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FALLACY OF THE LOG-NORMAL APPROXIMATION TO OPTIMAL PORTFOLIO DECISION-MAKING OVER MANY PERIODS

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FALLACY OF THE LOG-NORMAL APPROXIMATION

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I. Introduction

Thanks to the revival by von Neumann and Morgenstern, maximization of the expected value of a concave utility function of outcomes has for the last third of a century generally been accepted as the "correct" criterion for optimal portfolio selection. Operational theorems for the general case were delayed in becoming recognized, and it was appropriate that the seminal breakthroughs of the 1950's be largely preoccupied with the special case of mean-variance analysis.[1/] Not only could the fruitful Sharpe-Lintner-Mossin capital asset pricing model be based on it, but in addition, it gave rise to simple linear rules of portfolio optimizing. In the mean-variance model, the well-known Separation or Mutual-Fund Theorem holds; and with suitable additional assumptions, the model can be used to define a complete micro-economic framework for the capital market, and a number of empirically testable hypotheses can be derived. As a result, an overwhelming majority of the literature on portfolio theory have been based on this criterion.[2/]

Unfortunately, as has been pointed out repeatedly, the mean-variance criterion is rigorously consistent with the general expected-utility approach

only in the rather special cases of a quadratic utility function or of gaussian distributions on security prices--both involving dubious implications. Further, recent empirical work has shown that the simple form of the model does not seem to fit the data as well as had been previously believed,[3] and recent dynamic simulations[4] have shown that the behavior over time of some efficient mean-variance portfolios can be quite unreasonable.

Aside from its algebraic tractability, the mean-variance model has interest because of its separation property. Therefore, great interest inhered in the Cass-Stiglitz [7] elucidation of the broader conditions under which such a property must hold regardless of the probability distribution of returns. The special families of utility functions with constant-relative-risk aversions or constant-absolute-risk aversions further gained in interest.[5] But it was realized that real-life utilities need not be of so simple a form.

The desire for simplicity of analysis led naturally to a search for approximation theorems, particularly of the asymptotic type. Thus, even if mean-variance analysis were not exact, would the error in using it become small? A defense of it was the demonstration that mean-variance is asymptotically correct if the risks are "small" (i.e., for compact probabilities); closely related was the demonstration of the same asymptotic equivalence when the trading interval becomes small (i.e., continuous trading). More recently, it has been shown that as the number of assets becomes large, under certain conditions, the mean-variance solution is asymptotically optimal.[6]

A particularly tempting hunting ground for asymptotic theories was thought to be provided by the case in which investors maximize the expected utility of terminal wealth when the terminal date (planning horizon) is very far in the future. Recourse to the Law of Large Numbers, as applied to re-

peated multiplicative variates (cumulative sums of logarithms of portfolio changes) has independently tempted various writers, holding out to them the hope that one can replace an arbitrary utility function with all its intractability by the function $U(W_T) = \log(W_T)$: maximizing the geometric mean or the expected log of outcomes was hoped to provide an asymptotically exact criterion for rational action, implying as a bonus the efficiency of a diversification-of-portfolio strategy constant through time (i.e., a "myopic" rule, the same for every period, even when probabilities of different periods were interdependent!). So powerful did the max-expected-log criterion appear to be, that it seemed even to supercede the general expected utility criterion in cases where the latter was shown to be inconsistent with the max-expected-log criterion. For some writers, it was a case of simple errors in reasoning: they mistakenly thought that the sure-thing principle sanctified the new criterion.[7/] For others, an indefinitely large probability of doing better by Method A than by Method B was taken as conclusive evidence for the superiority of A.[8/] Still other converts to the new faith never realized that it could conflict with the plausible postulates of von Neumann maximizing; or, more sophisticatedly, have tried to save the approximation by appealing to bounded utility functions.[9/]

Except to prepare the ground for a more subtle fallacy of the same asymptotic genus, the present paper need not more than review the simple max-expected-log fallacy. It can concentrate instead on the asymptotic fallacy that involves, not primarily the Law of Large Numbers, but rather the Central Limit Theorem. It is well-known that uniform portfolio strategies give rise to a cumulative sum of logarithms of returns that do approach, when normalized, under specified conditions easily met, a gaussian distribution--suggesting heuristically a Log-normal Surrogate for the actual dis-

tribution. If one can replace the true distribution by its Log-normal Surrogate, only two parameters--each period's expected log of return, and variance of log of return--will become "asymptotically sufficient parameters" for efficient portfolio managing, truly an enormous simplification in that all optimal portfolios will lie on a new efficiency frontier in which the first parameter is maximized for each different value of the second. This frontier can be generated solely by the family $U(W_T) = (W_T)^{\gamma}/\gamma$, and for $\gamma \neq 0$, this leads away from the simple max-expected-log portfolio. So this new method does avoid the crude fallacy that men with little tolerance for risk are to have the same long-run portfolio as men with much risk tolerance. Furthermore, since the mean and variance of average-return-per-period are asymptotic surrogates for the log normal's first two moments, Hakansson's [17] average-expected-return seems to be given a new legitimacy by the Central-Limit Theorem.

One purpose of this paper is to show, by counterexamples and examination of illegitimate interchange of limits in double limits, the fallacies involved in the above-described asymptotic log-normal approximation. What holds for normalized variables is shown to be not necessarily applicable to actual terminal wealths. Then, constructively, we show that, not as the number of fixed-length periods goes to infinity because the planning horizon, T, goes to infinity, but rather as any <u>fixed</u> horizon planning interval is <u>subdivided</u> into a number of sub-interval periods that goes to infinity (causing the underlying probabilities to belong to a gaussian infinitely-divisible continuous-time probability distribution)--then the mean-log and variance-log parameters are indeed asymptotically sufficient parameters for the decision; so that one can prepare a (μ,σ) efficiency frontier that is quite distinct from the Markowitz mean-variance frontier (that had involved actual rather than logarithms of returns), but which now provides many of the same two-dimensional simplifica-

tion (such as separation properties).

There is still another kind of asymptotic approximation that attempts, as $T \to \infty$, to develop in Leland's [24] happy phrase a "turnpike theorem" in which the optimal portfolio proportions are well approximated by a uniform strategy that is appropriate to one of the special family of utility functions, $U(W_T) = (W_T)^\gamma/\gamma$, where $\gamma - 1$ is the limiting value of the elasticity of marginal utility with respect to wealth as $W_T \to \infty$. Letting γ then run the gamut from 1 to $-\infty$ generates a new kind of efficiency frontier, distinct from that of ordinary mean-variance or of [E{log W}, var{log W}], but which obviously generalizes the single-point criterion of the would-be expected-log maximizers--generalizes that criterion in that we now rationally trade off mean return against risk, depending on our own subjective risk tolerance parameter γ. There is some question[10] as to the robustness of the Leland theorem for utility functions "much different" from members of the iso-elastic family; but the main purpose of our paper is to uncover the booby traps involved in log-normal and other asymptotic approximations, and we do not examine this question in any depth.

II. Exact Solution

In any period, investors face n securities, 1,...,n. One dollar invested in the jth security results at the end of one period in a value that is $Z_j(1)$, a positive random variable. The joint distribution of these variables is specified as

$$\text{Prob}\{Z_1(1) \leq z_1,\ldots,Z_n(1) \leq z_n\} = F[z_1,\ldots,z_n] = F[z] \qquad (2.1)$$

where F has finite moments. Any portfolio decision in the first period is defined by the vector $[w_1(1),\ldots,w_n(1)]$, $\sum_1^n w_j(1) = 1$; if the investor begins with initial wealth of W_0, his wealth at the end of one period is given by the

random variable

$$W_1 = W_0[w_1(1)Z_1(1) + \ldots + w_n(1)Z_n(1)] \tag{2.2}$$

By the usual Stieltjes integration over $F[z_1,\ldots,z_n]$, the probability distribution of W_1 can be defined, namely

$$\begin{aligned}\text{Prob}\{\log(W_1/W_0) \leq x\} &= P_1[x;\ w_1(1),\ldots,w_n(1)] \\ &= P_1[x;\ w(1)].\end{aligned} \tag{2.3}$$

An investment program re-invested for T periods has terminal wealth, W_T, defined by iterating (2.2) to get the random variable

$$\begin{aligned}W_T &= W_0[\textstyle\sum_1^n w_j(1)Z_j(1)][\sum_1^n w_j(2)Z_j(2)]\cdots[\sum_1^n w_j(T)Z_j(T)] \\ &= W_T[w(1),\ldots,w(T)].\end{aligned} \tag{2.4}$$

It is assumed that the vector of random variables $[Z(t)]$ is distributed independently of $Z(t \pm k)$, but subject to the same distribution as $Z(1)$ in (2.1). Hence, the joint probability distribution of all securities over time is given by the product

$$\begin{aligned}&\text{Prob}\{Z(1) \leq z(1),\ldots,Z(T) \leq z(T)\} \\ &\qquad = F[z_1(1),\ldots,z_n(1)]\ldots F[z_1(T),\ldots,z_n(T)] \\ &\qquad = F[z(1)]\cdots F[z(T)].\end{aligned} \tag{2.5}$$

Since W_T/W_0 consists of a product of independent variates, $\log(W_T/W_0)$ will consist of a sum of independent variates. Therefore, its probability distribution is, for each T, definable recursively by the following convolutions,

$$\text{Prob}\{\log(W_T/W_0) \leq x\} = P_T[x;\ w(1),\ldots,w(T)] \tag{2.6}$$

$$P_2[x;\ w(1),\ w(2)] = \int_{-\infty}^{\infty} P_1[x-s;\ w(2)]P_1[ds;\ w(1)]$$

$$P_3[x;\ w(1),\ w(2),\ w(3)] = \int_{-\infty}^{\infty} P_1[x-s;\ w(3)]P_2[ds;\ w(1),\ w(2)]$$

$$\vdots \qquad\qquad\qquad \vdots$$

$$P_T[x;\ w(1),\ \ldots\ ,\ w(T)] = \int_{-\infty}^{\infty} P_1[x-s;\ w(T)]P_{T-1}[ds;\ w(1),\ldots,w(T-1)].$$

Here, as a matter of notation for Stieltjes integration, $\int_{-\infty}^{\infty} f(s)dg(s) = \int_{-\infty}^{\infty} f(s)g(ds)$.

