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Estimating The Variance Parameter From Noisy High Frequency Financial Data

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Estimating The Variance Parameter From Noisy High Frequency Financial Data

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Abstract

Many financial data are now collected at an ultra-high frequency, such as tick-by-tick. However, increasing the observation frequency while keeping the time span of the observation fixed does not always help in estimating parameters. A different type of consistency, the consistency of a estimator as the observation frequency goes to infinity, becomes important in studying high frequency data. In addition to the consistency, the deviation of a financial time series from a continuous process is also increasingly significant as the observation frequency increases. This deviation is not negligible and causes another difficulty in estimating parameters. This paper concentrates on constructing estimators of variance parameter using contaminated observations; i.e., observations from a continuous process with deviation at time of observation. The consistencies of these estimators, as the observation frequency goes to infinity, are analyzed.

Key Words: f-consistency; observation noise; quadratic estimator.

1 Introduction

I start with an example of estimating the mean parameter μ in a simple process $dB(t) = \mu dt + \sigma dW(t)$. For a fixed span of observation interval and $\sigma = 1$, does increasing the observation frequency help the estimation? This type of question has come up recently in studying high frequency financial time series. We now accumulate more and more financial data not because time goes fast, but because data are recorded more frequently. We have gone from quarterly data to monthly data to weekly, daily and now tick-by-tick data. How does more data help us in estimating parameters? It may surprise many people to know that increasing observation frequency while keeping the span of observation fixed does not always help in estimating the parameter. In the case of estimating the mean parameter, increasing observation frequency does not help the estimation at all. When σ is given, the minimum sufficient statistic for the mean parameter is the difference of the two end observation points. The difference of these two end observation points does not change as the observation frequency increases. However, when the variance parameter is interested, increasing observation frequency does help the estimation. The quadratic variation is a consistent estimator of the variance parameter as the observation frequency in the limit. This raises a new consistency problem, f-consistency, the consistency when the observation frequency goes to infinity while time span keeping fixed.

There is another issue associated with using high frequency data. When the observation frequency increases, the difference of financial data from a continuous process becomes increasingly significant. For example, as observation frequency increases, the variance of price increment does not approach zero. The first order autocorrelation of the increments is strongly negative. We often neglect such a difference in using low frequency data such as daily or monthly prices. This difference is not negligible in high frequency data. However, this should not keep us from using a continuous process for high frequency data. I suggested (Zhou 1991) that the high frequency financial data can be viewed as observations from a diffusion process with observation noises:

$$S(t) = P(t) + \epsilon_t, \quad t \in [a, b], \tag{1}$$

where $P(t)$ is a diffusion process

$$dP(t) = \mu(t) + \sigma(t) dW_t. \tag{2}$$

I call the diffusion process the signal process and the ϵ_t observation noise. The observation noise is the deviation of data from the continuous process and is assumed to be independent from the diffusion process. Many things contribute to this observation noise. In the currency market, for example, non-binding quoting error is part of the noise. In other markets, bid and offer difference also contributes to the observation noise. Many other micro-structural behaviors are all included in this so-called observation noise. For low frequency observations, the observation noise is overwhelmed by the signal change. When observation frequency increases, the signal change becomes smaller and smaller while the size of the noise remains the same. The noise totally dominates the price change in ultra-high frequency data. Viewing high frequency data as observation with noise certainly captures many basic characteristics of high frequency financial time series mentioned above.

In this paper, I concentrate on constructing the estimators of the variance parameter using noisy high frequency observations. The f-consistency is investigated for each estimator. Without loss of generality, I assume that the time span considered here is [0,1], which can be an hour or a month. The parameter to be estimated is

$$\sigma^2 = \int_0^1 \sigma^2(t)dt = \text{Var}(P(1) - P(0)). \tag{3}$$

This paper is organized as follows. In Section 2, I study the f-consistency of the maximum likelihood estimator under the assumption of Gaussian noise and a constant variance parameter. In Section 3, I explore the estimator by the method of moment under more relaxed assumptions. In Section 4, I construct an optimal quadratic estimator. In section 5, I investigate the sensitivity of each estimator to its assumptions and give an overall evaluation of each estimator.

2 F-consistency of The Maximum Likelihood Estimator

In this section, I assume that the process (1) has independent Gaussian noise with constant variance and the signal process (2) has $\mu(t) = \mu t$, $\sigma(t) = \sigma$. Under these assumptions, I can obtain a maximum likelihood estimator

(MLE). Solving the different equation (2), I have

$$S(t) = \mu t + \sigma W(t) + \epsilon_t, \quad t \in [0,1], \tag{4}$$

where $W(t)$ is a standard Winner process and ϵ_t are independent Gaussian random variables with mean zero and variance η^2. Taking $n+1$ equally spaced observations from [0,1], I have $\{S_{0,n}, S_{1,n}, ..., S_{n,n}\}$ such that

$$X_{i,n} = S_{i,n} - S_{i-1,n} = \frac{\mu}{n} + \frac{\sigma}{\sqrt{n}} Z_i + \epsilon_i - \epsilon_{i-1} \tag{5}$$

where Z_i is a standard Gaussian random variable. The joint distribution of $\{X_{1,n}, ..., X_{n,n}\}$ is a multivariate normal distribution with mean zero and variance matrix

$$\Sigma_n = \frac{\sigma^2}{n} I_n + \eta^2 A_n \tag{6}$$

where I_n is an identity matrix and

$$A_n = \begin{pmatrix} 2 & -1 & 0 & \dots & 0 & 0 \\ -1 & 2 & -1 & \dots & 0 & 0 \\ 0 & -1 & 2 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 2 & -1 \\ 0 & 0 & 0 & \dots & -1 & 2 \end{pmatrix} \tag{7}$$

The eigenvalues and eigenvectors of this matrix are known (Gregory and Karney, 1969)

$$\lambda_i = 4 \sin^2\left(\frac{i}{2(n+1)}\right) \tag{8}$$

and

$$v_i = (2/(n+1))^{1/2} \begin{pmatrix} \sin(i\pi/(n+1)) \\ \sin(2i\pi/(n+1)) \\ \vdots \\ \sin(ni\pi/(n+1)) \end{pmatrix}. \tag{9}$$

