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BAYESIAN ANALYSIS OF REDUCED FORM SYSTEMS

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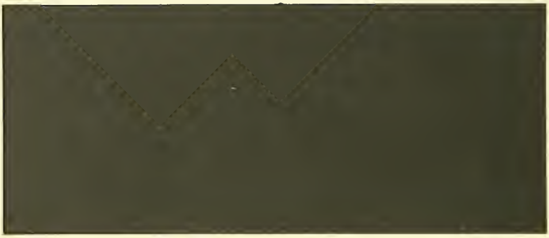

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Bayesian Analysis of Reduced Form Systems

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SUMMARY

Under the assumption that none of the parameters of a reduced form system are known with certainty, the natural conjugate family of prior densities for the joint distribution of these parameters is identified. Prior-posterior and preposterior analysis is done assuming that the prior is in the natural conjugate family, and some useful sampling distributions are derived. A procedure is presented for obtaining some non-degenerate joint posterior and preposterior distributions of all parameters even when the number of objective vector sample observations is less than the number of parameters of the process.

1. Introduction

The data generating process known as a simultaneous equations system among econometricians may be described in simplified form as follows: it is a set of stochastic equations

$$\underline{\underline{B}}\,\tilde{\underline{y}}^{(j)} + \underline{\underline{\Gamma}}\,\underline{z}^{(j)} = \tilde{\underline{u}}^{(j)} \quad , \qquad j=1,2,\ldots \tag{1a}$$

where $\underline{\underline{B}}$ and $\underline{\underline{\Gamma}}$ are (m x m) and (m x r) coefficient matrices, fixed for all j, $\underline{z}^{(j)}$ is a (r x 1) vector of predetermined variables and $\tilde{\underline{y}}^{(j)}$ and $\tilde{\underline{u}}^{(j)}$ are (m x 1) and (r x 1) random vectors respectively. It is often assumed that $\{\tilde{\underline{u}}^{(j)}, j=1,2,\ldots\}$ is a sequence of mutually independent, identically multivariate Normally distributed random vectors with mean $\underline{0}$ and variance-covariance matrix $\underline{\underline{\Sigma}} \equiv \underline{\underline{h}}^{-1}$. When, in addition it is assumed that $\underline{\underline{B}}$ is non-singular, there exists a wide variety of methods for estimating $\underline{\underline{B}}$ and $\underline{\underline{\Gamma}}$.

Drèze [3] has suggested that Bayesian methods be applied to the analysis of such systems when neither $\underline{\underline{B}}$, $\underline{\underline{\Gamma}}$, nor $\underline{\underline{h}}$ is known with certainty. In [3] he outlines some ways of treating the system when $\underline{\underline{h}}$, but not $\underline{\underline{B}}$ and $\underline{\underline{\Gamma}}$, is known with certainty. Zellner and Tiao [6] treat the reduced form system as defined in

section 1.1 below with parameters $\underline{\underline{h}}$ and $\underline{\underline{\Pi}} \equiv -\underline{\underline{B}}^{-1}\underline{\underline{\Gamma}}$ from a Bayesian point of view. They, in effect, do prior-posterior analysis of the data generating process defined in (1b) and derive the unconditional distribution of the $\tilde{\underline{y}}^{(j)}$ under the assumption that $(\tilde{\underline{\underline{\Pi}}}, \tilde{\underline{\underline{h}}})$ has a particular diffuse (and degenerate) prior density.

In this paper we identify a natural conjugate family for $(\tilde{\underline{\underline{\Pi}}}, \tilde{\underline{\underline{h}}})$, do prior-posterior analysis under the broader assumption that the prior is in the natural conjugate family, present sampling distributions unconditional as regards $(\tilde{\underline{\underline{\Pi}}}, \tilde{\underline{\underline{h}}})$, and do preposterior analysis. In addition, we show how Bayesian inference can be done even when the number of objective sample vector observations is less than the number of unknown parameters.

1. Definition of the Process

The reduced form data generating process is defined as one that generates independent (m x 1) random vectors $\tilde{\underline{y}}^{(1)},\ldots,\tilde{\underline{y}}^{(j)},\ldots$ according to the model

$$\tilde{\underline{y}}^{(j)} = \underline{\underline{\Pi}}\ \underline{z}^{(j)} + \tilde{\underline{v}}^{(j)} \quad , \tag{1b}$$

where $\underline{\underline{\Pi}}$--a matrix of dimension (m x r)--is a parameter whose value remains fixed. Initially we assume that $\underline{z}^{(j)}$ is a known (r x 1) vector which varies from observation to observation. The $\tilde{\underline{v}}^{(j)}$s are mutually independent (m x 1) random vectors identically distributed according to

$$f_N^{(m)}(\underline{v}|\underline{0},\ \underline{\underline{h}}) = (2\pi)^{-\frac{1}{2}m}\ e^{-\frac{1}{2}\underline{v}^t\underline{\underline{h}}\underline{v}}\ |\underline{\underline{h}}|^{\frac{1}{2}} \quad , \tag{2}$$

and so the density of $\tilde{\underline{y}}^{(j)}$ is

$$f_N^{(m)}(\underline{y}^{(j)}|\underline{\underline{\Pi}}\ \underline{z}^{(j)},\underline{\underline{h}})=(2\pi)^{-\frac{1}{2}m}\ e^{-\frac{1}{2}(\underline{y}^{(j)}-\underline{\underline{\Pi}}\ \underline{z}^{(j)})^t\underline{\underline{h}}(\underline{y}^{(j)}-\underline{\underline{\Pi}}\ \underline{z}^{(j)})}\,|\underline{\underline{h}}|^{\frac{1}{2}}. \tag{3}$$

Notice that if $\underline{\underline{B}}$ in (1a) is non-singular then premultiplying both sides of (1a) by $\underline{\underline{B}}^{-1}$ transforms (1a) into the form (1b) with $\underline{\underline{\Pi}} = -\underline{\underline{B}}^{-1}\underline{\underline{\Gamma}}$ and $\tilde{\underline{v}}^{(j)}=\underline{\underline{B}}^{-1}\tilde{\underline{u}}^{(j)}$.

1.1 Some Definitions

For future convenience, we make the following definitions:

$$\underline{\underline{Y}} = [\underline{y}^{(1)},\ldots,\underline{y}^{(n)}] = \begin{pmatrix} \underline{y}_1 \\ \vdots \\ \underline{y}_m \end{pmatrix} \quad , \quad \dim(m \times n)\ , \tag{3a}$$

$$\underline{\underline{Z}} = [\underline{z}^{(1)},\ldots,\underline{z}^{(n)}] = \begin{pmatrix} \underline{z}_1 \\ \vdots \\ \underline{z}_r \end{pmatrix} \quad , \quad \dim(r \times n)\ , \tag{3b}$$

$$\underline{\underline{\Pi}} = [\underline{\pi}^{(1)},\ldots,\underline{\pi}^{(r)}] = \begin{pmatrix} \underline{\pi}_1 \\ \vdots \\ \underline{\pi}_m \end{pmatrix} \quad , \quad \dim(m \times r)\ , \tag{3c}$$

Later, we shall have use for the matrix

$$\underline{\underline{P}} = [\underline{p}^{(1)},\ldots,\underline{p}^{(r)}] = \begin{pmatrix} \underline{p}_1 \\ \vdots \\ \underline{p}_m \end{pmatrix} , \qquad \dim(m \times r) . \tag{3d}$$

1.2 Likelihood of a Sample

The likelihood that the process defined in (1) will generate n successive values $\underline{y}^{(1)},\ldots,\underline{y}^{(j)},\ldots,\underline{y}^{(n)}$ is the product of the individual likelihoods:

$$(2\pi)^{-\frac{1}{2}mn}\, e^{-\frac{1}{2}\Sigma(\underline{y}^{(j)}-\underline{\underline{\Pi}}\,\underline{z}^{(j)})^t\underline{\underline{h}}(\underline{y}^{(j)}-\underline{\underline{\Pi}}\,\underline{z}^{(j)})}\,|\underline{\underline{h}}|^{\frac{1}{2}n} \quad . \tag{4}$$

If the process by which the $\underline{z}^{(j)}$s were generated is non-informative then this is the likelihood of the sample described by $(\underline{\underline{Y}}, \underline{\underline{Z}})$. The kernel of the likelihood (4) is

$$e^{-\frac{1}{2}\Sigma(\underline{y}^{(j)}-\underline{\underline{\Pi}}\,\underline{z}^{(j)})^t\underline{\underline{h}}(\underline{y}^{(j)}-\underline{\underline{\Pi}}\,\underline{z}^{(j)})}\,|\underline{\underline{h}}|^{\frac{1}{2}n} \quad . \tag{5}$$

Given $(\underline{\underline{Y}}, \underline{\underline{Z}})$ we may compute these statistics:

$$\begin{aligned}
&\underline{\underline{V}} \equiv \Sigma\, \underline{z}^{(j)}\underline{z}^{(j)^t} \equiv \underline{\underline{Z}}\,\underline{\underline{Z}}^t && , \quad \dim(r \times r),\ \nu \equiv n-m \text{ (redundant)},\\
&\underline{\underline{P}}^t \equiv (\underline{\underline{Z}}\,\underline{\underline{Z}}^t)^{-1}\,\underline{\underline{Z}}\,\underline{\underline{Y}}^t && , \quad \dim(r \times m),\\
&\underline{\epsilon}^{(j)} \equiv \underline{y}^{(j)}-\underline{\underline{P}}\,\underline{z}^{(j)} && , \quad \dim(m \times 1),\\
&\underline{\underline{\epsilon}} \equiv \Sigma\, \underline{\epsilon}^{(j)}\underline{\epsilon}^{(j)^t} && , \quad \dim(m \times m).
\end{aligned} \tag{6}$$

