

ON SLOWLY VARYING STOKES WAVES

BY

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ABSTRACT

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In this thesis investigations are made on the theory of a train of slowly modulated gravity waves propagating over uneven bottom topography. The primary object is to study the interplay of amplitude dispersion the frequency dispersion in waves on the surface of water where the depth is not too shallow compared to a typical wave length.

The solution of the wave train is expressed in expansions of the WKB type with a small parameter which is proportional to the wave steepness and the rate of modulation. A systematic perturbation scheme that may be carried out for all orders is presented. It is found that new terms directly representing modulation rate must be included to extend the scope of the Whitham's theory based on an averaged variational principle. Several specific examples are discussed and the following main conclusions are reached:

1. The Stokes waves of constant amplitude are unstable for all depths under three dimensional disturbances.
2. For quasi-steady waves normally incident on a mild beach, the local rate of depth variation is found to affect the wave phase which in turn gives additional dispersion effect as compared to Stokes waves of constant amplitude.
3. In deep water, there exist solutions of permanent wave envelopes which represent exact balance between amplitude dispersion and frequency dispersion.
4. Numerical computation on the transient development of the wave envelopes reveals that any wave group deviated from permanent form will eventually disintegrate into a sequence of peaks separated by nodal points (where amplitude is zero) and locally the wave group tends to approach the dynamically stable permanent form.

It is believed that basic studies of this kind may ultimately be of importance to ocean engineering and oceanography as they permit a closer look at the significance of the classical Stokes waves which have led to an oversimplified picture of reality.

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CHAPTER 1

INTRODUCTION

Because of mathematical convenience, most engineering approaches to water wave problems are essentially based on linearized approximation, and the nonlinear effect is usually considered in an intuitive manner. For example, one method in calculating wave amplitude in coastal waters is to make use of the geometical optics approximation, i.e., to assume energy conservation between wave rays. The nonlinear effect is then taken into account later by using Stokes higher order solution as being valid locally. In conjunction with some other additional assumptions this method has been extensively used to study problems such as wave force on coastal structures, shoaling and breaking of large amplitude waves etc.. Although systematic perturbation calculation for Stokes waves of uniform amplitude have been advanced to fifth order and even to seventh order with aid of computers (see e.g. Ippen (1966), p.127) it has not been questioned until recently whether Stokes waves of uniform amplitude can really exist except under carefully controlled laboratory conditions. It is therefore important to understand more about Stokes waves before such efforts are pertinent.

1.1 SURVEY OF LITERATURE

Stokes (1847) was the first to consider the effects of nonlinearity on periodic progressive waves on waters where the depth is not small compared with a typical wave length. On the assumption that a finite

amplitude wave of uniform amplitude exists, he expressed successive order of approximate solutions by coefficients of a perturbation expansion of small parameter which is proportional to the wave steepness. In the field of water waves at least, this now familiar method of approximation traditionally bear his name. Solution of Stokes waves of constant amplitude has since carried out by Stokes, Rayleigh (1917) and others to fifth order. Existence proofs for Stokes waves of uniform amplitude have been given by Levi-Civita (1925), Struik (1926) and more recently by Kraskovskii (1960,1961).

One important consequence of the recent study of nonlinear effects in surface waves is the idea of radiation stress first proposed by Longuet-Higgins & Stewart (1962). Analogous to the Reynolds stress in the theory of turbulence, the radiation stress is defined as the mean rate of transport of momentum induced by the oscillating wave field. By taking averages over a wave length and wave period this stress may be calculated explicitly, correct to the second order, from the linearized plane wave solution. The concept of radiation stress has since been applied to study many other nonlinear phenomena in water waves, for example, the change of mean sea level due to storm waves; the interaction of waves with currents; the generation of sea waves, long shore and rip currents etc. (see e.g. Longuet-Higgins & Stewart 1961,1962,1964; Longuet-Higgins 1969; Bowen 1969).

The superposition of several Stokes wave trains of constant amplitude produces the so-called resonance at the third order, if the wave numbers satisfy certain condition. Such resonance phenomena may be studied by regarding the wave amplitude to be a slowly varying

function of time. This problem is of considerable importance in oceanography, for it deals with the mechanism of energy exchange among different parts of the spectrum (see Phillips 1966).

Under a wide range of circumstances, real fluid effects, which often constitute formidable obstacles in fluid mechanical problems, play only a secondary role in the propagation of water waves. Many intriguing features possessed by water waves are due to the fact that they are basically nonlinear and dispersive. It is therefore natural that when these two factors: nonlinearity (amplitude dispersion) and (frequency) dispersion, are in strong contention, the physics becomes quite complicated. In shallow waters it is well known that nonlinearity tends to steepen the wave and frequency dispersion tends to flatten it out, and the two can be in complete balance with each other, giving rise to wave of permanent form such as solitary and cnoidal waves. For water depth that is not small compared to the wave length this fact is less well known. According to the linear theory any wave train which is different from exactly sinusoidal form (e.g., a line spectrum) will disperse into sinusoidal components each of which propagates with different phase and group velocities. As a result the wave group spreads out; this is a manifestation of frequency dispersion. Therefore, the existence of Stokes waves with permanent form is merely an indication that amplitude dispersion and frequency dispersion can be exactly in balance. the separate question of their instability appears to have been ignored for a long time. It is Benjamin & Feir (1967) and Benjamin (1967) who discovered, both theoretically and experimentally, that the exact balance between nonlinearity and frequency dispersion in Stokes waves is

in fact unstable subject to certain side-band disturbances. Given a pair of side-band modes, with wave frequencies and wave numbers fractionally different from the fundamental frequency and wave number, the whole system grows at a rate exponential in time and distance. They also performed experiments in a long tank in which a uniform wave train is generated at one side of the wave tank; it was then found that far downstream the wave disintegrated in a rather chaotic manner. It is thus concluded that Stokes waves are unstable to certain side-band disturbances and they cannot propagate over long distances without changing form. This work was further generalized by Benney and Newell (1967) for general weakly nonlinear dispersive systems and by Benney and Roskes (1969) for the study of three dimensional instability of Stokes waves.

In the instability analysis of Stokes waves initial disturbances are assumed to be small compared with the wave steepness. Therefore the conclusions are valid only for the initial period of growth. To understand the subsequent development, and to predict perhaps the eventual fate, it is necessary to inquire into the nonlinear evolution. To study the slow modulation of nonlinear dispersive waves in general, Whitham (1962, 1965 a,b) introduced the method of averaging in which the "microscopic" wave field is locally approximated by a plane wave of slowly varying amplitude and phase. For nearly periodic wave trains he deduced the basic equations governing the modulation of the amplitude, the wave number, etc., with respect to both space and time. He further introduced the alternative of assuming an averaged Lagrangian and showed

that these equations followed from a variational principle (1967 a,b). While Whitham's heuristic approach has since found wide applications, its justification by formal perturbation scheme has also been given by Luke (1966) for a nonlinear Klein-Gorden equation and by Hoogstraten (1968, 1969) for both deep and shallow water waves. In Hoogstraten's work two ordering parameters characterizing the wave steepness and the modulation rate were distinguished and Whitham's results were rederived.

1.2 SCOPE OF PRESENT INVESTIGATION

The primary objective of this thesis is to apply a very general perturbation scheme so as to study the slow evolution of Stokes waves over an uneven bottom topography. Special attention is given to the derivation of modulation equations similar to those obtained by Whitham's averaging technique. The result of this general theory is then applied to the study of three specific examples. First, we study the instability of Stokes waves under three dimensional side-band disturbances from a view somewhat different from Benney and Roskes (1969). In the second example, we consider a quasi-steady wave train propagating over a sloping beach in which we extend the existing investigations by geometrical optics approximation and the modified Stokes wave solution. Finally the general nonlinear evolution of deep water wave envelopes is considered in quite extensive details in order to illucidate the post-instability development.

1.3 ELEMENTARY IDEAS OF SLOW MODULATION OF WAVES

In this section we wish to bring out some essential ideas by considering a linear dispersive example. A slowly modulating wave train

may be described in two different ways. It can be considered either as a superposition of plane periodic waves of nearly equal frequencies (i.e. a narrow-banded spectrum) or as a wave train with wave amplitude, wave number and wave frequency being slowly varying functions of space and time. As an example, let us consider the superposition of two sinusoidal wave trains with equal wave amplitudes α but slightly different wave frequencies ω_1, ω_2 and wave numbers k_1, k_2 :

$$\begin{aligned} \eta(x,t) &= \alpha \cos(k_1 x - \omega_1 t) + \alpha \cos(k_2 x - \omega_2 t) \\ &= 2\alpha \cos\left[\left(\frac{k_1 - k_2}{2}\right)x - \left(\frac{\omega_1 - \omega_2}{2}\right)t\right] \cos\left[\left(\frac{k_1 + k_2}{2}\right)x - \left(\frac{\omega_1 + \omega_2}{2}\right)t\right] \end{aligned} \quad (1.1)$$

This is the classical example of beats; the envelope is much longer than the typical wave length and travels at the group velocity (Figure 1). Now (1.1) may be regarded as a simple wave train with slowly changing amplitude and phase. Rewriting (1.1) as,

$$\eta(x,t) = a(x,t) \cos \psi = |A(x,t)| \cos \psi \quad (1.2)$$

where $a(x,t)$ = the amplitude ,

$$A(x,t) = 2\alpha \cos\left[\left(\frac{k_1 - k_2}{2}\right)x - \left(\frac{\omega_1 - \omega_2}{2}\right)t\right] \quad (1.3)$$

and ψ = the wave phase

$$= \left(\frac{k_1 + k_2}{2}\right)x - \left(\frac{\omega_1 + \omega_2}{2}\right)t + \pi H(-A) \quad (1.4)$$

where $H(-A)$ is the heaviside step function. Note that as the wave profile $\eta(x,t)$ varies from one maximum point to the next maximum, i.e.,

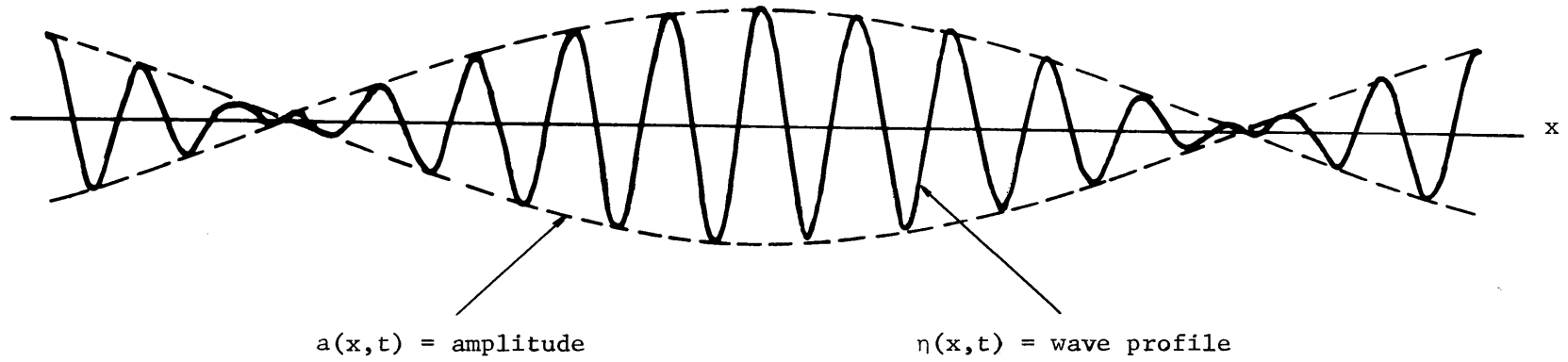


FIGURE 1 ADDITION OF TWO SINUSOIDS OF SLIGHTLY DIFFERENT WAVE FREQUENCIES

from one crest to the adjacent crest, the phase ψ changes by 2π so that a wave number and a wave frequency may be defined by the phase ψ as,

$$k = \frac{\partial \psi}{\partial x} \quad \text{and} \quad \omega = -\frac{\partial \psi}{\partial t} \quad , \quad (1.5 \text{ a,b})$$

and for this particular example,

$$k = \frac{1}{2} (k_1 + k_2) \quad \text{and} \quad \omega = \frac{1}{2} (\omega_1 + \omega_2) \quad . \quad (1.6 \text{ a,b})$$

By cross differentiating (1.5 a,b) we obtain

$$\frac{\partial k}{\partial t} + \frac{\partial \omega}{\partial x} = 0 \quad (1.7)$$

which is a conservation equation for wave crests. We stress that (1.5) fails to define k and ω at the nodal point where $k, \omega \rightarrow \infty$ because of the rapid change of phase by π there.

Now let us suppose that the dispersive relation for each of the plane waves is given by

$$\omega_{1,2} = f(k_{1,2}) \quad . \quad (1.8)$$

We would like to ask whether the wave number k and the wave frequency ω for the slowly varying wave train (1.6 a,b) would satisfy the same dispersion relation as the plane periodic wave given by (1.8). To be exact, for dispersive waves such as water waves, the answer is "no", because (1.8) is not linear. However, when $(k_1 - k_2)$ is small, i.e. the length of the group is long, the dispersion relation for ω and k may be approximated by

$$\omega = f(k) + \frac{(k_1 - k_2)^2}{8} \frac{\partial^2 f}{\partial k^2} + O[(k_1 - k_2)^3] \quad (1.9)$$

Referring to (1.3) we note that $(k_1 - k_2)$ is a small parameter characterizing the spatial rate of modulation of wave amplitude. It is thus clear from this simple example that the dispersion relation for a slowly modulating wave train is generally different from the plane wave one by a term proportional to the square of the rate of modulation.

It should be remarked here that in Whitham's averaging approach the local dispersion relation for the modulating wave train is taken to be the same as the plane periodic wave. For weakly nonlinear waves such as Stokes waves, the nonlinear term which enters the dispersion relation is second order in wave steepness. Unless the rate of modulation is very small compared to the wave steepness, the effect of nonlinearity can not be properly studied without including the direct effect of wave modulation into the dispersion relation. In an example aiming at testing Whitham's theory, Lighthill (1967) studied the nonlinear evolution of wave envelope pulses. He found that the effect of nonlinearity tends to steepen the wave envelope and hence to increase the rate of modulation in general. Thus the rate of modulation may be initially small compared with wave steepness, it may nevertheless catch up with the effect of nonlinearity and become equally important. An analogous situation exists in shallow water waves where nonlinearity and dispersion are both important for the eventual nonlinear evolution of wave profiles (Madsen and Mei 1969 a,b, 1970). One of the central ideas in this thesis is to allow the direct effect of modulation rate and the nonlinearity to be

equally important. As this allowance introduces new dispersive effects, phenomena not unlike those found in shallow water waves are found in the envelope of deep water waves.

CHAPTER 2

FORMULATION AND METHOD OF SOLUTION

2.1 BASIC FORMULATION

We consider a slowly modulating wave train over water where the depth changes slowly. As illustrated in Figure 2, we take $\vec{x} = (x, y)$ to be the horizontal coordinates and z the vertical coordinate. We also distinguish the three dimensional gradient vector $\nabla_3 = (\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z})$ from the horizontal gradient vector $\nabla_2 = (\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, 0)$. The water is modelled as a perfect fluid whose motion is irrotational. Accordingly there exists a velocity potential $\Phi(\vec{x}, z, t)$ defined by

$$\vec{u} = \nabla_3 \Phi \quad . \quad (2.1)$$

The potential Φ satisfies Laplace equation between the free surface $z = \eta(\vec{x}, t)$ and the bottom $z = -h(\vec{x})$; i.e.,

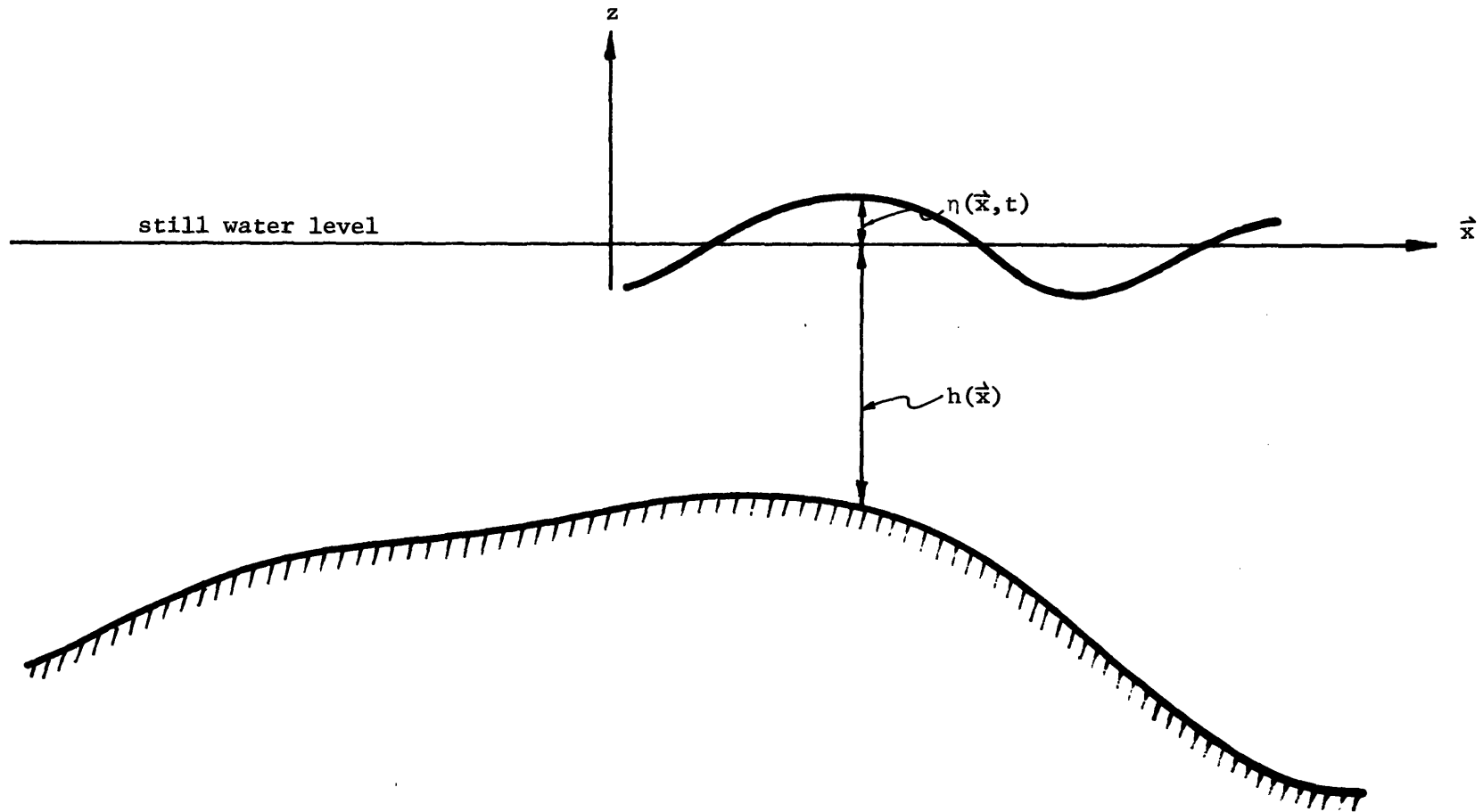
$$\nabla_3^2 \Phi = \nabla_2^2 \Phi + \Phi_{zz} = 0 \quad , \quad \eta(\vec{x}, t) \geq z \geq -h(\vec{x}). \quad (2.2)$$

The kinematical condition at the bottom is

$$\Phi_z + \nabla_2 \Phi \cdot \nabla_2 h = 0 \quad , \quad z = -h(\vec{x}) \quad . \quad (2.3)$$

Taking the total derivative, $\frac{\partial}{\partial t} + \vec{u} \cdot \nabla_3$, to the Bernoulli equation and using the kinematic boundary condition at the free surface we obtain a single free surface boundary condition for the velocity potential Φ , i.e.,

$$\Phi_{tt} + g\Phi_z + \left(\frac{\partial}{\partial t} + \frac{1}{2} \vec{u} \cdot \nabla_3\right) |\vec{u}|^2 = 0 \quad , \quad z = \eta(\vec{x}, t). \quad (2.4)$$



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FIGURE 2 DEFINITION SKETCH

For detailed derivation of equation (2.4), see e.g. Phillips (1966) p. 23. The free surface elevation η is given by the Bernoulli equation:

$$\eta = -\frac{1}{g} \left\{ \Phi_t + \frac{1}{2} |\vec{u}|^2 \right\}_{z=\eta(\vec{x},t)} \quad (2.5)$$

where the Bernoulli constant is absorbed in the velocity potential Φ . The boundary conditions (2.4) and (2.5) are nonlinear and are applied at a position unknown "a priori". Since the wave amplitude is generally small compared to the wave length we may expand the free surface boundary conditions (2.4) and (2.5) into Taylor series about $z=0$, yielding,

$$\sum_{\nu=0}^{\infty} \frac{\eta^\nu}{\nu!} \frac{\partial^\nu}{\partial z^\nu} \left\{ \Phi_{tt} + g \Phi_z + \left(\frac{\partial}{\partial t} + \frac{\vec{u}}{2} \cdot \nabla_3 \right) |\vec{u}|^2 \right\} = 0, \quad (2.6)$$

$$\eta = -\frac{1}{g} \sum_{\nu=0}^{\infty} \frac{\eta^\nu}{\nu!} \frac{\partial^\nu}{\partial z^\nu} \left\{ \Phi_t + \frac{1}{2} |\vec{u}|^2 \right\} \quad (2.7)$$

2.2 PERTURBATION EXPANSIONS

We now introduce the stretched variables,

$$\vec{X} = \varepsilon \vec{x} \quad \text{and} \quad T = \varepsilon t \quad (2.8)$$

to take care of the slow modulation of wave amplitude, wave number, etc.. Here ε is a small parameter characterizing the slow rate of modulation. For convenience we shall let the other small quantity, wave steepness, to be also of order ε . The solution for Φ and η are expanded into perturbation series of the small parameter ε as follows:

$$\Phi(\vec{x}, z, t) = \sum_{n=1}^{\infty} \varepsilon^n \sum_{m=-n}^{+n} \phi^{(n,m)}(\vec{x}, z, T) e^{im\psi(\vec{x}, T)/\varepsilon}, \quad (2.9)$$

$$\eta(\vec{x}, t) = \sum_{n=1}^{\infty} \varepsilon^n \sum_{m=-n}^{+n} \eta^{(n,m)}(\vec{x}, T) e^{im\psi(\vec{x}, T)/\varepsilon}. \quad (2.10)$$

In order to keep Φ and η to be real functions, $\phi^{(n,m)}$ and $\eta^{(n,m)}$ are defined to be the complex conjugates of $\phi^{(n,-m)}$ and $\eta^{(n,-m)}$. In the first order of approximation, i.e., $O(\varepsilon)$, the series contain only zero and first harmonic terms while in the higher order approximations higher harmonic terms are included in view of the product terms arising from the nonlinear boundary conditions. These expansions are essentially similar to Stokes perturbation series used for water waves over uniform depth (Wehausen and Laitone, 1960, p.654) except that now the idea of WKB (Wentzel-Kramers-Brillouin) expansion method is also incorporated, i.e. the amplitude functions $\phi^{(n,m)}(\vec{x}, z, T)$, $\eta^{(n,m)}(\vec{x}, T)$ and the phase function $\psi(\vec{x}, T)/\varepsilon$ are now taken to be slowly varying functions of space and time. The effect of stretching the horizontal coordinates and the idea of WKB expansion can be seen by examining the variation of a typical n th order term in the horizontal direction:

$$\nabla_2 \{ \phi^{(n,m)} e^{im\psi/\varepsilon} \} = \{ im(\nabla\psi) \phi^{(n,m)} + \varepsilon \nabla \phi^{(n,m)} \} e^{im\psi/\varepsilon}$$

where $\nabla = \varepsilon \nabla_2 = (\frac{\partial}{\partial X}, \frac{\partial}{\partial Y}, 0)$ is the horizontal gradient vector for the slow variable \vec{X} . Hence, any \vec{x} derivative consists of two terms

(see inside the bracket) of which the amplitude variation is the smaller. Thus, the slow modulation of the amplitude is incorporated in the formalism.

As remarked by Lighthill (1965) that if the wave length of the adjacent waves are assumed to differ by only a small fraction, it is very rare for a crest to cease being a crest, or to divide into two crests. Under this circumstance the wave number and the wave frequency may be defined by the phase function as follows:

$$\vec{k} = \nabla \psi \quad \text{and} \quad \omega = -\psi_T \quad . \quad (2.11 \text{ a,b})$$

By cross differentiation and adding up (2.11 a) and (2.11 b) we obtain a conservation law for wave crests; i.e.,

$$\frac{\partial \vec{k}}{\partial T} + \nabla \omega = 0 \quad . \quad (2.12)$$

Following Stokes (1847) we also expand the wave frequency into perturbation series; i.e.,

$$\omega = \sum_{n=0}^{\infty} \epsilon^{2n} \omega_{2n} \quad . \quad (2.13)$$

The expansion of ω into series of ϵ implies that the wave frequency depends not only on wave number but also on the wave amplitude and the rate of modulation.

Now upon substituting the perturbation expansion (2.9), (2.10) and (2.13) into the governing equation and boundary conditions (2.2), (2.3),

(2.6) and (2.7), different orders and harmonics may be separated, yielding a set of ordinary differential equations for each indices (n, m):

$$\begin{aligned} \phi_{z\bar{z}}^{(n,m)} - m^2 |\vec{k}|^2 \phi^{(n,m)} &= R^{(n,m)}(\vec{X}, z, T), \quad 0 \geq z \geq -h, \\ g \phi_{\bar{z}}^{(n,m)} - m^2 \omega_0^2 \phi^{(n,m)} &= G^{(n,m)}(\vec{X}, T), \quad z = 0, \\ \phi_{\bar{z}}^{(n,m)} &= F^{(n,m)}(\vec{X}, T), \quad z = -h, \\ \eta^{(n,m)} &= \frac{1}{g} \left\{ im \omega_0 \phi^{(n,m)}(\vec{X}, 0, T) - H^{(n,m)}(\vec{X}, T) \right\}. \end{aligned} \quad (2.14 \text{ a-d})$$

The functions $R^{(n,m)}$, $G^{(n,m)}$, $F^{(n,m)}$, and $H^{(n,m)}$ are in terms of solution lower than order n; their explicit forms are derived in Appendix A. Thus a kind of separation of variables is formally achieved and the task remains to solve the boundary value problem defined by system (2.14) in succession.

2.3 GENERAL METHOD FOR SOLUTION

Because of the increasing complexities at higher orders, it is helpful to study some general features before going into the details. We shall first present the formal solution to (2.14). In particular the zeroth harmonic is the simplest. By taking $m = 0$, we obtain a first order equation for $\phi_{\bar{z}}^{(n,0)}$ with two boundary conditions. The solution satisfying (2.14 a) and (2.14 c) may be given as

$$\phi_{\bar{z}}^{(n,0)} = \int_{-h}^z R^{(n,0)} d\bar{z} + F^{(n,0)} \quad (2.15)$$

but this give no further information on $\phi^{(n,0)}$. However, there is another boundary condition for $\phi_z^{(n,0)}$ at $z = 0$ yet to be satisfied. This imposes a condition on $R^{(n,0)}$, $F^{(n,0)}$, and $G^{(n,0)}$, which will be called a "solvability condition":

$$\int_{-h}^0 R^{(n,0)} d\bar{z} + F^{(n,0)} = \frac{1}{g} G^{(n,0)} \quad (2.16)$$

where

$$R^{(n,0)} = -\nabla^2 \phi^{(n-2,0)} \quad \text{and} \quad F^{(n,0)} = -\nabla h \cdot \left\{ \nabla \phi^{(n-2,0)} \right\}_{\bar{z}=-h} \quad (2.17 \text{ a,b})$$

as derived in equations (A.5) and (A.6) of Appendix A. It follows by using the Leibniz rule that (2.16) is equivalent to

$$\nabla \cdot \int_{-h}^0 \nabla \phi^{(n-2,0)} d\bar{z} = -\frac{1}{g} G^{(n,0)}, \quad (2.18)$$

and (2.15) then gives

$$\phi_z^{(n,0)} = -\nabla \cdot \int_{-h}^z \nabla \phi^{(n-2,0)} d\bar{z} \quad (2.19)$$

Equation (2.18), which is the consequence of the n th order, gives a relation for $\nabla \phi^{(n-2,0)}$; in other words, restriction on $\nabla \phi^{(n-2,0)}$ is found at two order later. Note the fact that we do not know

$\phi^{(n-2,0)}$ itself is not essential. Physically, $\varepsilon^n \phi_z^{(n,0)}$ contributes at the n th order to the vertical mean current while $\varepsilon^n \nabla_2 \phi^{(n,0)} = \varepsilon^{n+1} \nabla \phi^{(n,0)}$ contributes at the $(n+1)$ th order to the horizontal mean current.

For other harmonics, $m \neq 0$, the solution that satisfies (2.14 a) and (2.14 c) is formally,

$$\phi^{(n,m)} = A^{(n,m)} \cosh mQ + \frac{1}{m\bar{k}} F^{(n,m)} \sinh mQ + \frac{1}{m\bar{k}^2} \left\{ \sinh mQ \int_0^Q R^{(n,m)} \cosh mQ' dQ' - \cosh mQ \int_0^Q R^{(n,m)} \sinh mQ' dQ' \right\} \quad (2.20)$$

where $Q = \bar{k}(z+h)$ and $\bar{k} = |\vec{k}|$. Substituting (2.20) into the free surface condition (2.14 b) we have

$$\left(\bar{k} \sinh mq - m\bar{k}_\infty \cosh mq \right) \left\{ m A^{(n,m)} - \frac{1}{\bar{k}^2} \int_0^q R^{(n,m)} \sinh mQ dQ \right\} + \left(\bar{k} \cosh mq - m\bar{k}_\infty \sinh mq \right) \left\{ \frac{1}{\bar{k}} F^{(n,m)} + \frac{1}{\bar{k}^2} \int_0^q R^{(n,m)} \cosh mQ dQ \right\} = \frac{G^{(n,m)}}{g} \quad (2.21)$$

where $\bar{k}_\infty = \frac{\omega_0^2}{g}$ and $q = \bar{k}h$. In general for all $n, m = 2, 3, 4, \dots$, Equation (2.21) fixes the coefficient $A^{(n,m)}$ uniquely. When $n = 1$ the coefficient $A^{(n,1)}$ drops out explicitly from (2.21), which takes on special significance, as discussed below.

For $n = 1, m = 1$, it is easy to show that $R^{(1,1)} = F^{(1,1)} = G^{(1,1)} = H^{(1,1)} = 0$ (Appendix A), and that the system (2.14 a,b,c) is homogeneous, the solution is:

$$\phi^{(1,1)} = -ig \frac{A^{(1,1)}}{\omega_0} \cosh Q, \quad \eta^{(1,1)} = A^{(1,1)} \cosh q = a/2 \quad (2.22 \text{ a,b})$$

where a is defined as the first order amplitude. The classical dispersion relation may be obtained from (2.21), i.e.,

$$k \tanh q = k_{\infty} . \quad (2.23)$$

For $n = 1, m = 2, 3, \dots$, Equation (2.21) corresponds to the solvability condition for the inhomogeneous boundary value problem defined by (2.14 a,b,c); its specific form is

$$F^{(n,1)} + \frac{1}{k} \int_0^q R^{(n,1)} \cosh Q dQ = \frac{1}{g} G^{(n,1)} \cosh q . \quad (2.24)$$

Now $F^{(n,1)}$ and $R^{(n,1)}$ involve the terms $A^{(n-1,1)}$ and ω_{n-1} which are yet to be determined. Equation (2.24) therefore imposes an extra condition on them. Thus, for example, at the stage $n = 2$, (2.24) provides a condition for $A^{(1,1)}$; at $n = 3$ it provides a relation for ω_2 leaving $A^{(2,1)}$ arbitrary,...

CHAPTER 3

EXPLICIT SOLUTIONS AND EQUATIONS

GOVERNING SLOW MODULATIONS

In this chapter we shall carry out the solution of (2.14 a,b,c,d) explicitly to second order. Equations governing the modulation of a , \vec{k} , ω_0 , ω_2 , $\nabla\phi^{(1,0)}$ and $\eta^{(2,0)}$ are derived from the solvability conditions (2.18) and (2.24) using partly some result from third order. The explicit expressions for $R^{(n,m)}$, $F^{(n,m)}$, $G^{(n,m)}$ and $H^{(n,m)}$ are derived and simplified in terms of lower order results in Appendix A and B.

3.1 EXPLICIT SOLUTIONS

From Appendix A we have $R^{(1,0)} = F^{(1,0)} = G^{(1,0)} = H^{(1,0)}$. It follows immediately from (2.14) that for $n = 1, m = 0$,

$$\phi_{\vec{x}}^{(1,0)} = \eta^{(1,0)} = 0 \quad (3.1)$$

We recall that for $n = 1, m = 1$, the solution is given by (2.22) and (2.23). So far a (or $A^{(1,1)}$) and $\nabla\phi^{(1,0)}$ are arbitrary at this order and will be determined by the higher order solvability conditions.

For $n = 2, m = 0$, we have $R^{(2,0)} = F^{(2,0)} = G^{(2,0)} = 0$ and $H^{(2,0)} = \phi_T^{(1,0)} + (k_\infty^2 - k^2) \left\{ \phi_{\vec{x}=0}^{(1,0)2} \right\}$. Substituting into (2.19) and (2.14 d), we obtain

$$\phi_{\vec{x}}^{(2,0)} = 0 \quad (3.2 \text{ a})$$

and

$$g \eta^{(2,0)} = -\phi_T^{(1,0)} - \frac{1}{4} g k_\infty a^2 (\sigma^2 - 1) \quad (3.2 b)$$

where

$$\sigma = k / k_\infty = \coth g \quad (3.3)$$

Differentiating (3.2 b) with respect to \vec{X} ,

$$\frac{\partial}{\partial T} \{ \nabla \phi^{(1,0)} \} + \nabla \{ g \eta^{(2,0)} + \frac{1}{4} k_\infty (\sigma^2 - 1) g a^2 \} = 0 \quad (3.4)$$

Equation (3.4) is a momentum equation for the horizontal mean current $\nabla \phi^{(1,0)}$ and mean water level $\eta^{(2,0)}$. The last term in (3.4) is the mean transport of momentum due to the first order wave field.