The log-likelihood function of X is

$$l(\mu, \sigma^2, \eta^2; X) = -\frac{n}{2}\log(2\pi) - \frac{1}{2}\log|\Sigma_n| - \frac{1}{2}(X - \frac{\mu}{n})^T \Sigma_n^{-1} (X - \frac{\mu}{n}) \tag{10}$$

The derivatives of the log-likelihood function with respect to each parameter are

$$\frac{dl(\mu,\sigma^2,\eta^2;X)}{d\mu} = (X-\frac{\mu}{n})^T\Sigma_n^{-1}\frac{e}{n} \tag{11}$$

$$\frac{dl(\mu,\sigma^2,\eta^2;X)}{d\sigma^2} = -\frac{1}{2n}\text{tr}(\Sigma_n^{-1})+\frac{1}{2n}(X-\frac{\mu}{n})^T\Sigma_n^{-1}\Sigma_n^{-1}(X-\frac{\mu}{n}) \tag{12}$$

$$\frac{dl(\mu,\sigma^2,\eta^2;X)}{d\eta^2} = -\frac{1}{2}\text{tr}(\Sigma_n^{-1}A_n)+\frac{1}{2}(X-\frac{\mu}{n})^T\Sigma_n^{-1}A_n\Sigma_n^{-1}(X-\frac{\mu}{n}) \tag{13}$$

Rewrite matrix A as

$$A_n = V_n\Lambda_n V_n^T, \tag{14}$$

where e is a vector with all elements 1 and $V_n = (v_1, ..., v_n)$ is the matrix consisting of eigenvectors defined in (8) and Λ_n =diag(λ_i) is a diagonal matrix with diagnal elements being the eigenvalues defined in (9). The inverse of the covariance matrix

$$\Sigma_n^{-1} = V\text{diag}(\frac{1}{\sigma^2/n+\eta^2\lambda_i})V_n^T,$$

Let

$$(Y_1, ..., Y_n)^T = V_n^T X$$

Let $v_{\cdot i} = \sum_j v_{ij}$ and notice that $\sum_j v_{ij} = 1$. The MLE of μ, σ^2 and η^2 is the solution of equations:

$$0 = \sum_{i=1}^n \frac{Y_i v_{\cdot i}}{(\sigma^2/n+\eta^2\lambda_i)} - \frac{\mu}{n}\sum_{i=1}^n \frac{1}{(\sigma^2/n+\eta^2\lambda_i)} \tag{15}$$

$$0 = -\sum_{i=1}^n \frac{1}{2n(\sigma^2/n+\eta^2\lambda_i)} + \sum_{i=1}^n \frac{(Y_i-\mu/n)^2}{2n(\sigma^2/n+\eta^2\lambda_i)^2} \tag{16}$$

$$0 = -\sum_{i=1}^n \frac{\lambda_i}{2(\sigma^2/n+\eta^2\lambda_i)} + \sum_{i=1}^n \frac{(Y_i-\mu/n)^2\lambda_i}{2(\sigma^2/n+\eta^2\lambda_i)^2} \tag{17}$$

The Fisher information matrix of (μ, σ^2, η^2) is

$$I(\mu,\sigma^2,\eta^2) = \begin{pmatrix} \sum_{ij}[\Sigma^{-1}]_{ij}/n^2 & 0 & 0 \\ 0 & \frac{1}{2n^2}\text{tr}(\Sigma_n^{-1}\Sigma_n^{-1}) & \frac{1}{2n}\text{tr}(\Sigma_n^{-1}A_n\Sigma_n^{-1}) \\ 0 & \frac{1}{2n}\text{tr}(\Sigma_n^{-1}A_n\Sigma_n^{-1}) & \frac{1}{2}\text{tr}(\Sigma_n^{-1}A_n\Sigma_n^{-1}A_n) \end{pmatrix}$$

$$= \begin{pmatrix} \sum \frac{1}{n^2(\sigma^2/n+\eta^2\lambda_i)} & 0 & 0 \\ 0 & \sum \frac{1}{2n^2(\sigma^2/n+\eta^2\lambda_i)^2} & \sum \frac{\lambda_i}{2n(\sigma^2/n+\eta^2\lambda_i)^2} \\ 0 & \sum \frac{\lambda_i}{2n(\sigma^2/n+\eta^2\lambda_i)^2} & \sum \frac{\lambda_i^2}{2(\sigma^2/n+\eta^2\lambda_i)^2} \end{pmatrix}$$

The MLE of the mean is

$$\hat{\mu}_M = n(\sum_{i=1}^{n} \frac{Y_i v_{.i}}{(\sigma^2/n + \eta^2\lambda_i)})/(\sum_{i=1}^{n} \frac{1}{(\sigma^2/n + \eta^2\lambda_i)}) \tag{18}$$

Using the scoring method, the variance parameter and the variance of noise can be solved by the iteration of

$$\begin{pmatrix} \sigma_M^{2,(k)} \\ \eta_M^{2,(k)} \end{pmatrix} = \begin{pmatrix} \sigma_M^{2,(k-1)} \\ \eta_M^{2,(k-1)} \end{pmatrix} + I(\sigma_M^{2,(k-1)}, \eta_M^{2,(k-1)})^{-1} \begin{pmatrix} d_1^{(k-1)} \\ d_2^{(k-1)} \end{pmatrix}, \tag{19}$$

where

$$\begin{pmatrix} d_1^{(k-1)} \\ d_2^{(k-1)} \end{pmatrix} = \begin{pmatrix} \sum(-\frac{1}{2n^2(\sigma_M^{2,(k-1)}/n+\eta_M^{2,(k-1)}\lambda_i)} + \frac{(Y_i-\hat{\mu}/n)^2}{2n^2(\sigma_M^{2,(k-1)}/n+\eta_M^{2,(k-1)}\lambda_i)^2}) \\ \sum(-\frac{\lambda_i}{2(\sigma_M^{2,(k-1)}/n+\eta_M^{2,(k-1)}\lambda_i)} + \frac{(Y_i-\hat{\mu}/n)^2\lambda_i}{2(\sigma_M^{2,(k-1)}/n+\eta_M^{2,(k-1)}\lambda_i)^2}) \end{pmatrix}$$

and $I(\sigma^2, \eta^2)$ is the lower-right corner sub-matrix of the information matrix (18).