In terms of $\underline{\underline{V}}$, $\underline{\underline{P}}$, and $\underline{\underline{\epsilon}}$ we may write (5) as

$$e^{-\frac{1}{2}\text{tr}\ \underline{\underline{h}}\{[\underline{\underline{P}}-\underline{\underline{\Pi}}]\underline{\underline{V}}[\underline{\underline{P}}-\underline{\underline{\Pi}}]^t+\underline{\underline{\epsilon}}\}}\,|\underline{\underline{h}}|^{\frac{1}{2}(\nu+m)} \quad . \tag{7}$$

It is well known that† (7) is the kernel of the joint likelihood of $(\underline{\underline{P}}, \underline{\underline{\epsilon}})$ when $\nu > 0$ and $\underline{\underline{V}}$ is PDS and that given $\underline{\underline{V}}$, $\underline{\underline{\Pi}}$, and $\underline{\underline{h}}$, $\tilde{\underline{\underline{P}}}$ and $\tilde{\underline{\underline{\epsilon}}}$ are independent. It follows that the marginal likelihood of $\underline{\underline{\epsilon}}$ is Wishart with parameter $(\underline{\underline{h}}, \nu)$

†See T.W. Anderson [1], p. 183.

provided $\nu > 0$, and the marginal likelihood of $\underline{\underline{P}}$ multivariate Normal with kernel

$$e^{-\frac{1}{2}tr\{\underline{\underline{h}}[(\underline{\underline{P}}-\underline{\underline{\Pi}})\underline{\underline{V}}(\underline{\underline{P}}-\underline{\underline{\Pi}})^t]\}} |\underline{\underline{h}}|^{\frac{1}{2}} \quad . \tag{8}$$

We define

$$\underline{p} \equiv (\underline{p}_1, \ldots, \underline{p}_m)^t \quad , \quad \dim(mr \times 1) \ , \tag{9a}$$

$$\underline{\pi} = (\underline{\pi}_1, \ldots, \underline{\pi}_m)^t \quad , \quad \dim(mr \times 1) \ , \tag{9b}$$

where $\underline{p}_i$ is the ith row of $\underline{\underline{P}}$ and $\underline{\pi}_i$ the ith row of $\underline{\underline{\Pi}}$, and

$$\underline{\underline{H}} = \underline{\underline{h}} \otimes \underline{\underline{V}} \tag{10}$$

where $\otimes$ denotes the Kroenecker direct product of $\underline{\underline{h}}$ and $\underline{\underline{V}}$. Then the kernel (7) of the joint likelihood of $(\underline{\underline{P}}, \underline{\underline{\epsilon}})$ may be written as

$$e^{-\frac{1}{2}(\underline{p}-\underline{\pi})^t \underline{\underline{h}} \otimes \underline{\underline{V}}(\underline{p}-\underline{\pi})} |\underline{\underline{h}}|^{\frac{1}{2}} \cdot e^{-\frac{1}{2}tr\ \underline{\underline{h}}\ \underline{\underline{\epsilon}}} |\underline{\underline{h}}|^{\frac{1}{2}(\nu+m-1)} \quad . \tag{11}$$

The kernel of the marginal likelihood of $\tilde{\underline{p}}$ is

$$e^{-\frac{1}{2}(\underline{p}-\underline{\pi})^t \underline{\underline{H}}(\underline{p}-\underline{\pi})} |\underline{\underline{h}}|^{\frac{1}{2}} \quad . \tag{12}$$

Formula (11) is simply a rearrangement--not a transformation--of elements in the exponent of (9), as may be verified by writing out the trace in (8).

That the marginal likelihood of $\underline{p}$ is (12) follows from the fact that (12) is just a rearrangement of (9).

When the rank q of $\underline{\underline{Z}}$ is less than the column dimension r of $\underline{\underline{\Pi}}$ the statistic $\underline{\underline{P}}$ is not fully determined. In fact, m(r-q) of the mr elements of $\underline{\underline{P}}$ may be assigned arbitrary values, whereupon the remaining elements of $\underline{\underline{P}}$ are determined by the normal equations

$$(\underline{\underline{Z}}\ \underline{\underline{Z}}^t)\underline{\underline{P}}^t = \underline{\underline{Z}}\ \underline{\underline{Y}}^t \quad . \tag{13}$$

To facilitate discussion, we assume that the first q rows of $\underline{\underline{Z}}$ are linearly independent. Then we may partition $\underline{\underline{Z}}$ as

$$\underline{\underline{Z}} = \begin{bmatrix} \underline{\underline{Z}}_1 \\ \\ \underline{\underline{Z}}_2 \end{bmatrix} \quad \text{where} \quad \begin{matrix} \underline{\underline{Z}}_1 \text{ is } (q \times n), \\ \\ \underline{\underline{Z}}_2 \text{ is } ((r-q) \times n) , \end{matrix} \tag{14}$$

and $\underline{\underline{P}}$ as

$$\underline{\underline{P}} = [\underline{\underline{P}}_1 \ \underline{\underline{P}}_2] \text{ , where } \underline{\underline{P}}_1 \text{ is } (m \times q) \text{ and } \underline{\underline{P}}_2 \text{ is } (m \times (r-q)). \tag{15}$$

This allows us to write (13) as

$$\begin{bmatrix} \underline{\underline{Z}}_1 \underline{\underline{Z}}_1^t & \underline{\underline{Z}}_1 \underline{\underline{Z}}_2^t \\ \underline{\underline{Z}}_2 \underline{\underline{Z}}_1^t & \underline{\underline{Z}}_2 \underline{\underline{Z}}_2^t \end{bmatrix} \begin{bmatrix} \underline{\underline{P}}_1^t \\ \underline{\underline{P}}_2^t \end{bmatrix} = \begin{bmatrix} \underline{\underline{Z}}_1 \\ \underline{\underline{Z}}_2 \end{bmatrix} \underline{\underline{Y}}^t \quad . \tag{16}$$

The projection of each $\underline{y}_i$, i=1,2,...,m on the q-dimensional row space of $\underline{\underline{Z}}$ is unique even though there are an infinite number of linear combinations of the m rows of $\underline{\underline{Z}}$ in terms of which these projections may be expressed. We may express the projection of $\underline{y}_i$, i=1,2,...,m on this space as a set of unique linear combinations $\underline{\underline{P}}_1$ by arbitrarily specifying, for example, that $\underline{\underline{P}}_2=\underline{\underline{0}}$. As $\underline{\underline{Z}}_1\underline{\underline{Z}}_1^t$ is of rank q, $\underline{\underline{P}}_1$ has a unique value

$$\underline{\underline{P}}_1^t = (\underline{\underline{Z}}_1 \ \underline{\underline{Z}}_1^t)^{-1} \ \underline{\underline{Z}}_1 \ \underline{\underline{Y}}^t \quad . \tag{17}$$

By defining

$$\underline{\underline{P}} = [\underline{\underline{P}}_1 \ \underline{\underline{0}}] \ , \tag{18}$$

we have $\underline{\underline{P}} \ \underline{\underline{Z}} = \underline{\underline{P}}_1 \ \underline{\underline{Z}}_1$ so that $\underline{\underline{P}}$ as defined in (18) satisfies $\underline{\underline{P}}$ as defined in (6).

The definitions (6) of $\underline{\epsilon}^{(j)}$, $\underline{\underline{V}}$, and $\underline{\underline{\epsilon}}$ now go through without change, although $\underline{\underline{\epsilon}}$ will be singular if $n < m$.

In order to exploit the information in $(\underline{\underline{P}}, \underline{\underline{V}}, \underline{\underline{\epsilon}})$ even when $\nu < 0$, or $q < r$ or both we define

$$\delta = \begin{cases} 1 & \text{if } \underline{\underline{V}} \text{ is PDS} \\ 0 & \text{otherwise} \end{cases} , \quad \Phi = \begin{cases} m-1 & \text{if } \nu \leq 0 \\ 0 & \nu > 0 \end{cases} , \quad \underline{\underline{\epsilon}}^* = \begin{cases} \underline{\underline{\epsilon}} & \text{if } n \geq m \\ \underline{\underline{0}} & n < m \end{cases} ,$$

and write (11) as

$$e^{-\frac{1}{2}(\underline{p}-\underline{\pi})^t(\underline{\underline{h}} \otimes \underline{\underline{V}})(\underline{p}-\underline{\pi})} \; |\underline{\underline{h}}|^{\frac{1}{2}\delta} \; e^{-\frac{1}{2}\mathrm{tr}\; \underline{\underline{h}}\; \underline{\underline{\epsilon}}^*} \; |\underline{\underline{h}}|^{\frac{1}{2}(\nu+m-\Phi-1)} \; . \tag{12}$$

Notice that <u>even if</u> $q < r$, the joint likelihood of $(\underline{\underline{P}}_1, \underline{\underline{\epsilon}}) \equiv (\underline{p}_1, \underline{\underline{\epsilon}})$ may exist.

