Mean, Variance, and Temporal Coherence of the 3D Acoustic Field Forward Propagated through Random Inhomogeneities in Continental-shelf and Deep Ocean Waveguides

by

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Abstract

When an acoustic field propagates through a multimodal waveguide, the effect of variations in medium properties induced by 3D random inhomogeneities accumulates by multiple forward scattering over range. This causes significant random fluctuations in the received field and greatly affects underwater acoustic sensing and communication systems, such as Ocean Acoustic Waveguide Remote Sensing (OAWRS). In order to characterize this effect, analytical expressions are derived for the mean, variance and temporal covariance of the acoustic field forward propagated through an ocean waveguide containing internal waves, fish shoals, wind-generated bubble clouds and krill. These expressions account for the accumulated effects of multiple forward scattering through temporally and spatially varying scatter function densities of the 3D inhomogeneities. In order to quantify the statistics of the scatter function densities, physical models and statistical descriptions of these inhomogeneities are developed.

Acoustic field transmission through internal waves in both continental shelf and deep ocean waveguides is investigated. Stratified ocean models are used to describe physical and statistical internal waves properties. Simulations for a typical continental-shelf environment show that when the standard deviation of the internal wave displacement exceeds the acoustic wavelength, the acoustic forward field becomes so randomized that the expected total intensity is dominated by the variance field and lacks the coherent interference structure beyond moderate ranges. This leads to an effectively saturated field that decays monotonically. It is found that 3D scattering effects become pronounced when the acoustic Fresnel width exceeds the cross-range coherence length of the internal waves. This leads to frequency and range-dependent power losses in the forward field that explains some of the attenuation observed in acoustic transmission through typical continental shelf and deep ocean waveguides.

A general analytical expression is derived for the temporal coherence of an acoustic signal propagating through an ocean waveguide with random 3D inhomogeneities.
Advance knowledge of this coherence time scale is often essential in the design of ocean acoustic experiments and subsequent data analysis. This is because it determines the number of fluctuations in a given measurement period and the time window within which the coherent processing techniques essential to ocean acoustic data reduction and analysis can be applied. The analytic approach is found to explain the time scale of acoustic field fluctuations observed both at mega meters ranges in the deep ocean, as well as at kilometer ranges in continental shelf environments. The acoustic time scale is found to be much shorter than the coherence time scale of ocean internal waves. This is shown to be a consequence of multiple forward scattering of the acoustic waves through the internal waves.

Analytical expressions are derived for the attenuation and dispersion of the acoustic field forward propagated through fish shoals and wind-generated bubble clouds in an ocean waveguide. It is found that at swim bladder resonance, fish shoals may sometimes lead to measurable attenuation in the forward field. The attenuation at off-resonant OAWRS frequencies, however, is typically negligible as shown both by the present theory and experimental data. The modeled attenuation due to random wind-generated bubble clouds is found to be highly sensitive to the choice of cutoff radius, which determines whether resonant bubbles are included in the bubble spectra. It is also found that bubble clouds generated under high wind speeds lead to additional dispersion and attention of the transmitted signal. These expected distortions can significantly degrade standard coherent processing techniques in ocean acoustics, such as the match filter, if not taken into account.

Antarctic krill play a key role in the marine food chain as the primary source of sustenance for many species of whales, seals, birds, squid and fish. This makes knowledge of the distribution and abundance of krill essential to ecological research in the southern oceans. It is shown that swarms of Antarctic krill with typical packing densities can be instantaneously imaged by OAWRS over thousands of square kilometers in both deep and shallow water environments given properly designed experiments.

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Chapter 1

Introduction

Acoustics often provides the only available method for sensing and communication in the ocean [1, 2, 3]. When an acoustic wave propagates through an random ocean waveguide, random oceanographic fluctuations lead not only to significant fluctuations of the received signal, but also degradation in its coherence. Accurately accounting for these effects is often essential in many sensing and communication applications in the ocean.

The main focus of this thesis is to determine general modal solutions for the mean and covariance of the acoustic field forward propagated through 3-D random inhomogeneities in a stratified ocean waveguide [4]. There are three types of random inhomogeneities discussed in this thesis: (1) large and weak inhomogeneities such as internal waves. These have larger horizontal dimensions than the acoustic wave length, but intrinsic properties, such as sound speed and density are only slightly different from the surrounding medium; (2) small and strong inhomogeneities such as fish with swim bladders and wind-generated bubbles. For our application, these are much smaller than the acoustic wave length, but have internal properties that are much different from those of the surround medium. A ray based method will be used to study backscattering from small and weak inhomogeneities such as Antarctic krill, squid, zooplankton and other small organisms.

Internal waves are wide spread phenomena throughout most of stratified oceans in the world. They have a temporal fluctuation scales ranging from 1 minute to 1
day and spatial variability between 10 meter and serval kilometers, which fall within the spatio-temporal windows of most underwater acoustic applications. When an acoustic field propagates through an ocean waveguide containing random internal waves, the random variations in sound speed and density can have pronounced accumulated effects on acoustic propagation and significantly degrade the coherence of received acoustic signals. In Chap 3, analytical expressions for the mean and variance of the forward acoustic field propagated through 3-D random internal waves in a continental shelf waveguide are formulated by applying a general modal solution derived by Ratilal and Makris [4]. The formulations analytically describe the accumulated effect of multiple scattering on acoustic forward propagation. These effects lead to both dispersion and attenuation of the acoustic forward field and redistribute modal energy between the mean field (coherent field) and variance field (incoherent field). It is found that, for a typical continental shelf waveguide, the acoustic forward field is dominated by the mean or coherent field and still maintains range- and depth-dependent structure when the root-mean-square(rms) internal wave displacement is small compared to the acoustic wave-length. However, as the rms internal wave displacement approaches the acoustic wave-length, the acoustic field becomes effectively incoherent, decays monotonically, and no longer exhibits range-dependent modal interference structure, which makes standard processing techniques that rely upon modal coherence, such as matched field processing and the waveguide-invariant method for source range and depth localization, far less effective. It is also found that 2-D models for the mean and variance of the acoustic field propagated through a 3-D random internal wave field become inaccurate when the Fresnel width approaches and exceeds the cross-range coherence length of the internal wave field.

In Chap 4, an analytical expression is derived for the temporal coherence of an acoustic field after multiple forward scattering through random 3-D inhomogeneities in an ocean waveguide. This expression makes it possible to predict the coherence time scale of field fluctuations in ocean-acoustic measurements from knowledge of the oceanography. It is used to explain the time scale of acoustic field fluctuations observed at mega meters ranges in various deep ocean acoustic transmission experi-
ments. This time scale is found to be nonlinearly related to the much longer coherence time scale of deep ocean internal waves through a multiple forward scattering process. It is also shown that 3-D scattering effects become pronounced when the acoustic Fresnel width exceeds the cross-range coherence length of the deep ocean internal waves, which leads to frequency and range-dependent power losses in the forward field that may help to explain historic long range measurements.

When an acoustic wave is forward propagated through shoals of fish and wind-generated bubble clouds, they cause the sound to be scattered and potentially absorbed. This leads to attenuation of the acoustic field in the forward direction. In Chap 5, we derive an analytical expression for attenuation of the forward field propagated through a waveguide with fish shoals. It is found that at swim bladder resonance, fish shoals may sometimes lead to measurable attenuation in the forward field. The attenuation at off-resonant frequencies, however, is typically negligible as shown both by our theory and experimental data. Analytical expressions for the statistics of the acoustic forward field propagated through random bubble clouds are also presented for a given wind speed. These conditional statistical moments are then used to calculate the mean and standard deviation of acoustic intensity in decibels, given wind speed statistics. In our simulations, attenuation of the acoustic field propagated though bubble clouds given wind speed is calculated and compared with Weston’s experimental data. The fluctuations of acoustic intensity forward propagated through a shallow water waveguide are also calculated. It is found that fluctuation of acoustic intensity may be significant even under mean wind speed conditions at high frequencies. All these results are derived by applying a general modal solution for the statistical moments of an acoustic field propagating through 3-D random inhomogeneities by Ratilal and Makris [4]. The mean and variance of the acoustic forward field are expressed in terms of the spatial distribution, volume density and statistical moments of the scatter function density of fish shoals and bubble clouds.

In Chap 6, we show how ocean acoustic waveguide remote sensing (OAWRS) may be used to instantaneously detect and image krill populations over thousands of square kilometers in the Antarctic.
Chapter 2

Mean and second moment of the acoustic field forward propagated through a random medium in free space

2.1 Introduction

In order to elucidate the fundamental physics of the effect of random inhomogeneities on the acoustic wave forward propagation, we derive analytic expressions for the mean and second moment of the forward field propagated from a point source to a distant receiver in free space containing random inhomogeneities. This is done by first developing difference equations that describe the change in the mean and second moment forward field at the receiver due to scattering from an elemental spherical shell of inhomogeneities. Then, by integrating differential equations recast from the difference equations, the change of mean and second moment of the forward field are analytically marched through all shells to include the multiple scattering effect of random inhomogeneities between the source and receiver.
2.2 Mean and second moment of forward field

The origin of the spherical coordinate system is placed at the source position, while the receiver coordinate is given by \( r(r, 0, 0) \) that is in the forward direction from the source, as shown in Fig. 2-1.

![Diagram](source.png)

**Figure 2-1:** The geometry of the source, receiver and elemental shell containing inhomogeneities in free space.

It is assumed that the inhomogeneities are only confined within a spherical shell of radius \( r_t \) centered at the source and there are no inhomogeneities outside this spherical shell in the medium. Let \( \Phi_i(r|r_0) \) be the direct wave measured at receiver \( r \) from the source at \( r_0 \). The thickness of the spherical shell containing inhomogeneities is now augmented by a small amount, \( \Delta r_t \). Let \( \Phi_s(r|r_0, \Delta r_t(r_t)) \) be the scattered field at the receiver from inhomogeneities within this shell. Let the forward field at the receiver due to the existence of the spherical shell be \( \Phi(r|r_0, \Delta r_t(r_t)) \), which can be expressed as

\[
\Phi(r|r_0, \Delta r_t(r_t)) = \Phi_i(r|r_0) + \Phi_s(r|r_0, \Delta r_t(r_t)).
\]  

(2.1)
The mean forward field can be obtained by taking the expected value of Eq. (2.1),

$$\langle \Phi(r|\mathbf{r}_0, \Delta r_t(r_t)) \rangle = \Phi_i(r|\mathbf{r}_0) + \langle \Phi_s(r|\mathbf{r}_0, \Delta r_t(r_t)) \rangle. \quad (2.2)$$

As shown in Sec. 1.3, based on the stationary phase approximation, the expected scattered field from the shell is written as

$$\langle \Phi_s(r|\mathbf{r}_0, \Delta r_t(r_t)) \rangle = \Phi_i(r|\mathbf{r}_0)j\nu(r_t)\Delta r_t, \quad (2.3)$$

where \(\nu(r_t)\) is the wave-number change that depends on the expected scattering properties of the inhomogeneities distributed within the shell and will be explained in the following contents.

From Eq. 2.2, the change of the forward field as a result of scattering from the inhomogeneities within the shell is

$$\Delta\langle \Phi(r|\mathbf{r}_0) \rangle = \Phi_i(r|\mathbf{r}_0)i\nu(r_t)\Delta r_t, \quad (2.4)$$

where \(\Delta\langle \Phi(r|\mathbf{r}_0) \rangle = \langle \Phi(r|\mathbf{r}_0, \Delta r_t(r_t)) \rangle - \Phi_i(r|\mathbf{r}_0)\).

Difference equation. 2.2 describes the change of mean forward field to be the product of the incident field and the change of wave-number \(\nu\) when there is only one inhomogeneous slab between the source and receiver. As the acoustic wave propagates through the second slab in adjacent to the first shell, the change of the mean forward field is proportional to \(\nu\) as well the forward field that also depends on \(\nu\) due to the scattering from the first slab. It is easily to prove that the change of the mean forward field as a result of scattering from \(M^{th}\) slab centered at \(r_{mt}\) is

$$\Delta\langle \Phi(r|\mathbf{r}_0) \rangle = \langle \Phi(r|\mathbf{r}_0) \rangle i\nu(r_{mt})\Delta r_t, \quad (2.5)$$

where \(\langle \Phi(r|\mathbf{r}_0) \rangle\) is the mean forward field in the absence of the \(M^{th}\) slab. Equation
(2.5) can be recast as the integral equation

\[
\begin{align*}
\int_{\Psi_i} \frac{d\langle \Phi(r|r_0) \rangle}{\langle \Phi(\alpha)(r|r_0) \rangle} = i \int_0^\rho \nu(r_t)dr_t
\end{align*}
\] (2.6)

that marches the change of mean forward field through the inhomogeneous medium since the inhomogeneities in adjacent single-scatter shells are assumed to be uncorrelated with each other. This includes multiple forward scattering from source to receiver in a manner analogous to that used by Rayleigh [5, 83] and others [7, 8] in free space.

Integrating Eq. 2.14, we have

\[
\langle \Psi_T(r|r_0) \rangle = \Psi_i(r|r_0)\exp(i\int_0^\rho \nu(r_t)dr_t).
\] (2.7)

For range-independent inhomogeneities, the mean forward field is

\[
\langle \Psi_T(r|r_0) \rangle = \Psi_i(r|r_0)\exp(i\nu r).
\] (2.8)

where the extra phase term \(\exp(i\nu r)\) account for the scattering effect of the random inhomogeneities on the acoustic wave forward propagation. The real part of wave-number change \(\nu\) is the dispersion coefficient that accounts for the dispersion effect from the inhomogeneities distributed between the source and the received. The imaginary part of \(\nu\) is the attenuation coefficient that accounts for the attenuation effect from the inhomogeneities. Eq. 2.8 can also be understood from the effective medium wave-number method where the change of medium wave-number is due to the existence of inhomogeneities.

From Eq. 2.1, the second moment of the forward field propagating through an elemental spherical shell is found to be

\[
\begin{align*}
\langle |\Phi(r|r_0, \Delta r_t(r_t))|^2 \rangle = |\Phi_i(r|r_0)|^2 + \langle |\Phi_s(r|r_0, \Delta r_t(r_t))|^2 \rangle \\
+ \Phi_i(r|r_0)\langle \Phi^*_s(r|r_0, \Delta r_t(r_t)) \rangle + \Phi^*_i(r|r_0)\langle \Phi_s(r|r_0, \Delta r_t(r_t)) \rangle.
\end{align*}
\] (2.9)
As shown in Sec. 1.3, the second moment of the scattered field from the shell can be expressed as

\[ \langle |\Phi_s(r|r_0, \Delta r_t(r_t))|^2 \rangle = |\Phi_t(r|r_0)|^2 \mu(r_t) \Delta r_t, \quad (2.10) \]

where \( \mu \) is the variance coefficient. From Eq. 4.3, the two cross term arising from the interaction between the incident field and the scattered field is found to be

\[ \Phi_t(r|r_0)\langle \Phi^*_s(r|r_0, \Delta r_t(r_t)) \rangle + \Phi^*_t(r|r_0)\langle \Phi_s(r|r_0, \Delta r_t(r_t)) \rangle = -|\Phi_t(r|r_0)|^2 \Im(\nu(r_t)) \Delta r_t. \quad (2.11) \]

The change of the second moment due to the shell is

\[ \Delta \langle |\Phi(r|r_0)|^2 \rangle = |\Phi_t(r|r_0)|^2 (\mu(r_t) - 2\Im(\nu(r_t))) \Delta r_t \quad (2.12) \]

where \( \Delta \langle |\Phi(r|r_0)|^2 \rangle = \langle |\Phi(r|r_0, \Delta r_t(r_t))|^2 \rangle - |\Phi_t(r|r_0)|^2. \)

Following the same procedure used to derive the mean forward field, the change of the second moment of the forward field as a result of scattering from \( M^{th} \) slab centered at \( r_m \) is

\[ \Delta \langle |\Phi(r|r_0)|^2 \rangle = \langle |\Phi(r|r_0)|^2 \rangle \left( \mu(r_t) - 2\Im(\nu(r_t)) \right) \Delta r_t. \quad (2.13) \]

Recasting the difference equation to differential equation and integrating through the inhomogeneities from the source to the receiver

\[ \int_{|\Phi_t|^2}^{(\Psi_T)^2} \frac{d\langle |\Phi(r|r_0)|^2 \rangle}{\langle |\Phi(r|r_0)|^2 \rangle} = \int_0^r \left( \mu(r_t) - 2\Im(\nu(r_t)) \right) dr_t. \quad (2.14) \]

The second moment of the forward field is

\[ \langle (\Psi_T(r|r_0))^2 \rangle = |\Psi_t(r|r_0)|^2 \exp \left( \int_0^r [\mu(r_t) - 2\Im(\nu(r_t))] dr_t \right). \quad (2.15) \]

For the range-independent inhomogeneities, we can further simplify the second mo-
ment of the forward field

\[ \langle |\psi(t|r_0)|^2 \rangle = |\psi_t(r|r_0)|^2 \exp \left( (\mu(r_{mt}) - 2\Re(\nu(r_{mt})) |r \right). \] (2.16)

Note that \( \mu_n \) quantifies how the energy is transformed from the mean field to the covariance field via the *multiple* scattering from the inhomogeneities. The modal attenuation coefficient \( \Im(\nu(\rho_s)) \) determines how the forward field gets attenuated due to the energy scattered out of the forward direction. In Sec. 2.3, we will discuss the derivation of mean and second moment of the scattered field from an elemental shell, which is a key step leading to the mean and second moment of the forward field.

### 2.3 Mean and second moment of the scattered field from the elemental shell

#### 2.3.1 mean scattered field

The mean scattered field from an elemental shell containing random inhomogeneities can be expressed as

\[ \langle \Phi_s(r|r_0, \Delta r_t(r_t)) \rangle = \int \int \langle \Phi_t(r_t|r_0) \rangle \frac{4\pi}{k} S(r_t) G(r|r_t) dV_t \] (2.17)

where \( \Phi_t(r_t|r_0) = \frac{e^{ik|r_0-r_t|}}{r_0-r_t} \) is the incident field on the inhomogeneities within the shell, \( G(r|r_t) = \frac{e^{ik|r-r_t|}}{4\pi|r-r_t|} \) is the Green’s function from the inhomogeneities to the receiver. \( S(r_t) \) is the scatter function density that is defined to be scatter function \( S \), divided by unit volume. For discrete inhomogeneities such as bubbles, \( S(r_t) = N_v(r_t) S_{r_t} \), where \( N_v(r_t) \) is the volume density of the discrete scatterers. Assuming that the incident field at the inhomogeneities and the scatter function density are independent \( \langle \Phi_t(r_t|r_0) S(r_t) \rangle = \langle \Phi_t(r_t|r_0) \rangle \langle S(r_t) \rangle \), the mean scattered field can be expressed as

\[ \langle \Phi_s(r|r_0, \Delta r_t(r_t)) \rangle = \frac{e^{ikr}}{r} \int_{r_t+\Delta r_t}^{r_t+\Delta r_t} \int_0^{2\pi} \int_0^{\pi} e^{ikr_t|\sin \phi_t \cos \theta_t|} \frac{1}{kr_t} (S(r_t)) r_t^2 \sin \phi_t dr_t d\phi_t d\theta_t \] (2.18)
where $\phi_t$ is the elevation angle and $\theta_t$ is the azimuth angle, as shown in Fig. 2-2.

By applying the stationary phase approximation over elevation and azimuth angle, Eq. 2.18 is integrated to be

$$
\langle \Phi_s(r|r_0, \Delta r_t(r_t)) \rangle \approx \frac{e^{ikr}}{r} j \frac{2\pi}{k^2} \Delta r_t \left( \langle S(\theta_t = 0, \phi_t = \frac{\pi}{2}) \rangle - \exp(j2kr_t) \langle S(\theta_t = \pi, \phi_t = \frac{\pi}{2}) \rangle \right).
$$

(2.19)

There are two stationary points in azimuth direction where $\theta_t = 0, \pi$ and one stationary point in the elevation direction where $\phi_t = \frac{\pi}{2}$. The mean scattered field in Eq. 2.19 is the sum of the contributions from two Fresnel regions shown in Fig. 2-2: (1) the forward direction expressed in the first term, and (2) the backward direction expressed in the second term. The scattered arrivals from the inhomogeneities contained in the Fresnel region are coherent or in-phase at the receiver and then make main contribution to the scattered field, while the scattered arrivals from the inhomogeneities outside the Fresnel region are incoherent or its phase term varying so
much at the receiver that they cancel each other and then make little contribution to the mean scattered field. Compared to the forward scattered field, the backward scattered field need to travel extra path $2r_t$ to arrive at the receiver as expressed in the additional phase term $\exp(j2kr_t)$. This phase term oscillates at various shell positions and makes the total contribution of the backward scattering much smaller than the one from the forward scattering. We can approximately express the mean field as

$$\langle \Phi_s(r|r_0, \Delta r_t(r_t)) \rangle \approx \frac{e^{jkr}}{r} \frac{2\pi}{k^2} \Delta r_t \langle S(\theta_t = 0, \phi_t = \frac{\pi}{2}) \rangle$$

$$= \phi_t(r|r_0) j\nu \Delta r_t$$  \hspace{1cm} (2.20)

where $\nu = \frac{2\pi}{k^2} \langle S(0, \frac{\pi}{2}) \rangle = \frac{2\pi}{k^2} \langle N_v S(0, \frac{\pi}{2}) \rangle$.

The expression $\nu = \frac{2\pi}{k^2} \langle N_v S(0, \frac{\pi}{2}) \rangle$ was used by Lord Rayleigh in 1899 to explain why sun set is red and sky is blue. When light travels through atmosphere, it gets scattered by inhomogeneous particle in the atmosphere. Since the scatter function of blue light (higher frequency) is larger than the one of red light in the forward direction, the blue light gets more attenuation than the red light so that the sunset is red as our eyes are in the forward direction when looking at sunset. Most of the energy in blue light is scattered out of the forward direction and goes to other direction. This leads to the blue sky.

### 2.3.2 second moment of the scattered field

The second moment of the scattered field is

$$\langle |\Phi_s(r|r_0, \Delta r_t(r_t))|^2 \rangle$$

$$= \int \int \int \int \int \int \int \int \int \int \int \langle \Phi_t(r_t|r_0) \Phi_t^*(r_t|r_0) \frac{4\pi}{k^2} S(r_t) S^*(r_t) \rangle G(r|r_t) G(r|r_t) dV_t dV_t'$$

$$= \int \int \int \int \int \int \int \int \int \int \int \int \int \int \Phi_t(r_t|r_0) \Phi_t^*(r_t|r_0) \frac{4\pi}{k^2} S(r_t) S^*(r_t) \rangle G(r|r_t) G(r|r_t) dV_t dV_t'$$  \hspace{1cm} (2.21)
by applying the same assumption that the incident field and the scatter function
density is independent.

In order to compute the second moment of the scattered field, we need to first
quantify the second moment of scatter function density. From Ref. [4], the Fresnel
angle in forward and backward direction is defined to be \( \phi_F(r, r_t) = \sqrt{\frac{2\pi(r-r_t)}{krr_t}} \) and
\( \phi_B(r, r_t) = \sqrt{\frac{2\pi(r+r_t)}{krr_t}} \).

**Fully correlated inhomogeneities within the Fresnel angle**

When the radius of shell is small, the Fresnel angle is also small so that the inhomoge-
neities are assumed to be fully correlated over azimuth and elevation angle within
the Fresnel angle. This case is illustrated in Fig. 1(a) of Ref. [4]. Under this scenario,
the second moment of the scatter function can be assumed to be

\[
\langle S(r_t, \theta_t, \phi_t)S^*(r_t', \theta_t', \phi_t') \rangle \approx \langle S(r_t, \theta_t, \phi_t)S^*(r_t', \theta_t, \phi_t') \rangle. \tag{2.22}
\]

In radial direction, we assume that the second moment of scatter function density is
fully correlated when the separation \( r_t - r_t' \) is larger than the radial correlation length
\( \ell_{rt} \) defined in Appendix A of Ref. [4] and uncorrelated when the separation exceeds
the correlation length \( \ell_{rt} \). When the thickness of the shell is much larger than the
radial correlation length, we can delta-correlated the second moment of the scatter
function density

\[
\langle S(r_t, \theta_t, \phi_t)S(r_t', \theta_t, \phi_t) \rangle \\
\approx \langle |S(r_t, \theta_t, \phi_t)|^2 \rangle u(\ell_{rt} - |r_t - r_t'|) + \langle S(r_t, \theta_t, \phi_t)\rangle \langle S^*(r_t', \theta_t, \phi_t) \rangle u(|r_t - r_t'| - \ell_{rt}) \\
\approx (\langle |S(r_t, \theta_t, \phi_t)|^2 \rangle - |\langle S(r_t, \theta_t, \phi_t) \rangle|^2) \ell_{rt} \delta(r_t - r_t') + \langle S(r_t, \theta_t, \phi_t)\rangle \langle S^*(r_t', \theta_t, \phi_t) \rangle. \tag{2.23}
\]

where \( u \) is the step function.
Inserting the second moment of scatter function density into Eq. 2.21

\[
\langle |\Phi_s(r|0, \Delta r_t(r_t))|^2 \rangle = \frac{1}{r^2} \int \int \int \int \int \frac{e^{ikr^2[1 - \sin \phi \cos \theta - kr'_t[1 - \sin \phi \cos \theta']]}}{k^2 r_t r'_t} r^2 r'_t \sin \phi \sin \phi' \\
\operatorname{Var}(S(r_t, \theta_t, \phi_t)) = \ell_{rt} \delta(r_t - r'_t) dr_t dr'_t d\phi_t d\phi'_t d\theta_t d\theta'_t + \langle |\Phi_s(r|0, \Delta r_t(r_t))|^2 \rangle^2 \\
\quad = \frac{1}{r^2} (\frac{2\pi}{k^2})^2 \ell_{rt} \operatorname{Var}(S(\theta_t = 0, \phi_t = \frac{\pi}{2})) \Delta r_t + \langle |\Phi_s(r|0, \Delta r_t(r_t))|^2 \rangle^2 \\
= \Phi_t^2(r|0) \mu \Delta r_t + \langle |\Phi_s(r|0, \Delta r_t(r_t))|^2 \rangle^2,
\]

(2.24)

where \(\mu = (\frac{2\pi}{k^2})^2 \ell_{rt} \operatorname{Var}(S(0, \frac{\pi}{2}))\) is the variance coefficient.

