A FORMAL FRAMEWORK AND FUNDAMENTAL RESULTS
FOR SOCIAL ANALYSIS

by

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Behaviors are assumed to have preferences over social states (probability functions on the joint occurrence of actions of members of a collectivity of behaviors) and to behave so as to maximize their respective expected utility functions, where expectations are taken according to (subjective) probabilities which are estimated by impression functions continuous in social states. Processes, called social evolutions, are described, by which social states are mapped, via the described behavior, into sets of social states. A fixed environment is assumed.

It is proved, by use of Kakutani’s fixed point theorem, that every social evolution has a fixed point, called a static social contract. Brouwer’s fixed point theorem yields, incidentally, that for any behavior, there exists a social state which is invariant under the behavior’s impression function. A sequence of social states generated by a social evolution is called a history, and a social state is called historically attainable iff it is the limit point of some history. It is proved that a social state is historically attainable iff it is a static social contract. It is also shown that a social state which is repeated consecutively in a history is a static social contract. It is suggested as a blow to historicism that a history need not converge.

A postscript is added, where the existence of static social contract is shown to be an extension and generalization of Nash’s result that a finite game has an equilibrium point. Implications are suggested toward further research.

Preliminaries:

Let \( N = \{1, \ldots, n\} \) be a (non-empty and finite) collection, called a collectivity, of \( n \) behaviors, the latter being formally undefined for the present context, except for descriptive information conveyed about them by assumptions soon to be made. The collectivity \( N \) is to be imagined as having some surrounding, which we also leave as formally undefined.

Associated with the generic element \( j \) of \( N \) is a sure event \( X_j \), called a sure action, and associated with the surrounding is a sure event \( X_0 \), called the sure environ. We denote:

\[
N = \{0, \ldots, n\} ,
\]
\[ N^j = N \setminus \{j\} \]

and let \( m_k \) denote the (finite) cardinality of \( X_k \) (\( k \in \mathbb{N} \)), \( m = \prod_{k \in \mathbb{N}} m_k \), \( m^j = \prod_{k \in \mathbb{N}} j^k \). Then the Cartesian products

\[ X = \prod_{k \in \mathbb{N}} X_k \]

are also sure events, of cardinality \( m \) and \( m^j \), respectively. Elements of \( X_0, X_j, X, \) and \( X^j \) are called, respectively, elementary environs, elementary actions, elementary social events, and elementary \( j \)-exclusive social events. \( X \) is called the sure social event, and \( X^j \) is called the sure \( j \)-exclusive social event.

Since each of \( X_0, X_j, X, \) and \( X^j \) are finite, their power sets

\[ \mathcal{S}_0 = 2^{X_0} , \]
\[ \mathcal{S}_j = 2^{X_j} , \]
\[ \mathcal{S} = 2^X , \]
\[ \mathcal{S}^j = 2^{X^j} \]

are sigma-fields. Their elements are called environs, actions, social events, and \( j \)-exclusive social events, respectively; they are called basic if they are singleton. Thus, we can speak of probability functions defined on these sigma-fields. Let \( \mathcal{P}_0, \mathcal{P}_j, \mathcal{P}, \) and \( \mathcal{P}^j \) denote the sets of probability functions, called environments, behaviors, social states,
j-exclusive social states, respectively, defined on $S_0$, $S_j$, $S$, and $S^j$, respectively. It is clear that $P_0$, $P_j$, $P$, and $P^j$, which we may call environment space, behavior space (of $j$), social state space, and j-exclusive social state space, respectively, are nothing but the standard simplexes in $\mathbb{R}^0$, $\mathbb{R}^j$, $\mathbb{R}^m$, and $\mathbb{R}^{m^j}$, respectively. Thus, their extreme points are nothing but the standard basis vectors for the Euclidean space of corresponding dimensionality, and it is natural to call these extreme points basic. Denoting the standard basis of the Euclidean space associated with $P_0$, $P_j$, $P$, and $P^j$ by $B_0$, $B_j$, $B$, and $B^j$, respectively, it is equally natural to refer to these latter as basic environment space, basic behavior space (of $j$), basic social state space, and basic j-exclusive social state space, in the same order of reference.

Main Assumptions:

For any social state $p$, there are two kinds of marginals of $p$ which are of interest here. First we define these.

1. DEF. For any social state $p$, the j-exclusive marginal $f^j(p)$ of $p$ is defined by the j-exclusive marginal probability function $f^j$ as follows:

$$f^j: p \rightarrow p^j$$

such that, for any $E^j$ in $S^j$, if $f^j(p) = p^j$, then

$$p^j(E^j) = \int_{x_j \in X_j} p(E^j|x_j)p(x_j).$$
2. **DEF.** For any social state \( p \), the \( j \)-marginal \( f_j(p) \) of \( p \) is defined by the \( j \)-marginal probability function \( f_j \) as follows:

\[
f_j : P \rightarrow P_j
\]

such that, for any \( E_j \) in \( S_j \), if \( f_j(p) = p_j \), then

\[
p_j(E_j) = \int_{x_j \in X_j} p(E_j \mid \{x_j\})p(\{x_j\}).
\]

We may now state our main assumptions. The remarks immediately following each assumption should help to give an intuitive feeling about the content of the assumption.

3. **ASS. (A1).** For any behavior \( j \) in \( N \), there exists a unique continuous function \( I^j \),

\[
I^j : P^j(P) \rightarrow P,
\]

assigning a \( j \)-exclusive social state to each \( j \)-exclusive marginal of any social state (i.e., to any \( j \)-exclusive social state).

4. **DEF.** Let \( I^j \) be as above, and let \( * \) be the binary operation defined by

\[
p = p_j * p^j \quad \text{iff} \quad p(E_j E^j) = p_j(E_j)p^j(E^j), \quad (E_j \in S_j, E^j \in S^j).
\]

The impression function \( I_j \) of \( j \) is the (unique) function defined by
\[ I_j : P \rightarrow P, \]

\[ I_j(p) = p_j(p) \times I_j^1(p_j^1(p)) , \]

and, for any social state \( p \), \( I_j(p) \) is called \( j \)'s **impression** of \( p \).