1.2 Conjugate Distribution of $(\tilde{\underline{\underline{\Pi}}}, \tilde{\underline{\underline{h}}})$, $\tilde{\underline{\underline{\Pi}}}$, and $\tilde{\underline{\underline{h}}}$

When neither $\tilde{\underline{\underline{\Pi}}}$ nor $\tilde{\underline{\underline{h}}}$ are known with certainty but are regarded as random variables, the natural conjugate of (12) is

$$f_{NW}^{(m,r)}(\underline{\underline{\Pi}}, \underline{\underline{h}} | \underline{\underline{P}}, \underline{\underline{V}}, \underline{\underline{\epsilon}}, \nu)$$

defined as equal to

$$k(m,r,\nu)\; e^{-\frac{1}{2}\mathrm{tr}\; \underline{\underline{h}}[(\underline{\underline{\Pi}}-\underline{\underline{P}})V(\underline{\underline{\Pi}}-\underline{\underline{P}})^t]} \; |\underline{\underline{h}}|^{\frac{1}{2}\delta} \; e^{-\frac{1}{2}\mathrm{tr}\; \underline{\underline{\epsilon}}^*\underline{\underline{h}}} \; |\underline{\underline{h}}|^{\frac{1}{2}\nu-1} |\underline{\underline{\epsilon}}^*|^{\frac{1}{2}(\nu+m-1)} \tag{13a}$$

$$= \begin{cases} f_N^{(mr)}(\underline{\underline{\Pi}} | \underline{\underline{P}}, \underline{\underline{h}} \otimes \underline{\underline{V}}) \; f_W^{(m)}(\underline{\underline{h}} | \underline{\underline{\epsilon}}, \nu) & \text{if } \nu > 0 \text{ and } \underline{\underline{V}} \text{ and } \underline{\underline{\epsilon}} \text{ are PDS,} \\ 0 & \text{otherwise} \end{cases} \tag{13b}$$

where

$$k(m,r,\nu) \equiv [2^{\frac{1}{2}(\nu+r+m-1)} \; \pi^{m(m+2r-1)/4} \prod_{i=1}^{m} \Gamma(\tfrac{1}{2}[\nu+m-i])]^{-1} \tag{13c}$$

The conjugate family defined by (13) parallels that defined in (6a) of [2] for the Multinormal process.

We obtain the marginal prior on $\underline{\underline{\Pi}}$ by integrating (13a) with respect to $\underline{\underline{h}}$; if $\nu > 0$, and $\underline{\underline{V}}$ and $\underline{\underline{\epsilon}}$ are PDS, then

$$D(\underline{\underline{\Pi}} | \underline{\underline{P}}, \underline{\underline{V}}, \underline{\underline{\epsilon}}, \nu) = f_{Sg}^{(mr)}(\underline{\underline{\Pi}} | \underline{\underline{P}}, \underline{\underline{V}}, \underline{\underline{\epsilon}}, \nu) \propto |\underline{\underline{L}}+\underline{\underline{\epsilon}}|^{-\frac{1}{2}(\nu+m)} \quad , \tag{14a}$$

where

$$\underline{\underline{L}} \equiv [\underline{\underline{\Pi}}-\underline{\underline{P}}]\underline{\underline{V}}[\underline{\underline{\Pi}}-\underline{\underline{P}}]^t = \begin{bmatrix} (\underline{\pi}_1-\underline{p}_1)\underline{\underline{V}}(\underline{\pi}_1-\underline{p}_1)^t & \cdots & (\underline{\pi}_m-\underline{p}_m)\underline{\underline{V}}(\underline{\pi}_1-\underline{p}_1)^t \\ \vdots & & \vdots \\ (\underline{\pi}_1-\underline{p}_1)\underline{\underline{V}}(\underline{\pi}_m-\underline{p}_m)^t & \cdots & (\underline{\pi}_m-\underline{p}_m)\underline{\underline{V}}(\underline{\pi}_m-\underline{p}_m)^t \end{bmatrix} \tag{14b}$$

We will call a distribution of the form (14) a <u>non-degenerate generalized multivariate Student</u> distribution.

Proof: We integrate $\underline{\underline{h}}$ over the region $R_{\underline{\underline{h}}} \equiv \{\underline{\underline{h}} | \underline{\underline{h}} \text{ is PDS}\}$. If $\nu > 0$, and $\underline{\underline{V}}$ and $\underline{\underline{\epsilon}}$ are PDS,

$$D(\underline{\underline{\Pi}}|\underline{\underline{P}}, \underline{\underline{V}}, \underline{\underline{\epsilon}}, \nu) = \int_{R_{\underline{\underline{h}}}} f_N^{(mr)}(\underline{\underline{\Pi}}|\underline{\underline{P}}, \underline{\underline{V}}, h)\, f_W^{(m)}(\underline{\underline{h}}|\underline{\underline{\epsilon}}, \nu) d\underline{\underline{h}}$$

$$\int_{R_{\underline{\underline{h}}}} e^{-\frac{1}{2}\text{tr}\, \underline{\underline{h}}[(\underline{\underline{\Pi}}-\underline{\underline{P}})\underline{\underline{V}}(\underline{\underline{\Pi}}-\underline{\underline{P}})^t]}\, |\underline{\underline{h}}|^{\frac{1}{2}} \cdot e^{-\frac{1}{2}\text{tr}\, \underline{\underline{h}}\, \underline{\underline{\epsilon}}}\, |\underline{\underline{h}}|^{\frac{1}{2}\nu-1}\, d\underline{\underline{h}}$$

$$= \int_{R_{\underline{\underline{h}}}} e^{-\frac{1}{2}\text{tr}\, \underline{\underline{h}}\{[\underline{\underline{\Pi}}-\underline{\underline{P}}]\underline{\underline{V}}[\underline{\underline{\Pi}}-\underline{\underline{P}}]^t+\underline{\underline{\epsilon}}\}}\, |\underline{\underline{h}}|^{\frac{1}{2}(\nu+1)-1}\, d\underline{\underline{h}} \quad .$$

The integrand in the last expression is the kernel of a Wishart density with parameter $([\underline{\underline{\Pi}}-\underline{\underline{P}}]\underline{\underline{V}}[\underline{\underline{\Pi}}-\underline{\underline{P}}]^t + \underline{\underline{\epsilon}}, \nu+1)$, and so

$$D(\underline{\underline{\Pi}}|\underline{\underline{P}}, \underline{\underline{V}}, \underline{\underline{\epsilon}}, \nu) \propto |[\underline{\underline{\Pi}}-\underline{\underline{P}}]\underline{\underline{V}}[\underline{\underline{\Pi}}-\underline{\underline{P}}]^t + \underline{\underline{\epsilon}}|^{-\frac{1}{2}(\nu+m)} \quad ,$$

proving (14).

From (13b) it is obvious that the marginal distribution of $\tilde{\underline{\underline{h}}}$ is Wishart with parameter $(\underline{\underline{\epsilon}}, \nu)$.

Tiao and Zellner [6], have shown the important result that the conditional distribution of $\tilde{\underline{\pi}}_i$ given $\underline{\pi}_j$, $1 \leq j \leq m$, $j \neq i$, is multivariate Student, and that the marginal distribution of $\tilde{\underline{\pi}}_i$ is also multivariate Student.

1.3 Prior-Posterior Analysis

If a Normal-Wishart distribution with parameter $(\underline{\underline{P}}', \underline{\underline{V}}', \underline{\underline{\epsilon}}', \nu')$ is assigned to $(\tilde{\underline{\underline{\Pi}}}, \tilde{\underline{\underline{h}}})$ and if a sample then yields a statistic $(\underline{\underline{P}}, \underline{\underline{V}}, \underline{\underline{\epsilon}}, \nu)$ the posterior distribution of $(\tilde{\underline{\underline{\Pi}}}, \tilde{\underline{\underline{h}}})$ will be Normal-Wishart with parameter $(\underline{\underline{P}}'', \underline{\underline{V}}'', \underline{\underline{\epsilon}}^{*''}, n'', \nu'')$ where

$$\underline{\underline{V}}'' = \underline{\underline{V}}'+\underline{\underline{V}} \quad , \quad \delta'' = \begin{cases} 1 & \text{if } \underline{\underline{V}}'' \text{ is PDS} \\ 0 & \text{otherwise} \end{cases} \quad , \tag{16a}$$

$$\nu'' = \nu'+\nu+m+\delta'+\delta-\delta''-\Phi-1 \quad , \tag{16b}$$

$$\underline{\underline{P}}'' = [\underline{\underline{P}}'\, \underline{\underline{V}}' + \underline{\underline{P}}\, \underline{\underline{V}}]\underline{\underline{V}}''^{-1} \tag{16c}$$

and

$$\underline{\underline{\epsilon}}^{*''} = \begin{cases} \underline{\underline{\epsilon}}'+\underline{\underline{\epsilon}} + \underline{\underline{P}}'\underline{\underline{V}}'\underline{\underline{P}}'^{t} + \underline{\underline{P}}\ \underline{\underline{V}}\ \underline{\underline{P}}^{t} - \underline{\underline{P}}''\underline{\underline{V}}''\underline{\underline{P}}''^{t} \equiv \underline{\underline{\epsilon}}'' & \text{if } \underline{\underline{\epsilon}}'' \text{ is PDS} \\ \underline{\underline{0}} & \text{otherwise} \end{cases} \tag{16d}$$

When $\underline{\underline{\epsilon}}'$ and $\underline{\underline{\epsilon}}$ are both PDS, the prior density and the sample likelihood combine to give the posterior density in the usual manner. As was the case with the Multinormal process treated in [1], when either $\underline{\underline{\epsilon}}'$ or $\underline{\underline{\epsilon}}$ or both are singular, the prior density (13) or the sample likelihood (7) or both may not exist. Even in such cases, we wish to allow for the possibility that the posterior density may be well defined. To this end we define the parameter of the posterior density in terms of $\underline{\underline{\epsilon}}'$ and $\underline{\underline{\epsilon}}$ rather than $\underline{\underline{\epsilon}}^{*''}$ and $\underline{\underline{\epsilon}}^{*}$.

