ASYMPTOTIC STABILITY, IDENTIFICATION,
AND THE HORIZON PROBLEM

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ABSTRACT: Recent literature in discrete adaptive control has emphasized the importance of asymptotic stability of the adaptive controller in obtaining convergence of system parameter estimates to their true values. This paper studies the relationship between these results and the problem of the convergence of first period decisions in planning models as the planning horizon time increases. The primary results to date have been based on stationary and purely quadratic cost functions. This paper extends these results to cost functions containing linear terms and to discounted cost functions. The main results are a set of sufficient conditions on the nature of cost and system parameters under which first period decisions converge to a fixed value as the optimization time horizon increases. A characterization of the optimal asymptotic controls is given for the discounted and undiscounted cases.

I. INTRODUCTION

Consider the following linear system.

\begin{align*}
(1.1) \quad x_{k+1} &= Ax_k + Du_k + v_k \\
(1.2) \quad y_k &= Mx_k + w_k
\end{align*}

In (1.1), $x_k$ is a $p$-dimensional column vector which represents the state of the system at time $k$; $A$ is a $p \times p$ transition matrix; $D$ is a $p \times r$ control matrix; $u_k$ is an $r$-dimensional control vector; and $v_k$ represents a random disturbance. In (1.2), $y_k$ is a $q$-dimensional vector representing an observation made on the system at time $k$; $M$ is a $q \times p$ observation matrix; and $w_k$ represents noise. We will assume that $v_0, v_1, \ldots$ and $w_1, w_2, \ldots$ are independent sequences of zero mean, independent and identically distributed random vectors with covariance matrices $V$ and $W$ respectively and that $x_0$ is independent of the $v_i$'s and $w_j$'s and has finite covariance matrix.
Linear least squares prediction and filtering may be done for the system (1.1)-(1.2) using the Kalman Filter [1], which yields the projections \( x_t|k \) and \( y_t|k \) of \( x_t \) and \( y_t \) on the Hilbert subspace spanned by \( \gamma_1, \gamma_2, \ldots, \gamma_k \). These projections are given by

\[
\begin{align*}
\dot{x}_t|k &= (I - A_k)x_{t-1}|k-1 + A_k \gamma_k, \quad k \geq 1 \\
\dot{y}_t|k &= M_x \dot{x}_t|k, \quad t > k,
\end{align*}
\]

where \( I \) denotes the \( pxp \) identity matrix and \( x_0|0 = E[x_0] \). The weighting matrix \( A_k \) in (1.3) is determined by

\[
\begin{align*}
A_k &= S_k^{\frac{1}{2}} [MS_k'W + W]^+, \quad k \geq 1 \\
S_k &= AP_k^{-1}A' + V, \quad k \geq 1 \\
P_k &= [I - A_k]\gamma S_k', \quad k \geq 1
\end{align*}
\]

where \( \dagger \) denotes pseudo-inverse, \( ' \) denotes transpose, and \( P_0 \) is the covariance matrix of \( x_0 \).

Now consider the following optimization problem.

\[
(1.5) \quad \min E \{ \sum_{k=1}^{N} [x_k'Q_{1,k}x_k + u_{k-1}'Q_{2,k-1}u_{k-1} + G_{1,k}x_k + G_{2,k-1}u_{k-1}] \}
\]

subject to (1.1), (1.2), \( k=0,1,\ldots,N-1 \), where for \( k=1,2,\ldots,N \), \( Q_{1,k} \) and \( Q_{2,k-1} \) are symmetric positive semi-definite (psd) matrices and \( G_{1,k}, G_{2,k-1} \) are \( 1xp \) and \( 1xr \) row vectors respectively, and one of the following conditions is satisfied: (a) \( Q_{2,k-1} \) is positive definite (pd); (b) \( Q_{1,k} \) is pd and \( \text{rank}(D) = \min(p,r) \). Either (a) or (b) is required to insure the existence of a finite minimum of the performance criterion and to insure the existence of matrix inverses for the dynamic programming solution to the problem, (1.5).

