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CONTINUOUS TIME STOPPING GAMES

by

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ABSTRACT

We prove the existence of a Nash equilibrium of a class of continuous time stopping games. Under a strict monotonicity condition, there exists a unique symmetric equilibrium for a symmetric stopping game. Any Nash equilibrium of a stopping game is subgame perfect. The machinery employed in the analyses is the General Theory of Process. In the next version, we will provide conditions under which there is a unique Nash equilibrium for general stopping games and will give examples of our general theory.
Continuous Time Stopping Games

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Abstract

We prove the existence of a Nash equilibrium of a class of continuous time stopping games. Under a strict monotonicity condition, there exists a unique symmetric equilibrium for a symmetric stopping game. Any Nash equilibrium of a stopping game is subgame perfect. The machinery employed in the analyses is the General Theory of Processes. In the next version, we will provide conditions under which there is a unique Nash equilibrium for general stopping games and will give examples of our general theory.
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1. Introduction

This paper presents the theoretic findings on the continuous time version of the general stopping games. The game theoretic extension of the optimal stopping theory in the discrete time framework was initiated by Dynkin [1969] in analysis of a class of two person zero-sum stopping games. The modifications and extensions include Chaput [1974] in zero sum stopping games and Mamer [1986] in two person nonzero-sum stopping monotone games.

The widespread use or potential use of stopping games can be found in economics, finance, and management science. Examples include the entry and exit decisions of firms, job search, optimal investment in research and development, and the technology transition in industries. (See Fine and Li [1986], Mamer and McCardle [1985], Reinganum [1982] etc.) In fact, any stochastic dynamic game in which each player's strategy is a single dichotomous decision at each time can be formulated as a stopping time problem. Fine and Li [1986] also show that for this type of game the stopping time equilibria correspond to the subgame perfect equilibria in the natural extensive form game.

However, one of the major weaknesses of the discrete time stopping games is the multiplicity of equilibria. In an oligopolistic exit game, Fine and Li [1986] pointed out that the multiplicity of equilibria is a consequence of the fact that the industry demand process can jump from a point where both firms are viable as duopolists to a point where neither firm is viable as a duopolists, but each is viable as a monopolist. This problem might be avoided by assuming that the information evolves continuously, which could be valid only in a continuous time framework. This is one of the reasons that motivate this study.

The only work we are aware of dealing with game theoretic extension of continuous time optimal stopping problems is Chaput [1974]. He analyzed a two person zero sum stopping game proposed by Dynkin [1969].

Our study is composed of two parts. In this part, we start with a general setting of an $N$-person continuous time stopping game assuming the payoff processes are optional processes. A martingale approach (in contrast to the excessive function approach in the Markov setting) is adopted to show the existence of the reaction correspondences of players under fairly general conditions. The existence of a Nash equilibrium in games with a monotone payoff structures follows from Tarski's lattice theoretic fixed point theorem. We also show that there is a unique symmetric equilibrium if further structures are imposed. Moreover, any Nash equilibrium of the stopping game is a subgame perfect equilibrium as in the discrete time case.

This version is a preliminary and incomplete draft. The continuation of this work anticipates the uniqueness of an equilibrium when information is revealed continuously. We will also construct useful examples in economics, finance, and management science.
2. The formulation

Let \((\Omega, \mathcal{F}, P)\) be a complete probability space equipped with an increasing family of sub-sigma-field of \(\mathcal{F}\), or a filtration, \(\mathcal{F} = \{\mathcal{F}_t, t \in \mathbb{R}_+\}\), that satisfies the usual conditions:

1. complete: \(\mathcal{F}_0\) contains all the \(P\)-null sets;
2. right-continuous: \(\mathcal{F}_t = \bigcap_{s > t} \mathcal{F}_s, \forall t \in \mathbb{R}_+\).