<u>Proof</u>: Multiplying the kernel of the likelihood (7) by the kernel (13) of the prior we obtain

$$e^{-\frac{1}{2}\text{tr } h[\underline{\underline{\Pi}}-\underline{\underline{P}}']\underline{\underline{V}}'[\underline{\underline{\Pi}}-\underline{\underline{P}}]^{t}} \ |\underline{\underline{h}}|^{\frac{1}{2}\delta'} \ e^{-\frac{1}{2}\text{tr } \underline{\underline{\epsilon}}'\underline{\underline{h}}} \ |\underline{\underline{h}}|^{\frac{1}{2}\nu'-1}$$

$$\cdot\ e^{-\frac{1}{2}\text{tr } \underline{\underline{h}}(\underline{\underline{P}}-\underline{\underline{\Pi}})\underline{\underline{V}}(\underline{\underline{P}}-\underline{\underline{\Pi}})^{t}} \ |\underline{\underline{h}}|^{\frac{1}{2}\delta} \ e^{-\frac{1}{2}\text{tr } \underline{\underline{h}}\ \underline{\underline{\epsilon}}} \ |\underline{\underline{h}}|^{\frac{1}{2}(\nu+m-\Phi-1)}$$

$$= e^{-\frac{1}{2}S} \ |\underline{\underline{h}}|^{\frac{1}{2}\delta''} \ e^{-\frac{1}{2}\text{tr } \underline{\underline{h}}(\underline{\underline{\epsilon}}'+\underline{\underline{\epsilon}})} \ |\underline{\underline{h}}|^{\frac{1}{2}(\nu'+\nu+m+\delta'+\delta-\delta''-\Phi-1)-1}$$

where

$$S \equiv \text{tr } \underline{\underline{h}}\{(\underline{\underline{P}}-\underline{\underline{\Pi}})\underline{\underline{V}}(\underline{\underline{P}}-\underline{\underline{\Pi}})^{t} + (\underline{\underline{\Pi}}-\underline{\underline{P}}')\underline{\underline{V}}'(\underline{\underline{\Pi}}-\underline{\underline{P}}')^{t}\}$$

$$= \text{tr } \underline{\underline{h}}\{\underline{\underline{P}}\,\underline{\underline{V}}\,\underline{\underline{P}}^{t} - \underline{\underline{\Pi}}\,\underline{\underline{V}}\,\underline{\underline{P}}^{t} - \underline{\underline{P}}\,\underline{\underline{V}}\,\underline{\underline{\Pi}}^{t} + \underline{\underline{\Pi}}\,\underline{\underline{V}}\underline{\underline{\Pi}}^{t} + \underline{\underline{\Pi}}\,\underline{\underline{V}}'\underline{\underline{\Pi}}^{t} - \underline{\underline{P}}'\underline{\underline{V}}'\underline{\underline{\Pi}}^{t} - \underline{\underline{\Pi}}\,\underline{\underline{V}}'\underline{\underline{P}}'^{t} + \underline{\underline{P}}'\underline{\underline{V}}'\underline{\underline{P}}'^{t}\}$$

or since $\text{tr } \underline{\underline{h}}\{\underline{\underline{P}}\ \underline{\underline{V}}\ \underline{\underline{\Pi}}^{t}\} = \text{tr } \underline{\underline{h}}\{(\underline{\underline{\Pi}}\ \underline{\underline{V}}\ \underline{\underline{P}}^{t})^{t}\} = \text{tr } \underline{\underline{h}}\{\underline{\underline{\Pi}}\ \underline{\underline{V}}\ \underline{\underline{P}}^{t}\}$,

$$S = \text{tr } \underline{\underline{h}}\{\underline{\underline{\Pi}}\ \underline{\underline{V}}'\underline{\underline{\Pi}}^{t} + \underline{\underline{\Pi}}\ \underline{\underline{V}}\underline{\underline{\Pi}}^{t} - 2\underline{\underline{\Pi}}(\underline{\underline{V}}'\underline{\underline{P}}'^{t}+\underline{\underline{V}}\ \underline{\underline{P}}^{t}) + \underline{\underline{P}}'\underline{\underline{V}}'\underline{\underline{P}}'^{t} + \underline{\underline{P}}\ \underline{\underline{V}}\ \underline{\underline{P}}^{t}\} \ .$$

Defining $\underline{\underline{V}}''=\underline{\underline{V}}'+\underline{\underline{V}}$ as in (16a), $\underline{\underline{P}}''^{t}=\underline{\underline{V}}''^{-1}(\underline{\underline{V}}'\underline{\underline{P}}'^{t}+\underline{\underline{V}}\ \underline{\underline{P}}^{t})$ as in (16c), and completing the square, gives

$$S = \text{tr } \underline{\underline{h}}\{(\underline{\underline{\Pi}}-\underline{\underline{P}}'')\underline{\underline{V}}''(\underline{\underline{\Pi}}-\underline{\underline{P}}'')^{t} + \underline{\underline{P}}'\underline{\underline{V}}'\underline{\underline{P}}'^{t} + \underline{\underline{P}}\ \underline{\underline{V}}\ \underline{\underline{P}}^{t} - \underline{\underline{P}}''\underline{\underline{V}}''\underline{\underline{P}}''^{t}\} \ .$$

Defining ν'' as in (16b) and

$$\underline{\underline{\epsilon}}^* = \begin{cases} \underline{\underline{\epsilon}}'+\underline{\underline{\epsilon}} + \underline{\underline{P}}'\underline{\underline{V}}'\underline{\underline{P}}'^t + \underline{\underline{P}}\ \underline{\underline{V}}\ \underline{\underline{P}}^t - \underline{\underline{P}}''\underline{\underline{V}}''\underline{\underline{P}}''^t \equiv \underline{\underline{\epsilon}}'' & \text{if } \underline{\underline{\epsilon}}'' \text{ is PDS} \\ \underline{\underline{0}} & \text{otherwise} \end{cases}$$

we have (16d), completing the proof.

2. Sampling Distributions with Fixed n

We assume here that a sample of size n is to be drawn from an (m x r) dimensional reduced form process as defined in section 1 whose parameter $(\tilde{\underline{\underline{\Pi}}}, \tilde{\underline{\underline{h}}})$ is a random variable having a Normal-Wishart distribution with parameter $(\underline{\underline{P}}', \underline{\underline{V}}', \underline{\underline{\epsilon}}', \nu')$.

2.1 Conditional Joint Distribution of $(\tilde{\underline{\underline{P}}}, \tilde{\underline{\underline{\epsilon}}}|\underline{\underline{\Pi}}, \underline{\underline{h}}, \underline{\underline{V}})$

The conditional joint distribution of the statistic $(\tilde{\underline{\underline{P}}}, \tilde{\underline{\underline{\epsilon}}})$ given that the process parameter has value $(\underline{\underline{\Pi}}, \underline{\underline{h}})$ and given $\underline{\underline{V}}$ is, provided $\nu > 0$,

$$\begin{aligned} D(\underline{\underline{P}}, \underline{\underline{\epsilon}}|\underline{\underline{\Pi}}, \underline{\underline{h}}; \underline{\underline{V}}, \nu) &= f_N^{(mr)}(\underline{p}|\underline{\pi}, \underline{\underline{h}} \otimes \underline{\underline{V}})\ f_W^{(m)}(\underline{\underline{\epsilon}}|\underline{\underline{h}}, \nu) \\ &= f_N^{(mr)}(\underline{\underline{P}}|\underline{\underline{\Pi}}, \underline{\underline{h}} \otimes \underline{\underline{V}})\ f_W^{(m)}(\underline{\underline{\epsilon}}|\underline{\underline{h}}, \nu) \end{aligned} \tag{17}$$

2.1 Unconditional Joint Distribution of $(\tilde{\underline{\underline{P}}}, \tilde{\underline{\underline{\epsilon}}})$

The unconditional joint distribution of $(\tilde{\underline{\underline{P}}}, \tilde{\underline{\underline{\epsilon}}})$ as regards $(\tilde{\underline{\underline{\Pi}}}, \tilde{\underline{\underline{h}}})$, provided $\nu > 0$ and $\underline{\underline{V}}_u$ is PDS is

$$D(\underline{\underline{P}}, \underline{\underline{\epsilon}}|\underline{\underline{P}}', \underline{\underline{V}}', \underline{\underline{\epsilon}}', \nu'; n, \nu, \underline{\underline{V}})$$

$$\propto \frac{|\underline{\underline{\epsilon}}|^{\frac{1}{2}\nu-1}}{|[\underline{\underline{P}}-\underline{\underline{P}}']\underline{\underline{V}}_u[\underline{\underline{P}}-\underline{\underline{P}}']^t+\underline{\underline{\epsilon}}+\underline{\underline{\epsilon}}'|^{\frac{1}{2}(\nu''+m-1)}} \tag{18a}$$

where

$$\underline{\underline{V}}_u^{-1} \equiv \underline{\underline{V}}^{-1}\ \underline{\underline{V}}''\ \underline{\underline{V}}'^{-1} \quad , \quad \underline{\underline{V}}_u = \underline{\underline{V}}'\ \underline{\underline{V}}''^{-1}\underline{\underline{V}} = \underline{\underline{V}}\ \underline{\underline{V}}''^{-1}\ \underline{\underline{V}}' \ . \tag{18b}$$