The investor is postulated to act in order to maximize the expected value of (concave) terminal utility of wealth

$$\underset{\{w(t)\}}{\text{Max}}\ E\{U_T(W_T[w(1),\ldots,w(T)])\} = \underset{\{w(t)\}}{\text{Max}} \int_{-\infty}^{\infty} U_T(W_0 e^x)P_T[dx;\ w(1),\ldots,w(T)] \quad (2.7)$$

$$\equiv \overline{U}_T[w^{**}(1),\ldots,w^{**}(T);\ W_0].$$

Here, $U_T(\ )$ is a concave function that can be arbitrarily specified, and $[w_j(t)]$, for each t, is understood to be constrained by $\sum_1^n w_j(t) = 1$.

For a general U_T, the optimal solution $[\ldots,\ w^{**}(t),\ldots]$ will not involve portfolio decisions constant through time, but rather optimally varying in accordance with the recursive relations of Bellman dynamic programming, as discussed by numerous authors, as for example in Samuelson [42]. But, here, we shall for the most part confine our attention to uniform strategies

$$[w_1(t),\ldots,w_n(t)] \underset{t}{\equiv} [w_1,\ldots,w_n] \quad (2.8)$$

For each such uniform strategy, $\log (W_T/W_0)$ will consist of a sum of independent and identically distributed variates. We write the optimal uniform strategy as $w(t) \equiv w^*_T$, and abbreviate

$$P_T[x;w] \equiv P_T[x;\ w,\ldots,w], \quad T = 2,3,\ldots \quad (2.9)$$

$$W_T[w] \equiv W_T[w,\ldots,w]$$

$$\overline{U}_T[w^*;W_0] \equiv \overline{U}_T[w^*,\ldots,w^*;\ W_0].$$

Actually, for the special utility functions,

$$U_T(W) = W^\gamma/\gamma,\ \gamma < 1,\ \gamma \neq 0 \quad (2.10)$$

$$= \log W,\ \gamma = 0,$$

it is well known that $w^{**}(t) \equiv w^*$, independent of T, is a necessary result for full optimality.

III. Max-expected-log (Geometric Mean) Fallacy

Suppose Z_n represents a "safe security" with certain return

$$Z_n = e^r \equiv R \geq 1 \tag{3.1}$$

If the other risky securities are optimally held in positive amounts, together their uncertain return must have an expected value that exceeds R. Consider now the parameters

$$E\{\log(W_1/W_0)\} = E\{\log(\textstyle\sum_1^n w_j Z_j)\} = \mu(w_1,\ldots,w_n) = \mu(w) \tag{3.2}$$

$$\mathrm{Var}\{\log(W_1/W_0)\} = E\{[\log(\textstyle\sum_1^n w_j Z_j) - \mu(w)]^2\} = \sigma^2(w_1,\ldots,w_n) = \sigma^2(w).$$

For $w = (w_1,\ldots,w_n) = (0,0,\ldots,0,1)$, $\mu(w) = r \geq 0$. As w_n declines and the sum of all other w_j become positive, $\mu(w)$ must be positive. However, $\mu(w)$ will reach a maximum; call it $\mu(w^{\dagger\dagger})$, and recognize that $w^{\dagger\dagger}$ is the max-expected-log strategy.

As mentioned in section I, many authors fallaciously believe that $w^{\dagger\dagger}$ is a good approximation to w^* for T large, merely because

$$\lim_{T \to \infty} \mathrm{Prob}\{W_T[w^{\dagger\dagger}] > W_T[w]\} = 1, \quad w \neq w^{\dagger\dagger} \tag{3.3}$$

A by now familiar counter-example occurs for any member of the iso-elastic family, (2.10), with $\gamma \neq 0$. For a given γ, $w^*(t) \equiv w^*$, the same strategy, independent of T; since each γ is easily seen to call for a different w^*, it must be that $w^* \neq w^{\dagger\dagger}$ for all T and $\gamma \neq 0$, in as much as $w^* = w^{\dagger\dagger}$ only when $\gamma = 0$ and $U_T(W) = \log W$. Hence, the vague and tacit conjecture that w^* converges to $w^{\dagger\dagger}$ asymptotically as $T \to \infty$, is false.

Others, e.g. Markowitz [28, p.3], who are aware of the simple fallacy have conjectured that the max-expected-log policy will be "approximately" optimal for large T when $U_T(\ )$ is bounded (or bounded from above). I.e., if $U_T(\ )$ is bounded (or bounded from above), then the expected utility maximizer, it is argued, will be "approximately indifferent" between the $\{w^*\}$ and $\{w^{++}\}$ programs as T becomes large.

The exact meaning of "approximate indifference" is open to interpretation. A trivial meaning would be

$$\lim_{T \to \infty} EU_T(W_0 Z_T[w^*]) = M = \lim_{T \to \infty} EU_T(W_0 Z_T[w^{++}]) \tag{3.4}$$

where M is the upper bound of $U_T(\ )$. This definition merely reflects the fact, that even a sub-optimal strategy (unless it is too absurd) will lead as $T \to \infty$ to the upper bound of utility. For example, just holding the riskless asset with positive return per period $R = e^r > 1$, will, for large enough T, get one arbitrarily close to the bliss level of utility. Hence, even if (3.4)'s definition of indifference were to make the conjecture true, its implications have practically no eontent.

A meaningful interpretation would be indifference in terms of a "initial wealth equivalent." That is, let $\Pi_{ij}W_0$ be the initial wealth equivalent of a program $\{w^i\}$ relative to program $\{w^j\}$ defined such that, for each T,

$$EU_T(\Pi_{ij}W_0 Z_T[w^i]) \equiv EU_T(W_0 Z_T[w^j]) \tag{3.5}$$

$(\Pi_{ij} - 1)W_0$ is the amount of additional initial wealth the investor would require to be indifferent to giving up the $\{w^j\}$ program for the $\{w^i\}$ program. Thus, we could use $\Pi_{12}(T;W_0)$ as a measure[11/] of how "close" in optimality terms the $\{w^1\} \equiv \{w^{++}\}$ program is to the $\{w^2\} \equiv \{w^*\}$ program. The conjecture that max-expected-log is asymptotically optimal in this modified sense would be true only if it could be shown that $\Pi_{12}(T;W_0)$ is a decreasing function of T

and $\lim_{T \to \infty} \Pi_{12}(T;W_0) = 1$, or even if $\Pi_{12}(T;W_0)$ were simply a bounded function of T.

Consider the case when $U_T(\ ) = (\ )^\gamma/\gamma$, $\gamma < 1$. Then,

$$EU_T(W_0 Z_T[w^*]) = W_0^\gamma E[(Z_T[w^*])^\gamma]/\gamma \tag{3.6}$$

$$= W_0^\gamma \{E[(Z_1[w^*])^\gamma]\}^T/\gamma,$$

by the independence and identical distribution of the portfolio return in each period. Similarly, we have that

$$EU_T(\Pi_{12} W_0 Z_T[w^{\dagger\dagger}]) = (\Pi_{12} W_0)^\gamma \{E[(Z_1[w^{\dagger\dagger}])^\gamma]\}^T/\gamma \tag{3.7}.$$

Now, $\{w^*\}$ maximizes the expected utility of wealth over one period (i.e., for the iso-elastic family $w^{**} = w^*$), and since $w^* \neq w^{\dagger\dagger}$ for $\gamma \neq 0$, we have that

$$E[(Z_1[w^*])^\gamma] \gtreqless E[(Z_1[w^{\dagger\dagger}])^\gamma] \text{ as } \gamma \gtreqless 0 \tag{3.8}.$$

From (3.5), (3.6), and (3.7), we have that

$$\Pi_{12}(T;W_0) = [\lambda(\gamma)]^{T/\gamma} \tag{3.9}$$

where $\lambda(\gamma) \equiv E[(Z_1[w^*])^\gamma]/E[(Z_1[w^{\dagger\dagger}])^\gamma]$. From (3.8), $\lambda(\gamma) > 1$ and $T/\gamma > 0$, for $\gamma > 0$; $\lambda(\gamma) < 1$ and $T/\gamma < 0$, for $\gamma < 0$; and, since λ is independent of T and W_0, we have from (3.9) that, for $\gamma \neq 0$ and every $W_0 > 0$,

$$\partial\Pi_{12}(T;W_0)/\partial T > 0 \tag{3.10}$$

$$\lim_{T \to \infty} \Pi_{12}(T;W_0) = \infty.$$

Hence, even for $U_T(\ )$ with an upper bound (as when $\gamma < 0$), an investor would require an ever-larger initial payment to give up his $\{w^*\}$ program. Similar results obtain for $U_T(\ )$ functions which are bounded from above and below.[12] Therefore, the $\{w^{\dagger\dagger}\}$ program is definitely not "approximately" optimal for large T.

Further, the sub-optimal $\{w^{\dagger\dagger}\}$ policy will, for every finite T however large, be in a clear sense "behind" the best strategy, $\{w^*\}$. Indeed, let us

apply the test used in the Eisenhower Administration to compare U.S. and U.S.S.R. growth. How many years after the U.S. reached each real GNP level did it take the U.S.S.R. to reach that level? This defines a function $\Delta T = f(GNP_t)$, and some Kreminologists of that day took satisfaction that ΔT was not declining in time. In a new calculation similar to the initial wealth equivalent analysis above, let us for each level of $E[U_T(W_T)]$ define $\Delta T = T^{++} - T^*$ as the difference in time periods needed to surpass that level of expected utility, calculated for the optimal strategy $\{w^*\}$ and for the max-expected-log strategy $\{w^{++}\}$; then it is not hard to show that $\Delta T \to \infty$ as $T \to \infty$. Again the geometric mean strategy proves to be fallacious.

Finally, we can use the initial wealth equivalent to demonstrate that for sufficiently risk-averse investors, the max-expected-log strategy is a "bad" program: bad, in that the $\{w^{++}\}$ program will not lead to "approximate" optimality even in the trivial sense of (3.4) and hence, will be dominated by the program of holding nothing but the riskless asset.

