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EQUILIBRIUM EXISTENCE RESULTS FOR SIMPLE ECONOMIES AND DYNAMIC GAMES

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0. INTRODUCTION AND SUMMARY

The purpose of this paper is to bring some simplicity and generality to the investigation of equilibrium existence in certain "simple" dynamic games.[1] The way we deal with these objects is, essentially, by reducing them to what we term "simple economies", these latter being, from the viewpoint of equilibrium existence results, at once more general and less cumbersome. In §1, we exhibit some topological facts about a certain rather general class of simple economies, including the fact (1.7, Main Theorem) that they each possess a non-empty and compact set of equilibria. Then, in §2, we define the simple dynamic games of interest, immediately reducing them (2.5, Reduction Lemma) to the simple economies studied. In §3, we show (3.0) that the simple dynamic games in question have non-empty and compact sets of equilibria, using the fact (1.7) that the simple economies to which they reduce have such sets of equilibria. We then illustrate by example (3.3) that a general class of discrete-time, deterministic games with convex performance criteria is covered by the equilibrium existence results just described. This class includes dynamic games for which certain non-linearities in the next-state map are allowed, and for which controls are restricted to compact regions, these regions themselves varying as a function of state. Of course, we do not intend

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1) An excellent bibliography of previous work on dynamic games can be compiled through the references cited within [Kuhn and Szegö, 1971] and from the survey paper, [Ho, 1970].

that results particularized to this quite special example be understood as the main thrust of this paper.

Historically, the study of the existence of (competitive) general equilibrium in economies exhibits quite a long-standing and extensive literature. A crucial turning point in that literature is afforded by the Arrow & Debreu [1954] study, benefiting from Debreu's [1952] earlier investigation and using a FPT (fixed point theorem) of Eilenberg & Montgomery [1946]. This work is generalized in [Sertel, 1971] by use of the more powerful FPT's of [Prakash & Sertel, 1971]. Essentially, our present results can easily be demonstrated as corollaries to this last mentioned, but owing to the relative inaccessibility of both this work and the fixed point theory it employs, we restrict ourselves here to what can be done by using the relatively well-known FPT of Fan [1952] which [Prakash & Sertel, 1970] generalizes. Even with these handicaps, our main theorem generalizes the Arrow & Debreu [1954] equilibrium existence result.

We work in locally convex spaces. Such spaces include normed spaces and the conjugate space of the Banach space of all real-valued continuous functions on a given compact space. This conjugate space, in turn, is the natural habitat of probability measures under the weak (or w^*) topology (see Parthasarathy [1967]). Our working in locally convex spaces is motivated by the hope and conjecture that equilibrium existence results for stochastic dynamic games may also be obtained by reducing them to the simple economies introduced here.

0.0. <u>Notation and Conventions</u>: For the set of natural numbers we denote $\mathcal{N} = \{1,2,\ldots\}$. We fix $n \in \mathcal{N}$ and denote $N = \{1,2,\ldots,n\}$, $N_0 = N \cup \{0\}$. Given any set Z and any $m \in \mathcal{N}$, we consider the family $\{{}^i Z = Z \mid i = 1,\ldots,m\}$ and denote the Cartesian product $\prod_{i=1}^{m} {}^i Z = {}^1 Z \times \ldots \times {}^m Z$, by ${}_m Z$, generic elements being in lower case: ${}^m z \in {}^m Z$, ${}_m z \in {}_m Z$. When a given ${}_m z = ({}^1 z,\ldots,{}^m z)$ is understood, we denote projections and components according to $\pi_{({}_k Z)}({}_m z) = {}_k z$ and $\pi_{({}^k Z)}({}_m z) = {}^k z$, $k = 1,\ldots,m$.

Every Cartesian product of topological spaces is understood to carry the product topology.

Let Z be a set. Then $[Z]$ denotes the set of non-empty subsets of Z. When Z is equipped with a topology, $C(Z)$ denotes the set of non-empty closed subsets of Z. When Z lies in a real vector space, $\mathcal{Q}(Z)$ denotes the set of non-empty convex subsets of Z. We abbreviate $C(Z) \cap \mathcal{Q}(Z)$ to $C\mathcal{Q}(Z)$.

Finally, R denotes the set of reals; whenever considered as a topological space, R carries the usual topology.

1. SIMPLE ECONOMIES AND THEIR EQUILIBRIA

The main task of this section is to introduce the notion of a simple economy, define equilibria of such, and then establish some fundamental results --- including our main theorem giving sufficient conditions under which simple economies possess non-empty and compact sets of equilibria. Apart from the interest of these notions and results in their own right, altogether they provide the main framework and the tools for our analysis of the equilibrium existence question for the dynamic games formulated in the next section.

1.0. Definition: A simple economy is an ordered quadruplet

1.0.0. $S = \langle W, U, T, A \rangle$,

where

1.0.1. $W = \{X_\alpha \neq \emptyset \mid \alpha \in A\} \neq \emptyset$

is a non-empty family of non-empty sets X_α with generic elements $x_\alpha \in X_\alpha$, from which we define $X = \prod_A X_\alpha$, $X^\alpha = \prod_{A\setminus\{\alpha\}} X_\beta$, denoting $x \in X$ and $x^\alpha \in X^\alpha$ for generic elements $(\alpha \in A)$;

1.0.2. $U = \{u_\alpha : X \to R \mid \alpha \in A\}$

is an associated family of real-valued functions u_α on X;

1.0.3. $T = \{t_\alpha : X \rightarrow [X_\alpha] | \alpha \in A\}$

is an associated family of maps t_α assigning a non-empty subset $t_\alpha(x) \subset X_\alpha$ to each $x \in X$;

1.0.4. $\Lambda = \{\alpha : X^\alpha \rightarrow [X_\alpha] | \alpha \in A\}$

is a self-indexed family of maps defined, at each $x^\alpha \in X^\alpha$, by

$$\alpha(x^\alpha) = \{x_\alpha \in \hat{t}_\alpha(x^\alpha) | u_\alpha(x_\alpha, x^\alpha) \geq \sup_{\hat{t}(x^\alpha)} u_\alpha(., x^\alpha)\},$$

where $\hat{t}_\alpha : X^\alpha \rightarrow [X_\alpha]$ is defined by

$$\hat{t}(x^\alpha) = \{z_\alpha \in X_\alpha | z_\alpha \in t_\alpha(z_\alpha, x^\alpha)\}.$$

1.1. Remark: It is evident that in a simple economy wherein the maps t_α are each independent of x_α, we have precisely $t_\alpha(x_\alpha, x^\alpha) = \hat{t}_\alpha(x^\alpha)$ for all $x_\alpha \in X_\alpha$ and $x^\alpha \in X^\alpha$, i.e., $t_\alpha = \hat{t}_\alpha \circ \pi_{X^\alpha}$ $(\alpha \in A)$. The "economies" treated by Arrow & Debreu |1954| are all of this sort. For a discussion of some major deficiencies of such a notion of an economy, most of which are present also in that of a "simple economy" — hence, the qualifier 'simple'! — we refer the reader to [Sertel, 1971; pp. 18-22].

1.2. <u>Remark</u>: To check that 1.0 is not a self-contradiction, i.e., that simple economies are well-defined, we need only indicate that an object S satisfying 1.0.0. - 1.0.3. will permit 1.0.4. so long as, for each index $\alpha \in A$, the maps $t_\alpha(., x^\alpha) : X_\alpha \to [X_\alpha]$ have fixed points $z_\alpha \in t_\alpha(z_\alpha, x^\alpha)$ for each $x^\alpha \in X^\alpha$ and the function $u_\alpha(., x^\alpha)$ attains a supremum on each such set of fixed points. All this will obtain, e.g., when W consist of compact spaces, each $u_\alpha(., x^\alpha)$ is upper semi-continuous, $t_\alpha = \hat{t}_\alpha \circ \pi_{X^\alpha}$, and $t_\alpha(X)$ consists of compact spaces $(x^\alpha \in X^\alpha, \alpha \in A)$.

To aid in verbalizing thoughts, we borrow the following nomenclature of [Sertel, 1971].

1.3. <u>Nomenclature</u>: Let S be as in 1.0.

1.3.1. X_α is called the <u>behavior space</u> of α, x_α being called a <u>behavior</u> of α iff $x_\alpha \in X_\alpha$;

1.3.2. X^α is called the <u>α-exclusive behavior space</u> (of S), x^α being called an <u>α-exclusive behavior</u> (of S) iff $x^\alpha \in X^\alpha$;

1.3.3. X is called the <u>collective behavior space</u> (of S), x being called a <u>collective behavior</u> (of S) iff $x \in X$;

1.3.4. u_α is called the <u>utility function</u> of α;

1.3.5. t_α (resp., $\hat{t}_\alpha$) is called the <u>feasibility transformation</u> (resp., <u>effective feasibility transformation</u>) of α, $t_\alpha(X) = \{t_\alpha(x) | x \in X\}$ being called the <u>feasibility space</u> of α, and d_α being called a feasibility of α iff $d_\alpha \in t_\alpha(X)$;

1.3.6. A is called the personnel (of S), α being called a <u>behavor</u> (of S) iff $\alpha \in A$.

Regarding the following useful tool dealing with continuity properties of set-valued mappings, the definitions and facts in the appendix may be consulted.

1.4. <u>Theorem</u>: Let $u: P \times Q \to P$ be a continuous function on a non-empty compact Hausdorff space $P \times Q$, let $s: Q \to C(P)$ be continuous, and define a map α on Q by

$$\alpha(q) = \{p \in s(q) \mid u(p, q) \geq \sup_{s(q)} u(., q)\} \quad (q \in Q).$$

Then α maps Q upper semi-continuously into $C(P)$.

Proof: Given any $q \in Q$, $s(q)$ is compact, being closed in the compact P; hence, $s(q) \times \{q\}$ is compact, so that the continuous u attains a supremum on it. This shows that $\alpha(q) \neq \emptyset$ for each $q \in Q$. To establish the upper semi-continuity of α, we will repeatedly use the closed graph theorem A.2 and show that the graph $\Gamma_\alpha = \{(q, p) \mid q \in Q, \ p \in \alpha(q)\}$ of α is closed. From the closedness of Γ_α in the compact $Q \times P$, it will follow that Γ_α is compact, so that, for each $q \in Q$, $\alpha(q) = \pi_P\left((\{q\} \times P) \cap \Gamma_\alpha\right)$ is compact, hence, closed in P.

