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ENTRY AND EXIT: SUBGAME PERFECT EQUILIBRIA IN CONTINUOUS-TIME STOPPING GAMES

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Entry and Exit: Subgame Perfect Equilibria in Continuous-Time Stopping Games*

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Abstract

We study a class of continuous time entry-exit games in the extensive form, where the stochastically changing environment is modelled by a Brownian motion. There may be multiple subgame perfect equilibria. The equilibrium strategies which represent the bounds of all possible strategies in a subgame perfect equilibrium are explicitly characterized. A necessary and sufficient condition for the uniqueness of a subgame perfect equilibrium is also given.

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1 Introduction

One of the most important economic decisions faced by industrial firms is the decision to enter or to exit from an industry. This decision is of particular interest and importance since it determines the mode of competition and the economic life span of an industry. There is a vast economic literature on the entry and exit decisions of a firm, for which Wilson (1990) is a good recent survey. Some of the authors focus on the strategies firms can employ to gain or protect monopoly power; see, for example, Fudenberg and Tirole (1986), Fudenberg, Gilbert, Stiglitz, and Tirole (1983), and Milgrom and Roberts (1982a,b). The others focus on the effect of the market environment on the entry-exit decisions; see, for example, Fine and Li (1989), Ghemawat and Nalebuff (1985), and Londregan (1990).

Much of the literature in entry-exit decisions of a firm, however, uses discrete-time models. The notable exceptions are Fudenberg and Tirole (1986) and Ghemawat and Nalebuff (1985). Fudenberg and Tirole considered a model under certainty when there there exists asymmetry of information between the firms, and Ghemawat and Nalebuff focused on a model under certainty with complete information.

As the game theoretical extension of the optimal stopping theory, the literature of continuous time stopping games focuses almost exclusively on the normal form games (see, for example, Huang and Li (1990) and the references therein). There is now an emerging literature on continuous time extensive form games; see for example, Simon (1987) and Simon and Stinchcombe (1988). In these papers, however, continuous time games are analyzed by taking limits of the outcomes of discrete time games and there is no exogenous uncertainty.

The purpose of our paper is twofold. First, we extend some of the existing analyses of entry-exit games done either in continuous time under certainty or in discrete time under uncertainty to continuous time under uncertainty. Second, in so doing, we also contribute to the continuous time game theory by directly working with continuous time without taking limits of discrete time outcomes.

The rest of this paper is organized as follows. Section 2 formulates the entry-exit problem in continuous time for a single firm facing a stochastically changing demands modelled by a Brownian motion. The unique optimal entry-exit decisions are characterized as barrier policies: enters when the demand rises above a critical level and exits when the demand decreases to another critical level. Our analysis in this section overlaps somewhat with Dixit (1989). We assume that once a firm costly exits, it is prohibitively costly to reenter; while Dixit allows a firm to reenter with a finite cost after a costly exit. In addition, we allow the profit rate of a firm to be any bounded increasing function of the Brownian motion, while Dixit assumes that this function is exponential.

Section 3 analyzes an exit game of a duopoly. The two firms in the industry, one strong and one weak in a sense to be formalized, are seeking the best time to exit. One subgame perfect equilibrium is the one in which the strong firm does not exit unless the demand is such that it cannot sustain even as a monopoly and as a result the weak firm always exits before the strong firm does. We call this equilibrium the "natural equilibrium" for future reference.

We also identify other candidates for a subgame perfect equilibria by characterizing the exact upper bound and lower bound on each firm's subgame perfect equilibrium strategies. Some interesting phenomena occur in these equilibria. For example, either firm may exit when the demand has been on the average increasing. This happens for the strong firm, for example, because the weak firm plays tough and would not exits until the demand falls significantly. The strong firm then trades off the potential of becoming a monopolist after the weak firm exits against the current duopoly losses. An increasing demand increases the expected waiting time for the strong firm to become a monopolist and is a bad news. Thus it exits when the demand reaches a critical level from below.

Finally we give a set of necessary and sufficient conditions for a unique subgame perfect equilibrium. In such case, the unique equilibrium is the "natural equilibrium" discussed above.

We continue in Section 4 to consider a game of an incumbent versus a potential entrant. We exhibit a subgame perfect equilibrium. In this equilibrium, as expected, the existence of a potential entrant makes the life span of the incumbent shorter than that when it is a monopoly even though the incumbent may indeed remain as a monopoly throughout its lifetime. When the demand is low, the possibility of future duopoly competition limits the potential future monopoly profits. As a consequence the incumbent is less tolerant to the current monopoly losses than a monopoly facing no potential entrant and thus it exits earlier than a monopoly will even before the entrant enters. In addition, the entrant may not enter the industry even though the demand is above the level where both firms can be profitable as a duopoly. This is so because by waiting longer, the entrant may be able to enter after the incumbent exits. And the benefit from being a monopoly in the future outweighs the losses in the current duopoly profit.

Section 5 contains some concluding remarks and all the proofs are in the appendix.

2 Single Firm Problems

We consider a single firm's entry and exit decisions in this section. To begin, imagine that there is a firm in an industry facing a stochastic element, e.g., demand, that is subject to small and erratic random shocks. Formally we model the uncertain demand by taking the state space Ω to be the space of continuous functions of time from time 0 to time infinity. Each $\omega \in \Omega$ represents a complete description of one possible demand over the time horizon $[0, \infty)$ and $\omega(t)$ is the demand at time t if the state is ω. The state-dependent demand can then be modelled by a coordinate process $X(\omega, t) = \omega(t)$, that is, the demand at time t is $X(\omega, t)$ if the state is ω.

The information the firm has at t contains the historical demands from time 0 to time t. Mathematically, this information is represented by the smallest sigma-field on Ω with respect which $\{X(\omega, s) : 0 \leq s \leq t\}$ is measurable and is denoted by $\mathcal{F}_t^0$. The information the firm has at "infinity" from observing the demand over time is $\mathcal{F}^0 = \bigvee_{t \geq 0} \mathcal{F}_t^0$. Let P be the probability on the measurable space $(\Omega, \mathcal{F}^0)$ under which X is a (μ, σ)-Brownian motion starting from $X(0)$. That is, $X(t) = X(0) + \mu t + \sigma B(t)$, where B is a standard Brownian motion under P, $X(0)$ is a random variable independent of B, and μ and σ are constants. Note that in the above specification, we have used $X(t)$ and $B(t)$ to denote the random variables $X(\omega, t)$ and $B(\omega, t)$, respectively. Often, we will also use X_t interchangeably with $X(t)$.

To simplify the technical difficulty even more, we will "complete" the measurable space $(\Omega, \mathcal{F}^0)$ with respect to P and denote this space by $(\Omega, \mathcal{F})$.[1] Thus $(\Omega, \mathcal{F}, P)$ is a complete probability space.[2] We will also enlarge the information the firm has at time t to include not only the the historical realization of demand from time 0 to time t but also all the demand scenarios described in $(\Omega, \mathcal{F}, P)$ that will only happen with zero probability. This information is the "completion" of $\mathcal{F}_t^0$ with respect to the completed probability space and is denoted by $\mathcal{F}_t$. It follows from Chung (1982, corollary to theorem 2.3.4) that $\{\mathcal{F}_t : t \in [0, \infty)\}$, the increasing family of sub-sigma-fields of $\mathcal{F}$, is right-continuous in that $\mathcal{F}_t = \bigcap_{s > t} \mathcal{F}_s$ for all t. We assume that the probability measure P induces a family of conditional probability measures $\{P^x : x \in \Re\}$ such that, under P^x, X is a (μ, σ)-Brownian motion with initial state x or starting from x.

When the firm is already in the industry, it derives a profit rate of $\pi(X_t)$ at time t given the demand X_t. We assume that $\pi(\cdot)$ is increasing, nonconstant, and bounded from above and from below. In order to avoid the trivial case that π always takes positive values or negative values, we further assume that there is a y^0, $|y^0| < \infty$ such that $\pi(y) > 0$ for $y > y^0$ and $\pi(y) < 0$ for $y < y^0$. We are interested in two problems a firm faces. First, what is the optimal time to exit from the industry if the firm is already in it? Second, given that the firm chooses an optimal exit time once it is in the industry, what is then the optimal time to enter the industry?

Formally, the first problem the firm faces is to find an optimal exit time, to maximize the

[1] The procedure of completion with respect to P is to generated a sigma-field $\mathcal{F}$ using $\mathcal{F}^0$ and and all the subsets of P measure zero sets.

[2] A probability space is said to be complete if all the subsets of probability zero sets are measurable

expected discounted future profits conditional on $X_0 = x$:

$$\sup_{T \in \mathbf{T}} E_x \left[\int_0^T e^{-rt} \pi(X_t) dt \right], \tag{1}$$

where $\mathbf{T}$ denotes the collection of all optional times and $r > 0$ is the riskless interest rate. Note that, by the strong Markov property of X, the objective function of the firm can be written as

$$E_x \left[\int_0^T e^{-rt} \pi(X_t) dt \right] = f(x) - E_x[e^{-rT} f(X_T)],$$

where we understand that

$$E_x[e^{-rT} f(X_T)] \equiv \int_{\{T<\infty\}} e^{-rT(\omega)} f(X(\omega, T(\omega)) P_x(d\omega),$$

and where

$$f(x) \equiv E_x \left[\int_0^\infty e^{-rt} \pi(X_t) dt \right], \tag{2}$$

and $E_x[\cdot]$ is the expectation under P^x. By the fact that π is increasing and X is a Brownian motion, $f(x)$ is continuous and increasing.

Given the discussion above, (1) is equivalent to

$$\inf_{T \in \mathbf{T}} E_x[e^{-rT} f(X_T)]. \tag{3}$$

Putting

$$v(x) = f(x) - \inf_{T \in \mathbf{T}} E_x[e^{-rT} f(X_T)], \tag{4}$$

provided that a solution exists for (3), the second problem the firm faces is to find the optimal time to enter the industry knowing that it will behave optimally afterwards:

$$\sup_{S \in \mathbf{T}} E_x[e^{-rS} v(X_S)], \tag{5}$$

where we again used the strong Markov property of X.

We now show below that these two problems have unique solutions: There are two barriers y^* and y^e. The unique optimal exit time is the first time that the demand X is lower than y^* and the unique entry time is the first time that the demand is higher than y^e.

The next theorem shows the existence of a solution to (3) and this solution is a barrier policy.

Theorem 2.1 *There exists a $y^* \leq y^0$ so that*

$$T^* \equiv \inf\{t \geq 0 : v(X_t) = 0\} = \inf\{t \geq 0 : X_t \leq y^*\} \tag{6}$$

is a solution to (3) for all $x \in \Re$.

The next theorem shows that the barrier policy characterized in Theorem 2.1 is the unique optimal exit time.

Theorem 2.2 *The T^* defined in (6) is the unique solution to (3).*[3]

Using identical arguments, one can show that (5) has a unique solution, which is also characterized by a barrier: enter the industry if the demand X_t is greater than a given level y^e. It is clear that $y^e \geq y^0$ as otherwise the firm will be better off staying outside. We leave the proofs to the reader and record this result in the following theorem:

Theorem 2.3 *There exists a unique solution to (5). This unique solution is*

$$T^e = \inf\{t \geq 0 : \hat{v}(X_t) = v(X_t)\} = \inf\{t \geq 0 : X_t \geq y^e\}. \tag{7}$$

for some $y^e \geq y^0$, where

$$\hat{v}(x) = \sup_{T \in \mathrm{T}} E_x[e^{-rT} v(X_T)].$$

Combining Theorem 2.1 and 2.2, the optimal entry time and exit time for the firm given that the firm is currently outside the industry are recorded below:

Theorem 2.4 *Suppose that the firm is currently outside the industry. Then the unique optimal entry time for the firm is T^e defined in (7) and the unique optimal exit time for the firm is*

$$T^{ex} \equiv \inf\{t \geq T^e : X_t \leq y^*\}. \tag{8}$$

Besides the qualitative results reported above, the optimal barriers y^e and y^* and the associated expected discounted future profits can be calculated explicitly using Harrison (1985, chapter 3). These are recorded in the following proposition.