In an extension of Gunckel and Franklin's result [3] for the pure quadratic loss function, it can be shown (see [7]) that the optimal controls, \( u_k \), in (1.5) are given by

\[
(1.6) \quad u_k = -C_k x_k|k - H_k^{-1}z_k'/(1/2) \quad 0 \leq k \leq N-1;
\]
where $C_k$ and $H_k$ are determined by

\[(1.7a) \quad H_k C_k = D'(F_{k+1} + Q_{1,k+1})A \]
\[(1.7b) \quad H_k = Q_{2,k} + D'(F_{k+1} + Q_{1,k+1})D \]
\[(1.7c) \quad F_k = A'(F_{k+1} + Q_{1,k+1})A - C_k H_k C_k', \quad F_N = 0 \]
\[(1.8a) \quad Z_k = G_{1,k+1}D + G_{2,k} + B_{k+1}D \]
\[(1.8b) \quad B_k = (G_{1,k+1} + B_{k+1})A - Z_k C_k', \quad B_N = 0. \]

where it is shown in the dynamic programming solution to the problem that $F_{k+1}$ is psd and therefore, by the assumptions on $Q_{1,k+1}, Q_{2,k},$ and $D, H_k$ is nonsingular and $C_k$ is determined uniquely from (1.7a).

As Kalman has shown, there is a close relationship between equations (1.4) and (1.7) which allows results obtained for the filter equations, (1.4), to be applied to the optimization equations (1.7) and (1.8). In section II of this paper we use this relationship and the uniform asymptotic stability of the Kalman filter to show that the matrices $C_o = C_o(N)$ and $H_o^{-1}Z_o' = H_o^{-1}(N)Z_o'(N)$ in (1.6) converge to a fixed point as the time horizon $N$ increases, when the costs in (1.5) are stationary. This result is extended in section III to the case of discounted costs. In section IV we explore the implications of these results for the aggregate planning problem in production scheduling. Briefly, these results imply that when sales are generated by a linear autoregressive system, then the first period decision rules approach a fixed point as the time horizon is increased. Finally, generalizations to the above results are discussed in the concluding remarks.

The above results for the undiscounted case with pure quadratic loss function were originally proven by Kalman [5], although his results on the "separation theorem" were known earlier in another form as the certainty equivalence theorem, see Simon [8]. Kalman's results were applied and extended to the problem of aggregate planning and information system design.
by G. B. Kleindorfer [6]. Anderson et al. extended Kleindorfer's work in their study of the identification problem and they laid most of the groundwork for the analytical methods to be used here. The present work extends past results on the convergence of the first period decision rules to the case where the objective function contains linear terms in the state and control variables as well as extending known results to the discounted case.

II. THE UNDISCOUNTED CASE

We begin by studying the asymptotic behavior of (1.4). We may combine equations (1.4) to obtain

\[(2.1) \quad S_{k+1} = A(S_k - S_k M'[MS_k M' + W]^T MS_k)A' + V, \quad k \geq 1.\]

Moreover, it is clear from (1.4) that the matrices \(\{S_1, S_2, \ldots\}\) uniquely determine the corresponding sequences \(\{A_1, A_2, \ldots\}\) and \(\{P_1, P_2, \ldots\}\). We therefore restrict our attention to a study of (2.1) and note the following result proven in Anderson et al. [1].

Theorem 1: Let \(M = I\) and let \(V\) be pd and \(W\) psd; define \(\Phi\) on the set \(T\) of pd matrices by

\[(2.2) \quad \Phi(S) = AS(S + W)^{-1}WA' + V, \quad S \in T \]

so that \(S_k = \Phi(S_{k-1}), \quad k \geq 2\). Then \(\Phi\) has a unique pd fixed point \(S_0\), and \(\Phi^n(S) \rightarrow S_0\) uniformly on \(T\) as \(n \rightarrow \infty\), where \(\Phi^n\) denotes the \(n\)th iterate of \(\Phi\). Moreover, \(A_k \rightarrow A_0 = S_0(S_0 + W)^{-1}\), and \(P_k \rightarrow P_0 = S_0 - S_0(S_0 + W)^{-1}S_0\) as \(k \rightarrow \infty\).

This result was first proven originally by Kalman [5] under the assumption that the system (1.1) and (1.2) is completely observable and completely controllable. His proof also allows for \(M\), the observation matrix, to be non-square.

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1 Results of a similar nature are also contained in some unpublished research of Professor Lance Taylor of Harvard University.
Theorem 1 may also be easily generalized to include a non-square observation matrix.