As explained in Ref. [4], one of fundamental assumptions needed to apply the slab method is the scattered field from an elemental shell must be small in comparison to the incident field. Based on this assumption, the square of the mean scattered field from the shell \(|\Phi_s(r|0, \Delta r_t(r_t))|^2\) in Eq. 2.25, must be negligible in the second moment. The first term at the right hand of Eq. 2.25, the variance of the scattered field, is a statistical quantity that need not depend on the mean and not necessarily negligible. Eq. 2.25 turns to be

\[
\langle |\Phi_s(r|0, \Delta r_t(r_t))|^2 \rangle \approx \Phi_t^2(r|0) \mu \Delta r_t,
\]

(2.25)

**Uncorrelated inhomogeneities within the Fresnel angle**

When the radius of shell is large, the Fresnel angle also becomes large so that the inhomogeneities could be uncorrelated within the Fresnel angle over azimuth and elevation direction as illustrated in Fig. 1(b) of Ref. [4]. Here, we assume that the second moment of the scatter function density is fully uncorrelated when the angular separations \(\phi_t - \phi'_t\) and \(\theta_t - \theta'_t\) are larger than the correlation angle \(\alpha_c\) and fully correlated when the angular separations are smaller than \(\alpha_c\). The second moment of
the scatter function can be expressed as

\[
\langle S(r_t, \theta_t, \phi_t)S(r'_t, \theta_t, \phi_t) \rangle \\
\approx \langle |S(r_t, \theta_t, \phi_t)|^2 u(\ell_{r_t} - |r_t - r'_t|)u(\alpha_c - |\phi_t - \phi'_t|)u(\alpha_c - |\theta_c - \theta'_c|) \\
+ \langle S(r_t, \theta_t, \phi_t)S^*(r'_t, \theta'_t, \phi'_t) \rangle u(|r_t - r'_t| - \ell_{r_t})u(|\phi_t - \phi'_t| - \alpha_c)u(|\theta_c - \theta'_c| - \alpha_c) \\
\approx \langle (|S(r_t, \theta_t, \phi_t)|^2 - |S(r_t, \theta_t, \phi_t)|^2 \ell_{r_t} \alpha_c^2 \delta(r_t - r'_t) \delta(\phi_t - \phi'_t) \delta(\theta_t - \theta'_t) \\
+ \langle S(r_t, \theta_t, \phi_t)S^*(r'_t, \theta'_t, \phi'_t) \rangle \rangle (2.26)
\]

Inserting Eq. 2.26 into Eq. 2.21

\[
\langle |\Phi_s(r|r_0, \Delta r_t(r_t))|^2 \rangle \\
= \frac{1}{r^2} \int \int \int \int \int \int \frac{e^{i k r_t [1 - \sin \phi_t \cos \theta_t] - \int_{r_t}^{r'_t} \frac{2 \ell_{r_t}}{r'_t} \sin \phi_t \sin \phi'_t}}{r_t^2 r'_t} Var(S(r_t, \theta_t, \phi_t)) \ell_{r_t} \alpha_c^2 \delta(r_t - r'_t) \delta(\phi_t - \phi'_t) \delta(\theta_t - \theta'_t) \, dr_t \, dr'_t \, d\phi_t \, d\phi'_t \, d\theta_t \, d\theta'_t \\
+ \langle |\Phi_s(r|r_0, \Delta r_t(r_t))|^2 \rangle \\
= \frac{1}{(k^2)^2} \int \int \sin^2(\phi_t) Var(S(r_t, \theta_t, \phi_t)) \ell_{r_t} \alpha_c^2 dr_t d\phi_t d\theta_t + \langle |\Phi_s(r|r_0, \Delta r_t(r_t))|^2 \rangle \\
(2.27)
\]

As mentioned in Sec. 2.3.1, the main contribution to the scattered field is from the Fresnel region

\[
|\Phi_s(r|r_0, \Delta r_t(r_t))|^2 \\
= \frac{1}{(k r)^2} \int_{r_t-\Delta r_t/2}^{r_t+\Delta r_t/2} \int_{\phi_F/2}^{\pi+\phi_F/2} \sin^2(\phi_t) \int_{-\phi_F/2}^{\phi_F/2} Var(S(r_t, \theta_t, \phi_t)) \ell_{r_t} \alpha_c^2 \, dr_t \, d\phi_t \, d\theta_t \\
+ \langle |\Phi_s(r|r_0, \Delta r_t(r_t))|^2 \rangle \\
\approx \frac{1}{(k r)^2} \ell_{r_t} \alpha_c^2 \phi_F^2 Var(S(r_t, \theta_t = 0, \phi_t = \frac{\pi}{2}) \Delta r_t + \langle |\Phi_s(r|r_0, \Delta r_t(r_t))|^2 \rangle \\
= \frac{1}{(k r)^2} \frac{2 \pi (r - r_t)}{k \tau} Var(S(0, \frac{\pi}{2}) \Delta r_t + \langle |\Phi_s(r|r_0, \Delta r_t(r_t))|^2 \rangle \\
= \Phi_t^2(r|r_0) \mu \Delta r_t + \langle |\Phi_s(r|r_0, \Delta r_t(r_t))|^2 \rangle \\
\approx \Phi_t^2(r|r_0) \mu \Delta r_t, \quad (2.28)
\]
where $\mu(r_t) = V_c \frac{2\pi(r-r_0)}{k^3 r r_t} \text{Var}(S(0, \frac{x}{2}))$ and $V_c = \ell_r \ell_\theta \ell_\phi = \ell_r (r \alpha_c)^2$ is the coherent volume of the inhomogeneities within the Fresnel region. Noting that the variance coefficient depends on the locations of the receiver and the shell because of the range-dependent Fresnel angle $\phi_F$ in the forward direction.
Chapter 3

Mean and variance of the forward field propagated through three-dimensional random internal waves in a continental-shelf waveguide

3.1 Introduction

When an acoustic field propagates through a multimodal waveguide, random variations in medium properties can have a cumulative effect over range. This can drastically alter the delicate modal interference structure of the incident field, leading to significant randomization in the received field. Here, we model the mean, variance, and total intensity of the forward field propagated through an ocean waveguide containing temporally and spatially random 3-D internal waves using a general modal formulation described in Ref. [4]. This formulation is convenient because it takes into account the accumulated effects of multiple forward scattering on the mean and covariance of the forward propagated field. These include coherent, partially coherent and
incoherent interactions with the incident field, that lead to attenuation, dispersion, and exponential coefficients of field variance that describe mode coupling induced by the medium's inhomogeneities. An advantage of the formulation is that the first and second moments of the forward field can be analytically expressed in terms of the first and second moments of the inhomogeneous medium's spatially varying scatter function density. These inhomogeneities can be arbitrarily large relative to the acoustic wavelength and have arbitrary compressibility and density contrast from the surrounding medium.

Inhomogeneities arising from internal wave disturbances typically have relatively small differences in density and compressibility from the surrounding medium. A convenient approach for modelling their scattering properties is to apply the first-order Rayleigh-Born approximation to Green's theorem[9]. Internal wave scattering properties are then expressed in terms of the statistical variations in compressibility and density caused by the disturbance. This requires knowledge of the probability distributions of the compressibility and density variations, which can be expressed in terms of internal wave wavenumber spectra. The first-order Rayleigh-Born approximation leads to a purely real scatter function that can directly account for scattering-induced dispersion in the mean forward field but not attenuation, which requires the imaginary part. Attenuation in the mean forward field due to scattering is then determined from the waveguide extinction theorem[10, 11]. The waveguide extinction theorem for any given mode relates power loss in the forward azimuth to the total scattered power in all directions, which can be estimated with high accuracy using the first order Rayleigh-Born approximation.

Following the trend of ever increasing ocean utilization has come a greater interest in developing models to help understand and accurately predict the effect of internal wave fields on underwater acoustic transmission through continental shelf environments. Significant fluctuations have been observed [12, 13, 14] and predicted [15, 16] in signal transmission. These fluctuations lead to signal-dependent noise [15] that can significantly degrade sonar system performance. Since internal waves in continental shelf environments often have large displacements compared to the acoustic
wavelength and large slopes, standard perturbation theory methods for modeling the
effect of rough surface scattering [17, 18, 19, 20] on acoustic transmission through an
ocean waveguide may often be unsuitable. If the accumulated effects of multiple for-
ward scattering on dispersion and field variance are important in acoustic propagation
through extended internal wave fields, approaches that neglect them [17, 18, 19, 20]
may be inappropriate as noted in Ref. [21]. It is possible that acoustic transmission
through such complicated environments may be seriously altered by 3D scattering
effects beyond relatively short ranges. Two-dimensional models [110, 22, 20, 24, 13],
2D Monte-Carlo simulations [25] and adiabatic 3D models [26] may then also become
unreliable.

The present formulation is advantageous because the mean and variance of the
acoustic field multiply forward scattered through a 3D random waveguide can be
rapidly obtained from the compact analytic expressions of Ref. [4], given the mean
and spatial covariance of the internal wave displacement field, without restriction
on internal wave amplitude or slope. It can be readily applied to solve a variety of
underwater remote sensing and communication problems in continental shelf and deep
ocean environments. This includes the detection and localization of sources [27, 28]
and targets by passive and active sonar, as well as the estimation of biological [29, 30],
geological [31] and oceanographic parameters [32] by seismo-acoustic inverse methods.

We show that the accumulated effect of multiple forward scattering through ran-
dom internal wave fields typically must be included to properly model the statistical
moments of a forward propagated acoustic field in continental shelf environments. We
also show that 3-D multiple scattering effects can become important in both the mean
and variance of the forward field. This is because, as source-receiver range increases,
the Fresnel width of the forward field eventually exceeds the cross-range coherence
length of the internal waves, making out of plane scattering important when internal
wave amplitudes exceed the acoustic wavelength and slopes become higher. Out-of-
plane scattering cannot be accounted for in 2-D models. We illustrate this effect
by comparing the present 3-D model with a current standard approach, which is to
compute field moments by Monte-Carlo simulations with the 2-D parabolic equation.
We show that the acoustic field moments are highly dependent on both the rms displacements and coherence scales of the 3-D internal waves. In a waveguide where the rms internal wave height is small compared to the acoustic wavelength, the forward field remains coherent and exhibits the range and depth-dependent structure expected from the coherent interference between waveguide modes. The moderate dispersion and attenuation induced by multiple forward scatter through the internal wave disturbances still noticeably alters the mean field. Scattering in such a slightly random waveguide may not be strong enough to make 3-D effects noticeable.

When the rms internal wave height becomes larger than the acoustic wavelength, 3-D scattering effects become significant. The field variance or incoherent intensity is found to dominate the total intensity of the forward field beyond moderate propagation ranges. This causes the acoustic field to become fully saturated. In this case, the coherent modal interference structure in range and depth is lost, and the intensity of the forward field then decays monotonically. This makes standard processing techniques that rely upon model coherence, such as matched field processing and the waveguide invariant method [33] for source range and depth localization, far less effective.

The effects on acoustic transmission of random density fluctuations in the medium due to internal waves are also quantified. We show that internal-wave induced density effects can significantly affect acoustic propagation in specific environments, such as Arctic seas.

### 3.2 Formulation in a two-layer water column

Internal waves in mid-latitude continental shelf environments often occur at the interface between warm water near the sea surface and cooler water below[34, 14, 35]. In high latitude, the reverse is usually true where the cooler less dense water above comes from recently melted ice. Here, we model the internal wave field as disturbances propagating along the boundary between strata in a two-layer water-column, as illustrated in Fig. 3-1 Although many other internal wave models and parametrizations
Figure 3-1: Geometry of mid-latitude Atlantic continental shelf and Arctic environments with two-layer water column of total depth $H_w = 100$ m, and upper layer depth of $D = 30$ m. The bottom sediment half space is composed of sand. The internal wave disturbances have coherence length scales $\ell_x$ and $\ell_y$ in the $x$ and $y$ directions respectively and are measured with positive height $h$ measured downward from the interface between the upper and lower water layers.

3D Random Internal Wave Field in an Ocean Waveguide

![Diagram of 3D Random Internal Wave Field in an Ocean Waveguide]

could have been used to implement the general formulation of Ref. [4], the two-layer model is chosen here because it clearly illustrates the fundamental physics of internal waves in a continental shelf environment. This has made it probably the most frequently used model in the literature[36, 37, 38]. We focus on the baroclinic mode of the internal wave which has negligible displacement at the sea surface[37]. In the absence of internal waves, the boundary separating the upper medium with density $d_1$ and sound speed $c_1$ from the lower medium with density $d_2$ and sound speed $c_2$ is at a constant depth. In the presence of internal waves, a part of the lower medium protrudes into the upper medium and a part of the upper medium protrudes into the lower medium. We model protrusion of the lower medium into the upper medium as a volumetric inhomogeneity that scatters the sound field by Green's theorem, and vice versa for the lower medium.

To formulate the problem, we place the origin of the coordinate system at the sea surface. The $z$-axis points downward and normal to the interface between horizontal
strata. The water depth is $H$ and the boundary separating the upper and lower medium is at depth $z = D$. Let coordinates of the source be defined by $r_0 = (0, 0, z_0)$, and receiver coordinates by $r = (x, 0, z)$. Spatial cylindrical $(\rho, \phi, z)$ and spherical systems $(r, \theta, \phi)$ are defined by $x = r \sin \theta \cos \phi$, $y = r \sin \theta \sin \phi$, $z = r \cos \theta$ and $\rho = x^2 + y^2$. The horizontal and vertical wavenumber components for the $n$th mode are respectively $\xi_n = k \sin \alpha_n$ and $\gamma_n = k \cos \alpha_n$, where $\alpha_n$ is the elevation angle of the mode measured from the $z$-axis. Here, $0 \leq \alpha_n \leq \pi/2$ so that the down and upgoing plane wave components of each mode will then have elevation angles $\alpha_n$ and $\pi - \alpha_n$ respectively. The corresponding vertical wavenumber of the down and upgoing components of the $n$th mode are $\gamma_n$ and $-\gamma_n$ respectively, where $\Re\{\gamma_n\} \geq 0$. The wavenumber magnitude $k$ equals the angular frequency $\omega$ divided by the sound speed $c$ in the object layer so that $k^2 = \xi_n^2 + \gamma_n^2$. The azimuth angle of the modal plane wave is denoted by $\beta$, where $0 \leq \beta \leq 2\pi$. The geometry of spatial and wavenumber coordinates is shown in Ref. [39].

### 3.2.1 Statistical description of internal waves

**Joint spatial probability density of internal wave displacement**

The displacement $h(\rho_i)$ of the internal wave boundary at horizontal location $\rho_i$, as illustrated in Fig. 1, can be modeled as a Gaussian random process in space and time with mean $\langle h \rangle$, and variance $\eta_h^2 = \langle h^2 \rangle - \langle h \rangle^2$. Since the baroclinic internal wave displacement cannot penetrate the water surface or the sea bottom, we limit unphysical tails of the probability density function by windowing $h$. The probability density function of the internal wave displacement is,  

$$p_h(h) = \begin{cases} \frac{1}{\sqrt{2\pi\eta_h}} e^{-\frac{(h-\langle h \rangle)^2}{2\eta_h^2}} & \text{for } \langle h \rangle - h_1 \leq h \leq \langle h \rangle + h_2 \\ 0 & \text{elsewhere} \end{cases}$$

(3.1)

where $T = \frac{1}{P\left(\eta_h + 2h_1\right) - P\left(\eta_h - 2h_1\right)}$ is a normalization constant, and $P(b)$ is the cumulative distribution function,
Linear internal wave displacements are expected to follow a zero-mean circular complex Gaussian random process by central limit theorem given that they arise from the superposition of many statistically independent sources in space and time. The same model can also be used to describe nonlinear solitary internal waves in some cases when they are incompletely evolved[40] or broadly distributed in peak amplitude[41]. For nonlinear internal wave fields, non-zero mean displacements and much larger standard deviations are expected.

The joint probability density function of the internal wave displacement at horizontal locations \( \rho_t \) and \( \rho'_t \) can be expressed as,

\[
p(h(\rho_t), h(\rho'_t)) = \frac{1}{2\pi \eta h(\rho_t) \eta h(\rho'_t)} \left(1 - \rho^2\right)^{1/2} \exp\left(-\frac{[(h(\rho_t) h(\rho'_t) - 2h(\rho_t) h(\rho'_t) \eta h(\rho_t) \eta h(\rho'_t)]^2}{2 \eta^2 h(\rho_t) \eta^2 h(\rho'_t)} \left(1 - \rho^2\right)\right)
\]

where \( \rho \) is the correlation coefficient defined as,

\[
\rho = \frac{\langle h(\rho_t) h(\rho'_t) \rangle - \langle h(\rho_t) \rangle \langle h(\rho'_t) \rangle}{\sqrt{\left(\langle |h(\rho_t)|^2 \rangle - |\langle h(\rho_t) \rangle|^2\right) \left(\langle |h(\rho'_t)|^2 \rangle - |\langle h(\rho'_t) \rangle|^2\right)}}.
\]

Linear internal wave field as a stationary random process

For random internal wave fields that follow a stationary random process in space, the internal wave displacement correlation function and standard deviation \( \eta_h \) can be expressed in terms of the internal wave spectrum \( G(\kappa) \). For instance, the correlation function is

\[
\langle h(\rho_t) h(\rho'_t) \rangle = C_h(\rho_t - \rho'_t) = \frac{1}{(2\pi)^2} \int_0^{2\pi} \int_0^\infty G(\kappa) e^{i\kappa (\rho_t - \rho'_t)} d\kappa d\Theta,
\]

\[ (3.5) \]
where, \( \kappa = (\kappa_x, \kappa_y) = (\kappa \cos \Theta, \kappa \sin \Theta) \) is the internal wave wavenumber vector with magnitude \( \kappa \) and azimuthal direction \( \Theta \). The internal wave height standard deviation \( \eta_h \) is defined by,

\[
\eta_h^2 = C_{hh}(0), \tag{3.6}
\]

when \( \langle h \rangle = 0 \) as is the case for linear internal waves.

The internal wave disturbance has a horizontal coherence area given by

\[
A_c = \frac{(1/2\pi)^2 \int_0^{2\pi} \int_0^\infty |\mathcal{G}(\kappa)|^2 \kappa d\kappa d\Theta}{|(1/2\pi)^2 \int_0^{2\pi} \int_0^\infty \mathcal{G}(\kappa) \kappa d\kappa d\Theta|^2}_{\eta_h^4}, \tag{3.7}
\]

outside of which internal wave displacements can be assumed to be uncorrelated.

The corresponding coherence length scale of the internal wave disturbance \( \ell_c(\Theta) \) in any azimuthal direction \( \Theta \) is then given by,

\[
\ell_c^2(\Theta) = \frac{2(1/2\pi)^2 \int_0^\infty |\mathcal{G}(\kappa)|^2 \kappa d\kappa}{|(1/2\pi)^2 \int_0^{2\pi} \int_0^\infty \mathcal{G}(\kappa) \kappa d\kappa d\Theta|^2}_{\eta_h^4}. \tag{3.8}
\]

Here \( \ell_c(\Theta) \) defines a coherence shape function for the internal wave disturbance that is determined as its wavenumber spectrum spans the \( 2\pi \) azimuthal radians of \( \Theta \).

The full coherence lengths \( \ell_x \) and \( \ell_y \) of the internal wave disturbance in the \( x \) and \( y \) directions are then respectively,

\[
\ell_x = \ell_c(\Theta = 0) + \ell_c(\Theta = \pi) \tag{3.9}
\]

and

\[
\ell_y = \ell_c(\Theta = \pi/2) + \ell_c(\Theta = 3\pi/2). \tag{3.10}
\]

For an isotropic internal wave field, Eq. (3.5) reduces to
\[
\langle h(\rho_i)h(\rho'_i) \rangle = \frac{1}{2\pi} \int_0^\infty \mathcal{G}(\kappa)J_0(\kappa|\rho_i - \rho'_i|)\kappa d\kappa,
\]
(3.11)
since its wavenumber spectrum \(\mathcal{G}(\kappa)\) is independent of the azimuth angle \(\Theta\).

### 3.2.2 Scatter function of an internal wave inhomogeneity

To determine the plane wave scatter function of a coherence volume of internal wave inhomogeneity, we apply Green’s theorem[9],

\[
\Phi_s(r|r_0) = \iiint \left[ k^2 \Gamma_\kappa(r_t)\Phi(r_t|r_0)G(r|r_t) + \Gamma_d(r_t)\nabla \Phi(r_t|r_0) \cdot \nabla G(r|r_t) \right] dV_t,
\]
(3.12)

where \(\Gamma_\kappa\) is the fractional change in compressibility and \(\Gamma_d\) is the fractional change in density of the inhomogeneity centered at \(r_t\) relative to the original medium, \(G(r|r_t)\) is the free space Green function, and \(\Phi(r_t|r_0)\) is the total acoustic field in the volume of inhomogeneity. To integrate Eq. (3.12) analytically, we need to know the total field within the volume of the inhomogeneity. But this is the sum of the known incident and unknown scattered field in the volume. In the first order Rayleigh-Born approximation, the total field inside the inhomogeneity is approximated by the incident field. This is a good approximation when the scattered field within the inhomogeneity is small compared to the incident field as it typically is when the fractional compressibility and density changes are small, as they are in the present scenario. Green’s theorem evaluated using the first order Rayleigh-Born approximation then provides a first-order estimate of the scattered field from an inhomogeneity. From this we can obtain a first-order estimate of the inhomogeneity’s plane wave scatter function.

We first derive the plane wave scatter function for a coherent volume of internal wave inhomogeneity centered at horizontal location \(\rho_s\), where \(\rho_t = \rho_s + u_t\). For an incoming plane wave in the direction \(k_i = (k, \alpha_i, \beta_i) = (\xi_i, \gamma_i)\) and scattered plane wave in the direction \(k = (k, \alpha, \beta) = (\xi, \gamma)\), the first order scatter function of an internal wave is.
\[ \Re \{ S_{\rho_z}(\alpha, \beta, \alpha_i, \beta_i) \} = \iint_{A_c} \frac{k^3}{4\pi} \left[ \Gamma_{\kappa}(r_t) + \eta(k, k_i)\Gamma_d(r_t) \right] e^{i(\xi - \xi')u + (\gamma - \gamma')z_t} d^2u_d z_t, \tag{3.13} \]

by application of Green's theorem Eq. (3.12), where,

\[ \eta(k, k_i) = \frac{k_i \cdot k}{k^2} = \cos \alpha_i \cos \alpha + \sin \alpha_i \sin \alpha \cos (\beta_i - \beta), \tag{3.14} \]

is the cosine of the angle between the incident and scattered plane wave directions.

The fractional changes in compressibility and density depend on the displacement of the inhomogeneities at \( r_t \) and are given by

\[ \Gamma_{\kappa}(r_t) = \Gamma_{\kappa}(h(\rho_t), z_t) \]
\[ = \frac{\kappa_1 - \kappa_2}{\kappa_2} u[h(\rho_t) - (z_t - D)]u(z_t - D) + \frac{\kappa_2 - \kappa_1}{\kappa_1} u[(z_t - D) - h(\rho_t)]u(-(z_t - D)), \tag{3.15} \]
\[ \Gamma_d(r_t) = \Gamma_d(h(\rho_t), z_t) \]
\[ = \frac{d_1 - d_2}{d_1} u[h(\rho_t) - (z_t - D)]u(z_t - D) + \frac{d_2 - d_1}{d_2} u[(z_t - D) - h(\rho_t)]u(-(z_t - D)). \tag{3.16} \]

The areal scatter function density \( s_{\rho_z}(\alpha, \beta, \alpha_i, \beta_i) \) centered at \( (\rho_z, z_t) \) is related to the scatter function \( S_{\rho_z}(\alpha, \beta, \alpha_i, \beta_i) \) of Eq. (3.13) by,

\[ S_{\rho_z}(\alpha, \beta, \alpha_i, \beta_i) = A_c \int_0^H s_{\rho_z, z_t}(\alpha, \beta, \alpha_i, \beta_i) e^{i(\gamma - \gamma')z_t} \, dz_t, \tag{3.17} \]

where,
\[
s_{\rho_\alpha, z_t}(\alpha, \beta, \alpha_t, \beta_t) = \frac{1}{A_c} \int_A \int \frac{k^3}{4\pi} \left[ \Gamma_{\alpha}(r_t) + \eta(k, k_t) \Gamma_{d}(r_t) \right] e^{i(k\xi - \xi - \xi - \xi)} dt d^2u_t. \quad (3.18)
\]

Mean and correlation function of the scatter function density of internal wave inhomogeneities

The scatter function density \( s_{\rho_\alpha, z_t}(\alpha, \beta, \alpha_t, \beta_t) \) of Eq. (C.9) is a random variable since it depends on the internal wave displacement \( h(\rho_t) \). The mean of the scatter function density is,

\[
\langle s_{\rho_\alpha, z_t}(\alpha, \beta, \alpha_t, \beta_t) \rangle = \frac{1}{A_c} \int_A \int \frac{k^3}{4\pi} \left[ \Gamma_{\alpha}(h, z_t) + \eta(k, k_t) \Gamma_{d}(h, z_t) \right] p(h) dh e^{i(k\xi - \xi - \xi - \xi)} dt d^2u_t
\]

\[
= \frac{1}{A_c} \left\langle \frac{k^3}{4\pi} \left[ \Gamma_{\alpha}(h, z_t) + \eta(k, k_t) \Gamma_{d}(h, z_t) \right] \right\rangle \int_A e^{i(k\xi - \xi - \xi - \xi)} dt d^2u_t, \quad (3.19)
\]

For two inhomogeneities centered at \((\rho_t, z_t)\) and \((\rho', z_t')\) respectively, the correlation of their scatter function densities is,

\[
\langle s_{\rho_\alpha, z_t}(\alpha, \beta, \alpha_t, \beta_t) s_{\rho', \rho'}^*(\alpha', \beta', \alpha_t', \beta_t') \rangle = \left( \frac{1}{A_c} \right)^2 \int_A \int_A e^{i(k\xi - \xi - \xi - \xi - \xi - \xi - \xi - \xi)} p_{\rho_t}(\rho_t - \rho', z_t, z_t') d^2u_t d^2u_t', \quad (3.20)
\]

where

\[
C_{\rho\rho}(\rho_t - \rho', z_t, z_t') = \int_A \int_A \left( \frac{k^3}{4\pi} \right)^2 \left[ \Gamma_{\alpha}(h(\rho_t), z_t) + \eta(k, k_t) \Gamma_{d}(h(\rho_t), z_t) \right] \left[ \Gamma_{\alpha}(h(\rho_t'), z_t') + \eta(k', k_t') \Gamma_{d}(h(\rho_t'), z_t') \right] p(h(\rho_t), h(\rho_t')) dh(\rho_t) dh(\rho_t'), \quad (3.21)
\]

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following the notation of Ref. [4] Appendix A.

The joint probability function \( p(h(\rho_{i}), h(\rho_{f})) \) requires the internal wave spectrum to be specified, as can be seen from Eqs. (3) through (5).

**Isotropic internal waves**

We must calculate the mean and spatial covariance of the scatter function density \( s_{\rho_{i},\rho_{f}}(\alpha, \beta, \alpha_{i}, \beta_{i}) \) to determine the forward field moments. This requires the internal wave spectrum defined in Eq. (3.5) to be specified. Following Refs. [25] and [13], we assume the internal waves follow the isotropic Garret-Munk spectrum,

\[
\mathcal{G}(\kappa) = \frac{E_0}{2\pi^2} \frac{2\omega_c \sqrt{\kappa^2 c_g^2 - \omega_c^2}}{\kappa^2 c_g^2},
\]

for the relatively high wavenumbers of interest in our analysis, where \( E_0 \) is the average energy density that determines the strength of the internal wave fluctuation, \( \omega_c \) is the Coriolis frequency which is roughly \( 1.16 \times 10^{-5} \) Hz at 30° latitude, and \( c_g \approx 0.4 \) m/s is the phase speed of the internal waves in the two-layer ocean given by \( c_g^2 = g \frac{\rho_2 - \rho_1}{\rho_2} \frac{D(H-D)}{H} \) where \( g \) is the gravitational acceleration. The corresponding coherence radius \( \ell_c(\theta_i) = \ell_c \) is then independent of azimuth.

In this paper, we focus our analysis on internal wave fluctuations that occur within a measurement time scale \( T \). Within this time, only internal wave disturbances that occur with frequencies that are larger than \( f_{min} = 1/T \) or equivalently have wavenumbers larger than \( \kappa_{min} = 2\pi f_{min}/c_g = \frac{2\pi}{c_g T} \) can cause temporal fluctuation in the acoustic field. From Eq. (3.22), Eq. (3.8) and Eq. (7), the coherence radius is then \( \ell_c = \frac{\sqrt{2}}{\kappa_{min}} \), the \( x \) and \( y \) coherence lengths are \( \ell_x = \ell_y = \frac{2\sqrt{2}}{\kappa_{min}} \) and the coherence area is \( A_c = \pi \ell_c^2 = \frac{2\pi}{\kappa_{min}} \).

As noted in Ref. [4], the cross-range coherence length \( \ell_y \) of internal wave inhomogeneities can be greater or less than the Fresnel width \( Y_F(\rho, \rho_s) = \sqrt{\frac{2\pi \rho \rho_s}{\xi_{m,\rho}}} \), which depends on the range \( \rho \) from source to receiver and the range \( \rho_s \) from source to the inhomogeneities. When \( \ell_y < Y_F \), decorrelation in cross-range occurs within the Fresnel width or active region. But when \( \ell_y > Y_F \), internal wave inhomogeneities
are fully correlated across the active region. In this case, only the portion of the coherence area $A_F$ within the Fresnel width is important to forward scatter, where $A_F \leq A_c$.

### 3.2.3 Statistical moments of the forward propagated field

The mean field, variance, and expected total intensity of the forward field propagated through the ocean waveguide containing random internal waves can be expressed analytically using the formulation of Ref. [4]. We assume that the internal wave inhomogeneities obey a stationary random process in range. For a source at $r_0 = (-x_0, 0, z_0)$ and a receiver at $r = (x, 0, z)$, the mean forward field is given by Eq. (83) of Ref. [4] as

$$\langle \Psi_T(r|r_0) \rangle = \sum_{n=1}^{\infty} \Psi_i^{(n)}(r|r_0)e^{i\int_0^\rho \nu_n(\rho) d\rho},$$

(3.23)

where,

$$\Psi_i^{(n)}(r|r_0) = 4\pi \frac{i}{d(z_0)\sqrt{8\pi}} e^{-i\pi/4} u_n(z) u_n(z_0) \frac{e^{i\xi_n\rho}}{\sqrt{\xi_n\rho}},$$

(3.24)

is the incident field contribution from mode $n$ given no inhomogeneities in the medium, $u_n(z)$ is the modal amplitude at depth $z$, and $\nu_n$ is the horizontal wavenumber change due to multiple scattering from the inhomogeneities. The modal horizontal wavenumber change is complex and it leads to both dispersion and attenuation in the mean forward field. The real part $\Re\{\nu_n\}$ is the modal dispersion coefficient and the imaginary part $\Im\{\nu_n\}$ is the modal attenuation coefficient.

For the present case where the random inhomogeneities are due to internal waves,

$$\int_0^\rho \nu_n(\rho) d\rho = \int_0^\rho \frac{1}{\xi_n} \langle \Xi_h(n, n, 0, 0) \rangle d\rho,$$

(3.25)
from Eq. (60) of Ref. [4] where

\[ \Xi_n(m, n, \beta, \beta_1) = \int_0^H \frac{2\pi}{k(z_t)d(z_t)} \left[ N_m^{(1)} N_n^{(1)} e^{iR(\gamma_m+\gamma_n)z_t} s_{\rho_1, z_t}(\pi - \alpha_m, \beta; \alpha_n, \beta_1) \ight. \\
- N_m^{(2)} N_n^{(1)} e^{iR(-\gamma_m+\gamma_n)z_t} s_{\rho_1, z_t}(\alpha_m, \beta; \alpha_n, \beta_1) \\
- N_m^{(1)} N_n^{(2)} e^{iR(\gamma_m-\gamma_n)z_t} s_{\rho_1, z_t}(\pi - \alpha_m, \beta; \pi - \alpha_n, \beta_1) \\
+ N_m^{(2)} N_n^{(2)} e^{iR(-\gamma_m-\gamma_n)z_t} s_{\rho_1, z_t}(\alpha_m, \beta; \pi - \alpha_n, \beta_1) \left. \right] e^{-3\left(\gamma_m+\gamma_n\right)z_t} d\rho_1 \\
\] (3.26)

As discussed in Sec. 3.2.2, substituting Eq. (C.9) into Eq. (3.25) leads to a modal horizontal wavenumber change that is purely real. This accounts for the dispersion, but not the attenuation in the mean forward field due to scattering. The modal attenuation coefficients \( \Im\{\nu_n\} \) will be derived in Sec. 3.2.4 from the waveguide extinction theorem[10].