We also assume that \(\mathcal{F}_0\) is almost trivial in that it is generated by \(\Omega\) and all the \(P\)-null sets. We shall denote \(\bigvee_{t \in \mathbb{R}_+} \mathcal{F}_t\), the smallest sigma-field containing \(\{\mathcal{F}_t, t \in \mathbb{R}_+\}\) by \(\mathcal{F}_\infty\).

We interpret each \(\omega \in \Omega\) to be a complete description of the state of the world. The filtration \(\mathcal{F}\) models the way information about the true state of the world is revealed over time. In a discrete time finite state setting, a filtration can be thought of as an event tree.

Before we proceed, some definitions are in order.

A process \(X\) is a mapping \(X : \Omega \times \mathbb{R}_+ \to \mathbb{R}\) that is measurable with respect to \(\mathcal{F} \otimes \mathcal{B}(\mathbb{R}_+)\), the product sigma-field generated by \(\mathcal{F}\) and the Borel sigma-field of \(\mathbb{R}_+\). A process \(X\) is said to be adapted, if \(X(t)\) is measurable with respect to \(\mathcal{F}_t, \forall t \in \mathbb{R}_+\). A process is progressively measurable if \(\forall t \in \mathbb{R}_+\), the mapping \((\omega, s) \mapsto X(\omega, s)\) of \(\Omega \times [0, t]\) into \(\mathbb{R}\) is measurable with respect to the product sigma-field \(\mathcal{F}_t \otimes \mathcal{B}([0, t])\), where as usual \(\mathcal{B}([0, t])\) is the Borel sigma-field of \([0, t]\). The optional sigma-field, denoted by \(\mathcal{O}\), is the sigma-field on \(\Omega \times \mathbb{R}_+\) generated by adapted processes having right-continuous paths; cf., Chung and Williams [1983]. A process is optional if it is measurable with respect to \(\mathcal{O}\). Naturally, any adapted process with right-continuous paths is optional. It is also known that any optional process is adapted; cf., Chung and Williams [1983].

Let \(\mathbb{R}_+\) denote the extended positive real line. A Markov time \(T\) is a function from \(\Omega\) into \(\mathbb{R}_+\) such that

\[
\{\omega \in \Omega : T(\omega) \leq t\} \in \mathcal{F}_t, \quad \forall t \in \mathbb{R}_+.
\]

A Markov time \(T\) is said to be a stopping time if \(P\{T < \infty\} = 1\). A stopping time \(T\) is bounded if there exists a constant \(K \in \mathbb{R}_+\) such that \(P\{T \leq K\} = 1\).

Let \(T\) be a Markov time. The sigma-field \(\mathcal{F}_T\), the collection of events prior to \(T\), consists of all events \(A \in \mathcal{F}_\infty\) such that

\[
A \cap \{T \leq t\} \in \mathcal{F}_t, \quad \forall t \in \mathbb{R}_+.
\]

Let \(X\) be an optional process. Putting

\[
X(+\infty) \equiv \limsup_{t \to \infty} X(t).
\]

it follows from Dellacherie and Meyer [1978, Theorem IV.90] that \(X(+\infty)\) is an extended real-valued random variable. This is the convention we will use for any optional process to appear. It is also easy to check that the process \(\{X(t), t \in \mathbb{R}_+\}\) is optional. It then follows from similar arguments in
Theorem IV.64 of Dellacherie and Meyer [1978] that \( X(\omega, T(\omega)) \) is an extended real-valued random variable measurable with respect to \( \mathcal{F}_T \).

An optional process \( X \) satisfies condition \( A^- \) if

\[
E[\sup_{t \geq 0} X^-(t)] < \infty,
\]

where \( E[\cdot] \) denotes the expectation with respect to \( P \). It satisfies condition \( A^+ \) if

\[
E[\sup_{t \geq 0} X^+(t)] < \infty.
\]

Coming back to economics, we consider an \( N \)-player dynamic stopping game. Players are indexed by \( i = 1, 2, \ldots, N \). The payoff of this game is described by a family of adapted processes

\[
z_i(\omega, t; T_{-i}(\omega)); t \in \mathbb{R}_+, i = 1, 2, \ldots, N,
\]

where \( T_{-i} \) runs through all \((N - 1)\)-tuple of Markov times. Interpret \( z_i(\omega, t; T_{-i}) \) to be the payoff that player \( i \) receives in state \( \omega \) when his strategy is \( T_i \) and \( T_i(\omega) = t \), if \( T_{-i} \) are the strategies employed by players other than \( i \). We assume that payoff processes are optional processes satisfying condition \( A^- \).