Theorem 1 *The asymptotic behavior of the information matrix is*

$$I(\mu, \sigma^2, \eta^2) = \begin{pmatrix} \frac{1}{2\sigma^2\sqrt{\gamma n}} + o(1/\sqrt{n}) & 0 & 0 \\ 0 & \frac{\sqrt{n}}{8\sigma^4\sqrt{\gamma}} + o(\sqrt{n}) & \frac{\sqrt{n}}{8\sigma^4\sqrt{\gamma^3}} + o(\sqrt{n}) \\ 0 & \frac{\sqrt{n}}{8\sigma^4\sqrt{\gamma^3}} + o(\sqrt{n}) & \frac{n}{2\eta^4} + o(n) \end{pmatrix} \tag{20}$$

where $\gamma = \eta^2/\sigma^2$.

The proof can be found in the Appendix.

It is easy to prove that

$$\text{Var}(\hat{\mu}_M) = (\sum \frac{1}{n^2(\sigma^2/n + \eta^2\lambda_i)})^{-1} = O(\sqrt{n})$$

The variance diverges as the observation frequency increase. It is worse than the estimator $\hat{\mu} = S_{n,n} - S_{0,n}$, which has constant variance for any observation frequency. Instead of using the MLE of μ, $\hat{\mu} = S_{n,n} - S_{0,n}$ is used in this section as the estimator of mean parameter μ. The classical asymptotic results about MLE do not apply here because we are considering the observation frequency, rather than the time span, goes to infinity. To investigate the f-consistency of the estimator, I conduct a series of simulations for using different σ^2, signal-to-noise ratio $\gamma = \eta^2/\sigma^2$. For each observation frequency n, I simulate 100 series of noisy observations as in process (4). Then I calculate 100 MLE's and their sample mean and sample variance. The results are given in Table 1. Empirical results indicate that the MLE is f-consistent and the convergence rate of the variance of the MLE is simmilar to the inverse of the information matrix (20). That is

$$\mathrm{Var}(\hat{\sigma}_M^2) = \frac{8\sigma^4\sqrt{\gamma}}{\sqrt{n}} + o(\frac{1}{\sqrt{n}}) \tag{21}$$

$$\mathrm{Var}(\hat{\eta}_M^2) = \frac{2\eta^4}{n} + o(\frac{1}{n}) \tag{22}$$

When there is no observation noise, the MLE of the variance parameter is the quadratic variation $\hat{\sigma}^2 = \sum X_{i,n}^2 - (\sum X_{i,n})^2/n$. The variance of the quadratic variation estimator is σ^4/n. For both mean and variance parameter, the variances of MLE's converge $\sqrt{n}$ slower when there are observation noise.

Because the eigenvalues of matrix A_n are known, it is not too expensive to computing the MLE. However, there are several setbacks for this estimator. First, the variances of high frequency financial data are extremely unevenly distributed among all observations; i.e., the $\sigma_t = \sigma$ is often violated. Second, the noise in (4) is often not normally distributed and may be dependent. Third, the iteration (19) needs a reasonable initial guess. In the next section, I look for the estimator by the method of moment under more relaxed assumptions.

Table 1: Empirical Mean and Variance of MLE

γ	n	100 $E\hat{\theta}$	$var(\hat{\theta})$	500 $E\hat{\theta}$	$var(\hat{\theta})$	1000 $E\hat{\theta}$	$var(\hat{\theta})$
				$\sigma^2 = 1$			
0.1	$\hat{\sigma}^2$	0.953	0.365	1.04435	0.131	1.02014	0.0761
	$\hat{\eta}^2$	0.099	2.37e-4	0.10035	4.77e-5	0.10070	2.24e-5
0.01	$\hat{\sigma}^2$	0.976	0.092	0.99370	0.049	0.98798	0.0257
	$\hat{\eta}^2$	0.0104	6.96e-6	0.01000	7.23e-7	0.01006	3.34e-7
0.001	$\hat{\sigma}^2$	0.952	0.062	1.00262	0.017	1.00383	0.0094
	$\hat{\eta}^2$	0.00130	1.46e-6	0.00101	3.42e-8	0.00100	8.42e-9
				$\sigma^2 = 10$			
0.1	$\hat{\sigma}^2$	9.025	22.930	9.722	13.4747	9.793	7.1973
	$\hat{\eta}^2$	1.036	0.0326	0.995	4.92e-3	1.003	0.0020
0.01	$\hat{\sigma}^2$	10.625	10.002	10.022	4.2879	9.625	3.2054
	$\hat{\eta}^2$	0.095	4.91e-4	0.101	5.56e-5	0.101	2.28e-5
0.001	$\hat{\sigma}^2$	10.371	8.5926	9.973	1.4881	9.846	1.0128
	$\hat{\eta}^2$	0.0084	1.71e-4	0.0100	2.85e-6	0.0101	7.01e-7

For different parameters and 100 replications, this table gives empirical simulation of the mean and variance of MLE's in (16) and (17). The variance of the MLE of σ^2 decreases in order of $8\sigma^4\sqrt{\gamma}/\sqrt{n}$ and the variance of the MLE of η^2 decreases in order of $2\eta^4/n$.

3 Estimating the Variance Parameter by the Method of Moments

Assuming that the observation noises are independent with a finite fourth moment, I can construct a very simple estimator by the method of moment:

$$\hat{\sigma}^2 = \sum (X_{i,n}^2 + 2X_{i,n}X_{i-1,n}) - (\sum X_{i,n})^2/n. \tag{23}$$

The second term converges to zero and therefore is negligible as $n \to \infty$. For simplicity, I assume that $\mu = 0$. The estimator (23) simplifies to

$$\hat{\sigma}_{MM}^2 = \sum (X_{i,n}^2 + 2X_{i,n}X_{i-1,n}) \tag{24}$$

This estimator does not require any distribution assumption on the noises. The noise can be non-stationary. The estimator is nearly unbiased. The mean of this estimator is

$$\begin{aligned} E(\hat{\sigma}_{MM}^2) &= \sum (\sigma_{i,n}^2 + \eta_i^2 + \eta_{i-1}^2 - 2\eta_{i-1}^2) \\ &= \sigma^2 + \eta_n^2 - \eta_0^2, \end{aligned} \tag{25}$$

where $\sigma_{i,n}^2 = \int_{t_{i-1,n}}^{t_{i,n}} \sigma_t^2 dt$ and η_i^2 is the variance of observation noise ϵ_i. Unfortunately, this estimator is not f-consistent if the majority of noises is non-zero. The variance of the estimator is