We now show that $\Gamma_\alpha \subset Q \times P$ is closed. As s is continuous, it is upper semi-continuous, so that, by A.2, its graph $\Gamma_s = \{(q, p) \mid q \in Q, \ p \in s(q)\} \subset Q \times P$ is closed. As u is continuous on the compact $P \times Q$, $u(P \times Q) \subset R$ is compact and, of course, Hausdorff. By A.4, the function $v\colon Q \to R$ defined, at each $q \in Q$, by $v(q) = \sup_{s(q) \times \{q\}} u$ is continuous, so that its graph $\Gamma_v = \{(q, r) \mid q \in Q, \ r = v(q)\} \subset Q \times v(Q)$ $\subset Q \times u(P \times Q)$ is closed. Hence, $\Gamma_v \times P$ is closed. By continuity of u, so is the graph $\Gamma_u = \{(p, q, r) \mid (p, q) \in P \times Q, \ r = u(p, q)\} \subset P \times Q \times u(P \times Q)$ of u

closed. Thus, $(\Gamma_v \times P) \cap \Gamma_u$ is closed, hence compact. Therefore, its projection G into $P \times Q$ is compact, hence closed. Now Γ_α is nothing but $G \cap \Gamma_s$, which, clearly, is closed. This completes the proof.

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1.5. Corollary: In the last theorem, assume also that P is convex in a real vector space, that u is quasi-concave on $P \times \{q\}$ for each $q \in Q$, and that $s(Q) \subset \mathcal{CQ}(P)$. Then α maps Q upper semi-continuously into $\mathcal{CQ}(P)$.

Proof: Let $q \in Q$. All we need to show is that $\alpha(q)$ is convex. Now $\alpha(q)$ is nothing but the intersection of $s(q)$ with $\{p \in P | u(p, q) \geq v(q)\}$, the former of which is convex by assumption, and the latter of which is convex by quasi-concavity of u on $P \times \{q\}$. This shows that $\alpha(q)$ is convex, completing the proof.

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1.6. Definition: The evolution of a simple economy $S = \langle W, U, T, A \rangle$ is a map $E : X \to [X]$ defined, at each $x \in X$, by

$$E(x) = \prod_A \alpha(\pi_{X^\alpha}(x)) = \prod_A \alpha(x^\alpha).$$

A point $x \in X$ is called an equilibrium or contract of S iff $x \in E(x)$. The set C of contracts of S is called the contractual set of S.

This section culminates with the following:

1.7. Main Theorem: Let $S = \langle W, U, T, A\rangle$ be an ordered quadruplet such that, using the notation of 1.0.,

1.7.1. W is as in 1.0.1. with each X_α compact and convex in a locally convex Hausdorff topological vector space;

1.7.2. U is as in 1.0.2. and, for each $\alpha \in A$, u_α is continuous on X and quasi-concave on $X_\alpha \times \{x^\alpha\}$ for each $x^\alpha \in X^\alpha$;

1.7.3. T is as in 1.0.3. and, for each $\alpha \in A$, the graph

$T_\alpha = \{(x_\alpha, x^\alpha, x_\alpha') \mid (x_\alpha, x^\alpha) \in X, \ x_\alpha' \in t_\alpha (x_\alpha, x^\alpha)\}$

$\subset X \times X_\alpha$ of t_α is closed, with the graph

$T_\alpha(x^\alpha) = \{(x_\alpha, x_\alpha') \in X_\alpha \times X_\alpha \mid x_\alpha' \in t_\alpha(x_\alpha, x^\alpha)\}$ of

$t_\alpha(., x^\alpha) : X_\alpha \to [X_\alpha]$ convex for each $x^\alpha \in X^\alpha$; and

1.7.4. A is a self-indexed family of maps α such that α is defined, at each $x^\alpha \in X^\alpha$, by

$$\alpha(x^\alpha) = \{x_\alpha \in \hat{t}_\alpha(x^\alpha) \mid u_\alpha(x_\alpha, x^\alpha) \geq \operatorname*{Sup}_{\hat{t}(x^\alpha)} u_\alpha(., x^\alpha)\},$$

where $\hat{t}_\alpha$ is as in 1.0.4.

Then

(a) for each $\alpha \in A$, $\hat{t}_\alpha$ maps X^α upper semi-continuously into $CQ(X_\alpha)$;

(b) for each $\alpha \in A$, α maps X^α into $\mathcal{CQ}(X_\alpha)$; and

(c) S is a simple economy.

If, furthermore,

1.7.5. for each $\alpha \in A$, $\hat{t}_\alpha$ is lower semi-continuous,

then also

(d) for each $\alpha \in A$, α is upper semi-continuous; and

(e) the contractual set C of S is non-empty and compact.

<u>Proof</u>: (<u>ad</u> (a)): Let $\alpha \in A$ and $x^\alpha \in X^\alpha$. The graph $T_\alpha(x^\alpha)$ of the restriction $t_\alpha(., x^\alpha)$ of t_α to $X_\alpha \times \{x^\alpha\}$ is convex by assumption and it is closed since T_α is closed. Thus, $t_\alpha(., x^\alpha)$ is into $\mathcal{CQ}(X_\alpha)$; by A.2, it is also upper semi-continuous. Hence, by Fan's FPT (A.5), its graph $T_\alpha(x^\alpha)$ intersects the diagonal Δ_α of $X_\alpha \times X_\alpha$. Furthermore, this intersection is one of two closed and convex sets, and so is closed and convex. Being closed in the compact $X_\alpha \times X_\alpha$, it is compact. Moreover, its projection into X_α is nothing but $\hat{t}_\alpha(x^\alpha)$, showing that $\hat{t}(x^\alpha) \in \mathcal{CQ}(X_\alpha)$. Now the graph of $\hat{t}_\alpha$ is simply $\pi_{X^\alpha \times X_\alpha}(T_\alpha \cap \{(x_\alpha, y^\alpha, y_\alpha) \in X_\alpha \times X^\alpha \times X_\alpha | x_\alpha = y_\alpha\})$,

which is compact, hence closed. Thus, $\hat{t}_\alpha$ is upper semi-continuous. So much proves (a).

(ad (b)): From (a) we see that $\hat{t}_\alpha(x^\alpha)$ is closed, hence compact, so that the continuous restriction of u_α to $X_\alpha \times \{x^\alpha\}$ attains a supremum on a (non-empty) closed subset of $\hat{t}_\alpha(x^\alpha)$. Clearly, this subset is nothing but $\alpha(x^\alpha)$, and it is convex by quasi-concavity of u_α on $X_\alpha \times \{x^\alpha\}$, since $\hat{t}_\alpha(x^\alpha)$, by (a), is convex. This proves (b).

(ad (c)): As $[X_\alpha] \supset C\mathcal{Q}(X_\alpha)$, from (b) we see that $\alpha(X^\alpha) \subset [X_\alpha]$, i.e., that S is a simple economy (Cf. 1.2), proving (c).

For (d) and (e), now assume 1.7.5.

(ad (d)): Being upper semi-continuous by (a) and lower semi-continous by 1.7.5, $\hat{t}_\alpha$ is now continuous. Applying 1.4., (d) is proved.

(ad (e)): From (b) and (d), for each $\alpha \in A$, α maps X^α upper semi-continuously into $C\mathcal{Q}(X_\alpha)$. Thus, the evolution E of S maps X upper semi-continuously into $C\mathcal{Q}(X)$, so that, by Fan's FPT [1952], E has a fixed point. In other words, S has an equilibrium and $C \neq \emptyset$. Actually, since E is upper semi-continuous, its graph

$\Gamma_E = \{(x, x') \in X \times X | x' \in E(x)\}$ is closed, hence compact in the compact space $X \times X$. As $C = \pi_X(\Gamma_E \cap \Delta)$, where Δ is the diagonal of $X \times X$, C is actually compact. This completes the proof.

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2. SIMPLE DYNAMIC GAMES

The purpose of this section is to define simple dynamic games and their equilibria. The existence of these equilibria will be studied in the next section by reference to simple economies derived from simple dynamic games. The present section gives a constructive procedure for obtaining a unique derived simple economy from any given simple dynamic game.

2.0. Definition: A simple dynamic game (s.d.g.) is an ordered octuple,

2.0.0. $\dot{S} = < \dot{W}, \dot{Y}, \underline{\dot{Y}}, \dot{\delta}, \dot{\omega}, \dot{U}, \dot{T}, \dot{A} >$,

where

2.0.1. $\dot{W} = \{\dot{X}_{\dot{\alpha}} \neq \emptyset \mid \dot{\alpha} \in \dot{A}\} \neq \emptyset$ is a non-empty family of non-empty sets $\dot{X}_{\dot{\alpha}}$ with generic elements $\dot{x}_{\dot{\alpha}} \in \dot{X}_{\dot{\alpha}}$, from which we define $\dot{X} = \prod_{\dot{A}} \dot{X}_{\dot{\alpha}}$ and $\dot{X}^{\dot{\alpha}} = \prod_{\dot{A} \setminus \{\dot{\alpha}\}} \dot{X}_{\dot{\beta}}$, denoting $\dot{x} \in \dot{X}$ and $\dot{x}^{\dot{\alpha}} \in \dot{X}^{\dot{\alpha}}$ for generic elements $(\dot{\alpha} \in \dot{A})$;

2.0.2. $\dot{Y}$ is a non-empty set;

2.0.3. $\underline{\dot{Y}} \in [\dot{Y}]$;

2.0.4. $\dot{\delta} : \dot{X} \times \dot{Y} \to \dot{Y}$ is a function, from which we inductively define the derived functions ${}^{k}\dot{\delta} : {}_{n}\dot{X} \times \underline{\dot{Y}} \to \dot{Y}$ and ${}_{k}\dot{\delta} : {}_{n}\dot{X} \times \underline{\dot{Y}} \to \underline{\dot{Y}} \times {}_{k}\dot{Y}$ as follows:

$$\left.\begin{aligned} &{}^{k}\dot{\delta}({}_{n}\dot{x}, \underline{\dot{y}}) = \dot{\delta}({}^{k}\dot{x}, {}^{k-1}\dot{\delta}({}_{n}\dot{x}, \underline{\dot{y}})), \\ &\text{and} \\ &{}_{k}\dot{\delta}({}_{n}\dot{x}, \underline{\dot{y}}) = ({}^{0}\dot{\delta}({}_{n}\dot{x}, \underline{\dot{y}}), \ldots, {}^{k}\dot{\delta}({}_{n}\dot{x}, \underline{\dot{y}})) \end{aligned}\right\} \left\{\begin{aligned} &{}_{n}\dot{x} \in {}_{n}\dot{X}, \\ &\underline{\dot{y}} \in \underline{\dot{Y}}, \\ &k \in N, \end{aligned}\right.$$

where

${}^{0}\dot{\delta} : {}_{n}\dot{X} \times \underline{\dot{Y}} \to \dot{Y}$ is a map defined by

${}^{0}\dot{\delta}({}_{n}\dot{x}, \underline{\dot{y}}) = \underline{\dot{y}}$ $\quad({}_{n}\dot{x} \in {}_{n}\dot{X}, \underline{\dot{y}} \in \underline{\dot{Y}})$;

2.0.5. $\dot{\omega} = \{\dot{\omega}_{\dot{\alpha}} : \underline{\dot{Y}} \to [\dot{X}_{\dot{\alpha}}] \mid \dot{\alpha} \in \dot{A}\}$ is a family of maps;

2.0.6. $\dot{U} = \{\dot{u}_{\dot{\alpha}} : {}_{n}\dot{X} \times \dot{Y} \times {}_{n}\dot{Y} \to R \mid \dot{\alpha} \in \dot{A}\}$ is a family of real-valued utility functions.