For $n = 2$, $m = 1$, we have $R^{(2,1)} = -i \nabla \cdot \{ \vec{k} \phi^{(1,1)} \} - i \vec{k} \cdot \nabla \phi^{(1,1)}$,

$$F^{(2,1)} = -\nabla h \cdot \{ i \vec{k} \phi^{(1,1)} \}_{z=-h}, \quad H^{(2,1)} = \{ \phi_T^{(1,1)} \}_{z=0}, \quad \text{and}$$

$$G_T^{(2,1)} = \left\{ i (\omega_0 \phi^{(1,1)})_T + i \omega_0 \phi_T^{(1,1)} \right\}_{z=0}.$$

Upon integrating (2.20),

$$\phi^{(2,1)} = A^{(2,1)} \cosh Q - g \frac{A^{(1,1)}}{\omega_0} \{ \alpha_1 Q \cosh Q + \alpha_2 Q \sinh Q + \alpha_3 Q^2 \cosh Q \} \quad (3.5)$$

where

$$\alpha_1 = \frac{\vec{k} \cdot \nabla h}{k}, \quad \alpha_2 = \frac{\nabla \cdot \left\{ \frac{\vec{k}}{k} \left(\frac{A^{(1,1)}}{\omega_0} \right)^2 \right\}}{2 k \left(\frac{A^{(1,1)}}{\omega_0} \right)^2} \quad \text{and} \quad \alpha_3 = \frac{\vec{k} \cdot \nabla k}{2 k^3}. \quad (3.6 a, b, c)$$

The coefficient $A^{(2,1)}$ for the homogeneous problem will be chosen here so as to give the proper limits as $k h \rightarrow \infty$ (which can be worked out

independently); note that this requirement still does not give a unique choice for $A^{(2,1)}$. Using the fact that $Q \rightarrow q$ for $kh \rightarrow \infty$ we take

$$\phi^{(2,1)} = -g \frac{A^{(1,1)}}{\omega_0} \left\{ \alpha_1 (Q-q) \cosh Q + \alpha_2 (Q \sinh Q - q \tanh q \cosh Q) + \alpha_3 (Q^2 - q^2) \cosh Q \right\} \quad (3.7)$$

so that $\left\{ \phi^{(2,1)} \right\}_{z=0} = 0$. This leads to

$$\eta^{(2,1)} = \frac{1}{2} i \left(\frac{a}{\omega_0} \right)_T \quad (3.8)$$

The solvability condition (2.24) for $n = 2$, $m = 1$ is equivalent to

$$\begin{aligned} & -\nabla h \cdot \left\{ i \vec{k} \phi^{(1,1)} \right\}_{z=-h} + \int_{-h}^0 \left[-i \nabla \cdot (\vec{k} \phi^{(1,1)}) - i \vec{k} \cdot \nabla \phi^{(1,1)} \right] \cosh Q \, dz \\ & = \frac{1}{g} \cosh q \left\{ i (\omega_0 \phi^{(1,1)})_T + i \omega_0 \phi_T^{(1,1)} \right\}_{z=0} \end{aligned} \quad (3.9)$$

Multiplying both sides by $g \frac{A^{(1,1)}}{\omega_0}$ and making use of the Leibniz rule,

$$\frac{\partial}{\partial T} \left\{ \omega_0 \phi^{(1,1)2} \right\}_{z=0} + g \nabla \cdot \int_{-h}^0 \vec{k} \phi^{(1,1)2} \, dz = 0 \quad (3.10)$$

which can be further simplified by using (2.22) to give an energy equation:

$$\frac{\partial}{\partial T} \left\{ \frac{E}{\omega_0} \right\} + \nabla \cdot \left\{ \vec{c}_g \frac{E}{\omega_0} \right\} = 0 \quad (3.11)$$

where

E = energy density of the first order waves

$$= \frac{1}{2} g a^2 \quad (3.12)$$

and

$$\begin{aligned} \vec{C}_g &= \text{the linear group velocity} \\ &= \frac{\partial \omega_0}{\partial \vec{k}} = \frac{1}{2} \frac{\omega_0 \vec{k}}{k^2} \left(1 + 2q / \sinh 2q \right) . \end{aligned} \quad (3.13)$$

For $n = 2, m = 2$, we have $R^{(2,2)} = F^{(2,2)} = 0$, and from Appendix B, $G^{(2,2)} = -3i\omega_0 k_\infty^2 (\sigma^2 - 1) \{ \phi^{(1,1)2} \}_{z=0}$ and $H^{(2,2)} = \frac{1}{2} k_\infty^2 (3 - \sigma^2) \{ \phi^{(1,1)2} \}_{z=0}$.

Substituting into (2.22),

$$2A^{(2,2)} (k \sinh 2q - 2k_\infty \cosh 2q) = -\frac{3i\omega_0}{g} k_\infty^2 (\sigma^2 - 1) \{ \phi^{(1,1)2} \}_{z=0} \quad (3.14)$$

which, after some manipulation, gives $A^{(2,2)}$. Substituting into (2.20), and (2.14 d), we obtain

$$\phi^{(2,2)} = -i \frac{3}{16} \omega_0 a^2 (\sigma^2 - 1) \cosh 2Q$$

and

$$\eta^{(2,2)} = \frac{1}{8} k_\infty a^2 \sigma^2 (3\sigma^2 - 1). \quad (3.15 \text{ a,b})$$

The second order solutions are not yet complete. Additional relation between $\nabla \phi^{(1,0)}$ and $\eta^{(2,0)}$ and the second order correction for wave frequency ω_2 are still to be determined from the third order solvability condition.

Since $G^{(3,0)} = (g \eta^{(2,0)})_T - 2 \nabla \cdot \{ \omega_0 \vec{k} \phi^{(1,1)2} \}_{z=0}$ the solvability condition (2.18) leads to

$$g \nabla \cdot \int_{-h}^0 \nabla \phi^{(1,0)} dz + (g \eta^{(2,0)})_T - 2 \nabla \cdot \{ \omega_0 \vec{k} \phi^{(1,1)2} \}_{z=0} = 0 ,$$

or,

$$\frac{\partial \eta^{(2,0)}}{\partial T} + \nabla \cdot \left\{ \rho \nabla \phi^{(1,0)} + \frac{\bar{\rho}}{\omega_0} E \right\} = 0 \quad (3.16)$$

This is essentially an averaged continuity equation. The last term $\frac{\bar{\rho}}{\omega_0} E$ is the total Lagrangian rate of mass flux due to first order wave field.

For $n = 3, m = 1,$

$$R^{(3,1)} = -\nabla^2 \phi^{(1,1)} - i \nabla \cdot (\bar{\rho} \phi^{(2,1)}) - i \bar{\rho} \cdot \nabla \phi^{(2,1)},$$

$$F^{(3,1)} = -\nabla \rho \cdot \left\{ \nabla \phi^{(1,1)} + i \bar{\rho} \phi^{(2,1)} \right\}_{z=-h},$$

$$G^{(3,1)} = \left\{ -\phi_{TT}^{(1,1)} + 2\omega_0 \omega_z \phi^{(1,1)} - 2\omega_0 \bar{\rho} \phi^{(1,1)} \cdot \nabla \phi^{(1,0)} + g \eta^{(2,0)} \bar{\rho}^2 (\sigma^2 - 1) \phi^{(1,1)} + \frac{1}{2} \bar{\rho}^2 \phi^{(1,1)^3} \bar{\rho}^2 (9\sigma^4 - 10\sigma^2 + 9) \right\}_{z=0}.$$

Substituting into solvability (2.24), we have,

$$\begin{aligned} & -\nabla \rho \cdot \left\{ \nabla \phi^{(1,1)} + i \bar{\rho} \phi^{(2,1)} \right\}_{z=-h} - \int_{-h}^0 \left\{ \nabla^2 \phi^{(1,1)} + i \bar{\rho} \cdot \nabla \phi^{(2,1)} + i \nabla \cdot (\bar{\rho} \phi^{(2,1)}) \right\} \cosh \omega z dz \\ & = \frac{2\omega_0 \phi^{(1,1)}}{g} \cosh g \left\{ -\phi_{TT}^{(1,1)} / (2\omega_0 \phi^{(1,1)}) - \bar{\rho} \cdot \nabla \phi^{(1,0)} - \frac{1}{2} \omega_0 \bar{\rho} \eta^{(2,0)} (\sigma^2 - 1) \right. \\ & \quad \left. - \frac{1}{16} \omega_0 (\bar{\rho} a)^2 (9\sigma^4 - 10\sigma^2 + 9) \right\}_{z=0} \quad (3.17) \end{aligned}$$

After some manipulations equation (3.17) gives the second order correction ω_2 to the wave frequency which consists of five different types of terms; i.e.,

$$\omega_2 = \omega_2^S + \omega_2^U + \omega_2^D + \omega_2^X + \omega_2^T \quad (3.18)$$

where $\omega_2^S = \frac{1}{16} \omega_0 (\frac{p}{k} a)^2 (9\sigma^4 - 10\sigma^2 + 9),$

$$\omega_2^U = \frac{1}{k} \cdot \nabla \phi^{(1,0)},$$

$$\omega_2^D = \frac{1}{2} \omega_0 \frac{p}{k_0} (\sigma^2 - 1) \eta^{(2,0)},$$

$$\omega_2^T = \left(\frac{a}{\omega_0}\right)_{TT} / 2a ,$$

$$\omega_2^X = \frac{1}{a \cosh q} \left\{ \nabla h \cdot \left(-i \nabla \phi^{(1,1)} + \frac{1}{k} \phi^{(2,1)} \right)_{z=-h} + \int_{-h}^0 \left[-i \nabla^2 \phi^{(1,1)} + \frac{1}{k} \cdot \nabla \phi^{(2,1)} + \nabla \cdot \left(\frac{1}{k} \phi^{(2,1)} \right) \right] \cosh Q \, dz \right\} \quad (3.19 \text{ a-e})$$

Physically the terms ω_2^S , ω_2^D , ω_2^U , ω_2^X and ω_2^T represent respectively the effects of Stokes amplitude dispersion, mean depth change, mean current, spatial modulation and temporal modulation. To the order of ϵ^2 , the wave conservation equation (2.12) is now

$$\frac{\partial \frac{1}{k}}{\partial T} + \nabla \left\{ \omega_0 + \epsilon^2 (\omega_2^S + \omega_2^U + \omega_2^D + \omega_2^X + \omega_2^T) \right\} = O(\epsilon^3) \quad (3.20)$$

where ω_2 is given by (3.18) and (3.19).

Correct to the second order the solution for Φ and η are now

$$\Phi(\vec{x}, z, t) = \epsilon \phi^{(1,0)} + 2i\epsilon \phi^{(1,1)} \sin\left(\frac{\psi}{\epsilon}\right) + 2\epsilon^2 \phi^{(2,1)} \cos\left(\frac{\psi}{\epsilon}\right) + 2i\epsilon^2 \phi^{(2,2)} \sin 2\left(\frac{\psi}{\epsilon}\right) + O(\epsilon^3)$$

$$\eta(\vec{x}, t) = 2\epsilon \eta^{(1,1)} \cos\left(\frac{\psi}{\epsilon}\right) + \epsilon^2 \eta^{(2,0)} + 2i\epsilon^2 \eta^{(2,1)} \sin\left(\frac{\psi}{\epsilon}\right) + 2\epsilon^2 \eta^{(2,2)} \cos 2\left(\frac{\psi}{\epsilon}\right) + O(\epsilon^3) \quad (3.21 \text{ a,b})$$

where the solution of the amplitude functions of each harmonic such as $\phi^{(1,0)}$, $\phi^{(1,1)}$, $\eta^{(1,1)}$, $\phi^{(2,1)}$ has been given previously in this

paragraph. Inspecting (3.21), we notice that there are new terms $\varepsilon \phi^{(1,0)}$, $\varepsilon^2 \eta^{(2,0)}$, $2\varepsilon^2 \phi^{(2,1)} \cos(\frac{\psi}{\varepsilon})$ and $2i\varepsilon^2 \eta^{(2,1)} \sin(\frac{\psi}{\varepsilon})$ added to the Stokes second order solution. In Stokes waves of uniform amplitude the contribution due to $\varepsilon \phi^{(1,0)}$ and $\varepsilon^2 \eta^{(2,0)}$ may be ignored by redefining the potential Φ and the depth h . However, for a slowly varying wave train such mean quantities are coupled to the wave amplitude in a non-trivial way (see (3.4) and (3.16)). We point out that the terms $\phi^{(2,1)}$ and $\eta^{(2,1)}$ which is linear proportional to Q , may be obtained even if the free surface boundary conditions are linearized to begin with. Furthermore, $2\varepsilon^2 \phi^{(2,1)} \cos(\frac{\psi}{\varepsilon})$ and $2i\varepsilon^2 \eta^{(2,1)} \sin(\frac{\psi}{\varepsilon})$ are $\frac{\pi}{2}$ out of phase with the others, hence they can be replaced by a phase shift δ defined by

$$\begin{aligned} \varepsilon(2i\phi^{(1,1)}) \sin(\frac{\psi}{\varepsilon}) + \varepsilon^2(2\phi^{(2,1)}) \cos(\frac{\psi}{\varepsilon}) &= \varepsilon(2i\phi^{(1,1)}) \sin(\frac{\psi}{\varepsilon} + \varepsilon\delta) + O(\varepsilon^3) \\ \varepsilon(2\eta^{(1,1)}) \cos(\frac{\psi}{\varepsilon}) + \varepsilon^2(2i\eta^{(2,1)}) \sin(\frac{\psi}{\varepsilon}) &= \varepsilon(2\eta^{(1,1)}) \cos(\frac{\psi}{\varepsilon} + \varepsilon\delta_0) + O(\varepsilon^3) \end{aligned} \quad (3.22 \text{ a,b})$$

where $\delta = \phi^{(2,1)} / i\phi^{(1,1)} = -\{ \alpha_1(Q-q) + \alpha_2(Q \tanh Q - q \tanh q) + \alpha_3(Q^2 - q^2) \}$

and $\delta_0 = -i\eta^{(2,1)} / \eta^{(1,1)} = (\frac{a}{\omega_0})_T / a \quad (3.23 \text{ a,b})$

Now the solution for Φ and η are finally given by,

$$\begin{aligned} \Phi(\vec{x}, z, t) &= \varepsilon \phi^{(1,0)} + 2i\varepsilon \phi^{(1,1)} \sin(\frac{\psi}{\varepsilon} + \delta) + 2i\varepsilon^2 \phi^{(2,2)} \sin 2(\frac{\psi}{\varepsilon} + \delta) + O(\varepsilon^3) \\ \eta(\vec{x}, t) &= \varepsilon^2 \eta^{(2,0)} + 2\varepsilon \eta^{(1,1)} \cos(\frac{\psi}{\varepsilon} + \delta_0) + 2\varepsilon^2 \eta^{(2,2)} \cos 2(\frac{\psi}{\varepsilon} + \delta_0) + O(\varepsilon^3) \end{aligned} \quad (3.24 \text{ a,b})$$

Since the phase δ is a function of \vec{x} , the wave front is not strictly vertical.

3.2 EQUATIONS GOVERNING SLOW MODULATIONS

We have shown in the preceeding section that a slowly modulating wave train may be uniquely specified by four slowly varying wave parameters, namely, the wave amplitude a , the wave number \vec{k} , the mean current $\nabla\phi^{(1,0)}$, and the mean water level $\eta^{(2,0)}$. Equations governing these parameters are summarized as follows (cf (3.11), (3.16), (3.4) and (3.20)):

$$\begin{aligned} \frac{\partial}{\partial T} \left\{ \frac{\vec{E}}{\omega_0} \right\} + \nabla \cdot \left\{ \vec{C}_g \frac{\vec{E}}{\omega_0} \right\} &= 0 \\ \frac{\partial \eta^{(2,0)}}{\partial T} + \nabla \cdot \left\{ h \nabla \phi^{(1,0)} + \frac{\vec{k}}{\omega_0} \vec{E} \right\} &= 0 \\ \frac{\partial \nabla \phi^{(1,0)}}{\partial T} + \nabla \cdot \left\{ g \eta^{(2,0)} + \frac{1}{2} k_\infty (\alpha^2 - 1) \vec{E} \right\} &= 0 \\ \frac{\partial \vec{k}}{\partial T} + \nabla \cdot \left\{ \omega_0 + \varepsilon^2 (\omega_2^S + \omega_2^U + \omega_2^D + \omega_2^X + \omega_2^T) \right\} &= O(\varepsilon^3) \end{aligned} \quad (3.25 \text{ a-d})$$

where various contributions to ω_2 is given by (3.19). A similar set of equations has been derived previously by Whitham from an averaged variational principle. The first three equations (3.25 a,b,c) are completely equivalent to three of Whitham's equations (1967 a, equation (39), (40) and (42)) while (3.25 d) differs from Whitham's corresponding equation (1967 a, equation (41)) by two new terms, i.e. ω_2^X and ω_2^T . As can be seen from their definitions given in (3.19) both ω_2^X and ω_2^T involve terms twice differentiated with respect to space and time. Thus they represent direct effects of wave modulation. Only

when the modulation rate is very much less than the wave steepness (i.e. when modulation is characterized by scales much longer than $O(\frac{1}{\epsilon})$) can ω_2^X and ω_2^T be ignored, in which case (3.25 d) reduces to Whitham's exactly. We further remark that ω_2^X and ω_2^T , linear in a , may be obtained even if the free surface boundary condition are linearized to begin with. In fact ω_2^X and ω_2^T is the general second order correction to the dispersion relation of a plane wave as illustrated by the simple example in § 1.3 (cf the term $(\Delta k)^2 \frac{\partial^2 f}{\partial k^2}$ in (1.9)).

It was suggested by Whitham that Stokes waves are stable if the set of governing modulation equations is hyperbolic and unstable if elliptic. Now with the additional terms ω_2^X and ω_2^T involving second order derivatives, this classification is no longer appropriate, since higher order derivatives change the mathematical character of partial differential equations drastically.

Now let us go back to the discussion of (3.25). We note that equation (3.25 a) is exactly the same energy equation as derived from the linear theory. In the method of geometrical optics approximation (i.e. the first order linear approximation) the frequency dispersion relation is assumed to be the same as the linear periodic waves, so that the group velocity \vec{C}_g and the frequency ω_0 , uncoupled with wave amplitude, may be uniquely determined from (3.25 d) alone. Once the \vec{C}_g and ω_0 are determined the amplitude may be calculated from the energy equation (3.25 a). For linear problems, there are two types of factors which may affect the variation of \vec{C}_g and ω_0 and

consequently change the wave amplitude. The first is "frequency dispersion" which tends to sort out waves with longer waves propagating faster than the shorter waves; as a result the energy spreads out and amplitude reduces. The second factor is "refraction" in which the group velocity \vec{c}_g is affected by changing of depth; the amplitude usually (not always) tends to increase as waves propagate into shallower water. For nonlinear problems as described by (3.25), \vec{c}_g and ω_0 are coupled with the amplitude a and the mean quantities (or the long wave components) $\eta^{(2,0)}$ and $\nabla\phi^{(1,0)}$. Such nonlinear coupling effects between (\vec{c}_g, ω_0) and $(\eta^{(2,0)}, \nabla\phi^{(1,0)}, a)$ are all aspects of "nonlinear dispersion". In general it is impossible to isolate the dispersion effect due to nonlinearity from the effect of frequency dispersion and refraction because they are all coupled to each other. However, in a loose sense, we may consider the terms $\xi^2 \nabla (\omega_2^S + \omega_2^U + \omega_2^D)$ in equation (3.25 d) to represent the nonlinear amplitude dispersion since they are explicitly proportional to square of the amplitude. For the rest of the terms in (3.25 d), $\nabla\omega_0$ represents primarily the refraction and frequency dispersion while $\xi^2 \nabla (\omega_2^X + \omega_2^T)$, being affected directly by the modulation rate, represents the secondary effect of frequency dispersion.

So far we allow the total variation (not the rate) of wave amplitude to be $O(\xi)$, and that of wave number and wave frequency to be $O(1)$. For a more restricted class of problems such as two of the examples that we shall consider later in Chapters 4 and 6 the total initial variation

of wave number is further restricted to be not more than $O(\epsilon)$, i.e.,

$$\vec{k} = (\bar{k}, 0) + \epsilon (\mu, \nu) \quad (3.26)$$

where \bar{k} , being a constant, is the wave number of the primary wave train propagating along the x-direction and $\epsilon(\mu, \nu)$ is a small perturbation. In order that (3.26) remains true for large distance and time we must require $\vec{k}_T; \vec{k}_X \sim O(\epsilon)$. Referring to (3.25 d) we note that

$$\vec{k}_T \sim \frac{\partial \omega_0}{\partial h} \nabla h + \frac{\partial \omega_0}{\partial \bar{k}} \cdot \vec{k}_X + O(\epsilon^2) \quad (3.27)$$

hence we require in turn that the variation of bottom depth to be $O(\epsilon)$ on the present length scale $O(\frac{1}{\epsilon})$, i.e.,

$$\nabla h \sim O(\epsilon) \quad (3.28)$$

In other words, $\nabla_2 h \sim O(\epsilon^4)$ on the natural scale. With the condition (3.26) and (3.28) equation (3.25 a-d) may be further simplified as follows.

From definitions, we obtain,

$$k = \sqrt{(\bar{k} + \epsilon\mu)^2 + (\epsilon\nu)^2} = \bar{k} \left\{ 1 + \epsilon \left(\frac{\mu}{\bar{k}} \right) + \epsilon^2 \frac{1}{2} \left(\frac{\nu}{\bar{k}} \right)^2 \right\} + O(\epsilon^3)$$

$$\vec{C}_g = (\bar{C}_g, 0) + \epsilon \left(\mu \frac{d\bar{C}_g}{d\bar{k}}, \nu \frac{\bar{C}_g}{\bar{k}} \right) + O(\epsilon^2)$$

$$\omega_0 = \bar{\omega} + \epsilon\mu \bar{C}_g + \epsilon^2 \frac{\mu^2}{2} \left(\frac{\partial \bar{C}_g}{\partial \bar{k}} \right) + \epsilon^2 \frac{\nu^2}{2} \left(\frac{\bar{C}_g}{\bar{k}} \right) + O(\epsilon^3) \quad (3.29 \text{ a,b,c})$$

where $\bar{\omega}$ and \bar{C}_g are respectively the wave frequency and the linear group velocity of the primary wave train. In (3.29 c) second order terms

are kept because in (3.25 d) the term $\nabla \omega_0$ is $O(U)$ as compared to other terms being $O(\varepsilon^2)$. Now making use of (3.26) and (3.29 a,b,c) and introducing the symbols $d = \eta^{(2,0)}$, $\vec{U} = (U, 0) = \nabla \phi^{(1,0)}$ and $\vec{k} = (\mu, \nu)$ the set of modulation equations (3.25) is reduced to:

$$\begin{aligned} \frac{\partial a^2}{\partial T} + \bar{c}_y \frac{\partial a^2}{\partial X} + \varepsilon \left(\frac{\partial \bar{c}_y}{\partial k} \right) \frac{\partial}{\partial X} (\mu a^2) + \varepsilon \left(\frac{\bar{c}_y}{k} \right) \frac{\partial}{\partial Y} (\nu a^2) &= O(\varepsilon^2) , \\ \frac{\partial \vec{k}}{\partial T} + \nabla \left\{ \bar{c}_y \mu + \varepsilon \left(\frac{\partial \bar{c}_y}{\partial k} \right) \frac{\mu^2}{2} + \varepsilon \left(\frac{\bar{c}_y}{k} \right) \frac{\nu^2}{2} + \varepsilon \omega_2 \right\} + \frac{\partial \omega_0}{\partial h} \nabla h &= O(\varepsilon^2) , \\ \frac{\partial d}{\partial T} + \nabla \cdot h \vec{U} + \frac{\partial}{\partial X} \left(\frac{1}{2} g \frac{\bar{p}}{\bar{\omega}} a^2 \right) &= O(\varepsilon) , \\ \frac{\partial \vec{U}}{\partial T} + \nabla \left\{ g d + \frac{1}{2} g D_0 a^2 \right\} &= O(\varepsilon) \end{aligned} \quad (3.30 \text{ a - d})$$

where only $O(1)$ terms are kept in (3.30c,d) because d and \vec{U} are one order magnitude smaller than a and μ . The second order correction of wave frequency is much more simplified (see Appendix C), i.e.,

$$\omega_2 = \frac{1}{2} (\bar{\omega} \bar{k}^2 S_0) a^2 - \frac{1}{2} \left(\frac{\partial \bar{c}_y}{\partial k} \right) \frac{a_{XX}}{a} + \frac{1}{2} \left(\frac{\bar{c}_y}{k} \right) \frac{a_{YY}}{a} + \bar{k} U + (\bar{\omega} \bar{k} D_0) d \quad (3.31)$$

where

$$S_0 = \frac{1}{8} (9\bar{\sigma}^4 - 10\bar{\sigma}^2 + 9) \quad \text{and} \quad D_0 = \frac{1}{2} \left(\sigma - \frac{1}{\sigma} \right) . \quad (3.32 \text{ a,b})$$

We emphasize that in (3.30 b) the term $\frac{\partial \omega_0}{\partial h} \nabla h$, representing the refraction effect must be $O(\varepsilon)$ in order to be consistent with the condition (3.26). It is hoped that the system (3.30) may be used as a basis for studying the effect of depth variation on instability or non-linear evolution.

We remark that, in the linear instability analysis, the total variation of wave amplitude and wave number is further restricted, namely,

$$a = \bar{a} + \Delta a, \quad k = \bar{k} + \Delta k, \quad \omega = \bar{\omega} + \Delta \omega \quad (3.33)$$

where \bar{a} , \bar{k} , $\bar{\omega}$ are constant and Δa , Δk , $\Delta \omega$ are the small disturbances. More specifically, we require

$$\frac{\Delta a}{\bar{a}} \sim O(\epsilon) \quad \text{and} \quad \frac{\Delta k}{\bar{k}}, \frac{\Delta \omega}{\bar{\omega}} \sim O(\epsilon^2) \quad (3.34)$$

so that (3.30) can be linearized. A comparison of the limitation of different theories is summarized in Table 1.

	Total Variation of a, k & ω		Modulation Length Scale
	$\frac{\Delta a}{\bar{a}}$	$\frac{\Delta k}{\bar{k}}$ & $\frac{\Delta \omega}{\bar{\omega}}$	
Whitham's Averaged Equations	$O(1)$	$O(1)$	$O(\frac{1}{\epsilon^2})$
Equation (3.25)	$O(1)$	$O(1)$	$O(\frac{1}{\epsilon})$
Equation (3.30)	$O(1)$	$O(\epsilon)$	$O(\frac{1}{\epsilon})$
Linear Instability Theory of Benjamin & Feir	$O(\epsilon)$	$O(\epsilon^2)$	$O(\frac{1}{\epsilon})$

TABLE 1. RANGE OF VALIDITY FOR DIFFERENT THEORIES; (ϵ = wave steepness)

CHAPTER 4

LINEARIZED INSTABILITY THEORY OF STOKES WAVES

Benjamin and Feir (1967) considered the two-dimensional nonlinear interaction in deep water between a Stokes wave train and two wave trains with frequencies and wave numbers slightly different from the Stokes wave. These 'side-band disturbances' if within certain 'cutoff' limits grow exponentially with time and space. They also performed experiments in a long tank in which a train of deep water waves of constant amplitude was generated at one end of the tank. Far downstream it was observed that the wave eventually disintegrated into rather irregular manner. By artificial side-band disturbances measurement of the initial growth was found to agree reasonably well with their theoretical prediction. Thus, it is now firmly established that Stokes waves of constant amplitude are unstable in deep water.

This instability analysis was later extended to finite depth by Benjamin (1967). He found another cutoff limit at $kh = 1.36$; that is for $kh < 1.36$ the waves are stable and for $kh > 1.36$ the waves may be unstable. Whitham (1967), based on his modulation equations, arrived at the same cutoff limit of $kh = 1.36$ but could give no information on the cutoff limit of the side-band and the growth rate. Benney and Roskes (1969), using the method of multiple scales, made several generalizations, among which the modulation wave was taken to be oblique with respect to the primary Stokes wave train. New side-band cutoff limits were found.

We shall now demonstrate that the same problem can be equally well treated on the basis of (3.30) in the manner of Whitham (1967 a).

We take $h = \text{constant}$ and allow for small disturbances to a , \vec{k} , d and \vec{U} so that

$$a = \bar{a} + \varepsilon a' \quad , \quad \vec{k} = \varepsilon \vec{k}' = \varepsilon (\mu', \nu') \quad ,$$

$$d = \bar{d} + \varepsilon d' \quad , \quad \vec{U} = (U, 0) + \varepsilon \vec{U}' = (\bar{U}, 0) + \varepsilon (U', V') \quad . \quad (4.1 \text{ a-d})$$

Linearizing (3.30) with respect to primed quantities, and dropping the symbol $\bar{\quad}$ for the constant state (i.e. the Stokes waves with constant amplitude):

$$\left(\frac{2}{a}\right) a'_T + \left(\frac{2c_g}{a}\right) a'_X + \varepsilon \left(\frac{\partial c_g}{\partial k}\right) \mu'_X + \varepsilon \left(\frac{c_g}{k}\right) \nu'_Y = O(\varepsilon^2) \quad ,$$

$$\vec{k}'_T + \nabla \left\{ c_g \mu' + \varepsilon (\omega k a^2 S_0) a' - \varepsilon \left(\frac{\partial c_g}{\partial k}\right) \frac{a'_{XX}}{a} \right. \\ \left. - \varepsilon \left(\frac{c_g}{k}\right) \frac{a'_{YY}}{a} + \varepsilon k U' + \varepsilon (\omega k D_0) d' \right\} = O(\varepsilon^2) \quad ,$$

$$d'_T + h \nabla \cdot \vec{U}' + \left(g \frac{k}{\omega} a\right) a'_X = O(\varepsilon) \quad ,$$

$$\vec{U}'_T + \nabla \{ g d' + g D_0 a a' \} = O(\varepsilon) \quad . \quad (4.2 \text{ a-d})$$

The system of homogeneous linear partial differential equations (4.2) admits sinusoidal solutions (the eigenfunctions) represented by,

$$(a', \mu', \nu', d', U', V') = \vec{\Delta} \exp [i(\vec{k} \cdot \vec{X} - \Omega T)] \quad (4.3)$$

where $\vec{\Delta}$, $\vec{k} = (k_1, k_2)$ and Ω are the amplitude, wave number and frequency for the small disturbances. Substituting (4.3) into (4.2) we obtain a set of algebraic equations which can be written in matrix form as

$$\tilde{A} \vec{\Delta} = 0 \quad (4.4)$$

where

$$\tilde{A} = \begin{bmatrix} \frac{2}{a}(-\Omega + c_g k_1) & \varepsilon \frac{\partial c_g}{\partial k} k_1 & \varepsilon \left(\frac{c_g}{k}\right) k_2 & 0 & 0 & 0 \\ \varepsilon^2 A_0 k_1 & (-\Omega + c_g k_1) & 0 & \varepsilon^2 \omega k D_0 k_1 & \varepsilon^2 k k_1 & 0 \\ \varepsilon^2 A_0 k_2 & c_g k_2 & -\Omega & \varepsilon^2 \omega k D_0 k_2 & \varepsilon^2 k k_2 & 0 \\ g \frac{ka}{\omega} k_1 & 0 & 0 & -\Omega & h k_1 & h k_2 \\ g D_0 a k_1 & 0 & 0 & g k_1 & -\Omega & 0 \\ g D_0 a k_2 & 0 & 0 & g k_2 & 0 & -\Omega \end{bmatrix}, \quad (4.5 \text{ a,b})$$

$$A_0 = \omega k^2 a S_0 + \left(\frac{\partial c_g}{\partial k}\right) \frac{k_1^2}{2a} + \left(\frac{c_g}{k}\right) \frac{k_2^2}{2a}.$$

For non-trivial solution of $\vec{\Delta}$ we must have,

$$\det \tilde{A} = 0 \quad (4.6)$$

which may be solved by successive approximations and yield six eigen-frequencies (see Appendix D). Four of them are real to the order $O(\varepsilon)$, i.e.,

$$\Omega = \pm O(\varepsilon)$$

$$\text{and } -\Omega = \pm \sqrt{gh (k_1^2 + k_2^2)} + O(\varepsilon) \quad (4.7 \text{ a,b})$$

which corresponds to stable solutions. The remaining two may be complex:

$$\Omega/k_1 = c_g \pm \varepsilon \frac{1}{2} \frac{\omega_0}{k} \gamma^{\frac{1}{2}} \left[\left(\frac{k_1}{k}\right)^2 \gamma + 2(ka)^2 \mathcal{K} \right]^{\frac{1}{2}} + O(\varepsilon^2) \quad (4.8)$$

where

$$\mathcal{X} = S_0 + \frac{(g/\omega_0)^2 + 2(g/\omega_0)D_0 C_g + gh D_0^2 \sec^2 \theta}{c_g^2 - gh \sec^2 \theta}$$

$$\gamma = \frac{k C_g}{\omega_0} \left(\frac{k}{c_g} \frac{\partial C_g}{\partial k} + \tan^2 \theta \right)$$

(4.9 a,b,c)

$$\theta = \tan^{-1}(k_2/k_1)$$

The above result is completely equivalent to that of Benney & Roskes (1969) (see also Roskes (1969)).

The solution (4.8) is complex, and hence unstable, if

$$\gamma \left[\left(\frac{k_1}{k} \right)^2 \gamma + 2(ka)^2 \mathcal{X} \right] < 0$$

or, equivalently in the first quadrant,

$$\tan \theta < - \frac{k}{c_g} \frac{\partial C_g}{\partial k} \quad \left(\frac{k_1}{ka} \right)^2 > - \frac{2\mathcal{X}}{\gamma} \quad (4.10 \text{ a,b})$$

The regions of instability, that is the intersections of (4.10 a,b), are given in Figure 3 for three different depths $kh \rightarrow \infty$, $kh = 1.36$ and $kh = 1.0$. Figure 3(c) shows that the Stokes waves can be unstable for $kh < 1.36$ if the disturbances are allowed to be oblique to the primary wave train. In general the Stokes waves are unstable for all depths except for $kh = 0.38$ the waves are stable for disturbances in any direction (see Benney & Roskes (1969)). The growth rate for the unstable waves may be obtained by

$$\begin{aligned} & \left(\frac{\partial}{\partial T} + c_g \frac{\partial}{\partial X} \right) \ln \{ \Delta \exp[i(\vec{k} \cdot \vec{X} - \Omega T)] \} \\ & = \frac{1}{2} i \epsilon k_1 \frac{\omega_0}{k} \gamma \left[\left(\frac{k_1}{k} \right)^2 \gamma + 2(ka)^2 \mathcal{X} \right]^{\frac{1}{2}} + O(\epsilon^2) \end{aligned} \quad (4.11)$$

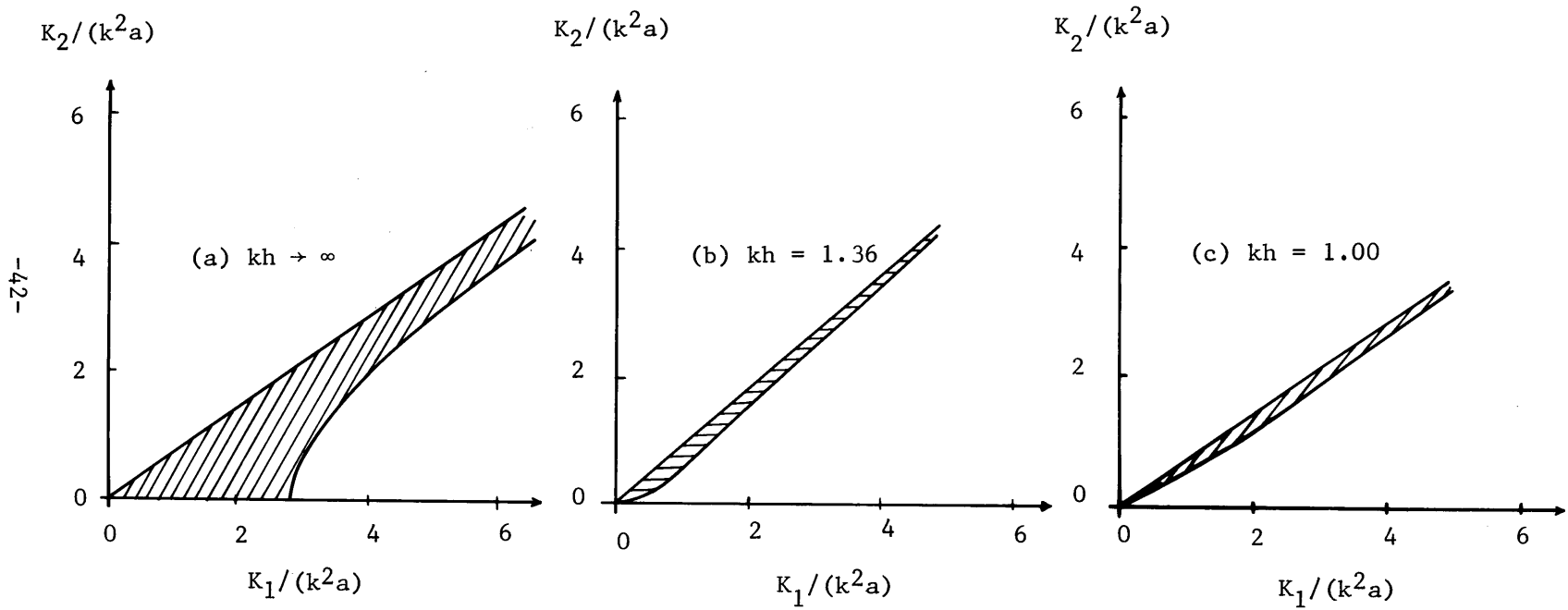


FIGURE 3 INSTABILITY REGIONS (SHADOWED) FOR STOKES WAVES

(regions in other quadrants can be obtained
by mirror reflection about K_1 and K_2 axes)

A more detailed discussion on the regions of instability and the direction for maximum growth in the $(K_1 - K_2)$ plane has been given by Benney & Roskes (1969) and will be omitted here. We point out that for $\Theta = 0$ (i.e., $K_2 = 0$) equations (4.8), (4.9) and (4.11) are reducible to Benjamin (1967, equation (46)) while Whitham's result (1967 a, equation (57)) is obtained by further letting $K_1 = 0$ (i.e. corresponding to zero side-band width).