Define $\{w^3\} \equiv [0, 0, 0, \ldots, 1]$ to be the program which holds nothing but the riskless asset with return per period, $R \geq 1$. Let $\{w^1\}$ be the max-expected-log program $\{w^{++}\}$ as before. Then, for the iso-elastic family, the initial wealth equivalent for the $\{w^1\}$ program relative to the $\{w^3\}$ program, $\Pi_{13}W_0$, is defined by

$$\frac{W_0^{\gamma}}{\gamma} \Pi_{13}^{\gamma} E[(Z_T[w^{++}])^{\gamma}] \equiv \frac{W_0^{\gamma}}{\gamma} R^{\gamma T} \tag{3.11}$$

or

$$\Pi_{13} = [\phi(\gamma)]^{-T/\gamma} \tag{3.12}$$

where $\phi(\gamma) \equiv E[(Z_1[w^{++}])^{\gamma}]/R^{\gamma}$.

To examine the properties of the $\phi(\gamma)$ function, we note that since $\{w^{++}\} \equiv [w_1^{++}, w_2^{++}, \ldots, w_n^{++}]$ does maximize $E \log(Z[w;1]) \equiv$

$E \log(\sum_1^{n-1} w_j[Z_j(1) - R] + R)$, it must satisfy

$$E\{[Z_j(1) - R]/Z[w^{\dagger\dagger};1]\} = 0, \quad j = 1,2,\ldots,n \tag{3.13}$$

Multiplying (3.13) by $w_j^{\dagger\dagger}$ and summing over $j = 1,2,\ldots,n$, we have that

$$E\{(Z[w^{\dagger\dagger};1] - R)/Z[w^{\dagger\dagger};1]\} = 0 \tag{3.14}$$

or that

$$E\{(Z[w^{\dagger\dagger};1])^{-1}\} = R^{-1} \tag{3.15}$$

From (3.12) and (3.15), we have that

$$\phi(\gamma) > 0; \quad \phi(0) = \phi(-1) = 1 \tag{3.16}$$

Further, by differentiation,

$$\phi''(\gamma) = E\{(Z[w^{\dagger\dagger};1])^{\gamma}\log^2(Z[w^{\dagger\dagger};1])\} > 0, \tag{3.17}$$

and so, ϕ is a strictly convex function with a unique interior minimum at γ_{min}. From (3.16), $-1 < \gamma_{min} < 0$, and therefore, $\phi'(\gamma) < 0$ for $\gamma < \gamma_{min}$. Hence, since $\phi(-1) = 1$, $\phi(\gamma) > 1$ for $\gamma < -1$.

But, from (3.12), $\phi(\gamma) > 1$ for $\gamma < -1$ implies that for any R, $W_0 > 0$, and $\gamma < -1$,

$$\lim_{T \to \infty} \Pi_{13}(T) = \infty, \tag{3.18}$$

and therefore, such risk-averse investors would require an indefinitely large initial payment to give up the riskless program for the max-expected-log one.

Further, in the case where $R = 1$ and the riskless asset is non-interest bearing cash, we have that for $\gamma < -1$, $E[(Z_1[w^{\dagger\dagger};1])^{\gamma}] > 1$ which implies that

$$\lim_{T \to \infty} EU_T(W_0 Z[w^{\dagger\dagger};T]) = \lim_{T \to \infty} W_0^{\gamma}\{E[(Z[w^{\dagger\dagger};1])^{\gamma}]\}^T/\gamma$$
$$= -\infty, \tag{3.19}$$

so that such a risk-averse person's being forced into the allegedly desirable max-expected-log strategy is just as bad for infinitely large T as having all his initial wealth taken away! Few people will opt to ruin themselves voluntarily once they understand what they are doing.[13/]

IV. A False Log-Normal Approximation

A second more subtle, fallacy has grown out of the more recent literature on optimal portfolio selection for maximization of (distant time) expected terminal utility of wealth.

Hakansson [17], after giving arguments based on maximizing expected average rate-of-return that imply myopic and uniform strategies, $w(t) \equiv w_T$, proceeds to use the Central-Limit theorem to argue that the asymptotic distribution as $T \to \infty$ for such portfolios will be log-normal. Hence, one is tempted to replace the true random variable portfolio return, $Z[w;T]$, with its associated log-normal random variable when T is large, prior to the maximization of the expected value of the particular isoelastic utility function. This done, Hakansson [17 p.871] is able to use the property that maximization of expected W^{γ}/γ under the log normal distribution reduces to a simple linear trade off relationship between the expected log of return and the variance of log return with γ being a measure of the investor's risk-return trade off. More generally, if a portfolio is known to have a lognormal distribution with parameters $[u,\sigma^2] \equiv [E \log(W_1/W_0), \text{Var} \log(W_1/W_0)]$, then for all utilities, it will be optimal to have a maximum of the first parameter for any fixed value of the second. While it is not true that for a fixed value of μ, all concave utility maximizers would necessarily prefer the minimum σ^2, it is true that for a fixed value of $\alpha \equiv \log[E(W_1/W_0)] = \mu + \frac{1}{2}\sigma^2$, all concave utility maximizers would optimally choose the minimum σ^2 [14]. Hence, the Hakansson derivation can suggest that there exists an asymptotic "efficient frontier" in either of the two related parameter spaces $[\mu,\sigma^2]$ or $[\alpha,\sigma^2]$. In the special case of the W^{γ}/γ family, the γ determines the point on that frontier where a given investor's optimal portfolio lies. One of us, independently, fell into this same trap.[15]

Unfortunately, substitution of the associated log-normal for the true distribution leads to incorrect results, as will be demonstrated by counter-example. The error in the analysis leading to this false conjecture results from an improper interchange of limits, as we now demonstrate.

For each uniform portfolio strategy $\{w\}$, define as in (2.2), the one-period portfolio return in period t by

$$Z[t;w] \equiv \sum_1^n w_i Z_i(t) = W_t[w]/W_{t-1} \qquad (4.1).$$

Given the distributional assumptions about asset returns in section II, the $Z[t;w]$ will be independently and identically distributed through time with

$$\text{Prob}\{\log(Z[t;w]) \leq x\} \underset{t}{\equiv} P_1(x;w) \qquad (4.2).$$

The T-period return on the portfolio is defined, for any $T \geq 1$, by

$$Z_T[w] \equiv \prod_{t=1}^{T} Z[t;w] = W_T[w]/W_0 \qquad (4.3)$$

with

$$\text{Prob}\{\log(Z_T[w]) \leq x\} \equiv P_T(x;w) \qquad (4.4)$$

as in (2.6) with the w's independent of time.

Define $Z_T^{\dagger}[w]$ to be a lognormally distributed random variable with parameters μT and $\sigma^2 T$ chosen such that

$$\mu T \equiv E\{\log(Z_T^{\dagger}[w]\} == E\{\log(Z_T[w])\} = TE\{\log(Z_1[w])\} \qquad (4.5)$$

$$\sigma^2 T \equiv \text{Var}\{\log(Z_T^{\dagger}[w])\} = \text{Var}\{\log(Z_T[w])\} = T\ \text{Var}\{\log(Z_1[w])\}.$$

We call $Z_T^{\dagger}[w]$ the "surrogate" lognormal to the random variable $Z_T[w]$; it is the lognormal approximation to $Z_T[w]$ fitted by equating the first two moments in the classical Pearsonian curve-fitting procedure. Note that by definition

$$\mu = \mu(w) = \int_{-\infty}^{\infty} xP_1(dx;w) \tag{4.6}$$

$$\sigma^2 = \sigma^2(w) = \int_{-\infty}^{\infty} (x-\mu)^2 P_1(dx;w).$$

Since each $\log(Z[t;w])$ in $\sum_1^T \log(Z[t;w])$ is an identically distributed, independent variate with finite variance, the Central-Limit theorem applies to give us the valid asymptotic relation[16]

<u>Central-Limit Theorem</u>. As T gets large, the normalized variable, $Y_T \equiv [\log(Z_T[w] - \mu(w)T]/\sigma(w)\sqrt{T}$, approaches the normal distribution. I.e.,

$$\lim_{T \to \infty} \text{Prob}\{Y_T \leq y\} = \lim_{T \to \infty} P_T[\sigma(w)\sqrt{T}\, y + \mu(w)T;w] \tag{4.7}$$

$$= N(y) \equiv (2\pi)^{-1/2} \int_{-\infty}^{y} e^{-\frac{1}{2}t^2}\, dt,$$

where N( ) is the cumulative distribution function for a standard normal variate.

Since $\log(Z_T[w]) = \log(W_T[w]/W_0)$, this is the valid formulation of the lognormal asymptotic approximation for the properly standardized distribution of terminal wealth.

In a more trivial sense, both $P_T(x;w)$ for $Z_T[w]$ and its surrogate, $\eta(x;\mu T,\sigma^2 T) \equiv N[(x - \mu(w)T)/\sigma(w)\sqrt{T}]$ for $Z_T^+[w]$, approach the same common limit, namely

$$\lim_{T \to \infty} P_T(x;w) = L = \lim_{T \to \infty} \eta(x;\mu T,\sigma^2 T) \tag{4.8}$$

where

$$\begin{aligned} L &= 1 \quad , \mu(w) < 0 \\ &= 0 \quad , \mu(w) > 0 \\ &= \tfrac{1}{2} \quad , \mu(w) = 0. \end{aligned} \tag{4.9}$$

But it is not the case that

$$\lim_{T \to \infty}\{[L - P_T(x;w)]/[L - \eta(x;\mu T,\sigma^2 T)]\} = 1 \tag{4.10}.$$

From the terra firma of the valid Central-Limit theorem, a false corollary tempts the unwary.