**Corollary 1:** Let $V$ and $W$ be pd and let $\text{rank}(M) = \min(q,p)$; define $\phi$ on the set $T$ of pd matrices by

$$
\phi(S) = A(S^{-1} + M'W^{-1}S)^{-1}A' + V, \quad S \in T.
$$

Then for any $S_1$ pd, $S_k = \phi(S_{k-1})$, $k \geq 2$. Moreover, $\phi$ has a unique pd fixed point, $S_0$, and $\phi^n(S) \to S_0$ uniformly on $T$ as $n \to \infty$, where $\phi^n$ denotes the nth iterate of $\phi$. Furthermore, $A_k + A_o = S_o M'[MS_oM' + W]^{-1}$, and $p_k \to P_o = S_o - S_o M'[MS_oM' + W]^{-1}MS_o$ as $k \to \infty$.

**Proof:** We first verify that for any $S_1$ pd, $S_k = \phi(S_{k-1})$, $k \geq 2$. By (2.1) and the fact that $[MS_kM' + W]^+ = [MS_kM' + W]^{-1}$ since $W$ is pd, we only need to show that

$$
(S^{-1} + M'W^{-1}S)^{-1} = S - SM'[MSM' + W]^{-1}MS, \quad S \in T.
$$

To demonstrate the validity of (2.4) consider the following calculations.

$$
(S^{-1} + M'W^{-1}S)(S - SM'[MSM' + W]^{-1}MS)
= I - M'[MSM' + W]^{-1}MS + M'W^{-1}MS - M'W^{-1}MS' + W^{-1}MS'[MSM' + W]^{-1}MS
= I + M'(W^{-1} - [MSM' + W]^{-1} - W^{-1}MS'[MSM' + W]^{-1})MS
$$

However,

$$
W^{-1}MS'[MSM' + W]^{-1} = W^{-1}(MSM' + W - W)[MSM' + W]^{-1}
= W^{-1}(I - W[MSM' + W]^{-1})
= W^{-1} - [MSM' + W]^{-1},
$$

so that substitution of (2.6) into (2.5) yields the desired result.

Corollary 1 now follows from theorem 1 since when $M$ is of full rank, $M'W^{-1}M$ is invertible and

$$
(S^{-1} + M'W^{-1}S)^{-1} = S[S + (M'W^{-1}S)^{-1}]^{-1}(M'W^{-1}S)^{-1}
$$

so that identifying $(M'W^{-1}S)^{-1}$ with $W$ in (2.2) yields the desired result.

We now consider the relationship between the control equations (1.7) and
the Kalman filter equations (1.4). In fact, equations (1.7) may be combined
to yield

\[(2.8) \quad F_k = A'(F_{k+1} + Q_{1,k+1})A - \\
A'(F_{k+1} + Q_{1,k+1})D[Q_2,k + D'(F_{k+1} + Q_{1,k+1})D]^{-1}D'(F_{k+1} + Q_{1,k+1})A \\
F_N = 0\]
or letting \(R_k = F_k + Q_{1,k}, \quad k = 1, 2, \ldots, N,\) we obtain

\[(2.9) \quad R_k = A'(R_{k+1} - R_{k+1}D[Q_2,k + D'R_{k+1}D]^{-1}D'R_{k+1})A + Q_{1,k}, \quad k = 1, \ldots, N-1; \\
R_N = Q_{1,N}\]

Comparing (2.9) with (2.1) we see that these difference equations are of
precisely the same form if we identify \(R \leftrightarrow S, \quad A \leftrightarrow A', \quad D \leftrightarrow M', \quad Q_{2,k} \leftrightarrow W, \quad Q_{1,k} \leftrightarrow V, \quad k \leftrightarrow N-k.\) Thus, when \(Q_{1,k} = Q_1, \quad Q_{2,k} = Q_2\) for all \(k,\) it is clear
that a study of the asymptotic behavior of \(F_1 + Q_1 = R_1 = R_1(N)\) as \(N \to \infty\) may
be obtained from corollary 1. Indeed, corollary 1 and the above remarks imply

