave, monotonically decreasing, with

$$A_{k-1}(p) \le A_k(p) \le A_{k+1}(p), \qquad \text{for all } p \in [0,1].$$



$I + A_{N-1}(p)$
$I + A_{N-2}(p)$
$I + A_{N-3}(p)$
C
I
0
α_{N-1}
α_{N-2}
α_{N-3}
$1 - \frac{I}{C}$
1
p