5. **REM.** According to Al, and in view of the last definition, each behavior \( j \) in \( N \) is free to "see" a social state \( p \) as a social state \( I_j(p) \), without it being required that \( I_j(p) = p \). Neither is it required, for any \( i, j \) in \( N \), that \( I_i(p) = I_j(p) \). In fact, all behaviors in \( N \) may hold different impressions of a social state and none of these impressions have to coincide with that social state. Thus, we are completely relaxing our system from (almost) any informational assumptions, let alone such assumptions as of the existence of perfect information (and perception) in the sense that, for all \( p \) in \( P \) and for all \( j \) in \( N \), \( I_j(p) = p \).

This relaxation is in order to allow maximal freedom to accommodate (almost) any information (and perception) system we may wish to specify and examine the consequences of.

Al requires only one simple requirement to be met, namely, that \( I^1 \) be a continuous function for all \( j \) in \( N \). Actually, this is required because it implies the following (obvious) corollary.

6. **COR.** The impression function \( I_j \) of any behavior \( j \) in \( N \) is continuous.

**Proof:** Each component of \( I_j \) is the product of the two continuous
functions $p_j$ and $I^j$, so is continuous. In $\mathbb{R}^m$, component-wise continuity is equivalent to continuity. Hence, $I^j$ is continuous.

7. REM. The very next steps we are about to take can usefully be anticipated here. The impressions of behaviors are to become the basis of their choice of behavior, given their preferences. The exact nature of this will very soon become clear. It is worth noting at this point, however, that a generic behavior $j$ "sees" social states as exhibiting, or allowing $j$, a certain independence.

In particular, $j$'s impression of any social state $p$ is a social state which is the $\ast$-product of $j$'s behavior $p_j(p)$ in that social state with a $j$-exclusive social state $I^j(p^j(p)) = q^j$ in $p^j$. Then, given the present $p$, which $j$ "sees" as $I_j(p) = p_j(p) \ast q^j$, if $q = q_j \ast q^j$ is preferred by $j$ to $I_j(p)$, at the next "moment" of choice, $j$ may choose the behavior $q_j = p_j(q)$. In fact, this is quite the sort of possibility we are designing our formal construction to yield, and we will soon define a "social evolution" which rests on this sort of process.

Note, however, that, should $j$'s behavior have interacted (in $j$'s "mind") with the $j$-exclusive social state $I^j(p^j(p))$ which $j$ "believes" to hold, so that changing from one behavior $p_j(p)$ to another behavior $q_j$ could (in $j$'s "opinion") affect the $j$-exclusive social state $q^j = I^j(p^j(p))$, then $j$'s choice of behavior would have to be more complicated. The complications would arise from the fact that $j$ could not take $q^j$
as a given in face of which to choose, since that very choice may alter $q^j$ (for example, from our point of observation, by altering $p$ in such a way that $p^j(p)$ is altered.)

Such complications are avoided by our definition of impressions, since, accordingly, a behavior $j$'s impression of a social state $p$ is such that actions $E_j$ of $j$ and $j$-exclusive social events $E^j$ are (mutually) independent: the probability $I_j(E_j E^j)$ of the joint (social) event $E_j E^j$ according to that impression is the product

$$I_j(E_j E^j) = p_j(E_j) p^j(E^j),$$

of the individual probabilities $p_j(E_j)$ and $p^j(E^j)$, where $I_j = I_j(p)$, $p_j = p_j(p)$, $p^j = p^j(p)$.

With the anticipation we now have of the use to which impressions are to be put in our framework, this might be the right moment at which to question the particular sort of independence, just described, with which we have defined impressions to be endowed. Assuming such independence is justifiable as long as behaviors in a collectivity — as they might do especially for large collectivities — typically consider themselves as "mere drops in a bucket" so far as typical other behaviors in the collectivity are concerned. But even if we are not justified from this angle, our main stand, namely, that behaviors choose behaviors according to continuous impressions of social states, may still be quite plausible for a possibly even more basic reason.
For suppose that behaviors were not "naive" and that they anticipated others' reactions in choosing their behaviors. If they anticipated these 'consciously" or otherwise) as if by methods of the usual kind of statistics, i.e., by use of continuous estimators (such as in projection by linear regression, etc.), given the history of the system up to the state $p$, the estimates would depend (continuously) on $p$. Thus, $I_j$ could be seen as such an estimator, and, formally speaking, nothing would change; our main result, below, would be unaffected.

At the risk of anticipating too much too early, we have tried to convince the reader that very little commitment was made by what we have assumed so far. The time should be ripe for our next assumption.
8. ASS. (A2) Associated with each behavior \( j \) in \( N \), there is a non-negative vector \( \mathbf{u}_j = (u_{j1}, ..., u_{jm}) \) in \( \mathbb{R}^m \) (where \( u_{jh} \geq 0 \) (\( h = 1, ..., m \)) is a "utility" assigned by \( j \) to the \( h^{th} \) basic social state \( b_h \)) in \( \mathbf{B} \), such that

\[
U_j(p) = <\mathbf{u}_j, \mathbf{I}_j(p)> = \sum_{h=1}^{m} u_{jh} I_{jh}(p)
\]

indicates a (complete, "preference") ordering \( (\preceq)_j \) of \( j \) on social state space \( \mathbf{P} \):

\[
U_j(p) \preceq U_j(q) \iff p \preceq q, \quad (p, q \in \mathbf{P}).
\]

9. REM. A2 is nothing but an assumption of "Subjective Probability and Behavior" (von Neumann (1944)-Ramsey (1926)-Savage (1954)). Accordingly, behaviors maximize expected utility, \( U_j \), in the sense that their impressions \( I_j \) yield the subjective probabilities with which expectations are to be taken. In an effort to guarantee clarity, it will be useful to go through this once in minimal detail.