$$\begin{aligned} \mathrm{Var}(\hat{\sigma}_{MM}^2) &= \sum (2\sigma_{i,n}^4 + 4\sigma_{i,n}^2\eta_i^2 + 4\sigma_{i-1,n}^2\sigma_{i,n}^2 + 4\sigma_{i-1,n}^2\eta_{i-1}^2 + 4\sigma_{i-1,n}^2\eta_i^2 \\ &\quad + 4\eta_i^2\eta_{i-2}^2 + 4\sigma_{i-1,n}^2\eta_{i-1}^2 + 4\eta_{i-1}^2\eta_{i-2}^2) + \eta_n^{(4)} + \eta_0^{(4)}, \end{aligned} \tag{26}$$

where $\eta_i^{(4)}$ is the fourth moment of noise ϵ_i. Suppose that η^2 is the minimum non-zero value of η_i^2, let m be the number of $\eta_i^2 > \eta^2$ and m be proportional to n, then

$$\mathrm{Var}(\hat{\sigma}_{MM}^2) \geq \sum (4\eta_i^2\eta_{i-2}^2) \geq 4 \sum \eta_i^2 \sum \eta_{i-2}^2/n = 4m^2/n\eta^4.$$

If all $\sigma_{i,n}^2 = \sigma^2/n$ and ϵ_i are independently and identically distributed (i.i.d.) with mean zero variance η^2, then

$$\mathrm{Var}(\hat{\sigma}_{MM}^2) = 6\sigma^4/n + 16\sigma^2\eta^2 + 8n\eta^4. \tag{27}$$

The optimal observation frequency is $n = \sqrt{3/4}/\gamma$. The minimum variance of the estimator is

$$\mathrm{Var}(\hat{\sigma}^2_{MM}) = 29.86\eta^2\sigma^2.$$

This estimator, after a certain point, is getting worse as the observation frequency goes to infinity. However, because of its simplicity, I want to investigate if there is an f-consistent estimator with this simplicity. In the next section, I construct an optimal quadratic estimator of the variance parameter.

4 Quadratic Estimator

In this section, I study the estimator in following quadratic form

$$\hat{\sigma}^2_Q = X^T Q X \tag{28}$$

where $X = (X_{1,n}, ..., X_{n,n})^T$ and Q is any $n \times n$ matrix. Similar to the last section, I assume $\mu = 0$. $\hat{\sigma}^2_Q$ has mean and variance:

$$E\hat{\sigma}^2_Q = \mathrm{tr}(Q\Sigma) \tag{29}$$

$$\mathrm{Var}(\hat{\sigma}^2_Q) = \mathrm{tr}(Q\Sigma Q\Sigma) \tag{30}$$

Assume that $\sigma^2_{i,n} = \sigma^2/n$ and $\eta_i^2 = \eta^2$.

$$\Sigma_n = \frac{\sigma^2}{n} I_n + \eta^2 A_n = V_n \mathrm{diag}(\frac{\sigma^2}{n} + \lambda_i \eta^2) V_n$$

where V_n and λ_i are defined in (8) and (9). V_n is symmetric, therefore $V_n^T = V_n$. Let

$$\tilde{Q}_n = V_n Q_n V_n, \tag{31}$$

then

$$E\hat{\sigma}^2_Q = \sum_i \tilde{q}_{ii}(\frac{\sigma^2}{n} + \lambda_i \eta^2) \tag{32}$$

$$\mathrm{Var}(\hat{\sigma}^2_Q) = \sum_{ij} \tilde{q}^2_{ij}(\frac{\sigma^2}{n} + \lambda_i \eta^2)^2 = \sigma^4 \sum_{ij} \tilde{q}^2_{ij}(\frac{1}{n} + \lambda_i \gamma)^2, \tag{33}$$

where $\gamma = \eta^2/\sigma^2$, the signal-to-noise ratio. Obviously, $\hat{\sigma}^2_Q$ is unbiased if and only if

$$\sum_i \tilde{q}_{ii} = n \quad \text{and} \quad \sum_i \tilde{q}_{ii}\lambda_i = 0 \tag{34}$$

Theorem 2 *For any given r, the solution of*

$$\min_{\tilde{Q}} \sum_{ij} \tilde{q}^2_{ij}(\frac{1}{n} + \lambda_i r)^2$$

$$\text{subject to} \quad \sum_i \tilde{q}_{ii} = n \quad \text{and} \quad \sum_i \tilde{q}_{ii}\lambda_i = 0$$

is $\tilde{q}_{ij} = 0$ for $i \neq j$ and

$$\tilde{q}_{ii} = \frac{\alpha + \beta\lambda_i}{2(1/n + \lambda_i r)^2} \tag{35}$$

where α and β satisfy

$$\begin{aligned} 2n &= \alpha \sum \frac{1}{(1/n + \lambda_i r)^2} + \beta \sum \frac{\lambda_i}{(1/n + \lambda_i r)^2} \\ 0 &= \alpha \sum \frac{\lambda_i}{(1/n + \lambda_i r)^2} + \beta \sum \frac{\lambda_i^2}{(1/n + \lambda_i r)^2} \end{aligned}$$

The proof is given in the Appendix.

For any given r, the best quadratic estimator is

$$\hat{\sigma}^2_Q = \sum_i \tilde{q}_{ii} Y_i^2, \tag{36}$$

where $\tilde{q}_{ii}$ is defined in (35).

Theorem 3 *For the optimal quadratic estimator $\hat{\sigma}^2_Q$, with Q defined in (31) and (35), the asymptotic convergence rate of*

$$\begin{aligned} \mathrm{Var}(\hat{\sigma}^2_Q) &= \sigma^4 \sum_{ij} \tilde{q}^2_{ij}(\frac{1}{n} + \lambda_i\gamma)^2 \\ &= (4\sqrt{r} + 2\frac{\gamma - r}{\sqrt{r}} + \frac{(\gamma - r)^2}{2\sqrt{r^3}})\frac{\sigma^4}{\sqrt{n}} + o(\frac{1}{\sqrt{n}}). \end{aligned}$$

The proof is again given in the Appendix.

From Theorem 3, for any given r, the quadratic estimator $\hat{\sigma}_Q^2$, with Q defined in (31) and (35) is f-consistent. The prefered value of r is γ, which is unknown. r should be chosen in order of γ, but should not be too small. When r is close to γ, the performance of the quadratic estimator is similar to MLE and it does not need any distribution assumption about noises. Otherwise, the variance does not decrease as the signal-to-noise ratio γ dereases.