Given the strategy of his opponents \( T_{-i} \), the objective of player \( i \) is to find a Markov time \( T_i \) that solves the following program:

\[
\sup_{T_i \in \mathcal{T}} E[z_i(T_i; T_{-i})].
\]  (2.1)

where \( \mathcal{T} \) is the collection of all Markov times.

A Nash equilibrium of the stopping game is an \( N \)-tuple of Markov times \( \{T_i; i = 1, 2, \ldots, N\} \) such that \( T_i \) solves (2.1) for all \( i = i, 2, \ldots, N \).

A Nash equilibrium of the stopping game \( \{T_i; i = 1, 2, \ldots, N\} \) is said to be subgame perfect if given a Markov time \( S \), putting \( A_i = \{T_i \geq S\} \), we have

\[
E[z_i(T_i; T_{-i})|\mathcal{F}_S] \geq E[z_i(\tau_i; T_{-i})|\mathcal{F}_S] \quad \text{a.s.}
\]

on the set \( A_i \) for all Markov times \( \tau_i \) such that \( \tau_i \geq S \) on the set \( A_i \), for all \( i = 1, 2, \ldots, N \).

3. Existence of reaction functions

In this section we shall show that given the strategy of his opponents, a player's best response always exists.

Some definitions are first given.

A supermartingale \( X \) is an optional process such that for all bounded stopping times \( T \) and \( S \) with \( T \leq S \) both

\[
E[X^-(S)] < \infty
\]  (3.1)
and
\[ E[X(S)|\mathcal{F}_T] \leq X(T) \quad a.s. \] (3.2)
are satisfied, where \( X^-(t) \equiv \max[-X(t),0] \), the negative part of \( X(t) \). A process is a martingale if both \( X \) and \(-X\) are supermartingales.

**Remark 3.1.** Traditionally, a *generalized supermartingale* is an adapted process, not necessarily optional, satisfying (3.1) and (3.2). Any generalized supermartingale in the traditional sense with right continuous paths is a supermartingale defined above.

Mertens [1969, Theorem 1] has shown that almost all of the paths of a supermartingale have right and left limits and are upper semi-continuous from the right.

A supermartingale \( X \) is regular if for every pair of stopping times \( T \leq S \) the expectation \( E[X(S)] \) exists and
\[ E[X(S)|\mathcal{F}_T] \leq X(T) \quad a.s. \]

A supermartingale \( \{X(t)\} \) is super regular if for every pair of Markov times \( T \leq S \) the expectation \( E[X(S)] \) exists and
\[ E[X(S)|\mathcal{F}_T] \leq X(T) \quad a.s. \] (3.3)

The following technical lemmas are keys to the main results in this section.

**Lemma 3.1.** Let \( \{X(t)\} \) be a super regular supermartingale and let \( T \) and \( S \) be two Markov times. Then
\[ E[X(S)|\mathcal{F}_T] \leq X(T) \quad a.s. \text{ on the set } \{S \geq T\} \]

**Proof.** Let \( A = \{S \geq T\} \). Suppose the assertion is not true. Then there exists \( A' \in \mathcal{F}_T \) with \( P(A') > 0 \) and \( A' \subset A \) such that
\[ E[X(S)|\mathcal{F}_T] > X(T) \quad a.s. \text{ on the set } A'. \]

Define two Markov times as follows:
\[
T'(\omega) = T(\omega) \text{ if } \omega \in A',
\]
\[ = +\infty \text{ otherwise;} \]
\[
S'(\omega) = S(\omega) \text{ if } \omega \in A',
\]
\[ = +\infty \text{ otherwise.} \]

Theorem IV.53 of Dellacherie and Meyer [1978] ensures that \( T' \) and \( S' \) are Markov times. By construction, almost surely, \( S' \geq T' \), \( S \leq S' \), and \( T \leq T' \). Hence \( \mathcal{F}_{T'} \subset \mathcal{F}_{S'} \).