Proof: From (13) and (17),

$$D(\underline{\underline{P}}, \underline{\underline{\epsilon}} | \underline{\underline{P}}', \underline{\underline{V}}', \underline{\underline{\epsilon}}', \nu'; n, \nu, \underline{\underline{V}})$$

$$= \int_{R_{\underline{\underline{h}}}} [\int_{R_{\underline{\underline{\Pi}}}} f_N^{(mr)}(\underline{\underline{P}}|\underline{\underline{\Pi}}, \underline{\underline{h}} \otimes \underline{\underline{V}}) f_N^{(mr)}(\underline{\underline{\Pi}}|\underline{\underline{P}}', \underline{\underline{h}} \otimes \underline{\underline{V}}') d\underline{\underline{\Pi}}]$$

$$\cdot f_W^{(m)}(\underline{\underline{\epsilon}}|\underline{\underline{h}}, \nu) f_W^{(m)}(\underline{\underline{h}}|\underline{\underline{\epsilon}}', \nu') d\underline{\underline{h}} .$$

The inner integral is $f_N^{(mr)}(\underline{\underline{P}}|\underline{\underline{P}}', \underline{\underline{h}} \otimes \underline{\underline{V}}_u)$. For since both $\tilde{\underline{\underline{\Pi}}}$ and $\tilde{\underline{\underline{B}}} \equiv \tilde{\underline{\underline{P}}} - \tilde{\underline{\underline{\Pi}}}$ are independent Multinormal random variables, it follows that $\tilde{\underline{\underline{P}}}$ is Multinormal with mean matrix

$$E(\tilde{\underline{\underline{P}}}) = E(\tilde{\underline{\underline{B}}}) + E(\tilde{\underline{\underline{\Pi}}}) = \underline{\underline{0}} + E(\tilde{\underline{\underline{\Pi}}}) = \underline{\underline{P}}'$$

and variance-covariance matrix

$$V(\tilde{\underline{\underline{P}}}) = V(\tilde{\underline{\underline{B}}}) + V(\tilde{\underline{\underline{\Pi}}}) = (\underline{\underline{h}} \otimes \underline{\underline{V}})^{-1} + (\underline{\underline{h}} \otimes \underline{\underline{V}}')^{-1}$$

$$= (\underline{\underline{h}}^{-1} \otimes \underline{\underline{V}}^{-1}) + (\underline{\underline{h}}^{-1} \otimes \underline{\underline{V}}'^{-1}) = \underline{\underline{h}}^{-1} \otimes (\underline{\underline{V}}^{-1} + \underline{\underline{V}}'^{-1}) .$$

Consequently the matrix precision of $\tilde{\underline{\underline{P}}}$ is

$$[\underline{\underline{h}}^{-1} \otimes (\underline{\underline{V}}^{-1} + \underline{\underline{V}}'^{-1})]^{-1} = \underline{\underline{h}} \otimes (\underline{\underline{V}}^{-1} + \underline{\underline{V}}'^{-1})^{-1} = \underline{\underline{h}} \otimes \underline{\underline{V}}_u$$

as

$$\underline{\underline{V}}'^{-1} + \underline{\underline{V}}^{-1} = \underline{\underline{V}}^{-1}(\underline{\underline{V}}' + \underline{\underline{V}})\underline{\underline{V}}'^{-1} = \underline{\underline{V}}^{-1} \underline{\underline{V}}'' \underline{\underline{V}}'^{-1} = \underline{\underline{V}}_u^{-1} .$$

Hence we may write the integral above as

$$\int_{R_{\underline{\underline{h}}}} f_N^{(mr)}(\underline{\underline{P}}|\underline{\underline{P}}', \underline{\underline{V}}_u, \underline{\underline{h}}) f_W^{(m)}(\underline{\underline{\epsilon}}|\underline{\underline{h}}, \nu) f_W^{(m)}(\underline{\underline{h}}|\underline{\underline{\epsilon}}', \nu') d\underline{\underline{h}}$$

$$\propto |\underline{\underline{\epsilon}}|^{\frac{1}{2}\nu - 1} \int_{R_{\underline{\underline{h}}}} e^{-\frac{1}{2}\mathrm{tr}\, \underline{\underline{h}}\{[\underline{\underline{P}} - \underline{\underline{P}}']\underline{\underline{V}}_u[\underline{\underline{P}} - \underline{\underline{P}}']^t + \underline{\underline{\epsilon}} + \underline{\underline{\epsilon}}'\}} |\underline{\underline{h}}|^{\frac{1}{2}(\nu' + \nu + m + \delta + \delta' - \delta'' - \Phi - 1) - 1} d\underline{\underline{h}}$$

Using the definition (16b) the last integral is

$$|\underline{\underline{\epsilon}}|^{\frac{1}{2}\nu - 1} \int_{R_{\underline{\underline{h}}}} e^{-\frac{1}{2}\mathrm{tr}\, \underline{\underline{h}}\{[\underline{\underline{P}} - \underline{\underline{P}}']\underline{\underline{V}}_u[\underline{\underline{P}} - \underline{\underline{P}}']^t + \underline{\underline{\epsilon}} + \underline{\underline{\epsilon}}'\}} |\underline{\underline{h}}|^{\frac{1}{2}\nu'' - 1} d\underline{\underline{h}} .$$

Since here $\nu > 0$, $\underline{\underline{V}}^* = \underline{\underline{V}}$, and $\delta'' = 1$, the integrand in the above integral is the kernel of a Wishart density with parameter $([\underline{\underline{P}} - \underline{\underline{P}}']\underline{\underline{V}}_u[P - P']^t + \underline{\underline{\epsilon}} + \underline{\underline{\epsilon}}', \nu'')$; hence

the kernel of the distribution of $(\tilde{\underline{\underline{P}}}, \tilde{\underline{\underline{\epsilon}}})$ unconditional as regards $\tilde{\underline{\underline{\Pi}}}$ and $\tilde{\underline{\underline{h}}}$ is (18a).

2.2 Unconditional Distribution of $\tilde{\underline{\underline{P}}}$

The distribution of $\tilde{\underline{\underline{P}}}$ unconditional as regards $\tilde{\underline{\underline{\Pi}}}$, $\tilde{\underline{\underline{h}}}$, and $\tilde{\underline{\underline{\epsilon}}}$ is, provided $\underline{\underline{V}}_u$ is PDS, the generalized multivariate Student distribution defined in (14). That is,

$$D(\underline{\underline{P}}|\underline{\underline{P}}', \underline{\underline{V}}', \underline{\underline{\epsilon}}', \nu'; n, \nu, \underline{\underline{V}}) \propto |[\underline{\underline{P}}-\underline{\underline{P}}']\underline{\underline{V}}_u[\underline{\underline{P}}-\underline{\underline{P}}']^t+\underline{\underline{\epsilon}}'|^{-\frac{1}{2}(\nu'+m)} \qquad (19)$$

<u>whether or not</u> $\nu \leq 0$.

<u>Proof</u>: From (9), the kernel of the marginal likelihood of $\tilde{\underline{\underline{P}}}$ given $(\underline{\underline{\Pi}}, \underline{\underline{h}})$ is

$$e^{-\frac{1}{2}\text{tr } \underline{\underline{h}}\{[\underline{\underline{P}}-\underline{\underline{\Pi}}]\underline{\underline{V}}[\underline{\underline{P}}-\underline{\underline{\Pi}}]^t\}} |\underline{\underline{h}}|^{\frac{1}{2}} \quad . \qquad (9)$$

If $\underline{\underline{P}}$ is defined as in (18) when $\nu \leq 0$, then (9) is the marginal likelihood of $\tilde{\underline{\underline{P}}}$ whether or not $\nu \leq 0$. Conditional on $\tilde{\underline{\underline{h}}}=\underline{\underline{h}}$, the prior distribution of $\tilde{\underline{\underline{\Pi}}}$ is, from (13), Multinormal with parameter $(\underline{\underline{P}}', \underline{\underline{h}} \otimes \underline{\underline{V}}')$.

Consequently, as shown in the course of the proof of (18a), the distribution of $\tilde{\underline{\underline{P}}}$ given $\tilde{\underline{\underline{h}}}=\underline{\underline{h}}$ but unconditional as regards $\tilde{\underline{\underline{\Pi}}}$ is Multinormal with parameter $(\underline{\underline{P}}', \underline{\underline{h}} \otimes \underline{\underline{V}}_u)$ where $\underline{\underline{V}}_u$ is defined in (18b). This implies that, the distribution of $\tilde{\underline{\underline{P}}}$ unconditional as regards $\tilde{\underline{\underline{\Pi}}}$, $\tilde{\underline{\underline{\epsilon}}}$, and $\tilde{\underline{\underline{h}}}$ is

$$\int_{R_{\underline{\underline{\epsilon}}}} \int_{R_{\underline{\underline{h}}}} f_N^{(mr)}(\underline{\underline{P}}|\underline{\underline{P}}', \underline{\underline{V}}_u, \underline{\underline{h}})\ f_W^{(m)}(\underline{\underline{\epsilon}}|\underline{\underline{h}}, \nu) \cdot f_W^{(m)}(\underline{\underline{h}}|\underline{\underline{\epsilon}}', \nu')d\underline{\underline{h}}\ d\underline{\underline{\epsilon}}$$

$$= \int_{R_{\underline{\underline{h}}}} f_N^{(mr)}(\underline{\underline{P}}|\underline{\underline{P}}, \underline{\underline{h}} \otimes \underline{\underline{V}}_u)\ f_W^{(m)}(\underline{\underline{h}}|\underline{\underline{\epsilon}}', \nu')d\underline{\underline{h}}$$

$$\propto \int_{R_{\underline{\underline{h}}}} e^{-\frac{1}{2}\text{tr } \underline{\underline{h}}\{[\underline{\underline{P}}-\underline{\underline{P}}']\underline{\underline{V}}_u[\underline{\underline{P}}-\underline{\underline{P}}']^t+\underline{\underline{\epsilon}}'\}} |\underline{\underline{h}}|^{\frac{1}{2}(\nu'+1)-1}\ d\underline{\underline{h}} \quad .$$

The integrand in the integral immediately above is the kernel of a Wishart density with parameter $([\underline{\underline{P}}-\underline{\underline{P}}']\underline{\underline{V}}_u[\underline{\underline{P}}-\underline{\underline{P}}']^t+\underline{\underline{\epsilon}}', \nu'+1)$. Hence (19) follows.