Without assumption of constant variances, it is very difficult to find an f-consistent estimator. In the next section, I empirically examine the sensitivity of both MLE and the quadratic estimator to the assumptions of constant variance and Gaussian noises.

5 Non-normal Noises and Unequal Variances

In applications to financial market, both assumptions of constant variance and Gaussian noises are not valid. The variance of the prices is changing over time, especially among high frequency observations. The noise is hardly Gaussian. In this section, I examine the sensitivity of both the MLE σ_M^2 and the quadratic estimator σ_Q^2 to their assumptions. The following six time series are simulated and used in estimating the overall variance:

Series I: $\sigma_{i,n}^2 = \sigma^2/n$ and the noise ϵ_i is i.i.d. ηt(5), a t random variable with a degree of freedom 5.

Series II: $\sigma_{i,n}^2 = \sigma^2/n$ and the noise ϵ_i is i.i.d. ηBernoulli(p) with $p = .5$.

Series III: $\sigma_{i,n}^2$ is sampled from uniform distribution $U(0,1)$ and then re-scaled such that $\sum \sigma_{i,n}^2 = \sigma^2$, the noise ϵ_i is i.i.d. ηt(5).

Series IV: $\sigma_{i,n}^2$is sampled from lognormal distribution $LN(0,1)$ and then re-scaled such that $\sum \sigma_{i,n}^2 = \sigma^2$, the noise ϵ_i is i.i.d. ηt(5).

Series V: $\sigma_{i,n}^2$is sampled from Bernoulli(p) with $p = 0.1$ and then re-scaled such that $\sum \sigma_{i,n}^2 = \sigma^2$ and the noise ϵ_i is i.i.d. ηBernoulli(p) with $p = .5$.

Series VI: $\sigma_{i,n}^2 = \sigma^2/n$ and the noise ϵ_i MA(1) with MA coefficient 0.5 and noise ηt(5).

In the simulation, following values are used for various parameters: $\sigma^2 = 1$, $\eta^2 = 0.01$ and $n = 100, 500$ and 1000. The empirical results are listed in Table 2.

The first two series have non-Gaussian noises. For t and Bernoulli random noises, both $\hat{\sigma}_M^2$ and $\hat{\sigma}_Q^2$ show the variance convergence rate of $1/\sqrt{n}$. The MLE takes advantages of smaller sigmal-to-noise ratio in series II and has smaller sample variance of the estimates.

The next three series have unequal variances over time. Again, both $\hat{\sigma}_M^2$ and $\hat{\sigma}_Q^2$ show the variance convergence rate of $1/\sqrt{n}$. The performances of two estimators are somewhat similar. The MLE is slightly better. For small n, both estimators slightly under estimate the variance. The bias disappears as the observation frequency increases. More variation in $\sigma_{i,n}^2$ causes more bias in both estimators. However, asymptotically, both estimators are not sensitive to this deviation from the assumption of equal variance. Many other simulations have confirmed above findings.

Series VI has correlated observation noises. The MLE clearly has significant bias that does not go away as observation frequency increases. However, the quadratic estimator performed much better. The bias, if any, is negligible. The variance of $\hat{\sigma}_M^2$ converges at rate of $1/\sqrt{n}$. For this set of data, the mean squared error of the quadratic estimator is about the same as ones using series I-V. Therefore, the quadratic estimator has advantages of being not sensitive to correlation among noises. The quadratic estimator, in other cases, can be used as an initial guess of the MLE. I end this section by giving a summary table (Table 3).

6 Discussion

A misleading perception is that the more data there is, the better. Increasing observation frequency while keeping time span constant does not always help parameter estimation. An estimator developed for low frequency data may not be usable for high frequency data. The observation noise, which does not decrease as the observation frequency increases, is the key obstacle. The name of observation noise is sometimes misleading in the financial

Table 2: Sensitivity of σ_M^2 and σ_Q^2 to Their Various Assumptions

	n	100		500		1000	
		$E\hat{\theta}$	$\text{var}(\hat{\theta})$	$E\hat{\theta}$	$\text{var}(\hat{\theta})$	$E\hat{\theta}$	$\text{var}(\hat{\theta})$
Series I	$\hat{\sigma}_M^2$	0.955	0.1246	1.008	0.0502	1.027	0.0412
	$\hat{\sigma}_Q^2$	0.985	0.1862	0.987	0.0797	1.003	0.0573
Series II	$\hat{\sigma}_M^2$	1.009	0.0847	0.984	0.0282	1.002	0.0161
	$\hat{\sigma}_Q^2$	0.961	0.2528	0.997	0.0938	1.047	0.0519
Series III	$\hat{\sigma}_M^2$	1.018	0.2038	1.001	0.0584	0.990	0.0397
	$\hat{\sigma}_Q^2$	0.944	0.1288	1.0239	0.104	0.976	0.0535
Series IV	$\hat{\sigma}_M^2$	0.946	0.1681	0.989	0.0560	0.998	0.0383
	$\hat{\sigma}_Q^2$	0.939	0.1621	0.981	0.0820	0.993	0.061
Series V	$\hat{\sigma}_M^2$	0.930	0.2085	0.972	0.0484	1.006	0.0300
	$\hat{\sigma}_Q^2$	0.892	0.2614	0.969	0.0757	0.991	0.0692
Series VI	$\hat{\sigma}_M^2$	1.958	0.8852	3.398	2.1119	3.734	1.9708
	$\hat{\sigma}_Q^2$	1.052	0.2176	1.105	0.0911	1.157	0.0591

For six different types of time series, this table lists empirical means and variances of MLE and the quadratic estimator. All series use $\sigma^2 = 1$, $\eta^2 = 0.01$. $r = 0.1$ is used in (36) for σ_Q^2. The variance of both estimators has the convergence rate $1/\sqrt{n}$.