Since \( X \) is a super regular supermartingale, we must have
\[ E[X(S')|\mathcal{F}_{T'}] \leq X(T') \quad a.s. \]

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Now taking conditional expectations with respect to $\mathcal{F}_T$ multiplying by $1_{A'}$ on both sides gives

$$1_{A'}E[X(S')|\mathcal{F}_T] \leq 1_{A'}E[X(T')|\mathcal{F}_T] \quad a.s.$$ 

Since $A' \in \mathcal{F}_T$ and since $1_{A'}X(S') = 1_{A'}X(S)$ and $1_{A'}X(T') = 1_{A'}X(T)$, we have

$$E[X(S)|\mathcal{F}_T] \leq X(T) \quad a.s. \text{ on the set } A',$$

a clear contradiction.

\begin{lemma}
Let $X$ be a supermartingale satisfying condition $A^-$. Then $X$ is super regular.
\end{lemma}

\textbf{Proof.} From the hypothesis, we know $\lim_{t \to \infty} X(t)$ exists and is finite, and $E[X^-(\infty)] < \infty$, where $X(\infty)$ is understood to be the limit; cf. Mertens [1969, Theorem 4]. If $X$ also satisfies condition $A^+$, the assertion follows from Dellacherie and Meyer [1982, Remark 5a of Appendix I], since $\{X(t): t \in \mathbb{R}_+\}$ is a supermartingale.

For the general case, let $T \leq S$ be two Markov times. For any constant $c$, we put $X' = X \wedge c$. Then the process $X'$ satisfies conditions $A^+$ and $A^-$. Also, since minimum is a concave function, $X'$ is a supermartingale. Using the arguments in the previous paragraph, we know

$$E[X'(S)|\mathcal{F}_T] \leq X'(T) \quad a.s.$$ 

Fatou's lemma implies that

$$E \left[ \lim_{t \to \infty} X'(S)|\mathcal{F}_T \right] \leq \lim_{t \to \infty} X'(T) \quad a.s.$$ 

Hence

$$E[X(S)|\mathcal{F}_T] \leq X(T) \quad a.s.$$ 

That is, $X$ is super regular.

\end{lemma}

The optional process $Y$ is said to lie above the optional process $W$, denoted by $Y \geq W$, if $Y(\omega,t) \geq W(\omega,t)$ for all $(\omega,t)$ outside a subset of $\Omega \times \mathbb{R}_+$ whose projection on $\Omega$ is of $P$-measure zero.

A regular supermartingale $Y$ is the minimum regular supermartingale (MRS) above an optional process $W$ if $Y \geq W$ and if $X \geq Y$ for any other regular supermartingale $X \geq W$.

The following proposition follows directly from Mertens [1969, Theorem 7] and Lemma 3.2.
Proposition 3.1. Given $T_{-i}$, there exists a MRS $Y_i(T_{-i})$ lying above $z_i(T_{-i})$. If $z_i(t; T_{-i})$ is right-continuous, $Y_i(t, T_{-i})$ is right continuous. Furthermore, $Y_i(T_{-i})$ is super regular and

$$\lim_{t \to \infty} Y_i(t; Y_{-i}) = \lim_{t \to \infty} \sup_{t \to \infty} z(t)$$

and

$$Y_i(0; T_{-i}) = \sup_{T_i \in T} E[z_i(T_i; T_{-i})].$$

Proof. The first assertion follows from the fact that $z_i(T_{-i})$ satisfies condition $A^-$ and Mertens [1969, Theorem 7]. The second assertion follows from Remark 23 of Appendix I of Dellacherie and Meyer [1982].