Suppose that a social state \( p \) obtains and that \( j \)'s impression of \( p \) is given by

\[
I_j(p) = p_j(p) \times I_j^j(p) = p_j \times q^j = q \in \mathbf{P}.
\]

Then, \( j \) chooses a behavior \( r_j \) in \( P_j \), such that

\[
U_j(r) = \max_{q_j \in \mathbf{P}_j} U_j(q_j \times q^j),
\]
where \( r = r_j^* q_j \).

We are about to give a name ("reaction") to the set of behaviors such as \( r_j \). However, since such name-giving would be slightly useless if the set were empty, i.e., if there existed no behavior with the characteristics of \( r_j \), we first establish a simple fact (10. COR.) which guarantees the non-emptiness of the mentioned set, i.e., the existence of a point such as \( r_j \).

10. COR. The (expected utility) function \( U_j \) associated with any behavior \( j \) in \( N \) is real-valued and continuous:

\[
U_j: P \rightarrow \mathbb{R}, \quad U_j \in C^0.
\]

Proof: \( U_j \) is clearly real-valued, since it is the inner product of two vectors in \( \mathbb{R}^m \) (as from 8. ASS. (A2)). Equally clearly, it is continuous, since \( I_j \) is continuous (as from 6. COR.).

11. DEF. The set

\[
Y_j(p) = \{ r_j \mid U_j(r) = \text{Max}_{q_j \in P_j} U_j(q_j^* q_j^j) \}, \quad (p \text{ in } P),
\]

where the notation of 9. REM. is used, is called the reaction of \( j \) to \( p \), and an element \( r_j \) of \( Y_j(p) \) is called an elementary reaction of \( j \) to \( p \).
The Cartesian product

\[ Y(p) = \prod_{k \in \mathbb{N}} Y_k(p) \]

is called the \textit{collective reaction} of \( \mathbb{N} \) to \( p \), and elements \( y = (r_0, \ldots, r_n) \) of \( Y(p) \) are called \textit{elementary collective reactions} of \( \mathbb{N} \) to \( p \), where we define (assume) \( Y_0(p) = \{p_0(p)\} = \{r_0\} \).

12. \textit{REM.} Thus, we assume the environment fixed.

13. \textit{COR.} Any (collective) reaction is non-empty.

\textbf{Proof:} Let \( p \) be any social state, let \( q^j = I^j(p^j(p)) \), and let the subset \( Q \) of \( P \) be defined by

\[ Q = \{q \mid q = q_j * q^j\}, \quad (q_j \text{ in } P_j). \]

\( Q \) is bounded, since \( P \) is bounded. Since \( P_j \) is closed, so is \( Q \) closed. Thus, \( Q \) is compact. Hence, the continuous (real-valued) function \( U_j \) attains a maximum on \( Q \). Thus,

\[ Y_j(p) = \{r \mid U(r) = \max_{q \in Q} U(q) \neq \emptyset, \quad (j \text{ in } N). \]

\( Y_0(p) \neq \emptyset \), by definition. Hence, also \( Y(p) \neq \emptyset \).
14. DEF. Let \( Y(p) \) be the collective reaction of \( N \) to the social state \( p \), and let \( y = (r_0, \ldots, r_n) \) denote the generic element of \( Y(p) \). A social state \( z \) such that

\[
(p_0(z), \ldots, p_n(z)) = y
\]

is called an elementary manifestation of \( y \), and the \( Z(y) \) of all such \( z \) is called the manifestation of \( y \). The set \( Z(Y(p)) \),

\[
Z(Y(p)) = \bigcup_{y \in Y(p)} Z(y),
\]

of all elementary manifestations of elementary collective reactions \( y \) in \( Y(p) \) is called the manifestation of \( Y(p) \).

15. DEF. A correspondence \( S \),

\[
S: P \longrightarrow 2^P,
\]

such that \( S(p) = Z(Y(p)) \), is called a social evolution.

16. DEF. A social state \( p \) such that \( p \in S(p) \) is called a static social contract (for \( N \), under \( S \)).

Main Result:

The main result to be proved is that, for any collectivity \( N \) of
behaviors, described as above, and, for any environment, there exists a static social contract. Toward this end, we will use the following:

17. THM. (Kakutani, 1941) Let \( P \) be a non-empty, compact and convex subset of \( \mathbb{R}^m \), and let \( S \),

\[
S: P \to 2^P,
\]

be a closed correspondence from \( P \) into \( P \), such that, for all \( p \) in \( P \), \( S(p) \) is non-empty, closed and convex. Then there exists a (fixed) point \( p \) in \( P \), such that

\[
p \in S(p).
\]

18. REM. A correspondence \( S \) defined on a domain \( P \) is a mapping which, to each point \( p \) in \( P \), assigns exactly one set \( S(p) \). In the case above, these sets are subsets of \( P \). Thus, \( S \) can be looked upon as a (point-to-set) function (sometimes called a "many-valued function", contrary to modern usage of 'function' as a non-(one-to-many) relation or mapping) whose range is contained in the power set \( 2^P \) of \( P \). (By saying that the function assigns a set \( S(p) \) to each \( p \), we mean that the correspondence may send \( p \) to any element \( q \) of \( S(p) \).)

A correspondence \( S \) from a set \( P \) into a set \( Q \) is closed iff its graph \( G(S) \),

\[
G(S) = \{ <p, q> | p \in P, q \in S(p) \}
\]
is closed.

Thus, our main result, namely, the existence of static social contract, will be established as a corollary to 17. THM. by the three lemmas below.

19. LEM. The social state space $P$ associated with any collectivity is a non-empty, compact and convex subset of $\mathbb{R}^m$.

Proof: Obvious. (P is the standard simplex in $\mathbb{R}^m$.)

20. LEM. The manifestation $Z(Y(p)) = S(p)$ of the collective reaction $Y(p)$ to any social state $p$ is non-empty, closed and convex.