**Theorem 2:** Let \(\text{rank}(D) = \min(p,r)\) and let \(Q_1\) and \(Q_2\) be pd. Define \(\psi\) on the
set of pd matrices \(T\) by

\[(2.10) \quad \psi(R) = A'(R - RD[Q_2 + D'RD]^{-1}D'R)A + Q_1, \quad \text{Re}T,\]

so that \(R_k = \psi(R_{k+1}).\) Then \(\psi\) has a unique pd fixed point \(R_\star\) and \(\psi^N(R) \to R_\star\)
uniformly on \(T\) as \(N \to \infty,\) where \(\psi^N\) denotes the \(N\)th iterate of \(\psi.\) Moreover,

\[H_\circ(N) = Q_2 + D'\psi^{N-1}(Q_1)D + Q_2 + D'R_\star D\quad \text{and} \quad C_\circ(N) = [Q_2 + D'\psi^{N-1}(Q_1)D]^{-1}D'\psi^{N-1}(Q_1)A \\
+ C_\star = [Q_2 + D'R_\star D]^{-1}D'R_\star A \quad \text{as} \quad N \to \infty.\]

Now let us consider the asymptotic behavior of \(B_1 = B_1(N)\) in (1.8) which
is required for the computation of \(Z_\circ(N),\) used with \(C_\circ(N)\) and \(H_\circ(N)\) in (1.6)
for the computation of the first period controls, \(u_0.\) From (1.8) we obtain

\[(2.11) \quad B_k = B_{k+1}(A - DC_k) + G_{1,k+1}(A - DC_k) + C_{2,k}C_k, \quad k = 1, 2, \ldots, N-1; \\
B_N = 0\]

In order to show that \(B_1 = B_1(N)\) converges to a fixed point as \(N \to \infty,\) we
will need the following lemmas, the first of which is due to Stein and is proven
Lemma 1: Let $Y$ be a square matrix and let $\rho(Y)$ be the spectral radius of $Y$. If there exists a pd matrix $L$ for which $L - Y'LY$ is pd then $\rho(Y) < 1$.

Lemma 2: Let $C_* = [Q_2 + D'R_*D]^{-1}D'R_*A$, where $R_*$ is the unique pd fixed point of $\Psi$. Then, $\rho(A - DC_*) < 1$.

Proof: By definition of $C_*$, we have

\begin{equation}
\begin{aligned}
(2.12) \quad A - DC_* &= A - D[Q_2 + D'R_*D]^{-1}D'R_*A \\
&= (I - D[Q_2 + D'R_*D]^{-1}D'R_*)A \\
&= R_*^{-1}(R_* - R_*D[Q_2 + D'R_*D]^{-1}D'R_*)A
\end{aligned}
\end{equation}

Now a calculation similar to the proof of (2.4) in corollary 1 shows that

\begin{equation}
\begin{aligned}
(2.13) \quad R_* - R_*D[Q_2 + D'R_*D]^{-1}D'R_* &= [R_*^{-1} + DQ_2^{-1}D']^{-1} \\
\text{Therefore, we have from (2.12)}
\end{aligned}
\end{equation}

\begin{equation}
\begin{aligned}
(2.14) \quad A - DC_* &= R_*^{-1}[R_*^{-1} + DQ_2^{-1}D']^{-1}A
\end{aligned}
\end{equation}

To prove the assertion it will suffice by lemma 1 to exhibit a pd matrix $L$ for which $L = (A - DC_*)'L(A - DC_*)$ is pd. But $L = R_*$ is such a matrix, for by definition, $R_* = \Psi(R_*),$ and therefore

\begin{equation}
\begin{aligned}
(2.15) \quad R_* - (A - DC_*)'R_*(A - DC_*) &= A'(R_* - R_*D[Q_2 + D'R_*D]^{-1}D'R_*)A + \\
&\quad Q_1 - (A - DC_*)'R_* (A - DC_*)
\end{aligned}
\end{equation}

Now using (2.13) and (2.14) in (2.15) we obtain

\begin{equation}
\begin{aligned}
(2.16) \quad R_* - (A - DC_*)'R_*(A - DC_*) &= A'(R_* - R_*D[Q_2 + D'R_*D]^{-1}D'R_*)A - \\
&\quad A'[R_*^{-1} + DQ_2^{-1}D']^{-1}R_*^{-1}R_*^{-1}[R_*^{-1} + DQ_2^{-1}D']^{-1}A + Q_1 \\
&= A'[R_*^{-1} + DQ_2^{-1}D']^{-1}(I - R_*^{-1}[R_*^{-1} + DQ_2^{-1}D']^{-1})A + Q_1 \\
&= A'[R_*^{-1} + DQ_2^{-1}D']^{-1}(I - (R_*^{-1} + DQ_2^{-1}D' - DQ_2^{-1}D'[R_*^{-1} + DQ_2^{-1}D']^{-1})A + Q_1 \\
&= A'[R_*^{-1} + DQ_2^{-1}D']^{-1}DQ_2^{-1}D'[R_*^{-1} + DQ_2^{-1}D']^{-1}A + Q_1 \quad \text{QED.}
\end{aligned}
\end{equation}