Proposition 2.1 *y^* is the unique number satisfying*

$$h(y^*) \equiv \int_{y^*}^{\infty} e^{-\alpha^* z} \pi(z) dz = 0. \tag{9}$$

and y^e is the unique number satisfying

$$\hat{h}(y^e) \equiv - \int_{y^*}^{y^e} e^{\alpha_* z} \pi(z) dz = 0, \tag{10}$$

[3] We used a different argument in an earlier version to prove this theorem. The current proof is suggested to us by an anonymous referee.

where

$$a_* \equiv \sigma^{-2}[\sqrt{\mu^2 + 2\sigma^2 r} + \mu], \tag{11}$$

$$a^* \equiv \sigma^{-2}[\sqrt{\mu^2 + 2\sigma^2 r} - \mu]. \tag{12}$$

Moreover, $y^e > y^0 > y^*$. *In addition,*

$$\hat{v}(x) = E_x[e^{-rT^e)}v(X_{T^e})] = \begin{cases} v(y^e)\theta(x, y^e) & \text{if } x \le y^e, \\ v(x) & \text{if } x > y^e; \end{cases} \tag{13}$$

$$v(x) = f(x) - E_x[e^{-rT^*}f(X_{T^*})] = \begin{cases} 0 & \text{if } x \le y^*, \\ f(x) - f(y^*)\theta(x, y^*) & \text{if } x > y^*; \end{cases} \tag{14}$$

where

$$\theta(x, y) = \begin{cases} exp[-a_*(x - y)] & \text{if } x \ge y, \\ exp[-a^*(y - x)] & \text{if } y \ge x; \end{cases} \tag{15}$$

Now we have completely solved the optimal entry and exit problem of a firm. The optimal policies are very simple. The firm should enter the industry if the demand rises above y^e and should exit afterwards when the demands falls below y^*. Note that since $y^* < y^0$, once in the industry, the firm will not exit the first time its instantaneous profit rate becomes negative. This is because there is a strictly positive probability that the demand will in the future be strictly higher than y^0. Thus the firm chooses to remain in the industry in anticipation of the rise in demand. Similarly, the firm does not enter the industry the first time its instantaneous profit rate becomes positive as there is a strictly positive probability that the demand will soon decline to make the profit negative. So the firm waits until the demand is sufficient high to enter.

The explicit expressions for y^* and y^e also allow us to derive the following comparative statics through direct computation:

Proposition 2.2 *Let* $\pi_1 > \pi_2$ *and let* y_1^* *and* y_2^* *be the corresponding optimal exit barriers for these two profit functions, respectively. Then* $y_1^* < y_2^*$. *Moreover,* y^* *is decreasing in* μ *and* σ *and increasing in* r *and* y^e *is increasing in* σ *and decreasing in* r.

A firm with a uniformly higher profit rate for all levels of demand than another firm will exit later. However, the optimal entry time for the firm with a uniformly higher profit rate may enter later as it anticipates that it will in the future be suffering from losses longer. Also, the higher the expected increase in the demand, the later a firm will exit; while the higher the interest rate, the higher the exit barrier and the earlier the firm will exit. The former is obvious and the latter follows since the firm does not exit immediately after the instantaneous profit becomes negative in the anticipation of future profits and an increase in the interest rate makes future profits less valuable. In addition, the larger the volatility of the demand, the lower the exit barrier. This is so since the exit option of the firm limits the downside risk of the uncertain demands and thus the added upside potential by an increase in σ makes the firm more willing to suffer current losses.

The comparative statics for the optimal entry time are less intuitive because a change of the parameters also affects the optimal exit time on which the optimal entry time depends. An increase in σ has two effects: the increase makes it more likely that the firm will suffer loss in the near future and in the meantime it depresses the exit barrier and thus increases the time span over which the firm will be making a negative instantaneous profits. In anticipation of the latter and because of the former, the firm increases its entry barrier and enters the industry later. So an increase in the volatility of the demands, may or may not increase the life span of the firm. An increase in the riskless interest rate also has two effects. First, it makes waiting to enter more costly for the firm. Second, it increases the exit barrier and thus makes the time span over which the firm will suffer losses shorter. The latter makes the firm afford to enter earlier and the former gives the firm incentive to enter earlier. The combined effects are that the optimal entry time is earlier. But since the optimal exit time is also earlier, it is unclear whether the total life span of the firm will be longer. There is no clear direction of change in the optimal entry time when μ increases. On the one hand, it makes waiting more costly. On the other hand, it increases the time span over which the firm will suffer losses by decreasing the exit barrier and thus creates an incentive for the firm not to enter until the demand is sufficiently high. These are two opposing effects.

We conclude the section by looking at the following simple example.

Example 2.1 *Let*

$$\pi(y) = \begin{cases} a, & \text{if } y \geq y^0; \\ -b, & \text{if } y < y^0, \end{cases}$$

be an increasing step function with $a, b > 0$ and $d \equiv a/b$. Solving equations (9) and (10), we have

$$y^* = y^0 - \frac{1}{a^*} \ln(1+d), \tag{16}$$

$$y^e = y^0 + \frac{1}{a_*} \ln\{1 + \frac{1}{d}[1 - (1+d)^{-a_*/a^*}]\}. \tag{17}$$

Suppose for $i = 1, 2$,

$$\pi_i(y) = \begin{cases} a_i, & \text{if } y \geq y_i^0; \\ -b_i, & \text{if } y < y_i^0, \end{cases}$$

where $a_1 > a_2 > 0, b_2 > b_1 > 0$, and $y_1^0 < y_2^0$. Then $\pi_1 > \pi_2, d_1 > d_2$, and hence $y_1^e < y_2^e$ since y^e is increasing in y^0 and decreasing in d. In this example, y^e is decreasing in π, i.e., the firm with higher profit rate enters the market earlier.

3 The Exit Game

Now we investigate the situation in which there are two firms in the industry. Denote by $\pi_{ij}(X_t)$ the profit rate for firm i if there are j firms in the market and the demand is X_t, where $i = 1, 2$ and $j = 1, 2$. These profit functions have the same characteristics as that in the single firm context of Section 2, that is, they are bounded, increasing, and nonconstant. And, there exist y^0_{ij} so that $\pi_{ij}(y) > 0$ for $y > y^0_{ij}$ and $\pi_{ij}(y) < 0$ for $y < y^0_{ij}$. Assume that the demand for the industry as a whole at time t is X_t. The profit for a firm in an duopoly situation is naturally less than that in a monopoly situation. Thus we assume that $\pi_{i1}(y) > \pi_{i2}(y)$ for all y. It then follows that $y^0_{i1} > y^0_{i2}$.

As the profit rate of a firm depends on whether it is a monopoly or a duopoly, its exit decision will certainly depend on the exit decisions made by the other firm. As a result, a gaming situation occurs. In this *exit game*, we will focus our attention on subgame perfect Nash equilibria in pure strategies. Thus we need an extensive form specification of the game. We will assume that once a firm exits from the industry, it is prohibitively costly to reenter. It follows that at any time t, one just has to specify the strategies employed by a firm depending on whether its opponent is in the industry. Note that since a firm becomes a monopoly once its opponent exits, its optimal strategy afterwards should simply be its unique optimal exit time established in Section 2 in a single firm context. Therefore, the game will be completely specified if we designate at any time t and in any state ω, the strategy a firm follows given that its opponent is still in the industry.

Formally, let $T_i : \Omega \times \Re_+ \mapsto \overline{\Re}_+$ be the strategy of firm i, where $T_i(\cdot, t) : \Omega \rightarrow \overline{\Re}_+$ is an optional time with $T_i(\omega, t) \geq t$ P^x-a.s. We also impose the regularity conditions:

1. the set
$$A = \{(\omega, s) \in \Omega \times \Re_+ : T(\omega, s) = s, \, s \in \Re_+\} \tag{18}$$
is progressively measurable;[4]

2. $T_i(\omega, t)$ is right-continuous in t.

For brevity of notation, we will often use $T_i(S)$ to denote $T_i(\omega, S(\omega))$ as a random variable for an optional time S. Our interpretation of T_i is as follows: At any optional time S, if firm i and its opponent are both in the industry and its opponent will continue to be in the industry, firm i will not exit immediately in the states where $T_i(S) > S$ and will exit immediately in the states where

[4] A process Y is progressively measurable if, as a mapping from $\Omega \times \Re_+$ to $\Re$, its restriction to the time set $[0, t]$ is measurable with respect to the product sigma-field generated by $\mathcal{F}_t$ and the Borel sigma-field of $[0, t]$. The progressive sigma-field is the sigma-field on $\Omega \times \Re_+$ generated by all the progressively measurable processes. A subset of $\Omega \times \Re_+$ is progressively measurable if it is an element of the progressive sigma-field. A good reference for these is Dellacherie and Meyer (1978).

$T_i(S) = S$. The purpose of the two regularity conditions will become clear later. Roughly, both are about how $T_i(t)$ changes over time. Denote by $\overline{\mathbf{T}}$ the space of all the mappings $T : \Omega \times \Re_+ \mapsto \overline{\Re}_+$ that satisfy these above conditions.

Given $T \in \overline{\mathbf{T}}$ and an optional time S, we put

$$\hat{T}(S) \equiv \inf\{t \geq S : T(\omega, t) = t\}. \tag{19}$$

In words, $\hat{T}(S)$ is the first time after S that the strategy T instructs the firm to exit prior to the exit time of its opponent.

To make our interpretation of $T(S)$ and $\hat{T}(S)$ precisely correct, however, we need to show that one is able to tell at each time t whether one should continue or exit according to $T(S)$ and $\hat{T}(S)$. That is, we need to show $T(S)$ and $\hat{T}(S)$ are optional times. This is among the subjects of the following proposition.

Proposition 3.1 *Suppose that $T \in \overline{\mathbf{T}}$ and S is an optional time. Then $T(S)$ is an optional time with $T(S) \geq S$ a.s., and if we define $\hat{T}$ according to (19), $\hat{T}(S)$ is an optional time. Moreover, $T(\hat{T}(S)) = \hat{T}(S)$ a.s.*

Note that the last assertion of the above proposition follows from the hypothesis that $T \in \overline{\mathbf{T}}$ is right-continuous in t. In words, it says that a firm will indeed exit at $T(\hat{T}(S))$ if it still remains at S. This is related to the kind of intertemporal consistency discussed by Perry and Reny (1990) and Simon and Stinchcombe (1988). To understand the necessity of this, it suffices to consider the following example. Let $T(t) = 1$ for all $t \in [0, 1/2]$ and $T(t) = t$ for all $t > 1/2$. That is, one should remain in the industry from time 0 to time 1/2 but exit immediately after time 1/2. This specification implies that $\hat{T} = 1/2$, but $T(\hat{T}) = 1 \neq \hat{T}$. At time 1/2, one is not sure what to do. His strategy at that time, $T(1/2) = 1$, instructs him to remain in the industry while his strategies after time 1/2 tell him, however, to exit immediately! The right-continuity of $T(t)$ in t eliminates this possibility.

We will use T_{-i} to denote firm i's opponent's strategy. Given T_{-i}, firm i solves the following program:

$$\sup_{T \in \overline{\mathbf{T}}} E_x \left[\int_0^{\hat{T} \wedge \hat{T}_{-i}} e^{-rt} \pi_{i2}(X_t) dt + 1_{\{\hat{T} > \hat{T}_{-i}\}} e^{-r\hat{T}_{-i}} v_{i1}(X_{\hat{T}_{-i}}) \right], \tag{20}$$

where v_{ij} is defined as in (14) with π replaced by π_{ij}.