CHAPTER 5

QUASI-STEADY WAVE TRAINS OVER CLOSED BEACH

In the coastal engineering literature the problem of wave shoaling has occupied an important position. Much has been attempted in developing a theory that would (i) correctly predict shoaling characteristics of finite amplitude waves (ii) hopefully lead to correct quantitative understanding of breaking. Both aspects are of practical significance in the inshore beach processes. Many publications have been devoted along the lines of Rayleigh's classical theory of refraction of infinitesimal waves i.e. by assuming (i) Stokes theory for a horizontal bottom to be valid locally and (ii) energy flux in Stokes waves is constant between rays. The more complete theory was largely the contributions of Longuet-Higgins and Stewart (1964) and Whitham (1962). Their approach is a systematic averaging over a distance much longer than a wave length and integrating over the depth. Since the problem is one of slow modulation caused by the slow variation of water depth, the present general theory should apply. We shall now show certain new second order features that have not been discussed heretofore.

Before going into the details, it would seem that Benjamin & Feir's and Benney & Roskes' theories of instability of Stokes waves would render the study of time periodic waves meaningless. However, firstly whether the uneven bottom may alter the conclusions of instability is

still to be studied. Secondly, over natural beaches refraction tends to turn all waves to normal incidence, reducing the likelihood of oblique side-band disturbances. Hence waves can still be stable in sufficiently shallow water. Thirdly, the growth of instability is a very slow process requiring a long distance for its manifestation. If the waves are of small amplitude compared to bottom slope, nonlinear effects will not be present until shallow water is reached, but then instability may not develop sufficiently fast to alter the basic picture before the occurrence of breaking near the shore. For these reasons it is still worthwhile to establish a good theory of shoaling of finite amplitude waves.

One important question is the reflection by the beach. This unfortunately cannot yet be answered before the mechanics of breaking is completely known. It is therefore necessary to take the simplifying yet reasonable assumption that all incident wave energy is absorbed at the breaker and no energy is reflected.

Now, for quasi-steady wave trains, the wave parameters such as a , $\frac{1}{k}$, ω , $\nabla\phi^{(1,0)}$ and $\eta^{(2,0)}$ are real and independent of t . The following classical results are immediate from equations (3.25 a,b,c,d):

$$\nabla \cdot \left(\vec{C}_g \frac{E}{\omega_0} \right) = 0 \quad , \quad (5.1)$$

$$h \nabla \phi^{(1,0)} = - \frac{1}{k} E + \text{constant} \quad (5.2)$$

$$\eta^{(2,0)} = - \frac{1}{9} \frac{k_{\infty}}{2} (\sigma^2 - 1) E \quad (5.3)$$

$$\omega = \omega_0 + \varepsilon^2 \omega_2 + O(\varepsilon^3) = \text{constant} \quad (5.4)$$

Equation (5.1) represents the well-known energy conservation in refracting waves which can be further integrated explicitly in two dimensions

$$\frac{a}{a_\infty} = \left[\frac{k}{k_\infty} \left(\frac{\sinh 2q}{\sinh 2q + 2q} \right) \right]^{\frac{1}{2}} \quad (5.5)$$

where a_∞ is the first order amplitude in infinitely deep water. Equation (5.2) states the constancy of mass flux due to waves, first obtained by Whitham (1962). In the case of a closed beach the constant vanishes and $\nabla \phi^{(1,0)}$ represents the return current. Equation (5.3) gives the mean sea level change first predicted by Longuet-Higgins & Stewart (1962) as a consequence of radiation stress.

In the rest of this section we restrict to the case of normal incidence over a closed beach of constant slope, hence $\vec{k} = (k, 0)$ and $p_{xx} = 0$ etc. Now the solutions for Φ and η may be rewritten from (3.24) as,

$$\Phi(x, z, t) = \varepsilon \phi^{(1,0)} + \varepsilon 2i \phi^{(1,1)} \sin\left(\frac{\psi}{\varepsilon} + \varepsilon \delta\right) + \varepsilon^2 2i \phi^{(2,2)} \sin 2\left(\frac{\psi}{\varepsilon} + \varepsilon \delta\right) + O(\varepsilon^3)$$

$$\eta(x, t) = \varepsilon^2 \eta^{(2,0)} + \varepsilon 2\eta^{(1,1)} \cos\left(\frac{\psi}{\varepsilon} + \varepsilon \delta_0\right) + \varepsilon^2 2\eta^{(2,2)} \cos 2\left(\frac{\psi}{\varepsilon} + \varepsilon \delta_0\right) + O(\varepsilon^3) \quad (5.6 \text{ a,b})$$

where the amplitude functions $\phi^{(1,0)}$, $\eta^{(2,0)}$, $\phi^{(1,1)}$, $\eta^{(1,1)}$, $\phi^{(2,2)}$ and $\eta^{(2,2)}$ are given by (5.2), (5.3), (2.22 a,b) and (3.15 a,b) respectively. One feature that is not known in existing theories is the phase lag

δ given in (3.23). For normal incidence, (3.23) can be further simplified to give,

$$\delta/p_x = q \left\{ -\left(\frac{z}{h}\right) + \frac{\sinh^2 2q}{(\sinh 2q + 2q)^2} \left[\left(1 + \frac{z}{h}\right) \frac{\tanh\left[\frac{q}{2}\left(1 + \frac{z}{h}\right)\right]}{\tanh q} - 1 \right] + \frac{q \left[2\left(\frac{z}{h}\right) + \left(\frac{z}{h}\right)^2 \right]}{(\sinh 2q + 2q)} \right\} \quad (5.7)$$

which has the following properties:

- (i) $(\delta/p_x) \rightarrow 0$ as $q \rightarrow 0$ for all (z/h) ,
- (ii) $(\delta/p_x) \rightarrow 0$ as $q \rightarrow \infty$ for all (z/h) ,
- (iii) (δ/p_x) vanishes identically for $(z/h) = 0$,
- (iv) $(\delta/p_x) = q^2 \frac{3 \sinh 2q + 2q}{(\sinh 2q + 2q)^2}$ (always positive) for $(z/h) = -1$

The phase lag (δ/p_x) as a function of $(p/k_0 h)$ & (z/h) is given in Figure 4(a).

Due to the z dependence of the phase lag δ a surface of constant phase is no longer vertical, but is given by

$$\psi = \int^x k dX + \varepsilon^2 \delta(X, z) - \omega_0 T_0 = \text{constant} \quad (5.8)$$

for fixed $T = T_0$. Equation (5.8) may be rewritten as

$$\begin{aligned} \psi &= \psi_0 = \text{constant} \\ &= \int^{x_0} k dX + \int_{x_0}^x \left\{ k(x_0) + (x-x_0) \frac{\partial k}{\partial x_0} + \dots \right\} dX \\ &\quad + \varepsilon^2 \delta(X, z) - \omega_0 T_0 \end{aligned} \quad (5.9)$$

With x_0 being defined by $\int^{x_0} k dX - \omega_0 T_0 = \psi_0$ the equal phase surface (or the wave front) can be written explicitly from (5.9) as:

$$F(X, z) = X - X_0 - \varepsilon^2 \frac{\delta(X_0, z)}{R(X_0)} = O(\varepsilon^4) \quad (5.10)$$

It is interesting to note that the wave front given by $F(X, z) = 0$ is perpendicular to the free surface $z = 0$ and the bottom $z + h(X) = 0$. To establish this property we have to prove in natural coordinates,

$$\left(\frac{\partial F}{\partial X}, \frac{\partial F}{\partial z} \right) \cdot \left(\frac{\partial z}{\partial X}, \frac{\partial z}{\partial z} \right) = 0 \quad \text{at} \quad z = 0, \quad (5.11 \text{ a,b})$$

and

$$\left(\frac{\partial F}{\partial X}, \frac{\partial F}{\partial z} \right) \cdot \left(\frac{\partial [z + h(X)]}{\partial X}, \frac{\partial [z + h(X)]}{\partial z} \right) = 0 \quad \text{at} \quad z = -h,$$

or, to prove equivalently, in strained coordinates,

$$\left(\varepsilon, \frac{\varepsilon^2}{R} \delta z \right) \cdot (0, 1) = \varepsilon^2 \frac{\delta z}{R} = 0 \quad \text{at} \quad z = 0, \quad (5.12 \text{ a,b})$$

and

$$\left(\varepsilon, \frac{\varepsilon^2}{R} \delta z \right) \cdot (\varepsilon h_x, 1) = \varepsilon^2 \left(h_x + \frac{\delta z}{R} \right) = 0 \quad \text{at} \quad z = -h.$$

Differentiating (3.23) with respect to z we obtain,

$$\delta z / R = - \left\{ h_x + \frac{A_x^{(1,1)}}{A^{(1,1)}} (\tanh Q + Q \operatorname{sech} Q) + \frac{R_x}{R^2} Q \right\} \quad (5.13)$$

It follows immediately $\delta z / R = -h_x$ at $z = -h$ so that (5.12 a) is true.

At $z = 0$,

$$\begin{aligned} \delta z / R &= \frac{1}{R A^{(1,1)2} \cosh^2 q} \left\{ A^{(1,1)2} \cosh^2 q \left(R h_x + \frac{R_x}{R} q \right) + \frac{A^{(1,1)2}}{4} (\sinh 2q + 2q) \right\} \\ &= \frac{\{ A^{(1,1)2} (\sinh 2q + 2q) \}_x}{4 R A^{(1,1)2} \cosh^2 q} = \frac{R_\infty (a^2 C_g)_x}{\omega_0 R A^{(1,1)2} \cosh^2 q} \equiv 0 \quad (5.14) \end{aligned}$$

since $(a^2 c_g)_x = 0$ from (5.1). Hence (5.12 b) is also established. Typical geometry of equal phase curves is given in Figure 4(b) & 4(c) which show that the wave fronts are indeed perpendicular to the free surface and the bottom (note that, with the normalized scale, it is shown in Figure 4(b) & 4(c) that $\frac{d(k_0 \delta / k h_x)}{d(k_0 z)} = 0$ at $z = 0$ and $\frac{d(k_0 \delta / k h_x)}{d(k_0 z)} = 1$ at $(\frac{z}{h}) = -1$ as given by (5.12 a,b)). Figure 4(b) & 4(c) also show the concavity of the wave fronts towards the shore; Battjes' (1968) speculation that they are circular is incorrect.

Referring to (5.4) $\omega_0 = \omega - \varepsilon^2 \omega_2 + O(\varepsilon^4)$ is not strictly a constant so that the second order correction for the wave number may be calculated by expanding $k = k_0 + \varepsilon^2 k_2 + \dots$ as follows. From (2.23) we have

$$[k_0 + \varepsilon^2 k_2 + \dots] \tanh[k_0 + \varepsilon^2 k_2 + \dots] h = \frac{(\omega_0 + \varepsilon^2 \omega_2 + \dots)^2}{g}$$

which, after expanding into Taylor's series, gives

$$k_0 \tanh k_0 h = \omega_0^2 / g \quad (5.15)$$

$$k_2 = -\omega_2 / \left[\frac{\omega}{2k} \left(1 + \frac{2g}{\sinh 2g} \right) \right] \quad (5.16)$$

Referring to (3.18)

$$k_2 = -\frac{1}{c_g} (\omega_2^S + \omega_2^U + \omega_2^D + \omega_2^X) \quad (5.17)$$

We point out that ω_2^X is new. The expression for ω_2^X is lengthy and is given in Appendix C. Various contributions to the second order

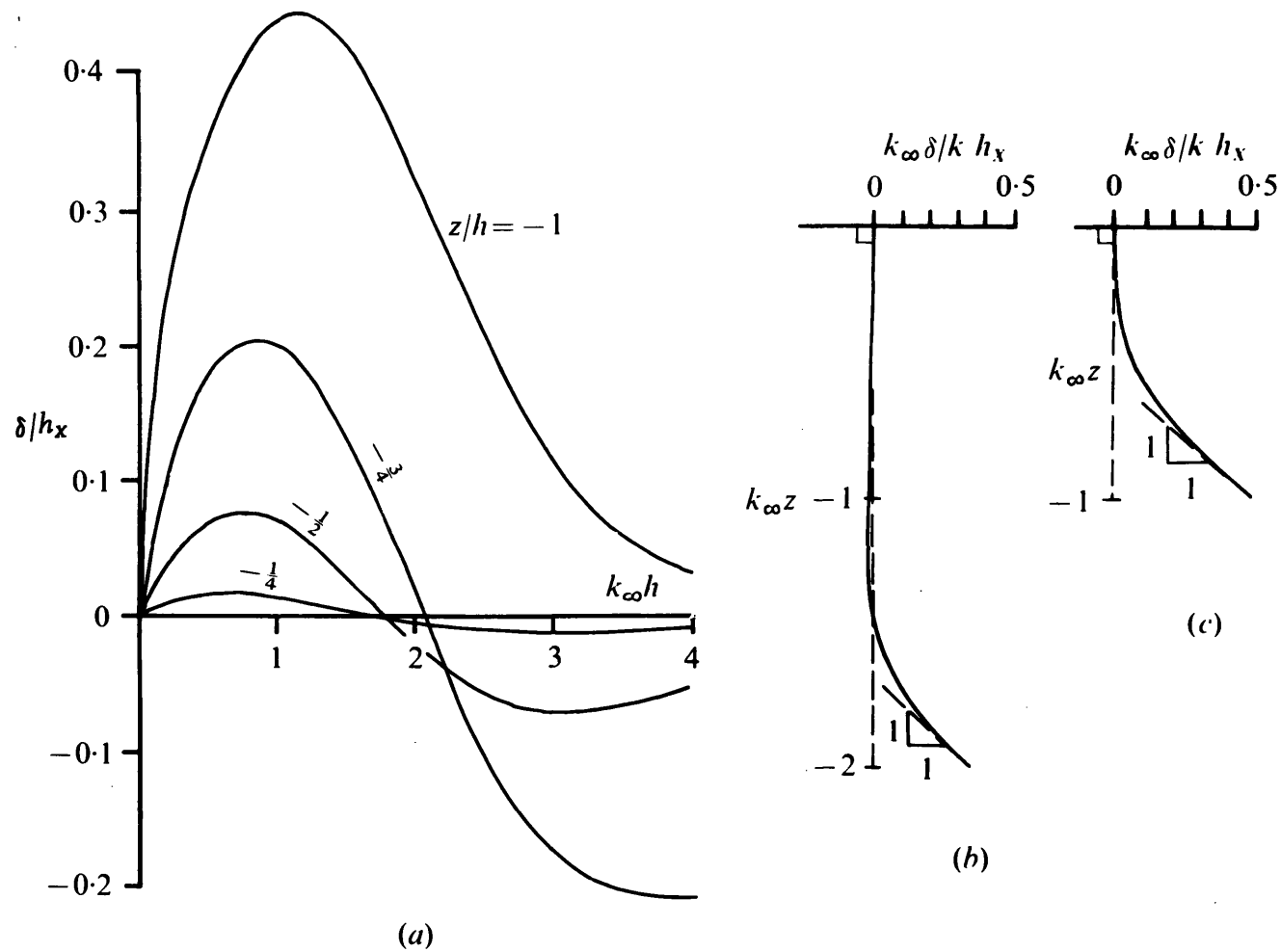


FIGURE 4 PHASE CHANGE ON A PLANE BEACH; NORMAL INCIDENCE

- (a) δ/h_x for $z/h = (0, -1/4, -1/2, -3/4, -1)$
- (b) wave front for $k_\infty h = 2$
- (c) wave front for $k_\infty h = 1$

correction to the local wave number are calculated and plotted in Figure 5.

It should be noted that over the beach, the Stokes' term $-\frac{\omega_2^S}{C_g}$ is always negative while $-\frac{\omega_2^D}{C_g}$, $-\frac{\omega_2^U}{C_g}$ and $-\frac{\omega_2^X}{C_g}$ are positive. In other words, $-\frac{\omega_2^S}{C_g}$ would tend to give a larger phase speed than that given by the first order linearized theory ($\frac{\omega_0}{k_0}$) whereas $-\frac{1}{C_g}(\omega_2^D + \omega_2^U + \omega_2^X)$ would do just the opposite. In particular the return current term $-\frac{\omega_2^U}{C_g}$ (now $\nabla \phi^{(1,0)} = -\frac{1}{2}ka^2$) nearly cancels the Stokes term $-\frac{\omega_2^S}{C_g}$. The two opposing trends seems to explain that in experimental measurements of wave speed on plane beaches, the first order linearized theory often gives a better prediction than an incomplete one using only the Stokes term, i.e., by $-\frac{\omega_2^S}{C_g}$ only (Eagleson, 1956). We also note that the contribution of $-\frac{\omega_2^S}{C_g}$ can be quite significant when the bottom slope is large compared to the wave steepness ka . The dependence of $\frac{(k_2/k_0)}{(k_0 a_0)^2}$ on $\frac{p}{k_0 h}$ and $\frac{hx}{(k_0 a_0)}$ is given in Figure 6. Although the modification to the wave number is only second order, for three dimensional problems the refraction diagram may be significantly modified (especially near the caustic) due to the cumulative second order effect along the wave orthogonal.

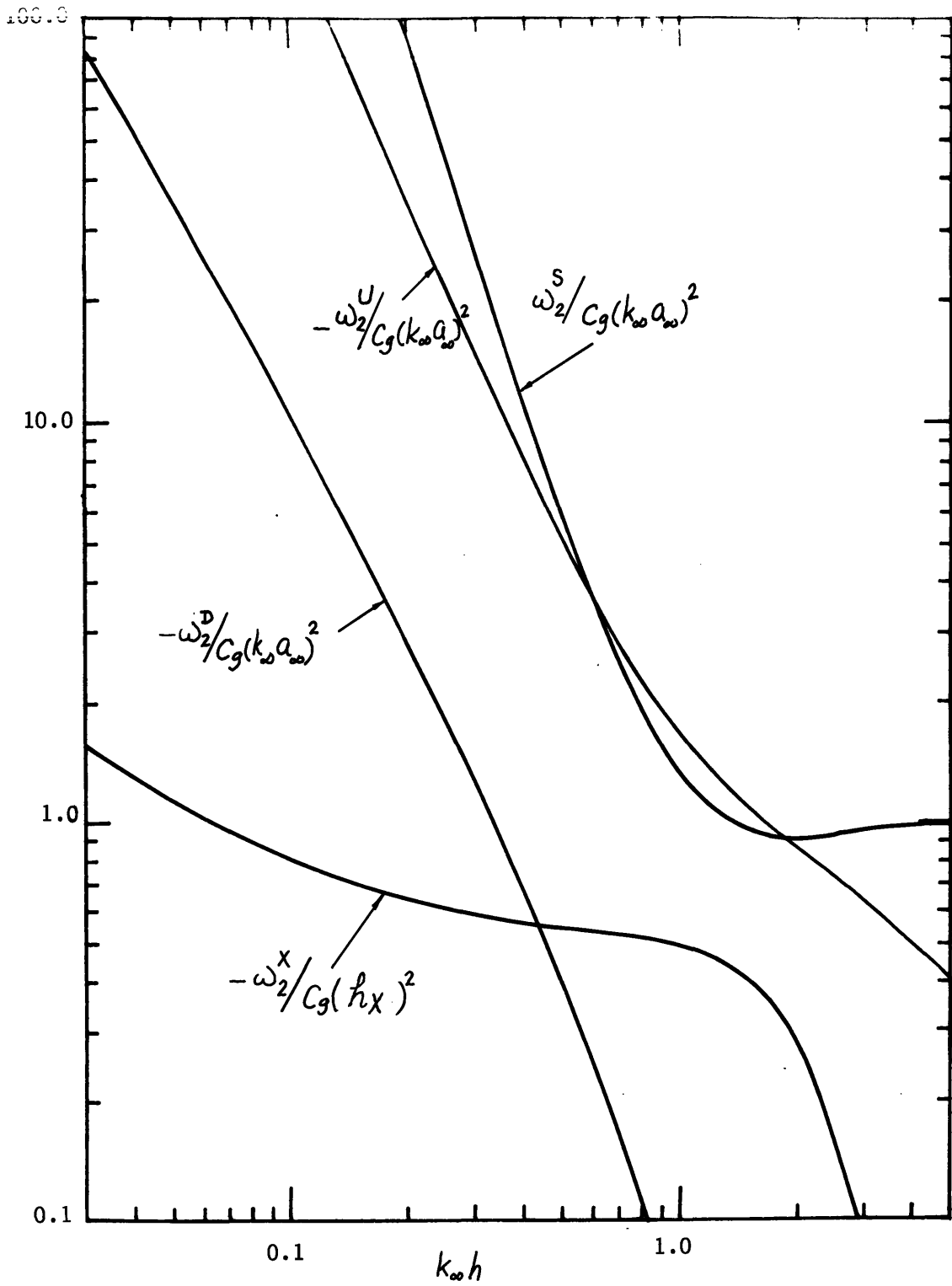


FIGURE 5 VARIOUS CONTRIBUTIONS TO SECOND ORDER MODIFICATION OF WAVE NUMBER; NORMAL INCIDENCE ON PLANE BEACH

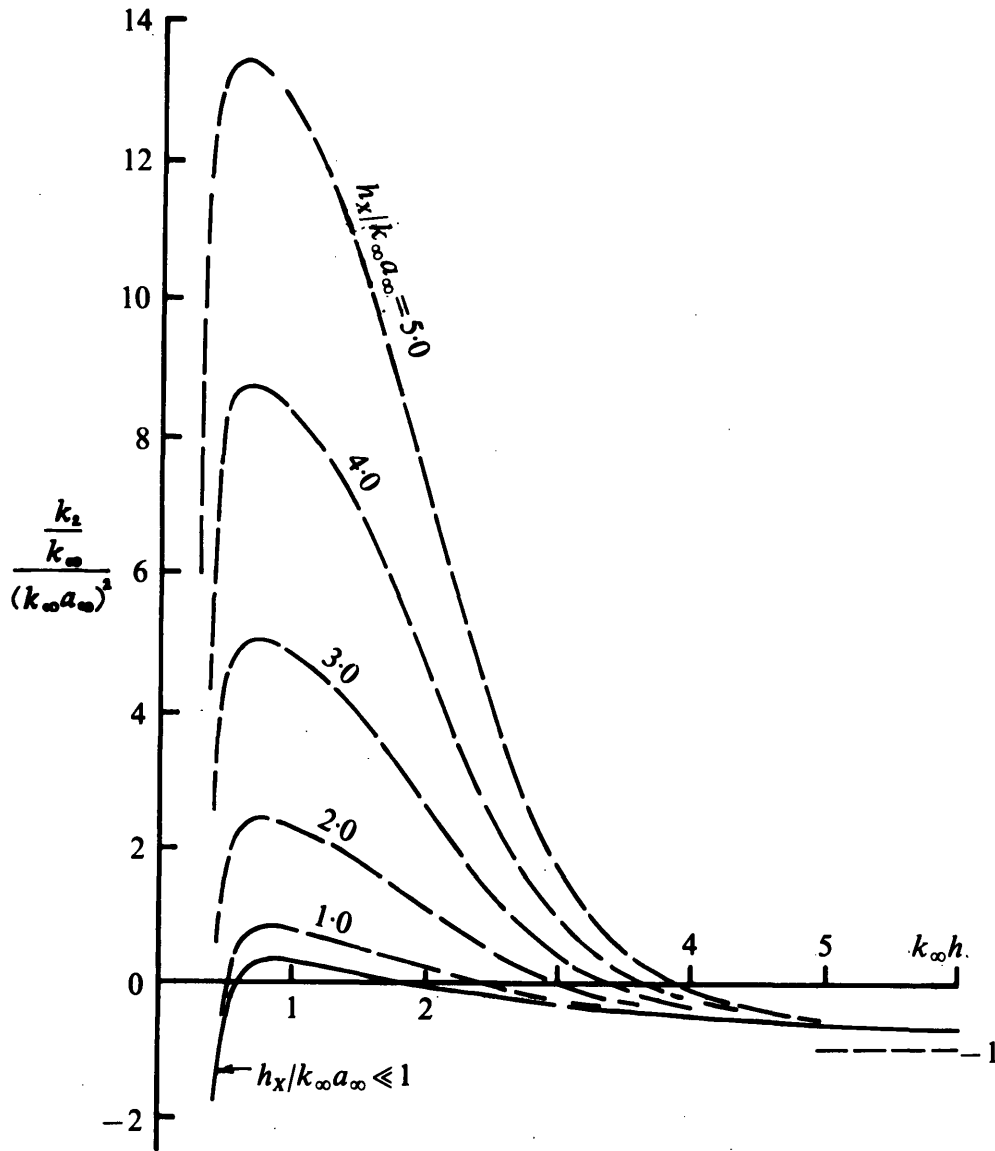


FIGURE 6 TOTAL SECOND ORDER MODIFICATION OF WAVE NUMBER; NORMAL INCIDENCE ON PLANE BEACH

CHAPTER 6

NONLINEAR EVOLUTION OF WAVE ENVELOPES IN DEEP WATER

In the linearized instability analysis of Stokes waves the initial disturbances are assumed to be small. The result is only valid for the initial growth of the unstable wave as long as it is still slightly deviated from an uniform wave train. Now it is natural to ask what is the subsequent development after the onset of instability. Would the waves settle down to some other stable situation or the waves would become so irregular that they cannot be considered as deterministic process? To understand the nonlinear process at large, it is therefore important to study the nonlinear evolution of the wave envelopes.

A particular example of nonlinear evolution of a wave group with non-uniform amplitude was studied by Lighthill (1965 b, 1967) based on Whitham's averaged variational principle. For a symmetrical envelope pulse (Figure 7) with initially uniform wave length, Lighthill found that the wave length decreases in the front of the group and increases behind the group, and energy tends to concentrate at the center. Eventually, the amplitude peaks up to a cusp at the center and the frequency changes rapidly, displaying a discontinuity in wave length at the center of the wave group. Such a discontinuity was speculated by Lighthill to be in some way analogous to the aerodynamic 'shock' in compressible flow. Using the same variational technique Howe (1967,1968) considered the non-

x_{50} = half width at $\bar{a}/\bar{a}_{\max} = 0.5$, $t = 0$

t_{crit} = critical time when the discontinuity occurs

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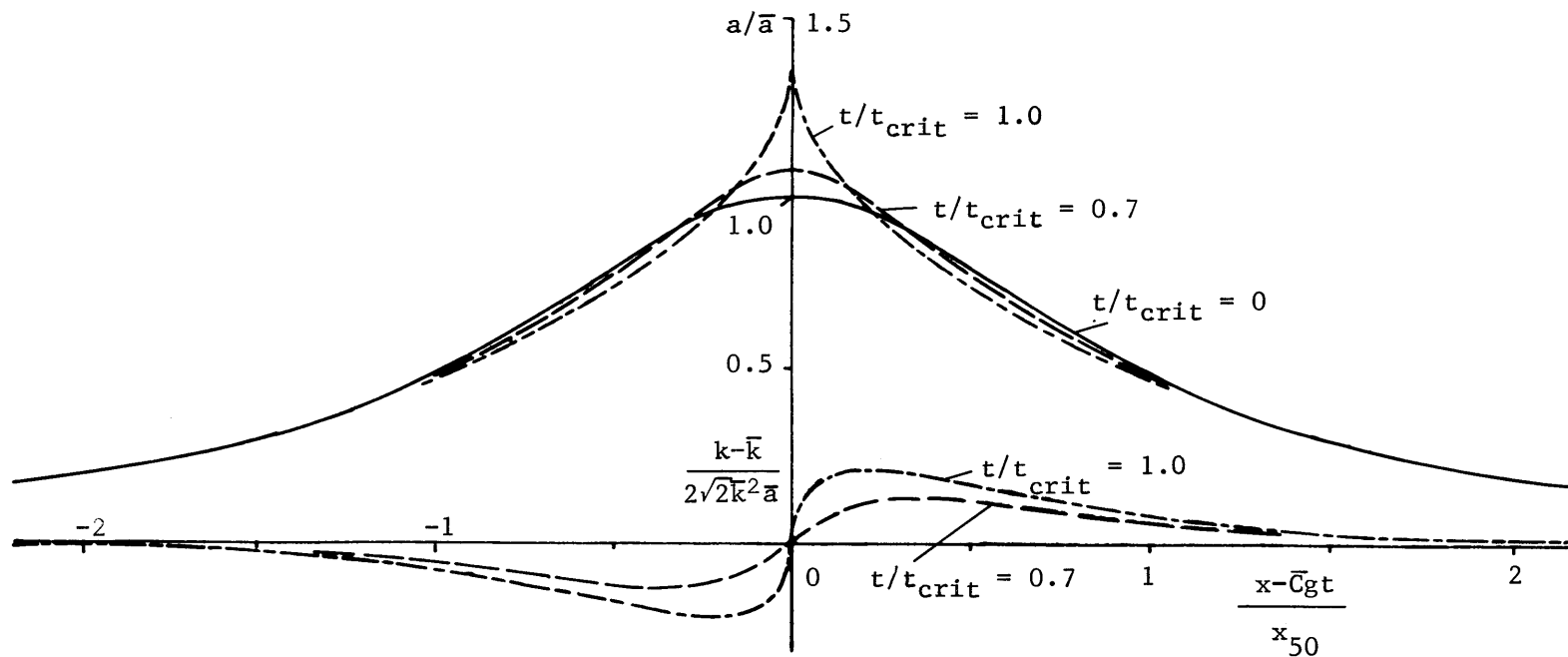


FIGURE 7 NONLINEAR EVOLUTION OF A WAVE PULSE IN DEEP WATERS

(taken from Lighthill, 1965)

linear evolution of waves generated by steady uniform flow passing a slowly modulated wavy wall (to simulate the diverging waves developed by a ship). A frequency 'shock', which he called a 'phase jump', was also obtained in his numerical calculations.

As pointed out before, Whitham's averaging method is valid only when the modulation rate is small compared to the wave steepness. Therefore the same restriction must apply to Lighthill's and Howe's analyses. As is indeed implied by Lighthill's result that modulation always increases gradually in certain part of the wave group; the modulation rate, inevitably, becomes comparable to the wave steepness, leading to the complete break-down of the theory. Thus, uniform validity of Lighthill's and Howe's analysis over indefinitely long time is out of the question. Indeed, long before the actual occurrence of the discontinuity, i.e. the 'shock' or the 'phase jump', the basic assumption of slow modulation is already violated.

Although experimental results (Feir 1967) agree qualitatively with some of Lighthill's predictions in the initial development of the wave pulse there appear to be essential discrepancies in the large time behavior, especially with regard to the frequency 'shock'. Based on the present more complete theory, the objective of this chapter is to extend the linear instability analysis and Lighthill's nonlinear analysis for large time. Special attention is given to the consideration of the interplay between amplitude dispersion and frequency dispersion.

For simplicity we shall consider only the two-dimensional wave propagation in deep water. In this case the mean current and the change

of water level are uncoupled with the wave amplitude so that the governing modulation equations reduce to two with only two dependent variables, i.e. the wave amplitude and wave number. Now the physical process is also the simplest in that the dispersion effect involves only frequency dispersion and the so-called amplitude dispersion due to nonlinear coupling of amplitude in the dispersion relation.

6.1 GOVERNING EQUATIONS

Consider the case that the variation of wave number is initially small so that (3.26) is satisfied and the governing modulation equations are given by (3.30). In deep water where $kh \rightarrow \infty$, $\sigma \rightarrow 1$ and $D_o \rightarrow 0$ the terms proportional to the square of wave amplitude drop out from equations (3.30 c,d) so that the long wave components d and \vec{U} are decoupled with the wave amplitude and wave number. Furthermore, we have $\frac{\partial U}{\partial X} \rightarrow 0$ from (3.30 c), so that the modulation equations (3.30 a,b) may be given, in two dimensions, as follows,

$$\begin{aligned} \left(\frac{\partial}{\partial T} + \bar{c}_g \frac{\partial}{\partial X} \right) a^2 + \varepsilon \left(\frac{d\bar{c}_g}{dk} \right) \frac{\partial}{\partial X} (\mu a^2) &= 0 \\ \left(\frac{\partial}{\partial T} + \bar{c}_g \frac{\partial}{\partial X} \right) \mu + \varepsilon \frac{\partial}{\partial X} \left\{ \left(\frac{d\bar{c}_g}{dk} \right) \frac{\mu^2}{2} + \frac{\bar{\omega}}{2} (\bar{k}a)^2 - \frac{1}{2} \left(\frac{d\bar{c}_g}{dk} \right) \frac{Q_{XX}}{a} \right\} &= 0 \end{aligned} \quad (6.1 \text{ a,b})$$

Notice now the changes of a^2 and μ following the group velocity are always $O(\varepsilon)$. We may therefore switch to a moving coordinate and rescale the time variable to account for the slow evolution, i.e.

$$\xi = X - \bar{c}_g T \quad \text{and} \quad \tau = \varepsilon T \quad (6.2 \text{ a,b})$$

With this moving coordinate system the modulation equations (6.1 a,b) may be more conveniently written as

$$\frac{\partial a^2}{\partial \tau} + \left(\frac{d\bar{c}_g}{d\bar{k}}\right) \frac{\partial}{\partial \xi} (\mu a^2) = 0 \quad (6.3 \text{ a,b})$$

$$\frac{\partial \mu}{\partial \tau} + \frac{\partial}{\partial \xi} \left\{ \left(\frac{d\bar{c}_g}{d\bar{k}}\right) \frac{\mu^2}{2} + \frac{\bar{\omega}}{2} (\bar{k} a)^2 - \frac{1}{2} \left(\frac{d\bar{c}_g}{d\bar{k}}\right) \frac{a_{xx}}{a} \right\} = 0$$

To simplify further we introduce the dimensionless variables:

$$A = a/\bar{a} \quad , \quad W = \frac{\mu}{2\bar{k}^2\bar{a}} = \frac{\omega_0 - \bar{\omega}}{\varepsilon\bar{\omega}(\bar{k}\bar{a})} \quad ,$$

$$X' = \bar{k}^2\bar{a}\xi \quad \text{and} \quad T' = (\bar{k}\bar{a})^2\bar{\omega}\tau \quad (6.4 \text{ a-d})$$

where \bar{a} is the typical initial wave amplitude. Since $\frac{\bar{k}}{\bar{c}_g} \frac{d\bar{c}_g}{d\bar{k}} = -\frac{1}{2}$ and $(\bar{k}/\bar{\omega}) \bar{c}_g = +\frac{1}{2}$ in deep water, the dimensionless modulation equations following the group velocity are given by,

$$\frac{\partial A^2}{\partial T'} + \frac{\partial}{\partial X'} \left(-\frac{W}{2} A^2 \right) = 0 \quad ,$$

$$\frac{\partial W}{\partial T'} + \frac{\partial}{\partial X'} \left(-\frac{W^2}{4} + \frac{A^2}{4} + \frac{A x x'}{16A} \right) = 0 \quad (6.5 \text{ a,b})$$

where A and W are respectively the normalized wave amplitude and wave frequency. We recall the meaning of the above equations. Equation (6.5 a) states the conservation of wave energy where the velocity of energy flux in the moving frame is $-\frac{W}{2}$. Equation (6.5 b) states the conservation of waves with the term $\frac{\partial}{\partial X'} \left(-\frac{A^2}{4} \right)$ representing the effect of amplitude dispersion and the term $\frac{\partial}{\partial X'} \left(-\frac{W^2}{4} + \frac{A x x'}{16A} \right)$ representing the effect of frequency dispersion. From this various other conservation laws can be derived in the manner of Whitham (1965 a) or others, but they do

not have any immediate physical meaning.