2.0.7. $\dot{T} = \{\dot{t}_{\dot{\alpha}} : \dot{X} \times \dot{Y} \to [\dot{X}_{\dot{\alpha}}] \mid \dot{\alpha} \in \dot{A}\}$ is a family of maps, whence, for each $\dot{\alpha} \in \dot{A}$, we inductively define the derived maps ${}^{k}\dot{t}_{\dot{\alpha}} : {}_{n}\dot{X} \times \underline{\dot{Y}} \to [\dot{X}_{\dot{\alpha}}]$ and ${}_{k}\dot{t}_{\dot{\alpha}} : {}_{n}\dot{X} \times \underline{\dot{Y}} \to [{}_{k}\dot{X}_{\dot{\alpha}}]$ as follows:[2]

(footnote 2) see next page)

$$\left.\begin{array}{l} {}^{k}\dot{t}_{\dot{\alpha}}({}_{n}\dot{x}, \dot{\underline{y}}) = \dot{t}_{\dot{\alpha}}({}^{k}\dot{x}, {}^{k-1}\dot{\delta}({}_{n}\dot{x}, \dot{\underline{y}})) \\ \text{and} \\ {}_{k}\dot{t}_{\dot{\alpha}}({}_{n}\dot{x}, \dot{\underline{y}}) = \prod_{j=o}^{k-1} {}^{j}\dot{t}_{\dot{\alpha}}({}_{n}\dot{x}, \dot{\underline{y}}) \end{array}\right\} \left\{\begin{array}{l} {}_{n}\dot{x} \in {}_{n}\dot{X} \\ \dot{\underline{y}} \in \dot{\underline{Y}}, \\ k \in N, \end{array}\right.$$

where

$^{o}\dot{t}_{\dot{\alpha}} : {}_{n}\dot{X} \times \dot{\underline{Y}} \rightarrow [\dot{X}_{\dot{\alpha}}]$ is a map defined by

$$^{o}\dot{t}_{\dot{\alpha}}({}_{n}\dot{x}, \dot{\underline{y}}) = \dot{\omega}_{\dot{\alpha}}(\dot{\underline{y}}) \quad ({}_{n}\dot{x} \in {}_{n}\dot{X}, \dot{\underline{y}} \in \dot{\underline{Y}});$$

2.0.8. $\dot{A}$ is a self-indexed family of maps $\dot{\alpha} : {}_{n}\dot{X}^{\dot{\alpha}} \times \dot{\underline{Y}} \rightarrow [{}_{n}\dot{X}_{\dot{\alpha}}]$ such that, for any $({}_{n}\dot{x}^{\dot{\alpha}}, \dot{\underline{y}}) \in {}_{n}\dot{X}^{\dot{\alpha}} \times \dot{\underline{Y}}$,

$$\dot{\alpha}({}_{n}\dot{x}^{\dot{\alpha}}, \dot{\underline{y}}) = \{{}_{n}\dot{x}_{\dot{\alpha}} \in {}_{n}\hat{\dot{t}}_{\dot{\alpha}}({}_{n}\dot{x}^{\dot{\alpha}}, \dot{\underline{y}}) \mid$$

$$\dot{u}_{\dot{\alpha}}({}_{n}\dot{x}_{\dot{\alpha}}, {}_{n}\dot{x}^{\dot{\alpha}}, {}_{n}\dot{\delta}({}_{n}\dot{x}_{\dot{\alpha}}, {}_{n}\dot{x}^{\dot{\alpha}}, \dot{\underline{y}})) =$$

$$\sup_{{}_{n}\hat{\dot{t}}_{\dot{\alpha}}({}_{n}\dot{x}^{\dot{\alpha}}, \underline{y})} \dot{u}_{\dot{\alpha}}(\cdot, {}_{n}\dot{x}^{\dot{\alpha}}, {}_{n}\dot{\delta}(\cdot, {}_{n}\dot{x}^{\dot{\alpha}}, \dot{\underline{y}}))\},$$

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2) (footnote of page 15) The reader may find it convenient to assume on first reading that $\dot{t}_{\dot{\alpha}}(\dot{x}^{\dot{\alpha}}, \dot{y}) = \dot{X}_{\dot{\alpha}}$ for all $(\dot{x}^{\dot{\alpha}}, \dot{y}) \in \dot{X}^{\dot{\alpha}} \times \dot{Y}$. This assumption implies, in particular, that ${}_{k}\dot{t}_{\dot{\alpha}}({}_{n}\dot{x}, \dot{\underline{y}}) = {}_{k}\dot{X}_{\dot{\alpha}}$ for all $({}_{n}\dot{x}, \dot{\underline{y}}) \in {}_{n}\dot{X} \times \dot{\underline{Y}}$, which, as will become clear below, corresponds to the case where the feasible control region is fixed.

where

$_n\hat{\dot{t}}_{\dot{\alpha}} : {}_n\dot{X}^{\dot{\alpha}} \times \dot{\underline{Y}} \to [{}_n\dot{X}_{\dot{\alpha}}]$ is defined for each $\dot{\alpha} \in \dot{A}$

and $({}_n\dot{x}^{\dot{\alpha}}, \dot{\underline{y}}) \in {}_n\dot{X}^{\dot{\alpha}} \times \dot{\underline{Y}}$ by

$${}_n\hat{\dot{t}}_{\dot{\alpha}}({}_n\dot{x}^{\dot{\alpha}}, \dot{\underline{y}}) = \{{}_n\dot{z}_{\dot{\alpha}} \in {}_n\dot{X}_{\dot{\alpha}} | {}_n\dot{z}_{\dot{\alpha}} \in {}_n\dot{t}_{\dot{\alpha}}({}_n\dot{z}_{\dot{\alpha}}, {}_n\dot{x}^{\dot{\alpha}}, \dot{\underline{y}})\}.$$

In order to facilitate discussion we introduce the following terminology.

2.1. Terminology: From here on, whenever we speak of a "dynamic game", we will always mean a s.d.g.. Let $\dot{S}$ be as in 2.0. The elements of $\dot{S}$ will be referred to as follows.

2.1.0. n will be called the planning horizon.

2.1.1. $\dot{X}_{\dot{\alpha}}$ $({}_n\dot{X}_{\dot{\alpha}})$ will be called the $\dot{\alpha}$-control space ($\dot{\alpha}$-plan space), and $\dot{x}_{\dot{\alpha}}$ $({}_n\dot{x}_{\dot{\alpha}})$ will be called an $\dot{\alpha}$-control ($\dot{\alpha}$-plan) iff $\dot{x}_{\dot{\alpha}} \in \dot{X}_{\dot{\alpha}}$ $({}_n\dot{x}_{\dot{\alpha}} \in {}_n\dot{X}_{\dot{\alpha}})$. $\dot{X}^{\dot{\alpha}}$ $({}_n\dot{X}^{\dot{\alpha}})$ will be called the $\dot{\alpha}$-exclusive control space ($\dot{\alpha}$-exclusive plan space), and $\dot{x}^{\dot{\alpha}}$ $({}_n\dot{x}^{\dot{\alpha}})$ will be called an $\dot{\alpha}$-exclusive control ($\dot{\alpha}$-exclusive plan) iff $\dot{x}^{\dot{\alpha}} \in \dot{X}^{\dot{\alpha}}$ $({}_n\dot{x}^{\dot{\alpha}} \in {}_n\dot{X}^{\dot{\alpha}})$. $\dot{X}$ $({}_n\dot{X})$ will be called the control space (plan space), and $\dot{x}$ $({}_n\dot{x})$ will be called a control (plan) iff $\dot{x} \in \dot{X}$ $({}_n\dot{x} \in {}_n\dot{X})$.

2.1.2. $\dot{Y}$ will be called the state space, and $\dot{y}$ will be called a state iff $\dot{y} \varepsilon \dot{Y}$.

2.1.3. $\dot{\underline{Y}}$ will be called the space of initial states, and $\dot{\underline{y}}$ will be called an initial state iff $\dot{\underline{y}} \varepsilon \dot{\underline{Y}}$.

2.1.4. $\dot{\delta}$ will be called the next-state map. ${}^{k}\dot{\delta}$ will be called the k-extended next state map. ${}_{k}\dot{\delta}$ will be called the k-string next-state map.

2.1.5. $\dot{\omega}$ will be called the initial control feasibility specification. $\dot{\omega}_{\dot{\alpha}}$ will be called the initial $\dot{\alpha}$-control feasibility specification, and $\dot{d}_{\dot{\alpha}} \varepsilon [\dot{X}_{\dot{\alpha}}]$ will be called an initial $\dot{\alpha}$-control feasibility for $\dot{\underline{y}} \varepsilon \dot{\underline{Y}}$ iff $\dot{d}_{\dot{\alpha}} = \dot{\omega}_{\dot{\alpha}}(\dot{\underline{y}})$.

2.1.6. $\dot{u}_{\dot{\alpha}}$ will be called the utility function of $\dot{\alpha}$.

2.1.7. $\dot{T}$ will be called the control feasibility specification. $\dot{t}_{\dot{\alpha}}$ will be called the $\dot{\alpha}$-control feasibility specification. ${}_{n}\dot{t}_{\dot{\alpha}}$ (${}_{n}\hat{\dot{t}}_{\dot{\alpha}}$) will be called the $\dot{\alpha}$-plan (effective) feasibility specification and ${}_{n}\dot{d}_{\dot{\alpha}}$ (${}_{n}\hat{\dot{d}}_{\dot{\alpha}}$) will be called an

$\dot{\alpha}$-plan (effective) feasibility for a plan $_n\dot{x}$ (an $\dot{\alpha}$-exclusive plan $_n x^{\dot{\alpha}}$) and an initial state $\dot{\underline{y}}$ iff $_n d_{\dot{\alpha}} = {}_n t_{\dot{\alpha}}\ ({}_n\dot{x},\ \dot{\underline{y}})$ $({}_n\hat{d}_{\dot{\alpha}} = {}_n\hat{t}_{\dot{\alpha}}\ ({}_n\dot{x}^{\dot{\alpha}},\ \dot{\underline{y}}))$.

2.1.8. $\dot{\Lambda}$ will be called the personnel of $\dot{S}$, while each $\dot{\alpha} \in \dot{\Lambda}$ will be called a player.

The connotation of the above denotation will become clear as we proceed; however, a few points can be made at this time. The key to understanding the above formulation of dynamic games is 2.0.8, which we now elucidate briefly.