We now show that $B_1(N)$ converges to a given finite vector, $B_*$. For convenience, we reverse the time index in (2.11) so that $k \leftrightarrow N-k$ and

\begin{equation}
\begin{aligned}
(2.17) \quad B_{t+1} &= B_t(A - DC_{t+1}) + C_1(A - DC_{t+1}) + C_2 C_{t+1}, \quad B_0 = 0.
\end{aligned}
\end{equation}
Theorem 3: Let $B_t$ be defined by (2.17). Then $B_t$ converges to $B_*$ given by

$$(2.18) \quad B_* = [G_1(A - DC_*) + G_2 C_*] [I - A + DC_*]^{-1}$$

where $C_*$ is given in lemma 2. Consequently, $B_1(N)$ converges to $B_*$ as $N \to \infty$.

**Proof:** Define the matrices $\Xi_t$ and $\Omega_t$ by

$$(2.19) \quad \Xi_t = A - D C_{t+1}$$

$$(2.20) \quad \Omega_t = G_1(A - D C_{t+1}) + G_2 C_{t+1}$$

so that (2.17) becomes

$$(2.21) \quad B_{t+1} = B_t \Xi_t + \Omega_t , \quad B_0 = 0 .$$

Now since $C_t \to C_*$, $\Xi_t \to \Xi_*$ = $A - D C_*$, and by lemma 2, $\rho(\Xi_*) < 1$, so that (2.21) is asymptotically stable. From (2.21) we obtain

$$(2.22) \quad B_{t+1} = B_0 \prod_{j=0}^{t} \Xi_j + \sum_{k=0}^{t} \Omega_k \prod_{j=k+1}^{t} \Xi_j$$

We now remark that since $\rho(\Xi_*) < 1$, $I - \Xi_*$ is invertible, and

$$(2.23) \quad (I - \Xi_*)^{-1} = \sum_{k=0}^{\infty} \Xi_k ,$$

so that letting $B_*^t = \sum_{k=0}^{t} \Omega_k \prod_{j=k+1}^{t} \Xi_j$, where $\Omega_* = G_1(A - D C_*) + G_2 C_*$, we have $B_*^t \to B_*$. Thus it suffices to show that $|B_t - B_*^t| \to 0$, where $|\cdot|$ is the Euclidean norm. To show this, we note that $\rho(\Xi_*) < 1$ implies (see Varga [10], p. 67) the existence of an integer $r \geq 1$ for which $|\Xi_*^r| < 1$, and since $\Xi_t \to \Xi_*$, there exist $\rho_0 < 1$ and an integer $k_0 \geq 1$ for which

$$(2.24) \quad \max \{|\Xi_*^s j|, |\Xi_*^r|\} < \rho_0 , \quad k \geq k_0 .$$

Since $\Omega_k$ is a convergent sequence, there exists a uniform upper bound, $U_1$, such that $|\Omega_k| \leq U_1$ for all $k$. Moreover, (2.24) implies for $s \geq 1$, and $k \geq k_0$ that

$$(2.25) \quad \max \{|\Xi_*^s j|, |\Xi_*^r|\} \leq \rho_0 \max \{|\Xi_*^k j|, |\Xi_*^r|\} \leq \rho_0 \max \{|\Xi_*^{k+1} j|, |\Xi_*^r|\}$$