By letting $A = 1 + \epsilon \alpha$, $W = \epsilon \beta$ equation (6.5) may be linearized to give

$$\begin{aligned} \beta_{\tau'} + \frac{1}{2} \alpha_{x'} + \frac{1}{16} \alpha_{x'x'x'} &= 0, \\ \alpha_{\tau'} - \frac{1}{4} \beta_{x'} &= 0, \end{aligned} \quad (6.6)$$

or, after eliminating β ,

$$\alpha_{\tau'\tau'} + \frac{1}{8} \alpha_{x'x'} + \frac{1}{64} \alpha_{x'x'x'x'} = 0 \quad (6.7)$$

which resembles the governing equation for an oscillating elastic column under axial load. With α being the lateral displacement, the terms $\alpha_{\tau'\tau'}$, $\alpha_{x'x'}$ and $\alpha_{x'x'x'x'}$ represent the effect of inertia, the effect of compression and the effect of shearing force respectively (see Morse (1948), p. 166).

6.2 PERMANENT WAVE ENVELOPES

For long wave in shallow water it is well known that nonlinearity and dispersion can be exactly in balance, giving rise to waves of permanent form such as solitary waves and cnoidal waves. Analogously, if amplitude dispersion and frequency dispersion in deep water waves are exactly in balance the wave envelopes will propagate without changing form. One trivial solution of (6.5 a,b) that has this property is

$$A = \text{constant} \quad \text{and} \quad W = \text{constant} \quad (6.8)$$

which is just the Stokes waves of constant amplitude. For other permanent wave envelopes, we let $\frac{\partial}{\partial T'} \equiv 0$ and integrate (6.5) with respect to X' ,

$$\begin{aligned} WA^2 &= C_1 = \text{constant} \quad , \\ -4W^2 + 4A^2 + \frac{AX'X'}{A} &= C_2 = \text{constant} \quad . \end{aligned} \quad (6.9 \text{ a,b})$$

Eliminating W from (6.9 a,b) and integrating once more,

$$[(A^2)_{X'}]^2 = -8A^6 + 4C_2A^4 + 8C_3A^2 - 16C_1^2 \quad , \quad (6.10)$$

or, in term of $E = A^2$,

$$[E_{X'}]^2 = -8E^3 + 4C_2E^2 + 8C_3E - 16C_1^2 \quad . \quad (6.11)$$

Equation (6.11) is well known to give solution of solitary or cnoidal type. The existence of solitary and cnoidal wave envelopes in weakly nonlinear dispersive system was first pointed out by Benney & Newell (1967). Now, (6.11) may be written in another way as,

$$[E_{X'}]^2 = f(E) = 8(E_{\max} - E)(E - E_{\min})(E - E_0) \quad (6.12)$$

where E_{\max} , E_{\min} are the maximum and minimum of E respectively.

Comparing (6.11) and (6.12) we have

$$C_1^2 = -\frac{1}{2} E_{\max} E_{\min} E_0 \quad (6.13)$$

which shown that E_0 must be negative. A sketch of the properties of (6.12) and its various possible solutions are given in Figure 8. The

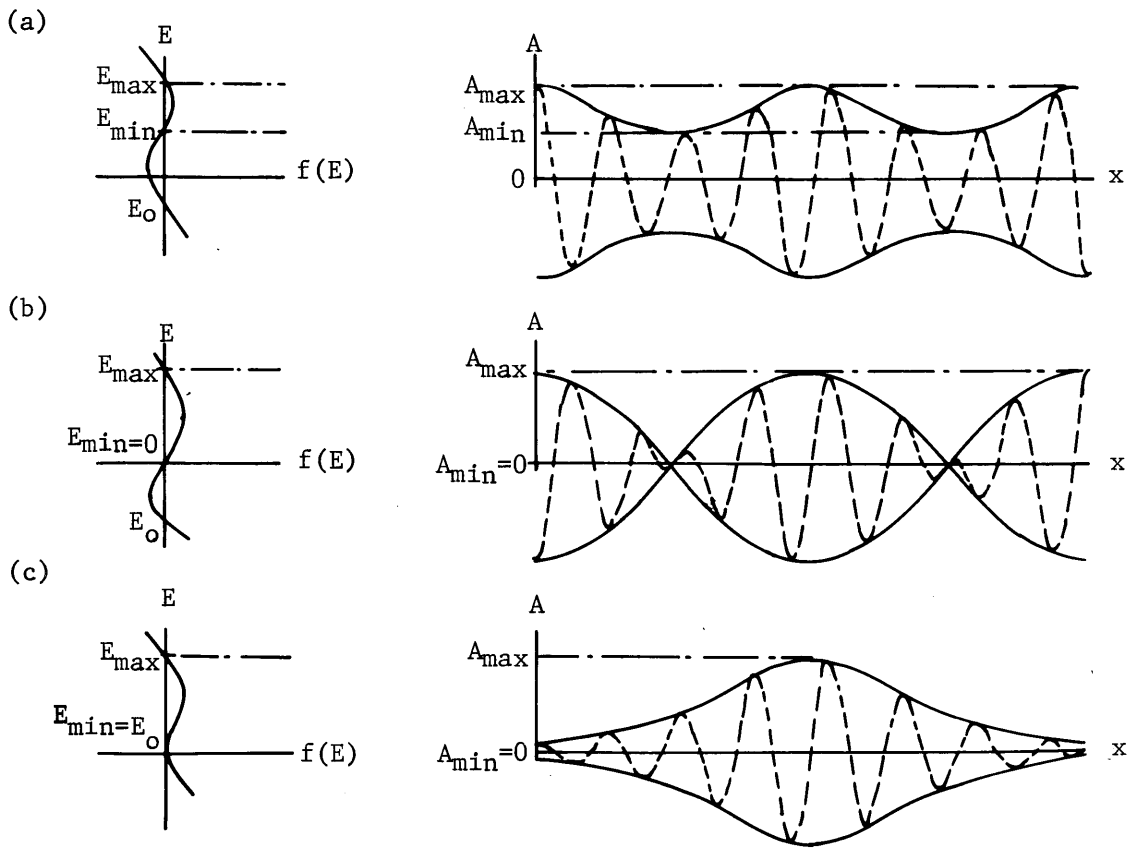
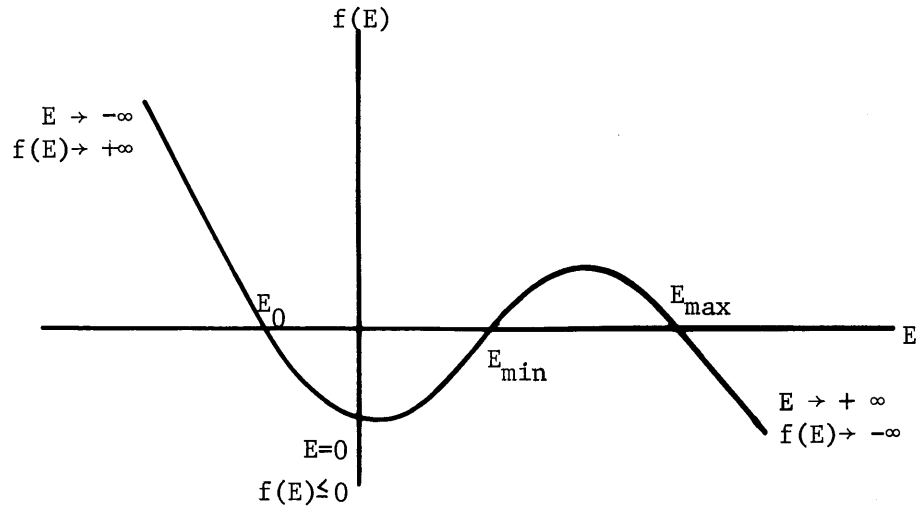


FIGURE 8 PERMANENT WAVE ENVELOPES

general solution of (6.12) is given by (see Lamb (1932), p. 426),

$$E = E_{min} + (E_{max} - E_{min}) \operatorname{cn}^2 \sqrt{2(E_{max} - E_{min})} \gamma X', \quad (\text{mod. } \gamma) \quad (6.14)$$

where $\operatorname{cn}(\)$ is the cnoidal function and its modulus γ is defined by

$$\gamma^2 = \frac{1 - E_{min}/E_{max}}{1 - E_0/E_{max}} \quad (6.15)$$

The wave length of the envelope (6.14) is given by

$$\lambda = \frac{\sqrt{2} \gamma}{\sqrt{E_{max} - E_{min}}} \int_0^{\pi/2} \frac{d\vartheta}{\sqrt{1 - \gamma^2 \sin^2 \vartheta}} \quad (6.16)$$

The frequency ω as evaluated from (6.9 a) is

$$\omega = \frac{c_1}{E} = \frac{\sqrt{-\frac{1}{2} E_{max} E_{min} E_0}}{E} \quad (6.17)$$

We note that the solutions (6.14) and (6.17) are uniquely determined once E_{max} , E_{min} and E_0 are specified.

There are no specific physical meaning attached to E_0 ; however, its relation to the wave length λ is given by (6.16) which may be rewritten as:

$$\gamma K(\gamma) = \delta = \lambda \sqrt{(E_{max} - E_{min})/2} \quad (6.18)$$

where $K(\gamma) = \int_0^{\pi/2} \frac{d\vartheta}{\sqrt{1 - \gamma^2 \sin^2 \vartheta}}$ is the complete elliptic integral of the first kind. The solution for γ may be obtained graphically from Figure 9.

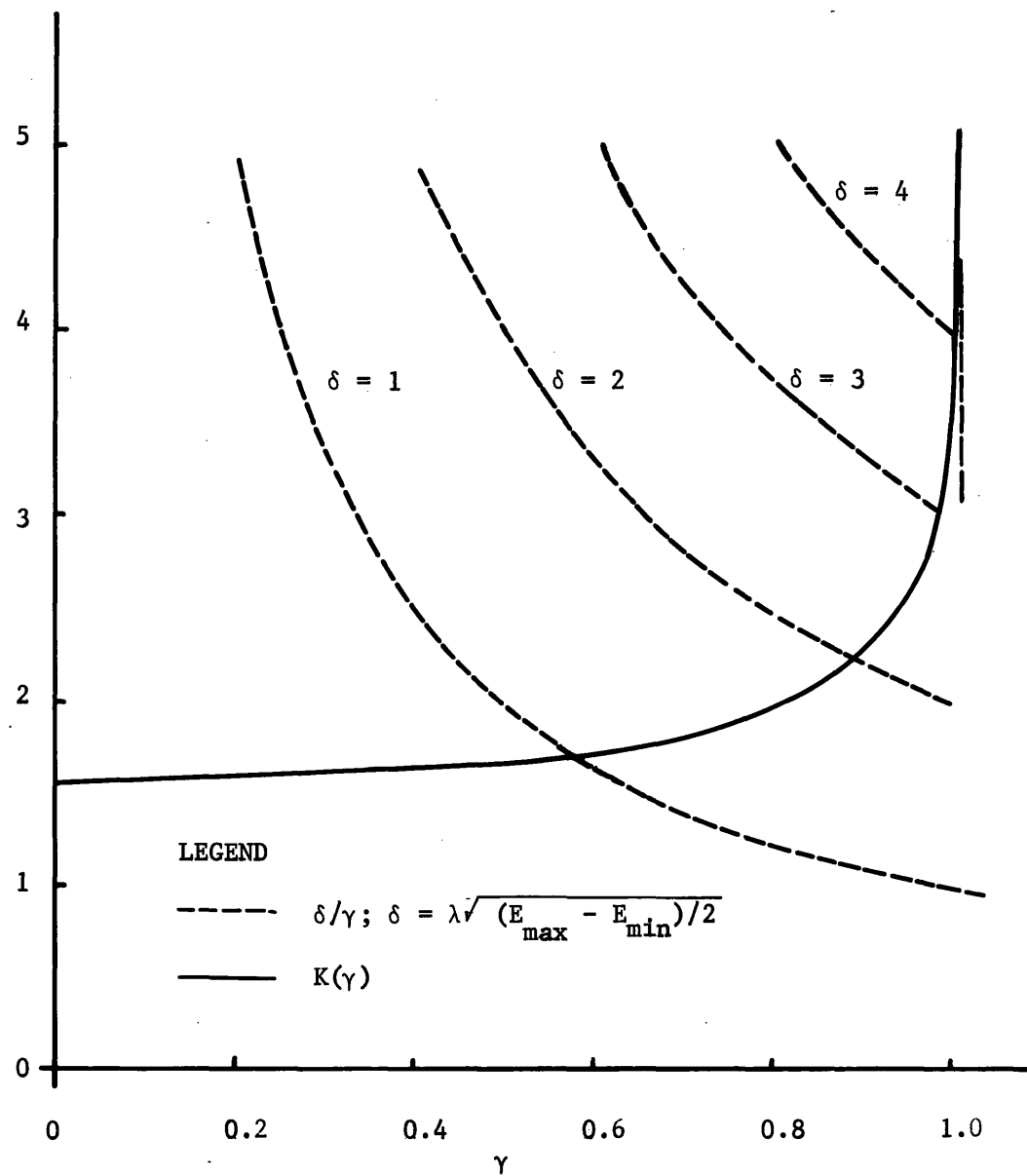


FIGURE 9 GRAPHICAL METHOD FOR DETERMINING THE WAVE LENGTH OF THE
 PERMANENT WAVE ENVELOPES

For the special cases (b) and (c) as described in Figure 8, $E_{\min} = 0$ so that $W \equiv 0$ and the wave length is uniform over the group. Now, from (6.12), we have $(A_X')^2 = 2 E_{\max} (-E_0)$ at $E = A = 0$. This implies that if $E_0 \neq 0$, $A_X' \neq 0$ at the nodes as shown in Figure 8(b). However, if we further let $E_0 \rightarrow 0$ (i.e. $\gamma \rightarrow 1$ or $\lambda \rightarrow \infty$) we obtain the solitary wave envelope as shown in Figure 8(c):

$$E/E_{\max} = \left\{ \operatorname{sech} \sqrt{2 E_{\max}} X' \right\}^2 .$$

or,

$$A/A_{\max} = \operatorname{sech} \sqrt{2} A_{\max} X' . \quad (6.19)$$

Note in particular as $X' \rightarrow \pm \infty$, $A \rightarrow 0$ so the solitary envelope is a pulse on zero background. The width $2 X'_{50}$ of the solitary pulse may be defined at $A/A_{\max} = 0.5$:

$$X'_{50} = \frac{0.935}{A_{\max}} . \quad (6.20)$$

so that the width of a solitary pulse is inversely proportional to wave steepness. Referring to Figure 9, we notice the solution of γ approaches very rapidly to unity for moderately large δ (say 3 or 4). It follows from (6.15) that $\gamma \rightarrow 1$ implies that $E_0 \rightarrow |E_{\min}| \rightarrow 0$. This is to say for large δ the permanent solution approach solitary pulse (6.19); i.e. case (c) as shown in Figure 8.

The general propagation speed of the cnoidal wave envelopes may be obtained from (6.17). We note from (6.17) that

$$W_{min} = \sqrt{-\frac{E_0}{2} \frac{E_{min}}{E_{max}}} , \quad (6.21)$$

or, by definition (6.4)

$$\frac{\omega_{omin}}{\bar{\omega}} - 1 = \varepsilon \bar{k} \bar{a} \sqrt{-\frac{E_0}{2} \frac{E_{min}}{E_{max}}} . \quad (6.22)$$

It follows immediately that the propagation speed is,

$$\bar{C}_g = \frac{g}{2\bar{\omega}} = \frac{g}{2\omega_{omin}} \left\{ 1 + \varepsilon \bar{k} a_{max} \sqrt{-\frac{1}{2} \frac{E_0}{E_{max}} \frac{E_{min}}{E_{max}}} \right\} . \quad (6.23)$$

When $E_{min} \rightarrow 0$, $\omega_{omin} = \bar{\omega}$ and $\bar{C}_g = \frac{g}{2\bar{\omega}}$. Therefore the propagation speeds for the permanent wave envelopes in Figures 7(b) and 7(c) are constant and independent of wave amplitude.

6.3 NUMERICAL METHOD

Beyond what is already discussed in § 6.2 analytical solution for (6.5) is in general very difficult. Therefore, the transient evolution of wave envelope has to be obtained through numerical computation. The numerical scheme used here is an explicit one. The finite difference equations for (6.5 a,b) are given as

$$A_{i,j+1}^2 = A_{i,j-1}^2 + \frac{1}{2} \left(\frac{\Delta T'}{\Delta X'} \right) [W_{i+1,j} A_{i+1,j}^2 - W_{i-1,j} A_{i-1,j}^2] ,$$

$$W_{i,j+1} = W_{i,j-1} + \frac{1}{4} \left(\frac{\Delta T'}{\Delta X'} \right) [(W_{i+1,j}^2 - W_{i-1,j}^2) - (A_{i+1,j}^2 - A_{i-1,j}^2)]$$

$$+ \frac{1}{16} \left(\frac{\Delta T'}{\Delta X'} \right)^3 \left[\frac{A_{i+2,j} - A_{i,j}}{A_{i+1,j}} - \frac{A_{i,j} + A_{i-2,j}}{A_{i-1,j}} \right] .$$

(6.24 a,b)

For computation convenience, a periodic boundary condition is applied for every N grid points over space; i.e.

$$A_{i+n \times N} = A_i \quad , \quad W_{i+n \times N} = W_i \quad ; \quad n = 0, \pm 1, \pm 2, \dots \quad (6.25)$$

Initial values of A & W are then prescribed.

Numerical stability analysis for (6.24) and (6.25) is difficult for their nonlinearity. However (6.7), i.e. the linearized model of (6.5 a,b), has been studied by an explicit scheme (see Richtmyer, 1964, p.185). in which $\frac{(\Delta T')}{(\Delta X')^2}$ is found to be the parameter deciding numerical stability. Using the linearized model as a guide our numerical stability criterion is obtained through trial and error on the computer and we found (6.24) and (6.25) are always stable if $\frac{(\Delta T')}{(\Delta X')^2} \leq 1$. As a further check of this numerical scheme we place a permanent wave envelope (6.19), as initially condition and choose $\Delta T' = 0.01$, $\Delta X' = 0.1$, $N = 200$. After $500(\Delta T')$ we found the permanent wave envelope essentially unchanged as predicted; the change of peak amplitude is 0.3% and the change of total energy is 0.02%. This accuracy is certainly satisfactory. In fact for all cases considered later the total energy is always conserved to within 0.5%.

6.4 TRANSIENT EVOLUTION OF WAVE ENVELOPES

Experiments on the nonlinear evolution of wave envelope in deep water has been done by Feir and reported by Benjamin (1967) and Feir (1967). The primary purpose of this section is to confirm several

important features in their records. With respect to the wave envelope pulse Lighthill (1965 b, 1967) has given analytical solutions based on Whitham's theory.

6.4.1 Wave Pulse Experiment

To isolated the effect of nonlinear dispersion we first use the modulation equations derived by Whitham (i.e., to neglect the dispersive term $(\frac{Ax'x'}{16A})_x$, from (6.5 a,b)) to study a wave pulse which at time $T' = 0$ is given by $A = \text{sech} \sqrt{2} X'$, $\omega \equiv 0$ (This is precisely Lighthill's problem). The numerical result is presented in Figure 10 which shows that the wave frequency tends to increase in the front and decrease behind the group, and, at the same time, the amplitude peaks up at the center of the group. As time increases further a rapid change of wave frequency and a sharp peak of amplitude is developed at the center. At this stage both theory and numerical solution break down. Essentially the same behavior has been analytically predicted by Lighthill (1965 b, 1967) and he speculated that the rapid change of frequency at the center may be in some way analogous to the aerodynamic 'shock'. Referring to (6.19) we note that the initial condition chosen for this calculation is a permanent wave envelope which according to the fuller theory with the dispersive term should propagate without change of form!

Now we use the complete nonlinear theory, i.e. equations (6.5), to calculate several pulse-shaped wave envelopes. For numerical calculations the distance between the boundaries are chosen to be large enough so that $Q \approx 0$ at the boundary and hence the periodic boundary conditions

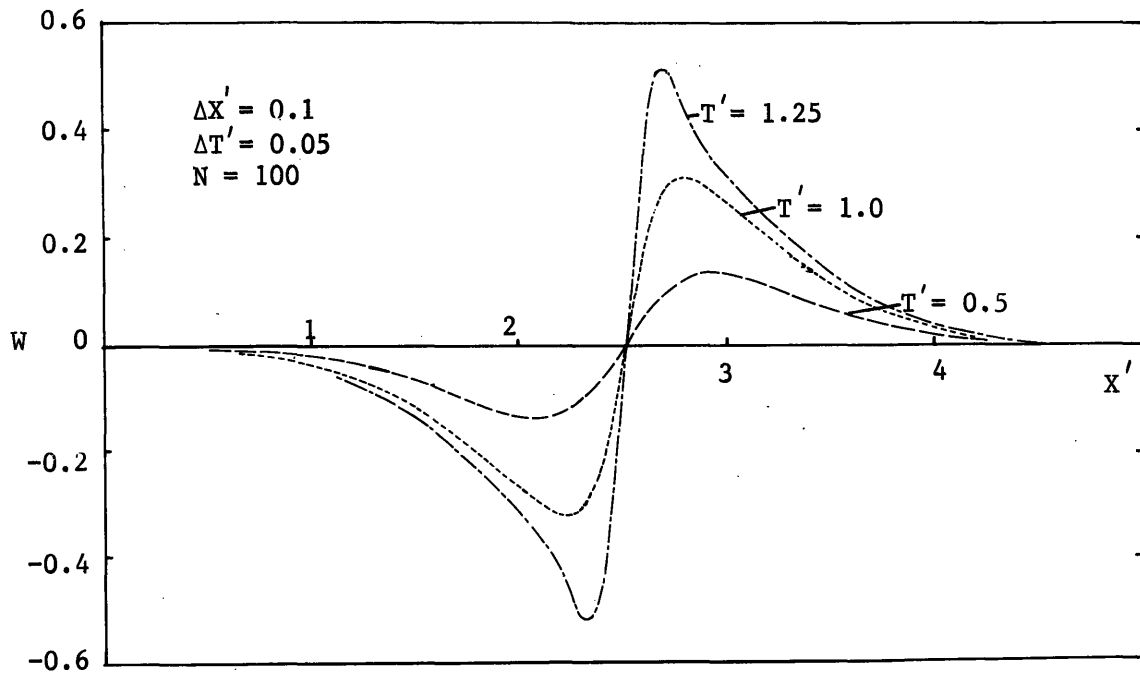
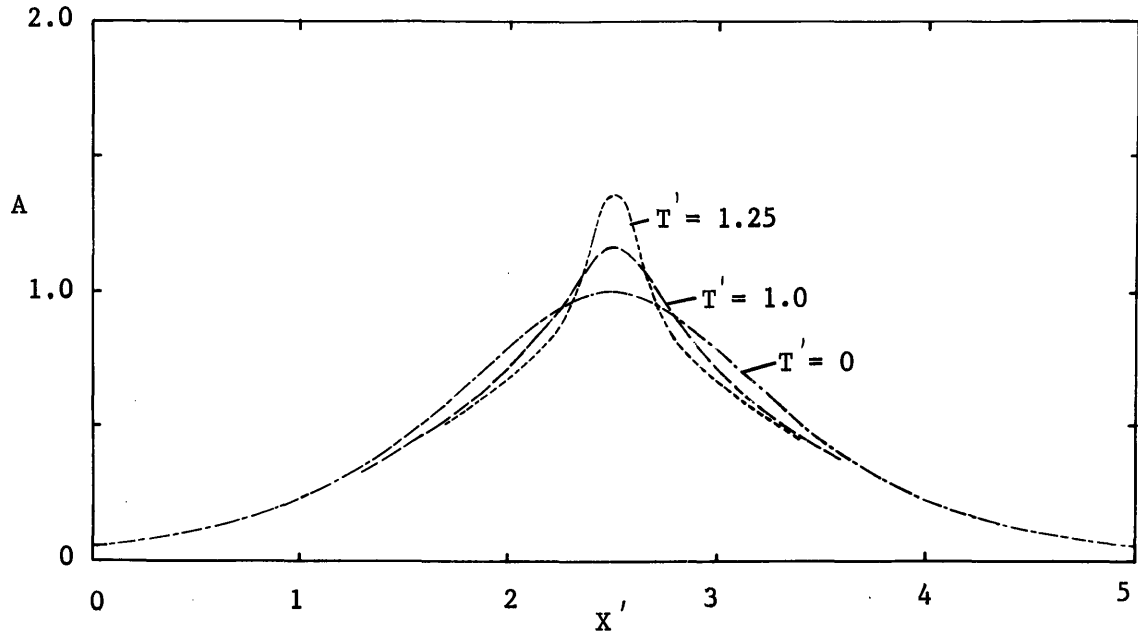


FIGURE 10 NONLINEAR EVOLUTION OF WAVE PULSE BASED ON WHITHAM'S THEORY

(Initial conditions: $A = \text{sech}\sqrt{2}X'$, $W = 0$)

do not affect the main region of interest. Let us consider first a wave pulse which is initially flatter than permanent form and the wave length is chosen to be uniform initially. In this case the amplitude dispersion is smaller than the frequency dispersion. Figure 11 shows that the wave amplitude starts to peak up and frequency increases in the front of the group and decreases behind it. However, as the time increases beyond $T' = 4.5$, the wave frequency at the center of the group tends to decrease back to zero and the amplitude gradually approaches a solitary envelope (6.19)*. This indicates that a local dynamic equilibrium between non-linearity and frequency dispersion is achieved. In Figure 12 where the maximum amplitude and the growth rate, $\frac{A_{T'}}{A}$, at the center of the wave group is plotted v.s. T' . It is shown that the growth rate gradually diminishes as the wave envelope approaches permanent form. This suggests that the wave envelope now settles down to a stable form. Another interesting aspect observed in Figure 11 is that the wave group has a tendency to disintegrate into groups separated by nodal points where the amplitude vanishes to zero and the frequency grows indefinitely. As the frequency becomes too large at the nodal point the theory breaks down and the numerical computation cannot be continued. Mathematically, a sharp peak of wave frequency at the nodal point simply implies a sudden change in the

*In Figures 8, 9, 10 & 11 the wave envelope is comparable with the solitary pulse solution (6.19) because $\gamma > 0.95$ for all these cases.

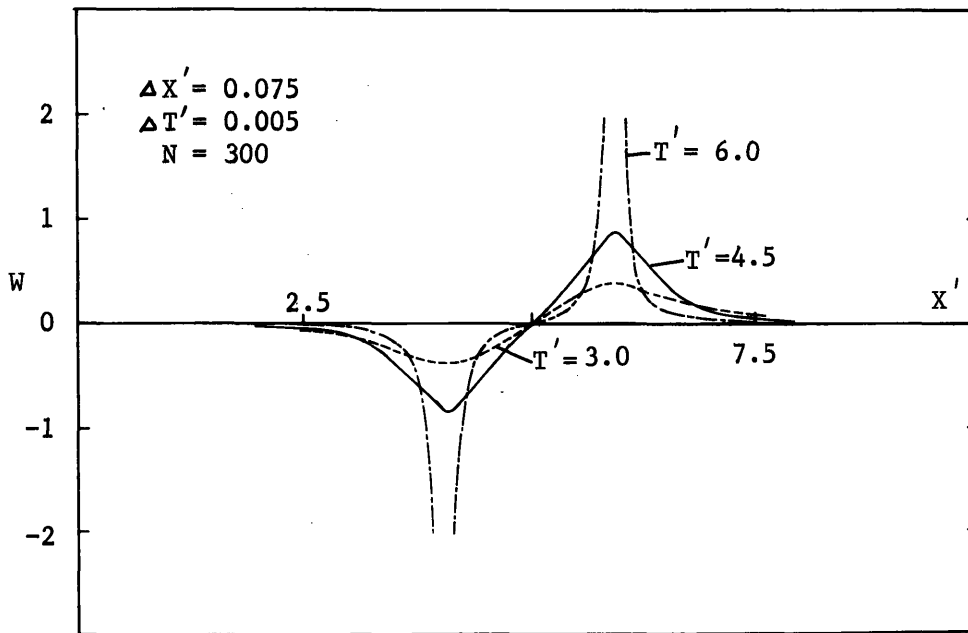
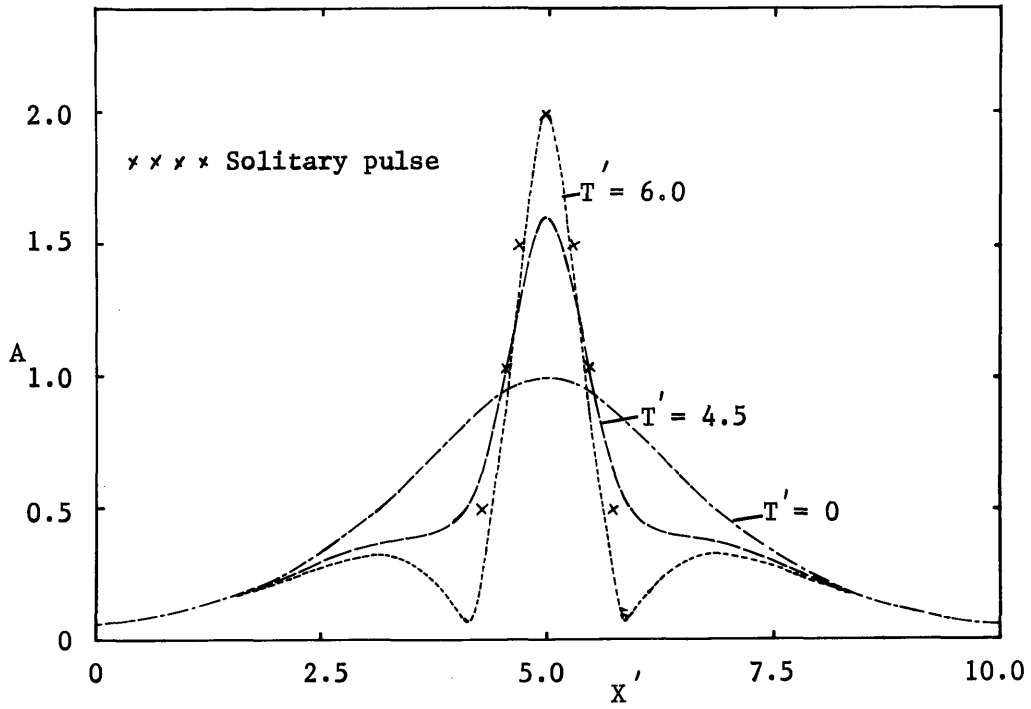


FIGURE 11 SYMMETRICAL PULSE INITIALLY FLATTER THAN PERMANENT WAVE ENVELOPE

Initial conditions: $A = \text{sech} \sqrt{2} X'/2$, $W = 0$

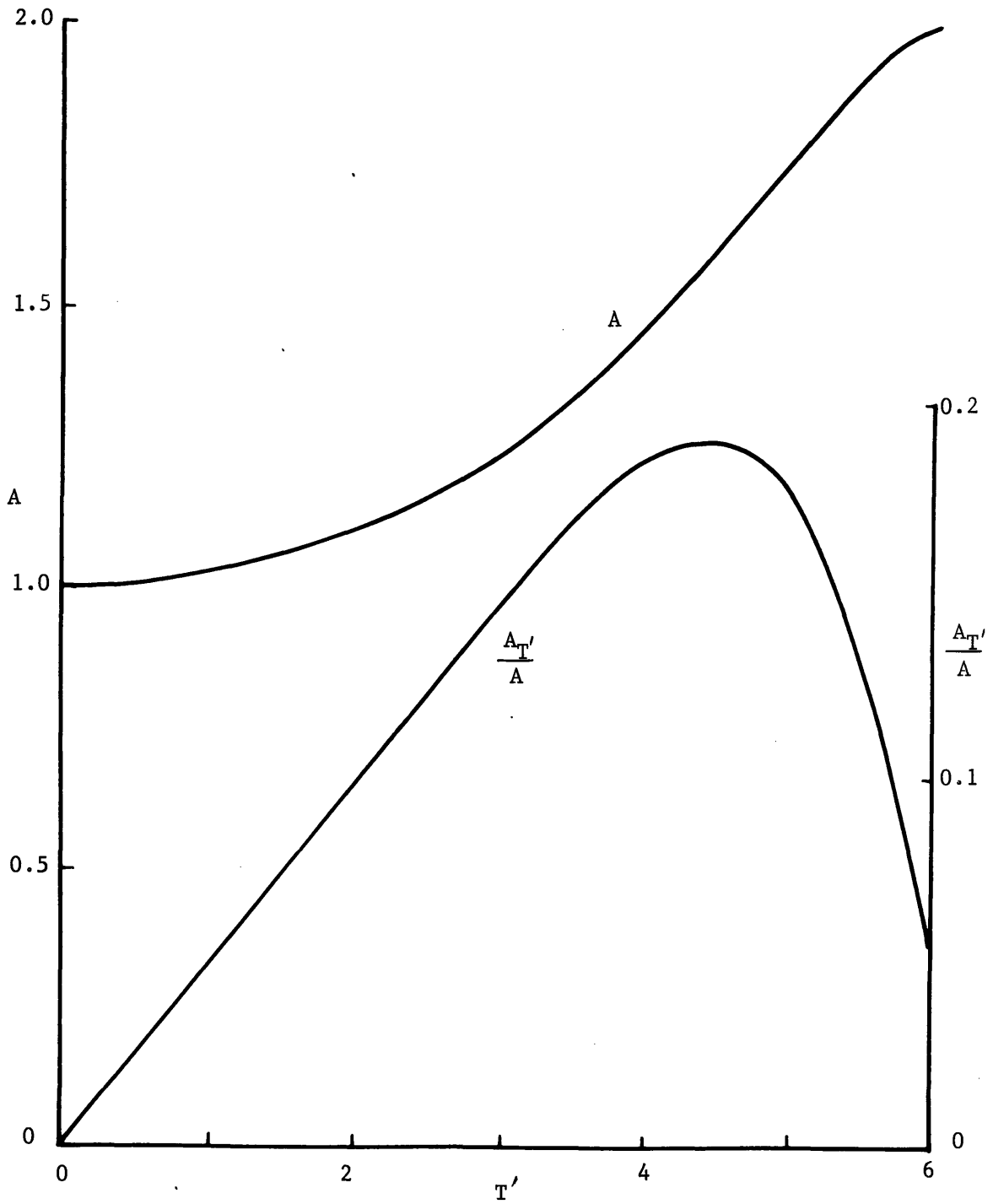


FIGURE 12 VARIATION OF WAVE AMPLITUDE AND GROWTH RATE AT THE CENTER OF THE WAVE GROUP

wave phase (see definition (2.11 a, b)) and the conservation law for wave crests breaks down at such points. Referring to the example given in § 1.3 a rapid change of phase at the nodes is in fact also present in the linear theory.

Several other situations have also been considered. In Figure 13 the wave pulse is initially steeper than a solitary pulse of equal maximum height. Again the wave pulse disintegrates into groups separated by nodes where the wave phase changes rapidly. The main wave group first approaches a solitary pulse. However, the frequency seems to reach different levels in different wave groups. Unfortunately, once the nodes are formed, the numerical calculation cannot proceed further and we cannot really be sure about the eventual situation.

In Figure 14 we consider an asymmetrical wave pulse, the front face is exactly the solitary pulse of equal height but the back is flatter. Similar to the previous cases, the wave pulse disintegrates. The central group first approaches permanent form while the leading group is smaller than the trailing group. This situation, as we shall see, is rather close to several cases of Feir's experimental observation.

In all previous cases the wave length is initially uniform (i.e. $W \equiv 0$). Now in Figure 15 we consider a symmetrical wave pulse with wavelength that are initially not uniform. In this case we find the wave envelope first distorted and then disintegrated. The largest group approaches solitary pulse and the side groups are not symmetrical.