Given any initial state $\dot{\underline{y}}$ and any $\dot{\alpha}$-exclusive plan $_n\dot{x}^{\dot{\alpha}}$, each player $\dot{\alpha}$ is assumed to develop an $\dot{\alpha}$-plan obeying 2.0.8. In doing so, player $\dot{\alpha}$ takes the given $({}_n\dot{x}^{\dot{\alpha}},\ \dot{\underline{y}})$ and computes the set $\dot{\alpha}({}_n\dot{x}^{\dot{\alpha}},\ \dot{\underline{y}})$ of solutions to the following optimal control problem:

$$\underset{{}_n\dot{X}_{\alpha} \times \{{}_n\dot{x}^{\dot{\alpha}},\ \dot{\underline{y}}\} \times {}_n\dot{Y}}{\text{Maximize}} \qquad \dot{u}_{\dot{\alpha}}({}_n\dot{x}_{\dot{\alpha}},\ {}_n\dot{x}^{\dot{\alpha}},\ \dot{\underline{y}},\ {}_n\dot{y})$$

subject to:

$${}^k\dot{y} = \dot{\delta}({}^k\dot{x}_{\dot{\alpha}}, {}^k\dot{x}^{\dot{\alpha}}, {}^{k-1}\dot{y}), \quad k \in N, \quad {}^o\dot{y} = \dot{\underline{y}},$$

$${}^{k+1}\dot{x}_{\dot{\alpha}} \in {}^k t_{\dot{\alpha}}({}_n\dot{x}_{\dot{\alpha}},\ {}_n\dot{x}^{\dot{\alpha}},\ \dot{\underline{y}}),\ k + 1 \in N$$

where ${}_n\dot{x} = ({}_n\dot{x}_{\dot{\alpha}},\ {}_n\dot{x}^{\dot{\alpha}}) = ({}^1\dot{x}, \ldots, {}^n\dot{x})$ and ${}_n\dot{y} = ({}^1\dot{y}, \ldots, {}^n\dot{y})$

are the sequences of controls and states over the planning horizon, left superscript being the "time" index. Note that, by 2.0.7, ${}^{1}\dot{x}_{\dot{\alpha}} \in {}^{o}t_{\dot{\alpha}}({}_{n}\dot{x}, \dot{\underline{y}}) = \dot{\omega}_{\dot{\alpha}}(\dot{\underline{y}})$, a given $\dot{\alpha}$-initial control feasibility, while, for $k = 1,\ldots,n - 1$, we have ${}^{k+1}\dot{x}_{\dot{\alpha}} \in t_{\dot{\alpha}}({}^{k}\dot{x}_{\dot{\alpha}}, {}^{k}\dot{x}^{\dot{\alpha}}, {}^{k-1}\dot{y})$, so that the feasible control region for ${}^{k}\dot{x}_{\dot{\alpha}}$ is determined by the initial state and the controls preceding the k^{th}.

We postpone a discussion of the conditions under which a s.d.g. $\dot{S}$ is well-defined until we have demonstrated that any s.d.g. may be reduced to a simple economy. Sufficient conditions for the existence of $\dot{S}$ will then be clear from similar conditions already stated (1.2. and 1.7(c)) for simple economies.

2.2. <u>Remark and Notation</u>: Let $\dot{S}$ be a s.d.g. specified as in 2.0, and suppose $\dot{\underline{Y}} \supset \dot{\underline{Y}}' \neq \emptyset$. Then clearly $\dot{S}' = < \dot{W}, \dot{Y}. \dot{\underline{Y}}', \dot{\delta}, \dot{\omega}, \dot{U}, \dot{T}, \dot{A} >$ is also a dynamic game. When $\dot{S}$ and $\dot{S}'$ are related in this fashion, we say that $\dot{S}'$ is a <u>restriction</u> of $\dot{S}$. When $\dot{\underline{Y}}' = \{\dot{\underline{y}}\} \subset \dot{\underline{Y}}$, $\dot{S}'$ will be called the restriction of $\dot{S}$ to the initial state $\dot{\underline{y}}$. When $\dot{S}$ is understood, we denote $\dot{S}'$ in this case by $\dot{S}(\dot{\underline{y}})$.

2.3. <u>Definition</u>: The <u>evolution</u> of a simple dynamic game $\dot{S} = < \dot{W}, \dot{Y}, \dot{\underline{Y}}, \dot{\delta}, \dot{\omega}, \dot{U}, \dot{T}, \dot{A} >$ is a map $\dot{E} : {}_{n}\dot{X} \times \dot{\underline{Y}} \rightarrow \lceil_{n}\dot{X} \times \dot{\underline{Y}}\rceil$ defined, at each

$({}_n\dot{x},\ \dot{\underline{y}}) \in {}_n\dot{X} \times \dot{\underline{Y}}$, by[3]

$$\dot{E}({}_n\dot{x},\ \dot{\underline{y}}) \;=\; \prod_{\dot{\Lambda}} \dot{\alpha}({}_n\dot{x}^{\dot{\alpha}},\ \dot{\underline{y}}) \times \{\dot{\underline{y}}\}.$$

A point $({}_n\dot{x},\ \dot{\underline{y}})$ will be called an <u>equilibrium</u> of $\dot{S}$ iff $({}_n\dot{x},\ \dot{\underline{y}}) \in \dot{E}({}_n\dot{x},\ \dot{\underline{y}})$. The set of equilibria of $\dot{S}$ is called the <u>contractual set</u> of $\dot{S}$.

2.4. <u>Remark</u>: It may be verified that the contractual set of $\dot{S}$ consists of precisely those points $({}_n\hat{\dot{x}},\ \hat{\dot{\underline{y}}}) \in {}_n\dot{X} \times \dot{\underline{Y}}$ which, for each $\dot{\alpha} \in \dot{\Lambda}$, satisfy

2.4.1. $$\dot{u}_{\dot{\alpha}}({}_n\hat{\dot{x}}_{\dot{\alpha}}, {}_n\hat{\dot{x}}^{\dot{\alpha}}, {}_n\dot{\delta}({}_n\hat{\dot{x}}_{\dot{\alpha}}, {}_n\hat{\dot{x}}^{\dot{\alpha}}, \hat{\dot{\underline{y}}}))$$

$$= \sup_{{}_n\dot{z}_{\dot{\alpha}} \in {}_n\dot{t}_{\dot{\alpha}}({}_n\dot{z}_{\dot{\alpha}}, {}_n\hat{\dot{x}}^{\dot{\alpha}}, \hat{\dot{\underline{y}}})} \dot{u}_{\dot{\alpha}}({}_n\dot{z}_{\dot{\alpha}}, {}_n\hat{\dot{x}}^{\dot{\alpha}}, {}_n\dot{\delta}({}_n\dot{z}_{\dot{\alpha}}, {}_n\hat{\dot{x}}^{\dot{\alpha}}, \hat{\dot{\underline{y}}}))$$

$$= \sup_{{}_n\dot{X}_{\dot{\alpha}} \times \{{}_n\hat{\dot{x}}^{\dot{\alpha}}, \hat{\dot{\underline{y}}}\} \times {}_n\dot{Y}} \dot{u}_{\dot{\alpha}}({}_n\dot{x}_{\dot{\alpha}}, {}_n\hat{\dot{x}}^{\dot{\alpha}}, \hat{\dot{\underline{y}}}, {}_n\dot{y})$$

s.t.

$${}^k\dot{y} = \dot{\delta}({}^k\dot{x}_{\dot{\alpha}}, {}^k\hat{\dot{x}}^{\dot{\alpha}}, {}^{k-1}\dot{y}),\ k \in N, \quad {}^0y = \hat{\dot{\underline{y}}},$$

$${}^{k+1}\dot{x}_{\dot{\alpha}} \in {}^k\dot{t}_{\dot{\alpha}}({}_n\dot{x}_{\dot{\alpha}}, {}_n\hat{\dot{x}}^{\dot{\alpha}}, \hat{\dot{\underline{y}}}), \quad k+1 \in N.$$

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3) Note that $\dot{E}$ is actually a map into $\prod_{\dot{\Lambda}} |{}_n\dot{X}_{\dot{\alpha}}| \times |\dot{\underline{Y}}| \subset |{}_n\dot{X} \times \dot{\underline{Y}}|$.

We now show that every simple dynamic game can be reduced to a simple economy whose evolution and, hence, whose contractual set are identical to those of the given simple dynamic game.

2.5. Reduction Lemma: Let $\dot{S} = < \dot{W},\dot{Y},\dot{\underline{Y}},\dot{\delta},\dot{\omega},\dot{U},\dot{T},\dot{A} >$ be a s.d.g..

Then

(a) there is a simple economy $S = < W,U,T,A >$, uniquely derivable from $\dot{S}$, with $X = {}_n\dot{X} \times \dot{\underline{Y}}$, and for which:

(b) $E(x) = \dot{E}(x) \quad (x \times X)$; and

(c) the contractual sets for S and $\dot{S}$ coincide.

Proof: (ad (a)): Let a s.d.g. $\dot{S}$ and an object $\dot{a} \notin \dot{A}$ be given. Denote by $\dot{A}_+$ the set $\dot{A} \cup \{\dot{a}\}$. Consider the following sequence of definitions which specify the elements of the simple economy S to be derived from $\dot{S}$.