Proof: By 13. COR., $Y(p)$ is non-empty. Let $y = (r_0, \ldots, r_n)$ be a point in $Y(p)$. Then, the (independent) joint probability distribution $s$, defined by

$$s(E) = r_0(E_0)\ldots r_n(E_n),$$

for any joint event $E = E_0\ldots E_n$ in $\mathcal{S}$ arising from events $E_k$ in $\mathcal{S}_k$ ($k \in \mathbb{N}$), is certainly a social state whose marginals $p_k(s) = r_k$, so that $s$ is an elementary manifestation of $y$, whereby $S(p)$ is non-empty.

To prove that $S(p)$ is closed, we establish that $Y(p)$ is closed. Since

$$Y(p) = \prod_{k \in \mathbb{N}} Y_k(p),$$

and since $Y_0(p)$ is closed (by the fact that, according to 11. DEF. and 12. REM.,
\(Y_0(p)\) is singleton), to establish that \(Y(p)\) is closed, all we need to show is that \(Y_j(p)\) is closed (\(j \in \mathbb{N}\)). Let \(\{y_j^c\}\) be a sequence of points in \(Y_j(p)\) converging to \(y_j\), and let \(\{U_j^c\} = \{U_j(y_j^c \ast I_j^j(f_j^j(p)))\}\) be the sequence of associated expected utilities. By 11. DEF., \(U_j^c = U_j^{c+1}\) (\(c = 1, \ldots\)). Since, by 10. COR., \(U_j^c\) is continuous, we have

\[
\lim_{c \to \infty} \{U_j^c\} = U_j(y_j \ast I_j^j(f_j^j(p))) = \bar{U}_j = U_j^c, \quad (c = 1, \ldots).
\]

Hence, \(\bar{y}_j \in Y_j(p)\), whereby \(Y_j(p)\) is closed, implying that \(Y(p)\) is closed.

Now we will use this result to show that \(S(p)\) is closed. Let \(\{s^@\}\) be a sequence of points in \(S(p) = Z(Y(p))\), let \(\{y^@\}\) be a sequence of points in \(Y(p)\), defined by

\[
y^@ = (f_0(s^@), \ldots, f_n(s^@)),
\]

let \(\{s^@\}\) converge to a point \(s^*\) (in \(P\), since \(P\) is closed, by 19. LEM.), and denote \(y^* = (f_0(s^*), \ldots, f_n(s^*))\). Suppose that \(s^* \notin S(p)\). Then, \(y^* \notin Y(p)\), for otherwise \(s^* \in S(p)\). But \(f_k\) is continuous in social states (\(k \in \mathbb{N}\)), so that \(f = (f_0, \ldots, f_n)\) is continuous on \(P\). This implies that \(y^* = \lim_{\@ \to \infty} \{y^@\}\), further implying, since \(Y(p)\) is closed, that \(y^* \in Y(p)\).

This contradicts that \(s^* \notin S(p)\), so that \(s^* \in S(p)\) and, hence, \(S(p)\) is closed.

Let \(p_j, p_j'\) be arbitrary behaviors in \(Y_j(p)\), and let \(c' = 1-c\), where \(c\) is an arbitrary real in the closed interval \([0, 1]\). Let \(\bar{p}_j = cp_j + c'p_j'\). Since \(U_j\) is linear (concave and convex) in \(p_j\) when \(I_j(p) = p_j \ast I_j^j(f_j^j(p))\) and when \(I_j^j(f_j^j(p))\) is fixed (as when \(p\) is fixed), denoting
we have

\[ \bar{U}_j = U_j (\tilde{p}_j * I_j^j (f_j^j (p))) \]

so that \( \tilde{p}_j \in Y_j (p) \), implying that \( Y(p) \) is convex, since \( Y_0 (p) \) is singleton and, consequently, convex. Let \( q, q' \in S(p) \), and let \( \bar{q} = cq + c'q' \), where \( c \) and \( c' \) are as above. Then \( f_k (q), f_k (q') \in Y_k (p), \) \( (k \in \mathbb{N}) \), so that \( \bar{f}_k = cf_k (q) + c'f_k (q') \in Y_k (p) \), since \( Y_k (p) \) is convex. But, since \( q, q' \) are probabilities, \( f_k (\bar{q}) = \bar{f}_k \). Hence \( \bar{q} \in S(p) \), i.e., \( S(p) \) is convex.

21. LEM. Any social evolution is closed.

Proof: With the notation used so far, let \( S = Z_0 Y : P \rightarrow 2^P \) be a social evolution, and let \( \{ p^c \} \) be a sequence of social states converging to (a social state) \( p^* \). To prove that the correspondence \( S \) is closed, we first prove that the correspondence \( Y \) is closed, where \( Y \) is the familiar mapping such that, for any \( p \) in \( P \), \( Y(p) \) is the collective reaction to \( p \). Let \( \{ y^c \} \) be a sequence of elementary collective reactions, such that \( y^c \in Y(p^c) \), and let \( y^* = \lim_{c \to \infty} y^c \). We need to show that \( y^* \in Y(p^*) \). Since \( f_j, f_j, I_j^j \) and \( U_j \) are all continuous functions of \( p \) \( (j \in \mathbb{N}) \), the sequence \( \{ u^c \} \),

\[ \{ u^c \} = \{(U_1 (f_1 (p^c)) * I_1^j (f_1^j (p^c))), \ldots, U_n (f_n (p^c)) * I_n^j (f_n^j (p^c)))\} \]
converges to the point \( U^* \),

\[
U^* = \left( U_1(f_1(p^*)) \ast \mathcal{I}^1(f_1(p^*)), \ldots, U_n(f_n(p^*)) \ast \mathcal{I}^n(f_n(p^*)) \right).
\]

But this fact is contradicted if \( y^* \not\in Y(p^*) \). Hence, \( y^* \in Y(p^*) \), and the correspondence \( Y \) is, therefore, closed.