In particular, $\lim_{t \to \infty} |\Xi_*^s j| = 0$ for every $j \geq 0$, so that

$$(2.26) \quad \lim_{t \to \infty} \sup_{\Omega_o} |B_{t+1} - B_*^t| \leq \lim_{t \to \infty} \sup_{\Omega_o} U_1 \{ |\Xi_*^s j| + |\Xi_*^r| \} = \lim_{t \to \infty} \sup_{\Omega_o} U_1 \{ |\Xi_*^{k+1} j| + \Xi_*^r \} \leq \lim_{t \to \infty} \sup_{\Omega_o} U_1 \{ |\Xi_*^{k+1} j| + \Xi_*^r \} , \quad j \geq k_0 .$$
But (2.25) implies for \( j_0 \geq k_0 \) and for \( i \geq 0 \)

\[
(2.27) \quad \sum_{k=j_0}^{t} \left| \Pi_{j=k+1}^{i+r} \right| \leq \sum_{p=0}^{j_0+i+k} \{ \sum_{k=j_0+i}^{j_0+i+k} \left| \Pi_{j=k+1}^{i} \right| \} \rho_o^p
\]

where \( \rho_o \) is the largest integer less than or equal to \( (t-j_0-1)/r \). Therefore,

\[
(2.28) \quad \lim_{t \to \infty} \sup \sum_{k=j_0}^{t} \left| \Pi_{j=k+1}^{i+r} \right| \leq U_2/(1-\rho_o)
\]

Then (2.26) and (2.28) imply

\[
(2.29) \quad \lim_{t \to \infty} \sup |B_{t+1} - B^*| \leq 2\rho_o^s U_1 U_2/(1-\rho_o)
\]

which may be made arbitrarily small by proper choice of \( s \). This completes the proof of theorem 2.

We may summarize the results of this section as follows. Let \( u_0(N) \) be the optimal first period controls given by (1.6) as

\[
(2.30) \quad u_0(N) = -C_0(N)x_0|_0 - 1/2 H_0^{-1}(N)Z_0'(N)
\]

Let \( D \) be of full rank and let \( Q_1 \) and \( Q_2 \) be \( \text{pd} \). Then \( u_0(N) \) converges to

\[
(2.31) \quad u^* = -C^*|_0 - 1/2 H^*-1 Z^*
\]

as \( N \to \infty \). The values of the parameters in (2.31) are given below and \( x_0|_0 \) is the linear least squares estimate of the system at the present time.

\[
(2.32) \quad R^* = A'(R^* - R^*D[Q_2 + D'R^*D]^{-1}D'R^*)A + Q_1
\]

\[
(2.33) \quad C^* = [Q_2 + D'R^*D]^{-1}D'R^*A
\]

\[
(2.34) \quad H^* = Q_2 = D'R^*D
\]

\[
(2.35) \quad Z^* = G_1D + G_2 + B^*D
\]

\[
(2.36) \quad B^* = [G_1(A - DC^*) + G_2C^*][I - A + DC^*]^{-1}
\]

It should be remarked that in practice one would determine \( R^* \) by iteration of (2.9) with \( k \in N-k \) until successive values of \( R^*_k \) and \( R^*_{k+1} \) were within a desired tolerance. In this process it should be recalled that convergence of \( R^*_k \) to \( R^* \) is uniform. In fact, it can easily be shown on the basis of the proof of the above theorem 1 in Anderson et. al. [1], that (with \( k \in N-k \)
\[(2.37) \quad R_N = \psi^N(Q_1) \leq \psi^N(A'[DQ_2^{-1}D']^{-1}A + Q_1) \leq A'[DQ_2^{-1}D']^{-1}A + Q_1 \]

where \(X \leq Y\) in (2.37) if \(Y - X\) is psd. Thus, since both \(\psi^N(Q_1)\) and \(\psi^N(A'[DQ_2^{-1}D']^{-1}A + Q_1)\) are converging to the fixed point \(R_*\), one can use the (some convenient) norm of the difference of these two quantities to obtain a precise bound on \(|\psi^N(Q_1) - R_*|\).

### III. THE DISCOUNTED CASE

Let \(\alpha\) be a given discount factor, \(0 < \alpha < 1\), and consider the minimization of (1.5) with

\[(3.1) \quad Q_{i,k} = \alpha^k Q_i, \quad G_{i,k} = \alpha^k G_i, \quad i = 1, 2; \quad k \geq 0.\]