2.5.1. Let $X_{\dot{a}} = \dot{\underline{Y}}$ and, for each $\dot{\alpha} \in \dot{A}$, let $X_{\dot{\alpha}} = {}_n\dot{X}_{\dot{\alpha}}$, defining $\bar{X} = \prod_{\dot{A}_+} X_{\dot{\alpha}}$ and $X^{\dot{\alpha}} = \prod_{\dot{A}_+ \setminus \{\dot{\alpha}\}} X_{\dot{\beta}} \quad (\dot{\alpha} \in \dot{A}_+)$;

2.5.2. Define the family of functions $\{u_{\dot{\alpha}} : \tilde{X} \to R \mid \dot{\alpha} \in \dot{A}_+\}$ by

$$\left.\begin{aligned} u_{\dot{\alpha}}(\tilde{x}) &= u_{\dot{\alpha}}(\pi_{X^{\dot{a}}}(\tilde{x}), {}_n\delta(\tilde{x})) \\ u_{\dot{a}}(\tilde{x}) &= \sqrt{2}, \end{aligned}\right\} \quad \tilde{x} \in \tilde{X};$$

2.5.3. Define the family of functions $\{t_{\dot{\alpha}} : \tilde{X} \to [X_{\dot{\alpha}}] \mid \dot{\alpha} \in \dot{A}_+\}$

by

$$\left.\begin{aligned} t_{\dot{\alpha}}(\tilde{x}) &= {}_n\dot{t}_{\dot{\alpha}}(\tilde{x}), \ \dot{\alpha} \in \dot{A} \\ t_{\dot{a}}(\tilde{x}) &= \pi_{X_{\dot{a}}}(\tilde{x}), \end{aligned}\right\} \quad \tilde{x} \in \tilde{X};$$

2.5.4. Define the family of maps $\{\alpha : X^{\dot{\alpha}} \to [X_{\dot{\alpha}}] \mid \dot{\alpha} \in \dot{A}_+\}$

by

$$\alpha(x^{\dot{\alpha}}) = \{x_{\dot{\alpha}} \in \hat{t}_{\dot{\alpha}}(x^{\dot{\alpha}}) \mid u_{\dot{\alpha}}(x_{\dot{\alpha}}, x^{\dot{\alpha}}) = \sup_{\hat{t}_{\dot{\alpha}}(x^{\dot{\alpha}})} u_{\dot{\alpha}}(\cdot, x^{\dot{\alpha}})\},$$

where $\hat{t}_{\dot{\alpha}} : X^{\alpha} \to [X_{\dot{\alpha}}]$ is defined by

$$\hat{t}_{\dot{\alpha}}(x^{\dot{\alpha}}) = \{z_{\dot{\alpha}} \in X_{\dot{\alpha}} \mid z_{\dot{\alpha}} \in t_{\dot{\alpha}}(z_{\dot{\alpha}}, x^{\dot{\alpha}})\}.$$

We now denote $A = \bigcup_{\dot{\alpha} \in \dot{A}_+} \{\alpha\}$ and, for $\alpha \in A$, we denote $X_{\alpha} = X_{\dot{\alpha}}$, $u_{\alpha} = u_{\dot{\alpha}}$, $t_{\alpha} = t_{\dot{\alpha}}$. These identifications yield

(from 1.0.1 - 1.0.4 and 2.5.1 - 2.5.4) a unique economy $S = < W,U,T,A >$.

(ad (b) and (c)) Clearly, if the evolutions of S and $\dot{S}$ are identical, their contractual sets will coincide (see 1.6 and 2.3). This leaves only (b) to be shown, and that is direct from definitions (compare 1.0.4 (2.0.8) and 1.6 (2.3) with 2.5.4).

2.6. Definition: Let $\dot{S}$ be a s.d.q. The simple economy S derived from $\dot{S}$ via the procedure given in the proof of 2.5 (a) will be called the derived (simple) economy of $\dot{S}$.

3. Equilibrium Existence Results for Dynamic Games

This section applies the equilibrium existence results obtained in section 1 for economies to the case of dynamic games. Given the reduction procedure of 2.5, the only concern is to demonstrate conditions on the elements of a given dynamic game $\dot{S}$ under which the elements of the derived economy of $\dot{S}$ satisfy the hypotheses of Theorem 1.7. This we now do.

3.0. Proposition: Let $\dot{S} = < \dot{W}, \dot{Y}, \underline{\dot{Y}}, \dot{\delta}, \dot{\omega}, \dot{U}, \dot{T}, \dot{A} >$ be specified as in 2.0, where

3.0.1. $\dot{X}_{\dot{\alpha}} \neq \emptyset$ is a compact and convex subset of a locally convex Hausdorff topological vector space $(\dot{\alpha} \in \dot{A})$ [so that the product $\dot{X}$, too, is compact and convex in a locally convex Hausdorff topological vector space];

3.0.2. $\dot{Y} \neq \emptyset$ is closed and convex subset of a locally convex Hausdorff topological vector space;

3.0.3. $\underline{\dot{Y}} \in [\dot{Y}]$ is compact and convex;

3.0.4. $\dot{\delta} : \dot{X} \times \dot{Y} \to \dot{Y}$ is continuous and, for each $\dot{x}^{\dot{\alpha}} \in \dot{X}^{\dot{\alpha}}$, linear on $\dot{X}_{\dot{\alpha}} \times \{\dot{x}^{\dot{\alpha}}\} \times \dot{Y}$;

3.0.5. For each $\dot{\alpha} \in \dot{A}$, $\dot{\omega}_{\dot{\alpha}} : \dot{\underline{Y}} \to |\dot{X}_{\dot{\alpha}}|$ is a continuous map with $\dot{\omega}_{\dot{\alpha}}(\dot{\underline{y}})$ convex for each $\dot{\underline{y}} \in \dot{\underline{Y}}$;

3.0.6. For each $\dot{\alpha} \in \dot{A}$, $\dot{u}_{\dot{\alpha}} : {}_n\dot{Y} \times \dot{\underline{Y}} \times {}_n\dot{Y} \to R$ is continuous and, for each $({}_n\dot{x}^{\dot{\alpha}}, \dot{\underline{y}}) \in {}_n\dot{X}^{\dot{\alpha}} \times \dot{\underline{Y}}$, quasi-concave on ${}_n\dot{X} \times \{{}_n\dot{x}^{\dot{\alpha}}, \dot{\underline{y}}\} \times {}_n\dot{Y}$;

3.0.7. For each $\dot{\alpha} \in \dot{A}$, $\dot{t}_{\dot{\alpha}} : \dot{X} \times \dot{Y} \to |\dot{X}_{\dot{\alpha}}|$ is a continuous map for which the section ${}^k\Gamma_{\dot{\alpha}}(\dot{x}^{\dot{\alpha}}) = \{(\dot{x}_{\dot{\alpha}}, \dot{x}^{\dot{\alpha}}, \dot{y}, \dot{x}'_{\dot{\alpha}}) \in \dot{X}_{\dot{\alpha}} \times \{\dot{x}^{\dot{\alpha}}\} \times {}^k\bar{\dot{D}} \times \dot{X}_{\alpha} | \dot{x}'_{\dot{\alpha}} \in \dot{t}_{\dot{\alpha}}(\dot{x}_{\dot{\alpha}}, \dot{x}^{\alpha}, \dot{y})\}$ is convex for each $\dot{x}^{\dot{\alpha}} \in \dot{X}^{\dot{\alpha}}$ and each $k \in N$, where ${}^k\bar{\dot{D}} \supset \dot{Y}$ is the closed convex hull of the set
${}^k\dot{D} = \{\dot{y} \in \dot{Y} |$ for some $({}_n\dot{x}, \dot{\underline{y}}) \in {}_n\dot{X}$ $\dot{\underline{Y}}$, $\dot{y} = {}^{k-1}\dot{\delta}({}_n\dot{x}, \dot{y})\}$.

Then

(a) $\dot{S}$ is a simple dynamic game;

(b) for each $\dot{\alpha} \in \dot{A}$, $\dot{\alpha}$ maps ${}_n\dot{X}^{\dot{\alpha}} \times \dot{\underline{Y}}$ upper semi-continuously into $CQ({}_n\dot{X}_{\dot{\alpha}})$; and

(c) the contractual set of $\dot{S}$ - in fact, of every restriction of $\dot{S}$ to a compact, convex $\dot{\underline{Y}}' \subset \dot{\underline{Y}}$ - is non-empty and compact.

Proof: (ad (a) and (b)): By Proposition 2.5, $\dot{S}$ will be a well-defined s.d.g. and $\dot{\alpha} \in \dot{A}$ will have the asserted continuity and convexity properties if the

derived economy of $\dot{S}$ fulfills the hypotheses (1.7.1-1.7.5) of Theorem 1.7. Let $S = \langle W,U,T,A\rangle$ be the derived economy of $\dot{S}$. We verify each of the hypothesis 1.7.1-5 separately.

(ad 1.7.1): 1.7.1 is fulfilled in view of 3.0.1 and 3.0.3.

(ad 1.7.2): 1.7.2 follows for $\alpha \in A\setminus\{a\}$ from 3.0.6., continuity of ${}_n\dot{\delta}$, and linearity of ${}_n\dot{\delta}$ on ${}_n\dot{X}_{\dot{\alpha}} \times \{{}_n\dot{x}^{\dot{\alpha}}\} \times \dot{\underline{Y}}$ for each ${}_n\dot{x}^{\dot{\alpha}} \in {}_n\dot{X}^{\dot{\alpha}}$, which, in turn, follows from the continuity and linearity properties of $\dot{\delta}$ (3.0.4). For $\alpha = a$, 1.7.2 is clearly satisfied since u_a is a constant map (2.5.2).

(ad 1.7.3): For $\alpha = a$, 1.7.3 follows trivially from 3.0.3. For $\alpha \in A\setminus\{a\}$, 1.7.3 requires for each $\dot{\alpha} \in \dot{A}$ that the graph

$${}_n\dot{T}_{\dot{\alpha}} = \{({}_n\dot{x}_{\dot{\alpha}}, {}_n\dot{x}^{\dot{\alpha}}, \dot{\underline{y}}, {}_n\dot{x}'_{\dot{\alpha}}) \mid ({}_n\dot{x}_{\dot{\alpha}}, {}_n\dot{x}^{\dot{\alpha}}, \dot{\underline{y}}) \in {}_n\dot{X} \times \dot{\underline{Y}},$$

$${}_n\dot{x}'_{\dot{\alpha}} \in {}_n\dot{t}_{\dot{\alpha}}({}_n\dot{x}_{\dot{\alpha}}, {}_n\dot{x}^{\dot{\alpha}}, \dot{\underline{y}})\} \subset {}_n\dot{X} \times \dot{\underline{Y}} \times {}_n\dot{X}_{\dot{\alpha}}$$

of ${}_n\dot{t}_{\dot{\alpha}}$ be closed and that the graph ${}_n\dot{T}_{\dot{\alpha}}({}_n\dot{x}^{\dot{\alpha}}, \dot{\underline{y}}) =$ $\{({}_n\dot{x}_{\dot{\alpha}}, {}_n\dot{x}'_{\dot{\alpha}}) \in {}_n\dot{X}_{\dot{\alpha}} \times {}_n\dot{X}_{\dot{\alpha}} \mid {}_n\dot{x}'_{\dot{\alpha}} \in {}_n\dot{t}_{\dot{\alpha}}({}_n\dot{x}_{\dot{\alpha}}, {}_n\dot{x}^{\dot{\alpha}}, \dot{\underline{y}})\}$ be convex for each $({}_n\dot{x}^{\dot{\alpha}}, \dot{\underline{y}}) \in {}_n\dot{X}^{\dot{\alpha}} \times \dot{\underline{Y}}$.

