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by

MURAT R. SERTEL

428-69

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MASSACHUSETTS INSTITUTE OF TECHNOLOGY
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Introduction:

Thirty odd years after the General Theory and Hicks' invention of the diagrammatics now so commonly known by the short-hand 'IS-LM', a cloud of ambiguity or confusion seems to surround this representation in the literature. On the one hand, there is ambiguity concerning the response of the equilibrium condition LM for the Keynesian money and (long-term) bond market to changes in the supply and transactions-velocity of money. On the other hand, there seems to be some confusion in the representation of Keynes' minimum rate of interest. It would appear not too early, therefore, to briefly demonstrate the more common pitfalls which can be fallen into in treating these elements of the Keynesian system. This is the task undertaken here.

Other writers' apparent oversights are dealt with mainly in footnotes. The main body of the note purports to give the rectified picture. In particular, the latter shows the following:

(A) that Keynes' minimum rate of interest is the limit of the r-intercept of LM as M → ∞ (where r is the rate of interest on consols and M is the quantity of money); and

(B) that the LM schedule will shift in different ways accordingly as it is associated with a change in M or a change in transactions-velocity and the active (transactions) demand function.

LM, Shifts in LM, and Keynes' Infinum:

With the usual sort of notation and assumptions, let Y denote communal income in money terms, r the rate of interest on long-term bonds,
the supply of money; assume prices (other than for bonds) fixed, and
the demand \( L \) for money additively separable into an active component
\( L_T \), independent of \( r \) and monotonically increasing linearly or piece-
wise linearly in \( Y \), and an idle component \( L_S \), independent of \( Y \) and
monotonically decreasing in \( r \). The equilibrium condition for the money
and bond (consol) market is thus given by

\[
L(Y, r) = L_T(Y) + L_S(r) = M,
\]

according to which the form of LM relation is determined by three sets
of data, namely, the forms of \( L_T \) and \( L_S \) and the value of \( M \).

In Figure 1, Plane I accommodates different forms of \( L_T \), Plane II
accomodates specification of \( L_S \), and the axis adjoining the two is
scaled to measure \( M \), as well as \( L \), \( L_T \) and \( L_S \), so that it also measures
the "heights" of the three-dimensional "hills" derived from data in
Planes I and II and the contours of which are projected as LM relations
onto Plane III of \((Y, r)\) points.

Take the case of a linear active demand, \( L_T = \frac{1}{v} Y \),
with constant velocity \( v > 0 \), as shown by the bold line in Plane II.
With \( L_S \) as in Plane II, if \( M = M_1 \), the equilibrium relation \( LM_1 \) emerges
on Plane III: if \( L_T \) is nil, for equilibrium, \( L_S = M_1 \) must hold, and this
happens at \((Y, r) = (0, \bar{r}_1)\); as \( r \) increases (the bond price \( \frac{1}{r} \) decreases),
preference increasingly favors bonds rather than idle cash balances,
i.e., \( L_S \) decreases, so that, for equilibrium, \( L_T \) must absorb increasing
portions of \( M_1 \), until all of it tends to be so absorbed at \((\bar{Y}_1, \infty)\).
Similarly, for \( M = M_0 < M_1 \), we should have derived an \( LM_0 \) everywhere
above \( LM_1 \), with \( \bar{r}_0 > \bar{r}_1 \) and \( \bar{Y}_0 < \bar{Y}_1 \). And it is thus that \( LM_2 \) appears with
\[ \tilde{r}_2 < \tilde{r}_1 \text{ and } \tilde{Y}_2 > \tilde{Y}_1, \text{ for } M = M_2 > M_1. \]

More generally, if \( L_T \) is monotonically increasing and \( L_S \) monotonically decreasing, \( L_M \) is one-to-one, so that \( r \) is a function \( r = L_M(Y;M) \) of \( Y \). In particular, denoting \( L_M(Y;M_1) = L_M(Y) \),

\[
(2) \quad \tilde{r}_i = \lim_{Y \to 0} (L_M(Y;M_1)) = \lim_{Y \to 0} (L_M(Y)), \quad (i = 1, 2, \ldots).
\]

Furthermore, induction on \( i \) carries us to the result that, with any increasing sequence \( (M_i) \) is associated a decreasing sequence \( (\tilde{r}_i) \) (and an increasing sequence \( (\tilde{Y}_i) \)). Thus, under these conditions, any increase in \( M \) brings about a shift in \( L_M \) which is unambiguously both rightward and downward.

