EQUILIBRIUM INTEREST RATE AND LIQUIDITY PREMIUM UNDER PROPORTIONAL TRANSACTIONS COSTS

Dimitri Vayanos
and
Jean-Luc Vila

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Massachusetts Institute of Technology

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Abstract

In this paper we analyze the impact of transactions costs on the rates of return on liquid and illiquid assets. We consider an infinite horizon economy with finitely lived agents along the lines of Blanchard (1985). In this economy agents face a constant probability of death, and the population is kept constant by an inflow of new arrivals. Agents start with no financial wealth and receive a decreasing stream of labor income over their lifetimes. In addition they can invest in long term assets which pay a constant stream of dividends. There are two such assets, the liquid asset and the illiquid asset. The liquid asset is traded without transaction costs, while trading the illiquid asset entails proportional transactions costs. Neither asset can be sold short. Agents buy and sell assets for life-cycle motives. In fact, they accumulate the higher yielding illiquid asset for long term investment purposes and the liquid asset for short term investment needs.

We find that when transactions costs increase, the rate of return on the liquid asset decreases, while the rate of return on the illiquid asset may increase or decrease. We also find, quite naturally, that the liquidity premium increases. The effects of transactions costs on the rate of return on the liquid asset and on the liquidity premium, are stronger the higher the fraction of the illiquid assets in the economy. Finally, transactions costs have first order effects on asset returns and on the liquidity premium.

We evaluate these effects for reasonable parameter values.

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1. INTRODUCTION

Although most of asset pricing theory assumes frictionless markets, transactions costs are ubiquitous in financial markets. Transactions costs can be decomposed into (i) direct transactions costs such as brokerage commissions, exchange fees and transactions taxes; (ii) bid-ask spread, (iii) market impact costs and (iv) delay and search costs. Aiyagari and Gertler (1991) report that typical (retail) brokerage costs for common stocks average 2% of the dollar amount of the trade while the bid-ask spread for actively traded stocks averages around .5%. Moreover, transactions costs vary across assets and over time. Money market accounts are clearly more liquid than stocks. In addition, deregulation as well as changes in information technology have reduced (but not eliminated) transactions costs.

Empirical work on transactions costs documents not only their magnitude but their important effect on rates of return. Amihud and Mendelson (1986) show that the risk-adjusted average return on stocks is positively related to their bid-ask spread. Even more direct evidence can be found by comparing two assets with exactly the same cash flows but different liquidity: (i) restricted ("letter") stocks which cannot be publicly traded for 2 years sell at a 35% discount below regular stocks and (ii) the average yield differential between Treasury Notes close to maturity and more liquid Treasury Bills is about .43%. More evidence is also presented in Boudoukh and Whitelaw (1991).

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1See Amihud and Mendelson (1991b).


3See Amihud and Mendelson (1991a).

4More evidence is also presented in Boudoukh and Whitelaw (1991).
returns will shed some light on some policy issues as well. Transactions taxes and differential taxation of long and short term capital gains both reduce liquidity and therefore affect assets' returns. As a result, investment decisions will change with additional welfare implications.

In this paper we analyze the impact of transactions costs on the rates of return on liquid and illiquid assets, in a general equilibrium framework. We are interested in questions such as: On what characteristics of the economy does the liquidity premium (the difference between the rates of return on illiquid and liquid assets) depend? How do transactions costs affect the rates of return on liquid and illiquid assets? Are these effects first or second order effects?

Despite their importance for asset pricing, these questions have so far not been satisfactorily addressed in the theoretical literature. A major reason is that to answer them, one has to move away from the basic model of asset pricing, namely the representative agent model. (One cannot understand the impact of trade frictions in a model where there is no trade.) Unfortunately, models with heterogeneous agents (and trade), tend to be quite intractable analytically.

Since our objective is to understand the effects of asset liquidity on asset pricing, we take risk out of the picture: All the assets that we consider (liquid or illiquid) pay a constant stream of dividends. The analysis of the joint effects of risk and liquidity is an interesting question that we leave for future research.

There are many ways to construct a deterministic economy with heterogenous agents. Agents may trade because of differences in preferences or endowments, that is, they may have different preferences for current versus future consumption, or they may have different labor income paths.\(^5\)

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\(^5\)In a stochastic economy, differential information together with liquidity shocks may also generate trade (see Wang (1992)).
In our economy both motives exist. More precisely, our economy is a tractable version of a multiperiod overlapping generations economy, the perpetual youth economy, first studied by Blanchard (1985). We believe, however that our results on the effects of transactions costs on asset returns could appear in other contexts as well.

In our model, agents face a constant probability of death (this is the key assumption that makes things tractable), and the population is kept constant by an inflow of new arrivals. Agents start with no financial wealth and receive a decreasing stream of labor income over their lifetimes. In addition they can invest in long term assets which pay a constant stream of dividends. There are two such assets, the liquid asset and the illiquid asset. The liquid asset is traded without transactions costs, while trading the illiquid asset entails proportional transactions costs. Neither asset can be sold short.

In this economy agents buy and sell assets for life-cycle motives. In fact, they accumulate the higher yielding illiquid asset for long term investment purposes and the liquid asset for short term investment needs.

We find that when transactions costs increase, the rate of return on the liquid asset decreases, while the rate of return on the illiquid asset may increase or decrease. We also find, quite naturally, that the liquidity premium increases. The effects of transactions costs on both the rate of return on the liquid asset and the liquidity premium, are stronger the higher the fraction of the illiquid asset in the economy. Finally, transactions costs have first order effects on asset returns and on the liquidity premium.

The reason why the rate of return on the liquid asset falls in response to an increase in transactions costs, can be briefly summarized as follows: Suppose that transactions costs increase from 0 to \( \epsilon \) and that the rate of return on the liquid asset stays the same in equilibrium. Then, in equilibrium, the rate of return on the illiquid asset must increase (by the liquidity premium) which implies that this asset
becomes cheaper. Agents now face better investment opportunities (they have the liquid asset at the same rate as before, and an additional investment opportunity), which leads to a uniform rise in their consumption. Moreover, they substitute consumption over time so that they buy more of the (cheaper) illiquid asset and hold it for a longer period. Thus, they will demand more securities for two reasons. The first reason is that they have to finance higher future consumption and pay transactions costs when selling the illiquid asset. The second reason is that, since the illiquid asset is cheaper, they buy more of it and hold it for a longer period. As a result, total asset demand goes up. The rate of return on the liquid asset has to fall to restore equilibrium. In addition, if there are more illiquid assets in the economy, total asset demand will increase more, and the rate of return on the liquid asset will have to fall by even more.

We cannot infer whether the rate of return on the illiquid asset will increase or decrease, by a similar reasoning. Indeed, suppose that transactions costs increase from 0 to $\varepsilon$ and that the rate of return on the illiquid asset in unaffected in equilibrium. The rate of return on the liquid asset falls by an amount equal to the liquidity premium. This time, agents face worse investment opportunities. The price of the illiquid asset is the same as before, but, in addition there are transactions costs, while the price of the liquid asset increases. Agents’ consumption shifts down uniformly. Furthermore, by substitution they accumulate less of the illiquid asset, but hold it for a longer period. They also accumulate less of the liquid asset. The effect on the total demand for securities is ambiguous. First, the future consumption to be financed is lower and the liquid securities are more expensive, but transactions costs have to be paid on the illiquid securities. Second, agents buy less of the liquid and illiquid assets, but hold the illiquid asset for a longer period.

Finally, the liquidity premium depends on the minimum holding period of the illiquid asset. It increases with the fraction of illiquid assets in the economy, since this period gets shorter.
There is a growing literature studying asset market frictions such as transactions costs, short sale constraints or borrowing constraints. This literature addresses three basic questions. The first question is to find the optimal consumption/investment policy given prices processes and imperfections. The objectives of the body of literature addressing this question are: (i) to derive the asset demand for a particular price process and (ii) to evaluate the cost that market imperfections impose upon market participants (given the price process.) The answer to (ii) sheds some light upon the 'equilibrium implications' of market frictions.\footnote{See for instance, Constantinides (1986), Davis and Norman (1990), Duffie and Sun (1989), Dumas and Luciano (1991), Fleming et al. (1992), Grossman and Vila (1992), Tuckman and Vila (1992), Vila and Zariphopoulou (1990), among many others.} The equilibrium determination of prices in markets with frictions, taking the financial structure (and the imperfections, in particular) as given, is the second question raised in the literature on market frictions. It is also the question addressed in the present paper. Finally, the third question addressed by the literature on market frictions is to endogenize the financial structure\footnote{See for instance Allen and Gale (1988), Boudoukh and Whitelaw (1992), Duffie and Jackson (1989) and Ohashi (1992).}. While we consider this question to be a fundamental one we do not address it in this paper i.e. we take the financial structure as given.

Most of the work about the equilibrium implications of market frictions, considers either a static framework along the lines of the Capital Asset Pricing Model (see among others, Brennan (1975), Goldsmith (1976), Levy (1978) and Mayshar (1979 & 1981) for a partial equilibrium analysis, and Fremault (1991) and Michaely and Vila (1992) for a general equilibrium treatment) or an overlapping generations economy where agents live for only two periods (Pagano (1989)). Although these models give us useful insights, they are not adequate for answering several of the questions we are interested in. In static models, assets are not sold but only liquidated. Moreover, in a static model (as well as in a two period overlapping generations model), agents cannot choose when to buy or sell assets, and...
the holding period is the same for all assets. It is thus clear that many of our results would not appear in that simplified framework.

In a context directly related to the present paper, Amihud and Mendelson (1986) consider a dynamic model where investors have different horizons. They argue that investors with, say, an investment horizon of 4 years who face a 2% roundtrip transactions cost when buying and selling assets will lose approximately \( \frac{1}{4} \% \) \((.5\%)\) per year because of the transactions cost. Hence, they will require a rate of return of .5\% higher on illiquid assets than on liquid assets. Consequently, the liquidity premium on assets which appeal to investors with a 4 year horizon must be approximately .5\%. The above reasoning implies that investors with longer horizons are less affected by transactions costs and would select higher yielding illiquid assets. By contrast, investors with shorter horizons select low yield liquid assets. This clientele effect explains the empirical fact that the cross-sectional relation between transactions costs and asset returns is concave.\(^6\) The analysis above, while insightful, takes investors’ horizons as given and does not explain how they change in response to an increase in transactions costs. Moreover since, as in the previous papers, the rate of return on the liquid asset is assumed for simplicity to be fixed, only the effect of transactions costs on the differentials of rates of return and not on their levels can be examined.

Two recent papers, one by Aiyagari and Gertler (1991), and one by Heaton and Lucas (1992) consider dynamic models where investors’ horizons are endogenous. In their models, agents are infinitely lived, face labor income uncertainty, and trade assets for consumption-smoothing purposes. These papers seek to solve the equity premium puzzle (see Mehra and Prescott (1985)), i.e. to explain the differential rates of return between the stock and the bond market. Aiyagari and Gertler (1991) argue that differential transactions costs between these two markets account for part of the

\(^6\)By contrast, if all investors had the same horizon this relation would be linear.
equity premium. In their model (as in ours), the 'stock' is riskless and therefore the equity premium is due to transactions costs and not to risk. Hence their model explains the fraction of the equity premium which is in fact a liquidity premium. They do not however analyze the effect of transactions costs on the level of rates of return as they take the rate of return on the liquid asset as given. By contrast with Aiyagari and Gertler (1991), Heaton and Lucas (1992) allow for a truly risky asset as well as for aggregate labor income uncertainty. They argue that transactions costs prevent investors from reducing the variability of their consumption by intertemporal smoothing thereby raising the equity premium. In addition, they endogenize the rate of return on the liquid asset and find that it falls in response to increased transactions costs.

While in our model agents save for life-cycle purposes rather than because of labor income uncertainty, our results are consistent with the numerical simulation results of the above two papers. We find in particular that transactions costs create a liquidity premium, as in Aiyagari and Gertler (1991), and that they cause the rate of return on the liquid asset to fall, as in Heaton and Lucas (1992). The contribution of our work is twofold: First, our closed form analysis allows us to precisely identify the different effects of transactions costs on asset demands and on rates of return. Second, we are able to easily perform and interpret various comparative statics.

The remainder of the paper is structured as follows: In part 2, we describe the model. We determine asset returns when there are no transactions costs in part 3. In part 4, we consider the case where there are transactions costs. In part 5, we illustrate our general results with some numerical examples. Part 6 contains concluding remarks and all proofs appear in the appendix.
II. THE MODEL

To analyze the impact of transactions costs on the return on assets and on the liquidity premium, we have adapted Blanchard's (1985) model of perpetual youth. A simplified exposition of the original model can be found in Blanchard and Fisher (1989).

We consider a continuous time overlapping generations economy with a continuum of agents with total mass equal to 1. An agent in this economy faces a constant probability of death per unit time, \( \lambda \). In addition, we assume that death is independent across agents and that agents are born at a rate equal to \( \lambda \). Therefore the population is stationary, with total mass equal to one and the distribution of age, \( t \), has a density function equal to \( \lambda e^{-\lambda t} \). Although agents can live arbitrary long lives in this economy, their life expectancy is bounded and equal to \( 1/\lambda \).

Agents are born with zero financial wealth and receive an exogenous labor income \( y \), over their lifetimes. We assume that \( y \) declines exponentially with age \( t \)

\[
y_t = \bar{y}e^{-\delta t}, \quad \delta > 0.
\] (1)

The aggregate labor income \( Y \) is constant and equal to

\[
Y = \int_{0}^{\infty} \lambda e^{-\lambda t}y_t \, dt = \frac{\lambda}{\lambda + \delta} \bar{y}.
\] (2)

The financial structure in this economy is given as follows. All assets in this economy are real perpetuities which pay a constant flow of dividends \( D \) per unit time. The total supply of perpetuities is normalized to one so that \( D \) is also the aggregate dividend. There are two such perpetuities. The liquid asset in total supply \( 1-k \) (\( 0 \leq k \leq 1 \)) can be exchanged without transactions costs. The price of the liquid asset is denoted by \( p \) and the rate of return on liquid assets is denoted by \( r = D/p \). The illiquid
asset, in total supply of \( k \), has a price equal to \( P \) and a rate of return equal to \( R = D/P \). Trading in the illiquid asset is subject to proportional transactions costs: when buying (or selling) \( x \) shares of the illiquid asset the agent must pay \( \epsilon x P \) transactions costs. Because of transactions costs, the rate of return on the illiquid asset and on the liquid asset will be different. The liquidity premium \( \mu \) is defined as

\[
\mu = R - r.
\]

(3)

Finally, none of these assets can be sold short.\(^9\)

Over the course of their lives, agents accumulate both assets. Since death is stochastic it imposes a financial risk upon the agents namely that of losing their accumulated holdings.\(^11\) We assume that this risk is fully and costlessly insurable in the following way: there exist insurance companies which pay shares of assets to the living participants in exchange for a claim on their estate. For example an insurance company that insures one share of say the liquid asset will pay a premium of \( \pi dt \) additional shares of the liquid asset per unit time \( dt \) to a living participant. Its compensation is to collect the share in the event of death. We assume that (i) the insurance market is perfectly competitive, (ii) insurance companies transfer assets costlessly\(^12\) and (iii) death is an idiosyncratic risk. As a result, the premium \( \pi dt \) must be equal to the probability of death \( \lambda dt \), for both the liquid and illiquid asset. Finally since, as previously indicated, agents do not derive any utility from their estate they will

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\(^9\)Note that we have defined \( R \) as the rate of return before transactions costs. The rate of return net of transactions costs depends upon the holding period and is therefore investor specific.