A *Nash equilibrium* of the extensive form exit game is a pair of strategies $(T_1, T_2) \in \overline{\mathbf{T}} \times \overline{\mathbf{T}}$ so that given T_{-i}, T_i solves (20) for $i = 1, 2$, for all $x \in \Re$.

Note that the action taken by firm i, or the *exit time* of firm i, in a Nash equilibrium (T_1, T_2) is an optional time $\hat{T}_i 1_{\{\hat{T}_i \leq \hat{T}_{-i}\}} + T^*_{i1} 1_{\{\hat{T}_i > \hat{T}_{-i}\}}$. That is, on the set $\{\hat{T}_i \leq \hat{T}_{-i}\}$, where firm i exits first, it exits at $\hat{T}_i$; while on the set $\{\hat{T}_i > \hat{T}_{-i}\}$, where firm $-i$ exits first, firm i behaves like a monopoly.

A Nash equilibrium (T_1, T_2) is a *subgame perfect equilibrium* if for any optional time S, $\hat{T}_i(S)$ solves, for all $x \in \Re$,

$$\sup_{T \in \overline{\mathbf{T}}} E_x \left[\int_S^{\hat{T}(S) \wedge \hat{T}_{-i}(S)} e^{-r(t-S)} \pi_{i2}(X_t) dt + 1_{\{\hat{T}(S) > \hat{T}_{-i}(S)\}} e^{-r(\hat{T}_{-i}(S) - S)} v_{i1}(X_{\hat{T}_{-i}}(S)) | \mathcal{F}_S \right], \quad (21)$$

where $E_x[\cdot|\mathcal{F}_S]$ denotes the expectation under P^x conditional on $\mathcal{F}_S$.

A subgame perfect equilibrium (T_1, T_2) is said to be unique if for any other subgame perfect equilibrium (T'_1, T'_2), we have

$$\hat{T}_i(S) 1_{\{\hat{T}_i(S) \leq \hat{T}_{-i}(S)\}} = \hat{T}'_i(S) 1_{\{\hat{T}'_i(S) \leq \hat{T}'_{-i}(S)\}} \quad P^x - a.s.$$

for all x and for all optional time S.

This definition of uniqueness seems rather weak. But it gives the right sense of uniqueness. The significance of $T_i(t)$ lies in the implied exit action taken by firm i in equilibrium. There can be two strategies T_i and T'_i which differ on a set of (ω, t) whose projection onto Ω is of a strictly positive probability. But they can imply the same exit times of the two firms in the subgame starting from an optional time S as long as $\hat{T}_i(S) 1_{\{\hat{T}_i(S) \leq \hat{T}_{-i}(S)\}} = \hat{T}'_i(S) 1_{\{\hat{T}'_i(S) \leq \hat{T}'_{-i}(S)\}}$ with probability one.

The optimizations of (20) and (21) look formidable as they are looking for a complicated mapping $T \in \overline{\mathbf{T}}$. The following proposition shows that (20) is equivalent to a much simpler optimization.

Proposition 3.2 *For every $\tau \in \mathbf{T}$, there exists a $T \in \overline{\mathbf{T}}$ so that $\tau = \hat{T}$ a.s. Thus, (20) is equivalent to*

$$\sup_{\tau \in \mathbf{T}} E_x \left[\int_0^{\tau \wedge \hat{T}_{-i}} e^{-rt} \pi_{i2}(X_t) dt + 1_{\{\tau > \hat{T}_{-i}\}} e^{-r\hat{T}_{-i}} v_{i1}(X_{\hat{T}_{-i}}) \right]. \quad (22)$$

We note that the optimization of (22) is performed by searching for an optional time τ directly rather than by indirectly looking for a $T \in \overline{\mathbf{T}}$.

The question of whether (21) can similarly be simplified is a more difficult one and is not addressed here. What is important for our purpose, however, is the fact that $\hat{T}(S)$ is an optional time for $S \in \mathbf{T}$ and $T \in \overline{\mathbf{T}}$ as reported in Proposition 3.1.

Before we turn our attention to the existence of a Nash equilibrium and a subgame perfect equilibrium, some notation is in order. Let y^*_{ij} be the unique optimal exit barrier in the single firm problem when the profit function is π_{ij}. Then y^*_{ij} is the optimal exit barrier for firm i when there are always j firms in the industry during its life span. For example, y^*_{22} is the optimal exit barrier for firm 2 when it has no chance to be a monopoly. Given the ordering on π_{i1} and π_{i2} assumed earlier, Proposition 2.2 shows that $y^*_{i1} < y^*_{i2}$. We assume further that $y^*_{11} < y^*_{21}$ and $y^*_{12} < y^*_{22}$. That is, firm 1 is a stronger firm than firm 2. Note that these last assumptions are insured by the hypothesis that $\pi_{1j} > \pi_{2j}$. But this is not necessary. Figure 1 depicts one possible relative positions of y^*_{ij}'s. For convenience, we use $T^*_{ij}(S)$ to denote $\inf\{t \geq S : X_t \leq y^*_{ij}\}$ for any optional time S. Also, let f_{ij} and h_{ij} be the functions defined in (2) and (9), respectively, with π replaced by π_{ij}. And let

$$g_{ij}(x, y) \equiv f_{ij}(x) - \theta(x, y) f_{ij}(y). \tag{23}$$

The following proposition shows that a subgame perfect equilibrium exists for the exit game by construction.

Proposition 3.3 *$(T_1, T_2) = (T_1(t), T_2(t); t \in \Re_+)$ is a subgame perfect Nash equilibrium, where*

$$\begin{aligned} T_1(t) &= T^*_{11}(t), \\ T_2(t) &= T^*_{22}(t). \end{aligned}$$

The expected discounted future profits in the equilibrium for the two firms, or equilibrium payoffs, are, respectively,

$$v_1^{ex}(x) = \begin{cases} g_{12}(x, y^*_{22}) + \theta(x, y^*_{22}) g_{11}(y^*_{22}, y^*_{11}) & \text{if } x \geq y^*_{22}, \\ v_{11}(x) & \text{if } x < y^*_{22}; \end{cases} \tag{24}$$

$$v_2^{ex}(x) = v_{22}(x), \tag{25}$$

where v_{ij} is defined in (4) with π replaced by π_{ij}.

In this equilibrium, the "stronger" firm (firm 1) acts like a monopoly throughout its life time and the "weaker" firm (firm 2) behaves like a duopoly throughout. Also, the "strong" firm always has a longer life time than the "weaker" firm. This equilibrium is a "stationary equilibrium" in that the strategies for the two firms at any time t is a "copy" of their strategies at time 0. In this case, one easily verifies that $\hat{T}_i(S) = T_i(S)$ with probability one.

One of the important results of this paper is a set of necessary and sufficient conditions for the equilibrium identified in Proposition 3.3 to be the unique subgame perfect equilibrium in the exit game. We will go about accomplishing this by first identifying candidates of other subgame perfect equilibria. The readers will find these candidate equilibria rather interesting. Then sufficient conditions for there not to exist candidates other than the one in Proposition 3.3 will then be given.

The following proposition records a useful restriction on all subgame perfect equilibrium exit times of the two firms. The exit time of firm i cannot be later than its monopoly exit time and cannot be earlier than its exit time when it is certain to remain a duopology throughout its life span.

Proposition 3.4 *Let* $(T_1, T_2) \in \overline{\mathbf{T}} \times \overline{\mathbf{T}}$ *be a subgame perfect equilibrium. Then for all optional time* S,

$$T^*_{i2}(S) \leq \hat{T}_i(S)1_{\{\hat{T}_i(S) \leq \hat{T}_{-i}(S)\}} + T^*_{i1}(S)1_{\{\hat{T}_i(S) > \hat{T}_{-i}(S)\}} \leq T^*_{i1}(S) \quad P^x - a.s. \forall x \in \Re.$$

The corollary below gives a trivial sufficient condition for $(T_1(t) = T^*_{11}(t), T_2(t) = T^*_{22}(t), t \in \Re_+)$ to be the unique subgame perfect equilibrium.

Corollary 3.1 *Suppose that* $y^*_{12} \leq y^*_{21}$. *Then* $(T_1(t) = T^*_{11}(t), T_2(t) = T^*_{22}(t), t \in \Re_+)$ *is the unique subgame perfect equilibrium.*

When $y^*_{12} \leq y^*_{21}$, the level of demand below which firm 2 cannot survive as a monopolist is even higher than that at which firm 1 cannot survive as a duopolist. So naturally, firm 2 exits always earlier than firm 1 in any subgame as firm 2 is much too much weaker than firm 1 and thus we have a unique subgame perfect equilibrium.

For the rest of the analysis, we therefore assume that $y^*_{21} < y^*_{12}$. In this case, when demand is between y^*_{21} and y^*_{12}, neither firm can survive as a duopolist and both can survive as a monopolist. Thus there may be more than one equilibrium.

The following proposition is instrumental for the main theorem of this section. It explicitly characterizes the unique optimal exit time of firm 1 when firm 2 does not exit until the demand is lower than y^*_{21} in every subgame.

Proposition 3.5 *Let* $T_2(t) = T^*_{21}(t)\ \forall t \in \Re_+$. *Then*

$$T_1(t) = \tau(t)1_{\{X_{\tau(t)} \in A^1_1\}} + T^*_{11}(t)1_{\{X_{\tau(t)} \in A^0_2 \setminus A^1_1\}} \tag{26}$$

is a solution to (21) given T_2, *where*

$$\begin{aligned}
\tau(t) &= \inf\{s \geq t : X_s \in A^1_1 \cup A^0_2\}, \\
A^1_1 &\equiv [y^1_1, \hat{y}^1_1] \cup (-\infty, y^*_{11}], \quad (27)\\
A^0_2 &\equiv (-\infty, y^*_{21}], \quad (28)\\
\hat{y}^1_1 &\equiv y^*_{12} \\
y^1_1 &\equiv \begin{cases} \inf\{y > y^*_{21} : m_1(y, y^*_{21}) \leq 0\} > y^*_{21}, & \text{if } \inf\{y > y^*_{21} : m_1(y, y^*_{21}) \leq 0\} > y^*_{21} \leq y^*_{12}; \\ \infty, & \text{otherwise,} \end{cases} \\
m_i(x, y) &= \int_y^x e^{a_\bullet z}(1 - e^{-(a^\bullet + a_\bullet)(z-y)})\pi_{i2}(z)dz + \int_{y^*_{i1}}^y e^{a_\bullet z}(1 - e^{-(a^\bullet + a_\bullet)(z-y)})\pi_{i1}(z)dz, \quad (29)
\end{aligned}$$

and where a_* and a^* are defined in (11) and (12), respectively. Moreover, if $T_1' \in \overline{\mathrm{T}}$ is another solution to (21), then

$$\hat{T}_1'(S)1_{\{\hat{T}_1'(S)\leq\hat{T}_2(S)\}} = \hat{T}_1(S)1_{\{\hat{T}_1(S)>\hat{T}_2(S)\}} \quad P^x - a.s.\ \forall x \in \Re.$$

Firm 1's behavior depicted in the above proposition can be described in words as follows. At any optional time τ, if firm 2 is still present, firm 1 will exit when X_t enters the set $[y_1^1, y_{12}^*]$ either from above or from below before it reaches $(-\infty, y_{21}^*]$. Otherwise, X_t will reach the set $(-\infty, y_{21}^*]$ first and firm 2 will exit and firm 1 becomes a monopoly and will follows its unique optimal strategy thereafter.