From Eq. (3.25), we see that the modal dispersion coefficients depend on the expected scatter function density in the forward azimuth. This is independent of the cross-range extent of the internal wave inhomogeneities, so the dispersion coefficients are also range independent.

The variance of the forward field at the receiver can be expressed by Eq. (84) of Ref. [4] as,

\[ \text{Var}(\Psi_T(r|r_0)) = \sum_n \frac{2\pi}{d^2(z_0)} \frac{1}{|\xi_n|} |u_n(z_0)|^2 |u_n(z)|^2 e^{-2\Im\{\xi_n\rho + \int_0^\rho \gamma_n(\rho) d\rho\}} \left( e^{\int_0^\rho \mu_n(\rho) d\rho} - 1 \right). \]

(3.27)

where \( \mu_n \) is defined in Ref. [4] as the exponential coefficient of modal field variance.

The variance of the forward field depends on the first and second order moments of
the scatter function density of the random medium. An analytic expression for $\mu_n$ for general surface and volume inhomogeneities is provided in Ref. [4].

For receiver ranges $\rho < \frac{4\xi^2}{\lambda}$ where the internal wave inhomogeneities are fully correlated within the Fresnel width, from Eq. (74) of Ref. [4], we have,

$$\int_{0}^{\rho} \mu_n(\rho_s) d\rho_s = \sum_{m=1}^{\infty} \rho_s \frac{1}{\xi_m |\xi_m|} \left[ \langle |\Xi_h(m, n, 0, 0)|^2 \rangle - \langle |\Xi_h(m, n, 0, 0)|^2 \rangle \right],$$

and for receiver ranges $\rho > \frac{4\rho_t}{\lambda}$ where the Fresnel width exceeds the internal wave cross-range coherence length, from Eq. (90) of Ref. [4], we have,

$$\int_{0}^{\rho} \mu_n(\rho_s) d\rho_s = \sum_{m=1}^{\infty} \left( e^{\frac{-\rho_s}{\rho_s^{\text{cor}}}} + A_c \sqrt{\frac{\rho}{2\pi\xi_m}} \sin^{-1} \sqrt{1 - \frac{\rho_s^{\text{cor}}}{\rho}} - \sin^{-1} \sqrt{\frac{\rho_s^{\text{cor}}}{\rho}} \right) \times \frac{1}{|\xi_m|} \left[ \langle |\Xi_h(m, n, 0, 0)|^2 \rangle - \langle |\Xi_h(m, n, 0, 0)|^2 \rangle \right],$$

where $\rho_s^{\text{cor}} = \rho/2(1 - \sqrt{1 - 4\xi^2/\rho \lambda})$. These equations show how the exponential coefficient of modal field variance couples energy in incident mode $n$ to all scattered modes $m$ after random multiple forward scattering through the waveguide.

The mean forward field of Eq. (3.23) is also called the coherent field, the magnitude square of which is proportional to the coherent intensity. The variance of the forward field in Eq. (3.27) provides a measure of the incoherent intensity. The total intensity of the forward field is the sum of the coherent and incoherent intensities. The coherent field tends to dominate at short ranges from the source and in slightly random media, while the incoherent field tends to dominate in highly random media. It should be noted that in a non-random waveguide $\mu_n = 0$ so that the variance of the forward
field is zero, from Eq. (3.27). This is expected since the field is fully coherent in this case.

3.2.4 Modal attenuation from generalized waveguide extinction theorem

Attenuation or extinction of the forward field arises from scattering by inhomogeneities and intrinsic absorption in the medium. As mentioned in Secs. 3.1 and 3.2.3, the purely real scatter function of Eq. (3.13) can only account for dispersion due to scattering. In order to include attenuation in the mean forward field, we apply the waveguide extinction theorem[10, 11]. The modal extinction cross-section of an object, for incident mode \( n \), is the ratio of the extinction \( \mathcal{E}_n \) or power loss caused by the object to the incident intensity \( I_{i,n} \)[10, 11],

\[
\sigma_n(x = 0) = \frac{\mathcal{E}_n(x = 0|r_0)}{I_{i,n}(z_h|r_0) \cdot i_x},
\]

(3.30)

where \( i_x \) is the propagation direction. The notation \( x = 0 \) means that the medium’s intrinsic absorption is set to zero during the calculation as described in Ref. [10] to properly determine the extinction cross-section of a given scatterer.

In a waveguide with random internal wave inhomogeneities, the modal attenuation coefficient \( \Im \{\nu_n\} \) can be expressed in terms of the modal extinction cross-section[10] \( \sigma_n(x = 0|h) \) of the inhomogeneities as

\[
\Im \{\nu_n(\rho_s)\} = \frac{1}{2A_c(\rho_s)} \left\langle \frac{1}{d(z_h)} \sigma_n(x = 0|h) \left| u_n(z_h) \right|^2 \right\rangle,
\]

(3.31)

from Eq. (60) of Ref. [4] and Eq. (20) of Ref. [10], where the expectation in Eq. (3.31) is taken over the height of the internal waves. Note that

\[
A_c(\rho_s) = \begin{cases} A_F & \ell_y > Y_F(\rho_s) \\ A_c & \ell_y < Y_F(\rho_s), \end{cases}
\]

(3.32)
is dependent on the range location $\rho_a$ of the inhomogeneity within the Fresnel width from source to receiver, where $A_F$ is defined in section 3.2.2.

For non-absorbing objects, the extinction $\mathcal{E}_m$ caused by the object is equal to the total scattered power $W_{s,n}$. By placing the object within a closed control surface, we can calculate $W_{s,n}$ as the total scattered power flux through the surface. A general analytic expression for the total scattered power flux from an object centered at depth $z_h$ within a closed cylindrical control surface with a radius of $x$ and height spanning the entire waveguide depth for incident mode $n$ is provided by Eq. (16) of Ref. [10]. We assume that the internal wave elements remove power from the incident field by scattering only so that intrinsic absorption losses from the inhomogeneities are negligible. This is a valid assumption since the internal wave inhomogeneities do not lead to absorption other than that already present in the medium. For a characteristic internal wave inhomogeneity given height $h$, the total scattered power flux is,

$$W_{s,n}(x|r_0, h) = \Re \left\{ \frac{1}{\omega d^2(z_0)} \frac{|u_n(z_0)|^2}{|\xi_n| |x_0|} \sum_{m=1}^{\infty} \frac{\xi_m}{|\xi_n|} A_m^2(\rho_m) e^{-2\Im(\xi_n x_0 + \xi_m x)} \int_0^{2\pi} |\Xi_n(m, n, \beta, 0)|^2 d\beta \right\}. \quad (3.33)$$

The $x$ component of incident intensity from mode $n$ on this inhomogeneity is,

$$I_{i,n}(z_0) \cdot i_x = \frac{2\pi}{\omega d^2(z_0) d(z_h) x_0} |u_n(z_0)|^2 |u_n(z_h)|^2 \Re \left\{ \frac{\xi_n^*}{|\xi_n|} e^{-2\Im(\xi_n) x_0} \right\}, \quad (3.34)$$

from Eq. (19) of Ref. [10]. Following Eq. (3.30), the modal extinction cross-section of the internal wave inhomogeneity is found from dividing Eq. (3.33) by Eq. (3.34) and setting $x = 0$.
\[
\sigma_n(x = 0|h) = \Re \left\{ \frac{1}{2\pi} d(z_h) \frac{1}{u_n(z_h)} \left[ \sum_{m=1}^{\infty} \frac{\xi_m^*}{|\xi_m|} A^2(\rho_s) \int_0^{2\pi} |\Xi_h(m, n, \beta, 0)|^2 d\beta \right] \right\}.
\]

(3.35)

The attenuation coefficient of mode \(n\) due to scattering in the random waveguide can then be found to be,

\[
\Im \{\nu_n(\rho_s)\} = \frac{1}{4\pi} A_c(\rho_s) \sum_{m=1}^{\infty} \Re \left\{ \frac{\xi_m^*}{|\xi_m|} \int_0^{2\pi} |\Xi_h(m, n, \beta, 0)|^2 d\beta \right\}
\]

(3.36)

by substituting Eq. (3.35) into Eq. (3.31).

### 3.3 Computing 2-D spatial realizations of a random internal wave field

To compare statistical moments of the forward field from the 3-D analytical formulation with Monte-Carlo simulations based on the 2-D parabolic equation, we must compute spatial realizations of the internal wave height \(h(x_t)\). We assume the Gaussian random internal wave field \(h(x_t)\) is a stationary random process that follows the correlation function shown in Fig. 3-2(b). The Fourier transform of internal wave height \(h(x_t)\) over a finite spatial window is then,

\[
H(\kappa) = \int_{-\frac{L}{2}}^{\frac{L}{2}} h(x_t) e^{-i\kappa x_t} dx_t
\]

where \(H(\kappa)\) is a zero-mean Gaussian random process due to its linear dependence on \(h(x_t)\). According to Parseval’s theorem

\[
\sigma_h^2 \approx \frac{1}{L} \int_{-\frac{L}{2}}^{\frac{L}{2}} \langle |h(x_t)|^2 \rangle dx_t = \frac{1}{L} \int \langle |H(\kappa)|^2 \rangle d\kappa
\]

(3.37)

56
while, from Eq. (3.6), $\eta_h^2 = C_{hh}(0)$ so that $\langle |H(\kappa)|^2 \rangle \approx \frac{L G(\kappa) \kappa}{2\pi}$. As $L$ becomes arbitrarily large, delta-function correlation is achieved across the wavenumber domain,

$$\langle H(\kappa)H(\kappa') \rangle \approx \frac{G(\kappa) \kappa}{2\pi} \delta(\kappa - \kappa'), \quad (3.38)$$

indicating that components of $H(\kappa)$ with wavenumber separations exceeding $2\pi/L$ are uncorrelated.

So, a random realization of the internal wave height $h(x_t)$ can be computed as the inverse Fourier transform of $H(\kappa)$ under the assumption that the $H(\kappa)$ are zero mean Gaussian random variables that are uncorrelated when sampled at wavenumber intervals of at least $2\pi/L$ and have variance $\frac{L G(\kappa) \kappa}{2\pi}$.

### 3.4 Illustrative examples

Here we provide examples illustrating the dependence of acoustic field moments on internal wave parameters. We study random and isotropic internal waves in typical continental shelf environments. In section 3.4.1, we investigate the effect of rms internal wave height and coherence length on acoustic transmission in a typical mid-latitude Atlantic continental shelf waveguide. In section 3.4.2, the coherent, incoherent, and total intensities of the forward acoustic field are compared when density fluctuations due to internal waves are first included and then neglected. We also compare our 3-D analytical model with 2-D Monte-Carlo simulations of the forward propagated acoustic field in section 3.4.3.

We examine the effect of internal waves on acoustic transmission over a measurement time period of approximately 10 minutes in our simulations. Over this time period, internal wave disturbances with wavenumbers larger than $\kappa_{\text{min}} = 0.028$ rads/m will undergo spatial-temporal variation. The normalized power spectrum of the internal wave field is shown in Fig. 3-2(a) and the correlation function in Fig. 3-2(b). The corresponding coherence length of the isotropic internal waves becomes
Figure 3-2: (a) Normalized spectrum of internal wave field over an observation period of approximately 10 mins with minimum wavenumber $\kappa_{min} = 0.028 \text{ rads/m}$. The spectrum is computed using Eq. (3.22) with a $\kappa^{-2}$ dependence at high frequencies. (b) Correlation function of the isotropic internal wave field with a coherence length of 100 m. The correlation function was obtained from the inverse Fourier transform of the internal wave spectrum plotted in (a).

\[ \ell_x = \ell_y = 100 \text{ m} \] from Fig. 3-2 and Eqs. (3.8), (3.9) and (3.10). The maximum Fresnel width $\frac{\sqrt{\ell_x \ell_y}}{2}$ then exceeds the cross-range coherence length for source-receiver separations larger than 11 km as shown in Fig. 3-3. These are the ranges where 3-D scattering effects can become pronounced when internal wave displacements exceed the acoustic wavelength.

### 3.4.1 Mid-latitude Atlantic continental shelf environment

Here, a water column of $H = 100 \text{ m}$ depth is used to simulate the geometry of a typical continental shelf environment, as shown in Fig. 1. The water column is comprised of a warm upper layer with density $d_1 = 1024 \text{ kg/m}^3$ and sound speed $c_1 = 1520 \text{ m/s}$ overlying a cool lower layer with density $d_2 = 1025 \text{ kg/m}^3$ and sound speed $c_2 = 1500 \text{ m/s}$. The bottom sediment half space is composed of sand with density $\rho_b = 1.9 \text{ g/cm}^3$ and sound speed $c_b = 1700 \text{ m/s}$. The attenuations in the water column and
Figure 3-3: (a) Fresnel half width for receiver ranges $p = 5$ km and $p = 50$ km. The Fresnel width $Y_F(p, \rho_s)$ is approximately equal to $\sqrt{\frac{\lambda(p-\rho_s)}{\rho}}$, where $\rho$ is the source-receiver separation and $\rho_s$ is the range from source to inhomogeneity. (b) The maximum Fresnel width $Y_{F,\text{max}} = Y_F(\rho, \rho/2) = \sqrt{\lambda \rho/2}$ as a function of source-receiver separation $\rho$.

We consider two scenarios. In the first scenario, the internal wave disturbances have a height standard deviation of $\eta_h = 1$ m which is smaller than the acoustic wavelength of $\lambda = 3.6$ m. For the ranges considered, we find the acoustic field to be slightly random. The internal wave spectrum has an amplitude of approximately $2 \text{ m}^2/\text{cph}$ at $\kappa_{\text{min}}$ for this example. In the second scenario, the internal wave disturbances have a height standard deviation of $\eta_h = 4$ m which is slightly larger than the acoustic wavelength. We find the acoustic field becomes highly random within a few kilometers of the source. The internal wave spectrum has an amplitude of approximately $30 \text{ m}^2/\text{cph}$ at $\kappa_{\text{min}}$ in this case. The internal wave correlation length and height standard
deviations modelled here are typical of those measured on the Strataform area of the New Jersey continental shelf. For instance they would correspond to the disturbances shown as Segment C of Fig. 24 in Ref. [14].

The acoustic field intensity is plotted as a function of range and depth in Fig. 3-4 for the shallow water waveguide of Fig. 1 when there are no internal waves present in the medium. The waveguide is static and undisturbed by inhomogeneities in this case. The forward acoustic field is then fully coherent since the variance in the forward field is zero everywhere in the medium. The acoustic intensity exhibits a range and depth dependent structure arising from coherent interference between the waveguide modes. Figure 3-5 shows the coherent, incoherent and expected total intensity for the slightly random waveguide. In this case, the incoherent intensity is small compared to the coherent intensity even at long ranges up to 50-km. The total field still maintains the range and depth dependent structure due to modal interference, but shows the effect of some moderate dispersion and attenuation. The situation changes, however, in the highly random waveguide as shown in Fig. 3-6. The coherent intensity decays rapidly as a function of range from the source due to severe attenuation arising from internal wave scattering. The incoherent component dominates the expected total intensity beyond the 11-km range where 3-D scattering begins to take effect. The expected total intensity eventually decays monotonically with range and no longer exhibits a significant coherent modal interference structure in range and depth beyond roughly 30 km in range.

The relative contributions of the waveguide modes to the depth-integrated total intensity of the forward field at the source location and at 1 km, 10 km and 50 km range from the source are shown in Fig. 3-7. There it can be seen that the highly random waveguide redistributes modal energies far more than the non-random waveguide and also has much less energy across the modal spectrum.

We next investigate the dependence of the forward field moments on the coherence lengths of the internal wave disturbances by letting them decrease to $\ell_x = \ell_y = 50$ m. Figures 3-8(a) and (b) show the coherent, incoherent and total acoustic intensities for internal wave height standard deviations of $\eta_h = 1$ m and $\eta_h = 4$ m respectively.
Figure 3-4: Acoustic field intensity at 415 Hz as a function of range and depth in the mid-latitude Atlantic continental shelf waveguide of Fig. 3-1 when there are no internal waves present so that the waveguide is undisturbed. The boundary between the warm and cool water is at the depth of 30 m from the water surface in this static waveguide. The source is at 50m depth with source level 0 dB re 1 μPa @ 1m. The acoustic intensity exhibits range and depth dependent variations due to coherent interference between waveguide modes.
Figure 3-5: Intensities of the (a) mean or coherent field, (b) variance or incoherent field, and (c) total field at 415 Hz as functions of range and depth in the mid-latitude Atlantic continental shelf waveguide of Fig. 3-1 when there is a random internal wave field present in the waveguide. The internal wave disturbances have a height standard deviation of $\sigma_h = 1 \text{ m}$ and coherence lengths of $\ell_x = \ell_y = 100 \text{ m}$. The source is at 50 m depth with source level 0 dB re 1 $\mu$Pa @ 1m. This medium is only slightly random and the total intensity in (c) is dominated by the coherent intensity out to 50 km range and exhibits range and depth dependent variations due to coherent interference between waveguide modes, similar to the static waveguide example in Fig. 3-4. Figure 3-5(d) shows the acoustic intensity as a function of range at a single receiver depth of 50 m for the fields shown in (a)-(c). For comparison, the acoustic intensity of the static waveguide is also plotted.
Figure 3-6: Similar to Fig. 3-5 but for a waveguide with an internal wave height standard deviation of $\eta_h = 4$ m. This medium is highly random and the total intensity in (c) is dominated by the variance or incoherent intensity beyond 11-km range. The total acoustic intensity decays monotonically as a function of range at sufficiently long ranges since the field is now completely incoherent and the waveguide loses the coherent range and depth dependent variations due to modal interference.
Figure 3-7: Contributions of the waveguide modes to the depth-integrated total intensity of the forward field at (a) the source location, (b) 1 km, (c) 10 km and (d) 50 km ranges from the source for a source strength of 0 dB re 1 μPa @ 1m. All values are absolute except those in (a) which are normalized by the maximum modal contribution.
Comparing the coherent field in Figs. 3-8(a) and (b) with Figs. 3-5(d) and 3-6(d), we find that attenuation due to scattering increases with increasing coherence length, especially in the highly random waveguide. This can be explained by noting that in Eq. (3.36) the attenuation coefficient is linearly related to the coherence area since the second moment of scatter function density is effectively independent of coherence area in the forward azimuth. The larger attenuation from multiple scattering leads to lower coherent intensity. The level of incoherent intensity, on the other hand, increases with increasing coherence area in the highly random waveguide. This is expected since in the limiting case where the cross-range coherence length exceeds the Fresnel width, incoherent intensity arises from a more multiplicative process of transmission through single-scatter shells decorrelated solely in range as in a 2-D multiple scattering scenario[44, 7].

3.4.2 Effect of internal wave density fluctuations on acoustic transmission in an Arctic environment

Internal waves cause not only sound speed but also density fluctuations in the water column that can sometimes affect acoustic transmission statistics. Density effects are expected to be pronounced in deep estuaries such as Norwegian Fjords where “fresh
Figure 3-9: Effect of (a) including and (b) neglecting internal wave density fluctuations on acoustic transmission in an Arctic waveguide with geometry described in Fig. 3-1. The internal wave field has coherence lengths of $\ell_x = \ell_y = 100$ m and a height standard deviation of $\eta_h = 4$ m. The acoustic intensity is plotted as a function of range for source and receiver at 50 m depth.

River water tends to move seawards above the heavier salt water [36]" before sufficient tidal motions occur to cause mixing [36] and in the Arctic seas where cold fresh water near the melting temperature of ice flows above warm and salty seawater[43].

Here we give an example of acoustic propagation through a 3-D linear internal wave field in an Arctic sea with 100 m water depth, as illustrated in Fig. (3-1). The interface between the cold fresh water above with density $d_1 = 1022$ kg/m³ and sound speed $c_1 = 1433$ m/s and the warm salty water below with density $d_2 = 1028$ kg/m³ and sound speed $c_2 = 1443$ m/s is also assumed to be at 30 m depth from sea surface[43]. In Fig. 3-9, we determine the effect of internal wave density fluctuations on the forward acoustic field by including and then neglecting them in our calculations. Significant differences occur between these two cases for both the coherent and incoherent field components. Attenuation due to scattering is reduced when density fluctuations are included. This is because scattering from density and compressibility inhomogeneities are exactly out of phase in the forward direction as can be seen by substituting sound speeds and densities for the two layers of the water column into $\Gamma_\kappa$ and $\Gamma_d$ in Eq. (C.9). Internal wave density fluctuations should then not be neglected in estimating acoustic transmission statistics in certain environments.
3.4.3 Comparison of 3-D analytic model with 2-D monte-carlo simulations

Comparisons of the coherent, incoherent and total acoustic field intensity determined by our 3-D analytic model and 2-D Monte-Carlo simulations using the parabolic equation for a slightly random waveguide with $\eta_h = 1$ m are illustrated in Figs. 3-10(a), (b) and (c). The coherent, incoherent and total intensities calculated with the 3-D and 2-D approaches match very well out to the 50-km ranges shown. Both the 3-D and 2-D simulations agree on the moderate dispersion, attenuation and low variance or incoherent intensity found at the longer ranges shown. This is a consequence of the weak scattering found for rms internal wave displacements small compared to the acoustic wavelength. The situation changes, however, in a highly random waveguide where rms internal wave displacements exceed the acoustic wavelength, with $\eta_h = 4$ m. As shown in Figs. 3-11(a), (b) and (c), the coherent and incoherent fields determined by the two approaches show a reasonable match within 11-km where the cross-range coherence is larger than the Fresnel width. This is the range within which the 2-D simulations are expected to be valid. Beyond 11 km range, the 2-D Monte-Carlo simulations of coherent and incoherent intensities are far less attenuated than those of the 3-D model and the 2-D coherent intensities are far less dispersed. This is because the 2-D parabolic equation cannot account for scattering of acoustic energy out of the forward direction and subsequent loss of coherence incurred by random cross-range variations in the 3-D medium. Figures 3-10(d) and 3-11(d) show 2-D Monte-Carlo simulations of the coherent, incoherent and total field for the slightly and highly random waveguides for comparison with the 3-D model examples of Figs. 3-5(d) and Fig. 3-6(d).

The expected values of depth-integrated total intensity for our 3-D model and 2-D Monte-Carlo simulations are compared in Fig. 3-12. They decay with range as a result of spreading, intrinsic absorption and the accumulated effect of multiple forward scattering through internal wave inhomogeneities in the medium. In the slightly random waveguide, the curves for the 3-D and 2-D models show close agreement.
Figure 3-10: Comparison of intensities from 2-D Monte-Carlo simulations and 3-D analytical model at the single receiver depth of 50 m in the presence of an internal wave field with height standard deviation of $\eta_h = 1$ m. A total of 1000 simulations were made using the parabolic equation to compute the 2-D Monte-Carlo field statistics. (a) coherent field comparison, (b) incoherent field comparison, (c) total field comparison, (d) only the 2-D Monte-Carlo simulated acoustic intensities of the coherent, incoherent and total fields used in (a)-(c).
Figure 3-11: Similar to Fig. 3-10 but for a waveguide with an internal wave height standard deviation of $\eta_h = 4$ m.
Figure 3-12: Total depth-integrated intensities for the waveguide used in Fig. 3-4. The static case with no internal waves in the medium is compared 3-D analytical model and 2-D Monte-Carlo simulations with internal wave height standard deviations of $\eta_h = 1\ m$ and $\eta_h = 4\ m$. The attenuation or power loss due to scattering is most significant in 3-D analytical model for the highly random waveguide.

<table>
<thead>
<tr>
<th>Depth–integrated intensity, source depth 50m</th>
</tr>
</thead>
<tbody>
<tr>
<td>dotted line: Undisturbed</td>
</tr>
<tr>
<td>solid line: 2-D Monte-Carlo, $\eta_h = 1m$</td>
</tr>
<tr>
<td>dash line: 3-D Model, $\eta_h = 1m$</td>
</tr>
<tr>
<td>dash-dot line: 2-D Monte-Carlo, $\eta_h = 4m$</td>
</tr>
<tr>
<td>dot dash line: 3-D Model, $\eta_h = 4m$</td>
</tr>
</tbody>
</table>

with eachother and with the curve for the undisturbed waveguide, as expected since scattering is weak. In the highly random waveguide, the 3-D model shows far greater power loss than the 2-D Monte-Carlo simulations. This is because the 2-D model cannot account for the out-of-plane scattering that must occur when the Fresnel width exceeds the cross-range coherence length of the internal waves.
3.5 Conclusion

Statistical moments of the acoustic field forward propagated through an ocean waveguide with random internal waves are modelled with a normal mode formulation that accounts for 3-D multiple forward scattering through medium inhomogeneities. The formulation analytically describes the accumulated effects of multiple forward scattering. These redistribute both coherent and incoherent modal energy, including attenuation and dispersion. Calculations for typical continental shelf environments show that the acoustic field becomes effectively incoherent at typical operational ranges when the rms internal wave height is on the order of the acoustic wavelength. It is found that two-dimensional models for the mean and variance of the acoustic field propagated through a 3-D random internal wave field then become inaccurate when the Fresnel width approaches and exceeds the cross-range coherence length of the internal wave field. Density fluctuations caused by internal waves may have a non-negligible effect on acoustic transmission in certain continental shelf environments, such as in Arctic seas.
Chapter 4

Acoustic field attenuation and temporal coherence after multiple forward scattering through 3-D deep-ocean internal waves

4.1 INTRODUCTION

Internal waves cause random compressibility and density fluctuations that can have a pronounced accumulated effect on acoustic signals propagating over long ranges in the deep ocean. The resulting random multiple forward scattering causes significant fluctuations in the acoustic field\cite{45, 46, 47, 48, 49, 50, 51, 52}, leads to degradation in the temporal coherence of acoustic signals\cite{53, 54, 55} and significant signal-dependent noise.\cite{56} Understanding the properties of this signal-dependent noise is often critical to effectively employ acoustics for ocean remote sensing and communication\cite{71, 57, 30, 31, 58, 59, 60}, as well as in the Acoustic Thermometry of Ocean Climate (ATOC).\cite{54} Knowledge of the coherence time scale of a received signal is essential in (1) reducing the error of any measurement or estimate obtained from fluctuating acoustic field data by stationary averaging, (2) applying the fundamental
coherent processing techniques of ocean acoustics, such as matched filtering, beamforming, matched-field and synthetic aperture processing. The coherence time scale, for example, is needed to determine the number of statistically independent samples of the received acoustic signal during a given measurement time, which can then be averaged to reduce the estimation error and the signal-dependent noise.[56, 61] Since coherent processing must typically be restricted to within the coherence time scale of field fluctuations, a good estimate of the coherence time scale is usually necessary to design an effective experiment.

In this paper, we derive a general analytical expression for the temporal correlation function, which can be used to predict the coherence time scale of the acoustic power and field propagated through 2-D and 3-D random inhomogeneities such as internal waves, bubbles, fish schools, eddies, and surface waves. From this expression, the coherence time scale is calculated for acoustic waves propagating through random internal wave inhomogeneities. It is shown that the coherence time scale of the acoustic field fluctuations observed at megameter ranges in various deep ocean acoustic transmission experiments[62, 54] can be explained by multiple forward scattering through linear internal waves in the deep ocean. The roughly 10-minute acoustic coherence time scale measured[62, 54] is shown to be nonlinearly related to the much longer 4-hour coherence time scale of the internal waves. The latter is derived from the Garrett-Munk (GM) spectrum.[63, 64] Analysis of the temporal coherence function at zero time lag, which corresponds to the expected intensity of the fluctuating signal, shows that 3-D scattering from internal waves can lead to power loss in low frequency long range propagation in the deep ocean. For a given receiver range, these losses begin to become pronounced as the frequency decreases to the point where the Fresnel length[4] of the acoustic measurements exceeds the cross-range coherence length[4] of the internal waves. This leads to out-of-plane scattering that removes energy from the forward direction. After reaching a maximum, power losses in the forward direction begin to decrease due to the weakening Rayleigh-Born scattering as the acoustic frequency decreases. This may explain the unexpected attenuation observed in some historic long range (megameter), low frequency (5 to 75 Hz) measurements.[45, 65]
Much related work in the 1970s and 1980s focused on using ray theory to work towards an estimate of temporal coherence by accumulating random phase fluctuations along isolated water-borne ray paths in the deep ocean. [46, 66] Due to perceived oversimplifications, many moved away from ray theory and began performing numerical Monte-Carlo simulations with the 2-D parabolic wave equation. For example, in Ref. [67], a 2-D Monte-Carlo approach based on a parabolic approximation for a specific deep ocean environment was used to estimate the temporal coherence function. Some disadvantages of the Monte-Carlo approach, however, are that (1) it does not provide a general analytical expression for the temporal coherence function but instead requires intensive numerical calculations for each specific case, (2) it is restricted to 2-D propagation and scattering scenarios.

In Sec. 4.2, analytical expressions for the temporal correlation functions of the acoustic power and field forward propagated through a slowly time-varying random medium are derived. In Sec. 4.3, the mean and temporal coherence of the scatter function are derived for internal wave inhomogeneities in the deep ocean. This is required to determine the temporal coherence function of the acoustic power and field. Differences between 2-D and 3-D scattering processes are reviewed in Sec. 4.4. Illustrative examples are provided in Sec. 4.5.