False Corollary. If the returns on assets satisfy the distributional assumptions of section II, then, as $T \to \infty$, the optimal solution to

$$\underset{\{w\}}{\text{Max}}\ E[U_T(W_T[w])] = \underset{\{w\}}{\text{Max}} \int_{-\infty}^{\infty} U_T[W_0 e^x] P_T(dx;w)$$

over all[17/] uniform strategies $\{w\}$, will be the same as the optimal solution to

$$\underset{\{w\}}{\text{Max}}\ E[U_T(W_0 Z_T^{\dagger}[w])] = \underset{\{w\}}{\text{Max}} \int_{-\infty}^{\infty} U_T[W_0 e^x] \eta'(x;\mu T,\sigma^2 T)\,dx$$

where $\eta'(x;\mu T,\sigma^2 T) = (2\pi\sigma^2 T)^{-1/2}\exp[-(x - \mu T)^2/2\sigma^2 T] = N'[(x - \mu(w)T)/\sigma(w)\sqrt{T}]/\sigma(w)\sqrt{T}$. I.e., one can allegedly find the optimal portfolio policy for large T, by "replacing" $Z_T[w]$ by its surrogate log-normal variate, $Z_T^{\dagger}[w]$.

Let us sketch the usual heuristic arguments that purport to deduce the false corollary. From (4.7) and (4.8), one is tempted to reason heuristically, that since Y_T is approximately distributed standard normal, then $\log(Z_T[w]) = \sigma\sqrt{T}Y_T + \mu T \equiv X_T$ is approximately normally distributed[18/] with mean μT and variance $\sigma^2 T$, for large T. Hence, if X_T has a density function $P_T'(x;w)$, then, for large T, one can write the approximation

$$P_T'(x;w) \approx \eta'(x;\mu T,\sigma^2 T) \tag{4.11}$$

and substitute the right-hand expression for the left-hand expression whenever T is large. As an example, from (4.11), one is led to the strong (and false!) conclusion that, for large T,

$$\int_{-\infty}^{\infty} U(W_0 e^x) P_T'(x;w)\,dx \approx \int_{-\infty}^{\infty} U(W_0 e^x)\eta'(x;\mu T,\sigma^2 T)\,dx \tag{4.12}$$

in the sense that

$$\lim_{T\to\infty} [\int_{-\infty}^{\infty} U(W_0 e^x) P_T'(x;w)dx - \int_{-\infty}^{\infty} U(W_0 e^x)\eta'(x;\mu T,\sigma^2 T)dx] \tag{4.13}$$

$$= \int_{-\infty}^{\infty} U(W_0 e^x) \lim_{T\to\infty} [P_T'(x;w) - \eta'(x;\mu T,\sigma^2 T)]dx = 0.$$

But, actually (4.13) is quite false as careful analysis of the Central-Limit theorem will show.

A correct analysis immediately shows that the heuristic argument leading to (4.12) - (4.13) involves an incorrect limit interchange. From (4.7), for each T, the random variable Y_T will have a probability function $F_T(y;w) = P_T(\sigma(w)\sqrt{T}y + \mu(w)T;w)$. By definition, we have that

$$\overline{U}_T = \int_{-\infty}^{\infty} U(W_0 e^x) P_T(dx;w) = \int_{-\infty}^{\infty} U(W_0 e^{\sigma\sqrt{T}y + \mu T}) F_T(dy;w) \tag{4.14}$$

and, for the surrogate-function calculation,

$$\int_{-\infty}^{\infty} U(W_0 e^x)\eta'(x;\mu T,\sigma^2 T)dx = \int_{-\infty}^{\infty} U(W_0 e^{\sigma\sqrt{T}y + \mu T}) N'(y)dy \tag{4.15}.$$

Further, from the Central-Limit theorem, we also have that, in the case where $P_T(\ ;w)$ and $F_T(\ ;w)$ have densities, $\partial P_T/\partial x = P_T'(x;w)$ and $\partial F_T/\partial y = F_T'(y;w)$,

$$\lim_{T\to\infty} F_T'(y;w) = N'(y) \tag{4.16}.$$

However, to derive (4.13) from (4.14) - (4.16), the following limit interchange would have to be valid for each y,

$$\lim_{T\to\infty} [U(W_0 e^{\sigma\sqrt{T}y + \mu T}) F_T'(y;w)] = [\lim_{T\to\infty} U(W_0 e^{\sigma\sqrt{T}y + \mu T})][\lim_{T\to\infty} F_T'(y;w)] \tag{4.17}$$

In general, as is seen from easy counter-examples, such an interchange of limits will be illegitimate, and hence, the False Corollary is invalid. In those cases where the limit interchange in (4.17) is valid (e.g., U is a bounded function), the False Corollary holds only in the trivial sense of (3.4) in section III. I.e., as already noted, in the limit as $T \to \infty$, there will exist an infinite number of portfolio

programs (including holding one hundred percent of the portfolio in the positive-yielding, riskless asset) which will give expected utility levels equal to the upperbound of U. As we now show by counter-example, it is not true that portfolio proportions, w^*, chosen to maximize expected utility over the $P_T(\ ;w)$ distribution will be equal to the proportions, w^+, chosen to maximize expected utility over the surrogate $\eta(\ ;\mu T,\sigma^2 T)$, even in the limit.

To demonstrate our counter-example to the False Corollary, first, note that for the isoelastic family, the expected utility level for the surrogate lognormal can be written as

$$E(W_0 Z_T^+[w])^\gamma/\gamma = W_0{}^\gamma \exp[\gamma\mu(w)T + \tfrac{1}{2}\gamma^2\sigma^2(w)T]/\gamma \qquad (4.18).$$

As Hakansson [17] has shown, maximization of (4.18) is equivalent to the maximization of

$$[\mu(w) + \tfrac{1}{2}\gamma\sigma^2(w)] \qquad (4.19)$$

Hence, from (4.19), the maximizing w^+ for (4.18) depend only on the mean and variance of the logarithm of one-period returns.

Second, note that because the portfolio returns for each period are independently and identically distributed, we can write the expected utility level for the true distribution as

$$E(W_0 Z_T[w])^\gamma/\gamma = W_0{}^\gamma\{E(Z_1[w]^\gamma)\}^T/\gamma \qquad (4.20)$$

Hence, maximization of (4.20) is equivalent to the maximization of

$$E(Z_1[w]^\gamma)/\gamma \qquad (4.21)$$

which also depends only on the one-period returns.[19/]

Consider the simple two-asset case where $Z[t;w] \equiv w(y_t - R) + R$ and where the y_t are independent, Bernoulli-distributed random variables with $\text{Prob}\{y_t = \lambda\} = \text{Prob}\{y_t = \delta\} = 1/2$ and $\lambda > R > \delta > 0$. Substituting into (4.21),

we have that the optimal portfolio rule, w*, will solve

$$\underset{\{w\}}{\text{Max}}\ \{[w(\lambda - R) + R]^{\gamma} + [w(\delta - R) + R]^{\gamma}\}/2\gamma \quad , \tag{4.22}$$

which by the usual calculus first-order condition implies that w* will satisfy

$$0 \;=\; (\lambda - R)[w^*(\lambda - R) + R]^{\gamma-1} + (\delta - R)[w^*(\delta - R) + R]^{\gamma-1} \tag{4.23}$$

Rearranging terms in (4.23), we have that

$$[(R - \delta)/(\lambda - R)] \;=\; [(w^*(\lambda - R) + R)/(w^*(\delta - R) + R)]^{\gamma-1} \tag{4.24}$$

or

$$w^* \;=\; w^*(\gamma) \;=\; (A - 1)R/[(\lambda - R) + A(R - \delta)] \tag{4.25}$$

where

$$A \;\equiv\; [(R - \delta)/(\lambda - R)]^{\frac{1}{\gamma-1}}.$$

From (4.6), the surrogate lognormal for $Z_T[w]$ will have parameters

$$\mu \;=\; \mu(w) \;\equiv\; 1/2\{\log[w(\lambda - R) + R] + \log[w(\delta - R) + R]\} \tag{4.26}$$

and

$$\sigma^2 \;=\; \sigma^2(w) \;\equiv\; 1/2\{\log^2[w(\lambda - R) + R] + \log^2[w(\delta - R) + R]\} - \mu^2 \tag{4.27}$$

The optimal portfolio rule relative to (4.19), $w^{\dagger}$, will be the solution to

$$0 \;=\; 1/2\{\frac{\lambda-R}{w^{\dagger}(\lambda-R)+R} + \frac{\delta-R}{w^{\dagger}(\delta-R)+R}\}(1 - \frac{\gamma}{2}\log[(w^{\dagger}(\lambda - R) + R)(w^{\dagger}(\delta - R) + R)])$$

$$+ \frac{\gamma}{2}\{\log[w^{\dagger}(\lambda - R) + R](\frac{\lambda-R}{w^{\dagger}(\lambda-R)+R}) + \log[w^{\dagger}(\delta - R) + R](\frac{\delta-R}{w^{\dagger}(\delta-R)+R})\} \tag{4.28}$$

which can be rewritten as

$$0 \;=\; (\frac{\lambda-R}{R-\delta})[1 + \frac{\gamma}{2}\log(B)] - B[1 - \frac{\gamma}{2}\log(B)] \tag{4.29}$$

where $B \equiv [w^{\dagger}(\lambda - R) + R]/[w^{\dagger}(\delta - R) + R]$. Note that since both w^* and $w^{\dagger}$ are independent of T, if the False Corollary had been valid, then $w^* = w^{\dagger}$. Suppose $w^* = w^{\dagger}$. From the definitions of A and B and from (4.24), we have that $B = A$ and $(\lambda - R)/(R - \delta) = A^{1-\gamma}$. Substitute A for B in (4.29) to get

$$H(\gamma) \equiv A^{1-\gamma}[1 + \frac{\gamma}{2} \log(A)] - A[1 - \frac{\gamma}{2} \log(A)] \tag{4.30}$$

For arbitrary λ, δ, and R, $H(\gamma) = 0$ only if $\gamma = 0$ (I.e., if the original utility function is logarithmic.). Hence, $w^* \neq w^{\dagger}$ for $\gamma \neq 0$, and the False Corollary is disproved. The reason that $w^* = w^{\dagger}$ for $\gamma = 0$ has nothing to do with the lognormal approximation or size of T since in that case, (4.19) and (4.21) are identities.

This effectively dispenses with the false conjecture that for large T, the lognormal approximation provides a suitable surrogate for the true distribution of terminal wealth probabilities; and with any hope that the mean and variance of expected-average-compound-return can serve as asymptotically sufficient decision parameters.