When firm 1 finds itself playing a duopoly game at τ with a demand $X_\tau \in (y_{21}^*, y_1^1)$, it knows that firm 2 will not exit until the demand falls below y_{21}^*. So staying in the industry, firm 1 will incur losses as the current demand is lower than y_{12}^*. But, remaining in the industry gives firm 1 the opportunity to become a monopolist if the demand falls to reach y_{21}^* and firm 2 exits. So firm 1 trades off this potential future gains with the current losses. Falling demand turns out to be a good news for firm 1 as it may become a monopolist sooner. On the other hand, a rising demand means firm 1 will suffer losses longer and is a bad news. On balance, when the demand rises to reach y_1^1, the prospect of the potential future monopoly profit becomes so dismal and firm 1 exits.

Similarly, if $X_\tau \geq y_{12}^*$, firm 1 exits when the demand falls below y_{12}^*. Continuing on, firm 1 will suffer too much loss to be balanced out by the prospect of becoming a monopoly in the future.

The proposition below records firm 2's exit time as the unique best response to (26), whose proof is very similar to that for Proposition 3.5 and is omitted.

Proposition 3.6 *Suppose that $y_1^1 < \infty$. Then*

$$T_2(t) = \tau(t)1_{\{X_{\tau(t)}\in A_2^1\}} + T_{21}^*(t)1_{\{X_{\tau(t)}\in A_1^1\setminus A_2^1\}} \tag{30}$$

is a solution to (21) given T_1 of (26), where A_1^1 is defined in Proposition 3.5,

$$\begin{aligned}
\tau(t) &= \inf\{s \geq t : X_s \in A_1^1 \cup A_2^1\}, \\
A_2^1 &\equiv [\hat{y}_2^1, y_{22}^*] \cup (-\infty, y_2^1], \\
\hat{y}_2^1 &\equiv \begin{cases} \inf\{y \geq y_{12}^* : m_2(y, y_{12}^*) \leq 0\}, & \text{if } \inf\{y \geq y_{12}^* : m_2(y, y_{12}^*) \leq 0\} \leq y_{22}^*; \\ \infty, & \text{otherwise;} \end{cases} \\
y_2^1 &\equiv \inf\{y \in [y_{21}^*, y_1^1] : n_2(y_1^1, y) \leq 0\}, \\
n_i(x,y) &= -\int_y^x e^{-a^* z}(1 - e^{-(a^*+a_*)(x-z)})\pi_{i2}(z)dz + \int_{y_{i1}^*}^x e^{-a^* z}(1 - e^{-(a^*+a_*)(x-z)})\pi_{i1}(z)dz
\end{aligned} \tag{31}$$

where m_2 is defined in (29), and a_* and a^* are defined in (11) and (12), respectively. Moreover, if $T_2'(\tau)$ be another solution to (21) given T_1 of (26), then

$$\hat{T}_2'(S)1_{\{\hat{T}_1'(S)\geq\hat{T}_2(S)\}} = \hat{T}_2(S)1_{\{\hat{T}_1(S)\geq\hat{T}_2(S)\}} \quad P^x - a.s.\ \forall x \in \Re.$$

In response to (26), firm 2 plays a stationary two-barrier policy at an optional time τ as a duopoly: exits when demand reaches A_2^1 before it reaches A_1^1; otherwise, exits when the demand falls below y_{21}^*. The interpretation of this two-barrier policy is similar to that for (26). When the demand is below y_{22}^* and firm 2 is a duopoly, it always trades off the potential of being a monopoly in the future against the current duopoly losses in its exit decision.

Note here that the exit game we are considering satisfies the monotone property studied in Huang and Li (1990a) in the sense that the longer its opponent stay in the industry, the earlier a firm will exit as the best response. From Proposition 3.4, we know that $T_{21}^*(S)$ is the upper bound of firm 2's exit time in any subgame perfect equilibrium starting at an optional time τ. As a consequence, the best response of firm 1 characterized in Proposition 3.5 is a lower bound on its subgame perfect equilibrium exit times. This lower bound is always tighter than $T_{12}^*(S)$ as $y_1^1 > y_{21}^*$. Moreover, if $y_1^1 = \infty$, then firm 1's exit time at any subgame starting from τ must be greater than $T_{11}^*(S)$. This together with Proposition 3.4 implies that firm 2's equilibrium exit time must be $T_{22}^*(S)$ and we have a unique subgame perfect equilibrium.

Corollary 3.2 *Let $(T_1, T_2) \in \overline{\mathbf{T}} \times \overline{\mathbf{T}}$ be a subgame perfect equilibrium and suppose that y_1^1 of Proposition 3.5 is equal to ∞, then $(T_1(t) = T_{11}^*(t), T_2(t) = T_{22}^*(t); t \in \Re_+)$ is the unique subgame perfect equilibrium.*

To search for the necessary and sufficient conditions for uniqueness, we first explicitly identify a subgame perfect equilibrium whose equilibrium exit time for firm 1 is the largest lower bound and that for firm 2 is the least upper bound of all subgame perfect equilibrium exit times. This result together with Proposition 3.4 gives a set of two exit times for each of the two firms, between which subgame perfect equilibrium exit times must lie. Finally, conditions for these two sets to be singleton sets are given. From Proposition 3.3, since $(T(t) = T_{11}^*(t), T_2(t) = T_{22}^*; t \in \Re_+)$ is always a subgame perfect equilibrium, it is then the unique subgame perfect equilibrium.

Our arguments for deriving the tighter bounds on the subgame perfect equilibrium exit times go as follows. Suppose that y_1^1 of Proposition 3.5 is not equal to infinity. Since (26) is a lower bound of firm 1's subgame perfect equilibrium exit times at any subgame, by the monotone property described in Huang and Li (1990), (30) becomes a new upper bound of its subgame perfect equilibrium exit times. We repeat this procedure to generate tighter and tighter upper bounds on firm 2's and tighter and tighter lower bounds on firm 1's subgame perfect equilibrium exit times. If this procedure has a fixed point other than the equilibrium of Proposition 3.3, this fixed point is itself a subgame perfect equilibrium and provides the exact largest lower bound and the exact least upper bound, respectively, for firm 1's and firm 2's subgame perfect equilibrium exit times.

Theorem 3.1 *Suppose that there exist y_1, y_2, $\hat{y}_1$ and $\hat{y}_2$ with $y^*_{21} < y_2 < y_1 \le \hat{y}_1 \le y^*_{12} < \hat{y}_2$ such that $m_1(y_1, y_2) = 0$, $n_2(y_1, y_2) = 0$, $n_1(\hat{y}_2, \hat{y}_1) = 0$, and $w(y_1, y_2, \hat{y}_2) \le 0$, where*

$$\hat{y}_2 = \begin{cases} \inf\{y > \hat{y}_1 : m_2(y, \hat{y}_1) \le 0\}, & \text{if } \inf\{y > \hat{y}_1 : m_2(y, \hat{y}_1) \le 0\} \le y^*_{22}; \\ \infty, & \text{otherwise}, \end{cases}$$

and where m_i and n_i are defined in (29) and (31), respectively, and $w(y_1, y_2, \hat{y}_2)$ is the expected profit for firm 1 as a duopoly at an optional time S with $X_S = y_1$ when it exits at $T^*_{11}(S)$ and firm 2 exits at (30) with y^*_{21} replaced by y_2 and $\hat{y}^1_2$ replaced by $\hat{y}_2$. Then

$$\begin{aligned} T_1(t) &= \tau(t)1_{\{X_{\tau(t)}\in A_1\}} + T^*_{11}(t)1_{\{X_{\tau(t)}\in A_2\setminus A_1\}}, \quad t \in \Re_+, \\ T_2(t) &= T^*_{21}(t)1_{\{X_{\tau(t)}\in A_1\}} + \tau(t)1_{\{X_{\tau(t)}\in A_2\setminus A_1\}}, \quad t \in \Re_+, \end{aligned} \tag{32}$$

is a subgame perfect equilibrium, where

$$\begin{aligned} \tau(t) &\equiv \inf\{s \ge t : X_s \in A_1 \textstyle\bigcup A_2\}, \\ A_1 &\equiv [y_1, \hat{y}_1] \textstyle\bigcup (-\infty, y^*_{11}], \\ A_2 &\equiv [\hat{y}_2, y^*_{22}] \textstyle\bigcup (-\infty, y_2]. \end{aligned}$$

Moreover, if (T'_1, T'_2) is another subgame perfect equilibrium, then

$$T_1(t) \le \hat{T}'_1(t)1_{\{\hat{T}'_1(t)\le \hat{T}'_2(t)\}} + T^*_{11}(t)1_{\{\hat{T}'_1(t)>\hat{T}'_2(t)\}} \le T^*_{11}(t)$$

and

$$T^*_{22}(t) \le \hat{T}'_2(t)1_{\{\hat{T}'_2(t)\le \hat{T}'_1(t)\}} + T^*_{21}(t)1_{\{\hat{T}'_2(t)>\hat{T}'_1(t)\}} \le T_2(t)$$

P^x-a.s. for all $x \in \Re$.

The equilibrium of Theorem 3.1 is a "stationary equilibrium" in the sense that every $T_i(t)$ is a "copy" of $T_i(0)$. We can thus describe the equilibrium by looking at $T_i(0)$. We take cases. First, if $X_0 \ge y^*_{22}$, firm 2 exits when the demand decreases to y^*_{22} and firm 1 continues to y^*_{11} as a monopoly. Second, if $X_0 \in [\hat{y}_2, y^*_{22}]$, firm 2 exits immediately and firm 1 is a monopoly throughout. Third, if $X_0 \in (\hat{y}_1, \hat{y}_2)$, firm 2 and firm 1 both stay on with the former making a negative profit. If the demand rises to reach $\hat{y}_2$, it is a bad news for firm 2 as firm 1 will be in the industry for a long time. Thus firm 2 exits and firm 1 becomes a monopoly. On the other hand, if the demand drops to reach

$\hat{y}_1$, which is lower than y^*_{12}, firm 1 knows that firm 2 will not exit until either the demand reaches $\hat{y}_2$ from below or reaches y^*_{21} from above, so firm 1 exits as the prospect of being a monopoly in the future is gloomy and firm 2 becomes a monopoly. Fourth, if $X_0 \in [y_1, \hat{y}_1]$, as firm 2 will continue until the demand decrease to y^*_{21}, firm 1 exits immediately. Fifth, if $X_0 \in (y_2, y_1)$, firm 1 exits when the demand increases to reach y_1 before it decreases to y_2 as the prospect of being a monopoly is gloomy. Otherwise, firm 2 exits when the demand reaches y_2 and firm 1 continues as a monopoly. Finally, if $X_0 \leq y_2$, firm 2 exits immediately and firm 1 plays its monopoly strategy henceforth. The relative positions of the barriers are depicted in Figure 1.

There are two interesting features of this equilibrium. First, either firm may exit when the demand has been increasing. This happens in two occasions. When demand is between $\hat{y}_1$ and $\hat{y}_2$, the weak firm (firm 2) exits as the demand drifts up to reach $\hat{y}_2$ before it reaches down to $\hat{y}_1$; or when the demand is between y_2 and y_1, the strong firm (firm 1) may exit as the demand goes up to reach y_1 before it decreases to reach y_2. Second, the strong firm exits before the weak firm does when the demand has been declining. This happens when the demand is between $\hat{y}_1$ and $\hat{y}_2$.

The equations that determine y_1, y_2, $\hat{y}_1$, and $\hat{y}_2$ are from the first order conditions for the two firms' optimization p oblems of (21). The existence of a solution to these equations with the desired order implies the existence of a stationary subgame perfect equilibrium in addition to the one in Proposition 3.3. Otherwise, A_1 becomes $(-\infty, y^*_{11}]$ and A_2 becomes $(-\infty, y^*_{22}]$. Then one concludes that the lower bound on firm 1's subgame perfect equilibrium exit time at any subgame starting from an optional time S is $T^*_{11}(S)$ and the upper bound for firm 2's exit time is $T^*_{22}(S)$. As a consequence, there exists a unique subgame perfect equilibrium. This fact is recorded in the corollary below:

Corollary 3.3 *$(T_1(t) = T^*_{11}(t), T_2(t) = T^*_{22}(t); t \in \Re_+)$ is the unique subgame perfect equilibrium, if and only if there do not exist y_1, y_2, $\hat{y}_1$, and $\hat{y}_2$ that satisfy the conditions of Theorem 3.1.*

We now demonstrate the possibility of a unique or multiple subgame perfect equilibria in the following example.