\(^10\)If short sales were costless no agent would sell the illiquid asset and would short the liquid asset instead. Our results do not change if we assume that the cost of short selling is higher than \( \epsilon \), which is a reasonable assumption (see for instance Boudoukh and Whitelaw (1991) and Tuckman and Vila (1992) for evidence on short sale costs).

\(^11\)Agents do not leave any heir behind and care only about themselves.

\(^12\)The introduction of insurance companies is a convenient way to close the model. As can easily be seen from our discussion in the introduction our results would carry through in a multi-period overlapping generations model with deterministic death. The latter would be much more complicated to solve analytically.
purchase full insurance.

We assume that agents maximize at time 0 the expected value of a time separable utility function of their consumption i.e.

\[ E \left[ \int_0^T u(c_t) e^{-\mu t} \, dt \right]. \]  

(4)

Since the only uncertainty comes from the possibility of death we can write (4) as\(^\text{13}\)

\[ \int_0^T u(c_t) e^{-(\delta + \lambda) t} \, dt. \]  

(5)

We also assume that the utility function exhibits a constant elasticity of substitution equal to \( \frac{1}{\lambda} \) i.e.

\[ u(c) = \frac{1}{1-A} c^{1-A}. \]  

(6)

In this paper we focus on the stationary equilibria of this economy. In a stationary equilibrium, the rates of return \( r \) and \( R \) are constant. We seek to understand the determination of \( r, R \) and \( \mu \) as functions of the parameters of the model: \( \epsilon, k, \lambda, \delta, \beta, A, Y \) and \( D \).

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\(^{13}\)See Blanchard and Fisher (1989) for details.

\(^{14}\)The case \( A = 1 \) corresponds to \( u(c) = \log c \).
III. THE NO TRANSACTIONS COSTS CASE

In this section, we analyze the determination of the interest rate in the benchmark case when transactions costs are equal to zero. In this case there is no difference between the liquid asset and the illiquid asset: \( r = R \) and \( \mu = 0 \).

3.1. The consumer's problem

The financial wealth \( w_t \) of the consumer at date \( t \) is defined as the value of the consumer's asset. That is if \( x_t \) is the number of shares that the consumer owns at date \( t \)

\[
w_t = px_t.
\]

At date \( t \), the investor receives a labor income \( y_t \) per unit time. His financial income (per unit time) entails \( D x_t \) in dividend income plus \( \lambda x_t \) shares worth \( \lambda px_t \). Since he consumes \( c_t \) per unit time the dynamics of his wealth are

\[
dw_t = D x_t dt + \lambda px_t dt + (y_t - c_t) dt = (r + \lambda)w_t dt + (y_t - c_t) dt; \quad c_t \geq 0.
\]

\[
w_0 = 0; \quad w_t \geq 0.
\]

(7)

From equations (5) and (7), the consumer's problem is the optimization problem of an infinitely lived consumer with discount factor \( \beta + \lambda \) and who faces a constant interest rate. This constant interest rate equals the rate of return on the perpetuity, \( r \), plus the premium paid by the insurance company, \( \lambda \).

Hence the consumer's problem can be written as

\[
\max \int_0^\infty u(c_t) e^{-(\delta + \lambda) t} dt = \max \int_0^\infty \frac{1}{1 - A} c_t^{1-A} e^{-(\delta + \lambda) t} dt
\]

(8)

s.t. \( \int_0^\infty c_t e^{-(\delta + \lambda) t} dt = \int_0^\infty y_t e^{-(\delta + \lambda) t} dt, \quad w_t \geq 0 \)

The problem (8) above admits the following solution\(^{15} \)

\(^{15}\)To calculate this solution we have assumed that the borrowing constraint is not binding, i.e. \( \delta > \omega = (\beta - \gamma)/A \), and that the maximum in (8) is finite, i.e. \( \psi = r + \lambda + \omega > 0 \). Both restrictions hold in equilibrium. (See appendix A for details.)
\[ c_t = \bar{y} \frac{\psi}{\varphi} e^{-\omega t} \quad \text{with} \quad (9) \]

\[ \omega = \frac{\beta - r}{A}, \quad (10a) \]

\[ \varphi = r + \lambda + \delta \quad \text{and} \quad (10b) \]

\[ \psi = r + \lambda + \omega. \quad (10c) \]

Finally, the consumer’s financial wealth at date \( t \) equals

\[ w_t = \bar{y} \frac{e^{-\omega t} - e^{-\lambda t}}{\varphi}. \quad (11) \]

3.2. Equilibrium

In equilibrium the aggregate financial wealth

\[ \int_0^\infty \lambda e^{-\lambda t} w_t \, dt \]

equals the market value of the perpetuities i.e. \( p = D/r \).

Hence from equation (11) and definitions (10), the equilibrium interest rate solves the equation

\[ \frac{r^* (\delta - \omega^*)}{\varphi^* (\lambda + \omega^*)} = \frac{D}{Y}. \quad (12) \]

where \( r^* \), \( \omega^* \), \( \varphi^* \) and \( \psi^* \) denote the equilibrium values of \( r \), \( \omega \), \( \varphi \) and \( \psi \), respectively.

Equation (12) determines the interest rate \( r^* \) uniquely. As expected the interest rate goes up with the discount factor \( \beta \), the probability of death \( \lambda \) and the ratio of aggregate financial income over aggregate labor income \( D/Y \). The interest rate goes down with the rate of decline of labor income, \( \delta \), since an increase in \( \delta \) leads to greater incentives to save. Finally if the coefficient of elasticity of substitution, \( 1/A \), goes up the interest rate goes down provided that \( r^* \) be greater than \( \beta \). Otherwise the interest rate goes up.

In equilibrium, agents use the financial markets to smooth their consumption over their lifetime: they
buy assets when they are young and begin selling assets at age $\tau^*$ where $\tau^*$ solves

$$ r^*w_{\tau^*} + y_{\tau^*} - c_{\tau^*} = 0.16 $$

From (11), $\tau^*$ is given by

$$ \tau^* = \frac{1}{\omega^* - \delta} \log \frac{\omega^* + \lambda}{\delta + \lambda}. \tag{13} $$

The aggregate dollar volume in this economy equals

$$ T = \int_0^\tau \lambda e^{-\lambda t} \left| rw_t + y_t - c_t \right| dt = 2\lambda w_\tau e^{-\lambda \tau^*} = 2\frac{\delta - \omega^*}{\phi^*} e^{-\tau^*}. \tag{14} $$

\footnote{Note that we do not consider the payment of shares by insurance companies to be a trade. Hence although the agent's portfolio may still be growing $(dW_t > 0)$, the agent is considered a seller if his portfolio grows at a rate lower than $\lambda$. In the absence of transactions costs, this assumption simply amounts to defining who is called a seller and who is not. With transactions costs, however, matters are different. Since we have assumed that insurance companies pay living participants shares of asset as opposed to cash and that this transfer is costless, our definition of a seller is the correct one.}
IV. TRANSACTIONS COSTS AND ASSETS’ RETURNS

In this section, we derive the main results of this paper namely the determination of the rate of return on the liquid asset, \( r \), the rate of return on the illiquid asset, \( R \), and the liquidity premium \( \mu \) in the presence of transactions costs. We will consider the case of small transactions costs and focus on their first order effect on equilibrium variables. For this purpose we write

\[
\begin{align*}
    r(e) &= r' + (b-m')e + o(e) \\
    R(e) &= r' + b\epsilon + o(e) \\
    \mu(e) &= m'\epsilon + o(e)
\end{align*}
\]

where \( b \) and \( m' \) are the first order equilibrium effects that we seek to calculate. We first consider the case where the supply of the illiquid asset, \( k \), is less than one so both assets are available to investors. As we shall see, the case where all assets are illiquid, i.e. \( k=1 \), is somewhat different. This case will be studied in section 4.4.

Before proceeding with the formal derivations, it is useful to show that in equilibrium the liquidity premium per unit of transactions costs \( \mu/\epsilon \) must be greater than the rate of return on liquid assets, \( r \), or equivalently \( R>r(1+\epsilon) \). This is because, since agents are born without any financial assets, in equilibrium they must buy the illiquid asset at some point in their lives. Now consider an agent who buys for one dollar worth of asset inclusive of transactions costs at date \( t \) and sells it \( \Delta t \) periods later.

If he buys the liquid asset his cash flows are

- 1 at date \( t \)
- \( rds \) for \( s \) between \( t \) and \( t+\Delta t \) and
- 1 at date \( t+\Delta t \).

If he buys the illiquid asset, given the transactions costs he will get \( 1/[P(1+\epsilon)] \) shares. Hence his cash flows are
- 1 at date t

\[ R/(1+\epsilon) \text{ds for s between } t \text{ and } t+\Delta t \text{ and} \]

\[ (1-\epsilon)/(1+\epsilon) \text{ at date } t+\Delta t. \]

If \( R \leq r(1+\epsilon) \), i.e. if \( \mu \leq r \epsilon \), then buying the liquid asset always dominates buying the illiquid asset. Hence

\[ \mu > r \epsilon. \]

In particular, this means that the effect of transactions costs on the liquidity premium is at least a first order effect. With this a priori information about equilibrium prices, we next characterize the investor's demand for liquid and illiquid assets when \( \mu > r \epsilon \).

### 4.1. The consumer's problem

With transactions costs, the consumer's financial wealth \( w_0 \) is the sum of the value of his liquid portfolio, denoted by \( a_0 \), and of the value of his illiquid portfolio, denoted by \( A_0 \). Denoting by \( i_t \) (respectively \( I_t \)) the incremental dollar investment in the liquid asset (respectively illiquid asset), the dynamics of \( a_t \) and \( A_t \) are given by

\[ da_t = \lambda a_t \text{dt} + i_t \text{dt}, \quad a_0 = 0, \quad a_t \geq 0 \quad (15a) \]

\[ dA_t = \lambda A_t \text{dt} + I_t \text{dt}, \quad A_0 = 0, \quad A_t \geq 0 \quad (15b) \]

\[ c_t = y_t + r a_t + R A_t - i_t - I_t - \epsilon |I_t|, \quad c_t \geq 0. \quad (15c) \]

From (15c) above, we can see that the agent's consumption equals the labor income \( y_t \) plus the dividend income \( r a_t + R A_t \), minus purchases of liquid assets \( i_t \), minus purchases of illiquid assets, minus transactions costs \( \epsilon |I_t| \).

With transactions costs, the consumer problem becomes far more complex. Proposition 4.1 (proven

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\(^{17}\)This lower bound is reached asymptotically when the holding period of illiquid assets goes to infinity, that is when the fraction of illiquid assets, \( k \), goes to zero.
in appendix B) describes the optimal policy of the consumer for small transactions costs and for a subset of values of \( r \) and \( R \) that are of interest, i.e. such that their equilibrium values belong to this subset.

**Proposition 4.1:** For \( \epsilon \) small and for \( r \) and \( R \) belonging to a subset of their possible values, the optimal policy has the following form: The consumer buys the illiquid asset until an age \( \tau_i \). He then buys the liquid asset. He next sells the liquid asset until an age \( \tau_i + \Delta \). At age \( \tau_i + \Delta \), he does not own any share of the liquid asset. He then start selling the illiquid asset until he dies.

We find that in equilibrium, agents will buy high yield illiquid assets for long term investment and low yield liquid assets for short term investment. This fairly intuitive result is consistent with the analysis of Amihud and Mendelson (1986). The clientele for the illiquid asset are the agents of age less that \( \tau_i \) while the clientele for the liquid asset are the agents of age between \( \tau_i \) and the age at which they begin to sell it. The marginal investor is the investor who buys the illiquid asset at date \( \tau_i \) and sells it at date \( \tau_i + \Delta \). As in Amihud and Mendelson (1986), this investor determines the liquidity premium (see below).

The liquid and illiquid portfolios as function of age \( t \) are plotted in figure 1.

Proposition 4.1 presents the qualitative properties of the optimal consumption/investment policy. In what follows, these qualitative properties will allow us to calculate the consumption at date \( t \), \( c_o \), as a function of the initial consumption, \( c_o \). We will also show how the intertemporal budget equation, properly modified to account for transactions costs, leads to the determination of the initial consumption \( c_o \). Finally, we will show how the parameter \( \Delta \) can be easily calculated as function of the rate of return on the liquid asset, \( r \), the liquidity premium, \( \mu \), and the level of transactions costs, \( \epsilon \). or the sake of the presentation, all technical details have been sent to appendix B.
Over the course of his life the agent faces three interest rates.

First until age $\tau_1$, the interest rate which is relevant for the consumption-savings decision is

$$R_L + \lambda = \frac{R}{1+\epsilon} + \lambda.$$  

Indeed, consider a consumer who at $t \in [0, \tau_1]$ decides to consume $\$1$ less, but wants to have the same wealth after $t+dt$. He then buys $\frac{1}{P(1+\epsilon)}$ illiquid securities. At $s$ between $t$ and $t+dt$, he consumes the extra dividend flows $(D/P(1+\epsilon))e^{\lambda s}$ and at $t+dt$, he consumes the proceeds from avoiding to buy $(1/P(1+\epsilon))e^{\lambda t}$ securities, i.e. $se^{\lambda t}$. Hence by foregoing $\$1$ at $t$ he gets $\$1 + \lambda dt + (R/(1+\epsilon))dt + o(dt)$ between $t$ and $t+dt$. Therefore, for $R$ given, higher transactions costs increase the desire to consume earlier rather than later. The reason is that the consumer has to buy an asset which is more expensive, but that pays the same dividend.