V. A Chamber of Horrors of Improper Limits

It is worth exposing at some length the fallacy involved in replacing the true probability distribution, $P_T(x;w)$, by its surrogate log-normal or normal approximation, $\eta(x;\mu T,\sigma^2 T) = N[(x-\mu T)/\sigma\sqrt{T}]$. First as a salutary warning against the illegitimate handling of limits, note that by the definition of a probability density, $P_T'(x;w)$ or $\partial P_T(x;w)/\partial x$, where such a density exists

$$\int_{-\infty}^{\infty} P_T'(x;w)dx \underset{x}{\equiv} 1 \tag{5.1}.$$

Also, as is well known when summing independent, identically-distributed,

non-normalized variates, the probabilities spread out and

$$\lim_{T\to\infty} P_T'(x;w)dx \underset{x}{\equiv} 0 \tag{5.2}$$

Combining (5.1) and (5.2), we see the illegitimacy of interchanging limits in the following fashion:

$$1 = \lim_{T\to\infty}\int_{-\infty}^{\infty} P_T'(x;w)dx = \int_{-\infty}^{\infty} \lim_{T\to\infty} P_T'(x;w)dx = \int_{-\infty}^{\infty} 0 \cdot dx = 0. \tag{5.3}$$

Similarly, as was already indicated for a non-density discrete-probability example in the previous section, from the following true relation,

$$\lim_{T\to\infty}[P_T'(x;w) - \eta'(x;\mu T,\sigma^2 T)] = 0, \tag{5.4}$$

it is false to conclude that, for $0 < \gamma < 1$,

$$\lim_{T\to\infty}\int_{-\infty}^{\infty} e^{\gamma x}[P_T'(x;w) - \eta'(x;\mu T,\sigma^2 T)]dx = \int_{-\infty}^{\infty} e^{\gamma x}\lim_{T\to\infty}[P_T'(x;w) - \eta'(x;\mu T,\sigma^2 T)]dx$$

$$= \int_{-\infty}^{\infty} e^{\gamma x} \cdot 0\, dx = 0 \tag{5.5}$$

Hence, trying to calculate the correct $\overline{U}_T[w]$, in (2.7) even for very large T, by relying on its surrogate

$$\int_{-\infty}^{\infty} e^{\gamma x}\eta'(x;\mu T,\sigma^2 T)dx = \exp[\gamma\mu(w) + 1/2\gamma^2\sigma^2(w)]T \tag{5.6}$$

leads to the wrong portfolio rules, namely to $w^{\dagger} \neq w^{*} \equiv w^{**}(t)$.

A specific example for the density case can demonstrate the treachery involved in replacing the actual probabilities by their asymptotic surrogate.

Consider

$$P_1(x) = 0 \quad , x \leq 0 \tag{5.7}$$

$$= 1 - e^{-2x} , x > 0$$

$$P_1'(x) = 0 \quad , x \leq 0$$

$$= 2e^{-2x} \quad , x > 0$$

$$\mu = \int_0^\infty 2xe^{-2x}dx = 1/2$$

$$\sigma^2 = \int_0^\infty (x - 1/2)^2 2e^{-2x}dx = 1/4.$$

By convolution, we easily deduce that the non-negative variates satisfy

$$P_2'(x) = 2e^{-2x}2x \quad , x > 0 \tag{5.8}$$

$$\vdots$$

$$P_T'(x) = 2e^{-2x}(2x)^{T-1}/(T-1)!, x > 0,$$

which is a gamma probability law. By the Central-Limit theorem, the limit of the probability density for the normalized variate $Y_T \equiv [X_T - \mu T]/\sigma\sqrt{T}$, $f_T(y) \equiv P_T'(\sigma\sqrt{T}y + \mu T;w)\sigma\sqrt{T}$, will be

$$\lim_{T\to\infty} f_T(y) = \lim_{T\to\infty} P_T'(1/2\sqrt{T}\, y + 1/2T;w) \cdot 1/2\sqrt{T} \tag{5.9}$$

$$= \lim_{T\to\infty} \{\sqrt{T} \exp[-\sqrt{T}\, y - T](\sqrt{T}\, y + T)^{T-1}/(T-1)!\}$$

$$= (2\pi)^{-1/2}e^{-1/2y^2} = N'(y)$$

as can be verified by Stirling's approximation to the factorial for large T.

However, replacing $2e^{-2x}(2x)^{T-1}/(T-1)!$ by its normal surrogate, $\eta'(x;1/2T,\ 1/4T)$, leads to disaster when computing $E[e^{\gamma x}]$. Consider the expected value of terminal money wealth itself, $E[W_T/W_o]$, as correctly computed and incorrectly computed by use of its log-normal surrogate: their discrepancy goes to infinity, not to zero, as T goes to infinity!

Thus, although

$$\int_0^\infty e^x \lim_{T\to\infty} [2e^{-2x}(2x)^{T-1}/(T-1)! - \eta'(x;1/2T,1/4T)]\ dx = 0, \quad (5.10)$$

it is still the case that

$$\lim_{T\to\infty} \int_0^\infty e^x\{[2e^{-2x}(2x)^{T-1}/(T-1)!] - \eta'(x;1/2T,1/4T)\}\ dx \quad (5.11)$$

$$= \lim_{T\to\infty} [(2)^T - e^{T(1/2 + 1/2 \cdot 1/4)}]$$

$$= \lim_{T\to\infty} [e^{T\log 2} - e^{5T/8}] = \infty$$

since $\log 2 \doteq .6931 > .625$.

Thus, we hope that the fallacy of the surrogate lognormal approximation with respect to optimal portfolio selection has been laid to rest.

In concluding this debunking of improper lognormal approximations, we should mention that this same fallacy pops up with monotonous regularity in all branches of stochastic investment analysis. Thus, one of us, Samuelson [39], had to warn of its incorrect use in rational warrant pricing.

Suppose a common stock's future price compared to its present price, $V(t + T)/ V(t) \equiv Z[T]$, is distributed like the product of T independent, uniform probabilities P[Z]. For T large, there is a lognormal surrogate for

$$\text{Prob}\{\log(Z[T]) \leq x\} \equiv \Pi_T(x) \tag{5.12}$$

Let the "rational price of the warrant" be given, as in the cited 1965 paper, by

$$F_T[V] = e^{-\alpha T} \int_{-\infty}^{\infty} \text{Max}[0, Ve^x - C]\Pi_T'(x)dx, \tag{5.13}$$

where C is the warrant's exercise price and $e^{\alpha} \equiv E\{V(t+1)/V(t)\}$. Although the density of the normalized variate $y \equiv (\log(Z[T]) - \mu T)/\sigma\sqrt{T}$ approaches $N'(y)$, it is false to think that, even for large T,

$$e^{\alpha T} \doteq \int_{-\infty}^{\infty} e^x \eta'(x;\mu T,\sigma^2 T)\, dx = \exp\{[\mu + 1/2\sigma^2]T\} \tag{5.14}.$$

Nor, for finite T, will we in other than a trivial sense, observe the identity

$$\begin{aligned} F_T[V] &\equiv F_T^{+}[V] \\ &= \exp\{[-\mu - 1/2\sigma^2]T\} \int_{-\infty}^{\infty} \text{Max}[0, Ve^x - C]\eta'(x;\mu T,\sigma^2 T)\, dx \end{aligned} \tag{5.15}$$

It is trivially true that $\lim_{T\to\infty} \{F_T[V] - F_T^{+}[V]\} = V - V = 0$, but untrue that $\lim_{T\to\infty} \{(V - F_T[V])/(V - F_T^{+}[V]\} = 1$. Of course, if $\Pi_1(x)$ is gaussian to begin with, as in infinitely-divisible continuous-time probabilities, there is no need for an asymptotic approximation and no surrogate concept is involved to betray one into error.

Similar remarks could be made about treacherous lognormal surrogates misapplied to the alternative 1969 warrant pricing theory of Samuelson and Merton.[20/]

To bring this chamber of horrors concerning false limits to an end, we must stress that limiting inequalities can be as treacherous as limiting equalities. Thus, consider two alternative strategies that produce alterna-

tive random variables of utility (or of money), written as U_I and U_{II} respectively and satisfying probability distributions $P_I(U;T)$ and $P_{II}(U;T)$. Too often the false inference is made that

$$\lim_{T\to\infty} \text{Prob}\{U_I > U_{II}\} = 1 \tag{5.16}$$

implies (or is implied by) the condition

$$\lim_{T\to\infty}\{E[U_I] - E[U_{II}]\} > 0 \tag{5.17}.$$

Indeed, as was shown in our refuting the fallacy of max-expected-log, yielding U_I rather than the correct optimal U_{II} that comes from use of $w^* \neq w^{\dagger\dagger}$, it is the case that (5.16) is satisfied and yet it is also true that

$$E[U_I - U_{II}] < 0 \quad \text{for all T however large} \tag{5.18}.$$

Or consider another property of the $w^{\dagger\dagger}$ strategy: Namely,

$$\text{Prob}\ \{Z_T[w^{\dagger\dagger}] \leq x\} \leq \text{Prob}\{Z_T[w] \leq x\} \text{ for all } x < M(T;w) \tag{5.19}$$

where $M(T;w)$ is an increasing function of T with $\lim_{T\to\infty} M(T;\ w) = \infty$. Yet (5.19) does not imply (5.17) nor does it imply asymptotic First Order Stochastic Dominance. I.e., it does not follow from (5.19) that, as $T \to \infty$,

$$\text{Prob}\{Z_T[w^{\dagger\dagger}] \leq x\} \leq \text{Prob}\{Z_T[w] \leq x\} \quad \text{for } \underline{\text{all}}\ x \text{ independently of T} \tag{5.20}.$$

The moral is this: Never confuse exact limits involving normalized variables with their naive formal extrapolations. Although $a = b$ and $b = c$ implies $a = c$; still $a \approx b$ and $b \approx c$ cannot reliably imply $a \approx c$ without careful restrictions put on the interpretation of "$\approx$."