The rest of the assertions follows from Lemma 3.2, Mertens [1969, Theorem 7], and the fact that $Y(\infty)$ is finite a.s. since $Y$ satisfies $A^-$ (see again Mertens [1969, Theorem 4]).

We shall henceforth use the notation introduced in Proposition 3.1.

The following result is also useful:

Proposition 3.2. For every Markov time $S$ we have almost surely

$$Y_i(S; T_{-i}) = \sup_{t \in T, t \geq S} E[z_i(t; T_{-i})|T_S].$$

Proof. See Theorem 22 of Appendix I of Dellacherie and Meyer [1982].

Now define a Markov time $T_i(T_{-i})$:

$$T_i(T_{-i}) = \inf\{t \in \mathbb{R}_+: Y(t) = z(t, T_{-i})\}.$$

Here is our main theorem of this section. It shows that under some conditions there always exists for player $i$ a best response to other players' strategies.

Theorem 3.1. Suppose that $z_i(T_{-i})$ is upper-semi-continuous from the right, quasi-upper-semi-continuous from the left, and satisfies condition $A^+$. Then $T_i(T_{-i})$ is the best response to $T_{-i}$.

Proof. Using Lemma 3.2, the assertion follows from similar arguments of Theorem 7.3 of Thompson [1971].

From now on we shall assume that conditions of Theorem 3.1 on the reward processes are satisfied.

The following propositions give some properties of an optimal reaction.
Proposition 3.3. Fix $T_{-i}$. Let $r_i$ be a best response for player $i$ and let $Y_i(T_{-i})$ be the MRS majoring $z_i(T_{-i})$. Then

$$Y_i(r_i; T_{-i}) = z_i(r_i; T_{-i}) \ a.s.,$$  \hspace{1cm} (3.4)

and

$$T_i(T_{-i}) \leq r_i \ a.s.,$$  \hspace{1cm} (3.5)

and

$$E[Y_i(T_i; T_{-i})] = Y_i(0; T_{-i}).$$  \hspace{1cm} (3.6)

Proof. By the fact that $Y_i(T_{-i})$ is super regular (cf. Lemma 3.2), that $r_i$ is a best response, and that $Y_i(T_{-i})$ lies above $z_i(T_{-i})$, we have

$$E[Y_i(r_i; T_{-i})] \leq Y_i(0; T_{-i}) = E[z_i(r_i; T_{-i})] \leq E[Y_i(r_i; T_{-i})].$$  \hspace{1cm} (3.7)

where the equality follows from Proposition 3.2. Hence

$$E[Y_i(r_i; T_{-i})] = E[z_i(r_i; T_{-i})].$$

It then follows from the fact that $Y_i(T_{-i}) \geq z_i(T_{-i}) \ a.s.$ and the fact that $z_i(T_{-i})$ satisfies $A^+$ and $A^-$

$$Y_i(r_i; T_{-i}) = z_i(r_i; T_{-i}) \ a.s.$$

This is (3.4)

By the definition of $T_i(T_{-i})$, we then have $T_i(T_{-i}) \leq r_i \ a.s.$, which is (3.5).

Arguments used to prove (3.7) proves (3.6), since $T_i$ is an optimal response.

4. Nash equilibria when $z_i$'s have a monotone structure

In this section we will show that there exists a Nash equilibrium when the reward processes of players satisfy a monotone structure.
The following assumption will be made throughout this section.

**Assumption 4.1.** For \( t, t' \in \mathbb{R}_+ \) and \( t > t' \), almost surely, \( z_i(\omega, t; T_{-i}(\omega)) - z_i(\omega, t'; T_{-i}(\omega)) \) is nonincreasing in \( T_{-i}(\omega) \).

Denoting by \( T^N \) the collection of \( N \)-tuple of Markov times, we define \( \Phi : T^N \mapsto T^N \) as

\[
\Phi(T_1, \ldots, T_N) = (T_i(T_{-i}))_{i=1}^N.
\]

By Theorem 3.1, \( \Phi \) is a reaction function for the \( N \) players. By Theorem 3.1, the mapping is well-defined.