Second, between ages $\tau_1$ and $\tau_1 + \Delta$, the consumer invests in the liquid assets and therefore faces the interest rate $r + \lambda$.

Third and finally, after age $\tau_1 + \Delta$, when the investor is divesting out of the illiquid assets, he faces a higher rate

$$R_g + \lambda = \frac{R}{1-\epsilon} + \lambda.$$  

Indeed, consider a consumer who at $t \in [\tau_1, \infty)$ decides to consume $\$1$ less but wants to have the same wealth after $t+dt$. He sells $\frac{1}{P(1-\epsilon)}$ less illiquid securities. At $s$ between $t$ and $t+dt$, he consumes the extra dividend flow $(D/P(1-\epsilon))e^{\lambda s}$ and at $t+dt$ he consumes the proceeds from selling $(1/P(1-\epsilon))e^{\lambda t}$ securities, i.e. $se^{\lambda t}$. Hence by foregoing $\$1$ at $t$ he gets $\$1 + \lambda dt + (R/(1-\epsilon))dt + o(dt)$ between $t$ and $t+dt$. For $R$ given, higher transactions costs increase the desire to consume later rather than earlier. The reason is that the consumer has to sell a cheaper asset that pays the same dividend.

We denote $\hat{\rho}(t)$ be the interest rate relevant for date $t$ i.e.
\[ \hat{p}(t) = R_l + \lambda \text{ for } t < \tau_1, \]
\[ \hat{p}(t) = r + \lambda \text{ for } \tau_1 \leq t < \tau_1 + \Delta, \]
\[ \hat{p}(t) = R_b + \lambda \text{ for } \tau_1 + \Delta \leq t \]
and by \( p(t) \) the discount rate between date 0 and date \( t \) i.e.
\[ p(t) = \int_0^t \frac{\hat{p}(s) \, ds}{t}. \]

In appendix B, we indeed show that the optimal consumption must satisfy
\[ c_i = c_0 \, e^{-\omega(t)} \quad \text{with} \quad \omega(t) = \frac{(\beta + \lambda) t - \hat{p}(t)}{A} \quad (16) \]
where the consumption at birth \( c_0 \) is derived from the intertemporal budget constraint presented below.

Given proposition 1 and equation (15), it can easily be shown that the consumption path \( c_i \) must satisfy the intertemporal budget equation
\[ \int_0^{\tau_1} \frac{y_i - c_i}{1 + \epsilon} \, e^{-\rho(t)} \, dt + \int_{\tau_1}^{\tau_1 + \Delta} (y_i + (R-r)A_i - c_i) \, e^{-\rho(t)} \, dt + \int_{\tau_1 + \Delta}^{t} \frac{y_i - c_i}{1 - \epsilon} \, e^{-\rho(t)} \, dt = 0 \quad (17a) \]
with
\[ A_i = A_{ij} \left( 1 + (t - \tau_1) \right) e^{k(t - \tau_1)} \int_0^{\tau_1} \frac{y_i - c_i}{1 + \epsilon} e^{(\rho(s) - \rho(i))} ds. \quad (17b) \]

Equation (17a) says that the Net Present Value of lifetime savings net of transactions costs must equal zero where we define savings as total income minus consumption minus what must be reinvested in order for financial wealth to grow at the rate \( \hat{p}(t) \). Between periods \([0, \tau_1]\) and \([\tau_1 + \Delta, +\infty]\), this latter quantity equals the dividend income and therefore savings equal \( y_i - c_i \). Between \( \tau \) and \( \tau + \Delta \), only a fraction \( rA_i \) of the dividends from the illiquid portfolio must be reinvested and thus savings equal labor income, \( y_0 \), minus consumption \( c_i \) plus excess dividends \( (R-r)A_t \). Finally savings are adjusted for transactions costs.
Some simple algebra shows that the intertemporal budget equation can be written as

\[
\left[ \int_{0}^{\tau_{1}} (y_t - c_t) e^{-\rho(t)} dt + \int_{\tau_{1} - \Delta}^{\tau_{1}} (y_t - c_t) e^{-\rho(t)} dt + \int_{\tau_{1}}^{\tau_{1} - \Delta} (y_t - c_t) e^{-\rho(t)} dt \right] + \\
A_{\tau_{1}} e^{-\rho(\tau_{1})} \left[ \frac{\mu}{\tau} (1 - e^{-\tau A}) - \epsilon (1 + e^{-\tau A}) \right] = 0.
\]

The expression above consists of two terms. The first is the Net Present Value of labor income minus the Net Present Value of consumption taken with discount rate \(\rho(t)\). The second term is the Net Present Value of investing in the illiquid asset, i.e. the present value of the excess return \(\mu\) in the period \([\tau_{1}, \tau_{1} + \Delta]\) minus the present value of the transactions costs. Since the investor maximizes the present value of his investment with respect to \(A_{\tau_{1}}\), this term must be equal to zero for an optimal choice of \(\tau_{1}\) and \(\Delta\), i.e.

\[
\frac{\mu}{\epsilon} = \frac{1 + e^{-\tau A}}{1 - e^{-\tau A}}.
\]

Equation (18) can also be easily derived as follows. Consider a consumer at age \(\tau_{1}\). Since \(\tau_{1}\) and \(\Delta\) are optimally chosen, this consumer must be indifferent between investing in the illiquid asset and not doing so. Given that he starts selling the illiquid assets at \(\tau_{1} + \Delta\) his change in utility if he buys one unit of the illiquid assets at \(\tau_{1}\) is

\[
- u' (c_{\tau_{1}}) (1 + \epsilon) P + u' (c_{\tau_{1} + \Delta}) e^{-\delta \Delta} (1 - \epsilon) P + \int_{\tau_{1}}^{\tau_{1} - \Delta} u' (c_{t}) D e^{-\delta (t - \tau_{1})} dt = 0.
\]

The equation above together with the equation determining \(c_{i}\) (16) gives us (18). Equation (18) has a simple interpretation. As it was said, the consumer buys the illiquid asset for long term investment and the liquid asset for short term investment. The minimum time between buying and selling the illiquid asset, \(\Delta\), will, obviously, be decreasing in its excess rate of return over the liquid asset, \(\mu\).

Equation (18) states precisely how \(\Delta\) depends on \(\mu\).

From equation (18) it follows that the intertemporal budget constraint, for an optimal choice of \(\tau\) and \(\Delta\), states that the Net Present Value of consumption equals the Net Present Value of income.
where the discount factor is \( p(t) \), i.e.

\[
\int_{0}^{\tau_f} (y_t - c_t) e^{-p(t)} dt + \int_{\tau_f}^{\tau_f + \Delta} (y_t - c_t) e^{-p(t)} dt + \int_{\tau_f + \Delta}^{\tau_f} (y_t - c_t) e^{-p(t)} dt = 0 .
\] (19)

For any value of \( \tau, \Delta \), the initial consumption can be derived from (16), (19).

Having characterized the solution to the consumer's problem we turn to the equilibrium determination of \( r \) and \( R \).

### 4.2. Equilibrium

In equilibrium, the demands for liquid and illiquid assets (in dollar),

\[
\int_{0}^{\tau_f} \lambda e^{-\lambda t} a_t dt \quad \text{and} \quad \int_{0}^{\tau_f} \lambda e^{-\lambda t} A_t dt
\]
equal their market value, \((1-k)D/r\) and \(kD/R\) respectively.

As we stated in the beginning of this section, we will consider small transactions costs (small values of \( \epsilon \)), and find their first order effects on \( r, R \) and \( \mu \). Recall that we have defined \( b \) and \( m^* \) as

\[
\begin{align*}
    r(\epsilon) &= r^* + (b-m^*)\epsilon + o(\epsilon) \\
    R(\epsilon) &= r^* + b\epsilon + o(\epsilon) \\
    \mu(\epsilon) &= m^*\epsilon + o(\epsilon)
\end{align*}
\] (20a)

and that \( r^*, w^*, f^* \) and \( \psi^* \) are the equilibrium values of \( r, w, f \) and \( \psi \) for \( \epsilon = 0 \).

We also define by \( \tau_1(\epsilon) \) and \( \Delta(\epsilon) \) as the equilibrium values of \( \tau_1 \) and \( \Delta \) as a function of \( \epsilon \), and by \( \tau_1^* \) and \( \Delta^* \) the respective limits of \( \tau_1(\epsilon) \) and \( \Delta(\epsilon) \) as \( \epsilon \) goes to zero (Note that when \( \epsilon \) equals zero, the liquid asset and the illiquid asset are the same asset and therefore the holdings \( a_t \) and \( A_t \) are not well defined.) In other terms, \( \tau_1^* \) and \( \Delta^* \) are the zero-th order effects of transactions costs on holding periods.
The next proposition characterizes the equilibrium values of $m$ and $b$.

**Proposition 4.2.** There exists an equilibrium where $r$, $R$ and $\mu$ have the form of equations (20a), (20b), (20c). In equilibrium the first order effect of transactions costs on the liquidity premium, $m'$, is positive while the first order effect on the rate of return on the liquid asset, $b-m'$, is negative. The first order effect on the rate of return on the illiquid asset, $b$, has an ambiguous sign.

The rigorous derivations of $b-m'$, $b$ and $m'$, as well as explicit formulas are presented in appendix C.

Proposition 4.2, that we discuss in detail next, states that transactions costs decrease the rate of return on the liquid asset but have an ambiguous effect on the rate of return on the illiquid asset.

We discuss the results of proposition 4.2 in subsections 4.2.1., 4.2.2. and 4.2.3. In subsection 4.2.1, we characterize the parameters $\tau_i^*$ and $\Delta^*$ i.e. the zero-th order effect of transactions costs on optimal consumption/savings policies. In subsection 4.2.2 we go over the determination of the liquidity premium, and, in the rather long subsection 4.2.3 we discuss the determination of the rates of return (or, more accurately, the first order effects of transactions costs on these variables.)

**4.2.1. Optimal switching times**

The age at which agents switch from the illiquid asset to the liquid asset, $\tau_i^*$ and the age at which they start selling the illiquid asset, $\tau_i^*+\Delta^*$, can be easily interpreted from the limit case as $\epsilon$ goes to zero. Indeed consider the accumulation equations (15) with $\epsilon=0$ and $r=R=r'$. Given the investor’s total wealth $w_t=a_t+A_t$, the values of the liquid and illiquid portfolios over time are given by:

For $t \leq \tau_i^*$ \quad $A_t = w_t$ \quad and \quad $a_t = 0$.

For $\tau_i^* \leq t \leq \tau_i^*+\Delta^*$ \quad $A_t = w_t e^{-\lambda(t-\tau_i^*)}$ \quad and \quad $a_t = w_t - w_{\tau_i^*} e^{-\lambda(t-\tau_i^*)}$.

For $t \geq \tau_i^*+\Delta^*$ \quad $A_t = w_t$ \quad and \quad $a_t = 0$. 
It follows that the value of \( \tau^* \) and \( \Delta^* \) can be calculated by noting (i) that total financial wealth \( w \) grows by \( e^{\lambda \Delta^*} \) between \( \tau^* \) and \( \tau^* + \Delta^* \).

\[
w_{\tau^* + \Delta^*} = w_{\tau^*} e^{\lambda \Delta^*} \quad (21a)
\]

and (ii) that aggregate liquid financial wealth must equal the supply of liquid assets i.e.

\[
\int_{\tau^*}^{\tau^* + \Delta^*} \lambda e^{-\lambda t} a_t dt = \int_{\tau^*}^{\tau^* + \Delta^*} \lambda e^{-\lambda t} (w_t - w_{\tau^*} e^{-\lambda (t-\tau^*)}) dt = (1 - k) \frac{D}{r^*}. \quad (21b)
\]

Using the expression for \( w_t \) from equation (11) as well as equations (21a) and (21b) above we obtain the values of \( \tau^* \) and \( \Delta^* \).

### 4.2.2. Liquidity Premium

In appendix B, we show that the first order effect of transactions costs on the liquidity premium, \( m^* \), is given by:

\[
m^* = r^* \frac{1 + e^{-r^* \Delta^*}}{1 - e^{-r^* \Delta^*}}. \quad (22)
\]

It is fairly easy to understand the determination of \( m^* \). Equation (18) implies that if \( m \) were different from its value in equation (22), (i.e. if \( \mu \) were different in the first order), there would be zero-th order change in \( \Delta^* \). Therefore, there would be a zero-th order change in the demand for liquid versus illiquid assets. (Although there would only be a first order change in total asset demand.)

### 4.2.3. Rates of Return

The reasoning for the rates of return is more involved. To determine the parameter \( b \) (and \( b - m^* \)), we study total asset demand and supply, i.e. we use the equation that states that total asset demand equals total asset supply:

\[
\int_{0}^{\tau^*} \lambda e^{-\lambda t} (a_t + A_t) dt = \int_{0}^{\tau^*} \lambda e^{-\lambda t} w_t dt = (1 - k) \frac{D}{r} + k \frac{D}{R}. \quad (23)
\]

The reason for doing that is that to calculate the parameter \( b \) using the equation describing
equilibrium in one of the two asset markets, we need to know \( \mu \) to the second order. (As stated previously, a first order change in \( \mu \) has a zero-th order effect in the demand for each type of asset, therefore a second order change has a first order effect.) This is not a problem if we use the equation that states that total asset demand equals total asset supply. (A first order change in \( \mu \) has only a first order effect in total asset demand, therefore a second order change in \( \mu \) has a second order effect in total asset demand.)

To determine \( b-m' \), we thus make the following exercise: We assume that for fixed \( r \), transactions costs increase. Moreover, to preserve equilibrium in the zero-th order in each of the two markets, \( R \) increases in the first order, as determined above (it increases by \( m'e \) for \( m' \) given by equation (22)). We then find by how much total asset demand and supply change in the first order and infer \( b-m' \). Since this exercise is useful for understanding why the rate of return on the liquid asset decreases when transactions costs increase, we go over it in some detail.

To determine total asset demand, we must first understand how the consumption path of the agent is modified by the change in transactions costs and asset returns that we are considering. Lemma 4.3 (proven in appendix C) gives us the consumption of the agent at age 0.