Let $\dot{\alpha} \in \dot{A}$ be given and note that, by compactness of ${}_n\dot{X} \times \dot{\underline{Y}}$ and ${}_n\dot{X}_{\dot{\alpha}}$, ${}_n\dot{T}_{\dot{\alpha}}$ is closed iff ${}_n\dot{t}_{\dot{\alpha}}$ is upper semi-continuous (see A.2). But ${}_n\dot{t}_{\dot{\alpha}}$ will be upper semi-continuous if each of its components ${}^k\dot{t}_{\dot{\alpha}}$ is so, $k + 1 \in N$. In fact,

since ${}^{k}\dot{t}_{\dot{\alpha}}$ is a composition (see 2.0.7) of $\dot{t}_{\dot{\alpha}}$ (continuous by 3.0.7), projection $\pi_{k\dot{X}}$, and ${}^{k-1}\dot{\delta}$ (continuous by 2.0.4, 3.0.4), ${}^{k}\dot{t}_{\dot{\alpha}}$ is continuous, hence upper semi-continuous, $k \in N$. Also, ${}^{o}\dot{t}_{\dot{\alpha}}$ is continuous by 2.0.7 and 3.0.5. Thus, ${}_{n}\dot{t}_{\dot{\alpha}}$ is upper semi-continuous and ${}_{n}\dot{T}_{\dot{\alpha}}$ is closed.

We turn now to the convexity of the sections ${}_{n}\dot{T}_{\dot{\alpha}}$ $({}_{n}\dot{x}^{\dot{\alpha}}, \dot{\underline{y}})$. Fix $({}_{n}\dot{x}^{\dot{\alpha}}, \dot{\underline{y}}) \in {}_{n}\dot{X}^{\dot{\alpha}} \times \dot{\underline{Y}}$. We first verify that the sections ${}^{k}\dot{T}_{\dot{\alpha}}$ $({}_{n}\dot{x}^{\dot{\alpha}}, \dot{\underline{y}})$ are convex $(k + 1 \in N)$, where ${}^{k}\dot{T}_{\dot{\alpha}}$ $({}_{n}\dot{x}^{\dot{\alpha}}, \dot{\underline{y}}) = \{({}_{n}\dot{x}_{\dot{\alpha}}, {}_{n}\dot{x}^{\dot{\alpha}}, \dot{\underline{y}}, \dot{x}'_{\dot{\alpha}}) \in {}_{n}\dot{X}_{\dot{\alpha}} \times \{{}_{n}\dot{x}^{\dot{\alpha}}, \dot{\underline{y}}\} \times \dot{X}_{\dot{\alpha}} \mid \dot{x}'_{\dot{\alpha}} \in {}^{k}\dot{t}_{\dot{\alpha}}$ $({}_{n}\dot{x}_{\dot{\alpha}}, {}_{n}\dot{x}^{\dot{\alpha}}, \dot{\underline{y}})\} \subset {}_{n}\dot{X} \times \dot{\underline{Y}} \times \dot{X}_{\dot{\alpha}}$. That ${}^{O}\dot{T}_{\dot{\alpha}}$ $({}_{n}\dot{x}^{\dot{\alpha}}, \dot{\underline{y}})$ is convex is direct from the definition (2.0.7) of ${}^{o}\dot{t}_{\dot{\alpha}}$ and 3.0.5. So, let $k \in \{1,\ldots,n - 1\}$. Using 2.0.4 and 3.0.4., it is straightforward to verify that ${}^{k-1}\dot{\delta}$, which is into ${}^{k}\bar{\dot{D}} \subset \dot{Y}$, is linear on ${}_{n}\dot{X}_{\dot{\alpha}} \times \{{}_{n}\dot{x}^{\dot{\alpha}}\} \times \dot{Y}$. Now the convexity of ${}^{k}\Gamma_{\dot{\alpha}}$ $(\dot{x}^{\dot{\alpha}})$ (see 3.0.7) allows the reader to compute that ${}^{k}\dot{T}_{\dot{\alpha}}$ $({}_{n}\dot{x}^{\dot{\alpha}}, \dot{\underline{y}})$ is convex, noting, of course, that projection (in particular, $\pi_{k\dot{X}}$) is linear and recalling the definition (in 2.0.7) of ${}^{k}\dot{t}_{\dot{\alpha}}$. To see that ${}_{n}\dot{T}_{\dot{\alpha}}$ $({}_{n}\dot{x}^{\dot{\alpha}}, \dot{\underline{y}})$ is convex, observe that

$$\{{}_{n}\dot{x}_{\dot{\alpha}}, {}_{n}\dot{x}^{\dot{\alpha}}, \dot{\underline{y}}\} \times \prod_{k=o}^{n-1} {}^{k}\dot{t}_{\dot{\alpha}} \; ({}_{n}\dot{x}_{\dot{\alpha}}, {}_{n}\dot{x}^{\dot{\alpha}}, \dot{\underline{y}})$$

$= \{{}_{n}\dot{x}_{\dot{\alpha}}, {}_{n}\dot{x}^{\dot{\alpha}}, \dot{\underline{y}}\} \times {}_{n}\dot{t}_{\dot{\alpha}}$ $({}_{n}\dot{x}_{\dot{\alpha}}, {}_{n}\dot{x}^{\dot{\alpha}}, \dot{\underline{y}})$ and use the fact that ${}^{k}\dot{T}_{\dot{\alpha}}$ $({}_{n}\dot{x}^{\dot{\alpha}}, \dot{\underline{y}})$ is convex for each $k + 1 \in N$.

(<u>ad</u> 1.7.4): 1.7.4 follows directly from 2.5.4.

(ad 1.7.5) : The map $\hat{t}_a$, being a constant map (see 1.0.4, 2.5.3), is certainly lower semi-continuous. Now, fix $\alpha \in \Lambda\backslash\{a\}$. To establish the lower semi-continuity of $\hat{t}_\alpha$, we first note that by 2.5.3, $\hat{t}_\alpha = {}_n\hat{\dot{t}}_{\dot{\alpha}}$. Now let $V = \prod_{k\in N} {}^kV \subset {}_n\dot{X}_{\dot{\alpha}}$ be any basic open set in the product topology of ${}_n\dot{X}_{\dot{\alpha}}$. It suffices to show that

$$U = \{({}_n\dot{x}^{\dot{\alpha}}, \dot{\underline{y}}) \mid V \cap {}_n\hat{\dot{t}}_{\dot{\alpha}}({}_n\dot{x}^{\dot{\alpha}}, \underline{y}) \neq \emptyset\}$$

is open. This being trivially so when $V = \emptyset$, we assume henceforth that $V \neq \emptyset$. It is useful to define $Q = {}_n\dot{X}^{\dot{\alpha}} \times \dot{\underline{Y}}$ and, using the definition (2.0.8) of ${}_n\hat{\dot{t}}_{\dot{\alpha}}$, to rewrite U as $U = \{q \in Q \mid$ for some ${}_n\dot{x}_{\dot{\alpha}} \in V,\ k \in N \Rightarrow {}^k\dot{x}_{\dot{\alpha}} \in {}^{k-1}\dot{t}_{\dot{\alpha}}({}_n\dot{x}_{\dot{\alpha}}, q)\}$.

We now proceed to show that U contains a nbd of each of its points. Toward that, suppose $\hat{q} \in U$. We will construct a set $\hat{U}$ and complete our proof by showing that $\hat{U}$ is a nbd of $\hat{q}$ contained in U.

Fix ${}_n\bar{\dot{x}}_{\dot{\alpha}} \in {}_n\dot{X}_{\dot{\alpha}}$, and define ${}^oP = \{{}_n\bar{\dot{x}}_{\dot{\alpha}}\}$ and, for each $k\in N$, ${}^kP = \dot{X}_{\dot{\alpha}}$, writing[4] ${}^oP \times {}^1P \times \ldots \times {}^kP = {}_kP$, with generic elements ${}^kp \in {}^kP$ and ${}_kp \in {}_kP$ $(k \in N_o)$. For each $k + 1 \in N$, define the function ${}^k\chi : {}_n\dot{X}_{\dot{\alpha}} \times {}_k\dot{X}_{\dot{\alpha}} \to {}_n\dot{X}_{\dot{\alpha}}$ by ${}^k\chi({}_n\dot{x}'_{\dot{\alpha}}, {}_k\dot{x}_{\dot{\alpha}}) = ({}_k\dot{x}_{\dot{\alpha}}, {}^{k+1}\dot{x}'_{\dot{\alpha}}, \ldots, {}^n\dot{x}'_{\dot{\alpha}})$, $({}_n\dot{x}'_{\dot{\alpha}}, {}_k\dot{x}_{\dot{\alpha}}) \in {}_n\dot{X}_{\dot{\alpha}} \times {}_k\dot{X}_{\dot{\alpha}}$, and the map ${}^k\tau : {}_kP \times Q \to [{}^{k+1}P]$ by

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4) The reader will, please, excuse our momentary departure, in defining ${}_kP$, from our usual notational convention as announced in 0.0.

$${}^{k}\tau({}_{k}p,q) = {}^{k}\dot{t}_{\dot{\alpha}}({}^{k}\chi({}_{k}p),q) \qquad ({}_{k}p,q) \in {}_{k}P \times Q.$$

From 2.0.7 we see that, for any $k+1 \in N$, and for any $({}_{n}\dot{x}_{\dot{\alpha}},q) \in {}_{n}\dot{X}_{\dot{\alpha}} \times Q$, by writing ${}_{k}p = ({}_{n}\bar{\dot{x}}_{\dot{\alpha}},{}_{k}\dot{x}_{\dot{\alpha}})$, one obtains ${}^{k}\dot{t}_{\dot{\alpha}}({}_{n}x_{\dot{\alpha}},q) = {}^{k}\tau({}_{k}p,q)$.

From this we are able once more to rewrite U as

$$U = \{q\epsilon Q \mid \text{ for some } ({}^{1}p,\ldots,{}^{n}p)\epsilon V, k\epsilon N \Rightarrow {}^{k}p \in {}^{k-1}\tau({}_{k-1}p,q)\}.$$

Since $\hat{q} \in U$, $\exists\ ({}^{1}\hat{p},\ldots,{}^{n}\hat{p}) \in V$ with ${}^{k}\hat{p} \in {}^{k-1}\tau({}_{k-1}\hat{p},\hat{q})$ for every $k \in N$. Also, for each $k+1 \in N$, ${}^{k}\tau$ is continuous, since ${}^{k}\dot{t}_{\dot{\alpha}}$ is so (as seen in the verification ad 1.7.3, above) and since ${}^{k}\chi$ is obviously so. Thus, for each $k+1 \in N$, ${}^{k}\tau$ is certainly lower semi-continuous.