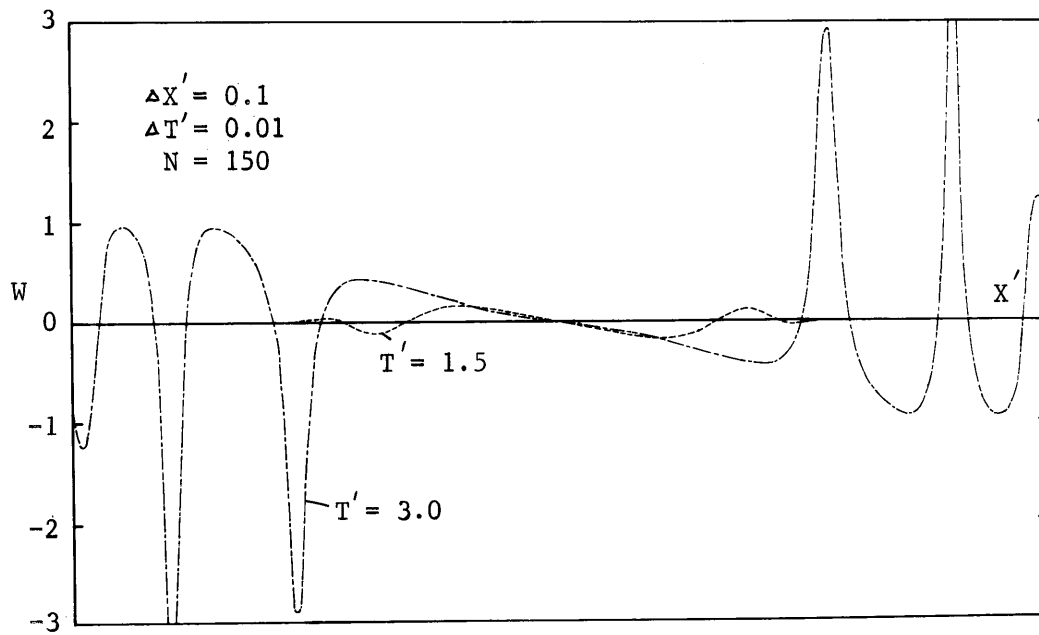
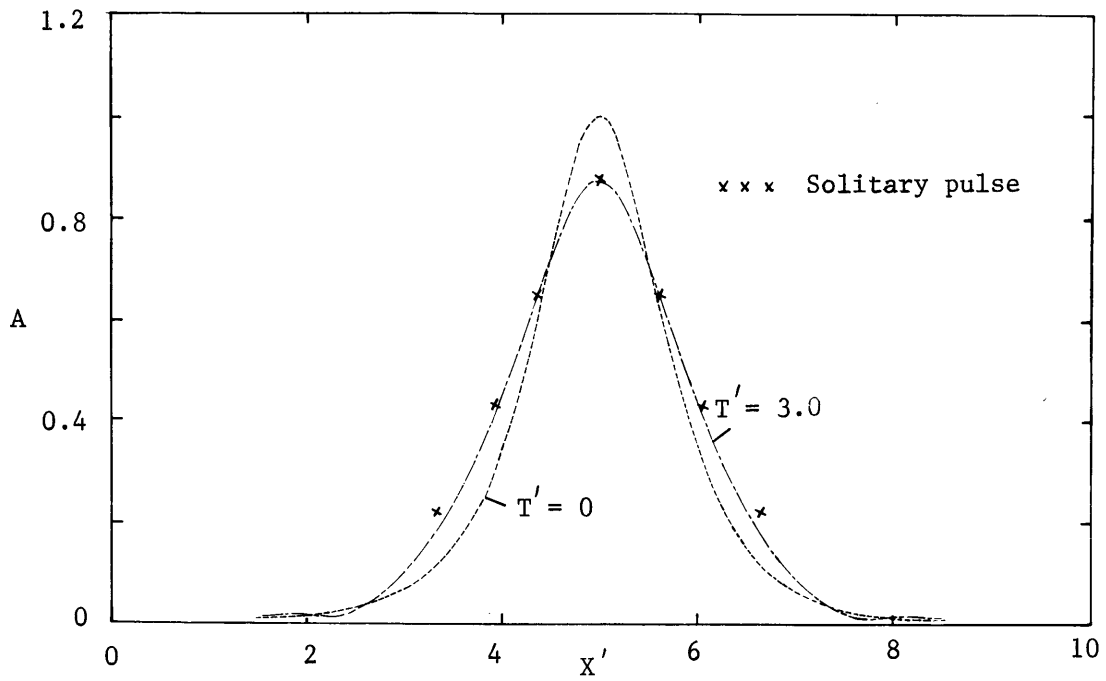


FIGURE 13 SYMMETRICAL PULSE INITIALLY STEEPER THAN PERMANENT WAVE ENVELOPE

Initial conditions: $A = \text{sech} 1.2\sqrt{2}X'$, $W = 0$

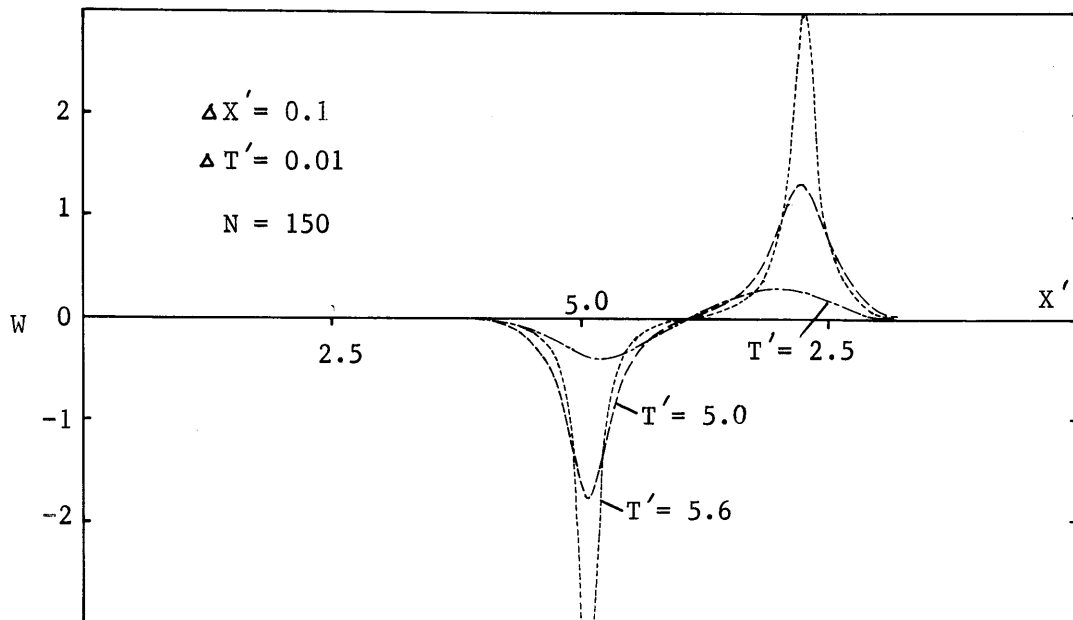
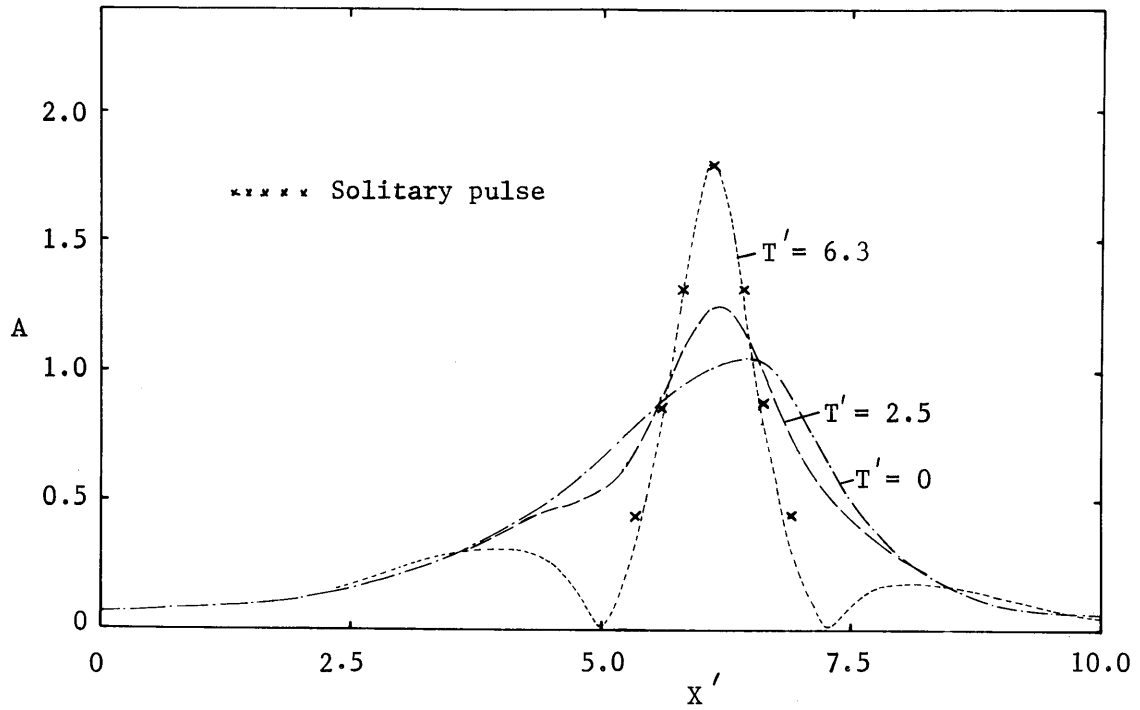


FIGURE 14 ASYMMETRICAL PULSE INITIALLY UNIFORM WAVE LENGTH

Initial conditions: $A = \text{sech}\sqrt{2}(X'-6.5)$, $X' \geq 6.5$; $W = 0$
 $\text{sech}\sqrt{2}(X'-6.5)/2$, $X' \leq 6.5$

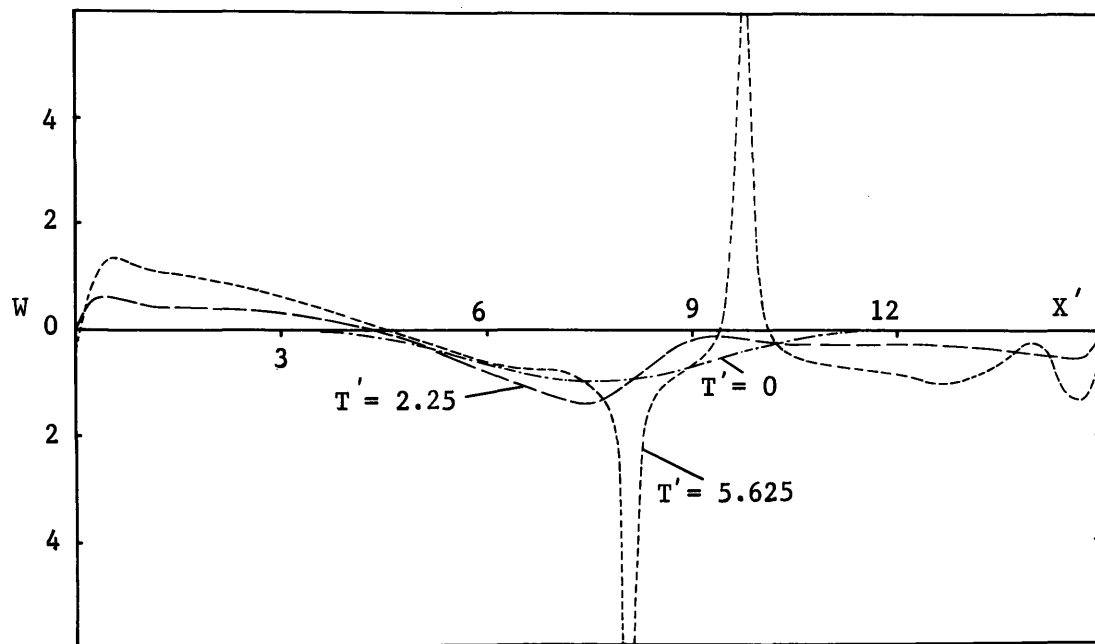
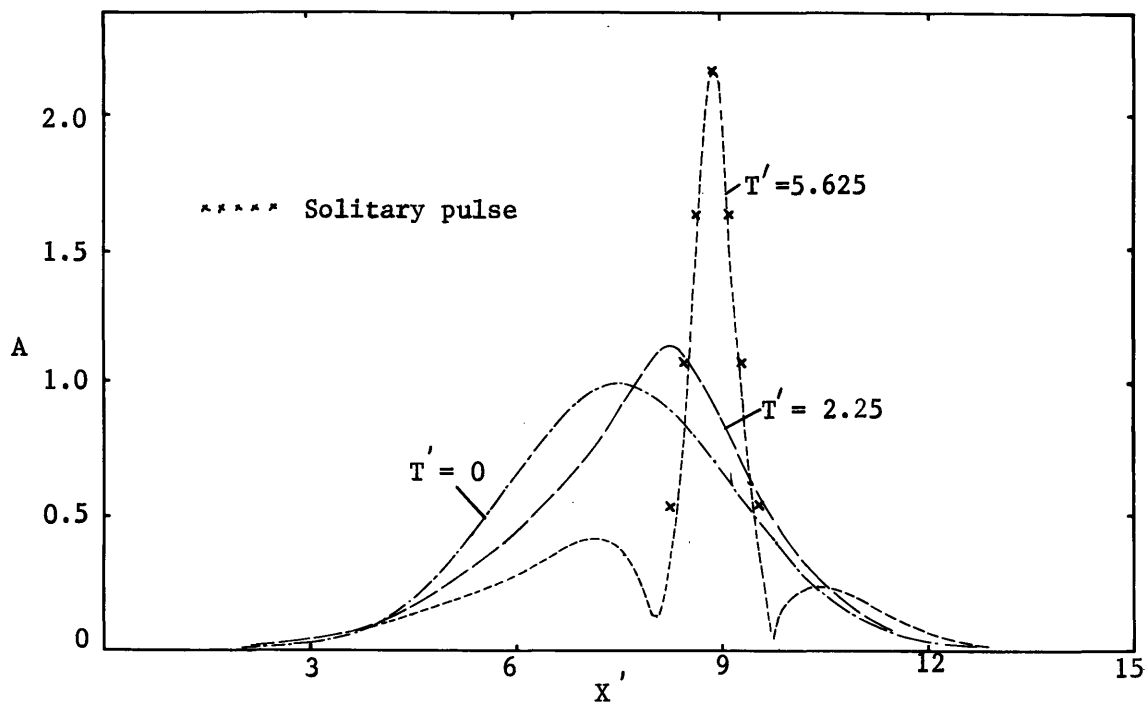


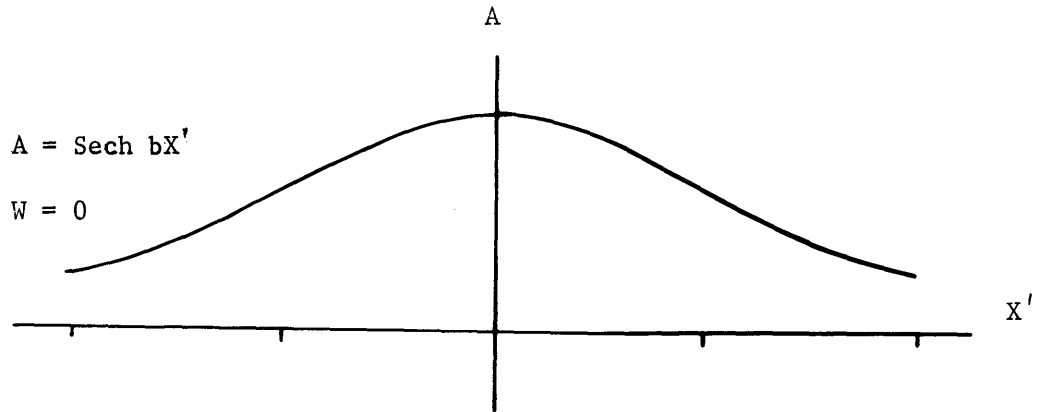
FIGURE 15 SYMMETRICAL PULSE WITH INITIAL FREQUENCY SPREAD

Initial conditions: $A = \exp(-.178 X'^2)$, $W = 0.5 \exp(-.178 X'^2)$

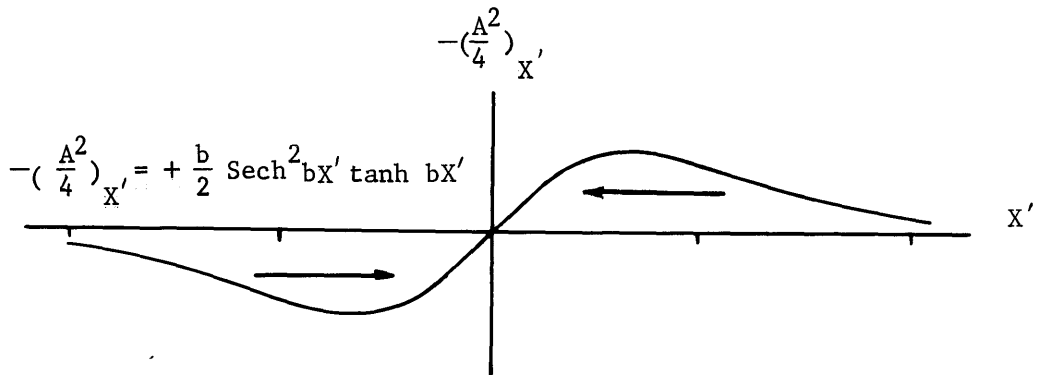
$\Delta X' = 0.15$, $\Delta T' = 0.005$, $N = 150$

6.4.2 Heuristic Discussion

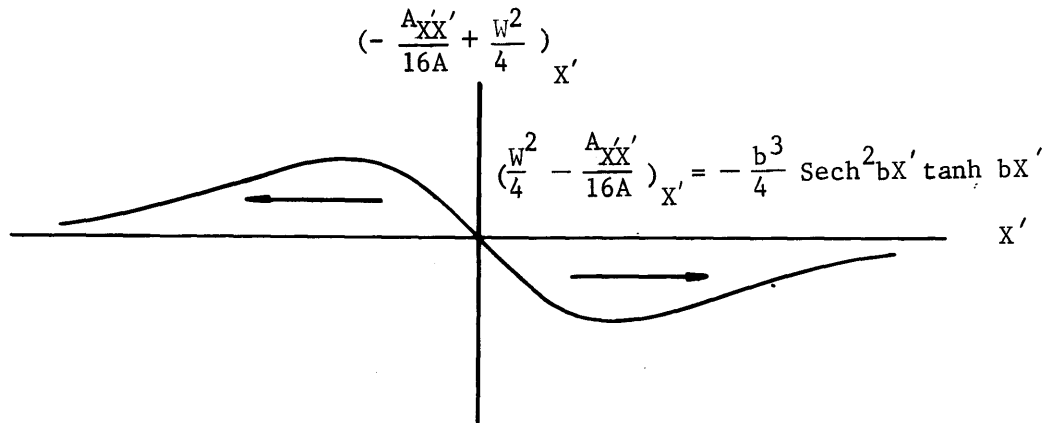
We shall attempt to give a qualitative discussion of the numerical results obtained in the previous section. The process of evolution of wave envelopes may be considered as a redistribution of wave energy within the wave group. The agent which does the redistribution is the group velocity. As we have mentioned previously in Chapter 3 that the group velocity, depending on the frequency (ω_0), may be affected by various types of dispersion effects. For deep water waves these dispersion effects are frequency dispersion and the so-called 'amplitude dispersion' due to the nonlinear coupling between the group velocity and the amplitude. Referring to the dimensionless equation (6.5 a) we note that $(-\frac{W}{2})$ is essentially the group velocity, while the terms $(\frac{A^2}{4} X')$ and $(-\frac{W}{4} + \frac{AXX'}{16A} X')$ in (6.5 b) represent the amplitude dispersion and the frequency dispersion respectively. To illustrate these two types of dispersion, we choose an initial pulse: $A = \text{Sech } bX'$, $W = 0$ and plot out the amplitude dispersion term and the frequency dispersion term in Figure 16(b) and 16(c). We found that the effect of amplitude dispersion is to increase W (i.e., to decrease the group velocity) in the front of the wave group and to decrease W behind it. Since energy is convected by $(-\frac{W}{2})$, the amplitude dispersion is to concentrate the energy (as directed by the arrows in Figure 16) into the center of the group and frequency dispersion tends to spread out the wave energy. For a group of waves which is initially flatter than the permanent wave envelope,



(a) Initial profile



(b) amplitude dispersion effect



(c) Frequency dispersion effect

FIGURE 16 SKETCH ILLUSTRATING EFFECTS OF AMPLITUDE DISPERSION AND FREQUENCY DISPERSION AT SMALL TIME.

the amplitude dispersion is initially stronger than the frequency dispersion. Therefore, as shown in Figure 17(a), amplitude will increase at the center until the amplitude dispersion and frequency dispersion reach equilibrium. On the other hand if the wave pulse is initially steeper than permanent wave the amplitude at the center would first decrease as illustrated in Figure 17(b).

The formation of the nodal points is probably due to the fact that for the first case the rate of energy convection from the two sides to the center (for the second case from center to two sides) is not uniform (Figure 17). The energy convection is much more rapid near the center than near the tails. Therefore between the center and the tails there is an energy deficiency and nodal points are formed.

6.4.3 Periodic Modulations

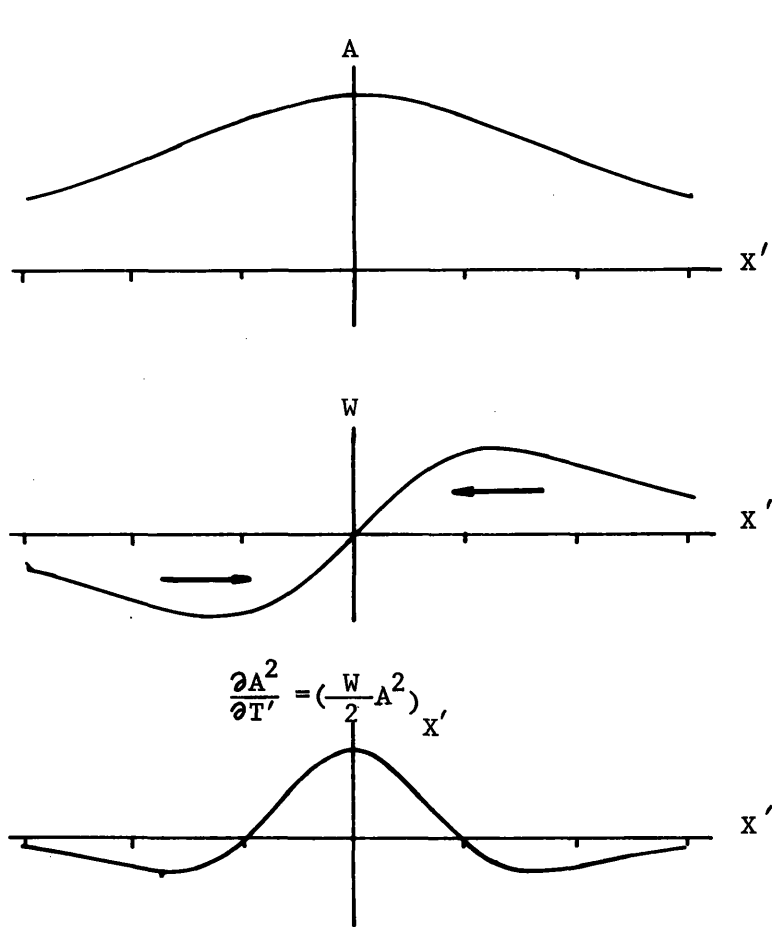
Since a periodic boundary condition is used, the present numerical scheme is very suitable for computing periodic modulation of wave envelopes, corresponding therefore to the problem studies by Benjamin and Feir (1967). For the sake of reference we first rederive the results for linear instability from (6.6) directly. Let us assume the following solutions:

$$(\alpha, \beta) = (\bar{\alpha}, \bar{\beta}) \exp [i(K'X' - \Omega'T')] \quad (6.26)$$

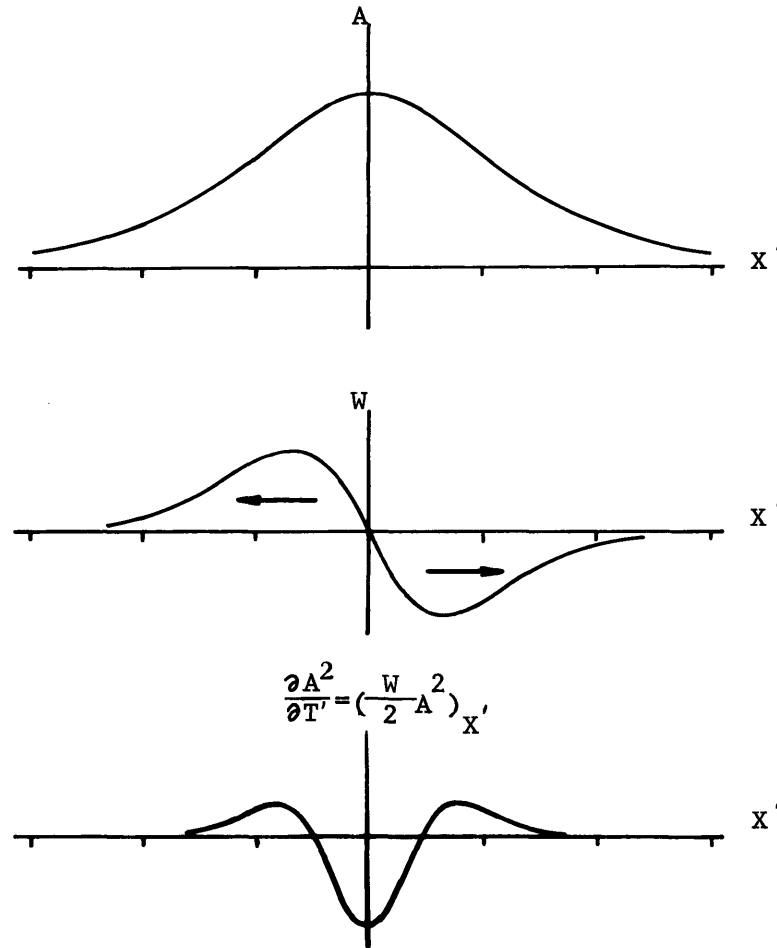
where K' and Ω' are the normalized wave number and normalized wave frequency of the side-band disturbances. The eigenvalue relation follows immediately,

$$\Omega' = \frac{1}{8} K' \sqrt{K'^2 - 8} \quad (6.27)$$

The disturbances (6.26) are unstable if $K' < 2\sqrt{2}$ and stable otherwise.



(a) Flatter than permanent form



(b) Steeper than permanent form

FIGURE 17 SKETCH ILLUSTRATING INITIAL NONLINEAR EVOLUTION OF WAVE PULSE

The growth rates for the unstable waves are given by

$$\frac{\alpha_{T'}}{\alpha} = \frac{\beta_{T'}}{\beta} = \frac{K'}{8} \sqrt{K'^2 - 8} \quad (6.28)$$

It is easily shown that at $K' = 2$ the growth rates are maximum; i.e., $\frac{\alpha_{T'}}{\alpha} = \frac{\beta_{T'}}{\beta} = \frac{1}{2}$. Substituting (6.26) into (6.6 b) we found for $K' = 2$:

$$\frac{\bar{\alpha}}{\beta} = i \frac{K'}{\Omega'} = i \quad (6.29)$$

So that α and β are equal in amplitude but out of phase by $\frac{\pi}{2}$.

We now consider the nonlinear problem and take the following as initial condition corresponding to the fastest growing case

$$A = 1 + (0.1) \cos 2X' \quad W = (0.1) \sin 2X' \quad (6.30)$$

The results of numerical calculation are shown in Figure 18. First we notice that the initial evolution of the disturbances is not exactly sinusoidal as the linearized instability theory predicted. There are sharper crest and flatter trough for the modulated envelope and this is indeed observed in one experimental record obtained by Feir (Benjamin 1967). As time proceeds further the wave envelope again disintegrates and develops into distinct groups separated by nodal points across which the phase changes rapidly. The frequency over the largest of the group returns to zero (Figure 18(b)). It is most significant that at a certain large time $T \cong 5.2$ the wave envelope approaches the cnoidal solution (6.14) and (6.17) with $E_{\min} = 0$ (each group reaching a cnoidal wave of amplitude equal to that of the group). This suggests

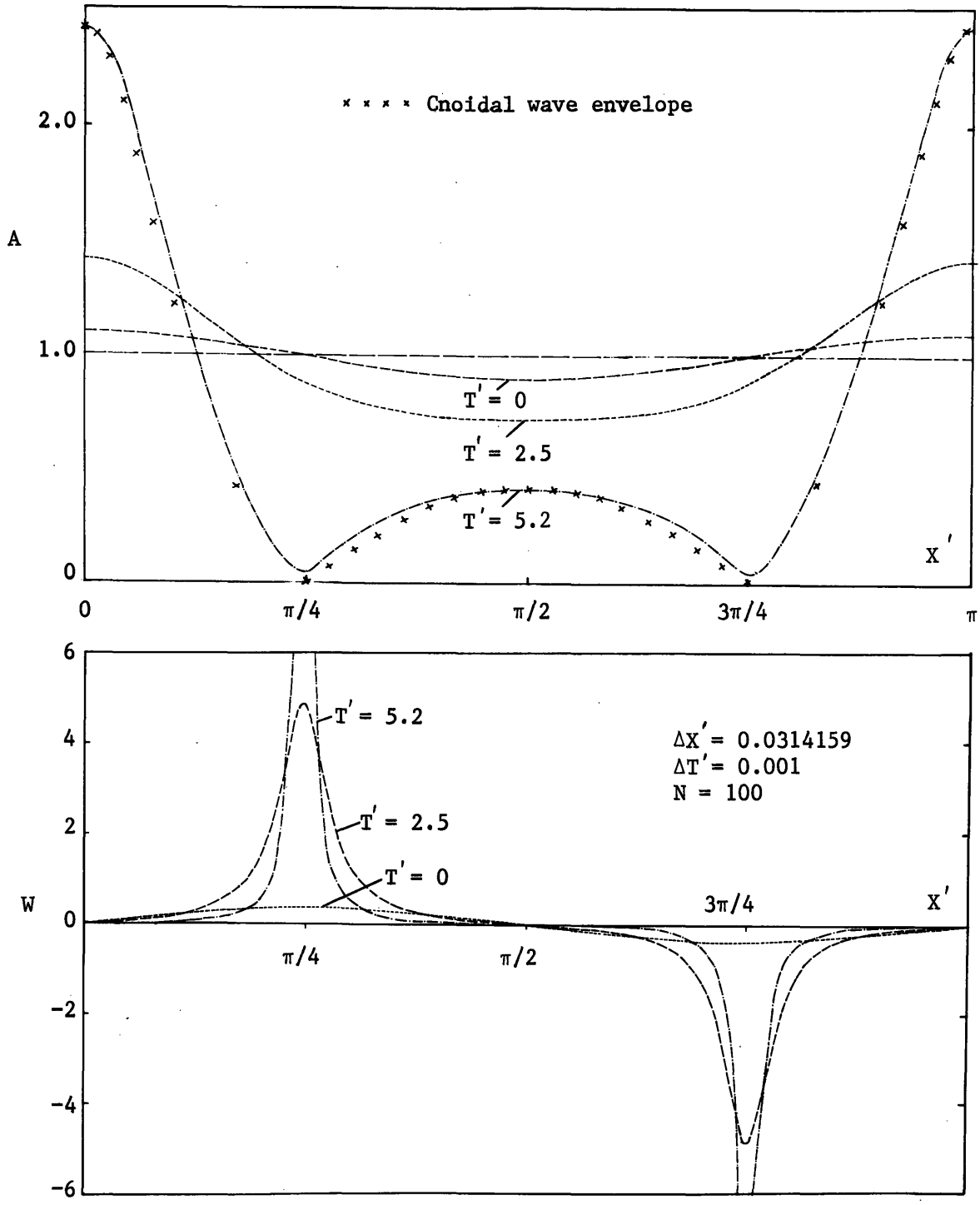


FIGURE 18 NONLINEAR EVOLUTION OF AN UNSTABLE STOKES WAVE
 (Initial condition: $A = 1 + (0.1)\cos 2X'$, $W = (0.1)\sin 2X'$)

that dynamic equilibrium between amplitude dispersion and frequency dispersion is reached. We further remark that the higher wave group tends very closely to solitary pulse solution given in (6.19) but not the lower group. This is because the parameter, $\gamma \cong 0.99$, is very close to unity for the higher group as compared to $\gamma \cong 0.3$ for the lower group (As discussed previously the cnoidal solution (6.14) approaches solitary solution (6.19) as $\gamma \rightarrow 1$). The growth rate of the disturbances is given in Figure 19. The crest of the envelope appears to grow faster than the trough. The trough is defined to be the mid-point between the two successive crests. Initially the growth rate for the height of the disturbance (i.e., the amplitude difference between the crest and the trough of the wave envelope) agrees with the linearized instability theory. It is interesting to observe that the growth rate finally reduce to zero as the wave envelope approaches permanent form. Although the numerical calculation cannot be proceeded further as soon as the nodal points are developed, due to the fact that the approach to permanent waves coincides with the diminishing growth rate, it appears quite certain that the permanent form is the final stage of evolution and no further changes can be expected. We recall that these cnoidal waves envelopes all travel at the same speed irrespective of their maximum amplitudes (at least to the present degree of approximation). These permanent envelopes of different amplitude will not separate from one another, which is a feature somewhat different from the comparable problem studied by Madsen and Mei (1969) for shallow water long waves.

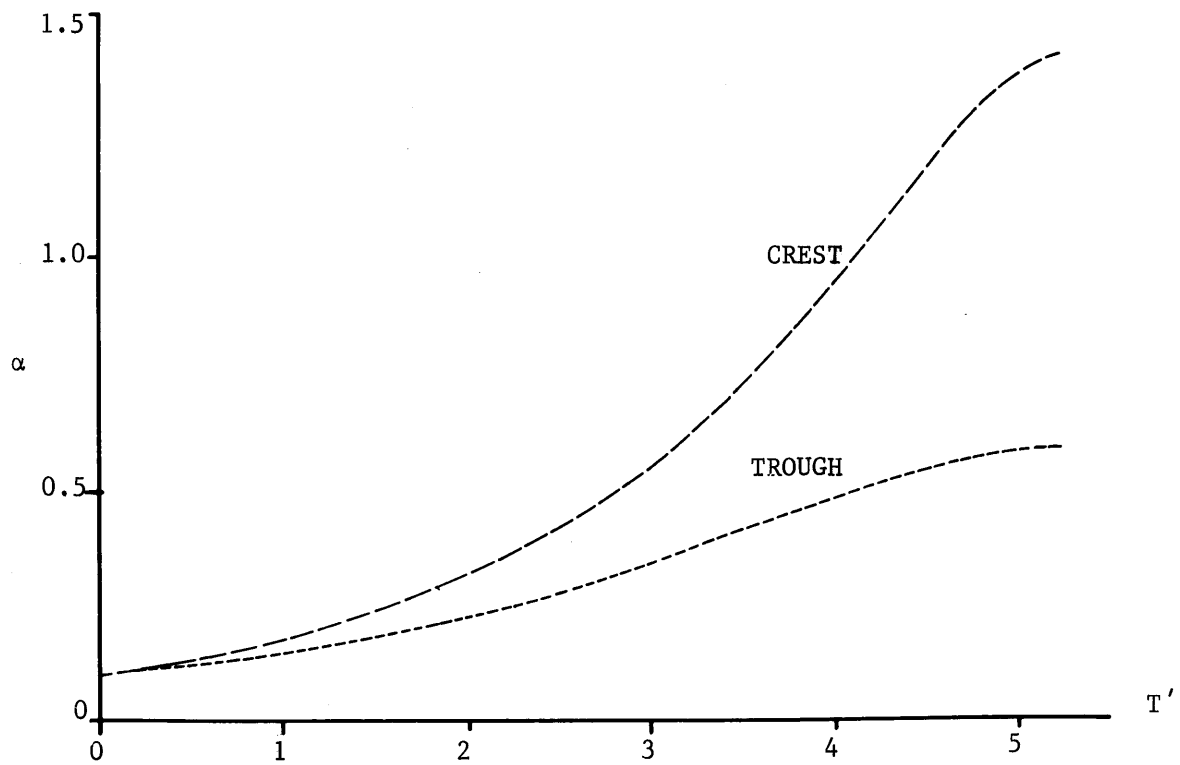
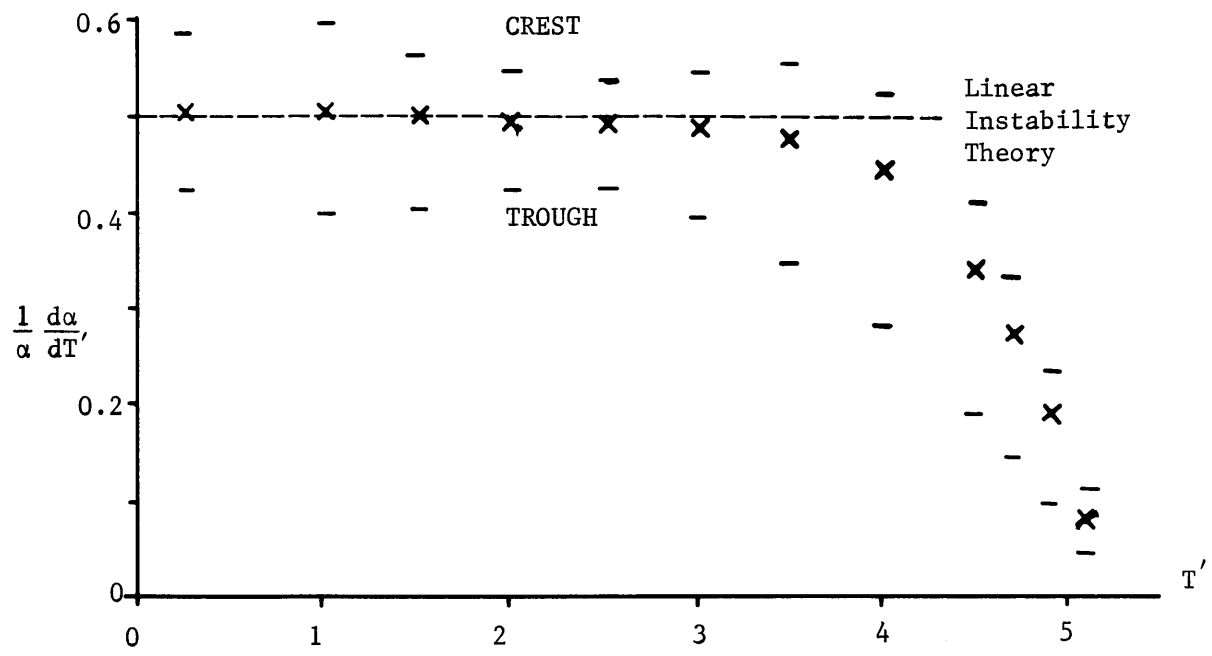


FIGURE 19 GROWTH RATE OF THE UNSTABLE STOKES WAVE

Another example of periodic modulation is considered in Figure 20. In this case the initial periodic modulation on both wave amplitude and wave frequency is chosen to be very large. After going through a rather complicated evolution the wave envelope is again found to disintegrate and settle down to permanent forms.