*Lemma 4.3.* The consumption at date zero, \( c_0 \) is given by:

\[
c_0 = \tilde{y} \frac{\psi^*}{\phi^*} + \varepsilon C_w + \varepsilon C_s + o(\varepsilon) \quad (24a)
\]

where \( C_w \) and \( C_s \) are given by
\[ C_w = \frac{\bar{y}}{\phi^*} \{ (\frac{m^*}{r^*} - 1) \int_{i}^{i^*} [(\lambda + \delta)e^{-\psi^*} - (\lambda + \omega^*)e^{-\psi^*}] dt \]

\[ + (\frac{m^*}{r^*} + 1) \int_{i^*}^{\Delta^*} [(\lambda + \delta)e^{-\psi^*} - (\lambda + \omega^*)e^{-\psi^*}] dt \} \tag{24b} \]

or, alternatively:

\[ C_w = \frac{\bar{y}}{\phi^*} \{ (m^* - r^*) \int_{i}^{i^*} [e^{\psi^*} - e^{-\psi^*}] dt + (m^* + r^*) \int_{i^*}^{\Delta^*} [e^{-\psi^*} - e^{-\psi^*}] dt \} \] \tag{24c}

and

\[ C_s = -\frac{1}{A} \frac{\bar{y}}{\phi^*} \{ (m^* - r^*) \int_{i}^{i^*} e^{-\psi^*} dt + (m^* + r^*) \int_{i^*}^{\Delta^*} e^{-\psi^*} dt \} \] \tag{24d}

The terms \( C_w \) and \( C_s \) have a very intuitive interpretation. First \( C_w \) can be interpreted as a wealth effect. For this we need to note that

\[ \frac{\bar{y}}{\phi^*} \frac{[(\lambda + \delta)e^{-\psi^*} - (\lambda + \omega^*)e^{-\psi^*}]}{\phi^*} \]

is the present value of the dollar amount of transactions between \( \tau \) and \( \tau + d\tau \) in the case \( \epsilon = 0 \). The consumer when buying the illiquid asset between 0 and \( \tau \), pays the transactions costs but pays a lower price. When selling the illiquid asset (from \( \tau \) until he dies), he pays the transactions costs and receives a lower price. The term \( C_w \) is equal to the present discounted value of these cash flows times \( \psi^* \). Clearly, since the rate of return in the liquid asset is kept constant, the consumer can only be better off compared to the case \( \epsilon = 0 \), and this term is positive as we can see in expression (24c).

The term \( C_s \) can be interpreted as a substitution effect. Since \( R/(1+\epsilon) > r \), saving is more attractive from 0 to \( \tau^*_1 \) (With dividends, agents buy the illiquid asset paying transaction costs but at a lower price which more than compensates them.) It is also clear that \( R/(1-\epsilon) > r \), therefore deferring consumption for later is also more attractive from \( \tau^*_1 + \Delta^* \) until death. Thus this term is negative.
Having interpreted the expression for $c_\infty$, we can briefly describe how the consumption path changes compared to the case $\epsilon = 0$. Because the consumer has better investment opportunities, (i.e. he has the liquid asset at the same price as before, and the illiquid asset), his consumption path goes up uniformly. (This is the wealth effect.) Because the illiquid asset is available at a lower price (that more than compensates transactions costs), and because the proceeds from selling it are lower (lower price plus transactions costs), the consumer changes the slope of the consumption path so that he buys more of the illiquid asset and holds it for a longer period. In other words, he buys more in the beginning of his life (he saves more) and he sells it at a lower rate (he defers consumption for later.) This is the substitution effect.

In lemma 4.4 (proven in appendix C) we determine total asset demand.

**Lemma 4.4.** Total asset demand is given by:

\[
\frac{\delta - \omega'}{(\lambda + \omega') \varphi^*} Y + \epsilon W_w + \epsilon W_s + o(\epsilon) \tag{25a}
\]

where

\[
W_w = \frac{\lambda}{\lambda + \omega'} \bar{\psi}^* \{ (m^*-r^*) \int_{\tau_1}^{\tau_2} e^{-\omega' t'} - e^{-\varphi^* t'} \, dt + (m^*+r^*) \int_{\tau_1}^{\tau_2} e^{-\omega' t'} - e^{-\varphi^* t'} \, dt \} + 2 \frac{\lambda \bar{Y}}{\varphi^* r^*} \{ e^{-(\omega^*+\lambda)} \tau_1^* - e^{-(\omega^*+\lambda)} \tau_2^* \} - \frac{m^*}{r^*} \int_{\tau_1}^{\tau_2} A_1 \lambda e^{-\lambda t} \, dt \tag{25b}
\]

and

\[
W_s = \frac{1}{A} \frac{\lambda}{\lambda + \omega'} \bar{\psi}^* \{ (m^*-r^*) \int_{\tau_1}^{\tau_2} (e^{-(1+\omega') t'} - e^{-\varphi^* t'}) \, dt + (m^*+r^*) \int_{\tau_1}^{\tau_2} (e^{-(1+\omega') t'} - e^{-\varphi^* t'}) \, dt \} \tag{25c}
\]

The term $W_w$ represents the additional demand for wealth (in the first order) of the consumer if the latter changes only the level but not the slope of his consumption path (i.e. if the wealth effect is present, but not the substitution effect) in response to the change in transaction costs and asset returns that we are studying. This additional demand for (dollar) wealth has an ambiguous sign,
because, on the one hand, higher future consumption to be financed and transactions costs to be paid when selling assets, require more wealth, but on the other hand, (illiquid) assets are cheaper.

The term $W_s$ corresponds to the substitution effect: Indeed as was said before the consumer changes the slope of his consumption path so that he buys more of the illiquid asset and holds it for a longer period. This implies more wealth accumulation. This term is positive and its magnitude depends on the elasticity of intertemporal substitution.

Finally total asset supply (in the first order) is:

$$\frac{D}{r^*} - \epsilon \frac{m^*}{r^*} \int_0^t e^{-\lambda t} A_t \, dt = \frac{D}{r^*} - \epsilon W_{\text{Supply}} \quad (26)$$

It decreases since the illiquid asset is cheaper.

The difference between total asset demand and supply is $W_w + W_s + W_{\text{Supply}}$ and is always positive. It is easy to understand why this is so, based on our earlier discussion. Higher future consumption to be financed, transactions costs to be paid in selling assets (and a lower price of the illiquid asset), require a larger number of securities to be held. (Although the dollar amount may be lower.) In addition, the consumer changes the slope of his consumption path in order to buy more of the illiquid asset and hold it for a longer period, making the imbalance between asset demand and supply even higher.

The value of $(b-m^*)$ is then easily deduced, and is negative.

The above discussion which explained why the rate of return on the liquid asset falls, can be summarized as follows: Suppose that transactions costs increase from 0 to $\epsilon$ and that the rate of return on the liquid asset stays the same in equilibrium. Then, (in equilibrium) the rate of return on the illiquid asset must increase (in the first order) by $m^* \epsilon$. Agents' consumption paths will shift
uniformly up, because they face more investment opportunities (wealth effect), and their slope will change so that the agents buy more of the illiquid asset and hold it for a longer period (substitution effect). The agents will demand more securities for two reasons. First, because they have to finance higher future consumption, pay transaction costs when selling assets and because assets are cheaper. Second because they change the slope of their consumption path so that they buy more of the illiquid asset and they hold more of it.

Although the first order effect of transactions costs on the rate of return on the liquid asset is unambiguous (b-m' is negative), the first order effect on the rate of return on the illiquid asset (i.e. the sign of b) is ambiguous. In what follows, we replicate (more briefly) the above exercise, assuming that this time, as transactions costs increase, R stays the same and r decreases in the first order, as determined above (r decreases by m*e).

This time, the consumer faces worse investment opportunities. The price of the illiquid asset is the same as before, but, in addition there are transactions costs, while the price of the liquid asset increases. The wealth effect implies then that his consumption path shifts down uniformly. On the other hand, the substitution effect implies that he changes the slope of his consumption path so that he accumulates less of the illiquid asset but he holds it for a longer period, and so that he accumulates less of the liquid asset. The effect on the total demand for securities is ambiguous. First, the future consumption to be financed is lower and the liquid securities are more expensive, but, on the other hand transactions costs have to be paid. Second, consumers buy less of the liquid and illiquid assets, but they hold the illiquid asset for a longer period. Therefore, both effects are ambiguous.
4.3. Comparative Statics

In this section we study how the effects of transactions costs on assets' returns depend on the parameters of the model. (More precisely, we find how \( m^* \), \( b \) and \( b - m^* \) depend on these parameters.) The parameter that is of greatest interest is \( k \), the fraction of illiquid assets to the total stock of assets. In lemma 4.5 (proven in appendix D) we examine how \( m^* \), \( b \) and \( b - m^* \) depend on \( k \).

**Lemma 4.5.** \( m^* \) increases in \( k \), \( b - m^* \) decreases in \( k \) while the dependence of \( b \) in \( k \) is ambiguous.

We briefly discuss the results of this Lemma.

The dependence of \( m^* \) on \( k \) is relatively simple to understand. More illiquid assets in the economy imply that the minimum holding period of an illiquid asset becomes shorter. The liquidity premium must increase so that consumers are willing to hold illiquid assets for shorter periods.

The dependence of \( b - m^* \) on \( k \) can be explained in light of the analysis of the previous subsection. There it was argued that to understand why the rate of return on the liquid asset falls in response to increased transactions costs, we could make the following experiment: We could suppose that transactions costs increased from 0 to \( e \) and that the rate of return on the illiquid asset had to increase (in the first order) by \( m^* e \). We could then study the difference between demand and supply of total wealth and infer the direction of change of the rate of return on the liquid asset. In fact, we can also infer the *magnitude* of change of the rate of return on the liquid asset, studying the magnitude of the difference between total asset demand and total asset supply.

As \( k \) increases, \( m^* \) increases, therefore both the wealth effect and the substitution effect are stronger. This implies that the difference between asset demand and supply is greater, and the first order effect on \( r \) (i.e. \( b - m^* \)) is bigger (in absolute value).
The effects of the other parameters on $m'$, $b$ and $b-m'$ are of less interest and are not reported here.
V. NUMERICAL EXAMPLES

In this section we present some numerical examples to illustrate the results of the previous sections. In all these examples we assume that $A=1$ ($u(c) = \log c$) and that the level of transactions costs $\epsilon$ equals 3% which is consistent with empirical evidence (see Aiyagari and Gertler (1991) for instance). In all figures 2 to 4, we plot various rates of return as a function of $k$, the supply of the illiquid asset. These figures are consistent with the results in proposition 4.2. and lemma 4.5, namely that (i) the liquidity premium is positive, (ii) the rate of return on the liquid asset goes down, (iii) the rate of return on the illiquid asset can go up or down, (iv) the effect of transactions costs on the liquidity premium and the rate of return on the liquid asset is large when $k$ is close to 1.

The main quantitative observations are as follows: (i) When $k$ is close to 1, the liquidity premium is significant (about 10% of the level of the rates of return). (ii) When $k$ is close to 1, transactions costs cause a non-trivial fall in the rate of return on the liquid asset while the rate of return on the illiquid asset remains almost constant.

These quantitative results have important practical applications. To understand the impact of a change in transactions costs in the economy, it is important to understand how assets are differently affected by this change. A technological change such as a reduction in computer cost can be assumed to reduce transaction costs for all assets which in our model corresponds to the case $k=1$. Our results suggest that rates of return will not change much. By contrast, a reduction of transactions costs on one single asset (e.g. by the introduction of a derivative security) will increase the price of this asset without any significant impact on the other assets. Finally, a transaction tax on a significant subset of existing assets (stocks, real estate ...) will lower their value by an amount less than suggested by a simple partial equilibrium analysis which takes the rates of return on the other assets as given.

INSERT FIGURES 2 TO 4
VI. CONCLUSION

In this work we have constructed a fairly simple general equilibrium model of an imperfect capital market. Our main result is that while transactions costs tend to push the rate of return on illiquid assets upward, there is a general equilibrium effect which tends to lower rates of return. The net result is that the rate of return on liquid assets goes down while the rate of return on illiquid assets may go up or down. We believe that these results are robust with respect to the specification of (i) the trading motives: life cycle, labor income shocks\(^{18}\) or taste shocks and (ii) the preferences.\(^{19}\)

Our model endogenously generates clienteles for assets with differential liquidity. This clientele effect is consistent with previous work by Amihud and Mendelson (1986). In fact, if we generalized our model to allow for many assets with different transactions costs, we would obtain the concave relationship between rates of return and transactions costs derived by these authors.

In this paper, we assumed that transactions costs were a pure destruction of resources. If instead, they are due to a transaction tax whose proceeds are distributed to the agents, the results are similar. Amihud and Mendelson (1990) argue that, holding the risk free rate constant, a .5% transaction tax would lower the market value of the NYSE stocks by 13.8%. While we do not dispute the fact that a small transaction tax will increase the liquidity premium significantly, our results suggest that the risk free rate will fall so that the stock price fall is likely to be somewhat smaller.

This line of research can be pursued in (at least) two directions. First, the interaction between risk

\(^{18}\)See for instance Amihud et. al. (1992).

\(^{19}\)The treatment of the perpetual youth model for a general utility function seems to us analytically untractable. In a companion note, we consider a two-period overlapping generations model similar in spirit to the model herein. This model is simpler but also much less rich. In particular, the holding period is the same for all assets and as a result the liquidity premium is fixed. In this simple model the results are independent of the functional form of the preferences.
and liquidity is not fully understood. It would be interesting to construct tractable models to analyze the interaction between transactions costs and risk and examine in particular whether, as it has been argued, illiquid markets are more volatile since investors find it more costly to absorb liquidity shocks. Second and more importantly, very little is known about the determination of the level of transactions costs as well as the financial structure created to deal with these transactions costs.
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This appendix considers the case when transactions costs are zero. We will first prove that in equilibrium $\delta > \omega$ and $\psi = r + \lambda + \omega > 0$. We will then prove that the equilibrium is unique.