Before closing the introduction, it is useful to first briefly review key steps in the derivation of Ratilal and Makris[4] that will also be needed here. Ratilal and Makris[4] combined the waveguide scattering theory and a differential marching procedure analogous to that used in free space optics by Rayleigh to derive the mean and power of the acoustic field forward propagated through 3-D random inhomogeneities in terms of waveguide modes. That analysis is for the mean, spatial covariance and temporal variance of the field. Since the temporal coherence function and the coherence time scale require the temporal covariance, it must be derived here, since the temporal variance and covariance are only equal at zero time lag. To obtain the temporal covariance, we follow the same marching procedure used in Ref. [4] to calculate the temporal coherence function of acoustic power and forward field propagated through 3-D inhomogeneities. We define the coherence time scale to be the e-folding time
scale at which the temporal coherence function falls to $1/e$ of its zero time lag value. Knowledge of this time scale is often essential for designing underwater experiments.

Since the inhomogeneities are assumed to follow a stationary random process over time, the mean field remains as the time invariant result derived in Ref. [4]. The process of propagating through a single differential slab of random inhomogeneities causes a change in the mean acoustic field. For each mode, this change can be expressed as a product of factors including the incident field, a complex modal wavenumber change induced by scattering, and the thickness of the slab. The mean field after multiple forward scattering through the inhomogeneous medium, obtained by integrating over differential slabs, takes the form of a product of the incident field and an exponential factor that involves the accumulated modal wavenumber changes over a range from the source to the receiver. These wavenumber changes determine the dispersive and attenuating effects of the inhomogeneities, through their real and imaginary parts respectively, and account for mode coupling in the scattering process.

Ratilal and Makris[4] use a similar marching procedure to derive the mean power of the forward field. The incremental change in acoustic power due to a single slab of inhomogeneities can be expressed in terms of the depth integral of the second moment of the scattered field, as well as cross terms between the scattered and incident fields. This change can then be expressed as the product of the incident power, the difference between the modal variance and attenuation coefficients, and the slab thickness. Modal variance coefficients, introduced by depth integration of the scattered field second moment, depend on the variance of intrinsic properties of the inhomogeneities, such as compressibility and density. The acoustic power at the receiver range is then obtained by marching the power change equation from the source to the receiver through direct integration. This power is expressed as a product of the incident power and an exponential factor involving the range integration of the difference between the modal variance and attenuation coefficients.
4.2 Analytical model for temporal coherence of the acoustic power and field forward propagated through a slowly time-varying random medium

In this section, we derive an analytical expression for the temporal correlation function of acoustic power and field forward propagated through a slowly time-varying random medium based on the normal mode method. We assume that (i) the random medium follows a stationary process and (ii) the medium in a single slab is static during the time period of acoustic wave propagation through it. This is a valid assumption since the correlation time scale of the medium fluctuation is much longer than the time it takes for the acoustic wave to propagate through a single slab. Note that the derivation of the temporal correlation function of the acoustic power (depth-integrated temporal correlation of acoustic total field) does not require an assumption of acoustic modal independence.\[4\] The assumption of modal independence is only used here to estimate the temporal correlation of the forward field at a specified receiver location $r$. It is approximately valid when the random component of the field becomes a circular complex Gaussian random variable,\[56\] since the independence of a few dominant modes is then necessary by the central limit theorem. Gaussian field fluctuations are typically in many ocean acoustic measurements,\[56\]

All the derivations in this section are based on a single frequency transmission. In the appendix, we extend our model of the temporal correlation function to a narrow-band signal, which can be approximately expressed in terms of the temporal correlation function of the forward field.
4.2.1 Mean scattered field for a slowly time-varying random medium

Here, we give the mean acoustic scattered field through slowly time-varying random inhomogeneities that are confined within a slab centered at \( \rho_a \) of thickness \( \Delta \rho_a \).

As shown in Fig. 4-1, the origin of the coordinate system is placed at the air-water interface with the positive z-axis pointing downward. The source is located at the horizontal origin \( r_0 = (0, 0, z_0) \), receiver coordinates are given by \( r = (x, y, z) \), and inhomogeneity coordinates are given by \( r_x = (x_x, y_x, z_x) \). Here, \( t \) will solely be used to denote time dependence (The notation \( t \) used in Ref. [4] is not for time dependence, but denotes target coordinates \( r_t = (x_t, y_t, z_t) \)).

Spatial cylindrical \((r, \theta, z)\) and spherical systems \((r, \phi, z)\) are defined by \( x = r \sin \theta \cos \phi, \ y = r \sin \theta \sin \phi, \ z = r \cos \theta \), and \( \rho^2 = x^2 + y^2 \). The horizontal and vertical wavenumber components for the \( n \)th mode are respectively \( \xi_n = k \sin \alpha_n \) and \( \gamma_n = k \cos \alpha_n \), where \( \alpha_n \) is the elevation angle of the mode measured from the z-axis. Here, \( 0 \leq \alpha_n \leq \pi/2 \) so that the down- and upgoing planewave components of each mode have elevation angles \( \alpha_n \) and \( \pi - \alpha_n \) respectively.

The corresponding vertical wavenumber of the down- and upgoing components of the \( n \)th mode are \( \gamma_n \) and \( -\gamma_n \) respectively, where \( \Re \{ \gamma_n \} \geq 0 \). The azimuth angle of the mode is denoted by \( \beta \). The wavenumber magnitude \( k \) equals the angular frequency \( \omega \) divided by the sound speed \( c \) in the object layer, where \( k^2 = \xi_n^2 + \gamma_n^2 \). The geometry of spatial and wavenumber coordinates is shown in Fig. 2 of Ref. [39].

The scattered field from random inhomogeneities confined within the slab is found by summing volumetric contributions from unit volumes of inhomogeneity as described in Eq. 1 of Ref. [4]. The scattered field discussed in the present paper is different from Eq. 1 of Ref. [4], since it varies as a function of time due to the slow time-varying random inhomogeneities in the slab according to,

\[
\Phi_\sigma(r|r_0, \Delta \rho_a(\rho_a), f, t) = \iiint_{\Delta V_a} \varphi_\sigma(r|r_0, r_x, f, t) dV_x, \tag{4.1}
\]

where \( \Delta V_a \) is the volume of the slab, \( \varphi_\sigma(r|r_0, r_x, f, t) \) is the scattered field per unit volume of inhomogeneities centered at \( r_x \) and \( t \) is the time at the receiver \( r \).
The scattered field from a unit volume of slowly varying inhomogeneity, which has been derived from Green's theorem in Ref. [69] and [11], can be expressed as

\[ \varphi_s(r|r_0, \Delta \rho_s(\rho_s), f, t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left( \frac{4\pi}{k} \right)^2 \left[ A_m(r-r_x)A_n(r_x-r_0)s_{r_x,t}(\pi - \alpha_m, \beta_s(\phi, \phi_x); \alpha_n, \phi_X) - B_m(r-r_x)A_n(r_x-r_0)s_{r_x,t}(\alpha_m, \beta_s(\phi, \phi_x); \alpha_n, \phi_X) - A_m(r-r_x)B_n(r_x-r_0)s_{r_x,t}(\pi - \alpha_m, \beta_s(\phi, \phi_x); \pi - \alpha_n, \phi_X) + B_m(r-r_x)B_n(r_x-r_0)s_{r_x,t}(\alpha_m, \beta_s(\phi, \phi_x); \pi - \alpha_n, \phi_X) \right], \]

(4.2)

where \( A_n(r_x-r_0) \) and \( B_n(r_x-r_0) \) are the amplitudes of the down- and upgoing modal plane wave components incident on the inhomogeneity at \( r_x \) defined in Eqs. 3 and 4 of Ref. [4], \( A_m(r-r_x) \) and \( B_m(r-r_x) \) are the amplitudes of up- and downgoing modal components scattered from the inhomogeneity defined in Eqs. 5 and 6 of Ref. [4], \( \beta_s(\phi, \phi_x) = \phi - \arcsin \left\{ \frac{\rho_s}{|\mathbf{p}_s|} \sin(\phi_x - \phi) \right\} \) is the receiver azimuth from the target and \( s_{r_x,t}(\alpha, \beta; \alpha_t, \beta_t) \) is the scatter function density[4] of inhomogeneities at \( r_x \).

Assuming that the inhomogeneities obey a stationary process in time, the mean
scatter function density is time-invariant. The mean scattered field from a single inhomogeneous slab can then still be expressed as

$$\langle \Phi_s(\mathbf{r}|\mathbf{r}_0, \Delta \rho_s(\rho_s), f, t) \rangle$$

$$= \langle \Phi_s(\mathbf{r}|\mathbf{r}_0, \Delta \rho_s(\rho_s), f) \rangle$$

$$= \sum_n \Phi''_n(\mathbf{r}|\mathbf{r}_0, f) i \nu_n(\rho_s) \Delta \rho_s,$$  \hspace{1cm} (4.3)

by following Eq. 59 of Ref. [4], where $\nu_n(\rho_s)$ is the time-invariant change of the complex wavenumber due to scattering from the slab.

### 4.2.2 Marching temporal correlation through a slowly varying random waveguide with difference and integral equations

In this section, a difference equation is derived for the depth-integrated temporal correlation of the total field scattered from inhomogeneities confined within a slab centered at $\rho_s$. The depth-integrated temporal correlation of the total field is then marched through the random waveguide to include multiple forward scattering from all inhomogeneities between the source and the receiver.

The total field scattered from inhomogeneities within the slab is

$$\Phi(\mathbf{r}|\mathbf{r}_0, \Delta \rho_s(\rho_s), f, t) = \Phi_i(\mathbf{r}|\mathbf{r}_0, f) + \Phi_s(\mathbf{r}|\mathbf{r}_0, \Delta \rho_s(\rho_s), f, t),$$  \hspace{1cm} (4.4)

where

$$\Phi_i(\mathbf{r}|\mathbf{r}_0, f) = 4\pi \frac{i}{d(z_0)\sqrt{8\pi}} e^{-i\pi/4} \sum_n u_n(z) u_n(z_0) \frac{e^{i\xi_n \rho}}{\sqrt{\xi_n \rho}}$$  \hspace{1cm} (4.5)

is the incident field expressed in terms of acoustic normal modes.

The mean total field does not depend on $t$ from Eq. 4.3 and is the same expression
in Eq. 83 of Ref. [4]

\[ \langle \Phi_T(r|\mathbf{r}_0, f, t) \rangle = \langle \Phi_T(r|\mathbf{r}_0, f) \rangle = 4\pi \frac{i}{d(z_o)\sqrt{8\pi}} e^{-i\pi/4} \sum_n u_n(z_0)u_n(z_0) \frac{e^{i\xi_n \rho}}{\sqrt{\xi_n \rho}} e^{i\int_0^\rho \nu_n(\rho_s)d\rho_s}, \]

(4.6)

where the additional exponential term accounts for the dispersion and attenuation induced by multiple scattering.

From Eq. 4.4, the depth-integrated temporal correlation of the total field can be written as

\[
\int_0^\infty \frac{1}{d(z)} \langle \Phi(r|\mathbf{r}_0, \Delta \rho_s(\rho_s), f, t)\Phi^*(r|\mathbf{r}_0, \Delta \rho_s(\rho_s), f, t') \rangle dz
\]

\[ = \int_0^\infty \frac{1}{d(z)} |\Phi_i(r|\mathbf{r}_0, f)|^2 dz
\]

\[ + \int_0^\infty \frac{1}{d(z)} \left( \langle \Phi_i(r|\mathbf{r}_0, f)\Phi_s^*(r|\mathbf{r}_0, \Delta \rho_s(\rho_s), f, t') \rangle + \langle \Phi_s^*(r|\mathbf{r}_0, f)\Phi_s(r|\mathbf{r}_0, \Delta \rho_s(\rho_s), f, t) \rangle \right) dz
\]

\[ + \int_0^\infty \frac{1}{d(z)} \langle \Phi_s(r|\mathbf{r}_0, \Delta \rho_s(\rho_s), f, t)\Phi_s^*(r|\mathbf{r}_0, \Delta \rho_s(\rho_s), f, t') \rangle dz. \]

(4.7)

The first term at the right hand side of Eq. 4.7 is the incident intensity

\[
\int_0^\infty \frac{1}{d(z)} |\Phi_i(r|\mathbf{r}_0, f)|^2 dz = W_i(\rho|\mathbf{r}_0) = \sum_n W_i^{(n)}(\rho|\mathbf{r}_0),
\]

(4.8)

where

\[
W_i^{(n)}(\rho|\mathbf{r}_0) = \frac{2\pi}{d^2(z_0)} |u_n(z_0)|^2 \frac{1}{\rho |\xi_n|} e^{-2\Im(\xi_n) \rho}.
\]

The second term of Eq. 4.7 is the depth-integrated cross term arising from interference between the incident and the scattered field. By inserting the mean scattered field of Eq. 4.3 into the second term of Eq. 4.7 and invoking modal orthogonality

\[
\int_0^\infty \frac{1}{d(z)} u_m(z)u_n(z) dz = \delta_{mn},
\]

(4.9)
integration leads to

\[
\int_0^\infty \frac{1}{d(z)} \left[ \langle \Phi_i(r|\mathbf{r}_0) \Phi_i^*(r|\mathbf{r}_0, \Delta \rho_s(\rho_s), f, t') \rangle + \langle \Phi_i^*(r|\mathbf{r}_0) \Phi_i(r|\mathbf{r}_0, \Delta \rho_s(\rho_s), f, t) \rangle \right] dz
= \int_0^\infty \frac{1}{d(z)} \left[ \Phi_i(r|\mathbf{r}_0) \langle \Phi_i^*(r|\mathbf{r}_0, \Delta \rho_s(\rho_s), f, t') \rangle + \Phi_i^*(r|\mathbf{r}_0) \langle \Phi_i(r|\mathbf{r}_0, \Delta \rho_s(\rho_s), f, t) \rangle \right] dz
= - \sum_n W_i^{(n)}(\rho|\mathbf{r}_0) \Theta \{\nu_n(\rho_s)\} \Delta \rho_s.
\] (4.10)

The last term of Eq. 4.7 is the depth-integrated temporal correlation of the field scattered from the slab. The temporal correlation of the scattered field

\[
\langle \Phi_s(r|\mathbf{r}_0, \Delta \rho_s(\rho_s), f, t) \Phi_s^*(r|\mathbf{r}_0, \Delta \rho_s(\rho_s), f, t') \rangle = \text{Corr}_{\Phi_s, \Phi_s}(\Delta \rho_s(\rho_s), f, \tau = t - t')
\]
depends on the temporal and spatial correlation of the scatter function density

\[
\langle s_{x,t}(\alpha, \beta, \alpha_i, \beta_i)s_{x',t'}^*(\alpha', \beta', \alpha_i', \beta_i') \rangle,
\]
which will be discussed in Sec. 4.2.3. When \( t = t' \), Corr_{\Phi_s, \Phi_s}(\Delta \rho_s(\rho_s), f, \tau) becomes the second moment of the scattered field \( \langle |\Phi_s(r|\mathbf{r}_0, \Delta \rho_s(\rho_s), f)|^2 \rangle \) of Eq. 61, Ref. [4]. From Sec. III.B of Ref. [4], the depth-integrated second moment of the scattered field is

\[
\int_0^\infty \frac{1}{d(z)} \langle |\Phi_s(r|\mathbf{r}_0, \Delta \rho_s(\rho_s), f)|^2 \rangle dz = \sum_{n=1}^{\infty} W_i^{(n)}(\rho|\mathbf{r}_0) \mu_n(\rho_s, \tau) |_{\tau=0} \Delta \rho_s,
\] (4.11)

where \( \mu_n(\rho_s, 0) \) is the variance coefficient that contains a modal sum to account for modal coupling due to the random scattering process. When \( t \neq t' \), we can apply the procedure used in the derivation of Eq. 4.11 to calculate the last term in Eq. 4.7,

\[
\int_0^\infty \frac{1}{d(z)} \langle \Phi_s(r|\mathbf{r}_0, \Delta \rho_s(\rho_s), f, t) \Phi_s^*(r|\mathbf{r}_0, \Delta \rho_s(\rho_s), f, t') \rangle dz
= \sum_{n=1}^{\infty} W_i^{(n)}(\rho|\mathbf{r}_0) \mu_n(\rho_s, \tau = t - t') \Delta \rho_s,
\] (4.12)

where \( \mu_n(\rho_s, \tau) \) is the temporal variance coefficient, which accounts for modal coupling and quantifies the modal energy transferred from the mean field to the covariance field. It also describes how the forward field de-correlates as the time lag \( \tau = t - t' \) increases after propagating through a slowly time-varying medium. The dependence of \( \mu_n(\rho_s, \tau) \) on time lag \( \tau \) is a consequence of its dependence on the temporal covariance of the
scatter function density \( \text{Cov}_{ss}(\rho_s, z_x, z'_x, \tau) \), which will be discussed in Sec. 4.2.3.

Inserting Eqs. 4.8, 4.10 and 4.12 into Eq. 4.7, the depth-integrated temporal correlation of the total field is found to be

\[
\int_0^\infty \frac{1}{d(z)} \langle \Phi(\mathbf{r}|\mathbf{r}_0, \Delta \rho_s(\rho_s), f, t)\Phi^*(\mathbf{r}|\mathbf{r}_0, \Delta \rho_s(\rho_s), f, t') \rangle dz
= \langle W_T(\rho|\mathbf{r}_0, \tau) \rangle
= \sum_n W_i^{(n)}(\rho|\mathbf{r}_0) \left( 1 + \mu_n(\rho_s, \tau) - 2\Im(\nu_n(\rho_s)) \right) \Delta \rho_s. \tag{4.13}
\]

This can be rewritten as a difference equation

\[
\Delta \langle W_T(\rho|\mathbf{r}_0, \tau = t - t') \rangle
= \sum_n \Delta \langle W_i^{(n)}(\rho|\mathbf{r}_0, \tau) \rangle
= \sum_n W_i^{(n)}(\rho|\mathbf{r}_0) \left( \mu_n(\rho_s, \tau) - 2\Im(\nu_n(\rho_s)) \right) \Delta \rho_s. \tag{4.14}
\]

Following the marching process described in Sec. III B of Ref. [4], we have

\[
\langle W_T(\rho|\mathbf{r}_0, \tau) \rangle = \sum_n \langle W_i^{(n)}(\rho|\mathbf{r}_0, \tau) \rangle = \sum_n W_i^{(n)}(\rho|\mathbf{r}_0) e^{\int_0^\rho \left( \mu_n(\rho_s, \tau) - 2\Im(\nu_n(\rho_s)) \right) d\rho_s}. \tag{4.15}
\]

Assuming independence between acoustic modes and following the derivation for the second moment of the forward field in Sec III C of Ref. [4], the temporal correlation of the forward field at receiver \( \mathbf{r} \) can be expressed as

\[
\langle \Phi(\mathbf{r}|\mathbf{r}_0, f, t)\Phi^*(\mathbf{r}|\mathbf{r}_0, f, t') \rangle
= \text{Corr}_{\Phi\Phi}(\mathbf{r}|\mathbf{r}_0, f, \tau)
= \sum_n |\phi_i^{(n)}(\mathbf{r}|\mathbf{r}_0)|^2 \exp \left( \int_0^\rho |\mu_n(\rho_s, \tau) - 2\Im(\nu_n(\rho_s))| d\rho_s \right). \tag{4.16}
\]

where \( |\phi_i^{(n)}|^2 \) is the incident intensity of acoustic mode \( n \) in a static ocean waveguide. Since both \( \mu_n(\rho_s, \tau) \) and \( \nu_n(\rho_s) \) depend on the position of inhomogeneities along the acoustic propagation path, range-dependence in the scattering process can be taken
into account.

### 4.2.3 Complex wavenumber change and temporal covariance coefficient

As explained in Sec. II A, the complex wavenumber change $\nu_n(\rho_s)$ is time-independent. The expression for $\nu_n(\rho_s)$ is given in Eq. 60a of Ref. [4] as

$$
\nu_n(\rho_s) = \int_0^\infty \frac{2\pi}{k} \frac{1}{\xi_n} d(\xi_n) \left[ (N_n^{(1)})^2 e^{2i\eta z_0} \langle \mathcal{S}_{\mathcal{R}_x}(\pi - \alpha_n, \phi; \alpha_n, \phi) \rangle - N_n^{(2)} N_n^{(1)} \langle \mathcal{S}_{\mathcal{R}_x}(\alpha_n, \phi; \alpha_n, \phi) \rangle \right] e^{-i2\pi z_0} \langle \mathcal{S}_{\mathcal{R}_x}(\pi \alpha_n, \phi; \alpha_n, \phi) \rangle dz_0. 
$$

(4.17)

At zero time lag, the temporal covariance coefficient $\mu_n(\rho_s, \tau)$ becomes the variance coefficient $\mu_n(\rho_s, \tau = 0)$. The variance coefficients for 2-D and 3-D scattering processes respectively are explicitly expressed in Eqs. 74 and 77 of Ref. [4]. The variance coefficient contains a term $C_{C, s}(\rho_s, z_x, z_x', m, n)$ found in Eq. 72 of Ref. [4], which is a function of spatial correlation of the scatter function density $Cov_{ss}(s_{\rho_x z_x}, s_{\rho_x z_x'}) = \langle s_{\mathcal{R}_x}(\alpha, \beta; \alpha_s, \beta_s) s_{\mathcal{R}_x^*}(\alpha', \beta'; \alpha_s', \beta_s') \rangle$.

For non-zero time lag, the time-dependence in $\mu_n(\rho_s, \tau)$ is introduced by time lag dependent term $C_{C, s}(\rho_s, z_x, z_x', m, n, \tau)$, which is obtained by replacing $Cov_{ss}(s_{\rho_x z_x}, s_{\rho_x z_x'})$ with $Cov_{ss}(\rho_b, z_x, z_x', \tau)$ in $C_{C, s}(\rho_s, z_x, z_x', m, n)$. The temporal covariance of the scatter function density $Cov_{ss}(\rho_s, z_x, z_x', \tau)$ is derived for 2-D and 3-D scenarios respectively in the following sections. Then, we give the expressions of the $\mu_n(\rho_s, \tau)$ by following the derivation of $\mu(\rho_s, \tau = 0)$ in Sec. III B of Ref. [4].

**Inhomogeneities fully correlated within the Fresnel width (2-D)**

Here, the cross-range coherence length $\ell_y$ of the random inhomogeneity is greater than the Fresnel width $Y_F$, which corresponds to an effectively 2-D scattering process for forward scatter. The temporal and spatial correlation of the scatter function density
is given by

\[
\langle s_{\mathbf{r},t}(\alpha, \beta, \alpha_i, \beta_i) s_{\mathbf{r}',t'}^*(\alpha', \beta', \alpha'_i, \beta'_i) \rangle
\]
\[
\approx \ell_z(\rho_s, z_{\chi}, z_{\chi'}) \left[ \langle s_{\rho_s, z_{\chi}, t}(\alpha, \beta, \alpha_i, \beta_i) s_{\rho_s, z_{\chi'}, t'}^*(\alpha', \beta', \alpha'_i, \beta'_i) \rangle - \langle \rho_s, z_{\chi}, t(\alpha, \beta, \alpha_i, \beta_i) \rangle \langle \rho_s, z_{\chi'}, t'(\alpha', \beta', \alpha'_i, \beta'_i) \rangle \right] \delta(x_{\chi} - x_{\chi'})
\]
\[
+ \langle \rho_s, z_{\chi}, t(\alpha, \beta, \alpha_i, \beta_i) \rangle \langle \rho_s, z_{\chi'}, t'(\alpha', \beta', \alpha'_i, \beta'_i) \rangle
\]
\[
= \ell_z(\rho_s, z_{\chi}, z_{\chi'}) \text{Cov}_{s8}(\rho_s, z_{\chi}, z_{\chi'}, \tau) \delta(x_{\chi} - x_{\chi'}) + \langle \rho_s, z_{\chi}, t(\alpha, \beta, \alpha_i, \beta_i) \rangle \langle \rho_s, z_{\chi'}, t'(\alpha', \beta', \alpha'_i, \beta'_i) \rangle,
\]

(4.18)

where \( \ell_z \) is the coherence length[4] of the random inhomogeneity in the direction of acoustic propagation. The temporal covariance of the scatter function density

\[
\text{Cov}_{s8}(\rho_s, z_{\chi}, z_{\chi'}, \tau = t - t')
\]
\[
= \langle \rho_s, z_{\chi}, t(\alpha, \beta, \alpha_i, \beta_i) s_{\rho_s, z_{\chi'}, t'}^*(\alpha', \beta', \alpha'_i, \beta'_i) \rangle - \langle \rho_s, z_{\chi}, t(\alpha, \beta, \alpha_i, \beta_i) \rangle \langle \rho_s, z_{\chi'}, t'(\alpha', \beta', \alpha'_i, \beta'_i) \rangle
\]

(4.19)

depends only on the time lag \( \tau = t - t' \) not on the absolute time \( t \) and \( t' \) as a consequence of the temporal stationarity assumption for inhomogeneities in the slab.

The temporal variance coefficient only depends on of \( \ell_z \) under 2-D scattering scenario

\[
\mu_n^{2-D}(\rho_s, \tau) = \sum_m \frac{1}{|\xi_m|} \int_0^\infty \! d\tilde{z}_{\chi} \int_0^\infty \! d\tilde{z}_{\chi'} \frac{\ell_z(\rho_s, \tilde{z}_{\chi}, \tilde{z}_{\chi'})}{\xi_m}
\]
\[
4\pi^2 \frac{k(\tilde{z}_{\chi})k(\tilde{z}_{\chi'})d(\tilde{z}_{\chi})d(\tilde{z}_{\chi'})}{C_{s,s}(\rho_s, \tilde{z}_{\chi}, \tilde{z}_{\chi'}, m, n, \tau)
\]

(4.20)
Inhomogeneities uncorrelated within Fresnel width

Here, $\ell_y < Y_F$ so that inhomogeneities contained within the Fresnel width lead to a 3-D scattering process and are uncorrelated, we have

$$
\langle s_{r_x,t}(\alpha, \beta; \alpha_i, \beta_i) s_{r_x',t'}^{*}(\alpha', \beta'; \alpha_i', \beta_i') \rangle \\
\approx A_c(\rho_s, z_x, z_x') \left[ \langle s_{\rho_s,z_x,t}(\alpha, \beta; \alpha_i, \beta_i) s_{\rho_s,z_x,t'}^{*}(\alpha', \beta'; \alpha_i', \beta_i') \rangle - \langle s_{\rho_s,z_x,t}(\alpha, \beta; \alpha_i, \beta_i) \rangle \langle s_{\rho_s,z_x,t'}^{*}(\alpha', \beta'; \alpha_i', \beta_i') \rangle \right] \delta(\rho_x - \rho_x') \\
+ \langle s_{\rho_s,z_x,t}(\alpha, \beta; \alpha_i, \beta_i) \rangle \langle s_{\rho_s,z_x,t'}^{*}(\alpha', \beta'; \alpha_i', \beta_i') \rangle \\
= A_c(\rho_s, z_x, z_x') \text{Cov}_{ss}(\rho_s, z_x, z_x', \tau) \delta(\rho_x - \rho_x') + \langle s_{\rho_s,z_x,t}(\alpha, \beta; \alpha_i, \beta_i) \rangle \langle s_{\rho_s,z_x,t'}^{*}(\alpha', \beta'; \alpha_i', \beta_i') \rangle,
$$

(4.21)

where $A_c(\rho_s, z_x, z_x')$ is the coherence area of the inhomogeneities[4].

The temporal covariance is a function of $A_c(\rho_s, z_x, z_x')$ under 3-D scattering scenario

$$
\mu^{3-D}_n(\rho_s, \tau) = \sum_m \sqrt{\frac{\rho}{2\pi\xi_m N_s(\rho - \rho_s) \lambda}} \int_{0}^{\infty} d\rho_x \int_{0}^{\infty} d\rho_x' A_c(\rho_s, z_x, z_x') \frac{4\pi^2}{k(z_x)k(z_x')d(z_x)d(z_x')}
$$

(4.22)

In summary, the temporal and spatial correlation of the scatter function density $\langle s_{r_x,t}(\alpha, \beta; \alpha_i, \beta_i) s_{r_x',t'}^{*}(\alpha', \beta'; \alpha_i', \beta_i') \rangle$ is expressed to be a function of the temporal covariance of the scatter function density $\text{Cov}_{ss}(\rho_s, z_x, z_x', \tau)$, which accounts for slow time variations of inhomogeneities via time lag $\tau$. 

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4.3 Statistics of scatter function density of internal wave inhomogeneities in a deep ocean environment

In this section, we derive expressions for the mean and temporal correlation of the scatter function density when random inhomogeneities are internal waves in a deep ocean environment. The scatter function density for a coherent volume of internal wave inhomogeneity centered at horizontal location $\rho_s$ is expressed in terms of the compressibility fraction $\Gamma_\kappa$ and the density fraction $\Gamma_\rho$ by applying the Rayleigh-Born approximation to Green's theorem, as shown in Sec. B of Ref. [68],

$$s_{\rho_s,z_x,t}(\alpha, \beta, \alpha_i, \beta_i) = \frac{1}{A_c(\rho_s, z_x)} \int \int \frac{k^3}{4\pi} \left[ \Gamma_\kappa(\rho_x, z_x, t) + \eta(k, k_i) \Gamma_d(\rho_x, z_x, t) \right] e^{i(\xi - \xi') \cdot u_x} d\xi' d^2u_x,$$

(4.23)

where $\rho_x = \rho_s + u_x$, and

$$\eta(k, k_i) = \frac{k_i \cdot k}{k^2} = \cos \alpha \cos \alpha_i \cos \alpha_i + \sin \alpha \sin \cos(\beta_i - \beta)$$

is the cosine of the angle between the incident and scattered plane wave directions.