We shall now summarize the entire picture of the instability of Stokes waves in deep water. When a side-band disturbance is within certain range $\frac{K}{k^2 a} < 2\sqrt{2}$ the maximum amplitude of the envelope grows exponentially. The envelope develops into a wave form with sharp peaks and flat troughs. Further onwards the growth rate diminishes while nodes develop at about $\frac{1}{4}$ wavelength from the crests of the envelope. Between nodes the envelope would seem to settle down to permanent cnoidal waves of different amplitudes appropriate for the wavelengths. Across the envelope nodes the frequency has sharp peaks signifying fast changes of phase. The envelope which consists of cnoidal waves of different amplitudes move at the same speed equal to the group velocity of the primary Stokes wave without further change of form. The situation is therefore quite analogous to other hydrodynamic instability problems when initial instability leads to a secondary steady state of finite amplitude.

6.4.4 Experimental Evidence

We shall attempt to discuss Feir's experiments (1967) based on our numerical results presented in the previous sections. In Feir's experiment wave pulses in the form of a half sine curve and with initially

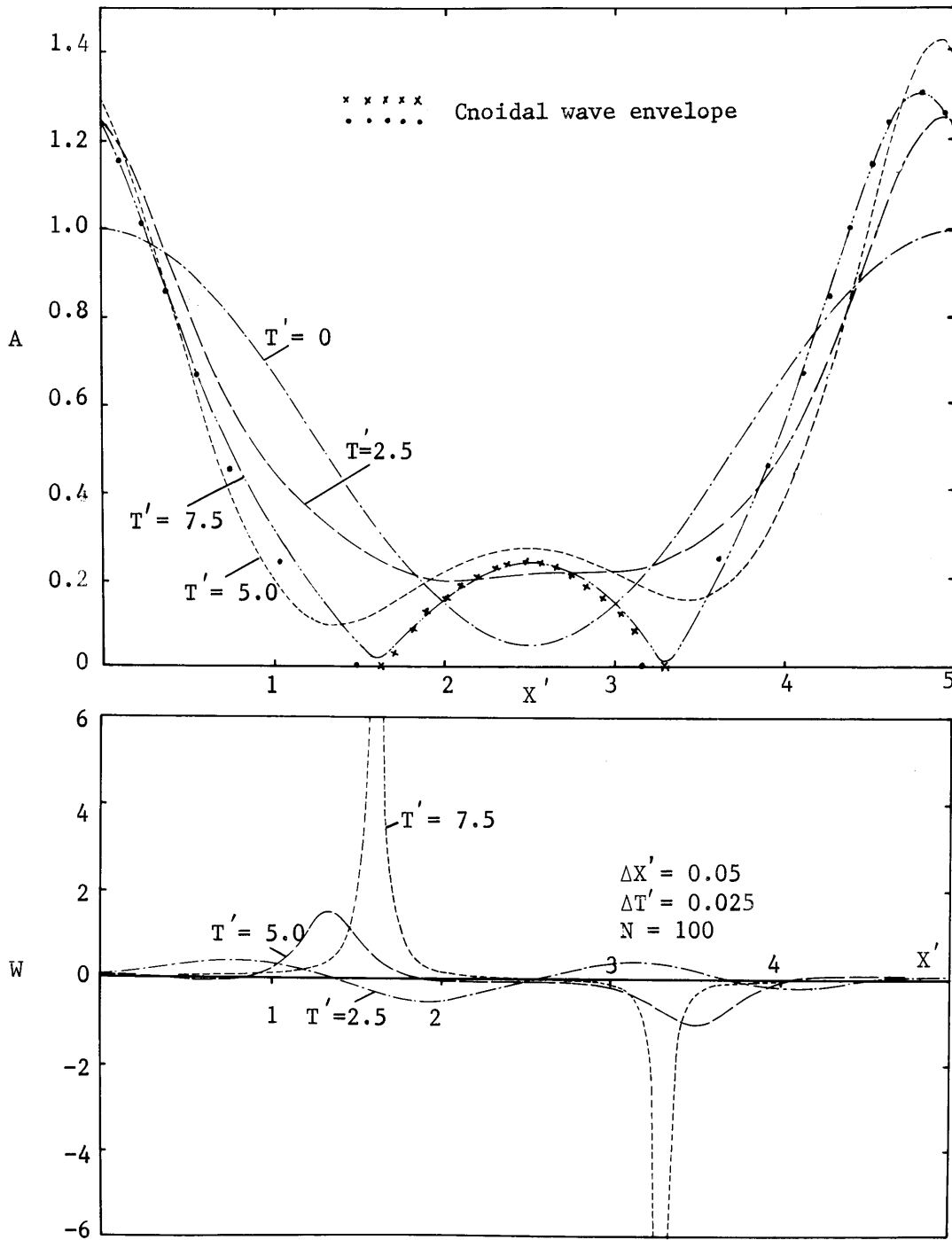


FIGURE 20 LARGE AMPLITUDE PERIODIC MODULATION

(Initial condition: $A = (1+0.9 \text{ Cos } 2\pi X'/5)/1.9$
 $W = 0.05 \text{ Cos } (2\pi X'/5)$)

constant frequency were generated by varying the wave maker amplitude but keeping its frequency constant. For a detailed discussion of the experimental set up see Feir (1967) The resulting surface displacement with respect to t was measured at two stations 4 ft and 28 ft down stream of the wave maker. Two records measured at 28 ft from the wave maker are reproduced in Figure 21. In order to get an idea on the relative effect of frequency dispersion and amplitude dispersion we compare the measured envelope with the envelope of permanent form, i.e. the solitary pulse. Notice that the wave records are now presented in dimensional variables. For comparison the solitary pulse (6.19) is rewritten in dimensional form:

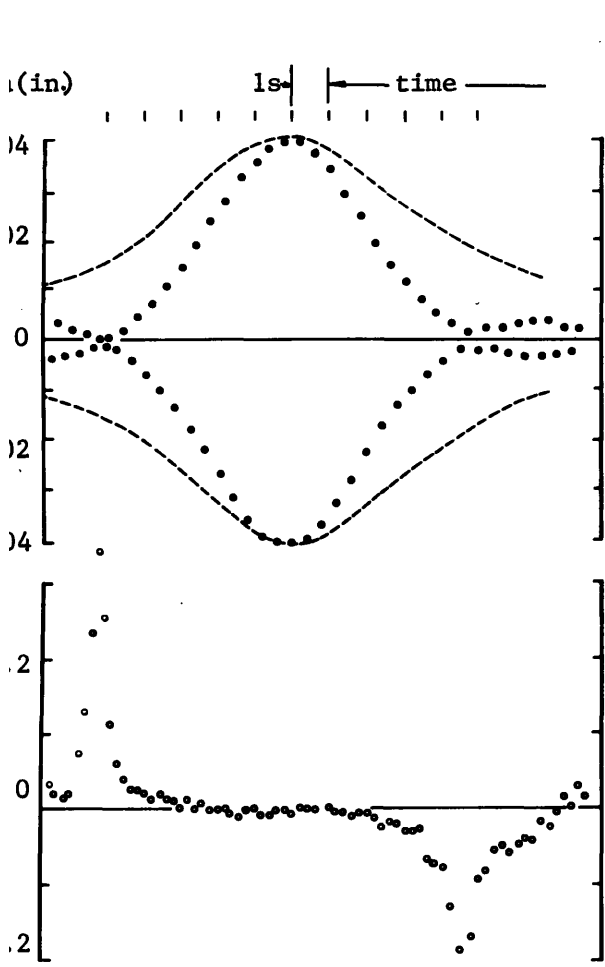
$$a/a_{max} = \operatorname{sech} \frac{\sqrt{2}}{2} (\bar{k} a_{max}) \bar{\omega} t \quad (6.31)$$

where $\chi=0$ is chosen at $a = a_{max}$. The width of the pulse $2t_{50}$, defined at $a/a_{max}=0.5$, is

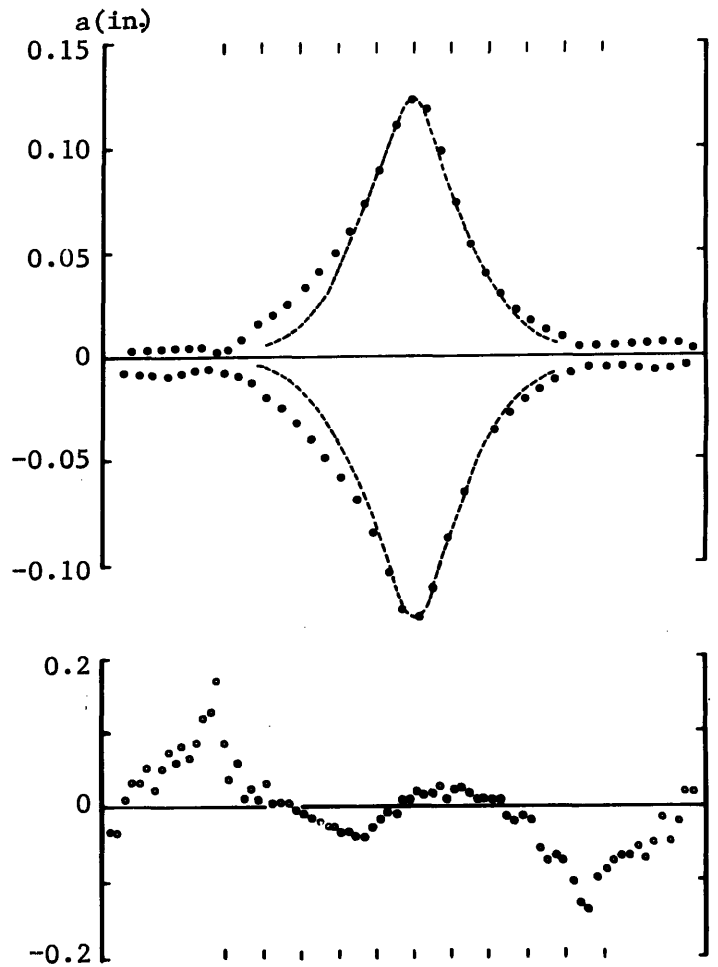
$$t_{50} = \frac{1.87}{\bar{\omega} \bar{k} a_{max}} \quad (6.32)$$

Comparing the measured envelope with the solitary pulse (6.31) we found that Figure 21(a) corresponds to wave envelope that is initially steeper than permanent form and Figure 21(b) corresponds to envelope that is flatter than permanent form. We now define T'_{28} to be the dimensionless time needed for the main group to travel from $x = 4'$ to $x = 28'$, i.e.,

$$T'_{28} = (\bar{k} \bar{a}) \frac{2\sqrt{2}}{g} (28' - 4') \quad (6.33)$$



(a) $\bar{k}a_{\max} \cong 0.0255, T_{28}' \cong 0.966$



(b) $\bar{k}a_{\max} \cong 0.112, T_{28}' \cong 4.5$

FIGURE 21 WAVE PULSE RECORDS (28 ft from the wave maker)

Experiment; Solitary pulse-----

In Figure 21(a) the wave steepness is very small so that the dimensionless time T'_{28} corresponding to a propagation of 24 ft is also small. In this case the frequency remains essentially uniform over the group as expected for small T' . For the present discussion let us ignore the narrow frequency peaks exhibited at the two ends of the group, which are resulted from the lack of smoothness of the initial pulse generated by the wave maker (for a discussion of this see Lighthill 1967). The record for a pulse initially flatter than solitary pulse is given in Figure 21(b). The corresponding dimensionless time T'_{28} increases to 4.5 as a result of increasing wave steepness. The frequency is no longer uniform over the group but becomes higher in the front and lower in the back; at the center, the amplitude steepens up to approach a solitary pulse. These features are essentially in agreement with the numerical calculations in Figure 11 for the same $T = 4.5$. At this stage T is still too small (or nonlinear effect is too weak) for the disintegration of the main group.

Another set of six records corresponding to different initial wave steepness is presented in Figure 22. It should be pointed out that in the second order, a is not the amplitude measured from the still water level, but $2a$ is the total wave height from crest to trough. We therefore replotted Feir's records so that the horizontal axis corresponds to the mid-line between crests and troughs. Comparison of each experimental record with permanent envelope is given in Figures 23, 24, 25, 26, 27 and 28. We first note that the viscous damping is very large in all these

experiments. By integrating the area in the Q^2 curve, it can be estimated that 50% of the total energy is lost as the waves propagated from 4 ft to 28 ft. It is interesting that in Figures 23 and 24 viscous damping reduces the maximum amplitude and at the same time reduces amplitude dispersion as compared to frequency dispersion, since it tends to make the wave envelope steeper than the permanent form. As may be expected from the small values of T'_{28} , in Figures 23 and 24, the nonlinear effect in the first two experiments is very small. Even if the viscous damping is neglected as in numerical calculation, it is not expected that the wave profile would reach dynamic equilibrium between amplitude dispersion and frequency dispersion at such small T'_{28} , (cf. Figure 11). However, as T'_{28} become larger and larger in Figures 25 to 28 the nonlinear effect becomes more important compared to viscous damping. Now, at the 28 ft station, the main group appears to approach gradually the permanent form. Disintegration of one wave group into multiple groups separated by nodal points is also observed in the last three cases (Figures 26, 27 and 28) when T'_{28} is large. Note that the disintegration in these experiments occurs only in the back of the main wave group. This is because the initial profiles at station 4 ft are very asymmetrical with the front being very close to permanent form and the back much flatter than the permanent form. Furthermore, if one studies carefully the profile near the nodal point for the last two cases (Figure 22) rapid change of wave phase is indeed observed.

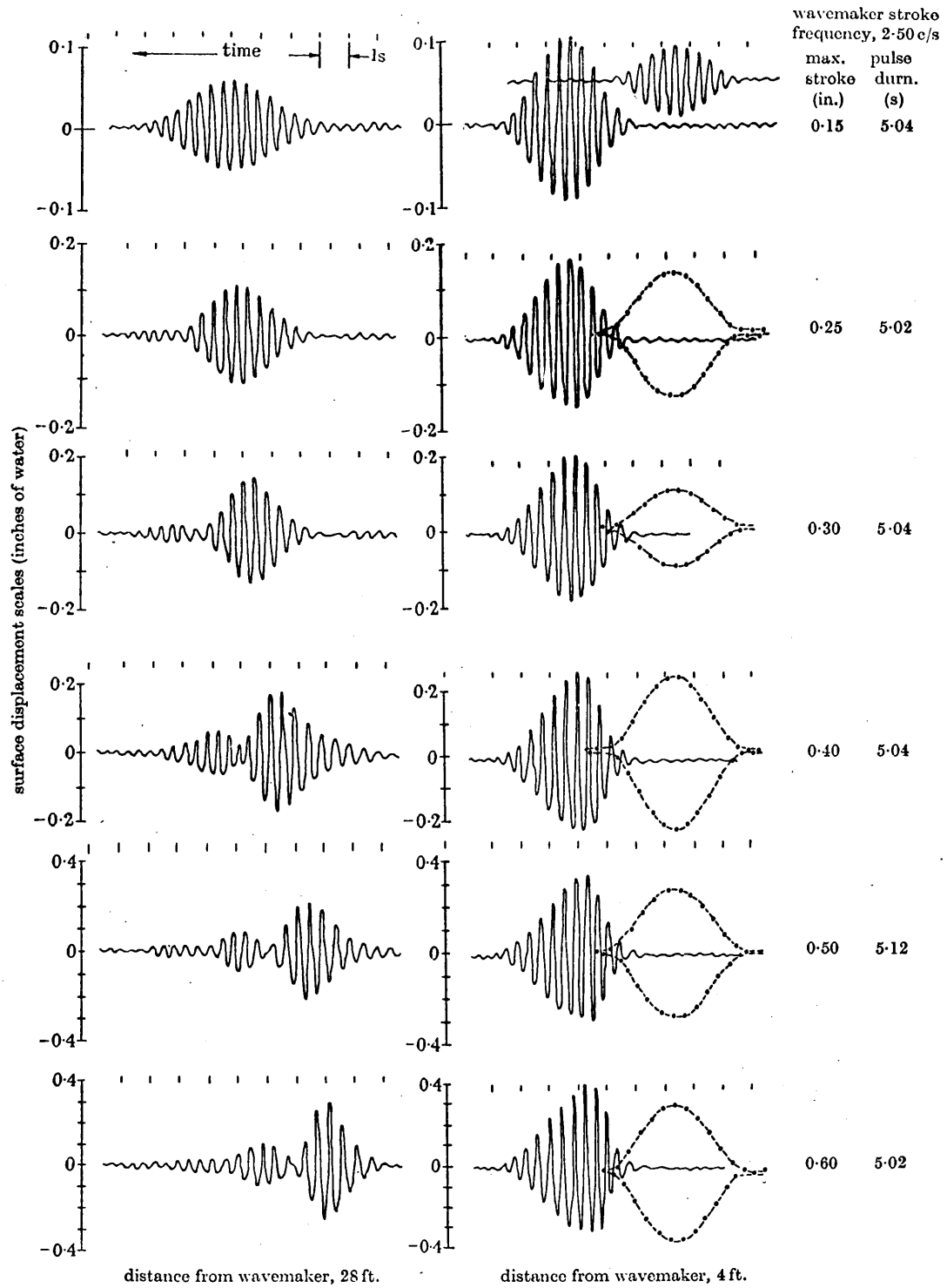
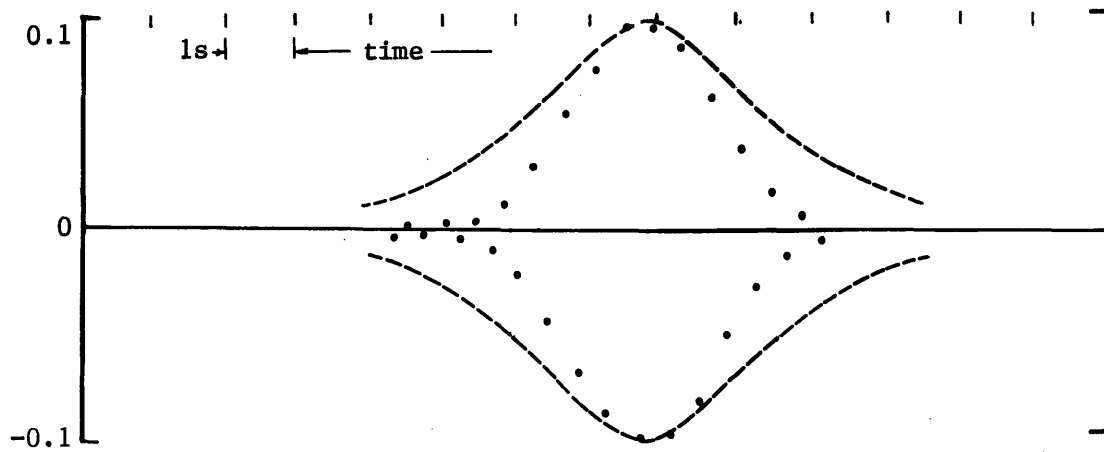
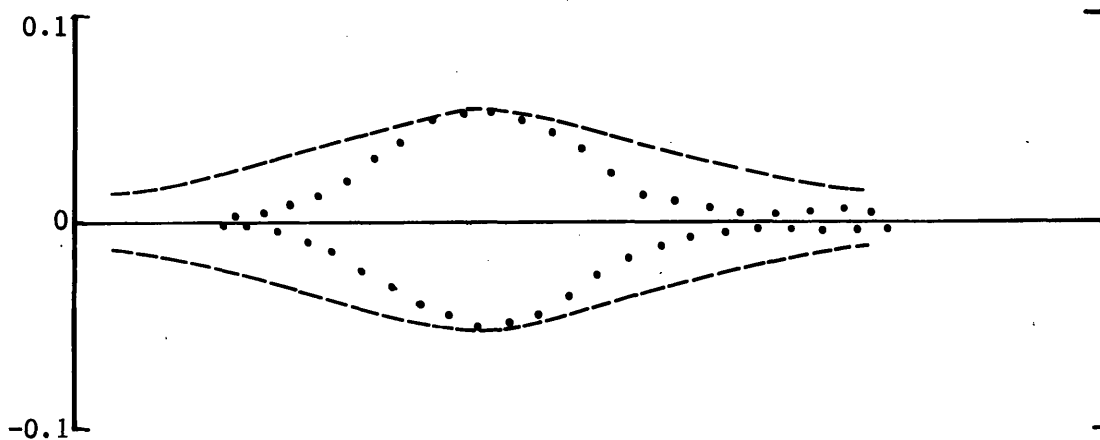


FIGURE 22 EXPERIMENTAL RECORDS OF WAVE PULSES

(taken from Feir, 1967)

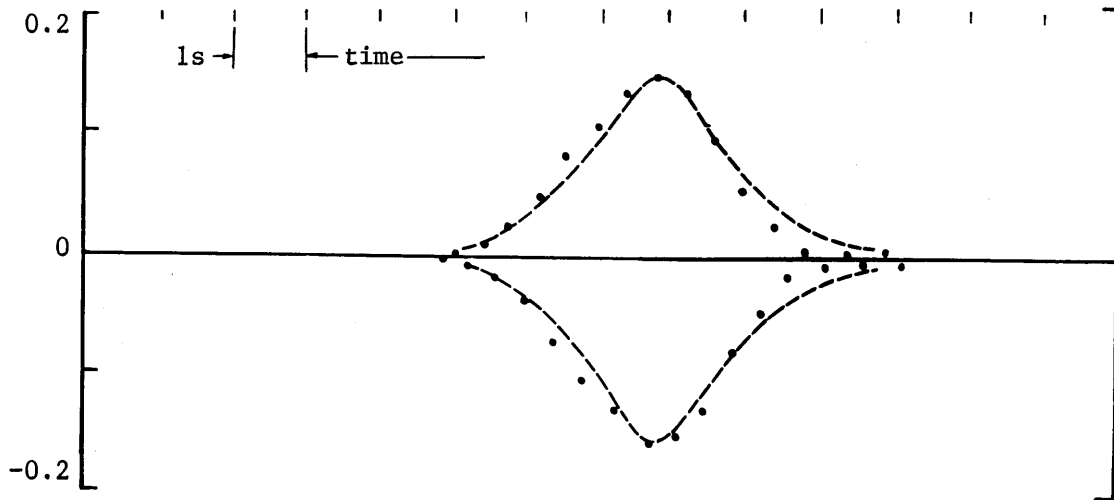


(a) 4 ft from the wave maker; $\bar{k}a_{\max} = 0.064$, $T' = 0$

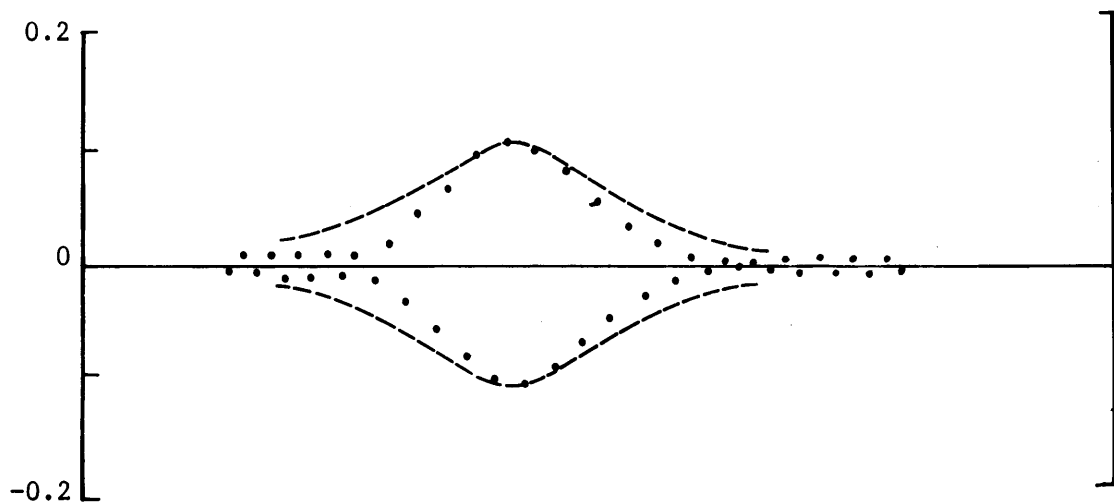


(b) 28 ft from the wave maker; $\bar{k}a_{\max} = 0.034$, $T' = 1.51$

FIGURE 23 WAVE PULSE RECORDS (Experiment, Solitary pulse -----)

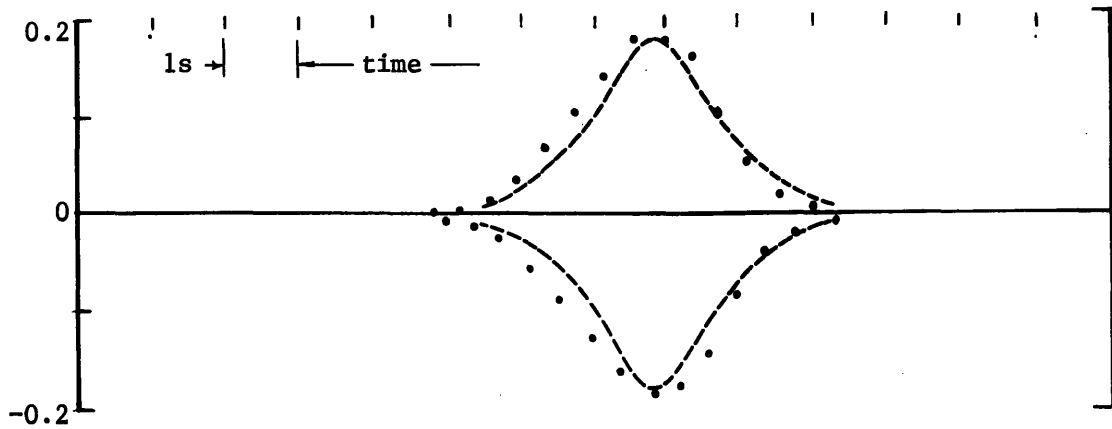


(a) 4 ft from the wave maker; $\bar{k}a_{\max} = 0.098, T' = 0$

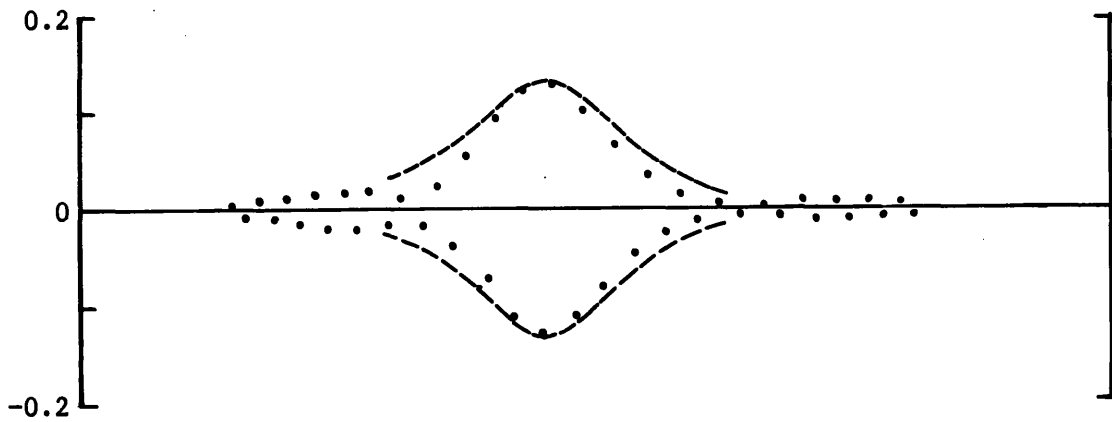


(b) 28 ft from the wave maker; $\bar{k}a_{\max} = 0.068, T' = 3.5$

FIGURE 24 WAVE PULSE RECORDS (Experiment ;Solitary pulse----)

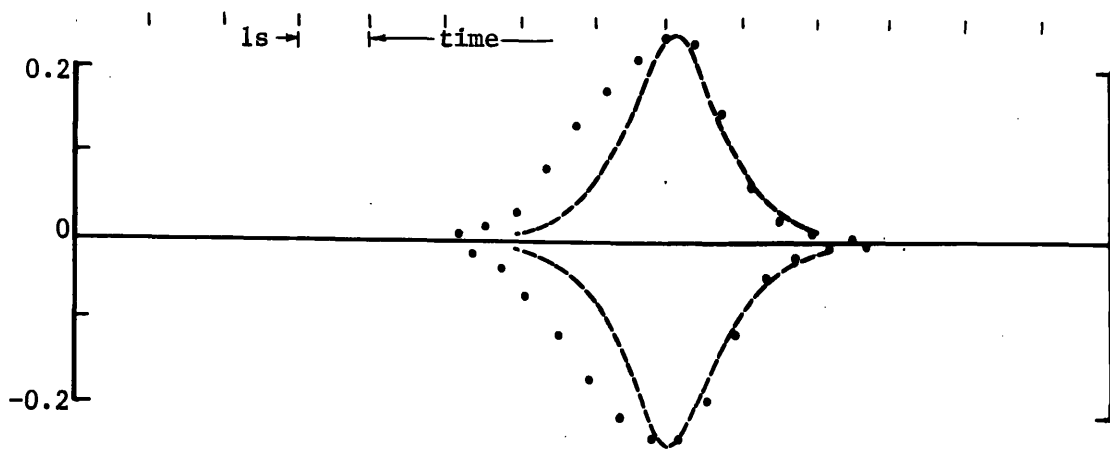


(a) 4 ft from the wave maker; $\bar{k}a_{\max} = 0.114$, $T' = 0$

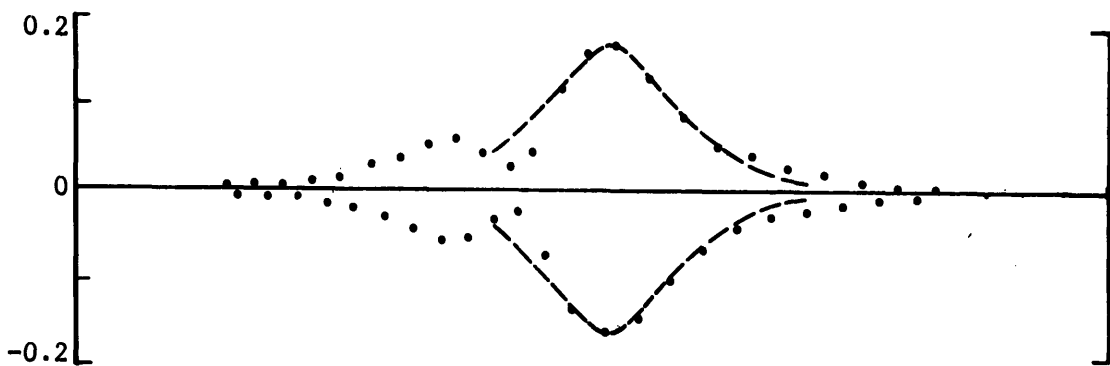


(b) 28 ft from the wave maker; $\bar{k}a_{\max} = 0.084$, $T' = 4.8$

FIGURE 25 WAVE PULSE RECORDS (Experiment; Solitary pulse ----)

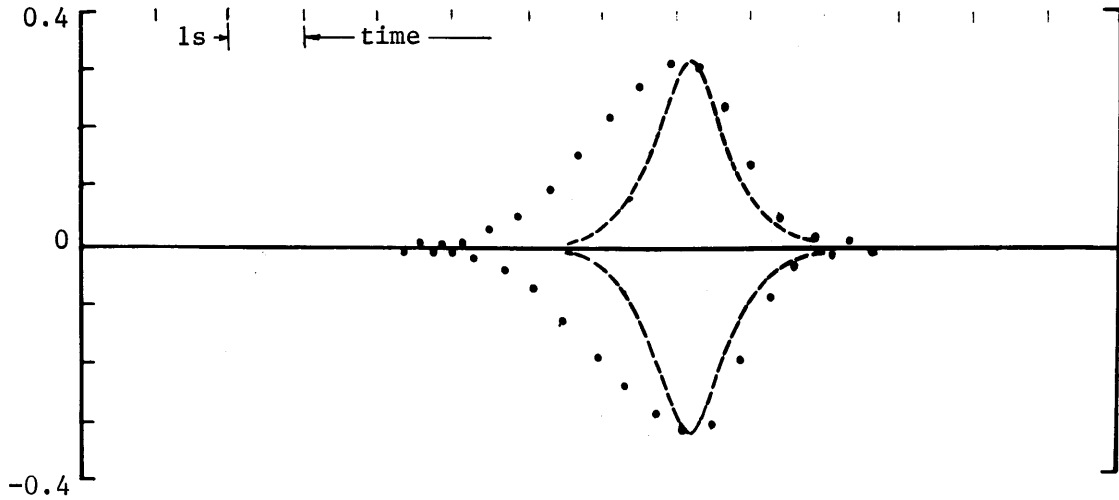


(a) 4 ft from the wave maker; $\bar{k}a_{\max} = 0.151, T' = 0$

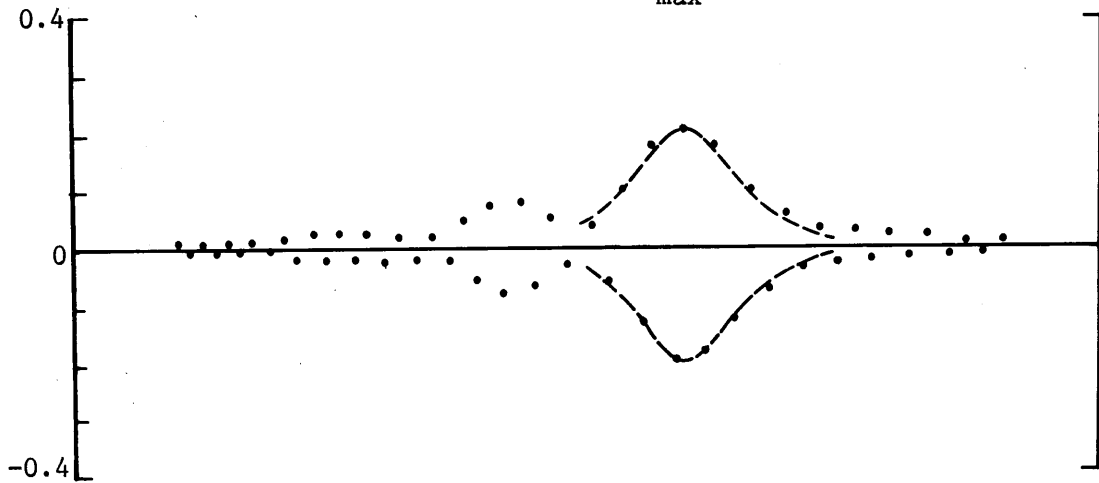


(b) 28 ft from the wave maker, $\bar{k}a_{\max} = 0.108, T' = 8.83$

FIGURE 26 WAVE PULSE RECORDS (Experiment ;Solitary pulse ----)