We must first calculate the optimal policy which entails solving

$$\max \int_{0}^{t} \frac{1}{1-A} c_{t}^{1-A} e^{-(\beta-\lambda)t} dt$$

s.t.

$$\int_{0}^{t} c_{t} e^{-(\beta-\lambda)t} dt = \int_{0}^{t} y_{t} e^{-(\beta-\lambda)t} dt, \quad w_{t} \geq 0.$$  \hspace{1cm} (8)

From He and Pages (1991), we know that a bounded value for (8) above exists provided that

$$\psi = r + \lambda + \omega > 0.$$ 

In that case, He and Pages show that

$$c_{t} = \frac{y_{t}}{\psi} e^{-\psi t} \quad \text{for every } t \text{ if } \omega = \frac{\beta - r}{A} \geq \delta$$

and

$$c_{t} = \frac{y_{t}}{\phi} e^{-\phi t} \quad \text{for every } t \text{ if } \omega < \delta \text{ with } \phi = r + \lambda + \delta.$$ \hspace{1cm} (A1)

If $\psi \leq 0$, then the value of (8) is $+\infty$. We show below that this cannot be the case in equilibrium.

In equilibrium, the resource constraint implies that

$$\int_{0}^{t} \lambda c_{t} e^{-\lambda t} \leq D + Y.$$ \hspace{1cm} (A2)

Hence the equilibrium utility of the representative agent is bounded by the solution to the program (A3) below

$$\max \int_{0}^{t} \frac{1}{1-A} c_{t}^{1-A} e^{-(\beta-\lambda)t} dt$$

s.t.

$$\int_{0}^{t} c_{t} e^{-\lambda t} dt = D + Y.$$ \hspace{1cm} (A3)

It is easy to show that (A3) has a finite value.

If $\omega \geq \delta$ then the agent does not buy any asset and therefore $w_{t} = 0$. Obviously, this is not consistent
with the equilibrium condition

\[ \int_0^\infty \lambda e^{-\lambda t} w dt = \frac{D}{r}. \]  

(A4)

Hence \( \delta \) must be larger than \( \omega \) in equilibrium.

From (A1) and (7) it follows easily that

\[ w_i = \frac{y}{\varphi} e^{-\omega t} - e^{-\delta t}. \]  

(A5)

Combining (A4) and (A5) yields the equilibrium condition

\[ \frac{r' (\delta - \omega')} {\varphi' (\lambda + \omega')} = \frac{r' (\delta - \frac{\beta - r'} {A})} {\frac{\varphi' (\lambda + \omega')} {r' + \lambda + \delta} (\lambda + \frac{\beta - r'} {A})} = \frac{D}{Y} \]  

(A6)

which must be solved for \( r' \).

Simple algebraic manipulations show that (A6) above has a unique positive solution \( r' \), that this solution is increasing in \( \beta \), \( \lambda \) and \( D/Y \) and that it is decreasing in \( \delta \). It is increasing in \( A \) if \( r' \) is greater than \( \beta \) and decreasing with \( A \) otherwise.
APPENDIX B

Proof of proposition 4.1

The method of proof is as follows: We first define the control variables. We then derive heuristic conditions for an optimal control. Next we construct a candidate optimal control and show that is indeed optimal.

Step 1: The control problem: Recall that \( i_t \) is the per unit time value of the liquid assets purchased at date \( t \) and that \( I_t \) is the per unit time value of the illiquid assets purchased at date \( t \). Recall also equations (15a), (15b) and (15c):

\[
\begin{align*}
\text{(15a)} & \quad da_t = \lambda a_t \, dt + i_t \, dt, \quad a_0 = 0 \text{ and } a_t \geq 0. \\
\text{(15b)} & \quad dA_t = \lambda A_t \, dt + I_t \, dt, \quad A_0 = 0 \text{ and } A_t \geq 0. \\
\text{(15c)} & \quad c_t = y_t + ra_t + RA_t - i_t - I_t - \epsilon |I_t|, \quad c_t \geq 0.
\end{align*}
\]

The control problem faced by the consumer is to maximize (5) with respect to the controls \( i_t \) and \( I_t \) subject to the dynamics of \( a_t \) and \( A_t \) (i.e. (B1) and (B2)).

Formally, we say that a control \((i(\cdot), I(\cdot))\) is admissible if it is (i) piecewise continuous (ii) it satisfies the no short sale constraints \( a_t \geq 0 \) and \( A_t \geq 0 \) as well as the constraint \( c_t \geq 0 \). We denote by \( \mathcal{C} \) the set of admissible controls and by \( J(i(\cdot), I(\cdot)) \) the payoff function, i.e. the utility that the consumer enjoys if he follows the controls \( i_t = i(t) \) and \( I_t = I(t) \). Using the fact that

\[
a_t = \int_0^t i_s e^{\lambda(t-s)} \, ds \quad \text{and} \quad A_t = \int_0^t I_s e^{\lambda(t-s)} \, ds
\]

the payoff function can be written as

\[
J(i(\cdot), I(\cdot)) = \int_0^\infty u(y_t + r \int_0^t i_s e^{\lambda(t-s)} \, ds + R \int_0^t I_s e^{\lambda(t-s)} \, ds - i_t - I_t - \epsilon |I_t|) e^{-(\lambda+\epsilon)t} \, dt. \quad \text{(B1)}
\]

Hence the control problem is to maximize \( J(i(\cdot), I(\cdot)) \) with respect to \((i(\cdot), I(\cdot))\) belonging to \( \mathcal{C} \).
Step 2: Heuristic optimality conditions: To derive heuristic conditions for an optimal control, we take a candidate optimal control \((i(\cdot), I(\cdot))\), denote by \(c(\cdot)\) the resulting consumption and consider the possible perturbations:

(i) Suppose first that \(a_i > 0\). Suppose that at \(t\), the consumer changes his consumption by \(\alpha\) dollars (per unit time) and invests \(\alpha\) more dollars in the liquid asset. At \(s\) between \(t\) and \(t + dt\), he consumes the extra dividend \(\alpha e^{kt}\) and at \(t + dt\) he sells \(\alpha e^{kt}\). He does not change his investment decision thereafter. His payoff will change by

\[
\alpha \left[ - u'(c_i) + e^{-(k+1)t} (e^{kt} + r dt) u'(c_{i, dt}) \right].
\]

Since, \((i(\cdot), I(\cdot))\) is optimal, the change in payoff must be non-positive for every \(\alpha\) and \(dt\) going to zero which yields the standard Euler equation

\[
\frac{du'(c_i)}{dt} = (\beta - r) u'(c_i). \quad (B2)
\]

(ii) Suppose now that \(a_i = 0\). Then the consumer can only increase his investment in the liquid asset and therefore condition \((B2)\) must be replaced by

\[
\frac{du'(c_i)}{dt} \leq (\beta - r) u'(c_i). \quad (B3)
\]

(iii) Consider a policy where \(A_t > 0\) for every \(t\), which will be the case in our candidate policy.

(iiiia) Suppose \(I_t > 0\). If between \(t\) and \(dt\), the consumer changes his consumption by \(\alpha (1 + \epsilon)\) dollars (per unit time), invests \(\alpha\) (with \(\alpha + I_t > 0\)) more dollars in the illiquid asset and does not change his investment decision thereafter, his payoff will change by

\[
\alpha \left[ -(1 + \epsilon) u'(c_i) + \int_t^\cdot R u'(c_i) e^{-k(s-t)} ds \right] dt
\]

which must be non-positive for any \(\alpha\) and so

\[
(1 + \epsilon) u'(c_i) = \int_t^\cdot R u'(c_i) e^{-k(s-t)} ds. \quad (B4)
\]

(iiiib) Suppose now that \(I_t < 0\). If between \(t\) and \(dt\), the consumer changes his consumption by \((1 - \epsilon)\alpha\)
dollars (per unit time), invests $\alpha$ (with $\alpha + I_i < 0$) more dollars in the illiquid asset and does not change his investment decision thereafter$^{20}$ his payoff will change by

$$\alpha \left[ - (1 - \epsilon) u'(c_i) + \int_I R u'(c_i) e^{-\beta(s-t)} ds \right] dt$$

which must be non-positive for any $\alpha$ and so

$$(1 - \epsilon) u'(c_i) = \int_I R u'(c_i) e^{-\beta(s-t)} ds .$$

(iii) If $I_i = 0$, then the first order condition reads as

$$(1 - \epsilon) u'(c_i) \leq \int_I R u'(c_i) e^{-\beta(s-t)} ds \ dt \leq (1 + \epsilon) u'(c_i) .$$

(B6)

Step 3: Construction of the candidate policy:

Given $c_o$, $\tau_1$ and $\Delta$ we define the candidate optimal consumption plan as in equation (16)

$$c_i = c_o e^{-\omega(t)} \text{ with } \omega(t) = \frac{(\beta + \lambda)t - \rho(t)}{A} .$$

(16)

This consumption plan will be completely defined once we specify $c_o$, $\tau_1$ and $\Delta$. We now motivate our definition of these parameters. To finance this consumption plan, the agent will buy shares of the illiquid asset from 0 to $\tau_1$. He will buy and then sell shares of the liquid asset between $\tau_1$ and $\tau_1 + \Delta$. Finally he will sell the illiquid assets from $\tau_1 + \Delta$ and $+\infty$. From equation (15a), (15b) and (15c) it follows that

$^{20}$Note that this perturbation is feasible only if $A_i > 0$ for every $t$. In the optimal policy, $A_i$ will indeed be positive for every $t$. However $a_i = 0$ for $t \leq \tau_1$ and $t \geq \tau_1 + \Delta$. For this reason, the optimality condition with respect to $i_i$ is written as an Euler equation (B2)-(B3) while the optimality condition with respect to $I_i$ is an integral condition.
\[ A_i = \int_0^t y_i - c_i \ e^{(\rho(t) - \rho(s))} \ ds + \int_{\tau_i}^t (y_i - c_i + RA_i) \ e^{(\rho(t) - \rho(s))} \ ds \quad \text{for } t \leq \tau_i, \]
\[ A_i = A_{i-1} e^{k(t-\tau_i)}; \quad \int_{\tau_i}^t (y_i - c_i + RA_i) \ e^{(\rho(t) - \rho(s))} \ ds \quad \text{for } \tau_i \leq t \leq \tau_i + \Delta, \] (B7)
\[ A_i = A_{i-1} e^{(\rho(t) - \rho(\tau_i + \Delta))} + \int_{\tau_i + \Delta}^t y_i - c_i \ e^{(\rho(t) - \rho(s))} \ ds; \quad a_i = 0 \quad \text{for } t \geq \tau_i + \Delta. \]

From (B7) together with the transversality condition
\[ \lim_{t \to -\infty} A_i e^{-\rho(t)} = 0 \]
we get
\[ \int_{\tau_i}^t y_i - c_i \ (1 + \varepsilon) e^{-\rho(t)} \ dt = e^{\tau_i} \int_{\tau_i + \Delta}^t (y_i - c_i + RA_i) \ e^{-\rho(t)} \ dt. \] (B8)

Finally
\[ a_{i-1} = 0 = e^{\rho(\tau_i - \Delta)} \int_{\tau_i}^{\tau_i + \Delta} (y_i - c_i + RA_i) \ e^{-\rho(t)} \ dt = 0. \] (B9)

We first define \( \Delta \) by (18) that is
\[ \frac{\mu}{\varepsilon} = \frac{R - r}{\varepsilon} = r \frac{1 + e^{-\tau}}{1 - e^{-\tau}} \] (18)
or equivalently
\[ e^{-\tau} = \frac{R - 1 - \varepsilon}{R - 1 + \varepsilon}. \] (B10)

Adding equations (B8) and (B9) gives the intertemporal budget equation (17a). We define \( c_0 \) (as a function of \( \tau, \varepsilon, r \) and \( R \)) from the intertemporal budget equation (17a) which together with (18) yields the simpler equation (19) below
\[ \int_{\tau_i}^{\tau_i - \Delta} (y_i - c_i) \ e^{-\rho(t)} \ dt + \int_{\tau_i}^{\tau_i + \Delta} (y_i - c_i) \ e^{-\rho(t)} \ dt + \int_{\tau_i}^{\tau_i + \Delta} (y_i - c_i) \ e^{-\rho(t)} \ dt = 0. \] (19)

Finally we define \( \tau_i \) by equation (B8).
We now show that this policy is well defined and admissible for $\epsilon$ small and for $r$ and $R$ belonging to a subset of their equilibrium values. We will also show that it varies smoothly with $\epsilon$, $r$ and $m$ where $m$ is defined by

$$R = r + m\epsilon.$$  \hspace{1cm} (B11)

We will show the following lemma B1 below:

Lemma B1: Consider $r$, $m$ and $m^*$ such that $m^*/r^*>1$ (recall that $r^*$ is the equilibrium interest rate when transactions costs are zero). There exists $\epsilon_0$ such that if

(i) $0<\epsilon<\epsilon_0$

(ii) $|r^*-r| \leq \epsilon_0$ / $|m^*-m| \leq \epsilon_0$

the consumption plan defined above, where $R=r+m\epsilon$ is well defined and admissible. Moreover, $\Delta$, $c_0$ and $\tau_1$ are infinitely differentiable $(C^\infty)$ in $(\epsilon, r, m)$.

Proof: The intuition for the proof is simple. If $\epsilon=0$, this consumption plan collapses to the optimal one for $\epsilon=0$. That plan is admissible. The admissibility of the candidate policy for $\epsilon>0$ will follow by continuity.

We first note that the equation defining $\Delta$ can be written as

$$e^{-r\Delta} = \frac{m-r}{m+r}.$$  \hspace{1cm} (B12)

Therefore $\Delta$ is uniquely defined, is $C^\infty$ in $(\epsilon,r,m)$ and verifies $0<\Delta<\bar{\Delta}<+\infty$ for $\epsilon_0$ sufficiently small.

The equation giving $c_0$ can be written as:

$$\bar{y} \left[ \frac{1-e^{-(R_L+\lambda+\delta)}}{R_L+\lambda+\delta} + e^{-(R_L+\lambda+\delta)\tau_1} \frac{1-e^{-(r+\lambda+\delta)\Delta}}{r+\lambda+\delta} + \frac{e^{-(R_L+\lambda+\delta)\tau_1} e^{-(r+\lambda+\delta)\Delta}}{R_B+\lambda+\delta} \right] =$$  \hspace{1cm} (B13)

$$c_0 \left[ \frac{1-e^{-(R_L+\lambda+\omega_L)}}{R_L+\lambda+\omega_L} + e^{-(R_L+\lambda+\omega_L)\tau_1} \frac{1-e^{-(r+\lambda+\omega)\Delta}}{r+\lambda+\omega} + \frac{e^{-(R_L+\lambda+\omega_L)\tau_1} e^{-(r+\lambda+\omega)\Delta}}{R_B+\lambda+\omega_b} \right]$$

with
\[ \omega_1 = \frac{\beta - R_L}{A} \quad \text{and} \quad \omega_b = \frac{\beta - R_b}{A}. \]

From equation (B13), it is obvious that \( c_0 \) is uniquely defined and is \( C^\infty \) in \( (\epsilon, r, m, \tau_1) \).