The mapping \( \Phi \) is said to be **monotone** if for any two Markov times \( \tau \geq S, \tau' = \Phi(\tau) \) and \( S' = \Phi(S) \) implies \( \tau' \leq S' \).

The following proposition shows that \( \Phi \) is monotone.

**Proposition 4.1.** \( \Phi \) is monotone.

**Proof.** Choose two Markov times \( \tau \geq S \). Let \( \tau' = \Phi(\tau) \) and \( S' = \Phi(S) \). Suppose the set

\[ A = \{ \tau'_i > S'_i \} \]

is of strictly positive measure for some \( i \). By Assumption 4.1 we know almost surely,

\[
[z_i(\tau'_i; T_{-i}) - z_i(S'_i; T_{-i})]1_A \\
\leq [z_i(\tau'_i; S_{-i}) - z_i(S'_i; S_{-i})]1_A.
\]

Taking conditional expectations with respect to \( \mathcal{F}_{S'_i} \) on both sides of the above relation gives

\[
0 \leq E[z_i(\tau'_i; T_{-i}) - z_i(S'_i; T_{-i})|\mathcal{F}_{S'_i}]1_A \\
\leq E[z_i(\tau'_i; S_{-i}) - z_i(S'_i; S_{-i})|\mathcal{F}_{S'_i}]1_A \leq 0,
\]

where we have used the fact that \( A \in \mathcal{F}_{S'_i} \) (cf. Dellacherie and Meyer [1978, Theorem IV.56]), where the first inequality follows from Proposition 3.4, and where the third inequality follows from Proposition 3.2. Thus

\[
E[z_i(\tau'_i; T_{-i})|\mathcal{F}_{S'_i}] = z_i(S'_i; T_{-i}) \quad \text{a.s. on the set } A. \tag{4.1}
\]

Now define \( \sigma_i = S'_i \land \tau'_i \) a.s. It is easily checked that \( \sigma_i \) is a Markov time and \( \sigma_i \leq \tau'_i, \sigma_i \neq \tau'_i \) on a set of strictly positive measure. We claim that \( \sigma_i \) is a best response to \( T_{-i} \). To see this, we note that

\[
E[z_i(\sigma_i; T_{-i})] = E[z_i(S'_i; T_{-i})1_A + z_i(\tau'_i; T_{-i})1_A'] \\
= E \left[ E[z_i(S'_i; T_{-i})1_A + z_i(\tau'_i; T_{-i})1_{A'}|\mathcal{F}_{S'_i}] \right] \\
= E[z_i(\tau'_i; T_{-i})],
\]

where we have used (4.1) and where \( A' \) denotes \( \Omega \setminus A \). This is a contradiction to the fact that \( \tau'_i = \Phi(T_{-i}) \) is the unique minimum best response to \( T_{-i} \). Thus \( A \) must be of measure zero.

The following is the first main theorem of this section:
**Theorem 4.1.** There exists a Nash equilibrium of the stopping game.

**Proof.** From Proposition 4.1 \( \Phi \) is a monotone mapping from \( T^N \) to itself. Proposition VI.1.1 of Neveu [1975] implies that \( T^N \) is a complete lattice. It then follows from Tarski's fixpoint theorem (cf. Tarski [1955]) that there exists a fixpoint for \( \Phi \). It is easily verified that the fixpoint is a Nash equilibrium.

The following theorem shows that if we have a symmetric game, that is, for any \((N - 1)\)-tuple of Markov times \( \tilde{T} \) we have \( z_i(\omega, t; \tilde{T}) = z_j(\omega, t; \tilde{T}) \) except on a subset of \( \Omega \times \mathbb{R}_+ \) whose projection to \( \Omega \) is of measure zero, then there exists a unique symmetric equilibrium, provided that the following assumption is satisfied.