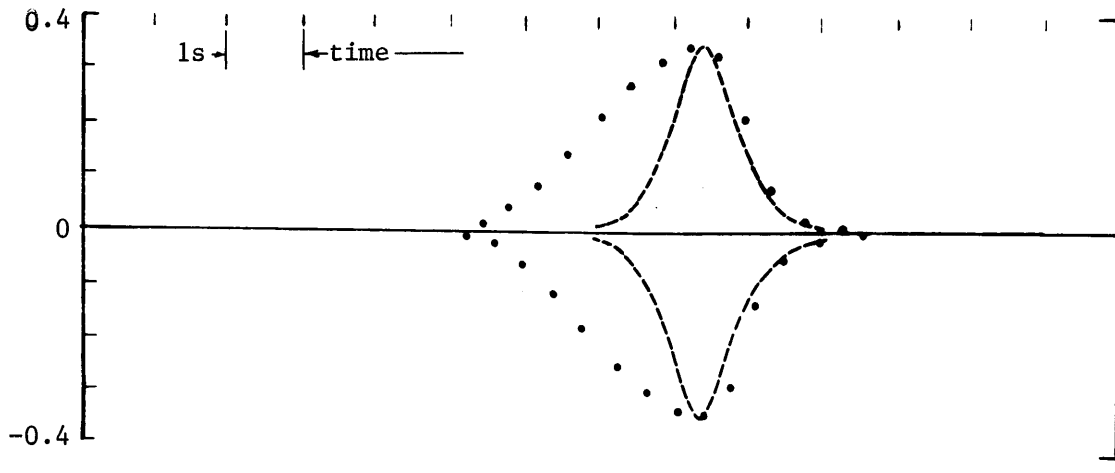


(a) 4 ft from the wave maker; $\bar{k}a_{\max} = 0.207$, $T' = 0$

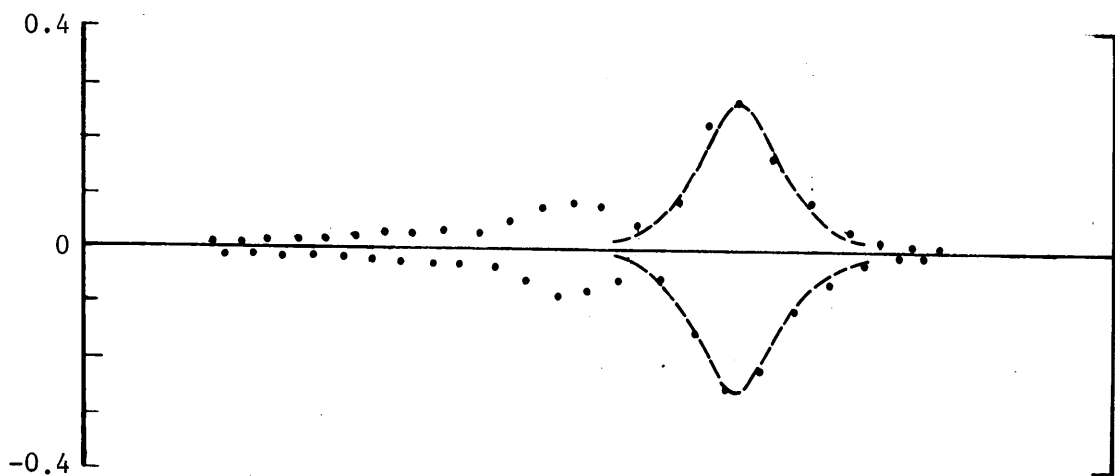


(b) 28 ft from the wave maker; $\bar{k}a_{\max} = 0.132$, $T' = 15.8$

FIGURE 27 WAVE PULSE RECORDS (Experiment; Solitary pulse ----)



(a) 4 ft from wave maker; $\bar{k}a_{\max} = 0.227$, $T' = 0$



(b) 28 ft from wave maker, $\bar{k}a_{\max} = 0.173$, $T' = 19.0$

FIGURE 28 WAVE PULSE RECORDS (Experiment; Solitary pulse ---)

CHAPTER 7

CONCLUSION

7.1 SUMMARY

The slow modulation of nearly periodic waves over uneven depth has been studied in this thesis. Improved results are obtained for quasi-steady waves over sloping bottoms. The general modulation equations are found to include further dispersion terms in comparison with Whitham's theory. As an example the difference between Whitham's and others' theory on the linear instability of Stokes waves is resolved. Envelopes of permanent forms are studied in detail. Finally, solitary and periodic envelopes are investigated numerically for their transient evolution. Many features revealed here resemble dispersive long waves in shallow water; in both cases the primary cause of the complex phenomena is the competition between amplitude and frequency dispersion.

In the numerical calculation of an unstable Stokes wave in deep water (Figure 18) the maximum wave amplitude actually increases to approximately 2.4 times larger than the initial amplitude of the Stokes wave. Other examples in Figures 11, 14, 15 and 20 all indicate increase of several fold in maximum wave amplitude. Such an increase of maximum wave amplitude is one particular nonlinear feature which would not be predicted from the linear theory with only frequency dispersion, because the effect of frequency dispersion always tends to spread out wave energy and decrease the wave amplitude. Since the maximum wave amplitude

is generally the most significant engineering design parameter, the study of nonlinear evolution of wave envelope is therefore important in showing that the classical theory may represent gross under-estimation.

Although viscous damping of wave energy in Feir experiments has been found as high as 50%, this is very likely due to side wall boundary layers. The viscous attenuation of gravity waves in ocean is usually small compared to the nonlinear growth. Take for example, a wave train in deep water, with a period of 10 sec and wave amplitude of 8 ft, the viscous attenuation of amplitude is only 0.002% over a distance of 32 miles (which equivalent to $T \cong 10.0$). As noted by Lighthill (1967 a) according to the linear theory the group velocity is not affected by the presence of small amount of dissipation. It is also known that in weakly nonlinear waves the dispersion relation is scarcely affected by viscosity. Hence the general nonlinear features will remain unchanged and the decay of wave amplitude may be estimated by the linear theory for periodic waves of the same wave length.

7.2 FUTURE PROSPECTS

7.2.1 Nonlinear Evolution Over Uneven Bottom

So far we have considered only the nonlinear evolution problem in deep water. The more interesting problem, especially in near shore oceanography, would be the study of evolution of unsteady wave train over uneven bottom (as opposed to the quasi-steady wave train studied in Chapter 5). The bottom slope in most coastal waters or of the continental

slopes may be considered as one order of magnitude smaller than the wave steepness. Under this condition, i.e. $\nabla_2 h \sim O(\epsilon^2)$, amplitude and frequency dispersions are as important as the refraction effect as discussed in Chapter 3. Now the governing equations (3.30) involve four dependent variables. This implies that a more complicated numerical scheme is to be developed to study this type of problems.

7.2.2 Simple Models of Ship Waves

Another extension to the present study would be to consider the three dimensional steady state evolution of wave envelope in deep water. One simple case that may be of interest to the Naval Architect is the waves generated by a uniform current passing a slowly modulated wavy wall, as studied by Howe (1967, 1968).

According to the linearized theory of Kelvin, that a ship generate two systems of waves in a wedge of half angle 19.5° . One system consists of diverging waves with crests roughly parallel to the ship's course; and the other is composed of lateral waves with crests roughly perpendicular to the ship's course. At high Froude numbers (e.g. at high speed) the former dominates while the latter is more visible at low Froude numbers. As may be observed from any aerial photograph of ship waves the divergent wave system is separated by nodes where sudden change of phase occurs (Figure 29). This phenomenon, unpredicted from linear theory, is quite similar to what we found in Chapter 5 that a two dimensional long crested wave group would tends to disintegrate and be separated by nodes where the phase changes rapidly.

Since the wavy wall creates waves crests inclined in the downstream direction it is useful to model the divergent waves for a narrow wave length spectrum (Figure 30). Now the wave pattern is steady, and the governing modulation equations involves only two independent variables, i.e., X and Y . The problem now is one of changing the numerical calculation of Chapter 5 from the (X, T) domain to (X, Y) domain, and no major difficulty can be foreseen at this stage.

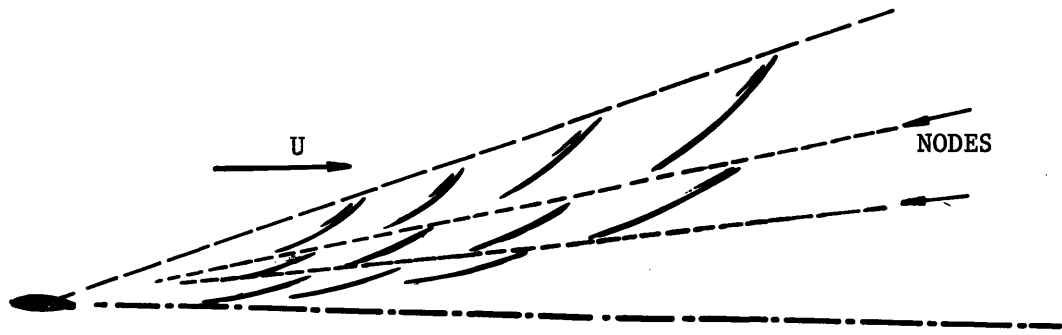


FIGURE 29 DIVERGENT WAVES GENERATED BEHIND A SHIP

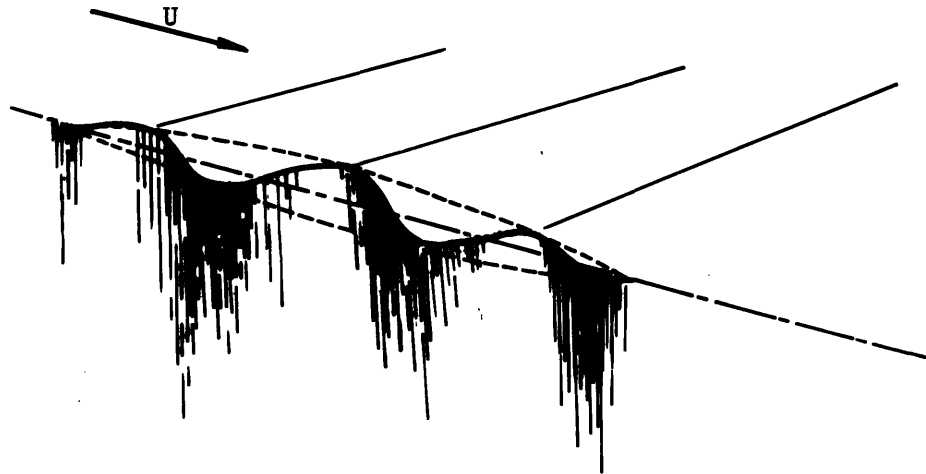


FIGURE 30 STEADY WAVE TRAINS GENERATED BY UNIFORM CURRENT PASSING WAVY WALL

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BIOGRAPHY

The author, Vincent Hwa-hing Chu, was born on July 27, 1942 in Canton, China. In September 1960 he entered the National Taiwan University to study Civil Engineering. He received his bachelor degree from that University in June, 1964.

The same year he decided to continue his graduate studies in the Department of Mechanical Engineering, University of Toronto where he was awarded the degree of Master of Applied Science in June 1966. He was a full time Laboratory Instructor from 1965 to 1966.

He came to M.I.T. in September 1966. While pursuing the doctoral program in the Department of Civil Engineering the author held a research assistantship.

The author has spent the summer of 1968 in the employment of Parsons, Brinckerhoff, Quade & Douglas, Inc., New York.

APPENDIX A

DERIVATION OF PERTURBATION EQUATIONS

We define the symbol $\langle \rangle^{(n,m)}$ by

$$\begin{aligned} \nabla f &= \sum_{n=0}^{\infty} \varepsilon^n \sum_{m=-n}^{+n} \langle \nabla f \rangle^{(n,m)} e^{im\psi/\varepsilon}, \\ \frac{\partial f}{\partial t} &= \sum_{n=0}^{\infty} \varepsilon^n \sum_{m=-n}^{+n} \langle \frac{\partial f}{\partial t} \rangle^{(n,m)} e^{im\psi/\varepsilon}, \\ fg &= \sum_{n=0}^{\infty} \varepsilon^n \sum_{m=-n}^{+n} \langle fg \rangle^{(n,m)} e^{im\psi/\varepsilon}. \end{aligned} \quad (\text{A.1 a,b,c})$$

For example, since by differentiation,

$$\nabla_2 \Phi = \varepsilon \nabla \Phi = \sum_{n=1}^{\infty} \varepsilon^n \sum_{m=-n}^{+n} \{ \varepsilon \nabla \phi^{(n,m)} + im \vec{k} \phi^{(n,m)} \} e^{im\psi/\varepsilon} \quad (\text{A.2})$$

we obtain

$$\langle \nabla_2 \Phi \rangle^{(n,m)} = \nabla \phi^{(n-1,m)} + im \frac{\vec{p}}{k} \phi^{(n,m)} \quad (\text{A.3})$$

where the first index in the superscript cannot be smaller than the second.

Thus $\langle \rangle^{(n,m)}$ is essentially an operator identifying orders and harmonics,

i.e. $\langle f \rangle^{(n,m)}$ is just the n th order and m th harmonic component of the

series $f = \sum_{n=0}^{\infty} \varepsilon^n \sum_{m=-n}^{+n} \langle f \rangle^{(n,m)} e^{im\psi/\varepsilon}$.

Now applying the operator $\langle \rangle^{(n,m)}$ to the bottom boundary condition (2.3)

$$\langle \Phi_{\vec{z}} \rangle^{(n,m)} = \langle -\nabla_2 \Phi \cdot \nabla_2 h \rangle^{(n,m)}, \quad \vec{z} = -\vec{h} \quad (\text{A.4})$$

or, upon making use of (A.3),

$$\phi_{z\bar{z}}^{(n,m)} = F^{(n,m)} = -\nabla h \cdot \left\{ \nabla \phi^{(n-2,m)} + im \frac{\vec{k}}{k} \phi^{(n-1,m)} \right\}. \quad (\text{A.5})$$

By differentiating (A.3) again we obtain $\langle \nabla_z^2 \bar{\Phi} \rangle^{(n,m)}$; and the Laplace equation (2.2) can be separated to give

$$\begin{aligned} \phi_{z\bar{z}}^{(n,m)} - m^2 |\vec{k}|^2 \phi^{(n,m)} &= R^{(n,m)} \\ &= -\nabla^2 \phi^{(n-2,m)} - im \nabla \cdot \left\{ \frac{\vec{k}}{k} \phi^{(n,m)} \right\} - im \frac{\vec{k}}{k} \cdot \nabla \phi^{(n-1,m)}. \end{aligned} \quad (\text{A.6})$$

For the nonlinear boundary conditions on the free surface, we have

$$\begin{aligned} g \phi_{z\bar{z}}^{(n,m)} - m^2 \omega_0^2 \phi^{(n,m)} &= G^{(n,m)} \\ &= g \phi_{z\bar{z}}^{(n,m)} - m^2 \omega_0^2 \phi^{(n,m)} - \left\langle \sum_{\nu=0}^{\infty} \frac{\eta^\nu}{\nu!} \frac{\partial^\nu}{\partial z^\nu} \left\{ \bar{\Phi}_{tt} + g \bar{\Phi}_z + \left(\frac{\partial}{\partial t} + \frac{\vec{u}}{2} \cdot \nabla \right) |\vec{u}|^2 \right\} \right\rangle^{(n,m)} \\ -g \eta^{(n,m)} + im \omega_0 \phi^{(n,m)} &= H^{(n,m)} \end{aligned} \quad (\text{A.7})$$

$$= im \omega_0 \phi^{(n,m)} + \left\langle \sum_{\nu=0}^{\infty} \frac{\eta^\nu}{\nu!} \frac{\partial^\nu}{\partial z^\nu} \left\{ \bar{\Phi}_t + \frac{1}{2} |\vec{u}|^2 \right\} \right\rangle^{(n,m)} \quad (\text{A.8})$$

A general compact expression such as (A.5) and (A.6) cannot be easily written down for $G^{(n,m)}$ and $H^{(n,m)}$. To keep track of the very involved computation of $G^{(n,m)}$ and $H^{(n,m)}$ it is helpful to refer to the diagram depicted in Figure 31. As a demonstration, let us consider the product

$\langle fg \rangle^{(4,1)} = \sum_{n_1+n_2=4}^{m_1+m_2=1} f^{(n_1,m_1)} g^{(n_2,m_2)}$; the pertinent elements $f^{(n_1,m_1)}$ and $g^{(n_2,m_2)}$ can be collected from the two opposite sides of the two rectangles

defined in Figure 31. Thus $\langle fg \rangle^{(4,1)}$ includes the following terms:

$$\begin{aligned} &f^{(3,0)} g^{(1,1)}, f^{(2,1)} g^{(2,2)}, f^{(1,-1)} g^{(3,2)}, f^{(2,0)} g^{(2,1)}, f^{(1,0)} g^{(3,1)}, \\ &f^{(1,1)} g^{(3,0)}, f^{(2,2)} g^{(2,-1)}, f^{(3,2)} g^{(1,-1)}, f^{(2,1)} g^{(2,0)}, f^{(3,1)} g^{(1,0)}. \end{aligned}$$

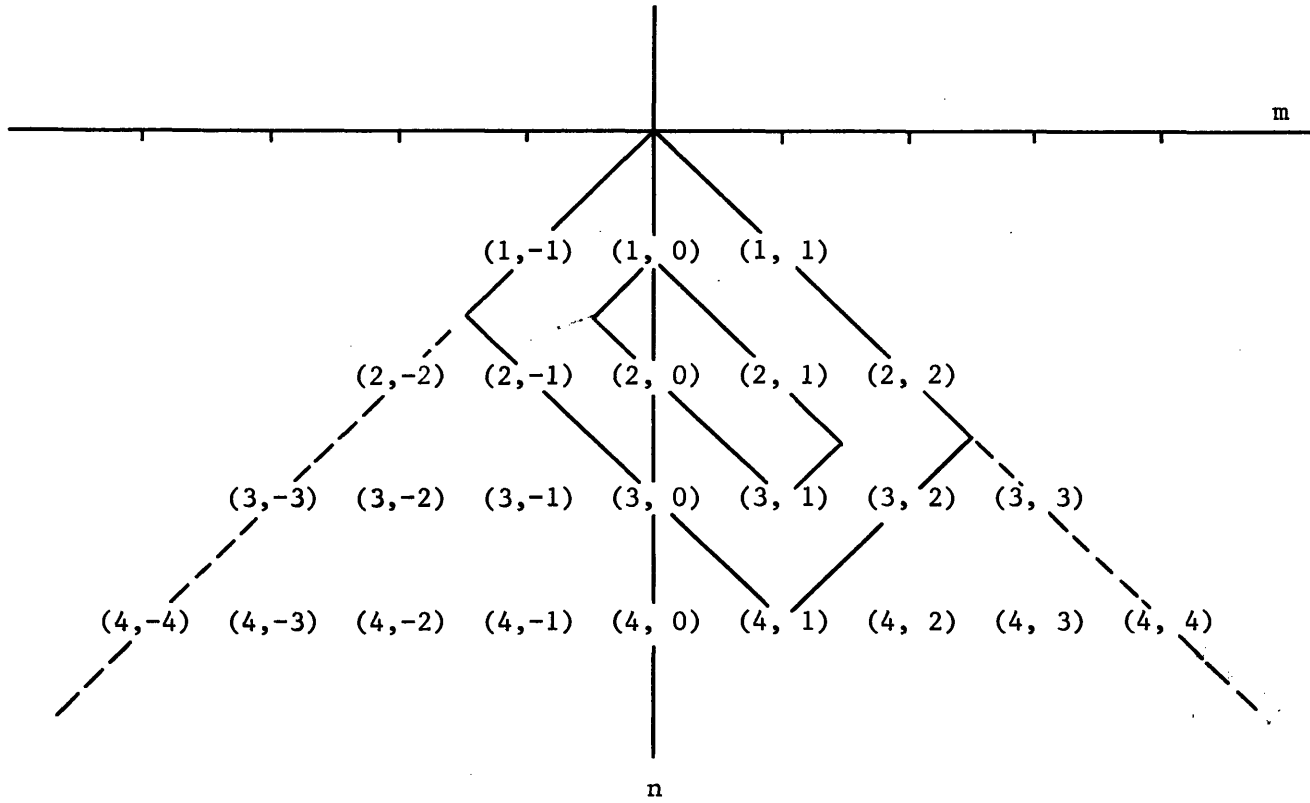


FIGURE 31 DIAGRAM FOR ASSISTING MULTIPLICATIONS OF TWO PERTURBATION SERIES

Using this method $G^{(n,m)}$ and $H^{(n,m)}$ are now calculated according to (A.7)

and (A.8) as follows:

1) For $\frac{1}{2}|\vec{u}|^2$ we define

$$f^{(n,m)} = \langle \frac{1}{2}|\vec{u}|^2 \rangle = \langle \frac{1}{2}|\nabla_2 \Phi|^2 + \frac{1}{2}\Phi_z^2 \rangle^{(n,m)}. \quad (\text{A.9})$$

Thus we have,

$$f^{(2,0)} = k^2 \phi^{(1,1)} \phi^{(1,-1)} + \phi_z^{(1,1)} \phi_z^{(1,-1)} + \frac{1}{2} \phi_z^{(1,0)} \phi_z^{(1,0)},$$

$$f^{(2,1)} = \phi_z^{(1,0)} \phi_z^{(1,1)}, \quad f^{(2,-1)} = \phi_z^{(1,0)} \phi_z^{(1,-1)},$$

$$f^{(2,2)} = -\frac{1}{2} k^2 \phi^{(1,1)2} + \frac{1}{2} \phi_z^{(1,1)2},$$

$$f^{(3,0)} = i\vec{k} \phi^{(1,1)} \cdot \{i\vec{k} \phi^{(2,-1)} + \nabla \phi^{(1,-1)}\} - i\vec{k} \phi^{(1,-1)} \cdot \{i\vec{k} \phi^{(2,1)} + \nabla \phi^{(1,1)}\},$$

$$f^{(3,1)} = 2k^2 \phi^{(2,2)} \phi^{(1,-1)} + i\vec{k} \phi^{(1,1)} \cdot \nabla \phi^{(1,0)} + \phi_z^{(2,1)} \phi_z^{(1,0)} + \phi_z^{(2,0)} \phi_z^{(1,1)} + \phi_z^{(2,2)} \phi_z^{(1,-1)}.$$

2) For $(\frac{\partial}{\partial t} + \frac{\vec{u}}{2} \cdot \nabla_3)|\vec{u}|^2$ we define

$$\begin{aligned} g^{(n,m)} &= \langle (\frac{\partial}{\partial t} + \frac{\vec{u}}{2} \cdot \nabla_3) |\vec{u}|^2 \rangle \\ &= \langle (2\frac{\partial}{\partial t} + \nabla_2 \Phi \cdot \nabla_2 + \Phi_z \frac{\partial}{\partial z}) \frac{1}{2} |\vec{u}|^2 \rangle^{(n,m)} \end{aligned} \quad (\text{A.10})$$

Thus we have,

$$g^{(2,0)} = 0,$$

$$g^{(2,1)} = -2i\omega_0 f^{(2,1)}, \quad g^{(2,-1)} = 2i\omega_0 f^{(2,-1)},$$

$$g^{(2,2)} = -4i\omega_0 f^{(2,2)},$$

$$g^{(3,0)} = 2f_T^{(2,0)} - ikf^{(2,1)} \cdot ik\phi^{(1,-1)} - ikf^{(2,-1)} \cdot ik\phi^{(1,1)} \\ + f_z^{(2,-1)} \phi_z^{(1,1)} + f_z^{(2,1)} \phi_z^{(1,-1)} + f_z^{(2,0)} \phi_z^{(1,0)},$$

$$g^{(3,1)} = 2f_T^{(2,1)} - 2i\omega_0 f^{(3,1)} - 2k^2 \phi^{(1,-1)} f^{(2,2)} \\ + f_z^{(2,2)} \phi_z^{(1,-1)} + f_z^{(2,1)} \phi_z^{(1,0)} + f_z^{(2,0)} \phi_z^{(1,1)}.$$

3) For $\frac{\eta^2}{2}$ we define

$$\xi^{(n,m)} = \left\langle \frac{1}{2} \eta^2 \right\rangle^{(n,m)} \quad (\text{A.11})$$

Thus

$$\xi^{(2,0)} = \frac{1}{2} \eta^{(1,0)2} + \eta^{(1,-1)} \eta^{(1,1)},$$

$$\xi^{(2,1)} = \eta^{(1,0)} \eta^{(1,1)}, \quad \xi^{(2,-1)} = \eta^{(1,0)} \eta^{(1,-1)},$$

$$\xi^{(2,2)} = \frac{1}{2} \eta^{(1,1)2}$$

4) For Φ_t we have

$$\langle \Phi_t \rangle^{(n,m)} = \phi_T^{(n-1,m)} - i\omega_0 m \phi^{(n,m)} - im\omega_2 \phi^{(n-2,m)} \quad (\text{A.12})$$

and for Φ_{tt} ,

$$\langle \Phi_{tt} \rangle^{(n,m)} = \phi_{TT}^{(n-2,m)} - im(\omega_0 \phi^{(n-1,m)})_T - im\omega_0 \phi_T^{(n-1,m)} \\ - im(\omega_2 \phi^{(n-3,m)})_T - im\omega_2 \phi_T^{(n-3,m)} - \dots \\ - m^2 \omega_0^2 \phi^{(n,m)} - 2m^2 \omega_0 \omega_2 \phi^{(n-2,m)} - \dots$$

(A.13)

5) Now from (A.8),

$$H^{(2,0)} = \phi_T^{(1,0)} + f^{(2,0)} + i\eta^{(1,1)} \omega_0 \phi_{\bar{z}}^{(1,-1)} - i\eta^{(1,-1)} \omega_0 \phi_{\bar{z}}^{(1,1)},$$

$$H^{(2,1)} = f^{(2,1)} + \phi_T^{(1,1)} - i\omega_0 \phi_{\bar{z}}^{(1,1)} \eta^{(1,0)},$$

$$H^{(2,2)} = f^{(2,2)} - i\eta^{(1,1)} \omega_0 \phi_{\bar{z}}^{(1,1)}.$$

6) From (A.7),

$$G^{(1,0)} = G^{(1,1)} = 0$$

$$G^{(2,0)} = -g^{(2,0)} + \omega_0^2 \phi_{\bar{z}}^{(1,-1)} \eta^{(1,1)} + \omega_0^2 \phi_{\bar{z}}^{(1,1)} \eta^{(1,-1)} \\ - g \phi_{\bar{z}\bar{z}}^{(1,1)} \eta^{(1,-1)} - g \phi_{\bar{z}\bar{z}}^{(1,-1)} \eta^{(1,1)} - g \phi_{\bar{z}\bar{z}}^{(1,0)} \eta^{(1,0)},$$

$$G^{(2,1)} = -g^{(2,1)} + i\omega_0 \phi_T^{(1,1)} + i(\omega_0 \phi_{\bar{z}}^{(1,1)})_T - \eta^{(1,1)} g \phi_{\bar{z}\bar{z}}^{(1,0)} \\ - \eta^{(1,0)} (g \phi_{\bar{z}\bar{z}}^{(1,1)} - \omega_0^2 \phi_{\bar{z}}^{(1,1)}),$$

$$G^{(2,2)} = -g^{(2,2)} - \eta^{(1,1)} (g \phi_{\bar{z}\bar{z}}^{(1,1)} - \omega_0^2 \phi_{\bar{z}}^{(1,1)}),$$

$$G^{(3,0)} = -\phi_{TT}^{(1,0)} - g^{(3,0)} - g_{\bar{z}}^{(2,1)} \eta^{(1,-1)} - g_{\bar{z}}^{(2,0)} \eta^{(1,0)} - g_{\bar{z}}^{(2,-1)} \eta^{(1,1)} \\ - \eta^{(2,0)} g \phi_{\bar{z}\bar{z}}^{(1,0)} - \eta^{(1,0)} g \phi_{\bar{z}\bar{z}}^{(2,0)} - \eta^{(2,-1)} (g \phi_{\bar{z}\bar{z}\bar{z}}^{(1,1)} - \omega_0^2 \phi_{\bar{z}}^{(1,1)}) \\ - \eta^{(2,-1)} (g \phi_{\bar{z}\bar{z}}^{(1,1)} - \omega_0^2 \phi_{\bar{z}}^{(1,1)}) - \eta^{(2,1)} (g \phi_{\bar{z}\bar{z}}^{(1,-1)} - \omega_0^2 \phi_{\bar{z}}^{(1,-1)} - \eta^{(2,0)} g \phi_{\bar{z}\bar{z}\bar{z}}^{(1,0)}) \\ - \eta^{(1,1)} (g \phi_{\bar{z}\bar{z}}^{(2,-1)} - \omega_0^2 \phi_{\bar{z}}^{(2,-1)} + i\omega_0 \phi_{T\bar{z}}^{(1,-1)} + i(\omega_0 \phi_{\bar{z}}^{(1,-1)})_T) \\ - \eta^{(1,-1)} (g \phi_{\bar{z}\bar{z}}^{(2,1)} - \omega_0^2 \phi_{\bar{z}}^{(2,1)} - i\omega_0 \phi_{T\bar{z}}^{(1,1)} - i(\omega_0 \phi_{\bar{z}}^{(1,1)})_T).$$

$$\begin{aligned}
G^{(3,1)} &= -\phi_{TT}^{(1,1)} + i(\omega_0 \phi^{(2,1)})_T + i\omega_0 \phi_T^{(2,1)} + 2\omega_0 \omega_z \phi^{(1,1)} \\
&\quad - g^{(3,1)} - g_{\bar{z}}^{(2,0)} \eta^{(1,1)} - g_{\bar{z}}^{(2,1)} \eta^{(1,0)} - g_{\bar{z}}^{(2,2)} \eta^{(1,-1)} \\
&\quad - \eta^{(2,0)} (g \phi_{\bar{z}\bar{z}}^{(1,1)} - \omega_0^2 \phi_z^{(1,1)}) - \eta^{(1,1)} g \phi_{\bar{z}\bar{z}}^{(2,0)} \\
&\quad - \eta^{(2,2)} (g \phi_{\bar{z}\bar{z}}^{(1,1)} - \omega_0^2 \phi_z^{(1,-1)}) - \eta^{(1,-1)} (g \phi_{\bar{z}\bar{z}}^{(2,2)} - 4\omega_0^2 \phi_z^{(2,2)}) \\
&\quad - \eta^{(2,1)} g \phi_{\bar{z}\bar{z}}^{(1,0)} - \eta^{(1,0)} \{ g \phi_{\bar{z}\bar{z}}^{(2,1)} - \omega_0^2 \phi_z^{(2,1)} - i(\omega_0 \phi_z^{(1,1)})_T - i\omega_0 \phi_{T\bar{z}}^{(1,1)} \} \\
&\quad - \xi^{(2,0)} (g \phi_{\bar{z}\bar{z}\bar{z}}^{(1,1)} - \omega_0^2 \phi_{\bar{z}}^{(1,1)}) - \xi^{(2,2)} (g \phi_{\bar{z}\bar{z}\bar{z}}^{(1,-1)} - \omega_0^2 \phi_{\bar{z}}^{(1,1)}) \\
&\quad - \xi^{(2,1)} g \phi_{\bar{z}\bar{z}}^{(1,0)}
\end{aligned}$$

APPENDIX B

SIMPLIFICATION FOR $G^{(n,m)}$ AND $H^{(n,m)}$

We first note that the functions $G^{(n,m)}$ and $H^{(n,m)}$ appeared in the free surface boundary conditions of (2.14 a-d) are evaluated at $z = 0$.

Therefore, all simplifications present in this Appendix will be restricted only for $z = 0$.