In this case equations (1.7) and (1.8) yield

\[(3.2) \quad R_k = A'(R_{k+1} - R_{k+1}D[D'R_{k+1}D + \alpha^k Q_2]^{-1}D'R_{k+1})A + \alpha^k Q_1, \quad 1 \leq k \leq N-1;\]

\[(3.3) \quad B_k = B_{k+1}(A - DC_k) + \alpha^{k+1} G_1(A - DC_k) + \alpha^k G_2 C_k, \quad B_N = 0, \quad 0 \leq k \leq N-1;\]

where \(C_k\) is determined by (1.7) and where

\[(3.4) \quad R_k = F_k + \alpha^k Q_1, \quad R_N = \alpha^N Q_1.\]

We now show that, by redefining the system parameters, (3.2) can be put in the form of (2.9) with stationary parameters, so that the desired asymptotic properties follow from theorem 2. We begin by defining the parameters

\[(3.5) \quad A = \beta A, \quad Q_1 = Q_1, \quad D = D, \quad Q_2 = (1/\alpha)Q_2, \quad \beta^2 = \alpha.\]

Then letting \(R_k = \alpha^k R_k\), we have from (3.2)

\[(3.6) \quad R_k = A'(\alpha^{-k} R_{k+1} - \alpha^{-k} R_{k+1} D[D'R_{k+1}D + \alpha^k Q_2]^{-1}D'R_{k+1})A + Q_1\]

\[= (\beta A)'(\alpha^{-k-1} R_{k+1} - \alpha^{-k-1} R_{k+1} D[D'(\alpha^{-k-1} R_{k+1} D + \alpha^{-1} Q_2]^{-1}D'\alpha^{-k-1} R_{k+1})(\beta A)\]

\[+ Q_1\]

or, using (3.5),

\[(3.7) \quad R_k = A'(R_{k+1} - R_{k+1} D[D'R_{k+1}D + Q_2]^{-1}D'R_{k+1})A + Q_1\]

Thus, (3.7) is precisely of the same form as (2.9) and theorem 2 therefore implies that \(R_1(N) = \alpha R_1(N)\) converges uniformly to a fixed point \(\alpha R_*\), as \(N \to \infty\). This also implies the convergence of \(C_o(N)\) and \(H_o(N)\) to \(C_* = H_*^{-1} D'\alpha R_* A\) and
\( H_* = Q_2 + D' R_* D \), respectively.

Similarly, the equation (2.11) for \( B_k \) becomes for the discounted case

\[
B_k = B_{k+1} (A - DC_k) + \alpha^{k+1} G_1 (A - DC_k) + \alpha^k G_2 C_k, \quad B_N = 0.
\]

Let \( B_k = \alpha^{-k} B_k \). Then (3.8) implies

\[
B_k = B_{k+1} \alpha (A - DC_k) + G_1 \alpha (A - DC_k) + G_2 C_k.
\]

If we identify \( \Xi_k = \alpha (A - DC_k) \) and \( \Omega_k = G_1 \alpha (A - DC_k) + G_2 C_k \), then we may proceed as in theorem 3 to prove the convergence of \( B_1 (N) \) to a finite vector \( B_* \) provided that \( \rho(\alpha [A - DC_*]) < 1 \), where \( C_* \) is the fixed point of \( C_0 (N) \) defined above. To show this we note that

\[
C_* = [Q_2 + D' \alpha R_* D]^{-1} D' \alpha R_* A
\]

\[= [\alpha^{-1} Q_2 + D' \alpha R_* D]^{-1} D' \alpha R_* A\]

\[= \beta [Q_2 + D' \alpha R_* D]^{-1} D' \alpha R_* A\]

and therefore, using (3.5), we obtain

\[
\alpha (A - DC_*) = \beta (A - DC_*)
\]

where \( C_* \) is the fixed point of the stationary system with parameters, \( A, D, Q_1 \), and \( Q_2 \), corresponding to \( C_0 (N) \). But \( \rho(\beta [A - DC_*]) = \beta \rho(A - DC_*) \) and

\( \rho(A - DC_*) < 1 \) by lemma 2, so that the desired result follows as in the proof of theorem 3. We may summarize the results of this section in the following manner.