Example 3.1 *Let*

$$\pi_{ij}(y) = \begin{cases} a_{ij}, & \text{if } y \geq y^0_{ij}; \\ -b_{ij}, & \text{if } y < y^0_{ij}, \end{cases}$$

be increasing step functions with $a_{i1} > a_{i2} > 0$, $b_{i2} > b_{i1} > 0$, and $y^0_{i1} < y^0_{i2}$ for $i, j = 1, 2$. Also assume that $\mu < 0$.

We choose the parameters, a_{ij}, b_{ij} and y^0_{ij}, through the following steps:

1. *Choose y^*_{11}, y^*_{21}, y^*_{12}, and y^0_{11} so that $y^*_{11} < y^*_{21} < y^*_{12}$, $y^*_{21} < y^0_{11}$, and $y^*_{12} - y^*_{21} < y^*_{21} - y^*_{11}$.*

2. *Let*

$$k(x) \equiv \frac{1}{a^*}e^{a^* x} + \frac{1}{a_*}e^{-a_* x}.$$

It can be shown that

$$k'(x) = e^{a^* x} - e^{-a_* x} \begin{cases} > 0, & \text{if } x > 0; \\ < 0, & \text{if } x < 0, \end{cases} \tag{33}$$

and for $\mu < 0$ *(or* $a^* > a_*$*),*

$$\frac{d}{dx}(k(x) - k(-x)) \begin{cases} > 0, & \text{if } x > 0; \\ < 0, & \text{if } x < 0. \end{cases} \tag{34}$$

Thus, for $\epsilon > 0$ *sufficiently small, the following will hold,*

$$k(y^*_{21} - y^*_{11} + \epsilon) > k(-(y^*_{12} - y^*_{21} - 2\epsilon)), \tag{35}$$

and

$$\frac{k(y^*_{12} - y^*_{21} - \epsilon) - k(0)}{k(y^*_{12} - y^*_{21} - 2\epsilon) - k(0)} < \frac{k(y^*_{12} - y^*_{21}) - k(0)}{k(-(y^*_{12} - y^*_{21})) - k(0)}. \tag{36}$$

since $k(y^*_{21} - y^*_{11}) > k(y^*_{12} - y^*_{21}) > k(-(y^*_{12} - y^*_{21}))$ *by (33) and (34), also noticing that* $y^*_{12} - y^*_{21} < y^*_{21} - y^*_{11}$ *as determined in step 1.*

Choose ϵ *that satisfies (35), (36) and* $0 < \epsilon < (y^*_{12} - y^*_{21})/2$. *Then, set* b_{ij} *so that*

$$\frac{b_{12}}{b_{11}} = \frac{k(y^*_{21} - y^*_{11} + \epsilon) - k(0)}{k(-(y^*_{12} - y^*_{21} - 2\epsilon)) - k(0)}, \tag{37}$$

$$\frac{b_{22}}{b_{21}} = \frac{k(y^*_{12} - y^*_{21} - \epsilon) - k(0)}{k(y^*_{12} - y^*_{21} - 2\epsilon) - k(0)}. \tag{38}$$

Note that $b_{12} > b_{11}$ *by (35) and* $b_{22} > b_{21}$ *by (33). Furthermore,*

$$\frac{b_{22}}{b_{21}} < \frac{k(y^*_{12} - y^*_{21}) - k(0)}{k(-(y^*_{12} - y^*_{21})) - k(0)} \tag{39}$$

by (36).

3. *Let* $y^*_{22} = y^*_{12} + \delta$, *where* $\delta > 0$ *is so chosen that*

$$\frac{b_{22}}{b_{21}} < \frac{k(y^*_{12} - y^*_{21}) - k(0)}{k(-(y^*_{22} - y^*_{21})) - k(0)}. \tag{40}$$

The existence of such a δ *follows from (39) since* $\lim_{\delta \downarrow 0} y^*_{22} = y^*_{12}$.

4. *Finally, we choose* a_{ij} *and* y^0_{ij} *to satisfy*

$$y^*_{ij} = y^0_{ij} - \frac{1}{a^*}\ln(1 + d_{ij})$$

where $d_{ij} \equiv a_{ij}/b_{ij}$, *for* $i, j = 1, 2$.

Let $y_2 = y_{21}^* + \epsilon$ *and* $y_1 = y_{12}^* - \epsilon$. *Then*

$$\begin{aligned} m_1(y_1, y_2) &= -b_{12}\int_{y_2}^{y_1} e^{a_\bullet z}(1 - e^{-(a^\bullet + a_\bullet)(z - y_2)})dz - b_{11}\int_{y_{11*}}^{y_2} e^{a_\bullet z}(1 - e^{-(a^\bullet + a_\bullet)(z-y_2)})dz \\ &= e^{a_\bullet y_2}(-b_{12}(k(-(y_1 - y_2)) - k(0)) + b_{11}(k(y_2 - y_{11}^*) - k(0))) = 0 \end{aligned}$$

by (37) and

$$\begin{aligned} n_2(y_1, y_2) &= b_{22}\int_{y_2}^{y_1} e^{-a^\bullet z}(1 - e^{-(a^\bullet + a_\bullet)(y_1 - z)})dz - b_{21}\int_{y_{21*}}^{y_1} e^{-a^\bullet z}(1 - e^{-(a^\bullet + a_\bullet)(y_1 - z)})dz \\ &= e^{-a^\bullet y_1}(b_{22}(k(y_1 - y_2) - k(0)) - b_{11}(k(y_1 - y_{21}^*) - k(0))) = 0 \end{aligned}$$

by (38). Also, for all $y \in (y_{12}^*, y_{22}^*]$,

$$m_2(y, y_{12}^*) \geq e^{a_\bullet y_2}(-b_{22}(k(-(y_{22}^* - y_{21}^*)) - k(0)) + b_{21}(k(y_{12}^* - y_{21}^*) - k(0))) > 0,$$

by (40). Therefore, $\hat{y}_2 = \infty$ *and* $\hat{y}_1 = y_{12}^*$. *So,* (T_1, T_2) *in (32) with above* y_i *and* $\hat{y}_i$ *is an equilibrium.*

It is much easier to construct the case in which there is a unique equilibrium. In the above example, after completion of step 1, we simply set

$$1 < \frac{b_{12}}{b_{11}} < \frac{k(y_{21}^* - y_{11}^*) - k(0)}{k(-(y_{12}^* - y_{21}^*)) - k(0)}.$$

This is doable since $k(y_{21}^* - y_{11}^*) > k(-(y_{12}^* - y_{21}^*))$. *Then, for any* $y \in [y_{21}^*, y_{12}^*]$

$$m_1(y, y_{21}^*) \geq m_1(y_{12}^*, y_{21}^*) = e^{a_\bullet y_{21}^*}(-b_{12}(k(-(y_{12}^* - y_{21}^*)) - k(0)) + b_{11}(k(y_{21}^* - y_{11}^*) - k(0))) > 0,$$

and firm 1 will not exit before firm 2's longest possible exit time T_{21}. *The uniqueness of the equilibrium follows from Corollary 3.2.*

4 Game of an Incumbent Versus a Potential Entrant

In this section, we investigate the situation where firm 1 is initially in the market and has a single option to exit, while firm 2 is not in the market at the beginning and has options to enter and then exit. This is thus a game of an incumbent versus a potential entrant, henceforth abbreviated as simply the *entry game*. Unlike in the previous section. our focus here is not to provide sufficient conditions for the uniqueness of a subgame perfect equilibrium, but to demonstrate that one such equilibrium exists. Using the ideas similar to those of Section 3, one can identify sufficient conditions for this equilibrium to be the unique subgame perfect equilibrium. This procedure, however, being tedious and highly computational, will not be repeated here and is left for the interested reader.

Before we proceed formally, we note the following. First, in any subgame perfect equilibrium, once the two firms are in the industry in the same time, therehence, they must be playing a subgame perfect equilibrium in the exit game discussed in Section 3. Second, if firm 1 exits before firm 2 enters, then afterwards, firm 2 must follow its unique optimal single firm entry and exit decisions characterized in Section 2. For simplicity, we assume that there exists a unique subgame perfect equilibrium for the exit game. By the analysis of Section 3, this unique equilibrium is the one in Proposition 3.3. Given this hypothesis and the two observations noted above, in analyzing the entry game with a focus on subgame perfect equilibria, we can restrict our attention on firm 1's exit decision before firm 2 enters and on firm 2's entry decision before firm 1 exits.

Let $T_1^x : \Omega \times \Re_+ \longmapsto \overline{\Re}_+$ be firm 1's strategy before firm 2 enters, where $T_1^x(t)$ is an optional time with $T_1^x(t) \geq t$ P^x-a.s. for all $x \in \Re$. In words, if at time t firm 2 has not entered and firm 1 has not exited, firm 1 will exit immediately if and only if $T_1^x(t) = t$. Similarly, let $T_2^e : \Omega \times \Re_+ \longmapsto \overline{\Re}_+$ be firm 2's strategy before firm 1 exits, where $T_2^e(t)$ is an optional time with $T_2^e(t) \geq t$ P^x-a.s. for all $x \in \Re$. If at time t, firm 1 has not exited and firm 2 has not entered, firm 2 enters immediately if and only if $T_2^e(t) = t$. We assume that T_1^x and T_2^e are right-continuous in t and satisfy the measurability condition stipulated for T_i in Section 3. Denote by $\overline{\mathbf{T}}_1$ and $\overline{\mathbf{T}}_2$ the space of all the possible T_1^x and T_2^e, respectively.

Define

$$\hat{T}_1^x(S) = \inf\{t \geq S : T_1^x(t) = t\}$$

and

$$\hat{T}_2^e(S) = \inf\{t \geq S : T_2^e(t) = t\}$$

for all $T_1^x \in \overline{\mathbf{T}}_1$ and $T_2^e \in \overline{\mathbf{T}}_2$. By Proposition 3.1, $\hat{T}_1^x(S)$ and $\hat{T}_2^e(S)$ are optional times.

A subgame perfect equilibrium of the entry game is composed of $(T_1^x, T_2^e) \in \overline{\mathbf{T}}_1 \times \overline{\mathbf{T}}_2$ so that $\hat{T}_1^x(S)$ solves, for all optional time S,

$$\sup_{T \in \overline{\mathbf{T}}} E_x \left[\int_S^{\hat{T}(S) \wedge \hat{T}_2^e(S)} e^{-r(s-S)} \pi_{11}(X_s) ds + e^{-r(\hat{T}_2^e(S)-S)} v_1^{ex}(X_{\hat{T}_2^e(S)}) 1_{\{\hat{T}_2^e(S) < \hat{T}(S)\}} | \mathcal{F}_S \right].$$

where v_1^{ex} is defined in (24), and $\hat{T}_2^e(S)$ solves

$$\sup_{T \in \overline{\mathbf{T}}} E_x \left[e^{-r(\hat{T}(S)-S)} v_2^{ex}(X_{\hat{T}(S)}) 1_{\{\hat{T}(S) < \hat{T}_1^x(S)\}} + e^{-r(\hat{T}_1^x(S)-S)} v_{21}(X_{\hat{T}_1^x(S)}) 1_{\{\hat{T}_1^x(S) \leq \hat{T}(S)\}} | \mathcal{F}_S \right].$$

where v_2^{ex} is defined in (25) and v_{21} is defined in (4) with π replaced by π_{21}.