To construct $\hat{U}$, set ${}^{k}V_{n} = {}^{k}V$ $(k \in N)$, and, for each $k+1 \in N$ (following the order $k = n-1, n-2,\ldots,0$), inductively define open sets ${}^{k}W$, ${}^{j}V_{k}$ $(j\epsilon\{1,\ldots,k\})$, and ${}^{k}U$, obeying:

$${}^{k}W = \{({}^{1}p,\ldots,{}^{k}p,q) \mid (\bigcap_{j=k+1}^{n} {}^{k+1}V_{j}) \cap {}^{k}\tau({}_{k}p,q) \neq \emptyset\}$$

with

$$({}^{1}\hat{p},\ldots,{}^{k}\hat{p},\hat{q}) \in (\prod_{j=1}^{k} {}^{j}V_{k}) \times {}^{k}U \subset {}^{k}W, \quad {}^{j}V_{k} \subset {}^{k}P, \quad {}^{k}U \subset Q.$$

[To check that this is possible, we use the following facts inductively (in the order $k = n-1, n-2,\ldots,0$) : (i) ${}^{k}\tau$ is

lower semi-continuous and $\bigcap_{j=k+1}^{n} {}^{k+1}V_j$ is open, so ${}^{k}W$ is open; (ii) ${}^{k+1}\hat{p} \in \bigcap_{j=k+1}^{n} {}^{k+1}V_j$, and ${}^{k+1}p \in {}^{k}\tau({}_{k}\hat{p},\hat{q})$, so $({}^{1}\hat{p},\ldots,{}^{k}\hat{p},\hat{q}) \in {}^{k}W$; (iii) using (i) and (ii), $\prod_{j=1}^{k} {}^{j}V_k \times {}^{k}U \subset {}^{k}W$ is chosen as a basic open set in the product topology of ${}^{1}P\times\ldots\times{}^{k}P \times Q$.

Set $\hat{U} = \bigcap_{k=o}^{n-1} {}^{k}U$. From its construction, it is clear that $\hat{U}$ is open and that $\hat{q} \in \hat{U}$. It remains to show only that $\hat{U} \subset U$. For this, we take any $q \in \hat{U}$ and show $q \in U$ by constructing a point $({}^{1}\tilde{p},\ldots,{}^{n}\tilde{p}) \in V$ such that ${}^{k+1}\tilde{p} \in {}^{k}\tau({}_{k}\tilde{p},\tilde{q})$ for every $k + 1 \in N$. Since $\tilde{q} \in \hat{U} \subset {}^{O}U \subset {}^{O}W$, we have $(\bigcap_{j=1}^{n} {}^{1}V_j) \cap {}^{O}\tau({}^{O}p,\tilde{q}) \neq \emptyset$. Choose ${}^{1}\tilde{p}$ as any point of this intersection. Now assume that, for $\tilde{k} + 1 \in N$, ${}^{k}\tilde{p} \in (\bigcap_{j=k}^{n} {}^{k}V_j) \cap {}^{k-1}\tau({}_{k-1}\tilde{p},\tilde{q})$ is chosen for every $k \in \{1,\ldots,\tilde{k}\}$. Then, ${}^{k}\tilde{p} \in \bigcap_{j=k}^{n} {}^{k}V_j$ $(k \in \{1,\ldots,\tilde{k}\})$ and $\tilde{q} \in \hat{U} \subset {}^{\tilde{k}}U$ being clear, it follows that $({}^{1}\tilde{p},\ldots,{}^{\tilde{k}}\tilde{p},\tilde{q}) \in (\prod_{j=1}^{\tilde{k}} {}^{j}V_k) \times {}^{\tilde{k}}U \subset {}^{\tilde{k}}W$. Now ${}^{\tilde{k}}W \neq \emptyset$, so that $(\bigcap_{j=\tilde{k}+1}^{n} {}^{\tilde{k}+1}V_j) \cap {}^{\tilde{k}}\tau({}_{\tilde{k}}\tilde{p},\tilde{q})$ is non-empty and affords an element ${}^{\tilde{k}+1}\tilde{p}$. This shows that the desired $({}^{1}\tilde{p},\ldots,{}^{n}\tilde{p})$ exists, so that $\hat{U} \subset U$, establishing that S satisfies 1.7.5, and completing the verification of the hypotheses of theorem 1.7.

(ad c) That $\dot{S}$ has a non-empty and compact contractual set equilibrium is now a direct consequence of theorem 1.7 and proposition 2.5 (b). That any restriction of $\dot{S}$ to a (non-empty) compact and convex set $\underline{Y}' \subset \underline{Y}$ has a non-empty and compact contractual set follows from 2.2 and a simple check that the restriction $\dot{S}'$ of $\dot{S}$ so obtained satisfies 3.0.1 - 3.0.7 (where only 3.0.3 is involved). This completes the proof.

3.1. Remark: If, in 3.0, one takes the restriction of $\dot{S}$ to the initial state $\dot{\underline{y}} \in \dot{\underline{Y}}$, then 3.0 (c) implies that $\dot{S}(\dot{\underline{y}})$ has an equilibrium for all $\dot{\underline{y}} \in \dot{Y}$, i.e., for each $\dot{\underline{y}} \in \dot{\underline{Y}}$, there exists $_n\hat{\dot{x}} \in {}_n\hat{\dot{X}}$ satisfying 2.4.1 with $\hat{\dot{\underline{y}}} = \dot{\underline{y}}$. Thus, under the hypotheses of the last proposition, there is an equilibrium plan for each feasible initial state. Proposition 3.0 also shows that there exist well-defined simple dynamic games, so that 2.0 is not a self-contradiction. Of course, this well-definedness holds under much weaker conditions than 3.0.1-7. Clearly, $_n\hat{t}_{\dot{\alpha}}$ will be a continuous map of $_n\dot{X}^{\dot{\alpha}} \times \dot{\underline{Y}}$ into $C(_n\dot{X}_{\dot{\alpha}})$ if $\dot{\omega}_{\dot{\alpha}}$, $t_{\dot{\alpha}}$, and $\dot{\delta}$ are only assumed continuous without any additional convexity or linearity assumptions. Given this, and recalling 1.2 and 2.5, the derived economy S of $\dot{S}$ (and, hence, $\dot{S}$ itself) will be well-defined if 3.0.1-3 hold, if $\dot{\delta}$ is continuous, and if, for each $\dot{\alpha} \in \dot{A}$, $\dot{\omega}_{\dot{\alpha}}$, $t_{\dot{\alpha}}$, and $\dot{u}_{\dot{\alpha}}$ are continuous.

3.2. <u>Remark</u>: In the simple dynamic games we have considered so far, the feasible control region to which the $\dot{\alpha}$-control ${}^{k+1}\dot{x}_{\dot{\alpha}}$ is constrained to belong depends on the previous control ${}^{k}\dot{x}$ and state ${}^{k-1}\dot{y}$. (See the discussion immediately preceding 2.2). The reader will note, however, that the case where the control region in question also depends on ${}^{k}\dot{y}$ falls naturally under the case that we consider generally. Below we illustrate this by considering a dependence of these regions on on ${}^{k}\dot{y}$, and, while this dependence is on ${}^{k}\dot{y}$ alone, the reader will easily be able to extend the simple idea involved to the general case where the dependence on ${}^{k}\dot{x}$ and ${}^{k-1}\dot{y}$ is not through ${}^{k}\dot{y}$ alone.

Suppose $\dot{S} = < \dot{W}, \dot{Y}, \dot{\underline{Y}}, \dot{\delta}, \dot{\omega}, \dot{U}, \dot{T}, \dot{A} >$ satisfies 3.0.1-4 and 3.0.6 and that, for each $\dot{\alpha} \in \dot{A}$, $\dot{\omega}_{\dot{\alpha}} = \Psi_{\dot{\alpha}}$ and $\dot{t}_{\dot{\alpha}} = \Psi_{\dot{\alpha}} \circ \dot{\delta}$, where $\Psi_{\dot{\alpha}} : \dot{Y} \rightarrow [\dot{X}_{\dot{\alpha}}]$ satisfies

3.2.1. $\Psi_{\dot{\alpha}}$ is a continuous map for which $\Psi_{\dot{\alpha}}(\dot{y}) \in CO(\dot{X}_{\dot{\alpha}})$ for all $\dot{y} \in \dot{Y}$, with ${}^{k}\Gamma(\Psi_{\dot{\alpha}}) = \{(\dot{x}_{\dot{\alpha}}, \dot{y}) \in \dot{X}_{\dot{\alpha}} \times {}^{k}\bar{D} \mid \dot{x}_{\dot{\alpha}} \in \Psi_{\dot{\alpha}}(\dot{y})\}$ convex $(k \in N)$, where ${}^{k}\bar{D}$ is as in 3.0.7.

The fashion in which the $\dot{\alpha}$-controls ${}^{k+1}\dot{x}_{\dot{\alpha}}$ are constrained in $\dot{S}$ to their respective control regions is, hence, as follows:

$${}^{k+1}\dot{x}_{\dot{\alpha}} \in \Psi_{\dot{\alpha}}({}^{k}\dot{y}), \quad k+1 \in N, \quad \dot{\alpha} \in \dot{A}.$$

Moreover, it is straightforward to check that $\dot{S}$ satisfies 3.0.5 and 3.0.7; so that the conclusions of Proposition 3.0 are valid.

The following example illustrates the application of 3.0 to a salient class of deterministic dynamic games.

3.3. <u>Example</u>: Let $\dot{A}$ be a finite set, and let $\{m_{\dot{\alpha}} \in N | \dot{\alpha} \in \dot{A}\}$, $\{\ell_{\dot{\alpha}} \in N | \dot{\alpha} \in \dot{A}\}$ and $m \in N$ be given. We specify the elements of a s.d.g. $\dot{S} = < \dot{W}, \dot{Y}, \underline{\dot{Y}}, \dot{\delta}, \dot{\omega}, \dot{U}, \dot{T}, \dot{A} >$ as follows:

3.3.1. $X_{\dot{\alpha}} \neq \emptyset$ is a compact, convex subset of ${}_{m_{\dot{\alpha}}}R$ $(\dot{\alpha} \in \dot{A})$;

3.3.2. $\dot{Y} = {}_{m}R$;

3.3.3. $\underline{\dot{Y}} \in [\dot{Y}]$ is compact and convex;

3.3.4. $\dot{\delta}$ satisfies 3.0.4;

3.3.5. $\dot{\omega}_{\dot{\alpha}} : \underline{\dot{Y}} \to [\dot{X}_{\dot{\alpha}}]$ is defined by $\dot{\omega}_{\dot{\alpha}}(\underline{\dot{y}}) = \Psi_{\dot{\alpha}}(\dot{y})$, $\underline{\dot{y}} \in \dot{Y}$, where $\Psi_{\dot{\alpha}} : \dot{Y} \to C(X_{\dot{\alpha}})$ is defined by $\Psi_{\dot{\alpha}}(y) =$

$\{\dot{x}_{\dot\alpha} \in \dot{X}_{\dot\alpha} | \psi_{\dot\alpha}(\dot{x}_{\dot\alpha},\dot{y}) \leq 0\}$, in which $\psi_{\dot\alpha} : X_{\dot\alpha} \times Y \to {}_{\ell_{\dot\alpha}}P$ is a given continuous function, $(\dot\alpha \in \dot{A})$;

3.3.6. $\dot{t}_{\dot\alpha} : \dot{X} \times \dot{Y} \to [\dot{X}_{\dot\alpha}]$ is defined for each $\dot\alpha \in \dot{A}$ by

$\dot{t}_{\dot\alpha}(\dot{x},\dot{y}) = \Psi_{\dot\alpha}(\dot\delta(\dot{x},\dot{y}))$, $(\dot{x},\dot{y}) \in \dot{X} \times \dot{Y}$;

3.3.7. $u_{\dot\alpha} : {}_nX \times \underline{Y} \times {}_nY \to P$ is defined for each $\dot\alpha \in \dot{A}$ and

$({}_n\dot{x},\dot{\underline{y}},{}_n\dot{y}) \in {}_n\dot{X} \times \dot{\underline{Y}} \times {}_n\dot{Y}$ by $\dot{u}({}_n\dot{x},\dot{\underline{y}},{}_n\dot{y}) =$

${}^nf_{\dot\alpha}({}^n\dot{y}) + \sum_{k=o}^{n-1} {}^kf_{\dot\alpha}({}^{k+1}\dot{x}, {}^k\dot{y})$, where ${}^o\dot{y} \in \dot{\underline{Y}}$, and where

$\{{}^k\dot{f}_{\dot\alpha} : \dot{X} \times \dot{Y} \to P | k + 1 \in N, \dot\alpha \in \dot{A}\}$ and $\{{}^nf_{\dot\alpha} : \dot{Y} \to P | \dot\alpha \in \dot{A}\}$

are each a family of continuous and concave functions;

3.3.8. $\dot\alpha$ satisfies 2.0.8 $(\dot\alpha \in \dot{A})$.