VI. A Digression on Asymptotic Constant-Relative-Risk-Aversion

If one knows that the relevant utility function belongs to the $U_T(W_T) = (W_T)^\gamma/\gamma$ family, some sweeping simplifications of portfolio analysis are possible without regard to the log-normal simplifications. Hence, much interest resides in recent work by Leland [24], suggesting that for large T, the maximizers of a broad class of utility functions will asymptotically behave as if they had isoelastic utility functions.

Figure 1a. illustrates the efficiency frontier generated by letting γ run from 1 through $-\infty$. Two risky assets and one safe asset are assumed; no log-normality properties are assumed, merely that each risky asset has zero probability of complete ruin. All optimal portfolios lie on the OA locus, which is a straight line through the origin by virtue of the Cass-Stiglitz [7] separation theorem. The point L is that chosen by the max-expected-log criterion. Any point between A and L corresponds to a positive fractional γ, as at B where $\gamma = 1/2$. Any point between O and L corresponds to a negative γ, as for a more cautious investor with $\gamma = -1$.

By contrast, in Figure 1b., the log-normal efficiency locus is drawn in the (μ,σ^2) space. If we combine with log-normality the W^γ/γ family, choice on this frontier comes where it is tangential to a contour of equal $(\mu + 1/2\sigma^2\gamma)$, with slope equal to $-\gamma/2$. Thus, (b',c',1') points of Figure 1b. correspond to $\gamma = (1/2,-1,0)$ points. Remember though that the (A,B,C) points in 1a. were generated for non-log-normal probabilities. Hence, the corresponding locus for 1b.'s log-normal probabilities, when plotted in 1a. will be the <u>new</u> locus OC'L'B'A'.

Just as this last lognormal locus is irrelevant in 1a. so in 1b. we can plot the (μ,σ^2) locus traced out for all γ along the true OA optimality

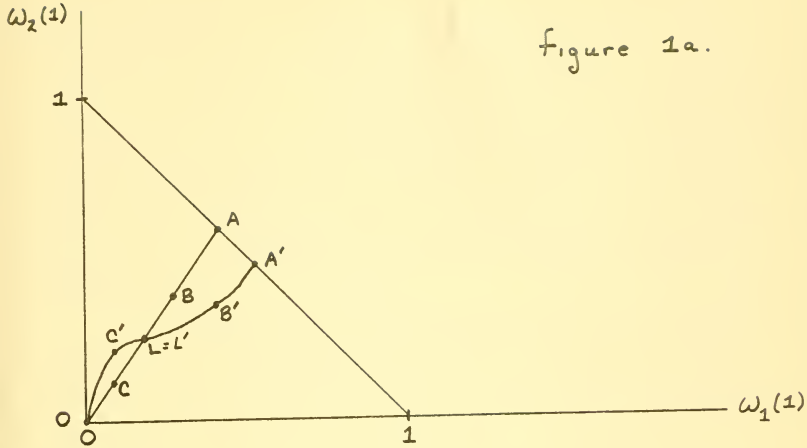


figure 1a.

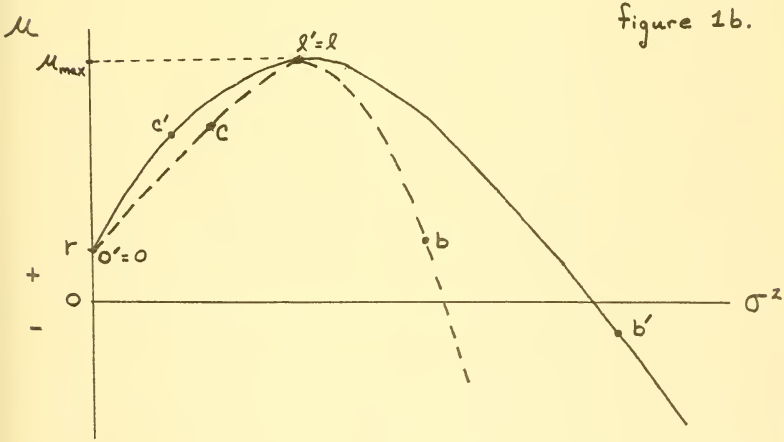


figure 1b.

portfolios. This is the broken locus oclb. Even though 1 and 1' coincide by definition of the lognormal surrogate, the two locuses would not agree elsewhere (except at a possible singular point of coincidence). And, definitely, the composite parameter $(\mu + \gamma\sigma^2/2)$ lacks relevance for choosing the best point on oclb corresponding to any given non-zero γ. (That is to say, the lognormal surrogate founders on the fact that the ratio $E(W_T^{\gamma})/\exp\{(\mu\gamma + 1/2\sigma^2\gamma^2)T\}$ does not tend to one as $T \to \infty$.) In short, Figure 1. confirms the invalidity of the lognormal surrogate (for every T however large!).

The size of T does affect the validity of Leland's purported isoelastic-utility approximation. Given a utility function that is not in the W^{γ}/γ family, the optimality locus in 1a. will generally depend on T; and for small T, it will not be a straight line. If, in a sense not to be analyzed here, $U_T(W_T)$ is "close" to $(W_T)^{\gamma}/\gamma$ for W_T very large (e.g., $U(W) = (W + \eta)^{\gamma}/\gamma$), then Leland is able to prove that the optimality locus -- call it $w_2(1) = f_T\{w_1(1)\}$ -- will approach the OA ray as $T \to \infty$: i.e., $\lim_{T\to\infty} f_T\{x\} \equiv ax$. (Warning: these diagrams are meant to be only indicative of a typical probability case. Whether some of the loci can have numerous inflection points and intersections is a question we do not attempt to analyze here.)

VII. Continuous-time Portfolio Selection

The enormous interest in the optimal portfolio selection problem for investors with distant time horizons was generated by the hope that an intertemporal portfolio theory could be developed which, at least asymptotically, would have the simplicity and richness of the static mean-variance theory without the well known objections to that model. While the analysis of previous sections demonstrates that such a theory does not validly obtain, we now derive an asymptotic theory involving limits of a different type which will produce the conjectured results.

We denote by T the time horizon of the investor. Now, we denote by N the number of portfolio revisions over that horizon, so that $h \equiv T/N$ will be the length of time between portfolio revisions. In the previous sections, we examined the asumptotic portfolio behavior as T and N tended to infinity, for a fixed h(=1). In this section, we consider the asymptotic portfolio behavior for a fixed T, as N tends to infinity and h tends to zero. The limiting behavior is interpreted as that of an investor who can continuously revise his portfolio. We leave for later the discussion of how realistic is this assumption and other assumptions as a description of actual asset market conditions, and for the moment, proceed formally to see what results obtain.

While the assumptions of previous sections about the distributions of asset returns are kept, we make explicit their dependence on the trading interval length, h. I.e., the per-dollar return on the jth asset between times $t-\tau$ and t when the trading interval is of length h, is $Z_j(t,\tau;h)$, where τ is integral in h.

Suppose, as is natural once time is continuous, that the individual $Z_j(\ )$ are distributed log-normally with constant parameters. I.e.,

$$\text{Prob}\{\log[Z_j(t,\tau;h)] \leq x\} = N(x;\mu_j\tau,\sigma_j^2\tau) \tag{7.1}$$

where

$$E\{\log[Z_j(t,\tau;h)]\} = \mu_j\tau \tag{7.2}$$

and

$$\text{Var}\{\log[Z_j(t,\tau;h)]\} = \sigma_j^2\tau,$$

with μ_j and σ_j^2 independent of h. Further, define

$$\alpha_j\tau \equiv (\mu_j + 1/2\sigma_j^2)\tau \tag{7.3}$$

$$= \log[E\{Z_j(t,\tau;h)\}]$$

and

$$\sigma_{ij}\tau \equiv \mathrm{Cov}\{\log[Z_i(t,\tau;h)],\log[Z_j(t,\tau;h)]\} \tag{7.4}$$

As in (3.1), the nth asset is riskless with $\alpha_n = r$ and $\sigma_n^2 = 0$.

We denote by $[w^*(t;h)]$ the vector of optimal portfolio proportions at time t when the trading interval is of length h, and its limit as $h \to 0$ by $w^*(t;0)$. Then the following separation (or "mutual fund") theorem obtains:

> Theorem VII.1 Given n assets whose returns are log-normally distributed and given continuous-trading opportunities (i.e. h=0), then the $\{w^*(t;0)\}$ are such that: (1) There exists a unique (up to a non-singular transformation) pair of "mutual funds" constructed from linear combinations of these assets such that independent of preferences, wealth distribution, or time horizon, investors will be indifferent between choosing from a linear combination of these two funds or a linear combination of the original n assets. (2) If Z_f is the return on either fund, then Z_f is log-normally distributed. (3) The fractional proportions of respective assets contained in either fund are solely a function of the α_i and σ_{ij},(i, j=1,2, . . .,n) an "efficiency" condition.

Proof of the theorem can be found in Merton [30, p. 384-6]. It obtains whether one of the assets is riskless or not. From the theorem, we can always work here with just two assets: the riskless asset and a (composite) risky asset which is log-normally distributed with parameters (α,σ^2).[21/] Hence, we can reduce the vector $[w^*(t;0)]$ to a scalar $w^*(t)$ equal to the fraction invested in the risky asset. If we denote by $\alpha^*(\equiv w^*(t)(\alpha-r) + r)$ and σ_*^2 $(\equiv w^{*2}\sigma^2)$, the (instantaneous) mean gain and variance of a given investor's optimal portfolio, an efficiency frontier in terms of (α^*,σ_*) or

(μ^*,σ_*) can be traced out as shown in either figures 2a or 2b. The (α^*,σ_*)-frontier is exactly akin to the classical Markowitz single-period frontier where a "period" is an instant. Although the frontier is the same in every period, a given investor will in general choose a different point on the frontier each period depending on his current wealth and T - t.