Now let \( \{ q^c \} \) be a sequence of social states, such that \( q^c \in S(p^c) \), and let \( q^* = \text{Lim} \{ q^c \} \). We need to show that \( q^* \in S(p^*). \) Suppose the contrary. Then

\[
y^* = (f_0(p^*), \ldots, f_n(p^*)) \not\in Y(p^*),
\]

contradicting that \( f_k \) is continuous for \( k \) in \( \mathbb{N} \) or that \( Y \) is a closed correspondence. Hence, \( q^* \in S(p^*), \) and \( S \) is, therefore, closed.

22. COR. (Fundamental): For any collectivity under any social evolution, there exists a social contract.

Proof: The result is implied directly by conjunction of 17. THM. and 19.-21. LEM.
23. REM. We have been able to construct a formalization of a social-scientifically interesting system which admits application of Kakutani's fixed point theorem (17. THM.) to yield the existence of a static social contract, regardless of impression or utility functions of behaviors, so long as the impressions are continuous and the behaviors maximize expected utility according to such impressions.

With his present hindsight as to the direction in which the analysis was proceeding at earlier stages, the reader may find it useful to read the highly anticipatory 7. REM. over again.

Further Results:

We have a number of further results flowing from our framework. On one hand, we have the result, relevant from a cognitive or information point of view, that, for any behavior $j$ in a collectivity $N$, there is a social state $p$ such that $I_j(p) = p$.

On the other hand, returning to the main theme of social contract, existence of static social contract is one thing, its attainability, uniqueness and stability (of various sorts) is quite another. So our next main result is that a social state is a static social contract iff it is historically attainable, where by historically attainable we mean a limit point of some history and where a history is a sequence $\{s^c\}$ of social states $s^c$ such that $s^{c+1} \in S(s^c)$. 
After establishing these results, we will be ready for some uniqueness and stability investigations concerning static social contracts. These will be major investigations in their own right and will be developed in a following study.

Toward proving our first mentioned further result, namely, the existence of a cognitively fixed point (social state), i.e., a social state which is fixed under impressions, we state the familiar fixed point theorem due to Brouwer (and which is a weaker result implied by Kakutani's more general fixed point theorem, stated above as 17. THM.).

24. THM. (Brouwer, 1909, 1910) Let \( P \) be a non-empty, compact and convex subset of \( R^n \), and let \( I, \)

\[
I: P \rightarrow P,
\]

be a continuous function from \( P \) into \( P \). Then there exists a (fixed) point \( p \) in \( P \), such that

\[
p \in I(p).
\]

Proof: \( I(p) \) is singleton, hence non-empty, closed and convex, for all \( p \) in \( P \). \( I \) is continuous, hence upper semi-continuous. The result follows by 17. THM..
25. DEF. A social state p associated with a collectivity N is said to be perfectly perceived by a behavior j in N iff the impression \( I_j(p) = p \).

26. COR. For any behavior j in a collectivity N, there exists a social state p associated with N (or N') which is perfectly perceived by j.

Proof: \( I_j \) is continuous from non-empty, compact and convex P into P, so that 24. THM. directly yields the desired result.

27. REM. It may be interesting to investigate under what conditions there exists a social state p such that \( I_j(p) = p \) for all j in \( N' \), where \( N' \) is a non-empty subset, called a sub-collectivity, of N.

28. DEF. Any sequence \( h = \{s^c\} \) of social states \( s^c \) associated with N is called a history of N iff \( s^{c+1} \in S(s^c) \). A social state is said to be historically attainable iff it is the limit point of some history.

29. COR. A social state is historically attainable iff it is a static social contract.

Proof: It is clear that any static social contract \( s^* \) is historically attainable, since \( s^* \in S(s^*) \) implies that \( h = \{s^c\} \), where \( s^c = s^* \ (c = 1, \ldots) \), is a history and that h converges to \( s^* \).

Let \( h = \{s^c\} \) be a history converging to s, and suppose that \( s \notin S(s) \). Then \( <s, s> \notin G(S) \) and, since the graph G(S) is closed, there exists an open ball with radius \( \delta > 0 \) about \( <s, s> \) whose intersection
with the graph $G(S)$ is empty. Since $<s^c, s^{c+1}> \in G(S)$, the distance

$$||<s^c, s^{c+1}>, <s, s>|| \geq \delta,$$

$(c = 1, \ldots)$,

contradicting that $h$ converges to $s$. Hence, $s \in S(s)$.

#

30. REM. Thus, not only does a static social contract exist (22. COR.), but, also, each such social state is attainable — in the special sense of that word ascribed by 28. DEF. and discussed further in this remark. In fact, no social state is historically attainable unless it is a static social contract. The result offered in 29. COR. thus gives the student much reason to investigate into the characteristics of historically attainable social states (static social contracts), such as their uniqueness and, in particular, their stability. It provides motivation also to study the processes by which they can be reached or approximated. This is a motivation to study histories — in our special sense, of course, which, nevertheless, may not be totally unsuggestive.

For example, a slight formal blow to historicism might ensue from the fact that a history need not converge. Since histories are sequences in the compact set $P$, necessarily each history has a converging subsequence. However, different subsequences may converge to different limit points, so that the history in question does not converge. Furthermore, a subsequence of a history need not be a history.
These, admittedly, are only suggestive statements, since we have kept environment fixed, ignored interaction of environment with impression functions and avoided many other complications. One such complication is the possibility that, supposing a stable static social contract is reached or approximated and that the environment changes, a certain socialization may occur as a result of the mentioned equilibrium (static social contract) reigning in that particular period of time. This certain socialization may be to the effect of altering preferences or impression functions. This would then make the system a dynamic one – equilibria affecting factors which determine the set of possible equilibria for the immediate future, and so on. The dynamic social analysis required to treat systems of this sort can be developed on the foundations provided here, once the analysis of stability is worked out for the static case. (The dynamic analysis will pose its own dynamic stability questions, the answering of which should be the next research target in a sequence of follow-ups on the present framework.)