**Theorem 4:** Let \( u_0 (N) \) be the optimal first period controls given by

\[
u_0 (N) = -C_0 (N) x_o |_o - 1/2 \ H_{-1}(N) Z_o' (N).
\]

Let \( D \) be of full rank and let \( Q_1 \) and \( Q_2 \) be pd. Then \( u_0 (N) \) converges to \( u_* \) given by

\[
u_* = -C_* x_o |_o - 1/2 \ H_{-1} Z_*' \]

as \( N \to \infty \). In (3.13) \( x_o |_o \) is the linear least squares estimate of the system state at the present time and the parameters \( C_*, \ H_*, \) and \( Z_* \) are determined by

\[
R_* = A' (R_* - R_* D [Q_2 + D' R_* D]^{-1} D' R_* A + Q_1
\]
\[ C_* = (Q_2 + D'R_*D)^{-1}D'R_*A \]  
\[ H_* = Q_2 + D'R_*D \]  
\[ Z_* = \alpha G_1D + G_2 + \alpha B_*D \]  
\[ B_* = [G_1\alpha(A - DC_*) + G_2C_*][I - \alpha(A - DC_*)]^{-1} \]

where \( A, D, Q_1, \) and \( Q_2 \) are given in (3.5).

IV. APPLICATIONS AND EXTENSIONS

Although the results of the preceding sections have wider applicability, we restrict ourselves to a brief exploration of their implications for the aggregate planning of production and work force (see Holt et al. [2]). This discussion will serve to highlight as well the limitations of the above analysis.

Following Holt et al. [2], we first assume the following model for the aggregate planning problem.

\[ \text{Min } E\{\sum_{k=1}^{N} f_k(I_{k+1}, P_k, W_k, U_k)\} \]

subject to

\[ I_{k+1} = I_k + P_k - S_k \]  
\[ W_{k+1} = W_k + U_k \]  
\[ I_0, W_0 \text{ given,} \]

where \( f_k \) is a quadratic-linear cost function in its arguments and represents period \( k \) costs. \( I_k \) is the inventory at the beginning of period \( k \), \( P_k \) is the aggregate production in period \( k \), \( S_k \) is the sales in period \( k \), \( W_k \) is the work force at the beginning of period \( k \), and \( U_k \) is the change in work force during period \( k \).

In order to reduce the above problem to the form of (1.1) and (1.2), we must assume that the sales are generated by a first order autoregressive scheme of the form
(4.5) \[ \xi_{k+1} = \Gamma \xi_k + v_k \]
(4.6) \[ \lambda_k = M \lambda_k + w_k \]
where \( \xi'_k = (S_k, S_{k-1}, \ldots, S_{k-n}) \). Then (4.5) and (4.6) could be incorporated into (4.2) to yield a system of the form of (1.1) and (1.2). Besides only being able to consider sales generated by an autoregressive scheme, the present results are also limited to costs which are separable in the state and control variables. This would rule out, for example, costs of the form (see [2])

\[
\text{Cost of Overtime}_k = c_1(p_k - c_2 w_k)^2 + c_3 p_k + c_4 w_k, \quad c_1, c_2 > 0,
\]

since such costs lead to terms of the form \(-2c_1c_2 p_k w_k\). For the above reasons it seems appropriate to generalize the fundamental model (1.1)-(1.2) to the form

\[
(4.7) \quad x_{k+1} = Ax_k + Du_k + S_k + v_k \\
(4.8) \quad y_k = Mx_k + w_k
\]

where all quantities above are defined as in (1.1)-(1.2) except \( s_k \) which is a deterministic \( p \)-vector.

Recent work (see [9]) in Kalman filter techniques has generalized the underlying model to which the Kalman filter is applicable to the form of (4.7)-(4.8). The fundamental filter equations (1.4) remain unchanged in this case and therefore the results on their asymptotic behavior are still applicable. Moreover, (4.7) and (4.8) are clearly directly related to the form of the aggregate planning problem (4.1)-(4.4). It remains to be determined whether the essential properties of equations (1.6)-(1.8) will hold for the system (4.7)-(4.8). It is my conjecture that the results of theorems 2, 3, and 4 hold for the system (4.7)-(4.8) whenever the cost function in (1.5) is a quadratic-linear function (including state-control cross-product terms) provided that the cost function is convex and strictly convex in the controls, \( u_k \), and when, in addition, the terms, \( s_k \), are bounded by a stationary linear system. Verification
of this conjecture involves first resolving the dynamic program leading to (1.6)-(1.8) with added cross product terms in $u_k$ and $x_k$ and subject to (4.7) and (4.8) instead of (1.1) and (1.2). The present work and that reported in [1] and [7] provides a foundation for further studies in this direction.
REFERENCES