2.3 Unconditional Distribution of $\tilde{\underline{\underline{\epsilon}}}$

The distribution of $\tilde{\underline{\underline{\epsilon}}}$ unconditional as regards $\tilde{\underline{\underline{P}}}$, $\tilde{\underline{\underline{\Pi}}}$, and $\tilde{\underline{\underline{h}}}$ provided that $\nu > 0$ and $\underline{\underline{\epsilon}}'$ is PDS is

$$D(\underline{\underline{\epsilon}}|\underline{\underline{\epsilon}}', \nu', \nu) \propto \frac{|\underline{\underline{\epsilon}}|^{\frac{1}{2}\nu-1}}{|\underline{\underline{\epsilon}}+\underline{\underline{\epsilon}}'|^{\frac{1}{2}(\nu''+m)}} \quad , \tag{20}$$

a non-standarized generalized inverted Beta distribution with parameter $(\frac{1}{2}\nu-1, \frac{1}{2}(\nu'+r+m)-1, \underline{\underline{\epsilon}}$ as defined in (9g) of [1].

Proof: The marginal likelihood of $\tilde{\underline{\underline{\epsilon}}}$ is $f_W^{(m)}(\underline{\underline{\epsilon}}|\underline{\underline{h}}, \nu)$ and so does not depend on $\underline{\underline{\Pi}}$. The marginal prior distribution of $\tilde{\underline{\underline{h}}}$ is $f_W^{(m)}(\underline{\underline{h}}|\underline{\underline{\epsilon}}', \nu')$. So, when $\nu > 0$, $\delta''=1$ and

$$D(\underline{\underline{\epsilon}}|\underline{\underline{\epsilon}}', \nu', \nu) = \int_{R_{\underline{\underline{h}}}} f_W^{(m)}(\underline{\underline{\epsilon}}|\underline{\underline{h}}, \nu)\, f_W^{(m)}(\underline{\underline{h}}|\underline{\underline{\epsilon}}', \nu')d\underline{\underline{h}}$$

$$\propto |\underline{\underline{\epsilon}}|^{\frac{1}{2}\nu-1} \int_{R_{\underline{\underline{h}}}} e^{-\frac{1}{2}\mathrm{tr}\ \underline{\underline{h}}\{\underline{\underline{\epsilon}}+\underline{\underline{\epsilon}}'\}}\ |\underline{\underline{h}}|^{\frac{1}{2}(\nu''+1)-1}\ d\underline{\underline{h}}$$

Since the integrand in the integral immediately above is the kernel of a Wishart density with parameter $(\underline{\underline{\epsilon}}+\underline{\underline{\epsilon}}', \nu''-1)$, (20) follows directly.

2.4 Unconditional Distribution of A Sample Observation $\tilde{\underline{y}}^{(j)}$ given $\underline{z}^{(j)}$

Suppose we wish to make a probability forecast of a sample observation $\tilde{\underline{y}}^{(j)}$ before the value it assumed is observed, but knowing $\underline{z}^{(j)}$. The conditional distribution of $\tilde{\underline{y}}^{(j)}$ given $\underline{\underline{\Pi}}$, $\underline{\underline{h}}$, and $\underline{z}^{(j)}$ is simply (3). However, if we regard $\tilde{\underline{\underline{\Pi}}}$ and $\tilde{\underline{\underline{h}}}$ as jointly distributed random variables to which we have assigned a Normal-Wishart prior distribution with parameter $(\underline{\underline{P}}', \underline{\underline{V}}', \underline{\underline{\epsilon}}', \nu')$, then the distribution of most interest to us is the distribution of $\tilde{\underline{y}}^{(j)}$ given $\underline{z}^{(j)}$ but unconditional as regards $(\tilde{\underline{\underline{\Pi}}}, \tilde{\underline{\underline{h}}})$. This distribution is, provided $\nu' > 0$, and $\underline{\underline{H}}_y$ is PDS,

$$D(\underline{y}^{(j)} | \underline{\underline{P}}', \underline{\underline{V}}', \underline{\underline{\epsilon}}', \nu'; 1, \underline{z}^{(j)}) = f_S^{(m)}(\underline{y}^{(j)} | \underline{\underline{P}}' \underline{z}^{(j)}, \underline{\underline{H}}_y, \nu') \qquad (21a)$$

where

$$k_z = 1 - \underline{z}^{(j)^t} \underline{\underline{V}}''^{-1} \underline{z}^{(j)} = |\underline{\underline{V}}''^{-1}| |\underline{\underline{V}}| \quad , \qquad (21b)$$

$$\underline{\underline{H}}_y = \nu' k_z (\underline{\underline{\epsilon}}' + \underline{\underline{P}}' [\underline{\underline{V}}' + \underline{\underline{V}}' \underline{\underline{V}}''^{-1} \underline{\underline{V}}' - k_z \underline{\underline{V}}] \underline{\underline{P}}'^t)^{-1}$$

Proof: We prove (21) by twice completing a square and then integrating. For notational simplicity, we drop the superscript on $\underline{y}^{(j)}$ and $\underline{z}^{(j)}$ throughout the proof. From (3) and (13),

$$D(\underline{y} | \underline{\underline{P}}', \underline{\underline{V}}', \underline{\underline{\epsilon}}', \nu'; 1, \underline{z}^{(j)})$$

$$= \int_{R_{\underline{\underline{\Pi}}}} \int_{R_{\underline{\underline{h}}}} f_N^{(m)}(\underline{y} | \underline{\underline{\Pi}}\, \underline{z}^{(j)}, \underline{\underline{h}})\, f_{NW}^{(m)}(\underline{\underline{\Pi}}, \underline{\underline{h}} | \underline{\underline{P}}', \underline{\underline{V}}', \underline{\underline{\epsilon}}', \nu') d\underline{\underline{\Pi}}\, d\underline{\underline{h}}$$

Dropping constants and completing the square in $\underline{\underline{\Pi}}$ in the exponent of the integrand above allows us to write this expression as proportional to

$$\int_{R_{\underline{\underline{h}}}} \Big[\int_{R_{\underline{\underline{\Pi}}}} e^{-\frac{1}{2} tr\, \underline{\underline{h}} \{ [\underline{\underline{\Pi}} - (\underline{yz}^t + \underline{\underline{P}}' \underline{\underline{V}}') \underline{\underline{V}}''^{-1}] \underline{\underline{V}}'' [\underline{\underline{\Pi}} - (\underline{yz}^t + \underline{\underline{P}}' \underline{\underline{V}}') \underline{\underline{V}}''^{-1}]^t \}} |\underline{\underline{h}}|^{\frac{1}{2}} d\underline{\underline{\Pi}} \Big] \qquad (21c)$$

$$e^{-\frac{1}{2} tr\, \underline{\underline{h}} \{ \underline{\underline{\epsilon}}' + \underline{y}\, \underline{y}^t + \underline{\underline{P}}' \underline{\underline{V}}' \underline{\underline{P}}'^t - [\underline{y}\, \underline{z}^t + \underline{\underline{P}}' \underline{\underline{V}}'] \underline{\underline{V}}''^{-1} [\underline{y}\, \underline{z}^t + \underline{\underline{P}}' \underline{\underline{V}}'] \}} |\underline{\underline{h}}|^{\frac{1}{2}\nu' - 1} d\underline{\underline{h}}$$

The integrand of the integral in square brackets is, aside from a multiplicative constant not involving $\underline{y}^{(j)}$, a Normal density, so that

$$D(\underline{y} | \underline{\underline{P}}', \underline{\underline{V}}', \underline{\underline{\epsilon}}', \nu'; 1, \underline{z}^{(j)}) \propto \int_{R_{\underline{\underline{h}}}} e^{-\frac{1}{2} tr\, \underline{\underline{h}}\, \underline{\underline{B}}} |\underline{\underline{h}}|^{\frac{1}{2}\nu' - 1} d\underline{\underline{h}}$$

where $\underline{\underline{B}}$ is the (m x m) matrix in curly brackets in the exponent outside the integral in square brackets in (21c). Since the integrand of the integral immediately above is a Wishart density with parameter ($\underline{\underline{B}}$, ν'),

$$D(\underline{y} | \underline{\underline{P}}', \underline{\underline{V}}', \underline{\underline{\epsilon}}', \nu'; 1, \underline{z}^{(j)}) \propto |\underline{\underline{B}}|^{-\frac{1}{2}(\nu' + r)} \quad ,$$

which is the kernel of a multivariate Student distribution.