The mean scatter function density is

$$\langle s_{\rho_s,z_x,t}(\alpha, \beta, \alpha_i, \beta_i) \rangle = \frac{1}{A_c(\rho_s, z_x)} \int \int \frac{k^3}{4\pi} \left[ \Gamma_\kappa(\rho_x, z_x, t) + \eta(k, k_i) \Gamma_d(\rho_x, z_x, t) \right] e^{i(\xi - \xi') \cdot u_x} d\xi' d^2u_x \int_{A_c}$$

(4.24)
and the temporal correlation of the scatter function is

\[
\langle s_{\rho, z, t}(\alpha, \beta, \alpha', \beta') \rangle \quad \text{s}_{\rho, z, t}(\alpha, \beta, \alpha', \beta')
\]

\[
= \frac{1}{A(\rho_s, z_t) A(\rho_s, z_t')} \int \int \int e^{i(\xi - \xi') \cdot (u - u')} \text{Cov}_{s, s'}(\rho_s, \rho_s', z_t, z_t', t, t') \, du \, du' \, d\xi \, d\xi',
\]

(4.25)

where

\[
\text{Cov}_{s, s'}(\rho_s, \rho_s', z_t, z_t', t, t')
\]

\[
= \left( \frac{k^3}{4\pi} \right)^2 \left[ \langle \Gamma_s(\rho_s, z_t, \xi) + \eta(\mathbf{k}, \mathbf{k}_t) \Gamma_d(\rho_s, z_t, \xi) \rangle \right] \left[ \langle \Gamma_s(\rho_s', z_t', \xi') + \eta(\mathbf{k}, \mathbf{k}_t) \Gamma_d(\rho_s', z_t', \xi') \rangle \right]
\]

(4.26)

is the temporal and spatial correlation of intrinsic scattering properties. To calculate Eq. 4.24 and Eq. 4.25, the statistical moments of fractional changes in compressibility \( \Gamma_s \) and density \( \Gamma_d \) are required.

Since the fluctuations of sound speed (\( \Delta c \)) and density (\( \Delta d \)) arising from random internal waves are much smaller than the unperturbed or local equilibrium sound speed and density, the fractional change of compressibility \( \Gamma_s \) and density \( \Gamma_d \) can be expanded up to second order in Taylor series

\[
\Gamma_s \approx \left[ -\frac{2\Delta c}{c_0} - \frac{\Delta d}{d_0} \right] + \left[ 3\left( \frac{\Delta c}{c_0} \right)^2 + \left( \frac{\Delta d}{d_0} \right)^2 + 2\frac{\Delta c \Delta d}{c_0 d_0} \right],
\]

\[
\Gamma_d \approx \frac{\Delta d}{d_0} - \left( \frac{\Delta d}{d_0} \right)^2.
\]

(4.27)

Fluctuations of sound speed and density, for practical purpose, are linearly dependent on the displacement of the internal wave\([46, 70]\) \( \xi(\rho_s, z_t, t) \) via,

\[
\frac{\Delta c(\rho_s, z_t, t)}{c_0} = \xi(\rho_s, z_t, t) \bar{G}(z_t)n^2(z_t)
\]

\[
\frac{\Delta d(\rho_s, z_t, t)}{d_0} = \xi(\rho_s, z_t, t) g^{-1}n^2(z_t)
\]

(4.28)
where $\tilde{G}(z\chi)$ is a function of the potential temperature and salinity\cite{46}, $g$ is the gravitational constant and $n(z\chi)$ is the buoyancy frequency or Brunt - Väisälä frequency,

$$n^2(z\chi) = -gd^{-1}\frac{\partial d_p(z\chi)}{\partial z\chi}, \quad (4.29)$$

where $d_p(z\chi)$ is the potential density\cite{46}.

The displacement $\xi$ of internal-waves at location $(\rho, z\chi)$ and time $t$ is taken to be a zero-mean Gaussian random variable\cite{46}

$$\xi(\rho, z\chi, t) = \sum_j \int H(\sigma, j)W_j(\sigma, z\chi)\exp(i[\sigma \cdot \rho - \Omega(\sigma, j)t])d^2\sigma, \quad (4.30)$$

where $H(\sigma, j)$ is a zero-mean Gaussian random variable that specifies the $j$th modal amplitude of internal waves at wave-number $\sigma$, and $W_j(\sigma, z\chi)$ is the $j$th modal shape of internal waves at depth $z\chi$. Internal wave angular frequency $\Omega$ is related to the magnitude of internal wave wave-number $\sigma$ via the dispersion relation given in Ref. [46].

Assuming the internal wave field follows a stationary random process in the horizontal space and in time, the spatial and temporal covariance of internal wave displacements at two measurement points $r_x(\rho, z\chi)$ and $r'_x(\rho, z\chi')$ and two times $t$ and $t'$, can be expressed as a function of the horizontal separation $R = \rho - \rho'$ and time lag $\tau = t - t'$\cite{63, 64, 46}, we have

$$\text{Cov}_{\xi}(R, \tau, z\chi, z\chi')$$

$$= \langle \xi(\rho, z\chi, t)\xi(\rho', z\chi', t') \rangle - \langle \xi(\rho, z\chi, t) \rangle \langle \xi(\rho', z\chi', t') \rangle$$

$$= \sum_j \int F_j(\sigma)W_j(\sigma, z\chi)W_j(\sigma, z\chi')\exp(i[\sigma \cdot R - \omega(\sigma, j)\tau])d^2\sigma, \quad (4.31)$$

where $F_j(\sigma) = \langle |H(\sigma, j)|^2 \rangle$ is the Garrett-Munk (GM) spectrum of internal waves, whose expressions and parameters can be found in page 56 of Ref. [46].

After inserting Eq. 4.28 into Eq. 4.27, and Eq. 4.27 into Eq. 4.26, the temporal
and spatial covariance of intrinsic scattering properties becomes

\[
\text{Cov}_{\mathcal{F}}(\mathbf{\rho}_x, \mathbf{\rho}_x', z_x, z_x', t, t') = \frac{k^6}{(4\pi)^2} h(z_x) h(z_x') \left\langle \xi(\mathbf{\rho}_x, z_x, t) \xi^* (\mathbf{\rho}_x', z_x', t') \right\rangle
\]

\[
= \frac{k^6}{(4\pi)^2} h(z_x) h(z_x') \text{Cov}_{\xi \xi}(R, \tau, z_x, z_x'),
\]

(4.32)

where \( h(z_x) = (2 \hat{G}(z_x) + \frac{1-\eta}{s}) n^2(z_x) \).

We approximate displacements of internal-waves at two horizontal positions \( \mathbf{\rho}_x \) and \( \mathbf{\rho}_x' \) within the coherence area \( A_c \) as being fully correlated[4, 68] such that

\[
\text{Cov}_{\mathcal{F}}(\mathbf{\rho}_x, \mathbf{\rho}_x', z_x, z_x', t, t') \approx \frac{k^6}{(4\pi)^2} h(z_x) h(z_x') \text{Cov}_{\xi \xi}(0, \tau, z_x, z_x').
\]

(4.33)

The mean scatter function density is found to be proportional to the second moment of the internal wave displacement from Eq. 4.24, C.5 and 4.30, such that Eq. 4.19 becomes

\[
\text{Cov}_{ss}(\rho_s, z_x, z_x', \tau) \approx \left\langle s_{\rho_s, z_x, t}(\alpha, \beta, \alpha_i, \beta_i) s^*_{\rho_s, z_x', t'}(\alpha', \beta', \alpha_i', \beta_i') \right\rangle.
\]

(4.34)

This is because the square of mean scatter function density proportional to the fourth power of internal wave displacement, is much smaller than \( \left\langle s_{\rho_s, z_x, t} s^*_{\rho_s, z_x', t'} \right\rangle \) which is on the order of the second moment of internal wave displacement.

Equation. 4.34 can be expressed as

\[
\text{Cov}_{ss}(\rho_s, z_x, z_x', \tau) \approx \frac{k^6 h(z_x) h(z_x')}{(4\pi)^2 A_c(z_x) A_c(z_x')} \text{Cov}_{\xi \xi}(0, \tau, z_x, z_x')
\]

\[
\int A_c \int A_c' \int e^{i(\xi, -\xi')(u_x - u_x')} d^2 u_x d^2 u_x'
\]

(4.35)

by substituting Eq. 4.31 into Eq. 4.33, then Eq. 4.33 into Eq. 4.25. By replacing
\[ \text{Cov}_{ss}(s_{\rho s, z}, s_{\rho s, z'}) \] with \( \text{Cov}_{ss}(\rho_s, z, z') \) in Eq. 72 of Ref. [4], we have

\[ C_{s,s}(\rho_s, z, z', m, n, \tau) = u_n(z)u^*_n(z')u_m(z)u^*_m(z')\text{Cov}_{ss}(\rho_s, z, z', \tau), \quad (4.36) \]

which leads to \( \mu_n(\rho_s, \tau) \) by inserting Eq. 4.36 into Eq. 4.20.

A purely real modal horizontal wavenumber change \( \nu_n(\rho_s) \) is obtained by substituting the mean scatter function density of Eq. 4.24 into Eq. 60a, Ref. [4]. This only accounts for dispersion, but not attenuation in the mean forward field. Assuming no power loss within the Fresnel region or in the forward direction for a 2-D scattering process, the depth-integrated intensity at zero time lag \( \langle W_T(\rho|r_0, \tau = 0) \rangle \) must equal the depth-integrated incident intensity \( \sum_n W_i^{(n)} \) of Eq. 4.15. This requires

\[ \Im(\nu_n^{2D}(\rho_s)) = \frac{1}{2} \mu_n^{2D}(\rho_s, \tau = 0). \quad (4.37) \]

Out-of-plane scattering becomes important in 3-D scenarios and leads to power loss in the forward direction, which requires an imaginary part in \( \nu_n(\rho_s) \). We apply the waveguide extinction theorem[10, 11] to calculate \( \Re(\nu_n(\rho_s)) \) in Eq. 36 of Ref. [68].

### 4.4 2-D and 3-D scattering processes

The Fresnel width in cross-range[4] is defined to be where the incident and forward scattered fields are highly coherent and have phase difference less than \( \pi/4 \). The Fresnel width, \( Y_F(\rho, \rho_s) = \sqrt{\frac{M(\rho - \rho_s)}{\rho}} \), depends on the range of the source, receiver and the scatterer[4, 68], where \( \lambda \) is the acoustic wave length. The maximum Fresnel width[4, 68] \( Y_{F_{max}}(\rho, \rho_s) = \sqrt{\lambda / 4} \) occurs at the mid-point between the source and receiver. When \( Y_F < \ell_y \), an internal wave inhomogeneity is correlated within the Fresnel width, which leads to an effective 2-D scattering process.

As the receiver range increases, \( Y_F \) exceeds \( \ell_y \), 3-D scattering initiates and uncorrelated internal-wave inhomogeneities appear within the Fresnel region. This leads to out-of-plane scattering that causes additional power loss in the forward direction.
4.5 Illustrative examples

4.5.1 Deep ocean waveguide

Here, a water column of $H = 4000$ m depth is used to simulate the geometry of a deep ocean waveguide. The bottom sediment half-space is composed of sand with density $d_b = 1.9g/cm^3$ and sound speed $c_b = 1700$ m/s. The attenuation coefficients in the water column and bottom are $\alpha = 6 \times 10^{-5}$ dB/\(\lambda\) and $\alpha_b = 0.8$ dB/\(\lambda\), respectively. A point source transmits acoustic waves at a frequency of 75 Hz. Both the source and the point receiver are located at 1000 m depth. We use a sound speed profile calculated from historical temperature and salinity data[54] corresponding to the Pacific ocean region between latitude 20°N and 31°N and longitude 123°W and 154°W, as shown in Fig. 4-2.

Figure 4-2: The sound speed profile in Pacific ocean region between latitude 20°N and 31°N and longitude 123°W and 154°W. The depth of sound-channel axis is roughly 700m.

The temporal correlation function of internal wave displacements at the depth of the sound channel-axis is shown in Fig. 4-3 following Eq. 4.31. The e-folding correlation time scale of the internal wave field is seen to be approximately 4 hours at that depth. As water depth increases, the coherence time scale of internal waves also increases because internal wave displacements decrease.
Figure 4-3: Temporal correlation function of Internal wave displacement at 700m as a function of time lag. The temporal correlation is the normalized spatial and temporal covariance of Eq. 28 at zero horizontal separation. The e-folding coherence time is approximately 4 hours. The solid horizontal line is plotted at the correlation function value $\frac{1}{e}$.

![Temporal correlation function of internal wave displacement at 700m depth](image)

The temporal correlation of depth-integrated intensity and acoustic forward field are shown in Fig. 4 following Eq. 4.15 and Eq. 4.16 respectively.

Uncertainty in internal wave energy level leads to variations in e-folding correlation time scale of depth-integrated intensity and forward scattered acoustic field. At three typical Garrett-Munk internal wave energy levels, for example, the e-folding correlation time scales vary between 7 to 14 minutes as also shown in Fig. 4. These time scales are more than an order of magnitude smaller than that of the internal waves. Scattering from a single slab containing internal waves causes only a small change in the temporal correlation of depth-integrated intensity. This is due to the weak scattering from a single slab and the very long time scale of the internal waves with respect to acoustic travel time through the slab. As acoustic power, or the depth-integrated intensity, is propagated through a series of uncorrelated slabs of inhomogeneities over range, accumulated multiple scattering dramatically degrades temporal correlation. This leads to coherence time scales for acoustic power and the forward field that are much shorter than that of the internal wave field. An assump-
Figure 4-4: Temporal correlation function of acoustic power and forward field for 3250km source-receiver separation as a function of time lag for various possible GM energy levels. The temporal correlation of acoustic power and forward field are the normalized temporal covariance of Eq. 17 and Eq. 18 respectively. The e-folding coherence time varies between 7 to 14 minutes depending on internal wave energy level. The temporal correlation function of acoustic power and forward field are plotted with dashed and solid curves respectively. The solid horizontal line is plotted at the correlation function value $\frac{1}{e}$.

4.5.2 Continental shelf waveguide

Similar to the deep ocean waveguide, a water column of $H = 80$ m depth is used to simulate the geometry of a continental shelf environment. A point source transmits acoustic waves at a frequency of 415 Hz. Both the point source and receiver are located at 40 m depth. The measured sound speed profile in the New Jersey continental shelf is used in the simulation, as shown in Fig. 4-5.

The temporal correlation function of internal wave displacements at 40m depth where the source is located, is shown in Fig. 4-6. The e-folding correlation time scale of the internal wave field in New Jersey continental shelf environment is approximately 1 hours at that depth. The temporal correlation of depth-integrated intensity and
Figure 4-5: Measured sound speed profile in New Jersey continental shelf waveguide

acoustic forward are shown in Fig. 4-7.

Figure 4-6: Temporal correlation function of internal wave displacement at 40m as a function of time lag. The e-folding coherence time is approximately 1 hour.

Similar to the deep ocean scenario, three typical Garrett-Munk internal wave energy levels are used in the simulations. The uncertainty in internal wave energy levels causes the time scales vary between 2.5 to 5.5 min. These time scales are more than an order of magnitude smaller than that of the internal waves, which is similar to the deep ocean scenario.
Figure 4-7: Temporal correlation function of acoustic power and forward field for 30km source-receiver separation as a function of time lag for various possible GM energy levels. The e-folding coherence time varies between 2.5 to 5.5 minutes depending on internal wave energy level. The temporal correlation function of acoustic power and forward field are plotted with dashed and solid curves respectively.

4.5.3 Acoustic power loss due to 3-D scattering

Acoustic power loss due to scattering is

$$PL(\rho|r_0) = 10\log(W_T(\rho|r_0, \tau = 0)) - 10\log W_i(\rho|r_0), \quad (4.38)$$

where $\langle W_T(\rho|r_0, \tau = 0) \rangle$ is the total acoustic power of Eq. 4.15 at zero time lag and $W_i(\rho|r_0)$ is the incident acoustic power of Eq. 4.8. We plot the acoustic power loss as a function of receiver range and acoustic frequency in Fig. 5.

The 2-D and 3-D scattering regions are separated by a black line showing where $(Y_F)_{max} = \ell_y$. In the 2-D region, there is no power loss in the forward direction as assumed in Sec. 4.4. In the 3-D region, power loss monotonically increases with the source-receiver separation for fixed frequencies. At a given source-receiver separation, however, power loss reaches a maximum in the low frequency range. This is because, two competing factors determine the total power loss: $Y_F$ and $\mu_n^{3-D}(\rho_s, 0)$. As frequency decreases, $Y_F$ becomes larger leading to more uncorrelated internal wave
inhomogeneities in the Fresnel region. This leads to more out-of-plane scattering and power loss. However, the $f^{-6}$ Rayleigh-born scattering roll off for any individual inhomogeneity results in an $f^{-2.5}$ roll off in variance coefficient $\mu_n(\rho_s, 0)$ at zero time lag, as seen in Eq. 4.15. This leads to less power loss.

4.6 Conclusions

We have derived an analytical expression for the temporal covariance of depth-integrated intensity or acoustic power propagated through 3-D random inhomogeneities, from which the coherence time scale of field or power fluctuations can be determined. Knowledge of the coherence time scale is typically essential in ocean acoustic re-
mote sensing. This is because it determines (1) the number of fluctuations in a given measurement period, which determines the amount of error reduction possible by stationary averaging in any ocean acoustic remote sensing measurement, and (2) the time window within which the coherent processing essential to ocean acoustic remote sensing, such as matched filtering, beamforming, matched filter processing and synthetic aperture processing, can be conducted. We have provided a general and rapid way of estimating the coherence time scale to aid in the design of ocean acoustic experiments and the interpretation of experimental measurements. We show by analysis that the time scale of acoustic power fluctuations after mega-meter range propagation through internal waves in the deep ocean is roughly 10 minutes, which matches experimental measurements, and is more than an order of magnitude smaller than the 240 minute correlation time scale of the internal wave field. This discrepancy between acoustic and internal wave time scales is explained by the present theory as the accumulated effect of multiple forward scattering through internal waves on acoustic waves.

We find that power loss due to 3-D scattering from internal wave inhomogeneities becomes pronounced when the Fresnel width exceeds the cross range correlation length of internal waves. As source-receiver separation increases, Fresnel width increases and contains more uncorrelated internal wave inhomogeneities, which leads to a monotonic increase in power loss. For a given source-receiver separation and decreasing frequency, power loss first increases as internal wave incoherence accumulates within the Fresnel width, and then decreases due to weakening Rayleigh-born scattering for any individual inhomogeneity. This may explain the unexpectedly high attenuations historically observed below 100Hz[45, 65].
Chapter 5

Attenuation due to fish shoals and wind-generated bubble clouds after multiple forward scattering through an ocean waveguide

5.1 INTRODUCTION

The ability to accurately model acoustic transmission loss is essential in many applications of ocean acoustics such as estimating source range, source level and target strength of submerged objects. In many ocean environments, attenuation from natural environmental scatterers such as fish and bubbles generated by sea surface agitations may significantly affect transmission loss.

In this paper, we present an analytical model to account for the attenuation and dispersion effects of fish shoals on the acoustic transmission. We then compare our simulations with the only available data of acoustic wave propagation a continental shelf waveguide through herring whose presence is simultaneously confirmed using independent Conventional Fish Finding Sonar (CFFS).

We find that, when the acoustic wavelength is larger than the swimbladder,
modeled attenuation of acoustic field forward propagated through herring shoals is negligible at frequencies off swimbladder resonance. This is consistent with experimental data of acoustic transmission through herring shoals. At frequencies close to resonance of the swimbladder (roughly 700 Hz), our model shows that fish shoals may sometimes lead to measurable attenuation in the forward field. Unfortunately, we do not have accurately calibrated experimental data at these frequencies, to compare with our modeled results.

All these results are calculated based on an analytical expression of attenuation and dispersion, which is expressed in terms of spatial distribution, volume density and statistical moments of the scatter function density of fish shoals. To constrain physical parameters such as the spatial distribution and volume density, we use both low frequency Ocean Acoustic Waveguide Remote Sensing (OAWRS) and high frequency CFFS data obtained during Main Acoustic Experiment 2003 [88]. To formulate the scatter function of swimbladder, we use Love’s model[73] assuming that the swimbladder and flesh surrounding it can be modeled as an air cavity and a viscous fluid, respectively.

Qiu et. al. find an additional attenuation of 3.3 dB/km for one-way acoustic transmission at 1300 Hz. They attribute this attenuation to anchovy. However, the presence of fish is mere speculation and lacks any simultaneous confirmation by an independent sensing system. Diachok et. al[98] use an empirical model to describe attenuation due to fish shoals by introducing an effective attenuation layer of certain thickness in the water column. The physical parameters of this layer such as thickness and the attenuation coefficient are estimated by matching the model with the experimental data. In contrast, our model is derived from Green’s theorem and gives physical insights into multiple scattering processes that lead to attenuation and dispersion effects.

In this paper, we also provide analytical expressions of mean, variance and second moment of the acoustic field forward propagated though bubble clouds for a given wind speed. We assume that (1) the positions and radii of bubbles are random and independent of each other and (2) the volume density and depth distribution of
bubble clouds are fully determined by the wind speed. The conditional statistical moments for a given wind speed are then used to calculate the mean and standard deviation of acoustic intensity under random wind speeds. In the simulations, we apply these expressions to calculate the fluctuations of acoustic intensity forward propagated though bubble clouds in a shallow water waveguide. We find that the fluctuation of acoustic intensity could be significant even under normal wind speed distributions. Simulations of the second moment of acoustic field given a wind speed based on our analytical model match with the ones from Monte-Carlo simulations by Norton et al.

Previous work discussing the effect of bubbles on the acoustic propagation in both free space and ocean waveguide focused on using ray theory and effective medium method. Urick and Ament[75] derived a complex propagation constant (wave-number change) by calculating the acoustic wave forward propagated through a slab, which containing uniformly distributed small spherical scatterers in a boundary-free medium. Command and Prosperetti[76] computed an attenuation coefficient for a medium containing uniformly distributed bubbles in free space. Sarkar and Prosperetti[78] applied WKB approximation to calculate the coherent (mean) and incoherent (variance) field in a half space medium containing bubbles. They, however, do not give explicit analytical expressions for the statistical moments of the forward field. Both ref[76] and ref [78] use ray theory, which may not be accurately describe the acoustic propagation in a continental shelf environment. Norton[79] calculated the transmission loss of the acoustic forward propagation through bubble clouds using Monte-Carlo simulations based on Parabolic equation method, which may require intensive computation.

Our analytical expressions for the attenuation and dispersion due to bubble clouds are derived based on normal mode method and depend on the statistics of scatter function density of bubble clouds. They can be directly applied to various ocean environments and are not restricted to particular spatial distribution of bubbles. Different from Monte-Carlo simulations based on PE, our expressions give insights into the multiple forward scattering leading to attenuation and dispersion. Our expressions also provide a quick and intuitive way to estimate the total fluctuation of acoustic inten-
sity due to the randomness of the position, radii and volume density of bubbles. To our knowledge, previous works calculate the statistics of acoustic intensity only for a given wind speed or a deterministic volume density of bubbles.

5.2 Statistics of acoustic field and intensity forward propagated through fish shoals and bubble clouds

In this section, we will give the analytical expression for mean, variance and second moment of the forward field propagated through a waveguide containing fish shoals and random bubble clouds. To formulate the problem, we place the origin of the coordinate system at the sea surface. The z-axis points downward and normal to the interface between horizontal strata. The water depth is $H$. Let coordinates of the source be defined by $r_0 = (0, 0, z_0)$, and receiver coordinates by $r = (x, 0, z) = (\rho, 0, z)$. Spatial cylindrical $(\rho, \phi, z)$ and spherical systems $(r, \theta, \phi)$ are defined by $x = r \sin \theta \cos \phi, \ y = r \sin \theta \sin \phi, \ z = r \cos \theta$ and $\rho = x^2 + y^2$. The horizontal and vertical wavenumber components for the $n$th mode are respectively $\xi_n = k \sin \alpha_n$ and $\gamma_n = k \cos \alpha_n$, where $\alpha_n$ is the elevation angle of the mode measured from the $z$-axis. Here, $0 \leq \alpha_n \leq \pi/2$ so that the down and upgoing plane wave components of each mode will then have elevation angles $\alpha_n$ and $\pi - \alpha_n$ respectively. The corresponding vertical wavenumber of the down and upgoing components of the $n$th mode are $\gamma_n$ and $-\gamma_n$ respectively, where $\Re \{\gamma_n\} \geq 0$. The wavenumber magnitude $k$ equals the angular frequency $\omega$ divided by the sound speed $c$ in the object layer so that $k^2 = \xi_n^2 + \gamma_n^2$. The azimuth angle of the modal plane wave is denoted by $\beta$, where $0 \leq \beta \leq 2\pi$. The geometry of spatial and wavenumber coordinates is shown in Fig. 2 of Ref. [39].

A general expression of the mean field forward propagated through random 3D inhomogeneities has been derived in Eq. 83 of Ref. [72]. We apply this analytical expression to find the mean forward field propagated through fish shoals and bubble
clouds in an ocean waveguide as

\[
\langle \Psi_T(r|r_0) \rangle = \sum_n \Psi^{(n)}_i(r|r_0) e^{i \int_{\rho_b}^{\rho_e} \nu_n(\rho) d\rho},
\]  

(5.1)

where

\[
\Psi^{(n)}_i(r|r_0) = 4\pi \frac{i}{d(z_0) \sqrt{8\pi}} e^{-i\pi/4} u_n(z) u_n(z_0) e^{i\xi_n \rho}
\]

is the incident field of the \(n^{th}\) mode in a static waveguide and \(u_n(z)\) is the modal amplitude at depth \(z\). The fish shoals and bubble clouds are assumed to be spatially confined between a horizontal range of \(\rho_b\) and \(\rho_e\).

The complex horizontal wavenumber change \(\nu_n\) accounts for the dispersion and attenuation effect due to the accumulated \textit{multiple forward scattering} from fish shoals and bubble clouds. The complex horizontal wavenumber change \(\nu_n\) for fish shoals and bubble clouds arbitrarily distributed in depth is obtained by applying Eqs. 60a and A14 of Ref. [72]

\[
\nu_n(\rho_s) = \int_0^H \frac{2\pi}{kd(z_t) \xi_n} |u_n(z_t)|^2 \langle s(\rho_s, z_t) \rangle dz_t
\]

\[
= \int_0^H \frac{2\pi}{kd(z_t) \xi_n} |u_n(z_t)|^2 \langle n_v(\rho_s, z_t) \rangle \langle S(\rho_s, z_t) \rangle dz_t,
\]

(5.2)

where \(s(\rho_s, z_t)\) is the scatter function density of fish shoals and bubble clouds at horizontal position \(\rho_s\) and depth \(z_t\), \(\langle n_v(\rho_s, z_t) \rangle\) is mean volume density, \(\langle S(\rho_s, z_t) \rangle\) is the mean scatter function of single swimbladder and bubble that will be discussed in Appendix A. The random variable \(n_v(\rho_s, z_t)\) is due to the randomness of volume density at the surface and \textit{efolding} depth, where both of them could be modeled to be fully determined by the wind speed. The randomness of \(S(\rho_s, z_t)\) is due to random bubble radii. The effect of the random position of individual fish and bubble is included when calculating the mean forward field expressed in Eq. 5.1.

The expression for the variance of the forward field propagated through fish shoals
and bubble clouds is

\[
\text{Var}(\Psi_T(r|r_0)) = \sum_n^{2\pi} \frac{d^2(z_0)}{|\xi_n|} |\mu_n(\rho_s)|^2 \left[ |u_n(z_0)|^2 |u_n(z)|^2 e^{-2\pi \xi_n(\rho_s)} \right]
\]

\[
e^{-2\pi \xi_n(\rho_s)} \int_0^{2\pi} |\mu_n(\rho_s)|^2 e^{-2\pi \xi_n(\rho_s)} d\rho_s - e^{-2\pi \xi_n(\rho_s)} \int_0^{2\pi} |\mu_n(\rho_s)|^2 e^{-2\pi \xi_n(\rho_s)} d\rho_s,
\]

(5.3)

by applying Eq. 84 of Ref. [72], where \( \mu_n(\rho_s) \) is the exponential coefficient of modal field variance, which is expressed as

\[
\mu_n(\rho_s) = \sum_{m=1}^{\infty} \sqrt{\frac{\rho_s}{2\pi}} \left( \frac{\rho}{\xi_m} \right) \int_0^\infty dz_t \frac{4\pi^2}{k^2 d^2(z_t)} |u_n(z_t)|^2 |u_m(z_t)|^2 V_c(z_t) \text{Var}(s(\rho_s, z_t)),
\]

(5.4)

by applying Eqs. 72, 94a and A24 of Ref. [72], where \( \text{Var}(\rho_s, z_t) = \frac{1}{V_c(z_t)} \text{Var}(\rho_s, z_t) + \text{Var}(n_v(\rho_s)) |s(\rho_s, z_t)|^2 \) is the variance of scatter function density of fish shoals and bubble clouds at position \( \rho_s, z_t \). The variance of the scatter function \( \text{Var}(s(\rho_s, z_t)) \) and will be discussed in Appendix A. The coherence volume of the scattered function density of fish shoals and bubble clouds \( V_c \) quantifies the spatial scale over which the scatter functions of two scatterers are correlated. Since the position and radii of individual fish swimbladder or bubble are independent of others, \( V_c \) is essentially the physical dimension of each scatterer. If the number density \( n_v \) is deterministic under a given wind speed, the term containing \( \text{Var}(n_v(\rho_s)) \) is zero. This leads to both \( \mu_n(\rho_s) \) of Eq. 5.4 and \( \text{Var}(\Psi_T(r|r_0)) \) of Eq. 5.3 to be independent of \( V_c \).