The rightward shifts in \( L_M \) brought about by increasing \( M \) are not also unambiguously downward, however, if \( L_S \) ceases to be a monotonically decreasing function of \( r \). The most interesting case in this category is where the graph \((r,L_S)\) coincides with the graph of \( r = \tilde{r} \) for \( L_S \geq M^* \), where \( M^* = \inf(L_S: L_S^{-1} = \tilde{r}) \). (In this case, \( L_S \) ceases to be a function, but \( L_S^{-1} \) is a function, since each \( L_S \) in the domain of \( L_S^{-1} \) is mapped by \( L_S^{-1} \) to at most one point \( r = L_S^{-1}(L_S) \) in the range, and the function \( L_S^{-1} \) is non-increasing.) Suppose the sequence \((M_i)\) is increasing and let \( M_k = \inf(M_i: M_i \geq M^*) \). (For example, suppose \( M_k = M^* \).) Then, shifts in \( L_M \) brought about by increases in \( M \) beyond \( M_k \) will bring about no fall in the \( r \)-intercept \( r_j(j \geq k) \), so that \( r_j = \tilde{r}(j \geq k) \). But, furthermore, for each \( r \), there will be a positive horizontal distance between any two curves \( L_M \) and \( L_M \) \((h \geq j)\), and, if \( L_T = \frac{1}{v} Y \), then this distance will be constant for all \( r \) and equal to \( \tilde{Y}_h - \tilde{Y}_j = v(M_h - M_j) \). Thus, all \( L_M \) will be horizontal for \( Y \leq v(M_j - M^*) \) and \( L_M \) will coincide
with $LM_h$ over this range, "kinking" into positive slope at the upper end of the range. These results are illustrated in Figure 1 by setting $M^* = M_k$, $k = 2$, $j = 3$, $h = 4$.\(^7\)

Under the broadest admissible conditions that $L_T$ is a monotonically increasing function and $L_S^{-1}$ is a non-increasing function such that either $L_S^{-1}$ is asymptotic to $r = \bar{r}$ or coincides with it for $L_S \geq M^*$, with any increasing sequence $(M_i)$ is associated a non-increasing sequence $(\bar{r}_i)$. Such an $(\bar{r}_i)$ will converge iff it is bounded from below. But under the stated conditions, $\bar{r}$ is a lower bound for such an $(\bar{r}_i)$, so that such a sequence does converge. In fact, for any $(M_i)$ which is unbounded from above (and increasing),

\[(3) \quad \bar{r} = \lim_{i \to \infty} (\bar{r}_i)\]

exists and is the "Keynes minimum" (which is actually an infimum), below which there is no demand for bonds.\(^8\)

In the literature, shifts in LM are pictured often in ways which we have so far not encountered here, so we turn now to understand these, which are shown in Figure 1 as shifts from bold $LM_1$ to dotted $LM_1'$ or $LM_1''$ curves. Such shifts are due, not to changes in $M$, as our notation is clear about, but, despite misleading presentations in the literature, to changes in the transactions-velocity $L_T/Y$.\(^9\) The reader may check that a shift from $v$ to $v'$, where the illustrative set-up of $\frac{1}{v} = \frac{1}{\bar{Y}} \geq \frac{1}{\bar{Y}_1} = \frac{1}{\bar{Y}_1}'$ has been chosen, brings about a shift of $LM_1$ to $LM_1'$, with no change in $M$, and similarly, that a shift from $L_T = \frac{1}{v}Y$ to $L_T'' = V(Y) = \min(\frac{1}{v}Y, \frac{1}{v}Y + c)$, where $c = (\frac{1}{v} - \frac{1}{v'})\bar{Y}_1$, leads to a shift from $LM_2$ to $LM_2''$, again with no change in $M$. Similarly for the couples $LM_3$ and $LM_3''$ and $LM_4$ and $LM_4''$.\(^9\)
The reader should compare these results with those of Hicks, Hansen, Eastham, Allen and Patinkin, for all but the last of which relevant references and remarks are furnished in footnotes so far. Since Patinkin does not use the LM framework, his representations are best examined separately, and a final moment of attention to this is now due. Very briefly, since, under the broadest admissible conditions, specified above and which are consistent with Patinkin's, the Keynes minimum is found as the double limit \( \bar{r} \) computed by sending \( Y \) to nil for each \( M_i \) in an unboundedly increasing sequence \((M_i)\) and then sending \( M_i \) (or \( i \)) to infinity, Patinkin's mistake emerges as reading off (as the Keynes minimum) the lower limit value of \( r \) (corresponding to the upper limit value of the bond price \( \frac{1}{r} \)) at which his demand \( B^d \) for bonds is vanishing, while \( Y \) is at (a presumable non-zero) full employment level. This value of \( r \) which he takes to be the Keynes minimum will always lie above the true Keynes minimum \( \bar{r} \) for any LM increasing in \( Y \) and positive \( Y \), so that, under any reasonable conditions, Patinkin's computation procedure will consistently over-estimate this true value.

\[
\begin{align*}
L_1 & \quad L_2 \quad L_3 \quad L_4 \\
M, L, L_t, L_s
\end{align*}
\]

Figure I
1 The author's thanks go to Professors Evsey D. Domar and Zenon S. Zannetos of M.I.T. and Richard C. Sutch of U. Cal., Berkeley, for helpful stylistic suggestions and encouragement to (re-)write this note.