From the viewpoint of a social engineer, an organizational designer, or one who would like to intervene in society, our framework should again be suggestive. Suppose that this agent has, not only preferences over social state space, but also means of influencing the basic elements of our framework, namely, preferences and impression functions of behaviors in the society (collectivity). Two main instruments which could be used to this end are suggested: incentive schemes and information system manipulation. Incentive schemes would be formally equivalent to payoff functions allocating payoffs (or probability distributions over payoffs) in return for behavior conditional upon behavior-exclusive social states.
These incentive schemes would be communicated through the next mentioned control variable: information systems. The latter variable should also allow phenomena such as classified information or secrets, propaganda, training and education, indoctrination, and the whole gamut. These phenomena are, in turn, of a nature which preferences and impression functions could be expected to be sensitive to. A third means which the agent in question may be able to resort to is changing the environment, which would have its effects without altering preferences or impression functions of behaviors (although the latter elements may be specified as environment-dependent in a dynamic model).

The significance of 29. COR. for an agent of the sort in question should be quite clear. Should he decide to attempt bringing about a particular social state of his liking, he should obviously pay attention to whether or not it is historically attainable. In this case, he should concentrate on contractual social states (static social contracts).

Secondly, it would be a matter of importance to such an agent whether or not his preferred contractual social states are stable, and, if they are stable, whether they are of neutral, global or local stability — questions and essential concepts which will have to be taken up in a follow-up of the present study.

It is the purpose of this remark only to suggest some directions of intellectual pursuit from the present framework, which appear to be suggested by that framework. So, we will not go into these matters at any greater length here. However, it will be appropriate at this point
to comment on the special use of 'attainable' as by 28. DEF..

Certainly, "attainable" does not mean "reachable" here as in ordinary parlance, for every social state has this property in the weak sense of everyday discourse. But the transitory passing through of a social state is not of great interest, for each social state is a point in some history — take the histories which initiate from that social state. This is not so important if the system \( N \) cannot be observed to be consistently close to the social state in question, such as if the current history is convergent to it. If such convergence is the case, then, for any arbitrarily small neighborhood about the limit point, all but a finite number of points in the convergent history will be within that neighborhood — possibly, the sequence actually "reaching" the limit point. This seems to be a more significant state of affairs for the (limit) point in question than its merely being a state which can exist in some "moment" of a sequence (history) of social states. This would appear to be sufficient elucidation — and defense, if that be the case — of the special sense in which 'attainable' has been used.

In closing this study, the simple but useful 31. COR. is offered as a means of detecting or identifying static social contracts.

31. COR. If a social state \( s \) occurs twice or more in a row in some history, then \( s \) is a static social contract.

**Proof:** By hypothesis, for some history \( h = \{ s^c \} \) and for some positive integer \( k \), we have \( s^{c+1} = s^c = s \), for \( c = k \). By 28. DEF., \( s^{c+1} \in S(s^c) \). Hence, \( s \in S(s) \).
Postscript:

After writing the above sections, it has come to the author's attention that a special case of the existence result 22. COR. had earlier been demonstrated by Nash (1950, 1951). It is only natural, therefore, that the two results be compared and put in their proper places vis a vis one another and in the perspective of the larger framework. This is what the present section is intended for. Without making undue further excursions into intellectual history, let us turn to Nash's result and its comparison with ours.³

Although Nash presents his theory in a game-theoretic context, this poses no language barrier, in that a natural translation into our terminology and a natural interpretation in the context of our present framework is easily afforded. "The notion of an equilibrium point is the basic ingredient in [Nash's] theory" (Nash, 1951, p.286), and the result that such a point exists is correspondingly fundamental. The proof of the result is by demonstration of the existence of a fixed point, using Kakutani's theorem in the earlier proof (Nash, 1950), and Brouwer's theorem in the later proof (Nash, 1951).

There are n players (behaviors) each having choice over a finite set of pure strategies (basic behaviors). For each player there is a payoff function assigning real numbers to each n-tuple of pure strategies. (Although a surrounding, or "nature", is not explicitly introduced, it could trivially be done so.) Each payoff function has an extension to a real-valued function defined on the set of all n-tuples of mixed strategies, and, under the special assumptions of Nash which we will immediately make explicit, this extension is a unique function which is linear in the mixed strategy of each player.
This unique n-linear extension is not so surprising since, although they are not defined or explicit, impression functions are implicitly introduced in a very restricted form. It is extremely important to understand the restrictions placed on impression functions, for these in turn induce restrictions on the payoff functions and are related in turn to a very important restriction on the form in which social states (in our terminology) take place in the theory. It will help our exposition here to start with the last-mentioned restriction.

In Nash's theory it is possible to consider only those social states which reflect an extreme type of independence between actions of players (behaviors). In particular, only those social states \( p \) can be considered according to which, denoting \( f_i(p) = f_i, f_j(p) = f_j \), (i, j in N),

\[
p(E_i,E_j) = f_i(E_i)f_j(E_j), \quad (i \neq j),
\]

i.e., one can consider as social states only those joint probability distributions according to which the actions \( E_i \) of any behavior \( i \) are distributed independently of actions \( E_j \) of any other behavior \( j \). It is clear that n-tuples of mixed strategies (i.e., elementary collective reactions) are in one-one correspondence with social states of this independent sort. If one is willing to restrict social states to this independent sort, then one might as well forget about social states and work simply with n-tuples of mixed strategies. This is what Nash does. This highly noisy world-view is the view which the players too entertain of their own world, in the sense that their play would be inconsistent with their observing non-independent distributions of others' actions unless they were indifferent between any two distributions as long as their j-marginals (for all j in N) were equal.
But this brings us to the particularly restrictive assumption implicitly placed on impression functions. This, in our earlier terminology, boils down to the case of assumed perfect information, in the sense that the impression functions are restricted to the form \( I_j(p) = p \) (for all \( j \) in \( N \)). By the just explained restriction of social states to the independent sort, this also guarantees that each player "sees" a given \( n \)-tuple of mixed strategies, i.e., a given elementary collective reaction, in a perfectly objective way, in the sense that all members of the collectivity see it as the same.