The mean intensity or the second moment of the forward field is the sum of the coherent intensity (mean field) and incoherent intensity (variance field)

\[
\langle |\Psi_T(r|r_0)|^2 \rangle = \langle |\Psi_T(r|r_0)|^2 \rangle + \text{Var}(\Psi_T(r|r_0)).
\]

(5.5)

The attenuation of the acoustic forward field is

\[
\text{Att} = |\Psi_T(r|r_0)|^2 - \langle |\Psi_T(r|r_0)|^2 \rangle
\]

(5.6)

Eq. 5.1 and Eq. 5.5 provide the conditional mean and second moment of acoustic field for a given wind speed. The conditional variance field of acoustic field expressed
in Eq. 5.3 quantifies the fluctuation of acoustic field due to the randomness of position and radii of bubbles. In an ocean environment, the wind speed is a random variable, we calculate the mean and standard deviation of acoustic intensity in decibel level to include the effect due to the randomness of wind speed. The mean intensity is

\[
\langle I(r|r_0) \rangle = \int_0^\infty 10\log_{10}\left(<\|\psi_T(r|r_0,W)\|^2>\right)p(W)\,dW
\]  

(5.7)

and the second moment of the acoustic intensity

\[
\langle I(r|r_0)^2 \rangle = \int_0^\infty \left[10\log_{10}\left(<\|\psi_T(r|r_0,W)\|^2>\right)\right]^2p(W)\,dW.
\]  

(5.8)

5.3 Attenuation of the forward field propagated through shoals of fish in a continental shelf

5.3.1 Attenuation due to fish shoals in New Jersey continental shelf environment

In this section, we will calculate the attenuation of the acoustic forward field propagated through fish shoals in the New Jersey continental shelf environment. Here, water depths of 85 m with a flat bottom is used to simulate the New Jersey continental shelf, as shown in Figs. 5-1. The bottom sediment half space is composed of sand with a sound speed of 1700 m/s. The attenuation in the water column and the bottom are \( \alpha = 6 \times 10^{-5}\text{dB/\lambda} \) and \( \alpha = 0.8\text{dB/\lambda} \), respectively.

During the Main Acoustic Experiment 2003 in the New Jersey continental shelf, a massive shoal of fish was instantaneously imaged using OAWRS, as shown in Fig 1 of Ref. [71]. Images obtained by OAWRS show that (1) the distance between the center of the shoal of fish and the source is approximate 12.5 km, and (2) the fish shoals span approximately 1 km and 5 km over range and cross-range direction. Echograms from Conventional Fish Finding Sonar (CFFS) in Fig 2 of Ref. [71] show that the fish shoals are mostly distributed between water depths of 75 m and 85 m.
Figure 5-1: The geometry of the source, the distribution of fish shoals and the measured sound speed profile in the New Jersey continental shelf environment. The receivers are located in the water column between 75 m to 85 m. In the simulations, the dimension of the fish shoals is assumed to be 1 km over range direction and larger than the cross-range resolution of OAWRS. The center of fish shoals is 12.5 km away from the source.

with averaged volume density 0.05 fish/m³. It is deduced that the shoals are mostly composed of Atlantic herring.[88] The length distribution of herring is deduced from in-situ target strength (TS) measurements made by CFFS at 38 kHz.[88] It is found that the mean and standard deviation of herring length is 28.6 cm and 4.29 cm respectively. Additionally, a neutral buoyancy depth of 78 m is found to best fit OAWRS target strength estimates[88] and consists with the echograms from CFFS. In the simulations, the dimensions of the fish shoals are assumed to be 1 km over range direction and larger than the cross-range resolution of OAWRS. The shoals of herring are assumed to be uniformly distributed over both range and depth. The source depth is 40 m and the receivers span over the whole fish layer. When calculating the attenuation of Eq. 5.6, the mean intensity of Eq. 5.5 is averaged over a 50Hz bandwidth and the depth of the receivers.
Fig. 5-2 shows the mean target strength of herring, where the resonant frequency of herring swim bladder is approximate 700 Hz. Fig. 5-3 shows the ratio between the extinction cross section $\sigma_{\text{ext}}$ and the scattering cross section $\sigma_{\text{sca}}$ in Eq. D.21 as a function of frequency. Since the attenuation is proportional to the extinction cross section, including and excluding the absorption effect greatly affects the attenuation especially at low frequencies. The attenuation coefficient of the acoustic forward field is shown in Fig. 5-4. The maximum attenuation coefficient is 5 dB/km and 1.7 dB/km that occurs at 750Hz when including and excluding absorption effect, respectively. We believe the actual attenuation coefficient could be a value between these two scenarios and Fig. 5-4 provides an upper and lower bound of the attenuation coefficient.

![Figure 5-2: Target strength data corresponding to the average scattering response of a shoaling fish in OAWRS 2003 constrained by local CFFS is shown as black circles, with standard deviations in black error bars, from Symonds[99]. The best fit of data to modeled target strength is the black curve where the buoyancy depth is 78 m. The swim bladder resonance is rough 700 Hz based on the black curve.](image.png)

Fig. 5-5 shows a 2D image of the acoustic received intensity obtained from the experiment in the New Jersey continental shelf.[71]. It shows two big shoals located roughly 20 km and 25 km south of the source. In order to compare the received
Figure 5-3: The ratio of the extinction cross section $\sigma_{\text{ext}}$ (Eq. D.21) and the scattering cross section $\sigma_{\text{sca}}$ (Eq. D.17) of swim bladder as a function of frequency when including and excluding due to viscous fluid absorption modeled by Love[73]. When neglecting the viscous fluid absorption, the absorption cross section turns to be zeros and $\frac{\sigma_{\text{ext}}}{\sigma_{\text{sca}}}$ equals to 1.

Intensity in the presence and absence of fish shoals, we pick two sectors, S1 and S2, bounded by the black lines shown in Fig. 5-5. Fig. 5-6 shows the comparison of the received acoustic intensity, averaged over S1, and the background reverberation level averaged over S2, at 415 Hz. If there was significant attenuation due to the fish shoal, we would be able to observe a sudden trend change in the received acoustic intensities before and after the fish shoal. We do not observe such a change in trend in Fig. 5-6, which leads us to believe that there is no significant attenuation due to the fish shoal. This is consistent with the theoretical prediction of the attenuation at 415 Hz (maximum 0.5 dB/km) shown in Fig. 5-4.
Figure 5-4: The attenuation coefficients of the acoustic forward field propagated through shoals of herring in the New Jersey continental shelf environment when including and excluding the viscous fluid absorption. At resonance, the attenuation including absorption is approximately three times larger than the one excluding absorption. This can be explained by noting that the extinction cross section is approximately three times larger than the scattering cross section at resonance when including the absorption effect, as shown in Fig. 5-3.

5.3.2 Attenuation due to fish shoals in Gulf of Maine continental shelf environment

Similar to the New Jersey continental shelf environment, a water column of 180 m with a flat bottom is used to simulate Gulf of Maine continental shelf environment. The bottom sediment half space is composed of sandy silt with a sound speed of 1700 m/s. The attenuation in the water column and the bottom are $\alpha = 6 \times 10^{-5} \text{dB}/\lambda$ and $\alpha = 6 \times 10^{-5} \text{dB}/\lambda$ respectively. The length of herring is assumed to be a Gaussian random variable with a mean and standard deviation of 26 cm and 3.9 cm, respectively, in Gulf of Maine. In the simulations, the shoals of fish is assumed to be uniformly distributed over 1 km over range direction and be large enough to cover the sonar footprint over the cross-range direction. Shoals of fish are also assumed to be uniformly distributed
Figure 5-5: A 2D image of the acoustic received intensity obtained from the experiment in the New Jersey continental shelf 2003. S1 sector shows acoustic transect with presence of two big fish shoals that are located roughly 20 km and 25 km south of source. S2 sector shows the background reverberation returns.

over depth from 150m to 180m in Gulf of Maine continental shelf. The neutral buoyancy depth of herring in Gulf of Maine is 105m. The source depth is 40m and the receivers are located within the fish layer. The mean intensity is averaged over 50Hz bandwidth and receivers’s depth to smooth the attenuation.

Fig. 5-7 shows the attenuation of acoustic forward field in Gulf of Maine. The maximum attenuation coefficient is only 0.45dB/km and 1.05dB/km at 1.2kHz near swimbladder resonance when including and excluding absorption effect, respectively. These are much smaller than the attenuation coefficient in New Jersey continental shelf and may not be measurable in the experiments.
5.3.3 Attenuation due to anchovies in Yellow Sea

In this section, we calculate the attenuation due to anchovies in Yellow sea continental shelf. A water column of 40 m with a flat bottom is used to simulate the yellow sea environment. The measured sound speed profile shown in Fig. 2 of Ref. [97] is used in the simulation. The bottom sediment half space is composed of sandy silt with a sound speed of 1584m/s. The attenuation in the water column and the bottom are $\alpha = 6 \times 10^{-5}\text{dB}/\lambda$ and $\alpha = 0.8\text{dB}/\lambda$ respectively. Both the source and receiver are located at 7 m water depth. In the simulations, the length of anchovy is assumed to be a Gaussian random variable with a mean of 10.6 cm and a standard deviation of 1.0 cm estimated from Fig 7 of Ref. [97]. The shoals of anchovy are assumed to be uniformly distributed in the layer from 0 m to 20 m with areal density $1/m^2$. The fish shoals are also assumed to be continuously distributed.
Figure 5-7: The attenuation coefficient of the acoustic forward field propagated through shoals of herring in the Gulf of Maine continental shelf environment. The maximum attenuation coefficient is only 1.05 dB/km at swimbladder resonance that may not be measurable in the experiments.

between the source and receiver.

The attenuation of the acoustic forward field propagated through shoals of anchovy is shown in Fig. 5-8. The maximum attenuation is 2.2 dB/km and 0.1 dB/m when including and excluding absorption effect. This large attenuation difference is because the equivalent radius of anchovy swim bladder is only 0.4 cm and the absorption cross section is approximately an order magnitude higher than the scattering cross section, as noted in Eq. D.21.

Qiu[97] et al found an extra 3.3 dB/km attenuation coefficient of one-way transmission at 1300Hz in their experiment. They attributed this attenuation to fish (probably Yellow Sea anchovies). Our simulated attenuation coefficient, even including absorption effect, is still smaller than Qiu’s measurements. Since Qiu[97] et al do not provide any explanation of how to obtain their transmission loss curves that are used to estimate the attenuation, it maybe not possible to make a consistent
comparison between their results and our theoretical calculations.

Figure 5-8: The attenuation coefficients of the acoustic field forward propagated through shoals of anchovies in Yellow sea for both including and excluding viscous fluid absorption scenarios.

5.4 Effect of random bubble clouds on the acoustic forward propagation

In this section, we show the statistics of acoustic field and intensity forward propagated through random bubble clouds. We also show the attenuation on the acoustic forward field when including and excluding resonant bubbles and compare it with the experiment data from Weston[74] et al.
5.4.1 Statistics of acoustic field and intensity forward propagation

It is well known that wind can break surface gravity waves into whitecaps in an ocean. This process generates a large number of random bubble clouds. When an acoustic wave propagates through these bubble clouds, the absorption and scattering due to bubbles lead to attenuation and dispersion effect, and then cause random fluctuations of the acoustic signal. In this section, we calculate the attenuation of the forward field based on two different expressions of bubble spatial distribution.

Different investigators\[89, 90, 91, 92, 77, 93, 94\] provide various expressions for the spatial distribution of bubble clouds and the spectra of their radii due to different experimental conditions and methods.

Keiffer, Novarini and Norton\[93\] give an expression for the distribution of bubbles in a persistent range-independent bubble layer by combining the work from Hall\[96\]

\[
N_v(a, z_t) = 1.6 \times 10^{10} p(a, z_t) \frac{W^3}{13 \text{m/s}} e^{-\frac{a}{z_e(W)}}
\]

where

\[
z_e(W) = \begin{cases} 0.4 \text{m}, & W \leq 7.5 \\ 0.4 \text{m} + 0.115(W - 7.5 \text{m/s}), & W > 7.5 \text{m/s} \end{cases}
\]

\[
p(a, z_t) = \begin{cases} \left[ \frac{a_{\text{ref}}(z_t)}{a} \right]^4 & a_{\text{min}} \leq a \leq a_{\text{ref}}(z_t) \\ \left[ \frac{a_{\text{ref}}(z_t)}{a} \right]^{x(z)} & a_{\text{ref}} < a \leq 1000 \mu \text{m} \end{cases}
\]

\[
a_{\text{ref}} = 54.4 + 1.984 z_t,
\]

and

\[
x(z) = 4.37 + \left( \frac{z_t}{2.55 \text{m}} \right)^2.
\]
Later, Novarini, Keiffer and Norton[94] use another expression to describe bubble spatial distribution that include not only the persistent bubble layer, but also temporally existent bubble plumes. The spatial separation, size, volume density, surface area, penetration depth of these bubble plumes, however, are range-dependent and vary from plume to plume[79]. In order to include the effects of these bubble plumes, we average the total number of bubbles in a plume over the inter-plume distance and added them to the existing persistent bubble layers.

In the simulations, a water column of 39 m with a flat bottom is used to simulate the Bristol channel where Weston et al.[74] did their experiments. An iso-velocity sound speed of 1500m/s is used.[74] The bottom sediment half space is composed of sand and shell with a sound speed of 1836m/s.[79] The attenuation in the water column and the bottom are $\alpha = 6 \times 10^{-5}\text{dB}/\lambda$ and $\alpha = 0.8\text{dB}/\lambda$ respectively. Both source and receiver are placed at the boundary between the water column and seabottom.[74] Due to lack of information about the transmitted source waveform and bandwidth used in the experiments, we use a 100 Hz bandwidth and apply Parseval sum to calculate the received acoustic intensity at 23 km receiver range. We assume the wind speed is a gaussian random variable with mean and standard deviation 9m/s and 5m/s, respectively.

Fig. 5-9 shows the unperturbed, mean, variance and second moment of the acoustic field forward propagated through bubble clouds under a given wind speed for 2kHz. The multiple scattering induced by random bubble clouds lead to moderate attenuation and dispersion on the acoustic forward field. The variance field, however, is negligible compared to the mean field. This suggests that the randomness of position and radii of bubbles may not lead to any measurable variation of acoustic field. Fig. 5-10 shows the mean and standard deviation of acoustic intensity as a function of range. The variation of the total volume density due to the randomness of wind speed leads to significant fluctuation of acoustic intensity. Fig. 5-11 show the standard deviation of acoustic field as a function of range. We find that the standard deviation approaches a constant after a certain propagation range. This is because the intensity field becomes fully saturated due to the randomness of bubble clouds. Fig. 5-12
show the comparison of the mean intensity based on our analytical expression and the Monte-Carlo simulations of Norton.[79] for 11 m/s wind speed. It is found that our model matches well with the representative Monte-Carlo points from Fig 4(c) in Ref. [79].

Figure 5-9: Unperturbed, mean, variance and second moment of acoustic field forward propagated through random bubble clouds under 10 m/s wind speed. These conditional mean, variance and second moments are calculated based on Eqs. 5.1, 5.3 and 5.5. The conditional variance is negligible compared to the conditional mean field that dominates the conditional second moment of the forward field.

5.4.2 Attenuation of acoustic forward field propagated through bubble clouds

Here, we compare the attenuation of acoustic forward field when including and excluding resonant bubbles based on Keiffer’s expression as well as Weston’s expression of bubble distribution. We also compare our simulations with the experimental data from Weston[74] et al.

Weston used the following distribution of bubble clouds to calculate the attenua-
Figure 5-10: Mean and standard deviation of acoustic transmission loss through bubble clouds under Gaussian random wind speeds. These statistical moments are calculated based on Eq. 5.7 and 5.8.

The expression for the transmission of acoustic transmission in Ref. [77]. His expression is based on the previous work from Johnoson and Cooke[89], Crawford and Farmer[90], Kolovayev[91] and Kirby[95]

\[ N_0(a, z_t) = 125 p(a) W^3 e^{(-\frac{a}{z_e})}, \]  

where \( W \) is the wind speed at 10 m height from sea surface and \( z_e = 1.2 \) m is the e-folding depth. The spectra of the bubble radii \( p(a) \) is

\[
p(a) = \begin{cases} 
0, & a < 1.7 \times 10^{-5} \\
2.41 \times 10^4 - 3.85 \times 10^{13} \times (a - 4.2 \times 10^{-5}), & 1.7 \times 10^{-5} < a < 5.8 \times 10^{-5} \\
1.55 \times 10^{-13} a^{-4} & 5.8 \times 10^{-5} < a,
\end{cases}
\]

where there is no cutoff radius in Eq. 5.15. The spectra of bubble radii is assumed to be independent of the depth, which is originally from Jing Wu[92]: "the bubble radius
spectra (probability density function of bubbles' radii) is invariant with either depth or wind speed”. Farmer also uses this assumption in his paper[90] where he states “that an equilibrium distribution exists in which the relative number of bubbles of a particular size is constant.” The total volume density $n_v(z_t)$ including all possible sizes of bubbles at depth $z_t$ is

$$n_v(z_t) = \int_0^\infty N_v(a, z_t)da = AW^n e^{-\frac{z_t}{z_e}}. \quad (5.16)$$

Compared to Weston’s expression, Keiffer’s expression has three major differences: (1) the e-folding depth is linearly proportional to wind speeds greater than 7.5 m/s. The higher the wind speed, the larger the e-folding depth and the deeper the penetration depth of bubble clouds, (2) The spectra of bubbles radii depends on the depth. The spectral slopes become steeper as the depth increases. There are fewer large bubbles in the deep water and (3) The spectra of the bubble radii has a cutoff radius
of 1000 μm.

Fig. 5-13 shows that the effect from resonant bubbles are much larger than that of off-resonant bubbles, although the number of resonant bubbles is much less than that of off-resonant bubbles. The choice of cutoff radii to include or exclude resonant bubbles, therefore, will greatly affect the simulation results. Although the 1000 μm cutoff radius is specified in Keiffer's expression, they do not provide any physical explanation why that cutoff radius is chose. To our knowledge, there is no conclusive value for the cutoff radius and 1000μ m and 4000μ m are chose as cutoff radii to exclude and include the resonant bubbles, respectively for Keiffer's expressions.

Fig. 5-14 shows the computed attenuation at 2kHz as a function of wind speed as well as Weston's 1968 and 1969 experimental data. For the 1000μ m cutoff radius and the wind speeds less than 10m/s, the computed attenuation based on Weston's expression with is larger than Weston's 1968 experimental data and the one based
Figure 5-13: (a) the spectra of bubble radii \( p(a) \) of Eq. 5.15, (b) the extinction cross section \( \sigma_{\text{ext}} = \frac{4\pi}{m^2} \Im(S) \) of bubbles at 2 kHz where \( \Im(S) \) is the imaginary part of a bubble’s scatter function of Eq. D.8, (c) the expected extinction cross section \( \langle \sigma_{\text{ext}} \rangle \). The attenuation effect due to resonant bubbles dominates even their volume densities are much lower than those of off-resonant bubbles.

on Keiffer’s expression. This maybe due to the non-dependence of e-folding depth on the wind speed in Weston’s expression, which has more bubbles compared to Keiffer’s expression and therefore leads to larger attenuation. When the wind speed is above 10m/s, the attenuation based on Keiffer’s expression increases more rapidly and seems to follow the trend of Weston’s 1969 experimental data. This can be explained by noting that in Eq. 5.2 the attenuation coefficient is proportional to the penetration depth of bubble clouds via the mean scatter function of bubble clouds. Since the e-folding depth of Keiffer’s expression linearly increases as wind speed increases, the deeper penetration depth of bubble clouds leads to larger attenuation coefficient. The simulated attenuations based on both Weston and Keiffer’s model are both smaller than the 1969 measurements, however, they are much larger than the 1968 measurements especially under higher wind speeds. Explanations of possible mechanisms resulting in the discrepancy between the 1968 and 1969 measurement data can be found in Ref. 120.
When the cutoff radius increases to 4000 \( \mu \text{m} \), the attenuations difference between including and excluding resonant bubbles using Weston's expression is as large as 30 dB resonant bubbles. The maximum attenuation difference based on Keiffer's expression, however, is only 10 dB. Because of the steeper power slope dependence in Keiffer's expression, there is less larger bubbles that leads to less attenuation. The simulated attenuation and experimental data at 1 kHz are shown in Fig. 5-15. The attenuation comparisons at different wind speeds are very similar to the ones under the 2 kHz scenario.

Figure 5-14: Attenuation of the acoustic field forward propagated through wind-generated bubble clouds based on (1) Weston's expression with cutoff radii 1000 \( \mu \text{m} \) and 4000 \( \mu \text{m} \) (2) Keiffer's expression with cutoff radii 1000 \( \mu \text{m} \) and 4000 \( \mu \text{m} \) at 2 kHz. For comparisons, Weston[74] et al experimental data in 1968 and 1969 are also plotted. The attenuation difference between including and excluding resonant bubbles could be larger than 30 dB based on Weston's expression while the difference is only 10 dB for Keiffer's expression.
5.5 Forward propagated field in time domain

In this section, we give the time domain expression for the mean forward field propagated through an ocean waveguide containing bubble clouds.

5.5.1 Mean field expression in time domain

The time domain response at the receiver is the Fourier transform of Eq. 5.1

\[
\Phi(r|r_0, t) = \int_{-\infty}^{\infty} 4\pi Q(\omega) \langle \Psi_T(r|r_0, \omega) \rangle e^{-i\omega t} d\omega, \quad (5.17)
\]

which can be integrated to

\[
\Phi(r|r_0, t) = \sum_m \Phi_m(r|r_0, t)
\]

\[
\approx \sum_m Q(\omega_m) \left[ i e^{-\left(\frac{1}{4} + \Psi(\varphi_m(\omega_m))\right)} \right] \times \frac{U_m(z_0, \omega_m) U_m(z, \omega)}{\sqrt{\xi(\omega_m)} \sqrt{\frac{1}{\varphi_m''(\omega_m)}}}
\]

\[
\times e^{(x+x_0)(\varphi_m(\omega_m) + \frac{\varphi_m''(\omega_m)}{2})} \quad (5.18)
\]
by assuming that the distance between the source and receiver is far enough so that
the stationary phase approximation (saddle point approximation) can be applied,
where \( \varphi_m(\omega) = \xi_m + \mathcal{R}(\nu_m) - \frac{\omega t}{x+x_0} \). The stationary point \( \omega_m \) must satisfy

\[
\frac{x + x_0}{V_m(\omega_m)} = t,
\]

where \( V_m \) is the group velocity for mode \( m \) in an ocean waveguide containing bubble
clouds. Chap 2 of Ref. [11] gives more explanation of the general stationary phase ap-
proximation (saddle point approximation) for the scattered field in time domain. The
stationary point \( \omega_m \) only exists when time \( t \) is longer than the shortest wave packet
travel time \( \frac{x + x_0}{(V_m)_{\text{max}}} \) from the source to receiver, where \( (V_m)_{\text{max}} \) is the maximum modal
group velocity within the frequency bandwidth of the transmitted source waveform.
For any time \( t \) earlier than \( \frac{x + x_0}{(V_m)_{\text{max}}} \), no stationary point \( \omega_m \) exists and \( \Phi_m(r|r_0,t) \approx 0 \),
which proves that Eq. 5.18 is causal. In section IV, we will use Titchmarsh’s theorem
to prove the causality of Eq. 5.18.

5.5.2 Simulations of acoustic signal propagated through bub-
ble clouds

In this section, simulations of the mean acoustic forward propagation through bubble
clouds in time domain are shown based on Keffer’s expression with 1000 \( \mu \) m cutoff
radius. A hanning-windowed continuous wave (CW) tone with 0.05 second width and
2 kHz carrier frequency is used as the source waveform, as shown in Fig. 5-16.

Figure. 5-17 shows the value of the waveform received at 20 km range under 10
m/s wind speed in the bubbly medium. For a comparison, a normalized received
waveform in the bubble-free medium is also shown. The scattering and absorption
effects due to bubble clouds not only cause the distortion of the signal waveform, but
also alter the structure of the received signal. The amplitudes of the later arrivals
propagated in the bubbly medium are found to be lower than those propagated in a
medium without bubbles due to the attenuation effect of the bubbles. The amplitude
of the early arrivals, however, are even larger than the ones without bubbles. In
Figure 5-16: The source spectrum and transmitted waveform. The central frequency (carrier frequency) is 2 kHz.

an ocean waveguide, the received acoustic signal is the *coherent superposition* of all modes that travel at different modal phase velocities. The coherent superposition of the modes with *opposite* sign in amplitudes, could lead to a very small received signal, while the superposition of closely arrived modes with the *same* sign could lead to a very large signal. Since the lower order modes travel more horizontally than the higher order modes in a continental shelf waveguide, the early arrivals are composed of the lower order modes and the late arrivals are composed of higher order modes.
The lower modes are less attenuated than their higher order neighbor modes with opposite sign, which leads to the larger first arrival peak in a bubbly medium. When the wind speed is increased to 20 m/s, the received waveform undergoes much more significant attenuation as shown in Fig. 5-18. Only first few modes survive at 20km receiver range.

![Received waveform at 20 km receiver range under wind speed 10 m/s at central frequency 2 kHz](image)

Figure 5-17: Received waveform at 20 km receiver range under 10 m/s wind speed. For a comparison, the normalized received waveform in a static waveguide is also plotted.

When processing data collected from underwater acoustic experiments, matched filter process is commonly used to increase the signal to noise ratio (SNR). Direct using the source waveform to correlate the received signal that contains the effect from random bubble clouds will degrade the matched filter performance. Therefore, the waveform that accounts for the effect of bubble clouds is necessary in the matched field process.
5.5.3 Causality and dispersion relation

In this section, we prove that the dispersion relationship obtained in a bubbly medium obeys Kramer-Kronig relation. We also show that our dispersion relationship is valid for $k \to 0$ when a proper physical model for bubbles (air-filled spheres) is used, but breaks down for an improper physical model for bubbles (vacuum spheres).

When an acoustic incident mode $\Psi^n_{in} = AQ(\omega)u(z_r)u(z_s)e^{ik_{n}p}$ is propagated through a single elemental slab of thickness $\delta_{\text{slab}}$ containing randomly distributed bubbles, the total field, which is the sum the incident field and the scattered field from bubbles, can be expressed as

$$\langle \Psi^n_T \rangle = \Psi^n_{in} + \Psi^n_s$$

$$= \Psi^n_{in} (1 + i\nu_n \delta_{\text{slab}})$$

$$= AQ(\omega)u(z_r)u(z_s)e^{ik_{n}p}(1 + i\frac{\nu_n}{k}\delta_{\text{slab}})$$

(5.19)

by applying Eq. 15 in Ref. [68], where $Q(\omega)$ is the source spectrum, $\nu_n$ is the change of
complex refractive index. In order to prove Eq. 5.19 obeys causality, we need to prove
that the term \( Q(\omega)(\frac{\nu_m}{k}) \) in Eq. 5.19 is causal and satisfies Kramers-Kronig relation.

To simplify the problem, we assume the bubbles are uniformly distributed in the
waveguide and express \( Q(\omega)(\frac{\nu_m}{k}) \) as

\[
Q(\omega)\frac{\nu_n}{k} = Q(\omega)\frac{2\pi n_v \langle S \rangle}{k^2 \xi_n} = Q(\omega)\frac{2\pi n_v}{\sqrt{(\frac{\nu}{c} + i\alpha)^2 - \gamma_n^2 (\frac{\nu}{c} + i\alpha)^2}},
\]

(5.20)

where \( \gamma_n \) is the vertical wavenumber of mode \( n \) and \( \alpha \) is the attenuation coefficient of
the medium. Eq. 5.20 is square integrable since it has no singularities in upper half plane

\[
\int_{-\infty}^{\infty} |Q(\omega)\frac{\nu_n(\omega)}{k}|^2 d\omega < C.
\]

(5.21)

According to Titchmarsh’s theorem, \( Q(\omega)\frac{\nu_m}{k} \) obeys causality and satisfies Kramers-Kronig relation. Therefore, Eq. 5.19 also obeys causality.

The dispersion relation can be expressed in terms of the mean scatter function of
the bubbles as

\[
\Re\left(\frac{\langle S \rangle}{k}\right) = \frac{k\xi_n}{\pi} \int_{-\infty}^{\infty} \frac{\Im(\langle S \rangle)}{k^2 \xi_n (\omega' - \omega)} d\omega'.
\]

(5.22)

by inserting \( \nu_n \) into the change of complex refractive index and applying Titchmarsh’s
theorem, where the real and imagine part of \( \frac{\nu_m}{k} \) are Hilbert transforms of each other.

We can also express the dispersion relation in terms of the extinction cross section

\[
\Re\left(\frac{\langle S \rangle}{k}\right) = \frac{k\xi_n}{4\pi^2} \int_{-\infty}^{\infty} \frac{\langle \sigma_e \rangle}{\xi_n (\omega' - \omega)} d\omega'.
\]

(5.23)

by applying the extinction theorem, where \( \sigma_e = \frac{4\pi}{k^2} \Im(\langle S \rangle) \) is the extinction cross section.