By completing the square in $\underline{y}$ in $\underline{\underline{B}}$ we may write this kernel as

$$|\underline{\underline{B}}|^{-\frac{1}{2}(\nu'+r)} = |k_z(\underline{y}-\underline{\mu}_y)(\underline{y}-\underline{\mu}_y) + \underline{\underline{\Phi}}|^{-\frac{1}{2}(\nu'+r)}$$

where

$$k_z = 1 - \underline{z}^t \underline{\underline{V}}''^{-1}\underline{z} \quad ,$$

$$\underline{\mu}_y^t = k_z^{-1}\, \underline{z}^t \underline{\underline{V}}''^{-1}\underline{\underline{V}}'\underline{\underline{P}}'^t \quad ,$$

$$\underline{\underline{\Phi}} = \underline{\underline{\epsilon}}' + \underline{\underline{P}}'\underline{\underline{V}}'\underline{\underline{P}}'^t + \underline{\underline{P}}'\underline{\underline{V}}'\underline{\underline{V}}''^{-1}\underline{\underline{V}}'\underline{\underline{P}}'^t - k_z\, \underline{\mu}_y\, \underline{\mu}_y^t \quad .$$

It remains to be shown that $\underline{\mu}_y = \underline{\underline{P}}'\underline{z}$ and that $\underline{\underline{H}}_y$ may be expressed as in (21b).

To this end observe that

$$k_z = 1 - \underline{z}^t\underline{\underline{V}}''^{-1}\underline{z} = |\underline{\underline{V}}''^{-1}| \begin{vmatrix} 1 & \underline{z} \\ \underline{z}^t & \underline{\underline{V}}'' \end{vmatrix} = |\underline{\underline{V}}''^{-1}|\,|-\underline{z}\,\underline{z}^t + \underline{\underline{V}}''| = |\underline{\underline{V}}''^{-1}|\,|\underline{\underline{V}}'| \; .$$

Thus

$$\underline{\mu}_y^t = k_z^{-1}\, \underline{z}^t\, \underline{\underline{V}}''^{-1}\underline{\underline{V}}'\underline{\underline{P}}'^t = k_z^{-1}\, \underline{z}^t\, \underline{\underline{V}}''^{-1}[\underline{\underline{V}}''-\underline{z}\,\underline{z}^t]\underline{\underline{P}}^t$$

$$= k_z^{-1}[\underline{z}^t - (\underline{z}^t\, \underline{\underline{V}}''^{-1}\, \underline{z})\underline{z}^t]\underline{\underline{P}}'^t = \underline{z}^t\, \underline{\underline{P}}'^t \quad .$$

It follows immediately that

$$\underline{\underline{\Phi}} = \underline{\underline{\epsilon}}' + \underline{\underline{P}}'(\underline{\underline{V}}' + \underline{\underline{V}}'\, \underline{\underline{V}}''^{-1}\, \underline{\underline{V}}' - k_z\, \underline{\underline{V}})\underline{\underline{P}}'^t$$

and that $\underline{\underline{H}}_y$ is as defined in (21b).

3. Preposterior Analysis with Fixed n > 0

We assume that a sample of fixed size $n > 0$ is to be drawn from an $(m \times 1)$ reduced form data generating process as defined in section 1. The parameters $(\tilde{\underline{\underline{\Pi}}}, \tilde{\underline{\underline{h}}})$ of the process are not known with certainty but are assumed to be random variables having a Normal-Wishart prior with parameter $(\underline{\underline{P}}', \underline{\underline{V}}', \underline{\underline{\epsilon}}', \nu')$ where $\nu' > 0$.

3.1 Joint Distribution of $(\tilde{\underline{P}}'', \tilde{\underline{\epsilon}}'')$ given $\underline{V}$

Provided $\nu > 0$, and $\underline{V}^*$ is PDS, the joint distribution of $(\tilde{\underline{P}}'', \tilde{\underline{\epsilon}}'')$ is

$$D(\underline{P}'', \underline{\epsilon}''|\underline{P}', \underline{V}', \underline{\epsilon}', \nu'; n, \nu, \underline{V}) \propto \frac{|\underline{\epsilon}''-\underline{\epsilon}'-[\underline{P}''-\underline{P}']\underline{V}^*[\underline{P}''-\underline{P}']^t|^{\frac{1}{2}\nu-1}}{|\underline{\epsilon}''|^{\frac{1}{2}(\nu''+m-1)}} \tag{22a}$$

where

$$\underline{V}^o = \underline{V}' \, \underline{V}^{-1} \, \underline{V}'' = \underline{V}'' \, \underline{V}^{-1} \, \underline{V}' \tag{22b}$$

and the range of $\tilde{\underline{\epsilon}}''$ is $R_{\underline{\epsilon}''} = \{\underline{\epsilon}''|\underline{\epsilon}''-\underline{\epsilon}'[\underline{P}''-\underline{P}']\underline{V}^o[\underline{P}''-\underline{P}']^t$ is PDS$\}$.

<u>Proof</u>: In a fashion similar to that used in establishing (12-20b) in [2], we can show that

$$[\underline{P}-\underline{P}']\underline{V}_u[\underline{P}-\underline{P}']^t = [\underline{P}''-\underline{P}'] \quad [\underline{P}''-\underline{P}']^t \quad , \tag{23a}$$

and that when $\underline{\epsilon}''$ is PDS,

$$\underline{\epsilon}'' = \underline{\epsilon}'+\underline{\epsilon}+\underline{P}'\underline{V}'\underline{P}'^t+\underline{P} \, \underline{V} \, \underline{P}^t-\underline{P}''\underline{V}''\underline{P}''^t = \underline{\epsilon}'+\underline{\epsilon}+[\underline{P}''-\underline{P}']\underline{V}^o[\underline{P}''-\underline{P}']^t \; . \tag{23b}$$

From (16b) and (16d) we have

$$(\underline{P}, \underline{\epsilon}) = ([\underline{P}''\underline{V}''-\underline{P}'\underline{V}']\underline{V}^{-1} \quad , \quad \underline{\epsilon}''-\underline{\epsilon}'-[\underline{P}''-\underline{P}']\underline{V}^o[\underline{P}''-\underline{P}']^t) \quad . \tag{24}$$

Letting $J(\underline{P}'', \underline{\epsilon}''; \underline{P}, \underline{\epsilon})$ denote the Jacobian of the integrand transformation from $(\underline{P}, \underline{\epsilon})$ and making this transformation in (18a) we obtain

$$D(\underline{P}'', \underline{\epsilon}''|\underline{P}', \underline{V}', \underline{\epsilon}', \nu'; n, \nu, \underline{V})$$

$$\propto \frac{|\underline{\epsilon}''-\underline{\epsilon}'-[\underline{P}''-\underline{P}']\underline{V}^o[\underline{P}''-\underline{P}']^t|^{\frac{1}{2}\nu-1}}{|\underline{\epsilon}''|^{\frac{1}{2}(\nu''+m-1)}} \cdot J(\underline{P}'', \underline{\epsilon}''; \underline{P}, \underline{\epsilon}) \; .$$

Since $J(\underline{P}'', \underline{\epsilon}''; \underline{P}, \underline{\epsilon}) = J(\underline{P}'', \underline{P}) \, J(\underline{\epsilon}'', \underline{\epsilon})$ and since both $J(\underline{P}'', \underline{P})$ and $J(\underline{\epsilon}'', \underline{\epsilon})$ are constants involving neither $\underline{P}$ nor $\underline{\epsilon}$, we may write the above kernel as (22a).

That the range of $\underline{\epsilon}''$ is as shown in (19b) follows from the definitions of $\underline{P}''$, $\underline{\epsilon}''$, $\underline{P}$, and $\underline{\epsilon}$.

3.2 Some Distributions of $\tilde{\underline{\underline{P}}}''$ and $\tilde{\underline{\underline{\epsilon}}}''$

The distribution of $\tilde{\underline{\underline{P}}}''$, unconditional as regards $\tilde{\underline{\underline{\Pi}}}$, $\tilde{\underline{\underline{h}}}$, and $\tilde{\underline{\underline{\epsilon}}}''$ is, provided that $\underline{\underline{V}}^{o} \equiv \underline{\underline{V}}'\underline{\underline{V}}^{-1}\underline{\underline{V}}''$ is PDS,

$$D(\underline{\underline{P}}''|\underline{\underline{P}}', \underline{\underline{V}}', \underline{\underline{\epsilon}}'; n, \nu, \underline{\underline{V}}) \propto |[\underline{\underline{P}}''-\underline{\underline{P}}']\underline{\underline{V}}^{o}[\underline{\underline{P}}''-\underline{\underline{P}}']^{t}+\underline{\underline{\epsilon}}'|^{-\frac{1}{2}(\nu'+m)} \qquad (25)$$

Proof: From (19), the unconditional distribution of $\tilde{\underline{\underline{P}}}$ is generalized multivariate Student with parameter $(\underline{\underline{P}}', \underline{\underline{V}}_{u}, \underline{\underline{\epsilon}}', \nu')$, and from (16c),

$$\underline{\underline{P}}'' = [\underline{\underline{P}}' \ \underline{\underline{V}}' + \underline{\underline{P}} \ \underline{\underline{V}}]\underline{\underline{V}}''^{-1} .$$

Rewriting the above as

$$\underline{p}'' = \underline{p}'(\underline{\underline{I}} \otimes \underline{\underline{V}}' \ \underline{\underline{V}}''^{-1} + \underline{p}(\underline{\underline{I}} \otimes \underline{\underline{V}} \ \underline{\underline{V}}''^{-1})$$

it is easy to see that the Jacobian of the transformation from $\underline{p}$ to $\underline{p}''$ is $|\underline{\underline{I}} \otimes \underline{\underline{V}} \ \underline{\underline{V}}''^{-1}|^{\frac{1}{2}} = |\underline{\underline{V}} \ \underline{\underline{V}}''^{-1}|^{\frac{1}{2}m}$, a constant as regards $\underline{p}''$. Thus substituting according to the dictates of (23a) into (19) yield the distribution of $\tilde{\underline{\underline{P}}}''$ as shown in (25).