It may be thought, incidentally, that this assumption of perfect information is not so strong as a variant of it, where, under certainty, each observer sees the "true" picture. The latter, however, may validly be called the weaker of the two assumptions, in that the former assumes, not only that the extreme points of (social) state space are seen by all as they are - as they are in "actuality", or in "the detached and objective scientist's view" - but that the whole of (social) state space is so seen.

In any case, given the assumption that social states are of what we have called the "independent" sort together with the assumption of perfect information, what emerges is that, for any social state, any two players have the same impression of it, and, a fortiori, that common impression is of the "independent" class. Hence, it is operationally immaterial whether players are actually indifferent between social states having the same \( j \)-marginals - between, for example, any two social states in the manifestation of an elementary collective reaction. For any two such social states are indistinguishable in the view of a Nash player, so that behavior is unaffected by anything but the \( n \)-tuple of \( j \)-marginals.
Thus, after all these restrictions, it is natural in Nash's theory to have payoff functions defined on the set of n-tuples of pure strategies (the basic social state space) uniquely extending to n-linear payoff functions on the set of n-tuples of mixed strategies (the social state space). For each player now computes the payoff from any given n-tuple of mixed strategies as the expected value of the payoff from all possible n-tuples of pure strategies, taking expectations according to an impression which is of the "independent" class. The expected utility (in our terminology) associated with a social state \( p \) is again, like in our case, the inner product of the vector \( u_j \) (whose components are the real numbers (utilities) assigned to each n-tuple of pure strategies by the "unextended payoff function") with the vector \( I_j(p) \) (which in this case can be taken to be the simple sort of vector (of the same dimensionality as \( u_j \) assigning probability \( I_j(p) (E_1E_2...E_n) = f_j(E_1)f_j(E_2)...f_j(E_n) \) to any collective action \( E_1E_2...E_n \)):

\[
U_j(p) = <u_j, I_j(p)>. 
\]

In our case, however, \( I_j(p) \) need not be of the independent class (since \( I^j(p) \) is allowed not to be of this class), and \( U_j \) need not be \((n-)\) linear in \( p \) (since \( I^j \) need only be continuous in \( p \)).

Having demonstrated the main differences between Nash's set-up and ours, we may now finally state the existence result that is so central to Nash's theory: For any n-person game (of the Nash variety), there exists an n-tuple of mixed strategies (behaviors), called an equilibrium point, such that, for any player \( j \in N \), given the mixed strategies of all other \((n-1)\) players (behaviors), the n-tuple has a mixed strategy, as its \( j^{th} \) element, maximizing the payoff function of player \( j \). There is no need to prove this fact here since it is clearly implied by 22.COR.
as a special case when social states, impression functions and payoff functions are restricted to the special variety used by Nash.

Having taken the reader through the trouble of reading this section up to here, the author may attempt to reciprocate by giving a brief hint of the more practical advantages, for the theoretical or applied researcher, of the framework presented here, as opposed to the more restricted one with which it was just compared. Some of these advantages may be listed as is done below.

1. Zannetos (1965) has given us sufficient motivation to be interested in the variance-covariance matrix associated with a given joint distribution of actions of a collectivity of actors, especially when those actors represent members or divisions of an organization. This matrix, or an associated matrix of (co-) variances between performances of the mentioned actors, is indicated to by Zannetos as reflective of organizational structure, especially centralizedness and decentralizedness, and of the complementarities and substitutivities between, e.g., divisional activities. It is also pointed to as a useful source of information in attempts to improve organizational performance by restructuring.

Now, if by definition the (co-) variance matrices associated with the joint probability distributions (social states) are all diagonal matrices with zeros off the main diagonal, the suggestions of Zannetos could be put to little use, which, in this author's estimation, would be a true waste. This waste could hardly be avoided if social states were assumed of the "independent" class.

2. Often it is useful to consider entities other than individuals as actors or behaviors. The metaphoric senses in which organizations act
or behave often may be indicative of a kind of entity-construct which, ontologically speaking, exists in a stronger and more real sense than one which can be decomposed into other "atomic" entities, that is, entities which have already gained our respect as existing in their own right as capable of behavior which cannot fruitfully be construed as decomposable into behaviors of more elementary entities. The behavior of a team, more in the sense of Allport (1961) than in the sense of Marschak (1954); Durkheim's (1895) very important argument that there are also social facts or phenomena (as opposed to merely sets of individual facts or phenomena); that which is often referred to as "collective mind" - all these are suggestive of entities existing in their own right as distinct from the individual elements of which they may be construed as having been constructed. And such entity-constructs are common referanda in social discourse.

Without getting too philosophical in a paper which is not intended to be philosophical, and without, therefore, being able to go into these matters in sufficient depth to allow us to make statements of precise meaning, it is possible to remark that viewing social states as being of the "independent" class would destroy at least one of the senses in which a whole may be different than the sum of its parts. Not only is the variance of two non-independently distributed random variables different from the sum of variances of the individual random variables, but also this has a tone which is suggestive that, if the individual random variables are associated with the occurrences of the actions of individual behaviors, the joint occurrence of these actions may be looked upon, in some sense which is not obviously unfruitful, as an action of the collectivity of the two actors in question. (This, again, is a take-off from Zannetos (1965).)
3. Making explicit a surrounding for a collectivity is mathematically trivial, but of significance for the social student. For the static contractual set, i.e., the set of static social contracts, for a collectivity may differ from environment to environment. This has implications both in the positive and in the normative realm, to use a partitioning in the economist's methodological jargon. On the positive side, a direct implication is that environment may explain why certain social contracts existing under one set of "environmental factors" do not exist under other "environmental factors", whatever the proper interpretation of these factors in the case under study. On the normative side, a direct implication is that the contractual set may be controllable through environment. And back again to the positive side, this provides an indirect implication that, in explaining why a certain contractual set obtains for a given collectivity, it may be fruitful to use the control behavior of certain "outsiders" (e.g., foreign organizations, or "social leaders" who are in controlling position vis a vis local collectivities) as explanatory variables. It is possible, going one step further, to suggest a framework employing the notions of a controllers' collectivity with its meta-contracts ruling over or controlling "lower level" contracts, the control being affected through the environment of the "lower level" collectivity. Without being too lengthy on "environmental control", we should pass over to consider a further variety of control which it is possible to explicitly consider due to the explicit introduction of impression functions into our framework.