Referring to Appendix A, we have $R^{(1,0)} = F^{(1,0)} = G^{(1,0)} = H^{(1,0)} = R^{(1,0)} = F^{(1,0)} = G^{(1,0)} = H^{(1,0)} = 0$. The first order relations at $z = 0$ are obtained immediately from (2.14 a-d); i.e.,

$$\begin{aligned} \phi_z^{(1,0)} &= \eta^{(1,0)} = 0 \quad , \\ \phi^{(1,1)} &= -\phi^{(1,-1)} \quad \phi^{(1,1)} \\ \eta^{(1,1)} &= \eta^{(1,-1)} = i \frac{\omega_0}{g} \phi^{(1,1)} = \frac{i k_{\omega_0}}{\omega_0} \phi^{(1,1)} \quad , \\ \phi_z^{(1,1)} &= \frac{\omega_0^2}{g} \phi^{(1,1)} = k_{\omega_0} \phi^{(1,1)} \\ \phi_{zz}^{(1,1)} &= k_{\omega_0}^2 \phi^{(1,1)} \end{aligned} \tag{B.1}$$

Making use of (B.1), the second order nonlinear terms given in Appendix A may be further simplified as follows:

For $n = 2, m = 0$:

$$\begin{aligned}
 f^{(2,0)} &= -(k^2 - k_\infty^2) \phi^{(1,1)2} \\
 g^{(2,0)} &= 0 \\
 \xi^{(2,0)} &= \eta^{(1,1)2} = -\frac{1}{g} k_\infty \phi^{(1,1)2} \\
 H^{(2,0)} &= \phi_T^{(1,0)} - (k^2 + k_\infty^2) \phi^{(1,1)2} + 2k_\infty \phi_z^{(1,1)} \\
 &= \phi_T^{(1,0)} + (k_\infty^2 - k^2) \phi^{(1,1)2} \\
 G^{(2,0)} &= 0
 \end{aligned} \tag{B.2}$$

For $n = 2, m = 1$:

$$\begin{aligned}
 f^{(2,1)} &= g^{(2,1)} = \xi^{(2,1)} = 0 \\
 H^{(2,1)} &= \phi_T^{(1,1)} \\
 G^{(2,1)} &= i\omega_0 \phi_T^{(1,1)} + i(\omega_0 \phi^{(1,1)})_T
 \end{aligned} \tag{B.3}$$

For $n = 2, m = 2$:

$$\begin{aligned}
 f^{(2,2)} &= \frac{1}{2} (k_\infty^2 - k^2) \phi^{(1,1)2} \\
 g^{(2,2)} &= -2i\omega_0 (k_\infty^2 - k^2) \phi^{(1,1)2} \\
 \xi^{(2,2)} &= -\frac{1}{2g} k_\infty \phi^{(1,1)2} \\
 H^{(2,2)} &= \frac{1}{2} (k_\infty^2 - k^2) \phi^{(1,1)2} + k_\infty \phi^{(1,1)} \phi_z^{(1,1)} = \frac{1}{2} (3k_\infty^2 - k^2) \phi^{(1,1)2} \\
 G^{(2,2)} &= 2i\omega_0 (k_\infty^2 - k^2) \phi^{(1,1)2} - i\omega_0 \phi^{(1,1)} (k^2 \phi^{(1,1)} - k_\infty^2 \phi^{(1,1)}) \\
 &= 3i\omega_0 (k_\infty^2 - k^2) \phi^{(1,1)2}
 \end{aligned} \tag{B.4}$$

Before going into third order, we first summarize the second order results at $z = 0$:

$$\begin{aligned}
\phi_z^{(2,0)} &= 0, \quad g\eta^{(2,0)} = -\phi_T^{(1,0)} + (k^2 - k_\infty^2)\phi^{(1,1)2}, \\
\phi^{(2,1)} &= \phi^{(2,-1)}, \quad \eta^{(2,1)} = -\eta^{(2,-1)} = \frac{1}{g} \{ i\omega_0 \phi^{(2,1)} - \phi_T^{(1,1)} \}, \\
\phi_z^{(2,1)} &= k_\infty \phi^{(2,1)} + \frac{i}{g} \{ \omega_0 \phi^{(1,1)2} \}_T / \phi^{(1,1)}, \\
\phi_{zz}^{(2,1)} - k^2 \phi^{(2,1)} &= -i \nabla \cdot (k \phi^{(1,1)2}) / \phi^{(1,1)}, \\
\phi^{(2,2)} &= -\phi^{(2,-2)} = i \frac{3}{4} \frac{\omega_0}{g} k_\infty (\sigma^4 - 1) \phi^{(1,1)2}, \\
\eta^{(2,2)} &= \eta^{(2,-2)} = -\frac{k_\infty^2}{g} \sigma^2 (3\sigma^2 - 1) \phi^{(1,1)2}, \\
\phi_z^{(2,2)} &= 4k_\infty \phi^{(2,2)} + \frac{1}{g} \Gamma^{(2,2)} = i 3 \frac{\omega_0}{g} k_\infty^2 \sigma^2 (\sigma^2 - 1) \phi^{(1,1)2}, \\
\phi_{zz}^{(2,2)} &= 4k^2 \phi^{(2,2)} = i 3 \frac{\omega_0}{g} k_\infty k^2 (\sigma^4 - 1) \phi^{(1,1)2}. \tag{B.5}
\end{aligned}$$

Now, for $n = 3, m = 0$:

$$\begin{aligned}
f^{(3,0)} &= 0, \quad f_z^{(2,1)} = 0, \\
g^{(3,0)} &= 2f_T^{(2,0)}, \quad g_z^{(2,1)} = 0, \\
G^{(3,0)} &= -2f_T^{(2,0)} - \phi_{TT}^{(1,0)} + 2g\eta^{(2,1)} (\phi_{zz}^{(1,1)} - k_\infty \phi_z^{(1,1)}) \\
&\quad - 2g\eta^{(1,1)} \left\{ \phi_{zz}^{(2,1)} - k_\infty \phi_z^{(2,1)} - \frac{i}{g} \frac{(\omega_0 \phi_z^{(1,1)2})_T}{\phi^{(1,1)}} \right\} \tag{B.6} \\
&= -(2f^{(2,0)} + \phi_T^{(1,0)})_T + 2(i\omega_0 \phi^{(2,1)} - \phi_T^{(1,1)}) (k^2 \phi^{(1,1)} - k_\infty^2 \phi^{(1,1)}) \\
&\quad - 2i\omega_0 \phi^{(1,1)} \left\{ \phi_{zz}^{(2,1)} - k_\infty^2 \phi^{(2,1)} - \frac{ik_\infty (\omega_0 \phi^{(1,1)2})_T}{g \phi^{(1,1)}} - \frac{i(\omega_0 k_\infty^2 \phi^{(1,1)2})_T}{g k_\infty \phi^{(1,1)}} \right\}
\end{aligned}$$

$$\begin{aligned}
G_T^{(3,0)} &= - (2f^{(2,0)} + \phi_T^{(1,0)})_T - 2i\omega_0 \phi^{(1,1)} \{ \phi_{zz}^{(2,1)} - k^2 \phi^{(2,1)} \} \\
&\quad - 2\phi^{(1,1)} \phi_T^{(1,1)} (k^2 - k_\infty^2) \\
&\quad - 2 \frac{\omega_0}{g k_\infty} \{ k_\infty^2 (\omega_0 \phi^{(1,1)})_T + (\omega_0 k_\infty^2 \phi^{(1,1)})_T \} \\
&= - (2f^{(2,0)} + \phi_T^{(1,0)})_T - 2\omega_0 \nabla \cdot \{ \vec{k} \phi^{(1,1)} \} - \{ \phi^{(1,1)} \}_T^2 (k^2 - k_\infty^2) \\
&\quad - \frac{2}{\omega_0} \{ k_\infty^2 (\omega_0 \phi^{(1,1)})_T + (\omega_0 k_\infty^2 \phi^{(1,1)})_T \} \\
&= \{ 2(k^2 + k_\infty^2) \phi^{(1,1)2} - (k^2 - k_\infty^2) \phi^{(1,1)2} + g \eta^{(2,0)} \}_T \\
&\quad - 2 \nabla \cdot \{ \omega_0 \vec{k} \phi^{(1,1)2} \} - 2 \vec{k} \phi^{(1,1)2} \cdot \nabla \omega_0 \\
&\quad - \{ (k^2 - k_\infty^2) \phi^{(1,1)2} \}_T + \phi^{(1,1)2} (k^2 - k_\infty^2)_T \\
&\quad - \frac{2}{\omega_0} \{ k_\infty^2 \omega_0 (\phi^{(1,1)2})_T + 2\omega_0 T k_\infty^2 \phi^{(1,1)2} + \omega_0 k_\infty^2 \phi^{(1,1)2} \}_T \\
&= \{ g \eta^{(2,0)} + 4k_\infty^2 \phi^{(1,1)2} \}_T - 2 \nabla \cdot \{ \omega_0 \vec{k} \phi^{(1,1)2} \} - k_\infty^2 T \phi^{(1,1)2} \\
&\quad - \{ 2k_\infty^2 (\phi^{(1,1)2})_T - (k_\infty^2)_T \phi^{(1,1)2} + 2(k_\infty^2 \phi^{(1,1)2})_T \} \\
&= (g \eta^{(2,0)})_T - 2 \nabla \cdot \{ \omega_0 \vec{k} \phi^{(1,1)2} \} \tag{B.6}
\end{aligned}$$

For $n = 3, m = 1$:

$$\begin{aligned}
f^{(3,1)} &= -2k^2 \phi^{(2,2)} \phi^{(1,1)} - \phi_z^{(2,2)} \phi_z^{(1,1)} + i \vec{k} \phi^{(1,1)} \cdot \nabla \phi^{(1,0)}, \\
f_z^{(2,0)} &= -2k^2 \phi^{(1,1)} \phi_z^{(1,1)} - 2\phi_z^{(1,1)} \phi_{zz}^{(1,1)} = \phi^{(1,1)2} (-4k^2 k_\infty), \\
f_z^{(2,2)} &= -k^2 \phi^{(1,1)} \phi_z^{(1,1)} + \phi_z^{(1,1)} \phi_{zz}^{(1,1)} \equiv 0, \\
g^{(3,1)} &= -2i\omega_0 f^{(3,1)} - 2k^2 \phi^{(1,1)} f^{(2,2)} + f_z^{(2,0)} \phi_z^{(1,1)}, \tag{B.7}
\end{aligned}$$

$$g_{\bar{z}}^{(2,0)} = g_{\bar{z}}^{(2,1)} = g_{\bar{z}}^{(2,2)} = 0$$

$$\begin{aligned}
G^{(3,1)} &= -\phi_{\tau\tau}^{(1,1)} + i(\omega_0 \phi^{(2,1)})_{\tau} + i\omega_0 \phi_{\tau}^{(2,1)} + 2\omega_0 \omega_2 \phi^{(1,1)} \\
&\quad + 2i\omega_0 f^{(3,1)} + 2k^2 \phi^{(1,1)} \frac{1}{2}(k^2 - k_{\infty}^2) + 4k^2 k_{\infty} \phi^{(1,1)2} k_{\infty} \phi^{(1,1)} \\
&\quad - g\eta^{(2,0)}(k^2 - k_{\infty}^2) \phi^{(1,1)} + g\eta^{(2,2)}(k^2 - k_{\infty}^2) \phi^{(1,1)} \\
&\quad - g\eta^{(1,1)} \left\{ \phi_{\bar{z}\bar{z}}^{(2,2)} - 4k_{\infty} \phi_{\bar{z}}^{(2,2)} \right\} \\
&= -\phi_{\tau\tau}^{(1,1)} + i(\omega_0 \phi^{(1,1)})_{\tau} + i\omega_0 \phi^{(2,1)} + 2\omega_0 \omega_2 \phi^{(1,1)} \\
&\quad - 2\omega_0 \bar{k} \phi^{(1,1)} \cdot \nabla \phi^{(1,0)} - g\eta^{(2,0)}(k^2 - k_{\infty}^2) \phi^{(1,1)} \\
&\quad - 4i\omega_0 k^2 \phi^{(2,2)} \phi^{(1,1)} - 2i\omega_0 \phi_{\bar{z}}^{(2,2)} \phi_{\bar{z}}^{(1,1)} - i\omega_0 \phi_{\bar{z}\bar{z}}^{(2,2)} \phi^{(1,1)} \\
&\quad + 4i\omega_0 k_{\infty} \phi_{\bar{z}}^{(2,2)} \phi^{(1,1)} + g\eta^{(2,0)}(k^2 - k_{\infty}^2) \phi^{(1,1)} \\
&\quad + \{4\sigma^2 + \sigma^2(1 - \sigma^2)\} k_{\infty}^4 \phi^{(1,1)3} \\
&= -\phi_{\tau\tau}^{(1,1)} + 2\omega_0 \omega_2 \phi^{(1,1)} - 2\omega_0 \bar{k} \phi^{(1,1)} \cdot \nabla \phi^{(1,0)} - g\eta^{(2,0)}(k^2 - k_{\infty}^2) \phi^{(1,1)} \\
&\quad - 4i\omega_0 k^2 \phi^{(2,2)} \phi^{(1,1)} - 2i\omega_0 \phi_{\bar{z}}^{(2,2)} \phi_{\bar{z}}^{(1,1)} - i\omega_0 \phi_{\bar{z}\bar{z}}^{(2,2)} \phi^{(1,1)} \\
&\quad + 4i\omega_0 k_{\infty} \phi_{\bar{z}}^{(2,2)} \phi^{(1,1)} + g\eta^{(2,2)}(k^2 - k_{\infty}^2) \phi^{(1,1)} \\
&\quad + k_{\infty}^4 \phi^{(1,1)3} \{ \sigma^2(1 - \sigma^2) + 4\sigma^2 \} \\
&= -\phi_{\tau\tau}^{(1,1)} + 2\omega_0 \omega_2 \phi^{(1,1)} - 2\omega_0 \bar{k} \phi^{(1,1)} \cdot \nabla \phi^{(1,0)} - g\eta^{(2,0)}(k^2 - k_{\infty}^2) \phi^{(1,1)} \\
&\quad + k_{\infty}^4 \phi^{(1,1)3} \left\{ 3(\sigma^4 - 1)\sigma^2 + 6\sigma^2(\sigma^2 - 1) + 3\sigma^2(\sigma^4 - 1) - 12\sigma^2(\sigma^2 - 1) \right. \\
&\quad \quad \left. - \frac{1}{2}\sigma^2(3\sigma^2 - 1)(\sigma^2 - 1) + \sigma^2(1 - \sigma^2) + 4\sigma^2 \right\} \\
&= -\phi_{\tau\tau}^{(1,1)} + i\omega_2 \omega_0 \phi^{(1,1)} - 2\omega_0 \bar{k} \phi^{(1,1)} \cdot \nabla \phi^{(1,0)} - g\eta^{(2,0)}(k^2 - k_{\infty}^2) \phi^{(1,1)} \\
&\quad + \frac{1}{2} k^2 \phi^{(1,1)3} k_{\infty}^2 (9\sigma^4 - 10\sigma^2 + 9)
\end{aligned} \tag{B.8}$$

APPENDIX C

SIMPLIFICATIONS OF ω_2^X FOR TWO SPECIFIC CASES

C.1 TWO-DIMENSIONAL WAVE TRAINS

We consider here a long crested wave propagated along the x-direction and the depth of water is assumed to be a function of x only, so that $\frac{\partial}{\partial y} = 0$ and $\frac{1}{k} = k \hat{x}$. Let us first differentiate $\phi^{(1,1)}$ and $\phi^{(2,1)}$ with respect to X :

$$\phi_X^{(1,1)} = -igk \left(\frac{A^{(1,1)}}{\omega_0} \right) \{ \alpha_2 \cosh Q + (2\alpha_3 Q + \alpha_1) \sinh Q \} , \quad (C.1)$$

where

$$\alpha_1 = h_X , \alpha_2 = \frac{(A^{(1,1)}/\omega_0)_X}{k(A^{(1,1)}/\omega_0)} , \alpha_3 = \frac{pk_X}{2k^2} . \quad (C.2)$$

$$\begin{aligned} \phi_{XX}^{(1,1)} = & -igk^2 \left(\frac{A^{(1,1)}}{\omega_0} \right) \{ (\alpha_1^2 + \alpha_2^2 + 2\alpha_1\alpha_3 + \frac{\alpha_2 X}{k}) e^{\cosh Q} \\ & + (2\alpha_1\alpha_2 + 4\alpha_1\alpha_3 + \frac{\alpha_1 X}{k}) \sinh Q + (4\alpha_1\alpha_3) Q \cosh Q \\ & + (8\alpha_3^2 + 4\alpha_2\alpha_3 + 2\frac{\alpha_3 X}{k}) Q \sinh Q + 4\alpha_3^2 Q^2 e^{\cosh Q} \} \end{aligned} \quad (C.3)$$

From (3.7) and (3.23) we write

$$\phi^{(2,1)} = \tilde{\phi}^{(2,1)} + i\phi^{(1,1)} \delta \quad (C.4)$$

where

$$\tilde{\phi}^{(2,1)} = -g \left(\frac{A^{(1,1)}}{\omega_0} \right) \{ \alpha_1 Q \cosh Q + \alpha_2 Q \sinh Q + \alpha_3 Q^2 \cosh Q \} \quad (C.5)$$

Differentiating (C.5),

$$\begin{aligned}
 \phi_X^{(2,1)} &= -g k \left(\frac{A^{(1,1)}}{\omega_0} \right) \left\{ (\alpha_1^2) \text{eosh } Q + (\alpha_1 \alpha_2) \sinh Q \right. \\
 &\quad + \left(2 \alpha_2 \alpha_3 + 4 \alpha_1 \alpha_3 + \frac{\alpha_1 X}{k} \right) Q \cosh Q \\
 &\quad + \left(\alpha_1^2 + \alpha_2^2 + 2 \alpha_2 \alpha_3 + \frac{\alpha_2 X}{k} \right) Q \sinh Q \\
 &\quad + \left(4 \alpha_3^2 + 3 \alpha_2 \alpha_3 + \frac{\alpha_3 X}{k} \right) Q^2 \cosh Q \\
 &\quad \left. + (3 \alpha_1 \alpha_3) Q^2 \sinh Q + 2 \alpha_3^2 Q^3 \sinh Q \right\}
 \end{aligned} \tag{C.6}$$

From (C.3), (C.5) and (C.6),

$$\begin{aligned}
 -\phi_{XX}^{(1,1)} - 2i k \phi_X^{(2,1)} - i k_X \phi^{(2,1)} &= i g k^2 \left(\frac{A^{(1,1)}}{\omega_0} \right) \left\{ \beta_1 \text{eosh } Q + \beta_2 \sinh Q \right. \\
 &\quad + \beta_3 Q \cosh Q + \beta_4 Q \sinh Q + \beta_5 Q^2 \text{eosh } Q + \beta_6 Q^2 \sinh Q - \\
 &\quad \left. + \beta_7 Q^3 \sinh Q \right\}
 \end{aligned} \tag{C.7}$$

where

$$\begin{aligned}
 \beta_1 &= \alpha_1^2 + \alpha_2^2 + 2 \alpha_3 \alpha_2 + \frac{\alpha_2 X}{k}, \\
 \beta_2 &= 4 \alpha_1 \alpha_2 + \alpha_1 \alpha_3 + \alpha_1 X / k, \\
 \beta_3 &= 4 \alpha_1 \alpha_2 + 14 \alpha_1 \alpha_3 + 2 \alpha_1 X / k, \\
 \beta_4 &= 2 \alpha_1^2 + 2 \alpha_2^2 + 8 \alpha_3^2 + 10 \alpha_2 \alpha_3 + 2 \frac{\alpha_2 X}{k} + 2 \frac{\alpha_3 X}{k} \\
 \beta_5 &= 14 \alpha_3^2 + 6 \alpha_2 \alpha_3 + 2 \alpha_3 X / k \\
 \beta_6 &= 6 \alpha_1 \alpha_3 \\
 \beta_7 &= 4 \alpha_3^2
 \end{aligned} \tag{C.8}$$

Differentiating α_1, α_2 and α_3 with respect to X , we get,

$$\begin{aligned}\alpha_{1X}/k &= h_{XX}/k = \alpha_4 \\ \alpha_{2X}/k &= (A^{(11)}/\omega_0)_{XX}/k^2(A^{(11)}/\omega_0) \\ &\quad - (A^{(11)}/\omega_0)_X \left\{ \frac{k_X}{k^3} \left(\frac{A^{(11)}}{\omega_0} \right) + \left(\frac{A^{(11)}}{\omega_0} \right)_X / k^2 \left(\frac{A^{(11)}}{\omega_0} \right)^2 \right\} \\ &= \alpha_6 - 2\alpha_2\alpha_3 - \alpha_2^2\end{aligned}\tag{C.9a}$$

where

$$\alpha_6 = \left(\frac{A^{(11)}}{\omega_0} \right)_{XX} / k \left(\frac{A^{(11)}}{\omega_0} \right)\tag{C.9b}$$

$$\alpha_{3X}/k = k_{XX}/2k^3 - (k_X)^2/k^4 = \alpha_5 - 4\alpha_3^2$$

where

$$\alpha_5 = \frac{k_{XX}}{2k^3}\tag{C.9c}$$

Now β_i defined in (C.8) may be further simplified to give,

$$\begin{aligned}\beta_1 &= 3\alpha_1^2 + \alpha_6, \\ \beta_2 &= 4\alpha_1\alpha_2 + 4\alpha_1\alpha_3 + \alpha_4, \\ \beta_3 &= 14\alpha_1\alpha_3 + 4\alpha_1\alpha_2 + 2\alpha_4, \\ \beta_4 &= 2\alpha_1^2 + 6\alpha_2\alpha_3 + 2\alpha_5 + 2\alpha_6 \\ \beta_5 &= 6\alpha_3^2 + 6\alpha_2\alpha_3 + 2\alpha_5 \\ \beta_6 &= 6\alpha_1\alpha_3 \\ \beta_7 &= 4\alpha_3^2\end{aligned}\tag{C.10}$$

Equation (3.19e) is now integrated as follows:

$$\begin{aligned}
\omega_2^X &= \frac{1}{a \cosh q} \left\{ h_X [-i \phi_X^{(1,1)} + \frac{1}{k} \tilde{\phi}^{(2,1)}]_{z=-h} + h_X [i \phi^{(1,1)} \vec{k} \delta]_{z=-h} \right. \\
&\quad + \frac{1}{k} \int_0^q [-i \phi_{XX}^{(1,1)} + 2k \tilde{\phi}_X^{(2,1)} + k_X \tilde{\phi}^{(2,1)}] \cosh Q dQ \\
&\quad \left. + \frac{1}{k} \int_0^q [(k i \phi^{(1,1)} \delta)_X + k (i \phi^{(1,1)} \delta)_X] \cosh Q dQ \right\} \\
&= \frac{1}{a \cosh q} \left\{ h_X [-i \phi_X^{(1,1)} + k \tilde{\phi}^{(2,1)}]_{z=-h} \right. \\
&\quad + \frac{1}{k} \int_0^q [-i \phi_{XX}^{(1,1)} + 2k \tilde{\phi}_X^{(2,1)} + k_X \tilde{\phi}^{(2,1)}] \cosh Q dQ \\
&\quad \left. + \frac{1}{k} \int_0^q 2ik \phi^{(1,1)} \delta_X \cosh Q dQ \right\} \\
&= -\alpha_1 \alpha_2 / \sinh q - \frac{1}{4} \left(\beta_1 - \frac{2\delta_X}{k} \right) (1 + 2q / \sinh 2q) \\
&\quad - \frac{1}{4} \beta_2 (\coth 2q - 1 / \sinh 2q) \\
&\quad + \frac{1}{8} \beta_3 (2q - \coth 2q + (1 + 2q^2) / \sinh 2q) \\
&\quad + \frac{1}{8} \beta_4 (2q \coth 2q - 1) \\
&\quad - \frac{1}{8} \beta_5 (2q^2 - 2q \coth 2q + 1 + 4q^3 / 3 \sinh 2q) \\
&\quad - \frac{1}{8} \beta_6 [(1 + 2q^2) \coth 2q - 2q - 1 / \sinh 2q] \\
&\quad - \frac{1}{16} \beta_7 [(4q^3 + 6q) \coth 2q - (6q^2 + 3)] \quad (C.11)
\end{aligned}$$

C.2 THREE-DIMENSIONAL WAVE TRAINS WITH SMALL FREQUENCY SPREAD

We let

$$\vec{k} = (\bar{k}, 0) + \varepsilon(\mu, \nu) \quad (\text{C.12})$$

where $\bar{k} = \text{constant}$. We further assume $\nabla^2 \bar{h} \sim O(\varepsilon)$ so that only $\nabla a \sim O(1)$.

From (3.6) we have

$$\alpha_1 \sim O(\varepsilon), \quad \alpha_2 \sim \frac{a_x}{ka}, \quad \alpha_3 \sim O(\varepsilon) \quad (\text{C.13})$$

and

$$\phi^{(2,1)} = -\frac{g a_x}{2\bar{\omega}\bar{k}} \left(\bar{Q} \frac{\sinh \bar{Q}}{\cosh \bar{q}} - \bar{q} \tanh \bar{q} \frac{\cosh \bar{Q}}{\cosh \bar{q}} \right) \quad (\text{C.14})$$

Differentiating (C.14) and (2.22a),

$$\nabla \phi^{(2,1)} = -\left(\frac{g \nabla a_x}{2\bar{\omega}\bar{k}} \right) \left(\bar{Q} \frac{\sinh \bar{Q}}{\cosh \bar{q}} - \bar{q} \tanh \bar{q} \frac{\cosh \bar{Q}}{\cosh \bar{q}} \right) \quad (\text{C.15})$$

$$\nabla^2 \phi^{(1,1)} = -i \left(\frac{g \nabla^2 a}{2\bar{\omega}} \right) \frac{\cosh \bar{Q}}{\cosh \bar{q}} \quad (\text{C.16})$$

Now, substituting (C.15) and (C.16) into (3.19e) and integrating:

$$\begin{aligned} \omega_2^X &= \frac{-g}{\bar{\omega} \cosh^2 \bar{q}} \int_{-h}^0 \left[\left(\frac{\nabla^2 a}{2a} \right) \cosh \bar{Q} + \left(\frac{a_{xx}}{a} \right) \left(\bar{Q} \sinh \bar{Q} - \bar{q} \tanh \bar{q} \cosh \bar{Q} \right) \right] \cosh \bar{Q} dz \\ &= -\bar{\omega} \left\{ \left(\frac{a_{xx}}{ka} \right) \frac{\bar{q} (1 - \bar{q} \tanh \bar{q})}{\sinh 2\bar{q}} + \frac{a_{xy}}{ka} \frac{(\sinh 2\bar{q} + 2\bar{q})}{4 \sinh 2\bar{q}} \right\} \end{aligned} \quad (\text{C.17})$$

Since

$$\frac{\partial \bar{c}_g}{\partial k} = \frac{\bar{\omega}}{2\bar{k}^2} \left\{ -\frac{1}{2} \left(1 + \frac{2\bar{q}}{\sinh 2\bar{q}} \right)^2 + \frac{4\bar{q} (1 - \bar{q} \tanh \bar{q})}{\sinh 2\bar{q}} \right\}$$

and $\bar{c}_g = \frac{1}{2} \frac{\bar{\omega}}{\bar{k}} \left(1 + \frac{2\bar{q}}{\sinh 2\bar{q}} \right)$, we have

$$\frac{\bar{q} (1 - \bar{q} \tanh \bar{q})}{\sinh 2\bar{q}} = \frac{\bar{k}^2}{2\bar{\omega}} \left(\frac{\partial \bar{c}_g}{\partial k} + \frac{\bar{c}_g^2}{\bar{\omega}} \right) \quad (\text{C.18})$$

Substituting (C.18) into (C.17), we obtain a simplified expression

for ω_2^X .i.e.,

$$\omega_2^X = \frac{1}{2} \bar{\omega} \left\{ \frac{a_{XX}}{a \bar{\omega}} \left(\frac{\partial \bar{c}_g}{\partial k} + \frac{\bar{c}_g^2}{\bar{\omega}} \right) + \frac{a_{YY}}{a \bar{\omega}} \left(\frac{\bar{c}_g}{k} \right) \right\} \quad (C.19)$$

For two-dimensional wave trains in deep water,

$$\frac{\partial}{\partial Y} = 0, \quad kh \rightarrow \infty, \quad \left(\frac{d \bar{c}_g}{dk} + \frac{\bar{c}_g^2}{\bar{\omega}} \right) \equiv 0$$

so that (C.19) reduces to

$$\omega_2^X \equiv 0 \quad (C.20)$$

APPENDIX D

THE ZERO OF THE CHARACTERISTIC DETERMINANT (4.5)

Referring to (4.5) and (4.6) we have,

$$\det. \tilde{A} = \begin{vmatrix} \frac{2}{\alpha}(-\Omega + C_g K_1) & \varepsilon \frac{\partial C_g}{\partial k} K_1 & \varepsilon \left(\frac{C_g}{k}\right) K_2 & 0 & 0 & 0 \\ \varepsilon^2 A_0 K_1 & (-\Omega + C_g K_1) & 0 & \varepsilon^2 \omega k D_0 K_1 & \varepsilon^2 k K_1 & 0 \\ \varepsilon^2 A_0 K_2 & C_g K_2 & -\Omega & \varepsilon^2 \omega k D_0 K_2 & \varepsilon^2 k K_2 & 0 \\ g \frac{ka}{\omega} K_1 & 0 & 0 & -\Omega & h K_1 & h K_2 \\ g D_0 a K_1 & 0 & 0 & g K_1 & -\Omega & 0 \\ g D_0 a K_2 & 0 & 0 & g K_2 & 0 & -\Omega \end{vmatrix} = 0 \quad (D.1)$$

Now assume

$$-\Omega/K = C_0 + \varepsilon C_1 + \dots \quad (D.2)$$

Substituting (D.2) into (D.1) and collecting zeroth order terms,

we obtain

$$\Omega^2 (\Omega - C_g K_1)^2 [-\Omega^2 + gh(K_1^2 + K_2^2)] = 0 \quad (D.3)$$

which give six roots:

$$\Omega = 0 \quad (\text{double roots})$$

$$\Omega / \sqrt{K_1^2 + K_2^2} = \pm \sqrt{gh} \quad (D.4a,b,c)$$

$$-\Omega = C_g \quad (\text{double roots})$$

The higher order corrections for the first four roots in (D.4) are always real. However the first order corrections for the last two roots can be complex as is found below:

$$\begin{vmatrix}
-\varepsilon \frac{2}{a} C_1 & \varepsilon \frac{\partial C_g}{\partial k} & \varepsilon \left(\frac{C_g}{k}\right) \tan \theta & 0 & 0 & 0 \\
\varepsilon A_0 & -\varepsilon C_1 & 0 & \varepsilon \omega_0 k D_0 & \varepsilon k & 0 \\
\varepsilon A_0 \tan \theta & C_g \tan \theta & -C_g & \varepsilon \omega_0 k D_0 \tan \theta & \varepsilon k \tan \theta & 0 \\
g \frac{ka}{\omega_0} & 0 & 0 & -C_g & h & h \tan \theta \\
g D_0 a & 0 & 0 & g & -C_g & 0 \\
g D_0 a \tan \theta & 0 & 0 & g \tan \theta & 0 & -C_g
\end{vmatrix} = 0 \quad (D.5)$$

Expanding (D.5),

$$\begin{aligned}
& C_g \left\{ \varepsilon \left(\frac{2}{a}\right) C_1 \varepsilon C_1 \left[-C_g (C_g^2 - gh \tan^2 \theta) + gh C_g \right] \right. \\
& + \varepsilon A_0 \varepsilon \left(\frac{\partial C_g}{\partial k}\right) \left[-C_g (C_g^2 - gh \tan^2 \theta) + gh C_g \right] \\
& + \varepsilon k \varepsilon \left(-\frac{\partial C_g}{\partial k}\right) \left[g D_0 a (C_g^2 - gh \tan^2 \theta) + g (gh D_0 a \tan^2 \theta + g \frac{ka}{\omega_0} C_g) \right] \\
& + \varepsilon \omega_0 D_0 \varepsilon \left(-\frac{\partial C_g}{\partial k}\right) \left[g D_0 a (C_g h) + C_g (gh D_0 a \tan^2 \theta + C_g g \frac{ka}{\omega_0}) \right] \\
& + C_g \tan \theta \left\{ \varepsilon A_0 \varepsilon \left(\frac{C_g}{k} \tan \theta\right) \left[-C_g (C_g^2 - gh \tan^2 \theta) + gh C_g \right] \right. \\
& + \varepsilon k \left(-\varepsilon \frac{C_g}{k} \tan \theta\right) \left[g D_0 a (C_g^2 - gh \tan^2 \theta) + g (gh D_0 a \tan^2 \theta + g \frac{ka}{\omega_0}) \right] \\
& \left. + \varepsilon \omega_0 D_0 \left(-\varepsilon \frac{C_g}{k} \tan \theta\right) \left[g D_0 a (C_g h) + C_g (gh D_0 a \tan^2 \theta + C_g g \frac{ka}{\omega_0}) \right] \right\} = O(\varepsilon^3)
\end{aligned} \quad (D.6)$$

After some manipulation, the first order corrections for $C = C_g + O(\varepsilon)$ is,

$$C_1^2 = \frac{\omega_0}{2} \left(\frac{\partial C_g}{\partial k} + \frac{C_g}{k} \tan^2 \theta \right) \left\{ \frac{a A_0}{\omega_0} + (ka)^2 \frac{\left[\left(\frac{g}{\omega_0}\right)^2 + 2 \left(\frac{g}{\omega_0}\right) D_0 C_g + gh D_0^2 \operatorname{sech}^2 \theta \right]}{[C_g^2 - gh \sec^2 \theta]} \right\} \quad (D.7)$$

or, in terms of \mathcal{X} and \mathcal{Y} defined in (4.9),

$$C_1 = \frac{1}{2} \frac{\omega_0}{k} \mathcal{Y}^{\frac{1}{2}} \left[\left(\frac{k_1}{k} \right) \mathcal{Y} + 2(ka)^2 \mathcal{X} \right]^{\frac{1}{2}} \quad (\text{D.8})$$

APPENDIX E

LISTING OF COMPUTER PROGRAM

The following computer program is developed for the numerical calculation of the finite difference equations (6.24 a,b) and (6.25). The program is written in 'Fortran IV'. The comment cards should make the program self-explanatory.


```

C ***** THIS PROGRAM USE AN EXPLICIT SCHEME TO
C      CALCULATE THE TRANSIENT WAVE ENVELOPE IN DEEP WATER *****
C ***** A(I,J), W(I,J) = AMPLITUDE AND FREQUENCY *****
C ***** N,M = NO. OF STEPS IN SPACE AND TIME *****
C ***** DX, DT = STEP SIZE IN SPACE AND TIME *****
C ***** L, LL = SPACE AND TIME INTERVALS FOR OUTPUT *****
      DIMENSION A(500,3), W(500,3)
      READ (5,112) N, M, L, LL, DX, DT
112  FORMAT (4I4, 2F10.7 )
      WRITE (6,333) N, M, DX, DT
333  FORMAT ( 29H NO. OF GRID PTS. IN SPACE = I4/
1      28H NO. OF GRID PTS. IN TIME = I4/
2      22H STEP SIZE IN SPACE = F10.7 /
3      21H STEP SIZE IN TIME = F10.7 / )
      J = 0
      JJ = -1
      NN = 3
      NHALF = (N+1)/2
      NNN = N-2
      E=0.0
C ***** INITIAL CONDITION *****
      READ (5,113) (A(I,1), I=NN,NNN)
113  FORMAT(5E15.8)
      READ (5,113) (W(I,1), I=NN,NNN)
      DO 77 I = NN,NNN
      E = F + A(I,1)**2
      77 CONTINUE
C ***** OUTPUT *****
250  JJ = JJ + 1
      IF (JJ) 261, 260, 260
260  CONTINUE
      WRITE (6,700) J
700  FORMAT ( /// 13 H TIME STEP = I4 )
      WRITE (6,601) (A(I,1), I=NN,NNN,L)
      WRITE (6,601) (W(I,1), I=NN,NNN,L)
601  FORMAT(5X,5E15.8)
      WRITE (6,602) E
602  FORMAT ( 13H SUM CHECK = E15.8 )
      JJ = -LL
261  CONTINUE
      E=0.0
      A(1,1) = A(N-4,1)
      A(2,1) = A(N-3,1)
      A(N-1,1) = A(4,1)
      A(N,1) = A(5,1)
      W(1,1) = W(N-4,1)
      W(2,1) = W(N-3,1)
      W(N-1,1) = W(4,1)
      W(N,1) = W(5,1)

```

```

C ***** FINITE DIFFERENCE EQUATIONS *****
  IF (J-1) 101,101,102
101 CONTINUE
  DO 259 I = NN, NNN
  AXXA =
  1((A(I,1)+A(I+2,1))*A(I-1,1)-(A(I,1)+A(I-2,1))*A(I+1,1))/(A(I+1,1)
  2*A(I-1,1))
  AXXA = AXXA/(4.*DX**2)
  W(I,2)=W(I,1)+DT*(W(I+1,1)**2-W(I-1,1)**2-A(I+1,1)**2+
  1A(I-1,1)**2-AXXA)/(8.*DX)
  A(I,2)=SQRT(A(I,1)**2+DT*(W(I+1,1)*A(I+1,1)**2
  1-W(I-1,1)*A(I-1,1)**2)/(4.*DX))
  E = F+A(I,1)**2
259 CONTINUE
  A(1,2) = A(N-4,2)
  A(2,2) = A(N-3,2)
  A(N-1,2) = A(4,2)
  A(N,2) = A(5,2)
  W(1,2) = W(N-4,2)
  W(2,2) = W(N-3,2)
  W(N-1,2) = W(4,2)
  W(N,2) = W(5,2)
  DO 2359 I = 1, N
  WW = W(I,2)
  W(I,2) = W(I,1)
  W(I,1) = WW
  AA = A(I,2)
  A(I,2) = A(I,1)
  A(I,1) = AA
2359 CONTINUE
  J = J + 1
  GO TO 250
102 CONTINUE
  DO 2459 I = 1, N
  WW = W(I,2)
  W(I,2) = W(I,1)
  W(I,1) = WW
  AA = A(I,2)
  A(I,2) = A(I,1)
  A(I,1) = AA
2459 CONTINUE
  DO 2259 I = NN, NNN
  AXXA =
  1((A(I,2)+A(I+2,2))*A(I-1,2)-(A(I,2)+A(I-2,2))*A(I+1,2))/(A(I+1,2)
  2*A(I-1,2))
  AXXA = AXXA/(4.*DX**2)
  W(I,1)=W(I,1)+DT*(W(I+1,2)**2-W(I-1,2)**2-A(I+1,2)**2+
  1A(I-1,2)**2-AXXA)/(4.*DX)
  A(I,1)=SQRT(A(I,1)**2+DT*(W(I+1,2)*A(I+1,2)**2
  1-W(I-1,2)*A(I-1,2)**2)/(2.*DX))
  E = E+A(I,1)**2

```

```
2259 CONTINUE  
    IF (J-M) 711,712,712  
711  J = J+1  
    GO TO 250  
712  CONTINUE  
    CALL EXIT  
    END
```

APPENDIX F

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