It is fairly straightforward to show that we can restrict \( \epsilon_0 > 0 \) such that

\[ | \frac{\partial c_0}{\partial \epsilon} | \leq \tilde{y} K_\epsilon \quad \text{in} \quad [0, \epsilon_0] \]

for all values of \( r \) and \( m \) in the set defined before and for all \( \tau_1 \epsilon [0, +\infty) \) where \( K_\epsilon \) is a positive constant.

We now turn to the equation defining \( \tau_1 \). This equation can be written as

\[
\frac{1}{1 + \epsilon} \left[ \frac{\tilde{y} - e^{-(R_L + \lambda + \delta)} \tau_1}{R_L + \lambda + \delta} - c_0 \frac{1 - e^{-(R_L + \lambda + \omega_L)} \tau_1}{R_L + \lambda + \omega_L} \right] = \frac{1}{1 - \epsilon} \left[ c_0 \frac{e^{-(R_L + \lambda + \omega_L)} \tau_1 e^{(1 + \omega)\Delta}}{R_b + \lambda + \omega_b} - \tilde{y} \frac{e^{-(R_L + \lambda + \delta)} \tau_1 e^{-(1 + \delta)\Delta}}{R_b + \lambda + \delta} \right].
\]

Straightforward algebra shows that we can rewrite this equation as

\[
e^{-(R + \lambda + \delta)\tau_1} - e^{-(R + \lambda + \omega)\tau_1} = e^{-(R + \lambda + \omega)\tau_1 - (1 + \omega)\Delta} - e^{-(R + \lambda + \omega)\tau_1 - (1 + \delta)\Delta} + f(\frac{c_0}{\tilde{y}}, r, m, \epsilon, \tau_1) \quad \text{(B14)}
\]

where \( f(\cdot, \cdot, \cdot, \cdot, \cdot) \) is a \( C^\infty \) function such that \( f=0 \) for \( \epsilon=0 \) and

\[ | \frac{\partial f}{\partial \epsilon} | \leq K_\tau \quad \text{in} \quad [0, \epsilon_0] \]

for all values of \( r \), \( m \) and \( \tau_1 \) (restricting \( \epsilon_0 \) if necessary). \( K_\tau \) is a positive constant.

Consider the equation for \( \epsilon=0 \)

\[
e^{-(R + \lambda + \delta)\tau_1} - e^{-(R + \lambda + \omega)\tau_1} = e^{-(R + \lambda + \omega)\tau_1 - (1 + \omega)\Delta} - e^{-(R + \lambda + \omega)\tau_1 - (1 + \delta)\Delta}. \quad \text{(B15)}
\]

The function \( g(\cdot): t \rightarrow e^{(k + 1)t} - e^{(\omega + 1)t} \) has the following graph (see figure A)

**INSERT FIGURE A**

Therefore this equation has a unique solution \( \tau_1 \) in \( ]0, \tau^*[. \) It is easy to show (given \( \Delta \gtrsim \Delta > 0 \)) that there exists \( \zeta > 0 \) such that \( \tau_1 \leq \tau^* - \zeta \) and \( \tau_1 + \Delta \gtrsim \tau^* + \zeta \). Using the implicit function theorem, the fact that \( +\infty \Delta \gtrsim \Delta \gtrsim \Delta > 0 \) and the fact that

\[ | \frac{\partial f}{\partial \epsilon} | \leq K_\tau \quad \text{in} \quad [0, \epsilon_0] \]
for all values of $r$, $m$ and $\tau$, we can show that we define a $C^\infty$ function $\tau(\epsilon,r,m)$ for $\epsilon \leq \epsilon_0$ and for all values of $r$ and $m$. Moreover, it is easy to show that

$$| \frac{\partial \tau}{\partial \epsilon} | \leq K_\epsilon \quad \text{in} \quad [0, \epsilon_0]$$

uniformly in $r$ and $m$ (restricting $\epsilon_0$ if necessary). This implies in particular that $\tau \leq \tau' - \frac{\epsilon}{2}$ and $\tau + \Delta \geq \tau' + \frac{\epsilon}{2}$, for $\epsilon$ small.

Summarizing our discussion above, it can be seen that $c_\epsilon(\epsilon,r,m)$ is close to its $\epsilon=0$ value. It is less straightforward to interpret the values of $\tau(0,r,m)$ and $\Delta(0,r,m)$. Indeed when $\epsilon=0$, all assets are liquid and the switching times do not have any particular meaning. However, as seen above $\tau(\epsilon,r,m)$ and $\Delta(\epsilon,r,m)$ have well-defined positive limits as $\epsilon$ goes to zero. As seen from (B12), given $m$ is easy to calculate $\Delta(0,r,m)=\Delta(\epsilon,r,m)$. Given $m$ (or $\Delta$), one can calculate $\tau(0,r,m)$ directly from the no transactions costs case. This value $\tau(0,r,m)$ is the time such that accumulated wealth (when $\epsilon=0$) grows at a rate $\lambda$ between $\tau(0,r,m)$ and $\tau(0,r,m)+\Delta$.

Therefore, the consumption plan is well-defined and $\Delta, c_\epsilon, \tau$ vary smoothly with $\epsilon, r, m$.

To show that it is admissible, we have to prove that $I, \geq 0$ in $[0, \tau]$, $I, \leq 0$ in $[\tau, +\Delta, +\infty[$ and $a, \geq 0$, $A, \geq 0$.

We recall that

$$I, = \frac{dA,}{dt} - \lambda A, .$$

We briefly sketch proofs of the above statements.

In $[0, \tau]$, we have

$$I, = \frac{RA, + y, - c,}{1 + \epsilon} .$$

It is easy to see that $I, = I,_{\epsilon=0} + g(\epsilon,r,m,t)$ where $g=0$ for $\epsilon=0$ and

$$| \frac{\partial g}{\partial \epsilon} | \leq K, .$$
uniformly. Since \(\tau_t \leq \tau' - \frac{\varepsilon}{\delta} \) and \(I_{i|a}=0 \geq 0 \) in \([0,\tau' - \frac{\varepsilon}{\delta}]\), it follows that for \(\varepsilon \leq \varepsilon_0\) (restricting again \(\varepsilon\)), \(I_i > 0\) which implies that \(A_i \geq 0\).

In \([\tau_i, \tau_i + \Delta]\) again, \(a_i\) and \(i_i\) are very close to their \(\varepsilon = 0\) counterparts.

We can easily show that:

- In \([\tau' - \frac{\varepsilon}{\delta}, \tau' + \frac{\varepsilon}{\delta}]\), \(a_i > 0\) by continuity,
- In \([\tau_i, \tau' - \frac{\varepsilon}{\delta}]\), \(i_i > 0\) by continuity and
- In \([\tau' + \frac{\varepsilon}{\delta}, \tau_i + \Delta]\), \(i_i < 0\) by continuity.

This implies that \(a_i \geq 0\) in \([\tau_i, \tau_i + \Delta]\)

In \([\tau_i + \Delta, +\infty]\), simple calculations using

\[
A_i = \int_i^t \frac{c_i - y_i}{1 - \varepsilon} e^{-\rho(s) - \rho(i)} ds
\]

show that

\[
I_i = \frac{1}{1 - \varepsilon} \left[ \frac{\lambda + \delta}{R_B + \lambda + \delta} \bar{y} e^{-\lambda t} - \frac{\lambda + \omega_B}{R_B + \lambda + \omega_B} c_0 e^{-\mu_j t} e^{-\omega e^{-\rho_{j\alpha}(1 - (\tau_j + \Delta))}} \right]
\]

The continuity argument can be applied in a compact set \([\tau' + \frac{\varepsilon}{\delta}, T]\) to show that \(I_i < 0\), while since \(\delta - \omega_B \geq \eta_0 / 2 > 0\) for \(t \in [T, +\infty]\), \(I_i\) will be negative if \(T\) is large enough.

Finally

\[
A_i = \frac{1}{1 - \varepsilon} \left[ \frac{c_0}{R_B + \lambda + \omega_B} e^{-\mu_j t} e^{-\omega e^{-\rho_{j\alpha}(1 - (\tau_j + \Delta))}} - \bar{y} \frac{\bar{y}}{R_B + \lambda + \delta} e^{-\lambda t} \right]
\]

and similar arguments show that \(A_i \geq 0\). Therefore the consumption plan is admissible. This ends the proof of lemma B1.

\(\blacksquare\)

**Step 4: Optimality of the control:**
Having shown that our candidate optimal control is well defined and admissible, we will show that it is indeed optimal. For this purpose, we first show that it satisfies (B2)-(B6).

**Lemma B2:** Our candidate control satisfies conditions (B2)-(B6).

**Proof:** It is obvious that

in \([0,T,]\]

\[
\frac{d \log u'(c)}{dt} = \beta - R_L \leq \beta - r ,
\]

that in \([\tau, \tau + \Delta]\]

\[
\frac{d \log u'(c)}{dt} = \beta - r ,
\]

and that in \([\tau + \Delta, +\infty[\]

\[
\frac{d \log u'(c)}{dt} = \beta - R_b \leq \beta - r .
\]

Hence (B2)-(B3) are satisfied.

For \(t \in [\tau, +\infty[\)

\[
\int_{t}^{\tau + \Delta} u'(c) e^{-\beta (s-t)} ds = u'(c) \int_{t}^{\tau + \Delta} e^{-R_b (s-t)} ds = \frac{u'(c)}{R_b} = \frac{u'(c)}{R} (1 - \epsilon)
\]

It follows that (B5) is satisfied.

For \(t \in [\tau, \tau + \Delta]\)

\[
\int_{t}^{\tau + \Delta} u'(c) e^{-\beta (s-t)} ds = u'(c) \left[ \int_{t}^{\tau + \Delta} e^{-\beta (s-t)} ds + \int_{\tau + \Delta}^{\tau + \Delta + r} e^{-\beta (s-t)} e^{-R_b (s-\tau - \Delta)} ds \right]
\]

\[
= u'(c) \left[ \frac{1 - e^{-\beta (\tau + \Delta - t)}}{r} + (1 - \epsilon) \frac{e^{-\beta (\tau + \Delta - t)}}{R} \right].
\]

In addition

\[
\frac{1 - \epsilon}{R} \leq \frac{1 - e^{-\beta (\tau + \Delta - t)}}{r} + (1 - \epsilon) \frac{e^{-\beta (\tau + \Delta - t)}}{R} \leq \frac{1 - e^{-r \Delta}}{r} + (1 - \epsilon) \frac{e^{-r \Delta}}{R}.
\]

From equation (B10), we get that

\[
\frac{1 - e^{-r \Delta}}{r} + (1 - \epsilon) \frac{e^{-r \Delta}}{R} = \frac{1 + \epsilon}{R}
\]

(B16)
Equation (B6) follows.

Finally, for \( t \in [0, \tau_i] \), we have:

\[
\int_t^\tau u'(c_i)e^{-R(t-t)} \, ds = u'(c_i) \left[ (1+\epsilon) \frac{1-e^{-R\tau_i}}{R} + e^{-R\tau_i} \frac{1-e^{-\epsilon \tau_i}}{\epsilon} + (1-\epsilon) \frac{e^{-R\tau_i}}{R} e^{-\tau_i} \right]
\]

\[= \frac{1+\epsilon}{R} \]

because of (B16).

This proves equation (B4) and ends our proof of lemma B2.

We now show that the candidate policy is indeed optimal.

**Lemma B3:** The candidate policy is optimal.

**Proof:** To derive this lemma we shall show that no perturbation of the candidate policy can be utility enhancing.

Consider an alternative policy \((i, + \delta i, I, + \delta I)\) which induces consumption \(c_i + \delta c_i\). We know from (B4) that

\[
\delta c_i = r \int_0^t \delta_i e^{\lambda t} \, dt + R \int_0^t \delta I, e^{\lambda t} \, dt - \delta I_i - \epsilon (e^{|I_i + \delta I_i| - |I_i|})
\]

Using the concavity of \(u(\cdot)\), it follows that

\[
u(c_i + \delta c_i) \leq u(c_i) + u'(c_i) \left[ r \int_0^t \delta_i e^{\lambda t} \, dt + R \int_0^t \delta I, e^{\lambda t} \, dt - \delta I_i - \delta I_i - \epsilon (e^{|I_i + \delta I_i| - |I_i|}) \right]
\]

We next multiply equation (B17) by \(e^{(t+\lambda)\tau_i}\), integrate from 0 to \(t\), and get

\[
\int_0^t u(c_i + \delta c_i)e^{-(t+\lambda)\tau_i} \, ds \leq \int_0^t u(c_i)e^{-(t+\lambda)\tau_i} \, ds + K_i(t) + K_i(t)
\]

with

\[
K_i(t) = \int_0^t u'(c_i) \left[ r \int_0^t \delta_i e^{\lambda (t-h)} \, dh - \delta I_i \right] e^{-(t+\lambda)\tau_i} \, ds
\]
\[ K_1(t) = \int_0^t u'(c) \left( R \int_0^s \delta I_b e^{\lambda(s-b)} \, dh - \delta I_a - \epsilon \left( |I_a + \delta I_a| - |I_a| \right) \right) e^{-(\beta + \lambda)s} \, ds. \]

We will show that when \( t \) is goes to infinity, \( K_1(t) \) and \( K_2(t) \) are asymptotically non-positive.

Integrating the second term of \( K_1(t) \) by parts, we get

\[
\int_0^t u'(c) \delta I_b e^{-(\beta + \lambda)s} \, ds = \left[ \int_0^t \delta I_b e^{-\lambda b} \, dh \right] u'(c) e^{\lambda s} - \int_0^t \delta I_b e^{-\lambda b} \, dh \int_0^t \delta I_b e^{-\lambda b} \, dh \, \frac{d}{ds} (u'(c) e^{-\lambda s}) \, ds.
\]

Therefore, the \( K_1(t) \) equals

\[
\int_0^t \left[ u'(c) \left( r - \beta \right) + \frac{d u'(c)}{ds} \right] \left( \int_0^s \delta I_b e^{-\lambda b} \, dh \right) e^{-\lambda s} \, ds - \left( \int_0^t \delta I_b e^{-\lambda b} \, dh \right) u'(c) e^{-\lambda t}.
\]

We note that

\[
\int_0^s \delta I_b e^{-\lambda b} \, dh = e^{-\lambda s} \delta a_i.
\]

If \( a_i > 0 \), then condition (B2) holds and the integrand in the first term of (B18) above must be zero.