Suppose, in addition, that the following condition holds.

3.3.9. For all $\dot\alpha \in \dot{A}$, $\Psi_{\dot\alpha}$ satisfies 3.2.1. [Note that these conditions on $\Psi_{\dot\alpha}$ will be satisfied if $\psi_{\dot\alpha}$ is linear or if $\psi_{\dot\alpha}$ satisfies: (i) $\psi_{\dot\alpha} : X_{\dot\alpha} \times \dot{Y} \to {}_{\ell_{\dot\alpha}}P$ is continuous; (ii) for each $\dot{y} \in \dot{Y}$, $\psi_{\dot\alpha}$ is convex on $\dot{X}_{\dot\alpha} \times \{\dot{y}\}$;

(iii) for all $\dot{y} \in \dot{Y}$, there exists $\dot{x}_{\dot{\alpha}} \in \dot{X}_{\dot{\alpha}}$ with $\psi_{\dot{\alpha}}(\dot{x}_{\dot{\alpha}}, \dot{y}) \leq 0$; and (iv) $\psi_{\dot{\alpha}}$ is convex on $X_{\dot{\alpha}} \times {}^{k}\bar{\dot{D}}$, $k \in N$. See, e.g. Hogan [1971], Theorems 10,12.]

It is easily verified that $\dot{S}$, as specified above, satisfies 3.0.1-3.0.7. Therefore, by 3.0, and recalling 2.4 and 3.1, for each $\hat{\dot{y}} \in \underline{\dot{Y}}$, there exists an ${}_{n}\hat{\dot{x}} \in {}_{n}\dot{X}$ satisfying, for all $\dot{\alpha} \in \dot{A}$, $u_{\dot{\alpha}}({}_{n}\hat{\dot{x}}, \hat{\underline{\dot{y}}}, {}_{n}\delta({}_{n}\hat{\dot{x}}, \hat{\dot{y}})) =$

$$\underset{{}_{n}\dot{X}_{\dot{\alpha}} \times {}_{n}\dot{Y}}{\text{Maximum}} \left({}^{n}f_{\dot{\alpha}}({}^{n}\dot{y}) + \sum_{k=0}^{n-1} {}^{k}f_{\dot{\alpha}}\left({}^{k+1}\dot{x}_{\dot{\alpha}}, {}^{k+1}\hat{\dot{x}}^{\dot{\alpha}}, {}^{k}\dot{y}\right)\right)$$

Subject to:

$${}^{k}\dot{y} = \dot{\delta}({}^{k}\dot{x}_{\dot{\alpha}}, {}^{k}\hat{\dot{x}}^{\alpha}, {}^{k-1}\dot{y}), \quad {}^{0}\dot{y} = \hat{\underline{\dot{y}}}, \qquad k \in N;$$

$$\psi_{\dot{\alpha}}({}^{k}\dot{x}_{\dot{\alpha}}, {}^{k-1}\dot{y}) \leq 0, \quad {}^{k}\dot{x}_{\dot{\alpha}} \in \dot{X}_{\dot{\alpha}}, \qquad k \in N.$$

4. Limitations and Extensions

Our main results are the equilibrium existence result for simple economies (1.7), the lemma (2.5) reducing simple dynamic games to simple economies, and the equilibrium existence result (3.0) for simple dynamic games.

The reader may have been struck by our somewhat curious obstinacy in always referring to the objects we treat as "simple". Here we try to indicate the deficiencies in the above formulation which gave rise to this standing qualification.

For simple economies, these deficiencies amount to our having ignored the following:

1. informational and/or perceptional imperfections on the part of behavors;

2. the usefulness of specifying "effective" preferences (utility functions) as dependent upon a design parameter which may be called an "incentive scheme";

3. the fact that, in practice, the feasibility transformations depend not only on the behavior chosen from a given feasibility, but also on the given feasibility itself - the resources one has tomorrow depend not only on what one does today, but also on the resources available today.

From the viewpoint of equilibrium existence, these ommissions turn out not to be crucial, since it can be shown, under fairly weak assumptions, that generalized economies (called "social systems" in [Sertel, 1971]), including the above features, retain the property of having non-empty, compact sets of equilibria (see [Sertel, 1971]). Nonetheless, from the viewpoint of social system design (e.g., selecting an appropriate incentive scheme), these characteristics are clearly essential elements of the problem.

Another element of simplicity in our work and the previous literature is the assumption that the collective feasibility is box-shaped, i.e., that what is currently feasible for α is independent of the current α-exclusive behavior. One would, of course, like to remove this restriction, as many of the problems of central planning (and control theory) are not completely decomposable on the constraint side, even by "dual" methods (incentive schemes).

Turning now to dynamic games, those treated here suffer from the same restrictions discussed above for simple economies. Moreover, there appear to be significant difficulties in interpreting the informational and behavioral connotations implied by the evolutions (and their equilibria) of these simple dynamic games. In fact, in the current formulation, each player $\dot{\alpha} \in \dot{A}$ computes his $\dot{\alpha}$-plan given an $\dot{\alpha}$-exclusive plan. This can be likened to the situation in an auction hall, where each player calls out his projected plan, and recomputes

his plan based on the projected (i.e. called out) plans of all other players. The equilibrium existence question posed, then, is simply whether there exists a plan which is repeatable in the sense of being called out twice in succession. We find it difficult, however, to interpret the equilibrium of this "auction" process, unless $n = 1$, in the case where plans are exhibited only as they are enacted; i.e., where equilibria are to be interpreted as repeatable in the sense of enactable twice in a row.[5]

Finally, as may be seen from 2.4, the equilibria whose existence is established here correspond to open loop equilibria of dynamic games as they are usually defined (see, e.g., [Ho, 1970]). However, the existence of closed loop equilibria can also be studied by similar methods. Essentially, one starts with a given simple dynamic game $\dot{S}$ and derives from it a dynamic game $\dot{S}'$ by replacing the $\dot{\alpha}$-control spaces ${}^{k}\dot{Y}_{\dot{\alpha}}$ of $\dot{S}$ by the function space of strategy maps (or control laws) from the observed state and control history (until time k) to ${}^{k}X_{\dot{\alpha}}$. The (non-trivial) issue to be resolved is the determination of conditions under which the dynamic game $\dot{S}'$ resulting from this transformation will inherit from $\dot{S}$ the convexity, linearity, continuity, and compactness assumptions required by Proposition 3.0 - or, more generally,

5) For a discussion of some aspects of the meaningfulness of equilibrium in dynamic games, see [Starr & Ho, 1969a, 1969b].

through an appropriate reduction, the assumptions required by Theorem 1.7. In principle, the same approach links the above results to stochastic dynamic games, though clearly much work remains to be done in specifying the transformations required for reducing these to the simple economies and dynamic games introduced here.

A. Appendix

In the interest of accessibility, here we collect a minimal amount of topological information concerning certain maps whose values or arguments are non-empty sets. In what follows, P and Q are topological spaces. Our first definition is extracted from [Michael, 1951], of course. (N.B. For lsf, usf and finite topologies on a hyperspace, see [Michael, 1951].)

A.1. Definition: Let $F : P \to [Q]$ be a map. We say that F is lower semi-continuous (lsc) [resp., upper semi-continuous (usc)] iff it is continuous when $[Q]$ is given the lower semi-finite (lsf) [resp., upper semi-finite (usf)] topology. [Equivalently, we say that F is lsc (resp., usc) iff the set

$$\{p \in P \mid F(p) \cap V\}$$

is open (resp., closed) in P whenever $V \subset Q$ is open (resp., closed).] We say that F is continuous iff it is continuous when $[Q]$ is given the finite topology. [Equivalently, we say that F is continuous iff it is both lsc and usc.]

The proof of the following "closed graph theorem" may be obtained, for example, by combining propositions 3.2 and 3.3 in [Prakash & Sertel, 1970], where 3.3 is also Lemma 2 of [Fan, 1952]:

A.2. Proposition: Assume that P and Q are both compact Hausdorff, and let $f : P \to Q$, $F : P \to C(Q)$ be two maps. Denote the graphs of f and F by g and G, respectively:

$$g = \{(p,q) \mid p \in P,\ q = f(p)\},$$
$$G = \{(p,q) \mid p \in P,\ q \in F(p)\}.$$

It is iff g (resp., G) $\subset P \times Q$ is closed that f is continuous (resp., F is usc).

The following fact, which is a part of corollary 3.1.4.3 in [Sertel, 1971], turns out to be a useful tool basic to optimization.

A.3. Proposition: Let $P \times Q$ be non-empty and compact Hausdorff, and let $u : P \times Q \to R$ be a continuous real-valued function. Define $w : C(P) \times Q \to R$ by

$$w(d,q) = \sup_{p \in d} u(p,q) \qquad (d \in C(P),\ q \in Q).$$

Equip $C(P)$ with the finite topology. Then w is continuous.

We have the following obvious

A.4. Corollary: Let all be as in A.3, and let $s : Q \to C(P)$ be continuous. Defining $v : Q \to R$ by

$$v(q) = w(s(q), q) \qquad (q \in Q),$$

v is then continuous.

A.5. <u>Fixed Point Theorem</u> Theorem 1; Fan 1952 : Given a locally convex Hausdorff topological vector space L, if $X \subset L$ is compact and convex and if $F : X \to CQ(X)$ is an usc transformation, then there exists a (fixed) point $x \in X$ such that $x \in F(x)$.

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