The following proposition reports a stationary subgame perfect equilibrium for the entry game. The proof of this proposition, being similar to that of Theorem 3.1, is omitted. In this equilibrium firm 2 sets a critical level y_2^e such that it enters the market the first time demand is above y_2^e before firm 1 exits and then plays the unique subgame perfect equilibrium of the exit game thereafter; otherwise it behaves optimally as a monopolist after firm 1 exits. The value y_2^e is determined so as to balance the marginal benefit of being a duopoly presently and the marginal benefit of being a future monopoly. Firm 1 sets a critical level y_1^x such that it exits if demand is below y_1^x before firm 2's entry; otherwise, it plays the unique subgame perfect equilibrium of the exit game after firm 2 enters.

We will use y_{21}^e and y_{22}^e to denote the entry barrier for firm 2 as a monopoly and as a duopoly, respectively, as described in Theorem 2.3. In addition, put

$$T_{21}^e(t) \equiv \inf\{s \geq t : X_s \geq y_{21}^e\}.$$

Proposition 4.1 *(T_1^x, T_2^e) is a subgame perfect equilibrium for the entry game, where*

$$T_1^x(t) = \tau(t)1_{\{X_{\tau(t)}\in A_1\}} + T_{11}^*(\tau(t))1_{\{X_{\tau(t)}\in A_2\}}, \tag{41}$$

$$T_2^e(t) = T_{21}^e(\tau(t))1_{\{X_{\tau(t)}\in A_1\}} + \tau(t)1_{\{X_{\tau(t)}\in A_2\}}, \tag{42}$$

$$\tau(t) = \inf\{s \geq t : X_s \in A_1 \cup A_2\}$$

with $A_1 = (-\infty, y_1^x]$ and $A_2 = [y_2^e, +\infty)$, and where $y_1^x \in (y_{11}^*, y_{12}^*)$ and $y_2^e \in (y_{22}^e, +\infty)$ are the unique solution to

$$\bar{m}_2(y_1, y_2) = 0,$$

$$\bar{n}_1(y_1, y_2) = 0,$$

with

$$\bar{m}_2(y_1, y_2) = -\int_{y_{22}^*}^{y_2} e^{a_* z}[1 - e^{-(a_*+a^*)(z-y_1)}]\pi_{22}(z)dz - e^{(a_*+a^*)y_1}\int_{y_{21}^*}^{y_{21}^e} e^{-a^* z}\pi_{21}(z)dz, \tag{43}$$

$$\bar{n}_1(y_1, y_2) = \int_{y_{11}^*}^{y_1} e^{-a^* z}\pi_{11}(z)[1 - e^{-(a_*+a^*)(y_2-z)}]dz + \frac{e^{-a^* y_2}}{\sqrt{\mu^2 + 2\sigma^2 r}}[g_{11}(y_2, y_{22}^*) - g_{12}(y_2, y_{22}^*)], \tag{44}$$

where g_{ij} is defined in (23).

Three interesting observations can be made. First, in the equilibrium, $y_1^x > y_{11}^*$. Thus the existence of a potential entrant may force the incumbent to exit earlier than when there is no potential entrant and hence the incumbent has a shorter economic life time. This occurs when demand decreases to reach y_1^x before it increases to reach y_2^e. The potential of becoming a duopolist at higher demand levels in the future, makes firm 1 less tolerant to the incurrence of current losses.

Second, the entrant sets its entry level y_2^e before firm 1 exits higher than its duopoly entry level y_{22}^e. This is due to the opportunity of its becoming a monopolist if it waits long enough for firm 1 to exit. Therefore the entrant sacrifices its current duopoly profits in exchange of the potential monopoly profit in the future.

Finally, there are two possible results of the incumbent versus entrant game: "peace" or "war". By "peace", we mean that the entrant enters after the incumbent exits and both firms share the market by different time segment. If the entrant enters before the incumbent exits, then they battle as a duopoly in the market.

5 Concluding Remarks

We have studied a class of entry-exit timing games in continuous time where the stochastically changing environment is modelled by a Brownian motion. In doing so, we work directly in continuous time with the extensive forms of the class of continuous-time entry-exit games. Given that the stochastically changing environment is stationary, we have identified some subgame perfect equilibria without too much difficulty. These equilibria are all stationary equilibria in that the strategy played by a firm at any time t is a copy of the strategy played at time $t = 0$. It will be interesting to investigate the existence a subgame perfect equilibrium in a general stochastic environment like the one of Huang and Li (1990).

There are many other economic applications of the continuous-time stochastic timing games besides the entry and exit decisions of firms analyzed in this paper. These include product and process technology choices, multiproduct and multimarket competition, and multiplant global competition. Indeed, any situation in which all players make dichotomous choices over time, and the players' well-beings are affected by time, by some stochastically changing elements, and by the choices made by other players can be formulated as a stopping game. Then the methods and technologies developed in the paper may apply.

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Appendix

A Proofs

PROOF OF THEOREM 2.1:[5]

PROOF. From the hypothesis that π is bounded, f is bounded. Theorem 1 of Huang and Li (1990) shows that there exists a solution to (3) and this solution is characterized by the first time that $e^{-rt}f(X_t)$ is equal to the largest regular submartingale dominated by it.[6] We claim that this largest submartingale is $e^{-rt}\phi(X_t) \equiv e^{-rt}(f(X_t) - v(X_t))$.

[5] We gratefully acknowledge that a referee suggested the current proofs for Theorems 2.1 and 2.2 to us.

[6] A regular submartingale $\{Y_t; t \in \Re_+\}$ is an optional process so that for any bounded optional time T, $E[Y_T^-] < \infty$ and for all optional times $S \geq T$, $E[Y_S|\mathcal{F}_T] \leq Y_T$ a.s., where an optional process is a process measurable with respect to the sigma-field on $\Omega \times [0, \infty)$ generated by all the processes adapted to **F** having right-continuous paths.

First we show that $e^{-rt}\phi(X_t)$ is a regular submartingale dominated by $e^{-rt}f(X_t)$. It is easy to see that $v(x) \geq 0$ for all x. Thus $e^{-rt}\phi(X_t) \leq e^{-rt}f(X_t)$ a.s. Note that for any optional times $S \geq T$,

$$\begin{aligned} E_x[e^{-rS}\phi(X_S)|\mathcal{F}_T] &= E_x\left[e^{-rS}\inf_{\tau\geq S} E_{X_S}[e^{-r(\tau-S)}f(X_\tau)|\mathcal{F}_T]\right] \\ &\geq e^{-rT}\inf_{\tau\geq T} E_x[e^{-r(\tau-T)}f(X_\tau)|\mathcal{F}_T] \\ &= e^{-rT}\phi(X_T), \quad a.s. \end{aligned}$$

where we have used the strong Markov property of X. Thus $e^{-rt}\phi(X_t)$ is a regular submartingale dominated by $e^{-rt}f(X_t)$.

Second we want to show that $e^{-rt}\phi(X_t)$ is the largest regular submartingale dominated by $e^{-rt}f(X_t)$. Suppose otherwise and let Y be a regular supermartingale dominated by $e^{-rt}f(X_t)$ and $Y(\omega,t) > e^{-rt}\phi(X(\omega,t))$ on a set of (ω,t) whose projection on to Ω is of a strictly positive probability. Define

$$S_n \equiv \inf\{t \geq 0 : Y(t) \geq e^{-rt}\phi(X_t) + \frac{1}{n}\}.$$

It is easily verified that S_n is an optional time and by the hypothesis there exists an $n > 0$ so that $P_x\{S_n < \infty\} > 0$. Since that $e^{-rt}f(X_t) \geq Y(t)$, we have

$$\begin{aligned} e^{-rS_n}\phi(X_{S_n}) = \inf_{T\in\mathbf{T},T\geq S_n} E_x[e^{-rT}f(X_T)|\mathcal{F}_{S_n}] &\geq \inf_{T\in\mathbf{T},T\geq S_n} E_x[Y(T)|\mathcal{F}_{S_n}] \\ &= Y(S_n) \quad a.s. \end{aligned}$$

However, $Y(S_n) \geq e^{-rS_n}\phi(X_{S_n}) + \frac{1}{n}$ with a strictly positive probability. Thus $e^{-rS_n}\phi(X_{S_n}) > e^{-rS_n}\phi(X_{S_n})$ with a strictly positive probability, which is a contradiction.

Finally, we want to show that

$$T^* \equiv \inf\{t \geq 0 : f(X_t) - \phi(X_t) \leq 0\}$$

is a barrier policy, that is, T^* is the first time that X_t is less than a level y^*. First we observe that $f(x) - \phi(x) = v(x)$ and it is obvious that $v(x)$ is increasing by the fact that X is spatial homogeneous and that π is increasing and nontrivial. Thus there must exist a y^* so that

$$T^* = \inf\{t \geq 0 : X_t \leq y^*\}.$$

The fact that $y^* \leq y^0$ follows from the observation that when $X_t > y^0$, it is better to stay in the industry as strictly positive profits are being generated. ▮

Proof of Theorem 2.2:

Proof. Proposition 2 of Huang and Li (1990) shows that if S is another optimal exit time, then $S \geq T^*$ a.s. Assume therefore $P_x\{S > T^*\} > 0$.

Put

$$T_n = \inf\{t \geq 0 : X_t \leq y^* - \frac{1}{n}\}.$$

It is well-known that $T_n \downarrow T^*$ a.s. (Only the case $X_0 = y^*$ is nontrivial. If $T^0 = \lim_{n\to\infty} T_n \neq 0$, the zero-one law implies that $T^0 > 0$ a.s. and thus $X_t \geq X_0$ for $0 \leq t \leq T^0$ a.s.. Since $X_{T^0} = y^*$, starting again from T^0 by using the strong Markov property and repeating the above arguments, we conclude that $T^0 = \lim_{n\to\infty} T_n < T^0$ a.s. if $T^0 = \infty$. Thus it must be that $T^0 = \infty$ a.s. This implies that $X_t \geq y^*$ for all $t \in \Re_+$. For a nontrivial σ, this is clearly impossible.) Thus $\{S > T^*\} = \bigcup_n \{S > T_n\}$. If we can show that, for every n, a.s. on $\{S > T_n\}$ we have

$$E_x[e^{-rS}\phi(X_S)|\mathcal{F}_{T_n}] > e^{-rT_n}\phi(X_{T_n}),$$

then we are done as this will imply

$$E_x[e^{-rS}\phi(X_S)|\mathcal{F}_{T^*}] > e^{-rT^*}\phi(X_{T^*}),$$

hence

$$E_x[e^{-rS}f(X_S)] \geq E_x[e^{-rS}\phi(X_S)] > E_x[e^{-rT^*}\phi(X_{T^*})] = E_x[e^{-rT^*}f(X_{T^*})] = \phi(x),$$

and S is suboptimal.