4. Since impression functions are explicitly present in our framework, and since they are allowed to vary, both from behavior to behavior and over time (if we wish to consider the latter alternative), with the only
proviso that they be continuous in social states and that each behavior have perfect information of his own behavior \((I_j(p) = f_j(p) * I_j(f_j(p)))\), impression functions are allowed quite an extent of freedom to be varied as control variables. The contractual set may be controlled, therefore, also by this additional type of control variable. (The reader should note that impression functions are fixed under the perfect information assumption, since it is required that \(I_j(p) = p\), and since there is a unique function \(I_j\) on \(P\) with this property. Under that assumption, impression functions cannot be considered as variables, so that the contractual set cannot be affected by varying them.)

The direct and indirect implications of this added possibility of control are easily imaginable in both the positive and the normative realm. So are meta-contracts between "informational" controllers imaginable without any aid that can be given here. What should be added here is that both "environmental" and "informational" control may take place simultaneously, opening the possibility of meta-contracts for a collectivity of controllers possessing, in general, both types of control. It should be interesting to follow out the implications of the possibility off the two types of control to interact. Complementarities and substitutivities between the two types would be of prime interest to investigate.

5. In order not to lengthen this list indefinitely, we may lastly mention how socialization processes may be viewed from the standpoint of our framework. Apart from environment and impression functions, the vectors \(u_j\), reflecting basic "tastes" of the behaviors, determine the contractual set. Thus, the n-tuple \(u = <u_1, \ldots, u_n>\) of utility vectors \(u_j\) \((j \in \mathbb{N})\) may be considered a third variable by which the contractual set may be controlled. This opens up two main types of pos-
sibilities.

Firstly, these n-tuples may be considered as controls available to behaviors in a "higher level" collectivity of controllers, where interaction of control through environment (environmental control), through impression functions (informational control), and through utility vectors (socializational control) may be investigated. Apart from the complementarities and substitutivities between these three types of control activity, it would be central to investigate the relationships existing between collectivities of different levels. The effect of moving from one contract to another at a higher level on the contract in a lower level would be of main interest here.

Secondly, the socialization process at \( t \), a given moment or period in time, may be seen as adding the incoming population of a collectivity as new entrants with \( u_j \) vectors of vintage \( t \), while that socialization process itself could be viewed as being determined by the history obtaining in that collectivity (now of changing membership, as some are added and others die out) up to \( t \). This begins to spin the web of a social dynamics. Past history, up to and including the present, determines the class of elementary school curricula, for example, and elementary school teaching, while this process "produces" a new generation with "tastes" (in a large sense) of present vintage. As senior members of the collectivity retire and new members of new-vintage tastes are added, the social evolution is changed, in turn changing the possible future courses of history and the contractual set. This, in turn, feeds back to the socialization process, and so on. Also, there is no reason why this sort of social dynamics should not take place at many levels of collectivity. Mobility of behaviors between levels in this dynamic picture would be a subject very pertinent to the corresponding analysis.
Short of beginning to actually do research in these directions, there is not much more to be said in the way of hinting at further research questions and areas which appear to afford formal treatment by extensions of the present framework. These extensions, static and dynamic, are open to anybody, and it is hoped that some of them will be taken up by the wide variety of social students to which they ought to appeal.
NOTES

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1 If $I^j(f^j(p))$ is fixed, then $U_j$ depends solely on $p_j$. Let $q$, $q'$, $\bar{q}$ be given by

$$q = p_j \ast I^j(f^j(p)),$$

$$q' = p'_j \ast I^j(f^j(p)),$$

$$\bar{q} = cq + c'q'.$$

Since $U_j$ is linear on $I_j(p)$ [see 8. ASS. (A2)], we have

$$U_j(\bar{q}) = cU_j(q) + c'U_j(q'),$$

so that $U_j$ is linear on $P_j$ when $I^j(f^j(p))$ is fixed.

2 By historicism is meant the claim that there is a "final" state (or "stage") toward which social history tends. Since a rigorous proof for this existence claim must not have been obvious to them, historicists (e.g., Hegel, Marx) have had to argue by describing particular ways in which history would "unfold" itself, and by trying to present evidence that actual history fits this description, the argument that the described process is a convergent one being left quite casual.
Since rigor does not typically bring popularity, the intellectual history of historicism is not altogether surprising. Often, however, belief in historicism appears to coincide with belief in the superiority, from a subjective preferential point of view, of the "final stage" claimed existent. It would seem more productive of such a final stage, on the other hand, for those who prefer it not to assume that it will come about anyway, but that creating a social state of their choice is very much the business of those who want it.

For a critique of historicism (in a very different vein than the present one) the reader is referred to Popper (1957).

Perhaps it would not be out of place to mention in this context that, although the results above were reached in ignorance of the fact that Nash had obtained what is claimed below to be a weaker, i.e., less general, existence result than 22. COR., that very result which must be recognized as due to Nash was conjectured also by the present writer, but ignored, upon obtaining the more general 22. COR. of which it is a special case, until this postscript was necessitated.

It is not too surprising to find that many social students whose strengths lie outside of technical mathematics are often repelled by mathematical works in their field, which, although they may present the world in a probabilistic model and bear titles indicative of an effort to "deal with the case under uncertainty", fail to convey a feeling of having taken account of the very aspects of their subject matter which arise out of the condition of uncertainty. Although Nash's mentioned works do not give the impression that they were meant to be contributions to social science per se, so that their author can hardly be criticized on that count, the same need not apply to those who would fit social phenomena into the Nash framework.
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