In the special case of iso-elastic utility,[22] $U_T = W^\gamma/\gamma$, $\gamma<1$, $w^*(t) = w^*$, a constant, and the entire portfolio selection problem can be presented graphically as in figure 3. Further, for this special class of utility functions, the distribution of wealth under the optimal policy will be lognormal for all t.[23]

Hence, from the assumptions of log-normality and continuous trading, we have, even for T finite and not at all large, a complete asymptotic theory with all the simplicity of classical mean-variance, but without its objectionable assumptions. Further, these results still obtain even if one allows intermediate consumption evaluated at some concave utility function, V(c).[24]

Having derived the theory, we now turn to the question of the reasonableness of the assumptions. Since trading continuously is not a reality, the answer will depend on "how close" $w^*(t;0)$ is to $w^*(t;h)$. I.e., for every $\delta>0$, does there exist an $h>0$, such that $||w^*(t;0) - w^*(t;h)|| < \delta$ for some norm $|| \ ||$, and what is the nature of the $\delta = \delta(h)$ function? Further, since lognormality as the distribution for returns is not a "known fact," what are the conditions such that one can validly use the lognormal as a surrogate?

Since the answer to both questions will turn on the distributional assumptions for the returns, we now drop the assumption of lognormality for the $Z_j(\ ;h)$, but retain the assumptions (maintained throughout the

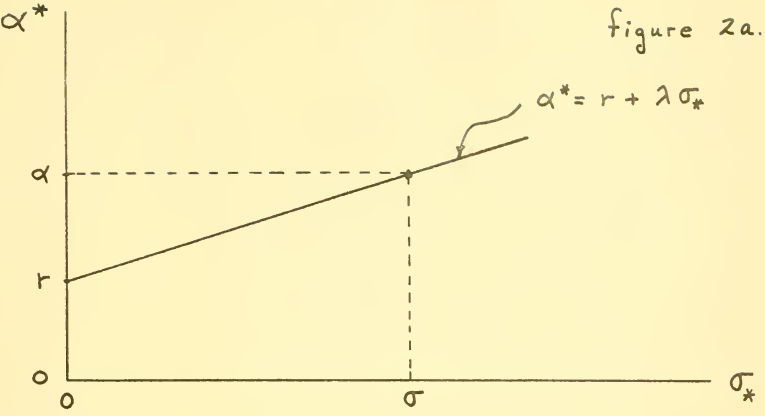


figure 2a.

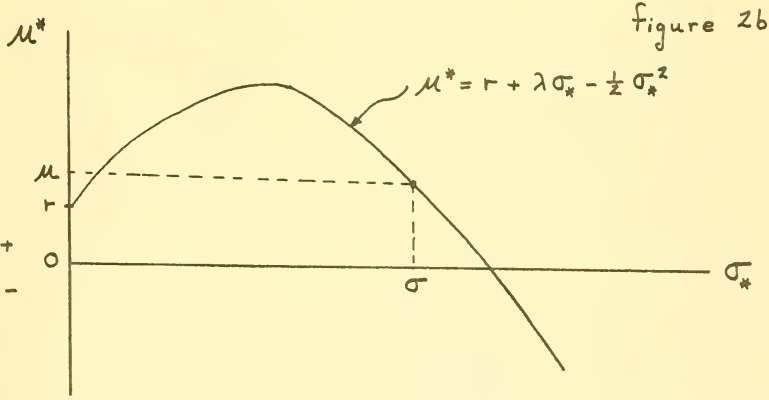
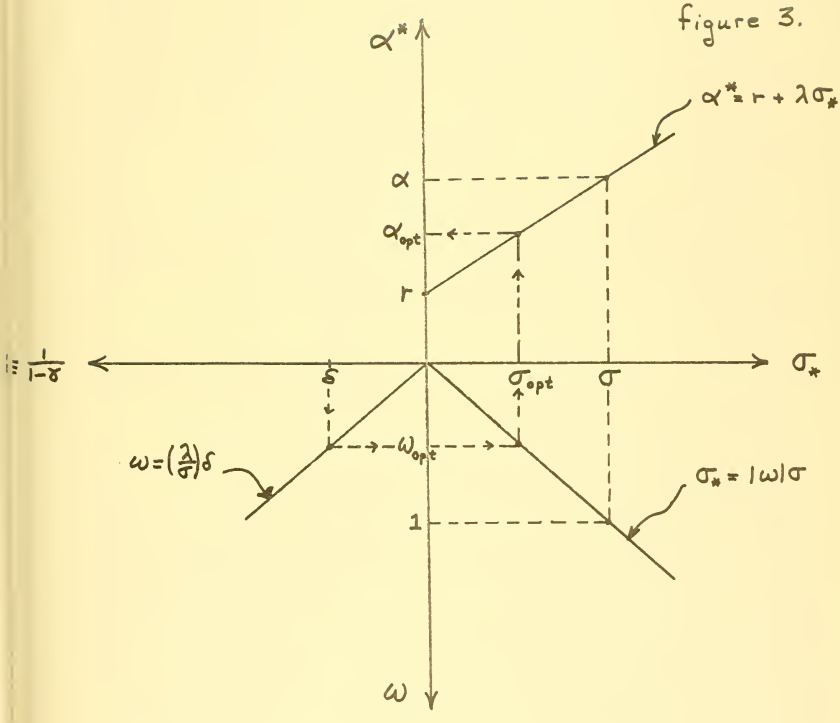


figure 2b.

figure 3.


α^*
$\alpha^* = r + \lambda\sigma_*$
α
α_{opt}
r
$\frac{1}{1-\gamma}$
δ
σ_{opt}
σ
σ_*
$\omega = \left(\frac{\lambda}{\sigma}\right)\delta$
$-\omega_{opt}$
$\sigma_* = |\omega|\sigma$
1
ω

paper) that, for a <u>given h</u>;[25] the one-period returns, $Z_j(\ ,h;h)$ have joint distributions identical through time, and the vector $[Z(t,h;h)] \equiv [Z_1(t,h;h), Z_2(t,h;h), \ldots, Z_n(t,h;h)]$ is distributed independently of $[Z(t+s,h;h)]$ for $s>h$.

By definition, the return on the jth security over a time period of length T will be

$$Z_j(t,T;h) \equiv \prod_{k=1}^{T} Z_j(kh,h;h) \tag{7.5}$$

Let $X_j(k,h) \equiv \log[Z_j(kh,h;h)]$. Then, for a given h, the $\{X_j(k,h)\}$ are independently and identically distributed with non-central moments

$$m_j(i;h) \underset{k}{\equiv} E\{[X_j(k,h)]^i\}, \quad i = 0, 1, 2, \ldots \tag{7.6}$$

and moment-generating function

$$\psi_j(\lambda;h) \underset{k}{\equiv} E\{\exp[\lambda X_j(k,h)]\}$$

$$= E\{[Z_j(kh,h;h)]^\lambda\} .$$

Define the non-central moments of the <u>rate of return</u> per period, $Z_j(kh,h;h) - 1$, by

$$M_j(1;h) = \alpha_j(h)h \underset{k}{\equiv} E\{Z_j(kh,h;h) - 1\} \tag{7.8}$$

$$M_j(2;h) = v_j^2(h)h + (\alpha_j(h)h)^2 \underset{k}{\equiv} E\{[Z_j(kh,h;h) - 1]^2\}$$

$$M_j(i;h) \underset{k}{\equiv} E\{[Z_j(kh,h;h) - 1]^i\}, \quad i = 3, 4, \ldots,$$

where $\alpha_j(h)$ is the expected rate of return per unit time and $v_j^2(h)$ is the variance of the rate of return per unit time.

The second author [43] has demonstrated that if the moments $M_j(\ ;h)$ satisfy

$$M_j(1;h) = 0(h) \tag{7.9}$$

$$M_j(2;h) = 0(h)$$

$$M_j(k;h) = o(h) \quad , \quad k > 2,$$

then $||w^*(t;0) - w^*(t;h)|| = 0(h)$ where "0" and "o" are defined by

$$g(h) = 0(h) \text{ if } (g/h) \text{ is bounded for all } h \geq 0 \tag{7.10}$$

and

$$g(h) = o(h) \text{ if } \lim_{h\to 0}(g/h) = 0 \tag{7.11}$$

Thus, if the distributions of returns satisfy (7.9), then, for every $\delta > 0$, there exists an $h > 0$ such that $||w^*(t;o) - w^*(t;h)|| < \delta$, and the continuous-time solution will be a valid asymptotic solution to the discrete-interval case. Note that if (7.9) is satisfied, then $\alpha_j(h) = 0(1)$ and $v_j^2(h) = 0(1)$, and $\alpha_j \equiv \lim_{h\to o}\alpha_j(h)$ will be finite as will $v_j^2 \equiv \lim_{h\to o} v_j^2(h)$.

Given that $M_j(\ ;h)$ satisfies (7.9), we can derive a similar relationship for $m_j(\ ;h)$. Namely, by Taylor series,[26]

$$m_j(1;h) = E\{\log(1 + [Z_j - 1])\} \tag{7.12}$$

$$= E\{\sum_{i=1}^{\infty} (-1)^{i-1}[Z_j - 1]^i/i\}$$

$$= M_j(1;h) - 1/2M_j(2;h) + o(h), \text{ from (7.9)},$$

$$= 0(h),$$

and

$$m_j(2;h) = E\{\log^2(1 + [Z_j - 1])\} \tag{7.13}$$

$$= E\{\sum_{p=1}^{\infty} \sum_{i=1}^{\infty} (-1)^{i+p}[Z_j - 1]^{i+p}/ip\}$$

$$= M_j(2;h) + o(h)$$

$$= O(h),$$

and, in a similar fashion,

$$m_j(k;h) = o(h) \quad \text{for } k > 2 \tag{7.14}.$$

Thus, the $m_j(\ ;h)$ satisfy (7.9) as well. If we define $\mu_j(h) \equiv m_j(1;h)/h$ as the mean logarithmic return per unit time and $\sigma_j^2(h) \equiv [m_j(2;h) - \mu_j^2(h)h^2]/h$ as the variance of the log return per unit time, then from (7.8), (7.12), and (7.13), we have that

$$\alpha_j \equiv \lim_{h\to 0}\alpha_j(h) = \lim_{h\to 0}[\mu_j(h) + 1/2\sigma_j^2(h)] = \mu_j + 1/2\sigma_j^2 \tag{7.15}$$

and

$$v_j^2 \equiv \lim_{h\to 0} v_j^2(h) = \lim_{h\to 0}\sigma_j^2(h) = \sigma_j^2 \tag{7.16}.$$

(7.15) demonstrates that the true distribution will satisfy asymptotically the exact relationship satisfied for all h by the log-normal surrogate; from (7.16), the variance of the arithmetic return will equal the variance of the logarithmic return in the limit. Hence, these important moment relationships match up exactly between the true distribution and its log-normal surrogate.