As \( k \) is close to 0, the limit of the left hand side of Eq. 5.23 becomes

\[
\lim_{k \to 0} \Re\left(\frac{\langle S \rangle}{k}\right) = 0
\]

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and the limit of the right hand side of Eq. 5.23 is

\[ \lim_{k \to 0} \frac{k \xi_n}{4\pi^2} \int_{-\infty}^{\infty} \frac{\langle \sigma_e \rangle}{\xi_n(\omega' - \omega)} d\omega' = 0 \]

Our dispersion relation for a waveguide with random distributed bubbles holds when wavenumber \( k \) is close to 0. However, if we model bubbles as pressure release spheres, the averaged scatter function is \( \langle S \rangle = -k \langle a \rangle + ik^2 \langle a^2 \rangle + O(ka) \). As \( k \to 0 \), the limit of left hand side of Eq. 5.23 approaches:

\[ \lim_{k \to 0} \Re \left( \frac{S}{k} \right) = \langle a \rangle \]

and the limit of right hand side of Eq. 5.23 is

\[ \lim_{k \to 0} \frac{k \xi_n}{4\pi^2} \int_{-\infty}^{\infty} \frac{\langle \sigma_e \rangle}{\xi_n(\omega' - \omega)} d\omega' = 0 \]

The limits are not in agreement as \( k \to 0 \). The relation breaks down because a pressure release sphere is not a proper physical model for bubbles.

## 5.6 Conclusion

An analytical expression of attenuation of the forward field propagated through a waveguide with shoals of fish and randomly distributed bubble clouds is formulated in terms of the spatial distribution and statistical properties of the scatter function density of these scatterers. Both simulations and experiment data show that shoals of fish have negligible effect on the forward propagated field at 415 Hz. Analytical expressions of the statistics of acoustic forward field are also presented for a given wind speed. The fluctuation of acoustic intensity could be significant even under moderate wind conditions. The attenuations due to bubble clouds are highly sensitive to the choice of cutoff radius in the spectra of bubble radii. The attenuation based on Keiffer’s expression approximately follows the trend of Weston’s experimental data at different wind speeds. When includes resonant bubbles, Keiffer’s expression gives
a much more reasonable attenuation than that based on Weston's expression as compared to Weston's experimental data. The expression for the mean forward field in time domain is also derived by applying stationary phase approximation. In the simulations, the bubble clouds generated under high wind speeds (>10 m/s) lead to additional time delay, distortion and attention on the transmitted signal. This could degrade the match filter performance if not properly accounted for. Titchmarsh's theorem is applied to prove that our expression of the dispersion relation in a continental shelf waveguide containing bubble clouds obeys causality and satisfies Kramers-Kronig relation. When wavenumber is close to 0, a proper physical model of bubbles is essential to validate our dispersion relation.
Chapter 6

Acoustic field propagated through small weak inhomogeneities

6.1 Introduction

In this section, we formulate the expression of the acoustic scattered intensity from small weak scatterers. This type of scatterers include creatures, such as krill, squid and zooplankton, whose compressibility and density are slightly different from the surrounding medium and dimension is much smaller than acoustic wave-length. The target strength of these scatterers is much lower than other biological scatterers such as fish that have swimbladders or air sacs. For example, at 1 kHz, the TS of herring is rough -30 dB while the TS of krill is only -160 dB. In order for OAWRS (Ocean Acoustic Waveguide Remote Sensing) to effectively detect these weak biological scatterers, the scattered intensity must at least 10 dB higher than the background reverberation level from sea bottom. An analytical expressions of the scattered intensity from these weak inhomogeneities is needed to determine under what condition (for example, volume density of the scatterers) OAWRS can image these weak scatterers.
6.2 Scattered field from weak scatterers based on Rayleigh-Born approximation

In this section, we derive the expression of the scattered field based on Rayleigh-Born approximation to Green's theorem.

The source and receiver are located at position \( r_0 \) and \( r \) respectively. The center of the scatterer is placed at \( r_c \), where \( r_t = r_c + u_t \). The scattered field based on Rayleigh-Born approximation in free space is expressed as

\[
\begin{align*}
    P_s(r|r_0) &= \iiint \{ k^2 \Gamma_\kappa(r_t) G(r|r_t) P_i(r_t|r_0) + \nabla G(r|r_t) \nabla P_i(r_t|r_0) \} \, dV_t \\
    &\approx P_i(r_c|r_0) \iiint k^2 \{ \Gamma_\kappa(r_t) - \eta(k, k_i) \Gamma_d(r_t) \} e^{i(k - k_i) \cdot u_t} \, dV_t \\
    &= P_i(r_c|r_0) \left( \frac{4\pi}{k} S \right) G(r|r_c),
\end{align*}
\]

(6.1)

where \( P_i(r_c|r_0) \) is the incident field on the center of the scatter, \( G(r|r_c) \) is the free space Green function. The scatter function of the scatter is expressed as

\[
S(k_i, k|r_c) = \iiint \frac{k^3}{4\pi} \{ \Gamma_\kappa(r_c) - \eta(k, k_i) \Gamma_d(r_c) \} e^{i(k - k_i) \cdot u_t} \, dV_t
\]

(6.2)

from Eq. 6.1, where

\[
\eta(k, k_i) = \frac{k_i \cdot k}{k^2} = \cos \alpha_i \cos \alpha + \sin \alpha_i \sin \alpha \cos(\beta_i - \beta)
\]

(6.3)

is the cosine of the angle between the incident and scattered plane wave directions. The scatter function can be further simplified to be

\[
S(k_i, k|r_c) \approx \frac{k^3}{4\pi} V_c(r_c) \{ \Gamma_\kappa(r_c) - \eta(k, k_i) \Gamma_d(r_c) \}
\]

(6.4)

for the scatterer whose dimension is much smaller than the acoustic wave length, where \( V_c(r_c) \) is the volume of the scatterer.
The total scattered intensity from scatterers contained in a sonar footprint is assumed to be incoherent summation of all scattered intensity from each individual scatterer

\[ I(\rho_{sc}) = \sum_{n=1}^{N} |P_s(n)|^2 \]

\[ = \int_{A} \int_{h_1}^{h_2} |P_s(r_c|r_0)(\frac{4\pi}{k}\mathcal{S})G(r|r_c)|^2D(r_c)dV, \]  \hspace{1cm} (6.5)

where \( \rho_{sc} \) is the center of sonar footprint, \( A \) is the area of sonar footprint and \( D(r_c) \) is the volume density of the scatterers at position \( r_c \). The scatterers are assumed to be distributed within the water layer between depth \( h_1 \) and \( h_2 \).

Eq. 6.5 can be approximately expressed as

\[ I(\rho_{sc}) \approx D_1^{\frac{S}{k}} \int_{A} \int_{h_1}^{h_2} |P_s(r_c|r_0)|4\pi G(r|r_c)|^2dV, \]  \hspace{1cm} (6.6)

based on the following assumptions: (1) the scatterers are uniformly distributed within sonar footprint, i.e., \( D \) is independent on the position \( r_c \). (2) The intrinsic properties of scatterer such as compressibility fraction \( \Gamma_c \), density fraction \( \Gamma_d \) and the dimension are independent on position \( r_c \).

We can further express Eq. 6.6 in term of the TS of scatterer as

\[ 10\log_{10}(I) \approx 10\log_{10}(D) + 10\log_{10}(\frac{\sigma_s}{4\pi}) \]

\[ + 10\log_{10}(\int_{A} \int_{h_1}^{h_2} |P_s(r_c|r_0)P(r|r_c)|^2dV), \]  \hspace{1cm} (6.7)

where \( P_s(r_c|r_0) \) and \( P(r|r_c) \) are the transmission loss from the source to the scatterer and the scatterer to the receiver respectively.
6.3 Illustrative examples: scattered intensity from Antarctic krill

Antarctic krill play a key role in the marine food chain of the Antarctic as the primary source of sustenance for many species of whales, seals, birds and squid\[100\]. They are widely distributed within the Antarctic, with high concentrations in the Scotia Sea off the Antarctic Peninsula\[101\].

The compressibility fraction $\Gamma_\kappa$ and density fraction $\Gamma_d$ depend on the density contract $g$ and sound speed contract $h$ between krill and the surrounding medium

$$
\Gamma_\kappa = \frac{1}{gh^2} - 1
$$

$$
\Gamma_d = 1 - \frac{1}{g},
$$

(6.8)

where $g$ and $h$ can be expressed as a function of the length of krill\[102\] $L$

$$
g = 5.439 \times 10^{-4}L(\text{mm}) + 1.002
$$

$$
h = 4.981 \times 10^{-4}L(\text{mm}) + 1.009.
$$

(6.9)

The shape of krill is modeled as a cylinder with diameter $d = \frac{L}{3^2}$\[102\]. The TS of krill is plotted in Fig. 6-1 as a function of frequencies and length of krill based on Eq. 6.4.

In the simulations, a water column with depth 200 m and 2000 m are used to simulate continental shelf and deep ocean environment in Antarctic area. The bottom sediment half-space is composed of sand with density $d_b = 1.9g/cm^3$ and sound speed $c_b = 1700m/s$. Since krill are mainly found in the upper water column ($< 50m$)\[103, 104, 105\], krill is assumed to be uniformly distributed within a layer between 0 m and 50 m. A typical swarm densities of Antarctic krill exceed $1000/m^3$ in Antarctic waters, and can reach up to $100,000/m^3$ in superswarms\[106\]. Here, we use $1000/m^3$ as krill volume density in the simulations.

A horizontal receiver array is placed at 50 m depth that has 3 m aperture and
Figure 6-1: Target strength of krill at different lengths as a function of acoustic frequency

The cross-range resolution of sonar footprint is 0.05r, where r is the range from krill to the receiver. The range resolution is 15 m that corresponds 0.05 seconds width of the transmitted pulse. For both deep ocean and continental shelf environments, a vertical source array is assumed to transmit sound within a ± 2.5 degree beam from the horizontal. Array side-lobe levels are designed to be 50 dB lower than the main lobe level. This simple design ensures propagation paths with very little bottom interference and enables OAWRS to detect krill swarms because significant bottom interaction only occurs when propagation angles exceed ± 7 degrees in shallow waters (200 m), as shown in Fig. 6-2, and ± 13 degrees in deeper waters (2000 m), with respect to the horizontal.

The scattered intensity from krill is calculated based on Eq. 6.7. Ray method[107] is used to compute the transmission loss from the source to krill and krill to receiver at 10 kHz acoustic frequency. In order to determine the feasibility of OAWRS detecting Antarctic krill, the reverberation from sea bottom is also computed based on Eq. 6.7 by replacing TS and volume density of krill with the scattering strength of the sea bottom inhomogeneities[108], which is approximately -30 dB at 10 kHz[109].

The scattered intensities from krill are compared with seafloor scattering for deep
Figure 6-2: Fig. 11. OAWRS detection of Antarctic krill employs ray paths and sound speed profile of the Antarctic environment. The vertical source array of an OAWR system is centered at 50 m depth and the total water depth is 200 m. Rays within the array main-lobe beam (+/- 2.5 degrees) are bounded by the red and blue lines and follow refracted surface-reflected paths that can image krill at long ranges. Rays at angles that exceed +/- 7 degrees (e.g. black line) are in the much weaker array side-lobes and follow surface-reflected bottom-reflected paths that are contaminated with seafloor scattering returns in shallow water. The figure is not to scale.

The scattered intensities from krill swarms are approximately 30 dB for deep ocean and 20 dB for continental shelf environments higher than the background reverberation level when the volume density of krill is 1000 m$^3$, as shown in Figs. 6-4 and 6-3. These dynamic range make it is highly feasible for OAWRS to image krill in both deep ocean and continental shelf environment. For smaller krill of 1 cm length, OAWRS imaging is expected to be less favorable in continental-shelf environment, unless densities are at least 10000 m$^3$, which is not uncommon in superswarms[106]. A source array with lower sidelobe levels may resolve this problem even for typical 1000 m$^3$ densities.

6.4 Conclusion

In this section, we formulated the scattered intensity for small weak scatterer based on Rayleigh-Born approximation. The scattered intensity is expressed in terms of
Figure 6-3: Scattered intensity from krill and sea-bottom for a deep ocean environment. Krill are assumed to have lengths that vary from 1-4 cm, a typical packing density of 1000 m$^3$, and to be uniformly distributed in a depth layer ranging from 0 m to 50 m.

TS, volume density of the weak scatterers as well as the transmission loss from the source to the scatterers and the scatterers to the receiver. In the simulations, we plot the scattered intensity from Antarctic krill with different sizes and compare them with the background reverberation noise. It is found that by using a simple designed source array, it is highly feasible for OAWRS to image krill in both shallow and deep water environment in Antarctic area.
Figure 6-4: Similar to Fig. 6-3 but for a continental shelf environment.
Chapter 7

Conclusion

In this thesis, analytical expressions for the mean, variance and temporal covariance of the acoustic field forward propagated through a stratified ocean waveguide containing 3D random inhomogeneities including internal waves, shoals of fish, wind-generated bubbles and Antarctic krill are developed. These expressions are expressed in terms of temporally and spatially varying scatter function densities, which are determined by intrinsic properties of these inhomogeneities. In order to calculate the statistical moments of the acoustic forward field, physical models and statistical descriptions of these inhomogeneities are required to quantify the statistics of the scatter function densities.

In Chap 3, a two-layer and continuous stratification models are applied to describe internal waves in a continental shelf waveguide and deep ocean environment, respectively. The statistics of the displacement of internal waves are formulated based on the Garret-Munk internal wave model. It is shown that, in a typical continental-shelf environment, when the standard deviation of the internal wave height exceeds the acoustic wavelength, the acoustic forward field becomes so randomized that the expected total intensity is dominated by the variance field and loses the coherent interference structure beyond moderate ranges. This leads to an effectively saturated field that decays monotonically. By comparing 2D Monte-carlo simulation with our 3D analytical model, we show that 3D scattering effects become important when the Fresnel width approaches and exceeds the cross-range coherence length of the internal
wave field.

In Chap 4, we formulate an analytical expression for the temporal coherence of an acoustic signal propagating in an ocean waveguide and use it to explain the time scale of acoustic field fluctuations observed at mega meter ranges in various deep ocean acoustic transmission experiments. This time scale is much shorter than that of internal waves in both deep ocean and continental shelf waveguides due to a multiple forward scattering process. When the acoustic Fresnel width exceeds the cross-range coherence length of the internal-waves, 3D scattering effects become pronounced and lead to frequency and range-dependent power losses in the forward field.

In Chap 5, we calculate the attenuation of acoustic field forward propagated though shoals of herring and wind-generated bubble clouds. The simulated attenuation due to fish shoals in New Jersey shelf is noticeable at the resonance of swim bladders. The attenuations at off-resonance frequencies, however, are negligible compared to that of other mechanisms such as wind-generated bubbles, as shown in the simulations and experimental data. Analytical expressions of the statistics of acoustic forward field are also presented for a given wind speed. The fluctuation of acoustic intensity could be significant even under moderate wind conditions. The attenuations due to bubble clouds are highly sensitive to the choice of cutoff radius in the spectra of bubble radii. This is because the effects due to resonant bubbles are much higher that that of off-resonant bubbles. We also show bubble clouds generated under high wind speeds (>10 m/s) could lead to additional time delay, distortion and attention on the transmitted signal. This could degrade the match filter performance if not properly accounted for. By applying Titchmarsh's theorem, we prove that our expression for the dispersion relation of acoustic wave propagation in a bubbly medium obeys casualty and satisfies Kramers-Kronig relation.

In Chap 6, the expression for the acoustic scattered intensity from weak scatterers is derived. In the simulations, the levels of the scattered intensity from Antarctic krill swarms are much higher than the background reverberation at typical volume density of krill swarms in both shallow water and the continental shelf waveguide. This makes krill swarms detectable by OAWRS near the Antarctic.
Appendix A

Temporal coherence of narrow-band acoustic signals

Here we discuss extension of the results in single-frequency transmission given in the main text to narrow-band signals. Let the acoustic signal measured at a receiver be

\[ \Psi(r|\text{r}_0, t) = \Psi_i(r|\text{r}_0, t) + \Psi_s(r|\text{r}_0, t), \]  

(A.1)

where \( \Psi_i(r|\text{r}_0, t) = \int_{-\infty}^{\infty} Q(f)\Phi_i(r|\text{r}_0, f)e^{-ij2\pi ft}df \) is the incident field, \( \Psi_s(r|\text{r}_0, t) = \int_{-\infty}^{\infty} Q(f)\Phi_s(r|\text{r}_0, f, t)e^{-ij2\pi ft}df \) is the scattered field and \( Q(f) \) is the source spectrum. The temporal coherence or autocorrelation function of the acoustic signal is

\[ \langle \Psi(r|\text{r}_0, t)\Psi^*(r|\text{r}_0, t') \rangle = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} Q(f)Q(f')\langle \Phi(r|\text{r}_0, f, t)\Phi(r|\text{r}_0, f', t') \rangle e^{-ij2\pi (f-t+f'-t')} df df'. \]  

(A.2)

In order to evaluate Eq. A.2, we need to calculate the temporal correlation of the total field \( \langle \Phi(r|\text{r}_0, f, t)\Phi^*(r|\text{r}_0, f', t') \rangle \) at two different frequencies \( f \) and \( f' \).

Derivations for the depth-integrated second moment of the scattered field in Sec. IV.B of Ref. [4], and the second term of Eq. 4.7 in Sec. 4.2.2 of the present paper,
rly on modal orthogonality

\[
\int_0^\infty \frac{1}{d(z)} u_m(f, z) u_n(f, z) dz = \delta_{nm}. \tag{A.3}
\]

It can be shown by the numerical simulations that modal orthogonality is still approximately valid for the acoustic modes at two different frequencies \(f\) and \(f'\)

\[
\int_0^\infty \frac{1}{d(z)} u_m(f, z) u_n(f', z) dz \approx \delta_{nm}, \tag{A.4}
\]

if the difference between \(f\) and \(f'\) is smaller than a few Hertz. Consequently, \(\langle \Phi(r|\tau_0, f, t)\Phi^*(r|\tau_0, f', t') \rangle\) can be approximated as \(\langle \Phi(r|\tau_0, f, t)\Phi^*(r|\tau_0, f, t') \rangle\) of Eq. 4.16 for sufficient narrow-band signals satisfying Eq. A.4. Equation A.2 is then approximately

\[
\langle \Psi(r|\tau_0, t)\Psi^*(r|\tau_0, t') \rangle \approx |Q(\bar{f})\Delta f|^2 \langle \Phi(r|\tau_0, \bar{f}, t)\Phi^*(r|\tau_0, \bar{f}, t') \rangle e^{-i2\pi f(t-t')}, \tag{A.5}
\]

where \(\bar{f}\) and \(\Delta f\) are the central frequency and bandwidth of the narrow-band signal.
Appendix B

Historical Notes: Dozier and Tappet’s 2-D model

In the late 1970’s, Dozier and Tappert published a paper[110] about a 2-D statistical theory of acoustic field forward propagation in a deep ocean waveguide containing random internal waves. They derived a random coupled mode equation (Eq. 35 in Ref. [110]) based on Markov approximation and paraxial approximation, which assumes the derivative of acoustic pressure over time is negligible compared to the derivation over space. A coupled power equation (Eq. 57) was also derived based on that random coupled mode equation. Eq. 35 quantifies how the amplitude of each individual acoustic mode changes due to a random sound speed fluctuation $\delta_c(r, z, t)$ induced by the internal waves. In this appendix, we will compare Dozier and Tappet’s coupled mode equation with our difference equation of mean field that is applied to the 2-D scenario.

In the paper[110], Dozier and Tappert expressed acoustic incident field to be

$$P(r, z) = \sum_{n=1}^{N} A_n(r) \phi_n(z) = \sum_{n=1}^{N} \frac{e^{iH_n r}}{\sqrt{t_n}} \psi_n(r) \phi_n(z) \quad (B.1)$$

where $\psi_n(r)$ is the amplitude of the acoustic modal $n$, $\phi_n(z)$ is the mode shape. The change of $\psi_n(r)$ due to the sound speed fluctuation caused by local internal wave inhomogeneity distributed between range $r$ and range $r + \Delta r$ is expressed in Eq. 35.
as following

\frac{\partial \psi_n}{\partial r}(r) = -i \sum_{m=1}^{N} R_{nm}(r)e^{i\ell_m r}\psi_m(r) \quad (B.2)

where \( R_{nm}(r) = \frac{\omega^2}{c_0^3(l_m l_n)^{1/2}} \int dz \frac{\Delta c(r, z)}{c_0} \phi_n(z)\phi_m(z) \), \( l_{mn} = l_m - l_n \) is the horizontal wave number difference between mode \( m \) and \( n \).

Eq. B.2 can be re-written as a difference equation since it was derived based on Markov approximation and was an initial-value problem

\frac{\partial \psi_n}{\partial r}(r) = \frac{\psi_n(r + \Delta r)|_{\Delta r = 0} - \psi_n(r)}{\Delta r}

= -i \sum_{m=1}^{N} R_{nm}(r')|_{r < r' < r + \Delta r}e^{i\ell_m r}\psi_m(r)

= -i \sum_{m=1}^{N} \frac{k^2}{\sqrt{l_m l_n}} e^{i\ell_m r}\psi_m(r) \int dz \frac{\Delta c(r', z)}{c_0} \phi_n(z)\phi_m(z), \quad (B.3)

In order to compare our expression of the mean change of acoustic field to Eq. B.2, we take the expectation of the Eq. B.3

\langle \psi_n(r + \Delta r) - \psi_n(r) \rangle = \langle \Delta \psi_n(r + \Delta r, r) \rangle

= -i \sum_{m=1}^{N} \frac{k^2}{\sqrt{l_m l_n}} e^{i\ell_m r}\psi_m(r) \Delta r \int dz \frac{\Delta c(r', z)}{c_0} \phi_n(z)\phi_m(z). \quad (B.4)

We now discuss our model of the change of mean acoustic field due to the scattering from random internal waves. The notations including horizontal wavenumber \( l_n \), range separation from source to receiver \( r \), mode's shape \( \phi_n(z) \) used by Dozier and Tappet are represented by \( \xi_n \), \( \rho \) and \( u_n(z) \), respectively, in our formulations shown below. From Eq. 53 of Ref. [4], the change of mean acoustic field after propagating
through a local internal wave inhomogeneity distributed within range \( \rho_t = \rho_s + u_t \) is

\[
\langle \Phi_s(r|r_0, \Delta \rho_s(\rho_s)) \rangle = \langle \Phi_n(r|r_0, \Delta \rho_s(\rho_s)) \rangle = \langle \Phi_n(r|r_0) - \Phi_s(r|r_0) \rangle = \langle \Delta \Phi_n(r|r_0, \Delta \rho_s(\rho_s)) \rangle
\]

\[
= \sum_{m=1}^{N} \int \frac{d \Delta \rho}{d \rho} \frac{2\pi}{k} \frac{d(z_0) d(z_t)}{\xi_m \sqrt{\xi - \xi_m}} \frac{\sqrt{2\pi}}{\xi_m} \frac{\Delta \rho_s(\rho_s)}{2} \sum_{m=1}^{N} \int \frac{d \Delta \rho}{d \rho} \frac{2\pi}{k} \frac{d(z_0) d(z_t)}{\xi_m \sqrt{\xi - \xi_m}} \frac{\sqrt{2\pi}}{\xi_m} \frac{\Delta \rho_s(\rho_s)}{2}
\]

\[
\times \sum_{m=1}^{N} \int \frac{d \Delta \rho}{d \rho} \frac{2\pi}{k} \frac{d(z_0) d(z_t)}{\xi_m \sqrt{\xi - \xi_m}} \frac{\sqrt{2\pi}}{\xi_m} \frac{\Delta \rho_s(\rho_s)}{2} \sum_{m=1}^{N} \int \frac{d \Delta \rho}{d \rho} \frac{2\pi}{k} \frac{d(z_0) d(z_t)}{\xi_m \sqrt{\xi - \xi_m}} \frac{\sqrt{2\pi}}{\xi_m} \frac{\Delta \rho_s(\rho_s)}{2}
\]

\[
\times [N_n^{(1)} N_n^{(1)} e^{i(\gamma_m + \gamma_n)z_t} \langle s_{\rho_s, z_t}(\alpha_m, \phi; \pi - \alpha_n, \phi) \rangle] \Delta \rho_s(\rho_s)
\]

where \( \rho_t \) is the center of the internal wave inhomogeneity, \( \Phi_n(r|r_0) \) is acoustic model defined in the following equation

\[
P(r|r_0) = \sum_{n=1}^{N} \Phi_n(r|r_0),
\]

where \( P(r|r_0) \) is the acoustic pressure field. Since the fractional change of density in the deep ocean is much smaller than that of compressibility from Eq. C.5, the mean scatter function density of Eq. 4.24 can be simplified

\[
\langle s_{\rho_s, z_t}(\alpha_m, \beta, \alpha_n, \beta) \rangle = \frac{1}{A_c} \frac{k^3}{4\pi} \left[ \Gamma_n(\rho_s, \alpha_n, \beta) \right] \int_{A_c} e^{i(\xi_n - \xi_m)u_s} d^2 u_s
\]

\[
= \frac{1}{A_c} \frac{k^3}{4\pi} \left[ \Gamma_n(\rho_s, \alpha_n, \beta) \right] \int_{-L_x/2}^{L_x/2} e^{i(\xi_n - \xi_m)z_t} d\zeta
\]

\[
= -\frac{k^3}{2\pi} \frac{(\Delta c(\rho_s, z_t))}{c_0} \text{sinc}[(\xi_n - \xi_m)L_x/2],
\]

which is independent on the incident and outgoing elevation angel \( \alpha_n \) and \( \alpha_m \). In-
serting the mean scattering function density into Eq. C.1

\[
\langle \Delta \Phi_n(r|r_0, \Delta \rho_s(\rho_s)) \rangle = \sum_{m} \int dz_t \frac{2\pi}{k} \frac{i}{d(z_0) d(z_t)} \frac{\sqrt{2\pi} e^{i\xi_n \rho}}{\sqrt{\xi_n \rho}} u_m(z) u_n(z_0) 
\times e^{i\pi/4} e^{i(\xi_n - \xi_m)\rho_s} \text{sinc} \left[ (\xi_n - \xi_m) \frac{\Delta \rho_s}{2} \right] u_m(z_t) u_n(z_t) s_{\rho_s, z_t} m, n \rangle \Delta \rho_s 
\]

\[
= -i \sum_{m=1}^{N} \frac{k^2}{\sqrt{\xi_n \xi_m}} e^{i(\xi_n - \xi_m)\rho_s} \Phi_m u_n(z_0) u_m(z_0) \text{sinc} \left[ (\xi_n - \xi_m) \frac{\Delta \rho_s}{2} \right] 
\times \text{sinc} \left[ (\xi_n - \xi_m) L \right] \int dz_t \frac{\langle \Delta c(\rho_t, z_t) \rangle}{c_0} \frac{u_m(z_t) u_n(z_t)}{d(z_t)} \Delta \rho_s 
(B.8)
\]

As \( \Delta \rho_s \to 0 \), \( L \to 0 \), the internal wave inhomogeneity is very close to the receiver and \( \rho_s \to \rho \). The difference equations of mean field of Eq. B.8 can be expressed as

\[
\langle \Delta \Phi_n(r|r_0) \rangle = -i \sum_{m=1}^{N} \frac{k^2}{\sqrt{\xi_n \xi_m}} e^{i(\xi_n - \xi_m)\rho_s} \Phi_m u_n(z_0) u_m(z_0) \int dz_t \frac{\langle \Delta c(\rho_t, z_t) \rangle}{c_0} \frac{u_m(z_t) u_n(z_t)}{d(z_t)} \Delta \rho 
(B.9)
\]

We express the acoustic pressure in term of modes as

\[
P(r|r_0) = \sum_{n=1}^{N} \Phi_n(r|r_0) = \sum_{n=1}^{N} \frac{e^{i\xi_n \rho}}{\sqrt{\xi_n}} \psi_n(\rho, z) u_n(z_0), 
(B.10)
\]

Comparing Eq. B.10 with Eq. B.1, we find that, in Dozier and Tappert’s expression of acoustic field, the modal amplitude \( \psi_n \) includes the mode shape \( u_n(z_0) \) at the source depth but exclude the mode shape at the receiver depth \( u_n(z) \). In our model, the modal amplitude \( \psi_n \) includes the mode shape at the receiver depth \( u_n(z) \) but exclude the mode shape at the source depth \( u_n(z_0) \).