The distribution of $\tilde{\underline{\underline{\epsilon}}}''$ unconditional as regards $\tilde{\underline{\underline{\Pi}}}$, $\tilde{\underline{\underline{h}}}$, but given $\underline{\underline{V}}$ and $\underline{\underline{P}}''$, is found in a similar fashion to be

$$D(\underline{\underline{\epsilon}}''|\underline{\underline{P}}', \underline{\underline{V}}', \underline{\underline{\epsilon}}'; n, \nu, \underline{\underline{V}}, \underline{\underline{P}}'') \propto \frac{|\underline{\underline{\epsilon}}''-\underline{\underline{\epsilon}}'-\underline{\underline{P}}'\underline{\underline{V}}'\underline{\underline{P}}'^{t}-\underline{\underline{P}} \ \underline{\underline{V}} \ \underline{\underline{P}}^{t}+\underline{\underline{P}}''\underline{\underline{V}}''\underline{\underline{V}}''^{t}|^{\frac{1}{2}\nu-1}}{|\underline{\underline{\epsilon}}''-\underline{\underline{P}}'\underline{\underline{V}}'\underline{\underline{P}}'^{t}-\underline{\underline{P}} \ \underline{\underline{V}} \ \underline{\underline{P}}^{t}+\underline{\underline{P}}''\underline{\underline{V}}''\underline{\underline{P}}''^{t}|^{\frac{1}{2}(\nu''+m)}} \qquad (26)$$

with range set $R_{\underline{\underline{\epsilon}}''}$.

4. Analysis When Rank $(\underline{\underline{Z}} \ \underline{\underline{Z}}^{t}) < r$

4.1 Inference When Rank $(\underline{\underline{Z}} \ \underline{\underline{Z}}^{t}) < r$

Even when the rank q of $\underline{\underline{Z}} \ \underline{\underline{Z}}^{t}$ is less than r, it is possible to do Bayesian inference on $(\tilde{\underline{\underline{\Pi}}}, \tilde{\underline{\underline{h}}})$ by appropriately structuring the prior so that the posterior on $(\tilde{\underline{\underline{\Pi}}}, \tilde{\underline{\underline{h}}})$ is non-degenerate. The procedure here is analagous to that suggested in section 3.3 of [5].

For example, suppose the data generating process is that of section 1.1, we assign a prior on $(\tilde{\underline{\underline{\Pi}}}, \tilde{\underline{\underline{h}}})$ with parameter $(\underline{\underline{0}}, \underline{\underline{0}}, \underline{\underline{0}}, 0)$, and then we observe a sample $(\underline{y}^{(1)},\underline{z}^{(1)}), (\underline{y}^{(2)},\underline{z}^{(2)}),\ldots,(\underline{y}^{(n)},\underline{z}^{(n)})$ where $n < m$. The posterior distribution of $(\tilde{\underline{\underline{\Pi}}}, \tilde{\underline{\underline{h}}})$ is degenerate in this case, as the posterior parameters (16) assume values

$$\nu''=\nu+m-\Phi-1=0 \quad , \quad \underline{\underline{P}}''=\underline{\underline{P}} \quad , \quad \underline{\underline{V}}''*=\underline{\underline{0}} \quad , \quad \underline{\underline{\epsilon}}''*=\underline{\underline{0}} \quad .$$

If, however, we require the prior on $(\tilde{\underline{\underline{\Pi}}}, \tilde{\underline{\underline{h}}})$ to be very diffuse, but are willing to introduce just enough prior information to make $\nu'' > 0$, $\underline{\underline{V}}''$ PDS, and $\underline{\underline{\epsilon}}''$ PDS, then a non-degenerate posterior will exist; e.g. assign $\nu'=1$, $\underline{\underline{V}}'=M\,\underline{\underline{I}}$, $M \gg 0$, and $\underline{\underline{\epsilon}}'=K\,\underline{\underline{I}}$, $K \gg 0$, so that $\nu''=1$, $\underline{\underline{V}}''=M\,\underline{\underline{I}} + \underline{\underline{V}}$, and $\underline{\underline{\epsilon}}''=K\,\underline{\underline{I}} + \underline{\underline{\epsilon}}$. The posterior on $(\tilde{\underline{\underline{\Pi}}}, \tilde{\underline{\underline{h}}})$ is then non-degenerate Normal-Wishart with parameter $(\underline{\underline{P}}'', \underline{\underline{V}}'', \underline{\underline{\epsilon}}'', \nu'') = (\underline{\underline{P}}, M\,\underline{\underline{I}}+\underline{\underline{V}}, K\,\underline{\underline{I}}+\underline{\underline{\epsilon}}, 1)$.

Notice that if $m < q < r$, $\underline{\underline{\epsilon}}$ is PDS, so that the posterior will be non-degenerate even if $\underline{\underline{\epsilon}}'=0$, so long as $\nu'' > 0$ and $\underline{\underline{V}}''$ is PDS.

To see that in fact a prior on $(\tilde{\underline{\underline{\Pi}}}, \tilde{\underline{\underline{h}}})$ with parameter $(\underline{\underline{0}}, M\,\underline{\underline{I}}, K\,\underline{\underline{I}}, 1)$ is extremely diffuse, we state the following result proved by Martin [4], who gives explicit formulas for calculating the means, variances, and covariances of elements of the variance-covariance matrix $\tilde{\underline{\underline{h}}}^{-1}$:

<u>Theorem</u>: If $\tilde{\underline{\underline{h}}}$ is (m x m) and Wishart distributed with parameter $(\underline{\underline{\epsilon}}, \nu)$ then $E(\tilde{\underline{\underline{h}}}^{-1}) = (1/\nu-2)\underline{\underline{\epsilon}}$ if $\nu > 2$; and letting $\tilde{h}^{-1}_{\alpha\beta}$ denote the $(\alpha\ \beta)$th element of $\tilde{\underline{\underline{h}}}^{-1}$, for $1 \leq \alpha, \beta, \gamma, \delta \leq m$

$$\mathrm{Cov}(\tilde{h}^{-1}_{\alpha\beta}, \tilde{h}^{-1}_{\gamma\delta}) = \frac{1}{3(\nu-2)(\nu-4)}\left[\frac{2(5-\nu)}{\nu-2}\,\epsilon_{\alpha\beta}\,\epsilon_{\gamma\delta} + \epsilon_{\alpha\nu}\,\epsilon_{\beta\delta} + \epsilon_{\alpha\delta}\,\epsilon_{\beta\gamma}\right]$$

if $\nu > 4$.

Since in this example $\nu''=1$, it is easy to show that none of the above moments exist. If we had specified $\nu'=5$, say, then first and second moments of $\tilde{\underline{\underline{h}}}^{-1}$ would exist and be equal to $E(\tilde{\underline{\underline{h}}}^{-1}) = \frac{M}{3}\,\underline{\underline{I}}$ and

$$\mathrm{Cov}(\tilde{h}^{-1}_{\alpha\beta}, \tilde{h}^{-1}_{\gamma\delta}) = \frac{1}{9}[\epsilon'_{\alpha\gamma}\,\epsilon'_{\beta\delta} + \epsilon'_{\alpha\delta}\,\epsilon'_{\beta\gamma}] \quad , \quad 1 \leq \alpha, \beta, \gamma, \delta \leq m ,$$

where $\epsilon'_{\ell k}=K$ if $\ell=k$ and 0 if $\ell \neq k$ for $1 \leq \ell, k \leq m$.

It is also important to observe that as the projection of each $\underline{y}_i$, $i=1,2,\ldots,m$ on the q-dimensional row space of $\underline{\underline{Z}}$ is unique, arbitrary specification of $(r-q)m$ elements of $\underline{\underline{P}}$ as equal to 0 does not influence the values assumed by the posterior parameters. That is, none of the posterior parameters $\underline{\underline{P}}''$, $\underline{\underline{V}}''$, $\underline{\underline{\epsilon}}''$ and ν'' depend on which $(r-q)m$ particular p_{ij}s are set equal to zero.

4.2 Probabilistic Prediction

If when $q < m$ we assign a prior to $(\tilde{\underline{\underline{\Pi}}}, \tilde{\underline{\underline{h}}})$ with parameter $(\underline{\underline{0}}, M\,\underline{\underline{I}}, K\,\underline{\underline{I}}, 1)$, then by (21) the unconditional distribution of the next sample observation given $\underline{z}^{(n+1)}$ is non-degenerate multivariate Student with parameter $(\underline{0}, k_z K\,\underline{\underline{I}}, 1)$ where $k_z = 1-\underline{z}^{(n+1)^t}(M\,\underline{\underline{I}} + \underline{\underline{V}})^{-1}\underline{z}^{(n+1)}$. Notice that setting $0 < \nu' \leq 2$ as we have done here implies that the second moment of $\tilde{\underline{y}}^{(n+1)}$ given $\underline{z}^{(n+1)}$ does not exist, so that the unconditional distribution of $\tilde{\underline{y}}^{(n+1)}$ is extremely diffuse in this particular example.

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