If \( a_i = 0 \), then from the short sale constraint \( \delta a_i \geq 0 \) and condition (B3) ensures that this integrand is non-positive. For \( t \geq \tau_i + \Delta_i \), \( a_i = 0 \) and thus \( \delta a_i \) must also be greater or equal than 0. This implies that the second term is non positive, for \( t \) large enough. We have therefore proven that \( K_1(t) \) is non positive for large \( t \).

Consider now \( K_2(t) \). Integrating by parts the first term, we get

\[
\int_0^t u'(c) \left( \int_0^s \delta I_b e^{\lambda(s-b)} \, dh \right) e^{-(\beta + \lambda)s} \, ds = \left[ \int_0^t u'(c) e^{-\beta b} \, dh \right] \left( \int_0^s \delta I_b e^{-\lambda b} \, dh \right) - \left[ \int_0^t u'(c) e^{-\beta b} \, dh \right] \delta I_a e^{-\lambda s} \, ds.
\]

Therefore \( K_2(t) \) is equal to

\[
\int_0^t \left[ \left( -u'(c) + R \int u'(c) e^{-\beta(b-s)} \, dh \right) \delta I_a - u'(c) \epsilon \left( |I_a + \delta I_a| - |I_a| \right) \right] e^{-(\beta + \lambda)s} \, ds - R \left( \int u'(c) e^{-\beta b} \, dh \right) \left( \int \delta I_a e^{-\lambda s} \, ds \right).
\]

If \( I_a > 0 \), \( |I_a + \delta I_a| - |I_a| \geq \delta I_a \), the integrand in (B19) is less or equal to

\[
[-u'(c)(1 + \epsilon) + R \int u'(c) e^{-\beta(b-s)} \, dh] \delta I_a = 0
\]

by condition (B4).
If \( I_1 < 0 \), \( |I_1 + \delta I_1| - |I_1| \geq -\delta I_1 \), the integrand is less or equal to

\[
[ -u'(c_1)(1-\epsilon) + R \int u'(c_b)e^{-\phi(b-t)}dh ] \delta I_1 = 0
\]
by condition (B5).

If \( I_1 = 0 \) and \( \delta I_1 > 0 \), the integrand is

\[
[ -u'(c_1)(1+\epsilon) + R \int u'(c_b)e^{-\phi(b-t)}dh ] \delta I_1 \leq 0
\]
by condition (B6).

If \( I_1 = 0 \) and \( \delta I_1 < 0 \), the integrand is

\[
[ -u'(c_1)(1-\epsilon) + R \int u'(c_b)e^{-\phi(b-t)}dh ] \delta I_1 \leq 0
\]
by condition (B6).

Therefore the first term in \( K_\delta(t) \) is less or equal to 0.

For the second term, note that

\[
\int_0^t \delta I_1 e^{-\mu t} ds = e^{-\mu t} \delta A_1
\]
and that

\[
\int u'(c_1)e^{-\phi t}ds = e^{-\phi t} u'(c_1) \frac{1-\epsilon}{R} = e^{-\phi t} \frac{1-\epsilon}{R} \quad \text{for} \quad t \geq \tau_1 + \Delta.
\]

Since \( \delta A_1_A_1 (\delta A_1 + A_1 \geq 0) \), \( A_1 \sim \exp(-\omega_b t) \) and \( \lambda + \omega_b > \eta_0/2 \) for \( \epsilon \) small, it is clear that the term

\[
- R \int u'(c_1) e^{-\mu t} ds \int_0^t \delta I_1 e^{-\mu t} ds
\]

is smaller than an arbitrary \( \xi > 0 \) for \( t \) large enough. It follows that \( K_\delta(t) \) is asymptotically non positive.

This concludes the proof of lemma B3.
Proof of Lemma 4.3:

Suppose that \( r = r' \) and that \( R = r' + m' \epsilon \). From appendix B, we know that for \( \epsilon \) small, the optimal consumption at time 0 is given by equation (19), which as we have seen be rewritten as

\[
\bar{y} \left[ \frac{1 - e^{-(R_L + \lambda + \delta) r_f}}{R_L + \lambda + \delta} + e^{-(R_L + \lambda + \delta) r_f} \frac{1 - e^{-(r_s + \lambda + \delta) \Delta}}{r' + \lambda + \delta} + \frac{e^{-(R_L + \lambda + \delta) r_f} e^{-(r_s + \lambda + \delta) \Delta}}{R_B + \lambda + \delta} \right] =
\]

\[
c_0 \left[ \frac{1 - e^{-(R_L + \lambda + \omega_L) r_f}}{R_L + \lambda + \omega_L} + e^{-(R_L + \lambda + \omega_L) r_f} \frac{1 - e^{-(r_s + \lambda + \omega) \Delta}}{r' + \lambda + \omega} + \frac{e^{-(R_L + \lambda + \omega_L) r_f} e^{-(r_s + \lambda + \omega) \Delta}}{R_B + \lambda + \omega_B} \right]
\]

The first term in brackets (divided by \( \bar{y} \)) can be written as:

\[
\frac{1}{R_L + \lambda + \delta} + e^{-(R_L + \lambda + \delta) r_f} \left( \frac{1}{r' + \lambda + \delta} - \frac{1}{R_L + \lambda + \delta} \right) + e^{-(R_L + \lambda + \delta) r_f} e^{-(r_s + \lambda + \delta) \Delta} \left( \frac{1}{R_B + \lambda + \delta} - \frac{1}{r' + \lambda + \delta} \right).
\]

Straightforward algebra shows that this term can be written as:

\[
\frac{1}{\varphi} \left[ 1 - \frac{m' - r'}{\varphi} \epsilon + \frac{m' - r'}{\varphi} e^{-\varphi r_f} + \frac{m' - r'}{\varphi} e^{-\varphi(\delta + \lambda)} + o(\epsilon) \right]
\]

or equivalently, as:

\[
\frac{1}{\varphi} \left[ 1 - \epsilon (m' - r') \int_0^{r_f} e^{-\varphi t} dt - \epsilon (m' - r') \int_{r_f + \delta} \varphi e^{-\varphi t} dt + o(\epsilon) \right].
\]

Similarly, the second can be written as:

\[
\frac{1}{\psi} \left[ 1 - \epsilon (1 - \frac{1}{A}) (m' - r') \int_0^{r_f} e^{-\psi t} dt - \epsilon (1 - \frac{1}{A}) (m' + r') \int_{r_f + \delta} \psi e^{-\psi t} dt + o(\epsilon) \right].
\]

Combining (C2) and (C3) we get equation (24a) i.e

\[
c_0 = \bar{y} \frac{\psi}{\varphi} + \epsilon C_w + \epsilon C_s + o(\epsilon)
\]

with \( C_w \) and \( C_s \) given by (24c) and (24d). Integrating (24c) by parts, we get
\[
(m' - r') \int_0^t [e^{\psi t} - e^{-\alpha t}] dt + (m' + r') \int_0^t [e^{-\omega t} - e^{-\alpha t}] dt
\]

\[
= (m' - r') \left[ -\frac{e^{-\omega t}}{1 - e^{-\omega r}} (e^{-(\omega + \lambda)t} - e^{-(\theta + \lambda)t}) \right]_0^t + (m' - r') \int_0^t \frac{e^{-\omega t}}{r'} ((\delta + \lambda)e^{-(\theta + \lambda)t} - (\omega + \lambda)e^{-(\theta + \lambda)t})
\]

\[
+ (m' + r') \left[ -\frac{e^{-\omega t}}{1 - e^{-\omega r}} (e^{-(\omega + \lambda)t} - e^{-(\theta + \lambda)t}) \right]_0^t + (m' + r') \int_0^t \frac{e^{-\omega t}}{r'} [(\delta + \lambda)e^{-(\theta + \lambda)t} - (\lambda + \omega)e^{-(\lambda + \omega)t}] dt.
\]

Using equations (21a) (substituting \( w_i \) from (11)) and (22), we find that the terms in brackets cancel out which yields equation (24b). This concludes the proof of lemma 4.3.

Proof of proposition 4.2:

Consider \( r' \) and \( m' \) where

\[
m' = r' \frac{1 + e^{-r' \alpha}}{1 - e^{-r' \alpha}}.
\]

The results of Appendix B imply that for \( \epsilon \) sufficiently small and for \( r \) and \( m \) belonging to a neighborhood of \( r' \) and \( m' \), the consumption plan defined in Proposition 4.1 is indeed well defined, admissible and optimal.

These results also imply that

\[
F(\epsilon, r, m) = \left( \int_0^t \lambda e^{-\delta t} \alpha_i dt, \int_0^t \lambda e^{-\delta t} A_i dt \right)
\]

is \( C^\infty \) in these variables.

We know that

\[
F(0, r', m') = ( (1 - k) \frac{D}{r'}, k \frac{D}{r'} ).
\]

It is also easy to see that

\[
\frac{\partial F}{\partial m}(0, r, m) = - \frac{\partial F}{\partial m}(0, r, m) \neq 0.
\]
(If $\epsilon=0$, $r$ is kept constant and $m$ changes, it is as if the total wealth is kept constant and the proportion of the wealth in liquid and illiquid assets changes.)

It is also clear that
\[
\frac{\partial F_1}{\partial r}(0,r,m) + \frac{\partial F_2}{\partial r}(0,r,m) = 0.
\]

(A change in interest rate has a non-zero effect on the total stock of wealth, at least in the $\epsilon=0$ case).

Therefore the Jacobian matrix
\[
\begin{bmatrix}
\frac{\partial F_1}{\partial r} & \frac{\partial F_2}{\partial r} \\
\frac{\partial F_1}{\partial m} & \frac{\partial F_2}{\partial m}
\end{bmatrix}
\]
is invertible and we can apply the implicit function theorem at the point $(0,r',m')$.

It is thus clear that for $\epsilon$ small there exists an equilibrium. Moreover $r$ and $m$ are $C^\infty$ in $\epsilon$ which establishes proposition 4.2.

We now calculate how total wealth changes when $\epsilon$ changes for fixed $r'$ and $m'$ i.e prove lemma 4.4.

**Proof of lemma 4.4:**

Adding the equations describing the evolution of $a_t$ and $A_t$ we get
\[
\frac{d}{dt}(a_t + A_t) = \lambda(a_t + A_t) + i_t + I_t
\]
\[
= \lambda(a_t + A_t) + ra_t + RA_t + y_t - c_t - \epsilon |I_t|,
\]
\[
= (\lambda + r)(a_t + A_t) + (R - r)A_t + y_t - c_t - \epsilon |I_t|.
\]
\[
\frac{d}{dt}(a_t + A_t) - \lambda(a_t + A_t) = r(a_t + A_t) + (R - r)A_t + y_t - c_t - \epsilon |I_t|.
\]

Multiplying by $e^{\lambda t}$, integrating from 0 to $\infty$ and using the fact (established in appendix B) that $(a_t + A_t)e^{\lambda t}$ goes to zero as $t$ goes to infinity, we get
\[
\int_0^t \lambda (\alpha_t + \mathbf{A}_t) \, e^{-\lambda t} \, dt = \frac{\lambda}{r} \int_0^t e^{-\lambda t} \left[ \mathbf{c}_t + \epsilon | I_t | - y_t - (R - r) \mathbf{A}_t \right] \, dt .
\] (C4)

Equation (C4) will allow us to calculate total wealth as a function of \( \epsilon \). We will assume that we are at \((\epsilon, \tau^*, \mathbf{m}^*)\)

\[
\int_0^t \epsilon | I_t | \, e^{-\lambda t} \, dt = \int_0^t \epsilon \left[ \frac{dA_t}{dt} - \lambda \mathbf{A}_t \right] e^{-\lambda t} \, dt - \int_{\tau^*+A}^t \epsilon \left[ \frac{dA_t}{dt} - \lambda \mathbf{A}_t \right] e^{-\lambda t} \, dt
\]

\[
= \epsilon A_{\tau^*} e^{-\lambda \tau^*} + \epsilon A_{\tau^*+A} e^{-\lambda (\tau^*+A)}
\] (C5)

\[
= 2 \epsilon A_{\tau^*} e^{-\lambda \tau^*} \quad \text{(since} \quad A_{\tau^*+A} = A_{\tau^*} e^{-\lambda A})
\]

\[
= 2 \epsilon \frac{\bar{y}}{\phi} \left[ e^{-\lambda (\mathbf{m}^*)_{\tau^*}} - e^{-\lambda (\mathbf{m}^*+r^* \mathbf{A})} \right] + o(\epsilon).
\]

\[
\int_0^t e^{-\lambda t} c_t \, dt = c_0 \int_0^t e^{-(\mathbf{m}^*+r^* \mathbf{A})t} \, dt + e^{-(\mathbf{m}^*+r^* \mathbf{A})\tau^*} \int_{\tau^*+A}^t e^{-(\mathbf{m}^*+r^* \mathbf{A})(t-\tau^*))} \, dt + e^{-(\mathbf{m}^*+r^* \mathbf{A})A} \int_{\tau^*+A}^t e^{-\lambda (\mathbf{m}^*-r^* \mathbf{A})}(t-\tau^*)) \, dt
\]

\[
= c_0 \left[ \frac{1 - e^{-(\mathbf{m}^*+r^* \mathbf{A})\tau^*}}{\lambda + \omega^*} + e^{-(\mathbf{m}^*+r^* \mathbf{A})\tau^*} \frac{1 - e^{-(\mathbf{m}^*+r^* \mathbf{A})A}}{\lambda + \omega^*} \right]
\]

\[
= \frac{c_0}{\lambda + \omega^*} \left\{ (1 + \epsilon \frac{\mathbf{m}^*-r^*}{\lambda + \omega^*}) + e^{-(\mathbf{m}^*+r^* \mathbf{A})\tau^*} \left[ 1 - (1 + \epsilon \frac{\mathbf{m}^*-r^*}{\lambda + \omega^*}) \right] + e^{-(\mathbf{m}^*+r^* \mathbf{A})(\tau^*+A)} \left[ 1 + \epsilon \frac{\mathbf{m}^*-r^*}{\lambda + \omega^*} \right] + o(\epsilon) \right\}
\]

\[
= \frac{c_0}{\lambda + \omega^*} \left\{ 1 + \epsilon \frac{\mathbf{m}^*-r^*}{\mathbf{A}} \int_0^{\tau^*} e^{-(\mathbf{m}^*-r^* \mathbf{A})t} \, dt + (\mathbf{m}^*+r^*) \int_{\tau^*+A}^t e^{-(\mathbf{m}^*+r^* \mathbf{A})dt} \right\} + o(\epsilon).
\] (C6)