Table 3: Summary and Comparason of σ_M^2 and σ_Q^2

Is the estimator sensitive to:		Gaussian noises?	equal spaced variances?	correlated noises?
MLE	Bias	No	Yes$\rightarrow 0$	Significant
	$\text{Var}(\hat{\sigma}_M^2)$	$O(\sqrt{\gamma/n})$	$O(\sqrt{\gamma/n})$	Not Converge
Q.E.	Bias	No.	Yes$\rightarrow 0$	Small
	$\text{Var}(\hat{\sigma}_Q^2)$	$O(1/\sqrt{n})$	$O(1/\sqrt{n})$	$O(1/\sqrt{n})$

The summary of this table is based on empirical simulations incuding ones not listed in this paper.

community. Currency spot quotes have widely recognized noises. However, a stock transaction price recorded precisely may still have so-called observation noise. The noise is simply the deviation of the price from an assumed underlying continuous process and may prefer to be called a different name in such application. The observation noise can include micro-activies of the market that is not interested in applications. If high frequency data is used in studying the macro-activity of the market such as daily volatility, the variance of daily price change, it is important not to be overwhelmed by micro-activities. The advantage of using high frequency data to estimate parameter such as daily volatility is that we can estimate the volatility day-by-day rather than an average. The MLE estimator developed here has been applied to many financial data such as the currency exchange rates and prices from futures market. The results are not listed here because that the true volatilities are unknown and no comparisons can be made. The estimators developed in this paper can be generalized to estimate the multi-dimensional covariance matrix. Treating micro-activities as observation noise, one can construct a consistent estimator of covariance matrix (Zhou 1994).

The f-consistency and observation noise issues also exist in many other applications. A lot of manufacturing processes have generating high frequency data series in last decade. To study such type data and to perform parameter estimation, one has to be aware of the observation noises and examine the consistency of high frequency.

Appendix:

i) Proof of Theorem 1, the asymptotic behavior of the Fisher information matrix (20):

From (18),

$$I(\mu,\sigma^2,\eta^2)=\begin{pmatrix}\sum\frac{v_{ij}^2}{n^2(\sigma^2/n+\eta^2\lambda_i)} & 0 & 0\\ 0 & \sum\frac{1}{2n^2(\sigma^2/n+\eta^2\lambda_i)^2} & \sum\frac{\lambda_i}{2n(\sigma^2/n+\eta^2\lambda_i)^2}\\ 0 & \sum\frac{\lambda_i}{2n(\sigma^2/n+\eta^2\lambda_i)^2} & \sum\frac{\lambda_i^2}{2(\sigma^2/n+\eta^2\lambda_i)^2}\end{pmatrix}.$$

First, I prove that the (2,2)-th element of the matrix

$$\sum\frac{1}{2n^2(\sigma^2/n+\eta^2\lambda_i)^2}=\frac{\sqrt{n}}{8\sigma^4\sqrt{\gamma}}+o(\sqrt{n}).$$

Recall that $\lambda_i=4\sin^2(\frac{i}{2(n+1)})$. We can easily prove that

$$\frac{1}{(1/n+4\gamma\sin^2(\pi x/2))^2},\quad \gamma>0$$

is a decreasing function of $x\in[0,1]$ and

$$\begin{aligned}\int_{\frac{1}{n+1}}^{1}\frac{1}{(1/n+4\gamma\sin^2(\pi x/2))^2}dx &< \frac{1}{n+1}\sum_{x=\frac{1}{n+1},\ldots,\frac{n}{n+1}}\frac{1}{(1/n+4\gamma\sin^2(\pi x/2))^2}\\ &< \int_0^{\frac{n}{n+1}}\frac{1}{(1/n+4\gamma\sin^2(\pi x/2))^2}dx. \qquad (37)\end{aligned}$$

The maximum value of $1/(1/n+4\gamma x^2)^2$ over [0,1] is n^2. Therefore

$$\begin{aligned}&\frac{1}{n+1}\sum_{x=\frac{1}{n+1},\ldots,\frac{n}{n+1}}\frac{1}{(1/n+4\gamma\sin^2(\pi x/2))^2}\\ =&\int_0^1\frac{1}{(\frac{1}{n}+4\gamma\sin^2(\frac{\pi}{2}x))^2}dx+O(n)\\ =&\left[\frac{(\frac{2}{n}+4\gamma)}{(\frac{1}{n^2}+4\frac{\gamma}{n})\pi\sqrt{\frac{1}{n^2}+4\frac{\gamma}{n}}}\text{Arctan}(\frac{(\frac{1}{n}+4\gamma)\tan(\frac{\pi}{2}x)}{\sqrt{\frac{1}{n^2}+4\frac{\gamma}{n}}})\right.\end{aligned}$$

$$
\begin{aligned}
&+\left.\frac{4\gamma\sin(\pi x)}{(\frac{1}{n^2}+4\frac{\gamma}{n})\pi(\frac{2}{n}+4\gamma-4\gamma\cos(\pi x))}\right]_{x=0}^{x=1}+O(n)\\
=&\ \frac{(\frac{2}{n}+4\gamma)}{(\frac{1}{n^2}+4\frac{\gamma}{n})\pi\sqrt{\frac{1}{n^2}+4\frac{\gamma}{n}}}\frac{\pi}{2}+O(n)\\
=&\ \frac{4\gamma}{4\frac{\gamma}{n}\sqrt{4\frac{\gamma}{n}}}\frac{1}{2}+O(n^{3/2})\\
=&\ \frac{n^{3/2}}{4\sqrt{\gamma}}+o(n^{3/2})
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\sum\frac{1}{2n^2(\sigma^2/n+\eta^2\lambda_i)^2} &= \frac{n+1}{2n^2\sigma^4}(\frac{n^{3/2}}{4\sqrt{\gamma}}+o(n^{3/2}))\\
&= \frac{\sqrt{n}}{8\sigma^4\sqrt{\gamma}}+o(\sqrt{n}).
\end{aligned}
$$

Next, I prove the (3,3)-th element of the matrix

$$
\sum\frac{\lambda_i^2}{2(\sigma^2/n+\eta^2\lambda_i)^2}=\frac{n}{2\eta^4}+o(n)
$$