If we define $Y_j(T;h) \equiv \log[Z_j(T,T;h)]$, then from (7.5), we have that

$$Y_j(T;h) \equiv \sum_{k=1}^{N} X_j(k,h) \text{ where } N \equiv T/h, \tag{7.17}$$

and from the independence and identical distribution of the $X_j(\ ;h)$, the moment-generating function of Y_j will satisfy

$$\begin{aligned} \emptyset_j(\lambda;h,T) &\equiv E\{\exp[\lambda Y_j(T;h)]\} \\ &= [\psi_j(\lambda;h)]^{T/h} \end{aligned} \tag{7.18}$$

Taking logs of both sides of (7.18) and using Taylor series, we have that

$$\begin{aligned} \log[\emptyset_j(\lambda;h,T)] &= (T/h)\log[\psi_j(\lambda;h)] \\ &= (T/h)\log[\sum_{k=0}^{\infty} \psi_j^{(k)}(0;h)\lambda^k/k!] \\ &= (T/h)\log[\sum_{k=0}^{\infty} m_j(k;h)\lambda^k/k!] \\ &= (T/h)\log[1 + m_j(1;h)\lambda + 1/2m_j(2;h)\lambda^2 + o(h)] \\ &= T[\lambda(m_j(1;h)/h) + \frac{\lambda^2}{2}(m_j(2;h)/h) + 0(h)]. \end{aligned} \tag{7.19}$$

Substituting $\mu_j(h)h$ for $m_j(1;h)$ and $\sigma_j^2(h)h + \mu_j^2(h)h^2$ for $m_j(2;h)$ and taking the limit as $h \to o$ in (7.19), we have that

$$\begin{aligned} \log[\emptyset_j(\lambda;0,T)] &= \lim_{h\to 0} \log[\emptyset(\lambda;h,T)] \\ &= \lambda\mu_j T + 1/2\sigma_j^2 T\lambda^2, \end{aligned} \tag{7.20}$$

and therefore, $\emptyset_j(\lambda;0,T)$ is the moment-generating function for a normally-distributed random variable with mean $\mu_j T$ and variance $\sigma_j^2 T$.

Thus, in what is essentially a valid application of the Central-Limit theorem, we have shown that the limit distribution for $Y_j(T,T;h)$ as h tends to zero, is under the posited assumptions gaussian, and hence, the limit distribution for $Z_j(T,T;h)$ will be log-normal for <u>all finite</u> T. Further from (7.15) - (7.16), the surrogate log-normal, fitted in the Pearsonian fashion of earlier sections, will, for smaller and smaller h, be in the limit, the true limit distribution for $Z_j(T,T;0)$.

It is straightforward to show that if the distribution for each of the $Z_j(\ ,h;h)$, $j = 1, 2, \ldots, n$, satisfy (7.9), then for bounded $w_j(t;h)$, the "single-period" <u>portfolio</u> returns, $Z[t;w(t),h]$, will for each t satisfy (7.9). However, unless the portfolio weights are constant through time (i.e., $w_j(t) \equiv w_j$), the resulting limit distribution for the portfolio over finite time, will not be log-normal.

How reasonable is it to assume that (7.9) will be satisfied? Essentially (7.9) is a set of sufficient conditions for the limiting continuous-time stochastic process to have a continuous sample path (with probability one). It is closely related to the "local markov property" of discrete-time stochastic processes which allows only movements to neighboring states in one period (e.g., the simple random walk).

A somewhat weaker sufficient condition (implied by [7.9]) [11, p.321] is that for every $\delta > 0$,

$$\text{Prob}\{-\delta \leq X_j(\ ;h) \leq \delta\} = 1 - o(h) \qquad (7.21),$$

which clearly rules out "jump-type"processes such as the Poisson. It is easy to show for the Poisson that (7.9) is not satisfied because $M_j(k;h) = O(h)$ for all k and similarly, (7.21) is not satisfied since $\text{Prob}\{-\delta \leq X_j(\ ;h) \leq \delta\} = 1 - O(h)$.

In the general case when (7.21) is satisfied but the distribution of returns are not completely independent nor identically distributed, the limit distribution will not be log-normal, but will be generated by a diffusion process. Although certain quadratic simplifications still occur, the strong theorems of the earlier part of this section will no longer obtain.

The accuracy of the continuous solution will depend on whether, for reasonable trading intervals, compact distributions are an accurate representation for asset returns and whether, for these intervals, the distributions can be taken to be independent. Examination of time series for common stock returns shows that skewness and higher-order moments tend to be negligible relative to the first two moments for daily or weekly observations, which is consistent with (7.9)'s assumptions. However, daily data tend also to show some negative serial correlation, significant for about two weeks. While this finding is inconsistent with independence, the size of the correlation coefficient is not large and the short-duration of the correlation suggests a "high-speed of adjustment" in the auto-correlation function. Hence, while we could modify the continuous analysis to include an Ornstein-Uhlenbeck type process to capture these effects,[27/] the results may not differ much from the standard model when empirical estimates of the correlation are plugged in.

VIII. Summary

In addition to demonstrating the fallacy, based on the Law of Large Numbers, that maximizing the expected log of return serves asymptotically to maximize expected terminal utility (even in the case of bounded utility), we have demonstrated the subtler fallacy, based on the Central-Limit theorem, that a Log-Normal Surrogate can be validly used and hence only the expected value and variance of the log of one-period returns need, for large enough T, be used to get a close approximation to the optimal expected-utility portfolio choice. Finally, in connection with limitingly often subdividing of time into more and more trading intervals, we have shown that Surrogate Log-Normal distributions are asymptotically valid, provided the key assumption concerning compactness of independent probabilities does hold with its implied log-normal continuous-time Wiener process.

FOOTNOTES

* We thank M. B. Goldman for scientific assistance. Aid from the National Science Foundation is gratefully acknowledged.

1. Notable exceptions/ that deal with the general case can be found in the works of Arrow [1], Rothschild and Stiglitz [37], and Samuelson[40]. Along with mean-variance analysis, the theory of portfolio selection when the distributions are Pareto-Levy has been developed and tested by Mandelbrot [26], Fama, [8], and Samuelson [41].

2. The literature is so extensive that we refer the reader to the bibliography. Additional references can be found in the survey articles by Fama [9] and Jensen [21] and the book by Sharpe [47].

3. See Black, Jensen, and Scholes [3] and Friend and Blume [12].

4. See Hakansson [18].

5. Cass and Stiglitz [7] showed that these utility functions were among the few which satisfied the separation property for arbitrary distributions. Other authors [15, 23, 29, 30, 42] have previously made extensive use of these functions.

6. For the first, see Samuelson [43]; for the second, see Merton [29, 30]; for the last, see Ross [35].

7. See, for example, Aucamp [2].

8. See Hakansson [16].

9. See Markowitz [27] and [28]

10. Ross [36] provides a rather simple example where the Leland result does not obtain.

11. $\Pi_{ij}(\; ; \;)$ could be used to rank portfolios because it provides a complete and transitive ordering. Note: if $\Pi_{ij}(\; ; \;) > 1$, then $\Pi_{ji}(\; ; \;) < 1$. However, $\Pi_{ij}(\; ; \;) \neq 1/\Pi_{ji}(\; ; \;)$.

12. See Goldman [13] for some examples.

13. Goldman [13] derives a similar result for a bounded utility function. Thus, the second author [44, p. 2495]conceded too much in his criticism of the geometric mean policy when he stated that such a policy would asymptotically outperform any other uniform policy for utility functions bounded from above and below.

14. See Merton [33 , appendix B] for a proof.

15. The neatness of the heuristic arguments and the attractiveness of the results has led a number of authors to assert and/or conjecture their truth. See Samuelson [44 , p.2496]where $\mu(w)$ and $\sigma^2(w)$ are misleadingly said to be "asymptotically sufficient parameters," an assertion not correct for $T \to \infty$, but rather for increasing subdivisions of time periods leading to diffusion-type stochastic processes.

16. To apply the Central-Limit theorem correctly, the choice for $\{w\}$ must be such that $Z[t;w]$ is a positive random variate with its logarithm well defined. Our later discussion of optimal policies is unaffected by this restriction since the class of utility functions considered will rule out $Z[t;w]$ with a positive probability of ruin. Cf. Hakansson [17 , p.868].

17. With reference to footnote 16, the corollary is false even if we restrict the set of uniform strategies considered to those such that $\text{Prob}\{Z_T[w] = 0\} = 0$ for all finite T.

18. The argument is that a constant times a normal variate plus a constant, is a normal variate.

19. This is an important point because it implies that if the conjecture that $\{w^\dagger\} = \{w^*\}$ could hold for the iso-elastic family for large T, then it would hold for T small or T = 1! I.e., $\mu(w)$ and $\sigma(w)$ would be sufficient parameters for the portfolio decision for any time horizon.

20. [45]. Since Black-Scholes [4] warrant pricing is based squarely on exact log-normal definitions, it is inexact for non-log-normal surrogate reasoning. See Merton [33] for further discussion.

21. See Merton [30 , p. 388] for an explicit expression for α and σ^2 as a function of the α_i and σ_{ij}, i, j = 1, 2, . . ., n.

22. See Merton [29 , p.251] and [30, p. 388-394].

23. See Merton [30 , p. 392].

24. Further, the log-normality assumption can be weakened and still some separation and efficiency conditions will obtain. See Merton [32].

25. It is not required that the distribution of one-period returns with trading interval of length h_1 will be in the same family of distributions as the one-period returns with trading interval of length $h_2(\neq h_1)$. I.e., the distributions need not be infinitely divisible in time. However, we do require sufficient regularity that the distribution of $X_j(T;h)$ is in the domain of attraction of the normal distribution, and hence, the limit distribution as $h \to 0$ will be infinitely divisible in time.

26. The series expansion is only valid for $0 < Z \leq 2$. However, as shown later in (7.21), for small enough h, Z will be in this range with probability one.

27. Cf. Merton [30 , p.401].

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