Now let $\tau = \inf\{t \geq T_n : X_t = y^*\}$ and set $S_0 = T_n$, $S_2 = S \bigvee T_n$, and $S_1 = S_2 \bigwedge \tau$. Then for $i = 1, 2$,

$$e^{-rS_i}\phi(X_{S_i}) \leq E_x[e^{-rS_{i+1}}\phi(X_{S_{i+1}})|\mathcal{F}_{S_i}], \tag{45}$$

since ϕ is a submartingale. Inequality (45) for $i = 0$ is strict on $\{S_1 > S_0\} = \{S > T_n\}$ since only negative profits are made between S_0 and S_1. To see this, we note that

$$\begin{aligned}
& e^{-rS_0}\phi(X_{S_0}) - E_x[e^{-rS_1}\phi(X_{S_1})|\mathcal{F}_{S_0}] \\
= \;& e^{-rS_0}f(X_{S_0}) - E_x[e^{-rS_1}f(X_{S_1})|\mathcal{F}_{S_0}] \\
= \;& E_x[\int_{S_0}^{\infty} e^{-rt}\pi(X_t)dt|\mathcal{F}_{S_0}] - E_x[\int_{S_1}^{\infty} e^{-rt}\pi(X_t)dt|\mathcal{F}_{S_0}] \\
= \;& E_x[\int_{S_0}^{S_1} e^{-rt}\pi(X_t)dt|\mathcal{F}_{S_0}] < 0,
\end{aligned}$$

where the first equality follows since on $\phi(X_{S_i}) = f(X_{S_i})$ for $i = 0, 1$, and the inequality follows since because $\pi(X_t) < 0$ for $t \in [S_0, S_1)$. The two inequalities in (45) ($i = 0$ and $i = 1$) imply that on $\{S > T_n\}$,

$$\begin{aligned}
E_x[e^{-rS}\phi(X_S)|\mathcal{F}_{T_n}] &= E_x[e^{-rS_2}\phi(X_{S_2})|\mathcal{F}_{S_0}] \\
&= E_x[E_x[e^{-rS_2}\phi(X_{S_2})|\mathcal{F}_{S_1}]|\mathcal{F}_{S_0}] \\
&\geq E_x[e^{-rS_1}\phi(X_{S_1})|\mathcal{F}_{S_0}] \\
&> e^{-rT_n}\phi(X_{T_n}),
\end{aligned}$$

which was to be proved. ∎

Proof of Proposition 3.1:

Proof. To prove that $T(S)$ is an optional time it suffices to prove that $T(S) \wedge t$ is $\mathcal{F}_t$-measurable, or $T(S) \wedge t \in \mathcal{F}_t$, for all $t \in \Re_+$; see Dellacherie and Meyer (1978, IV.49.3). First note that $S \wedge t \in \mathcal{F}_t$ as S is an optional time. By the right-continuity of $T(\omega, t)$ in t, we know $T(t)$ is Borel measurable in t. By the composition of two mappings, it follows that $T(\omega, S(\omega) \wedge t) \wedge t \in \mathcal{F}_t$. Next note that

$$\begin{aligned}
T(\omega, S(\omega)) \wedge t &= [T(\omega, S(\omega) \wedge t)]1_{\{S(\omega) \leq t\}}(\omega) + [T(\omega, S(\omega) \wedge t)]1_{\{S(\omega) > t\}}(\omega) \\
&= [T(\omega, S(\omega) \wedge t)]1_{\{S(\omega) \leq t\}}(\omega) + t\, 1_{\{S(\omega) > t\}}(\omega) \quad a.s.
\end{aligned}$$

By the fact that $\{S(\omega) > t\} \in \mathcal{F}_t$ as S is an optional time, we know $T(S) \wedge t \in \mathcal{F}_t$ as $T(S)$ is an optional time. The assertion that $T(S) \geq S$ a.s. is obvious.

Next we want to show that $\hat{T}(S)$ is an optional time. Observe that

$$\hat{T}(S(\omega)) = \inf\{t \geq 0 : (\omega, t) \in A \cap [S(\omega)\infty)\},$$

where A is defined in (18). By the hypothesis that A is a progressive set, and the fact that the stochastic interval $[S, \infty)$ is also a progressive set, then Dellacherie and Meyer (1982, IV.50) shows that $\hat{T}(S)$ is an optional time.

Finally, the last assertion follows from the hypotheses that $T(t)$ is right-continuous in t.

∎

Proof of Proposition 3.2:

Proof. Let $\tau \in \mathbf{T}$. Define

$$T(\omega, t) = 1_{[0,\tau(\omega)]}(\omega, t)\tau(\omega) + t1_{(\tau(\omega),\infty)}(\omega, t).$$

Then

$$A = \{(\omega, t) \in \Omega \times \Re_+ : T(\omega, t) = t,\, t \in \Re_+\} = [\tau(\omega), \infty).$$

It is known that the stochastic interval $[\tau(\omega), \infty)$ is progressively measurable. Thus $T \in \overline{\mathbf{T}}$. It is also easily verified that $\hat{T} = S$ P^x-a.s. for all x. ∎

Proof of Proposition 3.3:

Proof. First, we want to show that $T_i \in \overline{\mathbf{T}}$. By the definition of $T_i(t)$, we know that

$$A = \{(\omega, s) : T_i(\omega, s) = s,\ s \in \Re_+\} = \{(\omega, s) : X(\omega, s) \leq y^*_{ii}\}.$$

The (μ, σ)-Brownian motion is a progressively measurable process. Thus A is a progressively measurable subset and $T_i \in \overline{\mathbf{T}}$.

Second, we want show that (T_1, T_2) is a Nash equilibrium. It is obvious that $(T_1, T_2) \in \overline{\mathbf{T}} \times \overline{\mathbf{T}}$. Next, by definition, $\hat{T}_i = T^*_{ii}$ for $i = 1, 2$. Thus it suffices to show that T^*_{ii} solves, for all $x \in \Re$,

$$\sup_{\tau \in \mathbf{T}} E_x \left[\int_0^{\tau \wedge T^*_{jj}} e^{-rt}\pi_{i2}(X_t)dt + 1_{\{\tau > T^*_{jj}\}} e^{-rT^*_{jj}} v_{i1}(X_{T^*_{jj}})\right],$$

where $i \neq j$ and $i, j = 1, 2$.

Fix $x \in \Re$. In the above program, take $j = 1$ and let τ be a solution to the above program. We want to show that $\tau = T^*_{22}$ a.s. It is obvious that $\tau \geq T^*_{22}$. We first claim that $\tau \leq T^*_{11}$. Suppose otherwise. Then the expected discounted future profits for firm 2 is

$$\begin{aligned} & E_x\left[\int_0^{T^*_{11}\wedge\tau} e^{-rt}\pi_{22}(X_t)dt + \int_{T^*_{11}\wedge\tau}^{\tau} e^{-rt}\pi_{21}(X_t)dt\right] \\ < \quad & E_x\left[\int_0^{T^*_{11}\wedge\tau} e^{-rt}\pi_{22}(X_t)dt\right] + E_x\left[e^{-rT^*_{11}} v_{21}(X_{T^*_{11}})1_{\{\tau > T^*_{11}\}}\right] \\ = \quad & E_x\left[\int_0^{T^*_{11}\wedge\tau} e^{-rt}\pi_{22}(X_t)dt\right] + E_x\left[e^{-rT^*_{11}} v_{21}(y^*_{11})1_{\{\tau > T^*_{11}\}}\right] \\ = \quad & E_x\left[\int_0^{T^*_{11}\wedge\tau} e^{-rt}\pi_{22}(X_t)dt\right], \end{aligned}$$

where the strict inequality follows since at T^*_{11}, $X_{T^*_{11}} \leq y^*_{11} < y^*_{21}$, and the unique optimal exit time time for firm 2 in the monopoly case calls for exiting immediately. But $P^x\{\tau > T^*_{11}\} > 0$. Thus $\tau \leq T^*_{11}$ a.s.

Given that $\tau \leq T^*_{11}$ a.s., throughout firm 2's life span, it is a duopoly. Thus the unique optimal exit time for it is T^*_{22} and $\tau = T^*_{22}$.

Similar arguments establish that T^*_{11} also solves the above program when $j = 2$. Since x is arbitrary, (T_1, T_2) is a Nash equilibrium.

Third, we want to show that (T_1, T_2) is subgame perfect. Let S be any optional time. It is easily seen that $\hat{T}_1(S) = T^*_{11}(S)$ and $\hat{T}_2(S) = T^*_{22}(S)$. It suffices to show, given $\hat{T}_1(S)$, $\hat{T}_2(S)$ solves, for all $x \in \Re$,

$$\sup_{\substack{\tau\in \mathbf{T}\\ \tau\geq S}} E_x\left[\int_S^{\tau\wedge\hat{T}_1(S)} e^{-r(t-S)}\pi_{i2}(X_t)dt + 1_{\{\tau>\hat{T}_1(S)\}}e^{-r(\hat{T}_1(S)-S)}v_{i1}(X_{\hat{T}_1(S)})|\mathcal{F}_S\right],$$

and *vice versa*. By the strong Markov property of X and the fact that, conditional on X_S, $T^*_{11}(S)$ is independent of the values of X before S, the above program is equivalent to

$$\sup_{\substack{\tau\in \mathbf{T}\\ \tau\geq S}} E_x\left[\int_S^{\tau\wedge T^*_{11}(S)} e^{-rt}\pi_{i2}(X_t)dt + 1_{\{\tau>T^*_{11}(S)\}}e^{-rT^*_{11}(S)}v_{i1}(X_{T^*_{11}(S)})|X(S)\right],$$

where $E_x[\cdot|\mathcal{F}_S]$ is the expectation conditional on X_S under P^x. Then arguments identical to those used in showing that (T_1, T_2) is a Nash equilibrium show that $T^*_{22}(S)$ is a solution to the above program. The proof for $T^*_{11}(S)$ is identical.

Given that (T_1, T_2) is Nash equilibrium, the rest of the assertion can be verified by direct computation. ∎

PROOF OF PROPOSITION 3.4:

PROOF. It is clear that at S, firm i's strategy from then on must imply that it will stay on at least until it can not sustain as a duopoly when its opponent stays on forever. Similarly, firm i's strategy from S on must not stay longer than it can sustain as a monopoly. ∎

PROOF OF COROLLARY 3.1:

PROOF. Let (T_1, T_2) be a subgame perfect equilibrium. Given $y^*_{12} \leq y^*_{21}$, it follows from Proposition 3.4 that for any optional time S, $T_2(S) \leq T^*_{21}(S) \leq T^*_{12}(S) \leq T_1(S)$ P^x-a.s. for all x. So firm 1 will always stay longer than firm 2 in any subgame. It then follows that $T_i(S) = T^*_{ii}(S)$ P^x-a.s. for all x. ∎

PROOF OF PROPOSITION 3.5:

PROOF. We first record in the following lemma the explicit expressions of some functional that will be useful.

Lemma A.1 *Suppose $z < x < y, x, y, z \in \Re$, and $T \equiv \inf\{t \geq 0 : X_t \in \{y, z\}\}$. Define*

$$\psi(x, y, z) \equiv \frac{\theta(x, y) - \theta(x, z)\theta(z, y)}{1 - \theta(z, y)\theta(y, z)}. \tag{46}$$

Then

$$E_x[e^{-rT}; X_T = y] = v(x,y,z), \text{ and } E_x[e^{-rT}; X_T = z] = v(x,z,y).$$

Furthermore,

$$\begin{aligned}
\frac{\partial}{\partial y} v(x,y,z) &= -\frac{a^* + a_*\theta(z,y)\theta(y,z)}{1-\theta(z,y)\theta(y,z)} v(x,y,z), \\
\frac{\partial}{\partial y} v(x,z,y) &= \frac{(a_* + a^*)\theta(y,z)}{1-\theta(z,y)\theta(y,z)} v(x,y,z), \\
\frac{\partial}{\partial z} v(x,z,y) &= \frac{a_* + a^*\theta(y,z)\theta(z,y)}{1-\theta(y,z)\theta(z,y)} v(x,z,y), \\
\frac{\partial}{\partial z} v(x,y,z) &= -\frac{(a_* + a^*)\theta(z,y)}{1-\theta(y,z)\theta(z,y)} v(x,z,y).
\end{aligned}$$

PROOF. For the proof of the first part see Harrison (1985), and the partial derivatives obtain from direct computation. ∎

We proceed to prove Proposition 3.5.