It is easy to see that $x^2/(1/n+\gamma x)^2$ is an increasing function of x. Therefore,

$$
\frac{\sin^4(\pi x/2))}{(1/n+4\gamma\sin^2(\pi x/2))^2},\quad \gamma>0
$$

is an increasing function of x. The maximum value of above function is $1/(4\gamma)^2$. Using the similar technique as used above,

$$
\begin{aligned}
&\frac{1}{n+1}\sum_{x=\frac{1}{n+1},\dots,\frac{n}{n+1}}\frac{\sin^4(\pi x/2))}{(1/n+4\gamma\sin^2(\pi x/2))^2}=\int_0^1\frac{\sin^4(\frac{\pi}{2}x)}{(\frac{1}{n}+4\gamma\sin^2(\frac{\pi}{2}x))^2}dx+O(\frac{1}{n})\\
=&\left[-\frac{(\frac{2}{n^2}+12\frac{\gamma}{n})}{16\gamma^2(\frac{1}{n}+4\gamma)\pi\sqrt{\frac{1}{n^2}+4\frac{\gamma}{n}}}\mathrm{Arctan}(\frac{(\frac{1}{n}+4\gamma)\tan(\frac{\pi}{2}x)}{\sqrt{\frac{1}{n^2}+4\frac{\gamma}{n}}})\right.\\
&+\left.\frac{-\frac{2\pi x}{n^2}-\frac{12\gamma\pi x}{n}-16\gamma^2\pi x+16\gamma^2\pi x\cos(\pi x)+\frac{4\gamma}{n}(\pi x\cos(\pi x)-\sin(\pi x))}{(16\gamma^2(\frac{1}{n}+4\gamma)\pi(-\frac{2}{n}-4\gamma+4\gamma\cos(\pi x))}\right]_{x=0}^{x=1}+O(\frac{1}{n})
\end{aligned}
$$

$$
\begin{aligned}
&= -\frac{(\frac{2}{n^2}+12\frac{\gamma}{n})}{16\gamma^2(\frac{1}{n}+4\gamma)\pi\sqrt{\frac{1}{n^2}+4\frac{\gamma}{n}}}\frac{\pi}{2}+\frac{-\frac{2\pi}{n^2}-\frac{12\gamma\pi}{n}-16\gamma^2\pi-16\gamma^2\pi+\frac{4\gamma}{n}(-\pi)}{(16\gamma^2(\frac{1}{n}+4\gamma)\pi(-\frac{2}{n}-4\gamma-4\gamma)}+O(\frac{1}{n}) \\
&= -\frac{(12\frac{\gamma}{n})}{16\gamma^2(4\gamma)\sqrt{4\frac{\gamma}{n}}}\frac{1}{2}+\frac{-32\gamma^2}{16\gamma^2(+4\gamma)(-8\gamma)}+o(1) \\
&= \frac{1}{16\gamma^2}+o(1).
\end{aligned}
$$

Therefor

$$\sum\frac{\lambda_i^2}{2(\sigma^2/n+\eta^2\lambda_i)^2}=\frac{16^2(n+1)}{2\sigma^4}(\frac{1}{16\gamma^2}+o(1))=\frac{n}{2\eta^4}+o(n).$$

It is slightly tricky to prove the asymptotic result of (2,3)-th element

$$\sum\frac{\lambda_i}{2n(\sigma^2/n+\eta^2\lambda_i)^2}=\frac{\sqrt{n}}{8\sigma^4\sqrt{\gamma^3}}+o(\sqrt{n}).$$

Function

$$\frac{\sin^2(\pi x/2)}{(1/n+4\gamma\sin^2(\pi x/2))^2}$$

is not monotone. However, it is positive and has maximum one turning point. The maximum value of the function is $\frac{n}{16\gamma}$. The similar technique can be used here

$$
\begin{aligned}
&\frac{1}{n+1}\sum_{x=\frac{1}{n+1},\dots,\frac{n}{n+1}}\frac{\sin^2(\pi x/2)}{(1/n+4\gamma\sin^2(\pi x/2))^2}=\int_0^1\frac{\sin^2(\frac{\pi}{2}x)}{(\frac{1}{n}+4\gamma\sin^2(\frac{\pi}{2}x))^2}dx+O(1) \\
&= \Bigg[\frac{1}{(\frac{1}{n}+4\gamma)\pi\sqrt{\frac{1}{n^2}+4\frac{\gamma}{n}}}\text{Arctan}(\frac{(\frac{1}{n}+4\gamma)\tan(\frac{\pi}{2}x)}{\sqrt{\frac{1}{n^2}+4\frac{\gamma}{n}}}) \\
&\quad +\frac{\sin(\pi x)}{(\frac{1}{n}+4\gamma)\pi(-\frac{2}{n}-4\gamma+4\gamma\cos(\pi x))}\Bigg]_{x=0}^{x=1}+O(1) \\
&= \frac{1}{4\gamma\pi\sqrt{4\frac{\gamma}{n}}}\frac{\pi}{2}+o(\sqrt{n}) \\
&= \frac{\sqrt{n}}{16\sqrt{\gamma^3}}+o(\sqrt{n}).
\end{aligned}
$$

Therefor

$$\begin{aligned}\sum \frac{\lambda_i}{2n(\sigma^2/n+\eta^2\lambda_i)^2} &= \frac{4(n+1)}{2n}(\frac{\sqrt{n}}{16\sqrt{\gamma^3}}+o(\sqrt{n}))\\ &= \frac{\sqrt{n}}{8\sigma^4\sqrt{\gamma^3}}+o(\sqrt{n}).\end{aligned}$$

Now, I come back to the (1,1)-th element

$$\sum_{ij}[\Sigma_n^{-1}]_{ij}/n^2 = \frac{1}{\sigma^2}+o(1).$$

First,

$$\Sigma_n^{-1} = V_n \mathrm{diag}(1/(\sigma^2/n+\eta^2\lambda_i))V_n.$$

Therefore

$$\sum_{ij}[\Sigma_n^{-1}]_{ij} = \sum_{ij}(v_{i,j}^2/(\sigma^2/n+\eta^2\lambda_i)).$$

Since

$$\begin{aligned}\sum_j v_{i,j}^2 &= \frac{2}{n+1}\sum_j \sin^2(ij\pi/(n+1))\\ &= 1-\frac{1}{n+1}\frac{\cos(in\pi/(n+1))\sin(i\pi)}{\sin(i\pi/(n+1))}\\ &= 1,\\ \sum_{ij}[\Sigma_n^{-1}]_{ij}/n^2 &= \sum_i \frac{1}{n^2(\sigma^2/n+\eta^2\lambda_i)}.\end{aligned}$$