**Assumption 4.2.** For \( t, t' \in \mathbb{R}_+ \) and \( t > t' \), almost surely, \( z_i(\omega, t; T_{-i}(\omega)) - z_i(\omega, t'; T_{-i}(\omega)) \) is strictly decreasing in \( T_{-i}(\omega) \).

A definition is needed. Let \( \Psi \) be a mapping from \( T^N \) to all the subsets of \( T^N \) that gives all the best responses to an element of \( T^N \). By Theorem 3.1, \( \Psi \) is well-defined. We call \( \Psi \) the reaction correspondence. The reaction correspondence is said to be monotone if for any two Markov times \( r \geq S \) a.s., and \( r' \in \Psi(r) \), \( S' \in \Psi(S) \), we have \( r' \leq S' \) a.s.

The following proposition shows that \( \Psi \) is monotone, whose proof is similar to that of Proposition 4.1.

**Proposition 4.2.** \( \Psi \) is monotone if Assumption 4.2 is satisfied.

**Proof.** Choose two Markov times \( r \geq S \). Let \( r' = \Phi(r) \) and \( S' = \Phi(S) \). Suppose the set

\[
A = \{ r'_i > S'_i \}
\]

is of strictly positive measure for some \( i \). By Assumption 4.1 we know almost surely,

\[
[z_i(r'_i; r_{-i}) - z_i(S'_i; r_{-i})]1_A < [z_i(r'_i; S_{-i}) - z_i(S'_i; S_{-i})]1_A.
\]

Taking conditional expectations with respect to \( \mathcal{F}_{S'_i} \) on both sides of the above relation gives

\[
E[z_i(r'_i; r_{-i}) - z_i(S'_i; r_{-i})]1_A < E[z_i(r'_i; S_{-i}) - z_i(S'_i; S_{-i})]1_A,
\]

where we have used the fact that \( A \in \mathcal{F}_{S'_i} \); cf. Dellacherie and Meyer [1978, Theorem IV.56]. The left-hand-side of the relation is nonnegative almost surely by Proposition 3.4. Hence the right-hand-side is strictly positive and is a contradiction to Proposition 3.2. Thus \( A \) must be of measure zero.

Here is theorem:
**Theorem 4.2.** Suppose that the game is symmetric and that Assumption 4.2 is satisfied. Then there exists a unique symmetric Nash equilibrium for the stopping game.

**Proof.** Let

\[ D = \{ T \in T^N : T_i = T_j \text{ a.s. } \forall i, j \} \]

and let \( F(T) = D \cap \Phi(T) \) and \( F'(T) = D \cap \Psi(T) \) for all \( T \in T^N \).

Note that for each \( T \in D \), \( \Phi_i(T_{-i}) = \Phi_j(T_{-j}) \) a.s. for all \( i, j \), where \( \Phi_i(T) \) denotes the \( i \)-th component of \( \Phi(T) \). So \( F(T) \) is nonempty and \( F : D \to D \) is monotone since \( \Phi \) is. Thus \( F \) has a fixed point and there exists a symmetric equilibrium.

Let \( T' \) be a fixed point of \( F \) and therefore is one for \( F' \). Choose \( T \in D \) with \( T > T' \) a.s. Since \( T' \in \Psi(T^*) \), by monotonicity \( T' \leq T^* \) a.s. for all \( T' \in F'(T) \subset \Psi(T) \). So \( T' < T \) a.s. and \( T \notin \Psi(T) \).

Similarly, \( T < T' \) a.s. implies \( T \notin \Psi(T) \). Therefore, \( T^* \) must be the unique symmetric equilibrium.

Finally, we have

**Theorem 4.3.** Any Nash equilibrium of the stopping game is subgame perfect.

**Proof.** This follows from arguments similar to Lemma 3.1 and Proposition 3.2.

5. **Concluding remarks**

As we have mentioned in the introduction, this version is preliminary and incomplete. We demonstrated the existence of a Nash equilibrium in a class of continuous time stopping games. We are still working on showing the uniqueness of an equilibrium and on constructing examples of general interest.

**References**