Replacing for \( c_0 \), we get that

\[
\int_0^t e^{-\lambda t} c_t \, dt = \bar{y} \frac{\Psi}{\phi^* (\lambda + \omega^*)} \left\{ 1 + \epsilon \left[ (\mathbf{m}^*-r^*) \int_0^{\tau^*} (e^{-\lambda t} - e^{-\lambda t^*}) \, dt + (\mathbf{m}^*+r^*) \int_{\tau^*+A}^t (e^{-\lambda t} - e^{-\lambda t^*}) \, dt \right] + \right\}
\]

\[
= \frac{\epsilon}{\mathbf{A}} \left[ (\mathbf{m}^*-r^*) \int_0^{\tau^*} (e^{-\lambda \mathbf{m}^* t} - e^{-\lambda \mathbf{m}^* t^*}) \, dt + (\mathbf{m}^*+r^*) \int_{\tau^*+A}^t (e^{-\lambda \mathbf{m}^* t} - e^{-\lambda \mathbf{m}^* t^*}) \, dt \right] + o(\epsilon).
\] (C7)

Using (C4), (C5) and (C7), it is easy to find the sensitivity of total asset demand \( F_1 + F_2 \) to the transactions costs \( \epsilon \). The equations in lemma 4.4 follow. ■
We next derive the expression for $b-m^*$. For this purpose we differentiate the equilibrium condition

$$F_1 + F_2 = \int_0^\tau \lambda (a_i + A_i) e^{-at} dt = (1-k) \frac{D}{r} + k \frac{D}{r+m \epsilon}$$

with respect to $\epsilon$. Note that when $\epsilon$ changes, the equilibrium values of $r^*$ and $m^*$ will change. Hence we get

$$\left. \frac{\partial (F_1 + F_2)}{\partial \epsilon} \right|_{\epsilon=0} + \left. \frac{\partial (F_1 + F_2)}{\partial r} \right|_{\epsilon=0} (b-m^*) + \left. \frac{\partial (F_1 + F_2)}{\partial m} \right|_{\epsilon=0} \frac{dm}{d\epsilon} \bigg|_{\epsilon=0}$$

$$= - (1-k) \frac{D}{r} \bigg|_{\epsilon=0} (b-m^*) - k \frac{D}{(r+m \epsilon)^2} \bigg|_{\epsilon=0} (b-m^*) - k \frac{D \epsilon}{(r+m \epsilon)^2} \bigg|_{\epsilon=0} \frac{dm}{d\epsilon} \bigg|_{\epsilon=0} - k \frac{D m}{(r+m \epsilon)^2} \bigg|_{\epsilon=0}.$$

Using the fact that $F_1 + F_2$ is independent of $m$ for $\epsilon=0$, we have

$$\left. \frac{\partial (F_1 + F_2)}{\partial \epsilon} \right|_{\epsilon=0} + \left. \frac{\partial (F_1 + F_2)}{\partial r} \right|_{\epsilon=0} (b-m^*) = - \left. \frac{D}{(r^*)^2} (b-m^*) - \frac{m^*}{r^*} k \frac{D}{r^*}.$$

Using equations (25a)-(25c), we thus get

$$(b-m^*) \left( \left. \frac{\partial (F_1 + F_2)}{\partial r} \right|_{\epsilon=0} + \frac{D}{(r^*)^2} \right) = - \left[ \frac{\lambda}{\lambda + \omega} \psi' \left( (m^*-r') \int_0^{r' \lambda} (e^{-\omega t'} - e^{-\omega t}) dt + (m^* + r') \int_{r' \lambda}^{r' \lambda + \Delta} (e^{-\omega t'} - e^{-\omega t}) dt \right] + \frac{2 \lambda \bar{y}}{\phi' r^*} (e^{-\omega t'} - e^{-\omega t}) + \frac{1}{\lambda + \omega} \frac{\psi'}{\phi' r^*} \left( (m-r') \int_0^{r' \lambda} (e^{-\omega t'} - e^{-\omega t}) dt + (m + r') \int_{r' \lambda}^{r' \lambda + \Delta} (e^{-\omega t'} - e^{-\omega t}) dt \right)$$

We can calculate

$$\left. \frac{\partial (F_1 + F_2)}{\partial r} \right|_{\epsilon=0}$$

from

$$F_1 + F_2 = \frac{\delta - \omega}{(\lambda + \omega) \phi}$$

and derive $(b-m^*)$.

The right hand side is negative and it is easy to see that the coefficient of $(b-m^*)$ is positive.

Therefore $(b-m^*)$ is negative.
Appendix D

Proof of Lemma 4.5:

If \( k \) increases, obviously \( \Delta^* \) decreases. Moreover \( \tau_1^* \) increases and \( \tau_1^* + \Delta^* \) decreases. These can be seen from equations (21a) and (21b) with very simple algebra. Equation (22) implies then that \( m^* \) increases.

It is thus clear that \((b - m^*)\) which is proportional to

\[
\frac{\lambda}{\lambda + \omega} \bar{y} \frac{\Psi}{\varphi^*} \left[ \int_0^{\tau_1^*} (e^{-\varphi^* t} - e^{-\varphi^* t}) \, dt + \int_0^{\tau_1^*} (e^{-\varphi^* t} - e^{-\varphi^* t}) \, dt \right] + \frac{2\lambda \bar{y}}{\varphi^* \tau^*} (e^{-(\lambda - \omega) \tau_1^*} - e^{-(\lambda - \omega) \tau_1^*}) +
\]

\[
\frac{1}{A} \frac{\lambda}{\lambda + \omega} \bar{y} \frac{\Psi}{\varphi^* \tau^*} \left[ \int_0^{\tau_1^*} (e^{-(\lambda - \omega) t} - e^{-(\lambda - \omega) t}) \, dt + \int_0^{\tau_1^*} (e^{-(\lambda - \omega) t} - e^{-(\lambda - \omega) t}) \, dt \right]
\]

decreases, i.e. the effect of \( \epsilon \) on \( r^* \) is stronger (note that the function \( g(\cdot): t \to e^{(\omega + \lambda) t} - e^{(k + \lambda) t} \)

increases in \([0, \tau^*]\) and \( \tau_1 < \tau^* \).

\[\square\]
APPENDIX E

The Case k=1

The case of k=1 is slightly different. We find, as before, that transactions costs have a first order effect on the rate of return on the illiquid asset. The difference with the case (0<k<1) is that if we introduce a liquid asset in this economy in zero supply, that cannot be sold short, its return will be lower than the return on the illiquid asset by zero-th order term. (i.e. we have a zero-th order liquidity premium.) The reason for this result is that the minimum holding period has a first order length.

Since all of the consumer’s wealth is held in the form of the illiquid asset, we have:

\[ A_t = w_t. \]

The dynamics of \( w_t \) are described by:

\[ dw_t = \lambda w_t dt + I_t dt \] (E1a)

\[ c_t = y_t + R_t - I_t - \varepsilon |I_t|. \] (E1b)

Proposition E.1 describes the optimal policy of the consumer for small transactions costs and for a subset of values of \( R \) that are of interest, i.e. such that its equilibrium value belong to this subset.

**Proposition E.1:** For \( \varepsilon \) small and for \( R \) belonging to a subset of their possible values, the optimal policy has the following form: The consumer buys the (illiquid) asset until an age \( \tau_v \). He does nothing (i.e. he consumes his income \( y_t + R w_t \)) from \( \tau_v \) until an age \( \tau_v + \Delta \) when he starts selling the asset until he dies.

The proof of proposition E.1 is analogous to the proof of proposition 4.1 and is therefore omitted.

In what follows, we will (briefly) discuss the implications of proposition E.1. We will show how the
consumption $c_i$ as well as the width of the inaction period $\Delta$ can be derived.

In the case $k=1$, the expression for consumption $c_i$ must be changed from equation (16) to

$$c_i = c_0 e^{-\omega(t)} \quad t \leq \tau_i$$  \hfill (E2a)

$$c_i = R w_i + y_i \quad \tau_i \leq t \leq \tau_i + \Delta$$  \hfill (E2b)

$$c_i = c_{\tau_i - \Delta} e^{\omega(\tau_i - \Delta) - \omega(t)} \quad t \geq \tau_i + \Delta.$$  \hfill (E2c)

Again the initial consumption $c_0$ can be obtained from the intertemporal budget equation which in this case can be written as

$$\int_{0}^{\tau_i} \frac{y_i - c_i}{1 + \epsilon} e^{-\rho(t)} \, dt + \int_{\tau_i}^{\tau_i + \Delta} (y_i - c_i) e^{-\rho(t)} \, dt + \int_{\tau_i - \Delta}^{\tau_i} \frac{y_i - c_i}{1 - \epsilon} e^{-\rho(t)} \, dt = 0$$  \hfill (E3)

where

$$\hat{\rho}(t) = R_k + \lambda \quad \text{for} \quad t < \tau,$$

$$\hat{\rho}(t) = R + \lambda \quad \text{for} \quad \tau \leq t < \tau + \Delta,$$

$$\hat{\rho}(t) = R_b + \lambda \quad \text{for} \quad \tau + \Delta \leq t$$

and

$$\rho(t) = \int_{0}^{t} \hat{\rho}(s) \, ds.$$

This intertemporal budget equation is derived from the equation describing the evolution of $w_i$, using our results on the investment policy of the consumer. The analysis is very similar to the case $0 < k < 1$ and is omitted.

The parameter $\Delta$ condition is derived by the portfolio decision of a consumer of age $\tau_i$. This consumer is indifferent between investing in the illiquid asset and not doing so. Given that he starts selling the illiquid asset at $\tau_i + \Delta$, the change in his utility if he buys one unit of the illiquid asset at $\tau_i$ is:
\[-(1 + \varepsilon)P'u'(c_{t\tau}) + (1 - \varepsilon)P'u'(c_{t\tau + \Delta})e^{-\delta\Delta} + \int_{t\tau}^{t\tau + \Delta} Du'(c_t)e^{-\delta(t-t')}dt \]  
(E4)

and is equal to zero. From equations (E1a)\(^2\) and (E2) we get that the following relation between consumption at age \(\tau_i\) and consumption at age \(\tau_i + \Delta\)

\[ \frac{c_{\tau_i + \Delta} - y_{\tau_i + \Delta}}{R} = w_{\tau_i + \Delta} = w_{\tau_i}e^{\delta\Delta} = \frac{c_{\tau_i} - y_{\tau_i}}{R}e^{\delta\Delta}. \]

(E5)

Equations (E4) and (E5) yield the following relation which shows that the minimum period of holding the illiquid asset is of order \(\varepsilon\)

\[ \Delta = \frac{\psi^*}{(\lambda + \omega^*)(\delta - \omega^*)} \frac{2\varepsilon}{A} + o(\varepsilon). \]

(E6)

Having characterized the solution to the consumer's problem we turn to the equilibrium determination of \(R\).

In equilibrium, asset demand

\[ \int_0^\infty \lambda e^{-\lambda t}w_t dt \]

equals asset market value, \(D/R\).

As before, we will consider small transactions costs (small values of \(\varepsilon\), and find their first order effects on \(R\). We will thus write:

\[ R(\varepsilon) = r^* + b\varepsilon + o(\varepsilon) \]

(E7)

and calculate \(b\). Proposition E.2 gives us \(b\).

**Proposition E.2:** In equilibrium, \(R\) is uniquely determined. It has the form of equation (33), with \(b\) having an ambiguous sign.

\(^2\)Note that \(I_t=0\) between \(\tau_i\) and \(\tau_i + \Delta\).
The proof of proposition E.2 as well as the analytic expression for b are again omitted.

The discussion on the determination of b is similar to the discussion offered at the end of subsection 4.2.2, (the effects are similar) so we do not present it here. Instead, we focus on the determination of the rate of return on a liquid asset that is introduced in this economy in zero supply. Proposition E.3 gives us the rate of return on such an asset, as well as the implicit liquidity premium.

**Proposition E.3:** In equilibrium, transactions costs have a zero-th order effect on the rate of return on the liquid asset and on the liquidity premium.

Proposition E.3 states that we have a zero-th order liquidity premium. The reason for this result is that the minimum holding period has a first order length. We can show that

\[
    r = r^* - \frac{2\varepsilon}{\Delta} + o(1) = r^* - A \frac{(\lambda + \omega^*) (\delta - \omega)}{\psi} + o(1)
\]

\[
    \mu = A \frac{(\lambda + \omega^*) (\delta - \omega)}{\psi} + o(1) .
\]
Holdings of the liquid and illiquid assets

Figure 1
Figure 2

This figure plots the rates of return as a function of the fraction $k$ of illiquid assets. The solid line represents the benchmark case where transactions costs. The dotted line represents the rate of return on the illiquid asset while the dashed line represents the rate of return on the liquid asset. This example considers the following numerical values:

$$\lambda = 2\%; \delta = 4\%; \beta = .2\%; A = 1; D/Y = 50\%; \epsilon = 3\%.$$
Figure 3

This figure plots the rates of return as a function of the fraction $k$ of illiquid assets. The solid line represents the benchmark case where transactions costs are zero. The dotted line represents the rate of return on the illiquid asset while the dashed line represents the rate of return on the liquid asset.

This example considers the following numerical values:

$$\lambda = 20\%; \delta = 40\%; \beta = 2\%; A = 1; D/Y = 50\%; \epsilon = 3\%.\nonumber$$

Compared to figure 1, in this case the agent is more impatient and has therefore a shorter horizon. As a result, the interest rates and the liquidity premium are higher than in figure 1. Qualitative results are however unchanged.
Figure 4

This figure plots the rates of return as a function of the fraction $k$ of illiquid assets. The solid line represents the benchmark case where transactions costs are zero. The dotted line represents the rate of return on the illiquid asset while the dashed line represents the rate of return on the liquid asset. This example considers the following numerical values:

$$\lambda = 2\%; \delta = 4\%; \beta = .2\%; A = 1; D/Y = 300\%; e = 3\%.$$  

In this case, the importance of financial income compared to labor is such that, when $k$ is close to 1, the following paradoxical phenomenon occurs: transactions costs lower the rates of return on both assets.