It is easy to argue that

$$\begin{aligned}&\sum_i \frac{1}{n^2(\sigma^2/n+\eta^2\lambda_i)} = \frac{n+1}{n^2\sigma^2}\int_0^1 \frac{1}{\frac{1}{n}+4\gamma\sin^2(\frac{\pi}{2}x)}dx+O(1/n)\\ &= \frac{n+1}{n^2\sigma^2}\left[\frac{2}{\pi\sqrt{\frac{1}{n^2}+4\frac{\gamma}{n}}}\mathrm{Arctan}(\frac{(\frac{1}{n}+4\gamma)\tan(\frac{\pi}{2}x)}{\sqrt{\frac{1}{n^2}+4\frac{\gamma}{n}}})\right]_{x=0}^{x=1}+O(1/n)\\ &= \frac{1}{n\sigma^2}\frac{2}{\pi\sqrt{4\frac{\gamma}{n}}}\frac{\pi}{2}+o(\frac{1}{\sqrt{n}})\\ &= \frac{1}{2\sigma^2\sqrt{\gamma n}}+o(\frac{1}{\sqrt{n}})\end{aligned}$$

ii) Proof of Theorem 2, the optimal matrix $\tilde{Q}$ for the quadratic estimator:
For the optimization problem

$$\min_{\tilde{Q}} \sum_{ij} \tilde{q}_{ij}^2 (\frac{1}{n} + \lambda_i r)^2$$

$$\text{subject to } \sum_i \tilde{q}_{ii} = n \quad \text{and} \quad \sum_i \tilde{q}_{ii} \lambda_i = 0,$$

I define Lagrangian function

$$\Phi(Q; \alpha, \beta) = \sum_{ij} \tilde{q}_{ij}^2 (\frac{1}{n} + \lambda_i r)^2 - \alpha(\sum_i \tilde{q}_{ii} - n) - \beta(\sum_i \tilde{q}_{ii} \lambda_i)$$

Obviously, $\tilde{q}_{ij} = 0$ for $i \neq j$. For $i = j$, differentiate Φ with respect to each $\tilde{q}_{ii}$,

$$0 = 2\tilde{q}_{ii} (\frac{1}{n} + \lambda_i r)^2 - \alpha - \beta \lambda_i.$$

Conditions

$$\sum_i \tilde{q}_{ii} = n \quad \text{and} \quad \sum_i \tilde{q}_{ii} \lambda_i = 0$$

lead to follow equations

$$\begin{aligned} 2n &= \alpha \sum \frac{1}{(1/n + \lambda_i r)^2} + \beta \sum \frac{\lambda_i}{(1/n + \lambda_i r)^2} \\ 0 &= \alpha \sum \frac{\lambda_i}{(1/n + \lambda_i r)^2} + \beta \sum \frac{\lambda_i^2}{(1/n + \lambda_i r)^2} \end{aligned}$$

and

$$\tilde{q}_{ii} = \frac{\alpha + \beta \lambda_i}{2(1/n + \lambda_i r)^2}.$$

iii) Proof of Theorem 3, the convergence rate of the quadratic estimator $\hat{\sigma}_Q^2$:
From Theorem 1,

$$\sum \frac{1}{(1/n + \lambda_i r)^2} = \frac{n^{5/2}}{4\sqrt{r}} + o(n^{5/2})$$

$$\sum \frac{\lambda_i}{(1/n+\lambda_i r)^2} = \frac{n^{3/2}}{4r^{3/2}} + o(n^{3/2})$$
$$\sum \frac{\lambda_i^2}{(1/n+\lambda_i r)^2} = \frac{n}{r^2} + o(n).$$

Therefore,

$$\alpha = (2n)(\frac{n}{r^2} + o(n))/(\frac{n^{5/2}}{4\sqrt{r}}\frac{n}{r^2} + o(n^{7/2})) = \frac{8\sqrt{r}}{n^{3/2}} + o(n^{-3/2})$$
$$\beta = (2n)(-\frac{n^{3/2}}{4r^{3/2}} + o(n^{3/2}))/(\frac{n^{5/2}}{4\sqrt{r}}\frac{n}{r^2} + o(n^{7/2})) = -\frac{2r}{n} + o(\frac{1}{n}).$$

Rewrite the variance

$$\begin{aligned}
\mathrm{Var}(\hat{\sigma}_Q^2) &= \sigma^4 \sum_i \tilde{q}_{ii}^2 (\frac{1}{n} + \lambda_i \gamma)^2 \\
&= \sigma^4 \sum_i \tilde{q}_{ii}^2 [(\frac{1}{n} + \lambda_i r)^2 + 2(\gamma - r)(\frac{1}{n} + \lambda_i r) + (\gamma - r)^2 (\frac{1}{n} + \lambda_i r)^2]
\end{aligned}$$

It is easy to check that the first term

$$\begin{aligned}
\sum_i \tilde{q}_{ii}^2 (\frac{1}{n} + \lambda_i r)^2 &= \frac{\alpha^2}{4} \sum_i (\frac{1}{n} + \lambda_i r)^{-2} + O(\frac{1}{n}) \\
&= \frac{1}{4} (\frac{8\sqrt{r}}{n^{3/2}})^2 \frac{n^{5/2}}{4\sqrt{r}} + o(\frac{1}{\sqrt{n}}) \\
&= \frac{4\sqrt{r}}{\sqrt{n}} + o(\frac{1}{\sqrt{n}}).
\end{aligned}$$

The other two terms can be proved by similar techniques. It is a long and tedious calculus manipulation. Numerically, one can easily verify following equations:

$$\sum_i \tilde{q}_{ii}^2 (\frac{1}{n} + \lambda_i r)\lambda_i = \frac{1}{\sqrt{rn}} + o(\frac{1}{\sqrt{n}})$$
$$\sum_i \tilde{q}_{ii}^2 \lambda_i^2 = \frac{1}{2\sqrt{r^3}\sqrt{n}} + o(\frac{1}{\sqrt{n}})$$

Therefore

$$\begin{aligned}\mathrm{Var}(\hat{\sigma}_Q^2) &= \sigma^4 \sum_{ij} \tilde{q}_{ij}^2 (\frac{1}{n} + \lambda_i \gamma)^2 \\ &= (4\sqrt{r} + 2\frac{\gamma - r}{\sqrt{r}} + \frac{(\gamma - r)^2}{2\sqrt{r^3}})\frac{\sigma^4}{\sqrt{n}} + o(\frac{1}{\sqrt{n}}).\end{aligned}$$

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