Up inserting the acoustic mode \( \Phi_m(r|r_0) \) in Eq. B.10 into Eq. B.9

\[
\langle \Delta \Phi_n(r|r_0) \rangle = -i \sum_{m=1}^{N} \frac{k^2}{\sqrt{\xi_n \xi_m}} e^{i\xi_n \rho} \psi_m(\rho, z) u_n(z_0) \int dz_t \frac{\langle \Delta c(\rho_t, z_t) \rangle}{c_0} \frac{u_m(z_t) u_n(z_t)}{d(z_t)} \Delta \rho 
\]

\[
= -i \left( \frac{e^{i\xi_n \rho}}{\sqrt{\xi_n}} u_n(z_0) \right) \sum_{m=1}^{N} \frac{k^2}{\xi_m} \psi_m(\rho, z) \int dz_t \frac{\langle \Delta c(\rho_t, z_t) \rangle}{c_0} \frac{u_m(z_t) u_n(z_t)}{d(z_t)} \Delta \rho 
(B.11)
\]
Since the mode shape at source depth \( u_n(z_0) \) is not changed by internal waves, the change of \( \Phi_n(r|r_0) \) due to internal wave can be expressed as

\[
\Delta \Phi_n(r|r_0) = \frac{e^{i\xi_n\rho}}{\sqrt{\xi_n}} \Delta \psi_n(\rho, z) u_n(z_0)
\]

(B.12)

and Eq. B.8 can be written as

\[
\langle \Delta \psi_n(\rho, z) \rangle = -i \sum_{m=1}^{N} \frac{k^2}{\xi_m} \psi_m(\rho, z) \int dz_t \frac{\Delta c(\rho_t, z_t)}{c_0} \frac{u_m(z_t)u_n(z_t)}{d(z_t)} \Delta \rho_s
\]

(B.13)

Comparing Eq. B.13 with Eq. B.4, it is found that, for diagonal term where \( n = m \), two equations matches with each other exactly. For the off-diagonal term \( n \neq m \), beside the slight difference of the horizontal wave-number in the denominator, there is an extra phase term \( e^{i(\xi_n-\xi_m)\rho} \) in Dozier and Tappet’s expression. We think the change of mean modal amplitude \( \langle \Delta \psi_n(\rho, z) \rangle \) due to the local random inhomogeneity distributed between \( \rho \) and \( \rho + \Delta \rho \) should only be determined by the properties of these inhomogeneities. When the inhomogeneity induced sound speed fluctuations become small, the mean modal amplitude should also become small. This extra phase term \( e^{i(\xi_n-\xi_m)\rho} \) that appears in Dozier and Tappet’s expression of Eq. B.4, however, still remains constant and is totally independent on the properties of inhomogeneities. Our analysis, based on the first principle of scattering theory shows that it should not be included in \( \langle \Delta \psi_n(\rho, z) \rangle \).

The analytical expression of the mean modal amplitude change from Dozier and Tappet is a 2-D model and can not account for 3-D scattering processes (out-of-plan scattering). It can only be applied to weak inhomogeneities such as internal waves that cause small sound speed fluctuation. Our formulation of the mean modal amplitude change, as shown in Ref. [4], is derived from the first principle and can account for both 2-D and 3-D scattering process. It is also a general expression and can be applied to different types of inhomogeneities including internal waves, bubbles, fish shoals and sub-bottom anomalies. The only assumption needed in the our derivation is the single scattering approximation within the single shell between \( \rho \) and \( \rho + \Delta \rho \),
which is well satisfied and commonly used.
Appendix C

Numerical test for the assumption of diagonalization

One of the fundamental assumptions made in Ref. [4] is that the off-diagonal term in the scattered field

\[ \Phi_s(r|r_0, \Delta \rho_s) = \sum_n \sum \int_0^{\infty} \frac{d\chi}{k} \frac{2\pi}{d(z_0)d(\chi)} \frac{1}{\sqrt{\xi_m \xi_n}} \frac{i}{\xi_m^2 - \xi_n^2} u_m(z) u_n(z_0) e^{i\xi_m \rho} e^{i(\xi_n - \xi_m) \rho} \text{sinc} \left[ \left( \frac{\xi_n - \xi_m}{2} \right) \Delta \rho_s \right] \times \left[ N^{(1)}_m N^{(1)}_n e^{i(\gamma_m + \gamma_n) \chi} s_{\rho_s, \chi}(\alpha_m, \phi, \pi - \alpha_n, \phi) - N^{(2)}_m N^{(1)}_n e^{-i(\gamma_m + \gamma_n) \chi} s_{\rho_s, \chi}(\alpha_m, \phi, \alpha_n, \phi) - N^{(1)}_m N^{(2)}_n e^{i(\gamma_m - \gamma_n) \chi} s_{\rho_s, \chi}(\pi - \alpha_m, \phi, \pi - \alpha_n, \phi) + N^{(2)}_m N^{(2)}_n e^{-i(\gamma_m - \gamma_n) \chi} s_{\rho_s, \chi}(\pi - \alpha_m, \phi, \alpha_n, \phi) \right] \Delta \rho_s. \] (C.1)

is much smaller than the diagonal term when the shell thickness is large enough that \( \text{sinc} \left[ \left( \frac{\xi_n - \xi_m}{2} \right) \Delta \rho_s \right] \Delta \rho_s \Delta \rho_{\text{max}} \ll 1 \) or equivalently the condition,

\[ \Delta \rho_{\text{max}} \gg \lambda \left( \sin \alpha_n - \sin \alpha_m \right). \] (C.2)
is satisfied [11].

The mean scattered field from the shell, under condition (C.2), is then the single modal sum

\[
\Phi_s(r|\tau_0, \Delta \rho_s(\rho_s)) = \sum_n 4\pi \frac{i}{d(z_0)\sqrt{8\pi}} e^{-in/4} u_n(z)u_n(z_0) e^{i\xi_n\rho} \frac{d(z_0)}{\xi_n}
\]

\[
\times i \int \frac{1}{d(z_0) \xi_n} \left[ (N_n^{(1)})^2 e^{i2\gamma z_0} S_{\rho s, z x}(\pi - \alpha_n, \phi; \alpha_n, \phi) - N_n^{(2)} N_n^{(1)} S_{\rho s, z x}(\alpha_n, \phi; \alpha_n, \phi) \right] dz_0 \Delta \rho_{\text{max}}.
\]

(C.3)

In this appendix, we will prove the validity of this assumption by calculating the ratio between Eq. C.1 and Eq. C.3 when the inhomogeneities are random internal waves in both continental shelf and deep ocean environment.

The scatter function of an internal wave inhomogeneity is expressed in Eq.

\[
s_{\rho s, z x}(\alpha, \beta, \alpha_i, \beta_i) = \frac{1}{A_c(\rho_s, z_x)} \int k^3 4\pi \left[ \Gamma_s(\rho_s, z_x) + \eta(k, k_i) \Gamma_d(\rho_s, z_x) \right] e^{i(\xi - \xi) wz} d\xi d\eta.
\]

(C.4)

where \( \rho_x = \rho_s + u_x \) and

\[
\eta(k, k_i) = \frac{k_i \cdot k}{k^2} = \cos \alpha_i \cos \alpha + \sin \alpha_i \sin \alpha \cos(\beta_i - \beta)
\]

is the cosine of the angle between the incident and scattered plane wave directions.

Since the fluctuations of sound speed (\( \Delta c \)) and density (\( \Delta d \)) arising from random internal waves are much smaller than the unperturbed or local equilibrium sound speed and density, the fractional change of compressibility \( \Gamma_c \) and density \( \Gamma_d \) can be
expanded up to second order in Taylor series

\[
\Gamma_x \approx \left[-\frac{2\Delta c}{c_0} - \frac{\Delta d}{d_0}\right] + \left[3\left(\frac{\Delta c}{c_0}\right)^2 + \left(\frac{\Delta d}{d_0}\right)^2 + 2\frac{\Delta c \Delta d}{c_0 d_0}\right],
\]

\[
\Gamma_d \approx \frac{\Delta d}{d_0} - \left(\frac{\Delta d}{d_0}\right)^2.
\]

(C.5)

Fluctuations of sound speed and density, for practical purpose, are linearly dependent on the displacement of the internal wave[46, 70] \(\xi(\rho_x, z_x)\) via,

\[
\frac{\Delta c(\rho_x, z_x)}{c_0} = \xi(\rho_x, z_x)\tilde{G}(z_x)n^2(z_x)
\]

\[
\frac{\Delta d(\rho_x, z_x)}{d_0} = \xi(\rho_x, z_x)g^{-1}n^2(z_x),
\]

(C.6)

where \(\tilde{G}(z_x)\) is a function of the potential temperature and salinity[46], \(g\) is the gravitational constant and \(n(z_x)\) is the buoyancy frequency or Brunt – Väisälä frequency,

\[
n^2(z_x) = -g \frac{d}{dz} \rho_p(z_x),
\]

(C.7)

where \(\rho_p(z_x)\) is the potential density[46].

Since the fluctuation of sound speed is much than that of density from Eq. C.6, the scatter function of Eq. C.9 can be simplified to

\[
s_{\rho_x, z_x}(\alpha, \beta, \alpha_i, \beta_i) \approx \frac{1}{A_c(\rho_x, z_x)} \int \int \frac{k^3}{4\pi} \Gamma_x(\rho_x, z_x) e^{i(\xi - \xi_0) \cdot \mathbf{u}_x} d^2 \mathbf{u}_x,
\]

\[
\approx \frac{-2}{A_c(\rho_x, z_x)} \frac{k^3}{4\pi} \tilde{G}(z_x)n^2(z_x) \int \int \xi(\rho_x, z_x) e^{i(\xi - \xi_0) \cdot \mathbf{u}_x} d^2 \mathbf{u}_x
\]

(C.8)

Eq. C.9 can even be further simplified to be

\[
s_{\rho_x, z_x}(\alpha, \beta, \alpha_i, \beta_i) \approx (-2) \frac{k^3}{4\pi} \tilde{G}(z_x)n^2(z_x)\xi(\rho_x, z_x) \text{sinc}[(\xi - \xi_0)L_x/2]
\]

(C.9)
if we assume that the internal wave displacement is not various much within the
coeherence area $A_c$, where $L_z$ is the correlation length over range direction.

![Figure C-1: Numerical value of $\lambda/\sin(\alpha_1 - \sin \alpha_m)$ as a function of mode number $m$, where $\lambda$ is the acoustic wave length.](image)

Fig. C-1 shows the numerical value of $\lambda/\sin(\alpha_1 - \sin \alpha_m)$ of Eq. C.2, when $n = 1, m$ varies from 2 to 25. It shows that $\Delta \rho_{\text{max}}$, for first few dominant mode ($m < 5$), need to be greatly larger than 5km in order to satisfy Eq. C.2. Fig. C-2 shows the ratio between Eq. C.1 and Eq. C.3 as a function of source-receiver separation in a shallow water environment where the correlation radius[4] of internal wave is 500 m. When $\Delta \rho_{\text{max}}$ is the same with the internal wave correlation length, there only one internal wave inhomogeneities contained in the shell. This could lead to rough 20% error when neglecting the off-diagonal term. As $\Delta \rho_{\text{max}}$ increases from 1000m to 5000m, the ratio becomes more and more close to 1. It is not necessary for $\Delta \rho_{\text{max}}$ to be much larger than 5000m to make off-diagonal term negligible. This is because the scatter function density of internal waves also have a sinc function $\text{sinc}[(\xi_x - \xi_x)L_z/2]$ that can also de-couple modes in the forward direction. The total effects of the two sinc functions make the assumption of diagonalization approximately valid even when
\( \Delta \rho_{\text{max}} \) is on the same order of \( \lambda / (\sin \alpha_n - \sin \alpha_m) \).

Figure C-2: Ratio of the scatter field with and without off-diagonal terms for internal wave inhomogeneities in a shallow water environment. The internal wave correlation length is 500 m.

Fig. C-3 show the ratio in a deep ocean environment where the correlation length of internal waves is approximate 7 km. It shows that the ratio is close to 1 when \( \Delta \rho_{\text{max}} \) is larger than 2000 m. The assumption of diagonalization is a good approximation under this condition. When \( \Delta \rho_{\text{max}} \) is close the internal wave correlation length and only one internal wave inhomogeneity contained in the shell, the error to neglect off-diagonal term could be as large as 100%.

In summary, the thickness of shell should be larger than correlation length of the inhomogeneity to validate the diagonalization approximation. At the same time, it should also be small enough in order to make sure that the single scattering approximation is valid.
Figure C-3: Similar to Fig. C-2 but for a deep ocean waveguide. The correlation length of internal waves is 7 km.
Appendix D

Statistics of scatter function
density of wind-generated bubble
clouds and fish shoals containing
swim-bladder

D.0.1 Scatter function of an air-filled bubble

In this section, we give the scatter function of a single bubble using two different physical models. Then, we review the effective medium method, which could be used to calculate effective medium wavenumber change due to the presence of bubbles. The scatter function of a single bubble then can be calculated from the effective medium wavenumber change.

Modeling a single bubble as a non-absorptive air-filled penetrable sphere

In this model, the scattering process due to a single bubble is assumed to following an adiabatic process without any acoustic energy loss (non-absorptive). The acoustic pressure and normal velocity are also assumed to be continuous across the boundary of the bubble (penetrable).

The scatter function of a non-absorptive air-filled penetrable sphere is expressed
as \[ S(\alpha, \beta; \alpha_i, \beta_i) = S(\eta(\alpha, \beta; \alpha_i, \beta_i)) = \sum_{n=0}^{\infty} f(n) P_n(\eta(\alpha, \beta; \alpha_i, \beta_i)), \] (D.1)

where \( f(n) = i(-1)^n(2n + 1)A_n \) and the coefficient \( A_n \)

\[
A_n = \frac{j_n'(ka) - (\rho c/\rho_a c_a)[j_n'(kaa)/j_n(kaa)]j_n(ka)}{h_n'(ka) - (\rho c/\rho_a c_a)[j_n'(kaa)/j_n(kaa)]h_n(ka)}
\]

is determined by the boundary conditions at the sphere's surface given internal air density \( \rho_a \), sound speed \( c_a \), wavenumber \( k_a = \omega/c_a \), the surrounding medium’s density \( \rho \), sound speed \( c \) and wavenumber \( k \). For a compact sphere where \( ka \ll 1 \), its scatter function is approximately equal to the first term of Eq. D.1

\[
S(\alpha, \beta; \alpha_i, \beta_i) \approx f(0) P_0(\eta(\alpha, \beta; \alpha_i, \beta_i)) = (ka)^3 \left( \frac{\rho c^2}{3\rho_a c_a^2} \right) + (ka)^6 \left( \frac{\rho c^2}{3\rho_a c_a^2} \right)^2.
\]

Note that this zero order term is omnidirectional.

We can also express Eq. D.2 in terms of Minnaert resonant frequency

\[
S(\alpha, \beta; \alpha_i, \beta_i) = (ka) \left( \frac{\omega}{\omega_0} \right)^2 + i(ka)^2 \left( \frac{\omega}{\omega_0} \right)^4,
\] (D.2)

where \( \omega_0 = \frac{1}{a} \sqrt{\frac{3\gamma \rho_b}{\rho}} = \frac{1}{a} \sqrt{\frac{3\rho_a c_a^2}{\rho}} \) is the Minnaert resonance frequency for a bubble with radius \( a \), \( \gamma = \frac{\rho_a c_a^2}{F_0} \approx 1.4 \) is the ratio of the heat capacity of air at constant pressure to that at constant volume. The scatter function \( S(\alpha, \beta; \alpha_i, \beta_i) \) of Eq. D.2 is only applicable for a non-absorptive air-filled sphere and accounts for the scattering effect (radiation damping) but not thermal and viscous damping. The resonant effect of bubble is also not included in Eq. D.2.
Modeling a single bubble as a damped forced oscillator

Here, a single bubble is modeled as a damped oscillator that pulsates under the incident acoustic pressure. This damped oscillator behaves as a "new source" and radiates acoustic waves in all directions.

The displacement $R_e$ of the bubble boundary from its equilibrium position under the influence of an incident acoustic pressure is\[81\]

$$R_e = -\frac{P_A}{\rho_A} \frac{\sin \theta e^{\delta_{tot} t}}{\omega^2 \sqrt{\left(\frac{\omega^2}{\omega_0^2} - 1\right)^2 + \delta_{tot}^2}},$$

where $P_A$ is the incident acoustic pressure at the center of the bubble, $\cos \theta = \frac{\sqrt{\left(\frac{\omega^2}{\omega_0^2} - 1\right)^2 + \delta_{tot}^2}}{\sqrt{\left(\frac{\omega^2}{\omega_0^2} - 1\right)^2 + \delta_{rad}^2}}$ and $\delta_{tot}$ is the total damping coefficient given by

$$\delta_{tot} = \delta_{rad} + \delta_{th} = k a + d_{th} \left(\frac{\omega_0}{\omega}\right)^2,$$

where $\delta_{rad}$ and $\delta_{th}$ are the radiation and thermal damping coefficients respectively. The viscous damping coefficient is much smaller compared to thermal and radiation damping coefficients when the acoustic frequency is not much larger than bubble's resonant frequency\[81\]. The dimensionless thermal damping coefficient $d_{th}$ in Eq. D.4 is \[82\]

$$d_{th} = \frac{3(\gamma - 1)(a/\ell_D)|\sinh(a/\ell_D) + \sin(a/\ell_D)| - 2|\cosh(a/\ell_D) - \cos(a/\ell_D)|}{(a/\ell_D)^2|\cosh(a/\ell_D) - \cos(a/\ell_D)| + 3(\gamma - 1)(a/\ell_D)|\sinh(a/\ell_D) + \sin(a/\ell_D)|},$$

where $\kappa$ is the polytropic index which varies from $\gamma$ (adiabatic case) to unity (isothermal case). The width of thermal boundary layer $\ell_D$ equals to $\sqrt{D_a/2\omega}$. The thermal diffusivity of air $D_a$ is $2.3 \times 10^{-5}$ m$^2$/sec at 20°C.

The rate of mass flow of the medium around the oscillating bubble can be written
\[ n(t) = 4\pi a^2 \rho \tilde{R}_e = \frac{4\pi a P_A e^{i\theta} e^{i\omega t}}{\sqrt{\left(\frac{\omega_0^2}{2} - 1\right)^2 + \delta_{tot}^2}}. \]  

\text{(D.6)}

The radiated wave at range \( r \) is then

\[ P_s = \frac{n(t - \frac{r}{c})}{r} = P_A \left(\frac{4\pi}{k}\right) \frac{e^{i\theta} k a}{\sqrt{\left(\frac{\omega_0^2}{2} - 1\right)^2 + \delta_{tot}^2}} \frac{e^{i\omega (t - \frac{r}{c})}}{4\pi r}, \]

\text{(D.7)}

where the scatter function is found to be

\[ S = \frac{\left(\frac{\omega_0^2}{2} - 1\right) k a}{\left(\frac{\omega_0^2}{2} - 1\right)^2 + \delta_{tot}^2} + i \frac{\delta_{tot} k a}{\left(\frac{\omega_0^2}{2} - 1\right)^2 + \delta_{tot}^2}. \]

\text{(D.8)}

Comparing Eq. D.2 with Eq. D.8, we find that the two scatter functions equal each other under the following conditions: (i) the acoustic frequency \( \omega \) is much smaller than the bubble's resonant frequency \( \omega_0 \). (ii) the thermal damping is excluded in Eq. D.8 since the first model does not account for this effect. The expressions given in Eqs. D.3 and D.6 are valid only when \( ka \ll 1 \). This is because we assume that the incident pressure in Eq. D.6 is uniformly distributed over the entire bubble surface and is given by the value at the center of bubble. This assumption breaks down when the radius of the bubble is comparable to the acoustic wavelength and the bubble can no longer be modeled as an omnidirectional scatterer. In this case, the first model should be applied to calculate the scatter function and higher order terms of Eq. D.1 should be included to account for the azimuthal dependence of the scatter function.
Scatter function derived from effective-medium theory

In the effective-medium theory, a bubbly medium is modeled as an effectively homogeneous continuum with decreased density and compressibility. The effective wavenumber[81] of a medium containing bubbles that are identical and uniformly distributed over space is expressed as

\[
    k_{\text{eff}} = \frac{\omega}{c} \left( 1 + 2\pi c^2 n_v a \left( \frac{\omega^2}{\omega_0^2} - \frac{\omega^2}{\omega_0^2} + i\delta_{\text{tot}} \omega^2 \right) \right) \tag{D.9}
\]

based on the effective bulk modulus. The change in wavenumber due to the presence of bubbles is

\[
    \nu = k_{\text{eff}} - k = \frac{2\pi}{k^2} n_v \left( \frac{\left( \frac{\omega^2}{\omega_0^2} - 1 \right) k a}{\left( \frac{\omega^2}{\omega_0^2} - 1 \right)^2 + \delta_{\text{tot}}^2} + i \frac{\delta_{\text{tot}} k a}{\left( \frac{\omega^2}{\omega_0^2} - 1 \right)^2 + \delta_{\text{tot}}^2} \right). \tag{D.10}
\]

The wavenumber change due to the existence of small uniform discrete scatterers is related to the scatter function of each individual scatterer in free space [5, 83] as

\[
    \nu = \frac{2\pi}{k^2} n_v S, \tag{D.11}
\]

where \( S \) is the scatter function of the individual scatterer.

The scatter function of a single bubble \( S \) is then found to be

\[
    S = - \frac{\left( \frac{\omega^2}{\omega_0^2} - 1 \right) k a}{\left( \frac{\omega^2}{\omega_0^2} - 1 \right)^2 + \delta_{\text{tot}}^2} + i \frac{\delta_{\text{tot}} k a}{\left( \frac{\omega^2}{\omega_0^2} - 1 \right)^2 + \delta_{\text{tot}}^2} \tag{D.12}
\]

by comparing Eq. D.10 with Eq. D.11. The scatter function \( S \) calculated using the effective medium method is same as the scatter function in Eq. D.8 that is derived from modeling the bubble as a damped oscillator. This is because the same equation for the displacement of a bubble boundary in Eq. D.3 is applied in both methods. The effective wavenumber for an unbounded medium can only be calculated if the spatial distribution of the small scatterers is uniform in the entire medium. Hence, the effective medium method cannot be directly applied to calculate the wavenumber.
change in an ocean waveguide containing bubbles whose spatial distribution is a function of depth.

**D.0.2 Scatter function of a fish swim-bladder**

The scatter function of a fish swim-bladder is modeled differently from that of an air-filled bubble because (i) the swim-bladder is prolate spheroidal rather than spherical and (ii) the flesh around the swim-bladder[84] affects its resonant response.

The resonant frequency of a prolate spheroid shaped swim-bladder is found to be

\[ \omega_{\text{res}} = \frac{1}{a} \sqrt{\frac{3\gamma P}{\rho}}, \quad (D.13) \]

where \( a = \left( \frac{3}{4\pi} V \right)^{1/3} \) is the equivalent radius[85] of the prolate spheroid shaped swim-bladder with volume \( V \), \( \zeta \) is the correction term given by Weston[86] as

\[ \zeta = \frac{\sqrt{2} (1 - \epsilon^2)^{1/4}}{\epsilon^{1/3}} \left\{ \ln \left[ \frac{1 + \sqrt{1 - \epsilon^2}}{1 - \sqrt{1 - \epsilon^2}} \right] \right\}^{-1/2}, \quad (D.14) \]

where \( \epsilon \) is the ratio of the semi-minor axis \( a \) and semi-major axis \( b \) of a prolate spheroid. The major axis \( 2b \) remains fixed and approximately equals to 26-33\% of the total length of fish[87]. The semi-minor axis \( a \) at depth \( z \) is expressed as

\[ a(z) = \left( \frac{P(0)V(0)}{P(z)b} \right)^{1/2} \frac{3}{4\pi}, \quad (D.15) \]

by assuming that the volume of the swim-bladder follows Boyle’s Law, where \( P(z) \) is the pressure at depth \( z \) and \( V(0) \) is the volume of fish swim-bladder at surface, which can be related to the total fish weight via

\[ V(0) = 0.05(D_{\text{nb}}/10 + 1) \times W_{\text{fish}}. \quad (D.16) \]

Here, the volume of swim-bladder in \( cm^3 \) is approximately 5\% of the total weight of fish in grams at neutral buoyancy depth \( D_{\text{nb}} \). The fish weight can be determined
from the total fish length $L_{\text{fish}}$, for example, the weight of Atlantic herring$^{[87]}$ is approximately $0.0033L_{\text{fish}}^3$. 

The damping coefficient due to flesh around the swim-bladder, $\delta_{\text{flesh}}$, is given in the acoustic scattering cross-section$^{[84]}$ in the backscattered direction as

$$\sigma_{\text{scat}} = \frac{4\pi \bar{a}^2}{(\omega_{\text{res}} - \omega^2)^2 + (\delta_{\text{rad}} + \delta_{\text{flesh}})^2}$$

$$= \frac{4\pi \bar{a}^2}{(\omega_{\text{res}} - \omega^2)^2 + (\omega_{\text{res}})^2 \rho_{\text{fish}}^2}.$$  \hspace{1cm} (D.17)

The constant $H$ is given by

$$\frac{1}{H} = \frac{2\pi \bar{a} f^2}{\omega_{\text{res}} c} + \frac{\xi}{\pi \bar{a}^2 f_{\text{res}} \rho_{\text{fish}}}$$

$$= \frac{2\xi}{\omega_{\text{res}} c} + \frac{2\xi}{\bar{a}^2 \omega_{\text{res}} \rho_{\text{fish}}}$$

$$= \frac{\omega}{\omega_{\text{res}}} (\delta_{\text{rad}} + \delta_{\text{flesh}}),$$  \hspace{1cm} (D.18)

where $\xi$ is the viscosity of the fish flesh in Pa s. The damping coefficient due to the flesh is found to be

$$\delta_{\text{flesh}} = \frac{2\xi}{\bar{a}^2 \omega_{\text{res}} \rho_{\text{fish}}}. \hspace{1cm} (D.19)$$

The scatter function of the swim-bladder can be expressed as

$$S = \frac{(\omega_{\text{res}}^2 - \omega^2)}{(\omega_{\text{res}}^2 - \omega^2)^2 + \delta_{\text{tot}}^2} k\bar{a} + i \frac{\delta_{\text{tot}} k\bar{a}}{(\omega_{\text{res}}^2 - \omega^2)^2 + \delta_{\text{tot}}^2}.$$  \hspace{1cm} (D.20)

by inserting the total damping coefficient $\delta_{\text{tot}} = \delta_{\text{rad}} + \delta_{\text{flesh}}$, equivalent radius and resonant frequency of the swim-bladder into the scatter function of a damped forced
oscillator of Eq. D.8. The extinction cross section of swim bladder is expressed as

$$
\sigma_{\text{ext}} = \sigma_{\text{sca}} + \sigma_{\text{obs}}
= \frac{4\pi}{k^2} \frac{\tilde{\alpha}_{\text{tot}} k \bar{a}}{(\omega_r^2 - 1)^2 + \delta_{\text{tot}}^2}
= \frac{4\pi \bar{a}^2}{(\omega_r^2 - 1)^2 + \delta_{\text{tot}}^2} + \frac{4\pi \bar{a}^2 (\delta_{\text{flesh}} / \delta_{\text{rad}})}{(\omega_r^2 - 1)^2 + \delta_{\text{tot}}^2}
$$

(D.21)

by applying extinction theory

$$
\sigma_{\text{ext}} = \frac{4\pi}{k^2} \Im(S(0,0,0,0)).
$$

(D.22)

The extinction cross section of Eq. D.21 is composed of the scattering cross section $\sigma_{\text{sca}}$ and absorption cross section $\sigma_{\text{obs}}$ that account for energy removed from the forward direction due to radiation effect of swim bladder and energy loss, if any, due to the absorption effect of flesh, respectively. Note that $\sigma_{\text{sca}}$ in Eq. D.21 is consistent with $\sigma_{\text{sca}}$ of Eq. D.17. When Love[73] calculated $\sigma_{\text{sca}}$, the flesh surrounding fish swim bladder, is modeled as a viscous and heating conducting fluid and its viscosity is empirically determined. The actual flesh is elastic solid and modeling it as a viscous fluid maybe not properly account for its real effect on the acoustic wave propagation. This parameterized and linear model by Love[73] probably oversimplifies the role of flesh in the scattering process. In our understanding, the damping coefficient $\delta_{\text{flesh}}$, probably only decreases the amplitude of radial displacement of swim bladder at resonance, thereby decreasing the efficiency of the scattering. It is not a “real” damping coefficient and there is no energy absorption by the flesh when the acoustic wave is propagated through the swim bladder. The actual effect on scattering, due to flesh, is probably between the two extreme cases of (1) including absorption effect due to damping coefficient $\delta_{\text{flesh}}$, as modeled by Love and (2) excluding absorption effect, where the flesh only restricts expansion and contraction of swim bladder.

For a non-absorptive swim bladder, the scattering cross section equals the extinction cross section. The extinction theorem is applied to calculate the imaginary part.
of the scatter function of swim bladder in the forward direction

\[ \Im(S(0,0,0,0)) = \frac{(k\bar{a})^2}{(\omega_e^2 - 1)^2 + \delta^2_{\text{tot}}} = \frac{\delta_{\text{rad}} k\bar{a}}{(\omega_e^2 - 1)^2 + \delta^2_{\text{tot}}} \]  

(D.23)

by inserting \( \sigma_{\text{sca}} \) into Eq. D.22. Since the swim bladder can be modeled as an omnidirectional scatterer \((k\bar{a} << 1)\) even at the resonance, the scatter function of swim bladder is

\[ S = \frac{(\omega_e^2 - 1)k\bar{a}}{(\omega_e^2 - 1)^2 + \delta^2_{\text{tot}}} + i \frac{\delta_{\text{rad}} k\bar{a}}{(\omega_e^2 - 1)^2 + \delta^2_{\text{tot}}} \]  

(D.24)

by replace the imaginary part of scatter function in Eq. D.20 with Eq. D.23. Compared to the scatter function of Eq. D.20, the numerator of imaginary part of scatter function in Eq. D.24 only depends on the radiation damp coefficient \( \delta_{\text{rad}} \) in stead of \( \delta_{\text{tot}} \) to exclude the absorption effect from flesh. In the simulations, we will calculate the attenuation based on these two scatter functions of Eqs. D.20 and D.24.

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