3 Some of these assumptions are relaxed very soon. The additive separability of L into $L_T$ and $L_S$, however, is an analytical simplification which proves too useful for clarity to be abandoned. Thus, the possibility of interest-elasticity of transactions-motivated $L_T$ at high rates of interest where a given $M$ has already been completely drawn into the active sphere (see, e.g., Alvin Hansen, Monetary Theory and Fiscal Policy, McGraw-Hill, (1944), 66-70, 77-80) is ignored here, for the main points to be made are better put without confounding $L_T$ with a dependence on r. The symptoms of the mentioned phenomenon are kept, however, by specifying $L_S$ as asymptotic to $L_S = 0$ and allowing the transactions-velocity to increase with $Y$, so that r is pushed up higher and higher as greater portions of $M$ are drawn into transactions use, and, although this allows financing of transactions suitable for higher levels of $Y$, it is accompanied by a greater efficiency of transactionary use of money as the transactions-velocity of money increases with $Y$. The plausibility of an $L_S$ asymptotical to the r axis should be apparent when this is associated with the tendency of demand $L_S$ for idle cash to vanish as the bond (consol) price $1/r$ tends to zero.

4 The apparent complexity of Figure 1 is motivated not only by economy of space, but also by a desire to yield many results comparable within a single framework.

5 Thus, for small enough $M_0$, LM will be so high that it is above IS (not drawn, but insertable by the reader) for all $Y > 0$, so that an intersection determining r is not afforded. Hence, in stating the Keynesian complete system, it should be recognized as a necessary condition for the existence of a solution that $M$ be large enough to yield an r-intercept of the corresponding LM below that of IS. This point does not come out if it is thought, erroneously, that each LM has an r-intercept equal to the Keynesian minimum rate of interest. (Cf. R.G.D. Allen, MacroEconomic Theory, MacMillan, (1967), 134, Exercise 7.2.)

6 Allen somehow gets LM curves, drawn for different values of $M$, "hinged on" an r to which L is asymptotic (ibid., 122). This is impossible even with the form of $L_T$ (with $dL_T/dY > 0$, $d^2L_T/dY^2 > 0$) he specifies. (The effect of an $L_T$ with diminishing slope is treated later in the note, where the reader is brought to compare, e.g., $LM_2$ and $LM_2''$.)
Actually, Allen's error seems to be as old as can be in so far as it is rooted in passages (of Hicks) where Hicks contends that "If the supply of money is increased, the curve LM moves to the right..., but the horizontal parts of the curve are almost the same." and that "It should be observed that this minimum to the rate of interest applies not only to one curve LM (drawn to correspond to a particular quantity of money) but to any such curve." (op.cit., 155). Thus, all the curves would hinge on the same r-intercept, the "minimum to the rate of interest", with horizontal parts that were almost the same. We already know that this is impossible with an $L_S$ asymptotic to that r-intercept $\bar{r}$. However, in this asymptotic case the curves would not have horizontal parts. Horizontal parts would occur if $L_S$ actually turned infinitely elastic at some finite value of $L_S$, which case is examined next in the note. But in this case, any two curves drawn for $M$ greater than the value of $L_S$ at which infinite elasticity sets in are not just almost the same, but absolutely the same, as will be seen shortly, over the horizontal portion of the curve drawn for the smaller $M$. For anyone who has been lucky enough, as I have been, to have been a student of Professor Hicks, it is difficult not to feel the greatest of intellectual respect for him; but it seems that, on the mentioned points, an oversight had crept in. (Incidentally, his mentioned article was reprinted, intact with this oversight, in his Critical Essays in Monetary Theory, Oxford, (1967).)

7 See Footnote (6).

8 See Footnote (6) to see how Hicks and Allen have (mis-) identified the "Keynes minimum".

9 Since $M$ and $\bar{r}$ are not changed, the point at which LM,$''$ kinks is the same as that at which $LM_1$, kinks while the asymptote $\overline{Y}$, moves to $\overline{Y}''$. The combined effect of an increase from $M_1$ to $M$, and an increase from $\delta L_m/y = u$ to $\delta L_m/y = u''$ is found to be the same whether LM,$''$ is computed first and then shifted to $LM_1$, or LM, is computed first and then modified to account for the increase from $u$ to $u''$ (assuming that the slope of $LM_1$ (before and after the increase in $M$) is constant over the interval the end points of which are the Y values at which $LM_1$ and $LM_2$ exhibit their respective kinks).

Examples of how the effects of change in $M$ were confused with changes in velocity are Hicks' Figure 2 (op. cit., 155), Hansen's Figures 12 and 13 (op. cit., 77,79, respectively) and his Figure 16 in A Guide to Keynes, McGraw-Hill, (1953), 146, and Allen's Figures 7.4B and 7.4C (op. cit., 121, 122, respectively). (Hansen's last mentioned figure shows a shift very much like that from $LM_2$ to $LM_1$ in Figure 1 here. However, this shift is meant to be "due either to an increase in the money supply and/or to a decrease in the underlying liquidity preference schedule" (loc.cit.), of which the first cannot be.) With greatest respect for my teacher, Prof. J.K. Eastham, I must add Figure 8 on p.293 of his Graphical Economics (The English Universities Press, London, (1960)) to this list of graphics.

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