First we restrict our attention to finding the solution to

$$\sup_{\substack{\tau\in\mathbf{T}\\ \tau\geq S}} E_x\left[\int_S^{\tau\wedge T_{21}^*(S)} e^{-r(t-S)}\pi_{12}(X_t)dt + 1_{\{\tau>T_{21}^*(S)\}}e^{-r(T_{21}^*(S)-S)}v_{11}(X_{T_{21}^*(S)})|\mathcal{F}_S\right], \tag{47}$$

so that τ is a barrier policy. Then arguments similar to those of Theorem 2.2 will show that any other solution to (47) will be equal to this barrier policy. This barrier policy is $T_1(S)$ in the assertion. Then one easily verifies that (26) is an element of $\overline{\mathbf{T}}$ using similar arguments used in Proposition 3.3.

We begin with $X_S = x \in (y_{21}^*, y_{12}^*]$. Given that firm 2 plays $T_{21}(S)$, firm 1's payoff by playing (26) for some y is

$$u_1(x,y,y_{21}^*) = f_{12}(x) - v(x,y,y_{21}^*)f_{12}(y) - v(x,y_{21}^*,y)f_{12}(y_{21}^*) + v(x,y_{21}^*,y)v_{11}(y_{21}^*), \tag{48}$$

by (21) and Lemma A.1. Differentiate u_1 with respect to y gives:

$$\begin{aligned}
\frac{\partial u_1(x,y,y_{21}^*)}{\partial y} &= \frac{(a^*+a_*)v(x,y,y_{21}^*)}{1-\theta(y,y_{21}^*)\theta(y_{21}^*,y)}(f_{12}(y) - \theta(y,y_{21}^*)f_{12}(y_{21}^*)) \\
&\quad -(1-\theta(y,y_{21}^*)\theta(y_{21}^*,y))h_{12}(y) + \theta(y,y_{21}^*)v_{11}(y_{21}^*)) \\
&= \frac{Ae^{-a_*y}v(x,y,y_{21}^*)}{1-\theta(y,y_{21}^*)\theta(y_{21}^*,y)} m_1(y,y_{21}^*),
\end{aligned}$$

where A is strictly positive constant. Observe that the sign of the partial derivative depends upon the sign of m_1. Direct computation yields:

$$\frac{\partial m_1(y,y_{21}^*)}{\partial y} = e^{a^*y}\pi_{12}(y)(1-e^{-(a^*+a_*)(y-y_{21}^*)})\begin{cases} >0, & \text{if } y > y_{12}^0; \\ <0, & \text{if } y_{21}^* < y < y_{12}^0, \end{cases}$$

where we recall that y_{12}^0 is such that $\pi_{12}(y_{12}^0) = 0$.

Note that $m_1(y_{21}^*, y_{21}^*) = v_{11}(y_{21}^*) > 0$ and that m_1 is continuous in y. If $m_1(y_{12}^0, y_{21}^*) \geq 0$, then $y = \infty$ is the upper barrier independent of the initial state. Suppose otherwise, $m_1(y_{12}^0, y_{21}^*) < 0$.

By the continuity of m_1 there exists a unique $\bar{y} \in (y^*_{21}, y^0_{12})$ such that $m_1(\bar{y}, y^*_{21}) = 0$. Now, either $y = \bar{y}$ or $y = +\infty$ maximizes u_1. Note that

$$\theta(x, y^*_{21}) = \psi(x, y^*_{21}, y) + \psi(x, y, y^*_{21})\theta(y, y^*_{21}),$$

or

$$\theta(x, y^*_{21}) - \psi(x, y^*_{21}, y) = \psi(x, y, y^*_{21})\theta(y, y^*_{21}).$$

Thus,

$$\begin{aligned} & u_1(x, \infty, y^*_{21}) - u_1(x, \bar{y}, y^*_{21}) \\ = \;& \psi(x, \bar{y}, y^*_{21})f_{12}(\bar{y}) - (\theta(x, y^*_{21}) - \psi(x, y^*_{21}, \bar{y}))f_{12}(y^*_{21}) + (\theta(x, y^*_{21}) - \psi(x, y^*_{21}, \bar{y}))v_{11}(y^*_{21}) \\ = \;& \psi(x, \bar{y}, y^*_{21})f_{12}(\bar{y}) - \theta(\bar{y}, y^*_{21})\psi(x, \bar{y}, y^*_{21})f_{12}(y^*_{21}) + \theta(\bar{y}, y^*_{21})\psi(x, \bar{y}, y^*_{21})v_{11}(y^*_{21}) \\ = \;& \psi(x, \bar{y}, y^*_{21})(f_{12}(\bar{y}) - \theta(\bar{y}, y^*_{21})f_{12}(y^*_{21}) + \theta(\bar{y}, y^*_{21})v_{11}(y^*_{21})) \\ = \;& \psi(x, \bar{y}, y^*_{21})(1 - \theta(\bar{y}, y^*_{21})\theta(y^*_{21}, \bar{y})h_{12}(\bar{y}) \begin{cases} \geq 0, & \text{if } \bar{y} \geq y^*_{12}; \\ < 0, & \text{otherwise}. \end{cases} \end{aligned}$$

Then, $\bar{y}$ is the optimal upper barrier if $\bar{y} < y^*_{12}$, otherwise, the infinity is. Suppose $\bar{y} < y^*_{12}$. For $x \leq y_1$, firm 1 exits when y_1 is reached before firm 2 exits. For $x \in (y_1, y^*_{12}]$, firm 1 quits immediately. To summarize, firm 1's exit time is the entry time of X in $[y_1, y^*_{12}]$ before 2's exit and is the entry time of X in $(-\infty, y^*_{11}]$ after the exit of firm 2. ∎

PROOF OF THEOREM 3.1:

PROOF. Using arguments similar to those of Proposition 3.5, firm 1's exit time as the unique best response to (30) is a two-barrier policy characterized by y^2_1 and $\hat{y}^2_1$. Like in Proposition 3.5, if $y^2_1 = \infty$, $(T_i(t) = T^*_{ii}(t); i = 1, 2, t \in \Re_+)$ is the unique subgame perfect equilibrium. Otherwise, we have $y^2_1 \leq \hat{y}^2_1 \leq y^*_{12}$: firm 1 exits as a duopoly when the demand enters the set $[y^2_1, \hat{y}^2_1]$. Then firm 2's exit time as the unique best response is also characterized by a two-barrier policy like (30) with y^1_2 and $\hat{y}^1_2$ replaced by y^2_2 and $\hat{y}^2_2$ with $y^*_{21} < y^2_2 < y^2_1$. Let y^n_i and $\hat{y}^n_i$ be the optimal barriers for firm i in the n-th iteration. Repeat the iteration as long as $y^n_1 < \infty$. By the monotonicity of the game, y^n_1 and y^n_2 are increasing in n and $\hat{y}^n_1$ and $\hat{y}^n_2$ are decreasing in n. Therefore, they have a limit point. By the hypothesis of the theorem, this limit point is a fixed point of the above iterative procedure and is a Nash equilibrium at any optional time τ. The equilibrium strategies are stationary strategies and thus thus are subgame perfect.

The second assertion follows from the fact that the fixed point has been reached by starting from an upper bound of all of firm 2's subgame perfect equilibrium exit times and the monotonicity of the game. ∎

PROOF COROLLARY 3.3:

PROOF.

We take cases. Case 1: suppose that there do not exist y_1 and y_2 that satisfy $m_1(y_1, y_2) = 0$ and $n_2(y_1, y_2) = 0$ with the desired order of the y_i's. This implies that a lower bound of firm 1's subgame perfect equilibrium exit times is $T^*_{11}(t)$ for all t. Proposition 3.4 then implies that the unique subgame perfect equilibrium must be the one in Proposition 3.3.

Case 2: Suppose that there are y_i's and $\hat{y}_i$'s satisfy the desired conditions except that $w(y_1, \hat{y}_2, y_2) > 0$. Then again, a lower bound on firm 1's perfect equilibrium exit times is $T^*_{11}(t)$ for all t. So we have a unique subgame perfect equilibrium.

Case 3: Suppose that there exist y_i's and $\hat{y}_i$'s so that $m_1(y_1, y_2) = 0$, $n_2(y_1, y_2) = 0$, and $w(y_1, \hat{y}_2, y_2) \leq 0$, with $y_{21}^* < y_2 < y_1 \leq y_{12}^*$. Then the violation of the hypothesis of Theorem 3.1 must come from the $\hat{y}_i$'s. Suppose first that $\hat{y}_2 = \infty$. Then it is easily seen that $\hat{y}_1 = y_{12}^*$ and $n_1(\infty, y_{12}^*) = 0$. Hence the conditions of Theorem 3.1 are satisfied. Therefore, it must be that $\hat{y}_2 \leq y_{22}^*$.

There are several possibilities:

1. $n_1(\hat{y}_2, \hat{y}_1) = 0$ and $\hat{y}_1 < y_1$. Then a lower bound of firm 1's subgame perfect equilibrium exit times is $T_{11}^*(t)$ for al t and Proposition 3.4 implies that there exists a unique subgame perfect equilibrium.

2. $n_1(\hat{y}_2, \hat{y}_1) = 0$ and $\hat{y}_1 = \hat{y}_2$. Then at the subgame starting at the optional time S with $X_S = \hat{y}_1$, both firms exit immediately, which clearly is not a Nash equilibrium. So this cannot be the case.

3. There do not exist $\hat{y}_1$ and $\hat{y}_2$ so that $n_1(\hat{y}_2, \hat{y}_1) = 0$. Then a lower bound of firm 1's subgame perfect equilibrium exit times is $T_{11}^*(t)$ for all t. Then by Proposition 3.4, there is a unique subgame perfect equilibrium.

   The proof for necessity part uses similar arguments.

∎

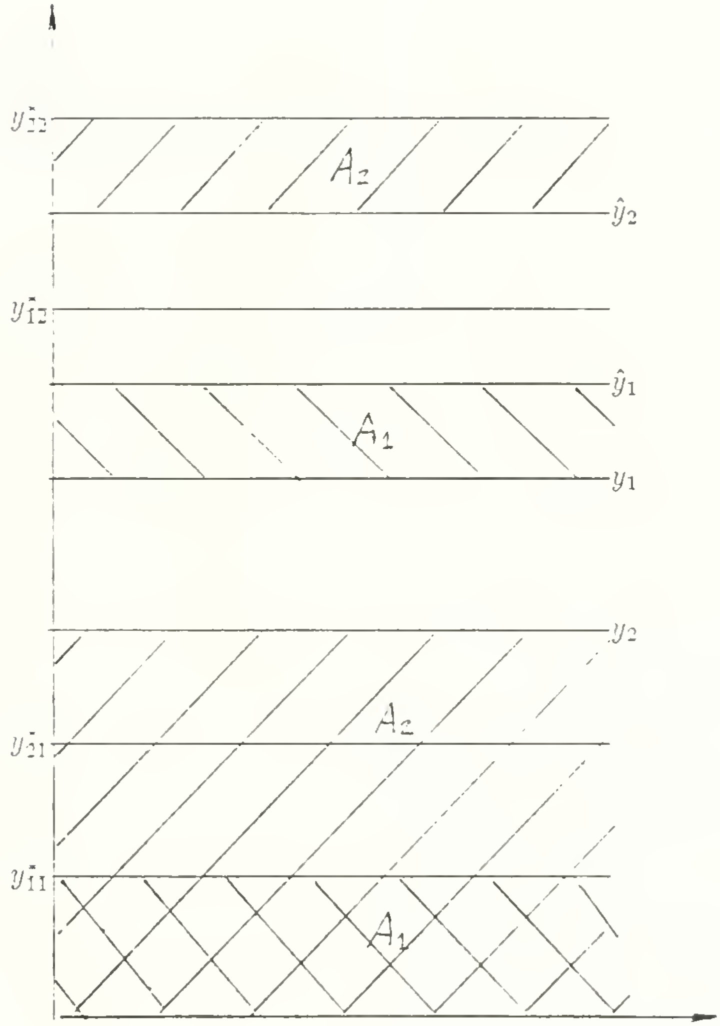

Figure 1: Relative Positions of the Barriers